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On measure rigidity of unipotent subgroups of semisimple groups

by

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1. Introduction

This paper represents part II in our three part series on Raghunathan's measure conjecture (see [R4] for part I).

More specifically, let $\mathbf{G}$ be a real Lie group (all groups in this paper are assumed to be second countable), $\mathbf{\Gamma}$ a discrete subgroup of $\mathbf{G}$ and $\pi: \mathbf{G} \to \mathbf{\Gamma}\backslash\mathbf{G}$ the projection $\pi(\mathbf{g}) = \mathbf{\Gamma g}$. The group $\mathbf{G}$ acts by right translations on $\mathbf{\Gamma}\backslash\mathbf{G}$, $(x, \mathbf{g}) \to x\mathbf{g}, x \in \mathbf{\Gamma}\backslash\mathbf{G}, \mathbf{g} \in \mathbf{G}$. Let μ be a Borel probability measure on $\mathbf{\Gamma}\backslash\mathbf{G}$. Define

$$(*) \qquad \mathbf{\Lambda}(\mu) = \mathbf{\Lambda}(\mathbf{G}, \mathbf{\Gamma}, \mu) = \{\mathbf{g} \in \mathbf{G}: \text{the action of } \mathbf{g} \text{ preserves } \mu\}.$$

The set $\mathbf{\Lambda}(\mu)$ is a closed subgroup of $\mathbf{G}$. The measure μ is called *algebraic* if there exists $\mathbf{x} = \mathbf{x}(\mu) \in \mathbf{G}$ such that $\mu(\pi(\mathbf{x})\,\mathbf{\Lambda}(\mu)) = 1$. In this case $\mathbf{x\Lambda}(\mu)\mathbf{x}^{-1} \cap \mathbf{\Gamma}$ is a lattice in $\mathbf{x\Lambda}(\mu)\,\mathbf{x}^{-1}$.

Definition 1. Let $\mathbf{U}$ be a subgroup of $\mathbf{G}$. We say that the action of $\mathbf{U}$ on $\mathbf{\Gamma}\backslash\mathbf{G}$ is *measure rigid* if every ergodic $\mathbf{U}$-invariant Borel probability measure on $\mathbf{\Gamma}\backslash\mathbf{G}$ is algebraic. The group $\mathbf{U}$ is called *measure rigid* in $\mathbf{G}$ if its action on $\mathbf{\Gamma}\backslash\mathbf{G}$ is measure rigid for every lattice $\mathbf{\Gamma} \subset \mathbf{G}$. An element $\mathbf{u} \in \mathbf{G}$ is measure rigid if the group $\{\mathbf{u}^k: k \in \mathbf{Z}\}$ is measure rigid. $\mathbf{U} \subset \mathbf{G}$ and $\mathbf{u} \in \mathbf{G}$ are called *strictly measure rigid* if their action on $\mathbf{\Gamma}\backslash\mathbf{G}$ is measure rigid for *every* discrete subgroup $\mathbf{\Gamma}$ of $\mathbf{G}$.

A subgroup $\mathbf{U}$ of $\mathbf{G}$ is called unipotent if for each $\mathbf{u} \in \mathbf{U}$ the map $\mathrm{Ad}_{\mathbf{u}}$ is a unipotent automorphism of the Lie algebra of $\mathbf{G}$.

RAGHUNATHAN'S MEASURE CONJECTURE. *Every unipotent subgroup of a connected Lie group* $\mathbf{G}$ *is measure rigid.*

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Various versions of this conjecture were stated in [D2] and [M]. It was shown in [F1] and [P] that when **G** is nilpotent the conjecture is true. Also we showed in [R4] that every unipotent subgroup of a solvable **G** is strictly measure rigid. As to semisimple **G** it was shown in [F2] and [D1] that for $\mathbf{G}=SL(2,R)$ the conjecture is true. To the best of my knowledge this is the only case of semisimple groups **G** for which the conjecture has been settled, although there has been a number of papers [B], [D2], [EP], [F2], [R3], [V] which established measure rigidity for certain unipotent subgroups of semisimple **G**.

This paper represents the first part in our proof of the Raghunathan's measure conjecture for semisimple **G**. To state our results we need to introduce some notations and definitions. Let **G** be a Lie group with the Lie algebra $\mathfrak{G}$ and let $\mathbf{g}\in\mathbf{G}$. Suppose that **G** acts on $\mathbf{\Gamma}\backslash\mathbf{G}$ with $\mathbf{\Gamma}$ being a discrete subgroup of **G**. We say that the **g**-orbit of $\pi(\mathbf{x})\in\mathbf{\Gamma}\backslash\mathbf{G}, \mathbf{x}\in\mathbf{G}$ *diverges* when $n\to\infty$ if there are $\mathbf{e}\neq\gamma_n\in\mathbf{\Gamma}, n=1,2,\ldots$ such that $(\mathbf{x}\mathbf{g}^n)^{-1}\gamma_n(\mathbf{x}\mathbf{g}^n)\to\mathbf{e}$, when $n\to\infty$. Note that if $\mathbf{\Gamma}$ is a lattice in **G** then a **g**-orbit diverges if and only if it eventually leaves every compact subset of $\mathbf{\Gamma}\backslash\mathbf{G}$. Obviously, in our definition of divergence the "if" part of this statement does not necessarily hold when $\mathbf{\Gamma}$ is not a lattice. Define

$$D(\mathbf{g}) = \{x\in\mathbf{\Gamma}\backslash\mathbf{G}\colon \text{the } \mathbf{g}\text{-orbit of } x \text{ diverges when } n\to\infty\}.$$

It is clear that if $D(\mathbf{g})\neq\varnothing$ for some $\mathbf{g}\in\mathbf{G}$ then $\mathbf{\Gamma}\backslash\mathbf{G}$ is not compact.

We shall call an element $u\in\mathfrak{G}$ nilpotent if $\mathrm{ad}_u\colon\mathfrak{G}\to\mathfrak{G}, \mathrm{ad}_u(v)=[v,u]$ is a nilpotent linear transformation of $\mathfrak{G}$. An element $g\in\mathfrak{G}$ is called R-diagonalizable if ad_g is diagonalizable over R. Also we denote by $\mathbf{Z}(\mathbf{G})$ the center of **G**.

Definition 2. Let **G** be a Lie group with the Lie algebra $\mathfrak{G}$ and $\mathbf{\Gamma}$ a discrete subgroup of **G**.

(1) A nilpotent element $u\in\mathfrak{G}$ is called *horocyclic* if there is an R-diagonalizable element $g\in\mathfrak{G}$ and a nilpotent element $u^*\in\mathfrak{G}$ such that $\mathrm{ad}_{u^*}(u)=g, \mathrm{ad}_g(u)=-2u$, $\mathrm{ad}_g(u^*)=2u^*$. In this case we say that u is "*horocyclic for* g" and "g is diagonal for u".

(2) An element $\mathbf{u}\in\mathbf{G}$ is called *horocyclic* if $\mathbf{u}=\exp u$ for some horocyclic element $u\in\mathfrak{G}$.

(3) An element $\mathbf{u}\in\mathbf{G}$ is called $\mathbf{\Gamma}$-horocyclic if $\mathbf{u}=\mathbf{z}\exp u$ for some horocyclic $u\in\mathfrak{G}$ and some $\mathbf{z}\in\mathbf{Z}(\mathbf{G})$ with $\mathbf{z}^k\in\mathbf{\Gamma}$ for some $k\in\mathbf{Z}$. An element $\mathbf{g}\in\mathbf{G}$ is diagonal for **u** if $\mathbf{g}=\exp g$ with g being diagonal for u.

Our terminology in Definition 2 is motivated by the fact that u, g and u^* generate a Lie subalgebra $sl_2(u,g)$ of $\mathfrak{G}$ isomorphic to $sl(2,R)$. It is a fact (see [J]) that if **G** is a connected semisimple Lie group then all nontrivial elements of one-parameter unipotent subgroups of $\mathbf{G}$ are horocyclic. Also (see [BM]) if $\mathbf{\Gamma}$ is a lattice in a semisimple $\mathbf{G}$ which projects densely into the maximal compact factor of $\mathbf{G}$ (such $\mathbf{\Gamma}$ is called *compatible* with $\mathbf{G}$) then every noncentral unipotent element of $\mathbf{G}$ is $\mathbf{\Gamma}$-horocyclic. Note that every $\mathbf{u}\in\mathbf{Z}(\mathbf{G})$ is strictly measure rigid by [R4, Corollary 1].

Let g be an R-diagonalizable element of $\mathfrak{G}$, $\mathbf{g}_p=\exp pg$, $\mathbf{g}=\mathbf{g}_1$, $\mathfrak{E}_\lambda$ the eigenspace of ad_g with the eigenvalue λ, $\mathfrak{E}^-(g)=\Sigma\{\mathfrak{E}_\lambda, \lambda<0\}$ and $\mathbf{E}^-=\mathbf{E}^-(\mathbf{g})=\exp\mathfrak{E}^-(g)$. It is clear that if g is diagonal for u, then so is $\mathbf{c}g\mathbf{c}^{-1}$ for every $\mathbf{c}\in\mathbf{C}(u)$—the centralizer of $\{\exp tu, t\in R\}$ in $\mathbf{G}$. In fact, an element $\bar{g}\in\mathfrak{G}$ is diagonal for u if and only if $\bar{g}=\mathbf{c}g\mathbf{c}^{-1}$ for some $\mathbf{c}\in\mathbf{E}^-(\mathbf{g})\cap\mathbf{C}(u)$ (see Proposition 1.1 and Theorem 1.1). In this case $\mathfrak{E}^-(\bar{g})=\mathfrak{E}^-(g)$. It is clear that $D(\mathbf{g})=D(\bar{\mathbf{g}})$ and $x\mathbf{E}^-(\mathbf{g})\subset D(\mathbf{g})$ whenever $x\in D(\mathbf{g})$.

We show below (Theorem 1.1) that if g is diagonal for a horocyclic element $u\in\mathfrak{G}$ then every eigenvalue of ad_g is an integer. Write $\mathfrak{E}^{-2}=\Sigma\{\mathfrak{E}_\lambda: \lambda\leq-2\}$. Clearly $u\in\mathfrak{E}_{-2}$. Now let $\mathbf{\Lambda}=\mathbf{\Lambda}(\mu)\subset\mathbf{G}$ be as in (*) and let $\mathfrak{L}(\mathbf{\Lambda})$ be the Lie algebra of $\mathbf{\Lambda}$ ($\mathfrak{L}(\mathbf{\Lambda})$ might be trivial). Define $\mathfrak{N}(\mathbf{\Lambda})=\mathfrak{L}(\mathbf{\Lambda})\cap\mathfrak{E}^{-2}$, $\mathbf{N}_\mathbf{u}(\mathbf{\Lambda})=\{\mathbf{u}^k\exp\mathfrak{N}(\mathbf{\Lambda}): k\in\mathbf{Z}\}$, where $\mathbf{u}=\mathbf{z}\exp u\in\mathbf{\Lambda}$ for some $\mathbf{z}\in\mathbf{Z}(\mathbf{G})$.

THEOREM 1 (The Main Theorem). *Let* $\mathbf{G}$ *be a Lie group and* $\mathbf{\Gamma}$ *a discrete subgroup of* $\mathbf{G}$ *(not necessarily a lattice). Let* $\mathbf{u}=\mathbf{z}\exp u$, $\mathbf{z}\in\mathbf{Z}(\mathbf{G})$, $u\in\mathfrak{G}$ *be a* $\mathbf{\Gamma}$*-horocyclic element of* $\mathbf{G}$ *and* $g\in\mathfrak{G}$ *a diagonal element for* u, $\mathbf{g}_p=\exp pg$, $\mathbf{g}_1=\mathbf{g}$. *Let* μ *be a Borel probability measure on* $\mathbf{\Gamma}\backslash\mathbf{G}$ *such that* $\mathbf{u}\in\mathbf{\Lambda}=\mathbf{\Lambda}(\mu)$ *and the action of* $\mathbf{N}_\mathbf{u}(\mathbf{\Lambda})$ *on* $(\mathbf{\Gamma}\backslash\mathbf{G},\mu)$ *is ergodic. Then either* (1) $\mu(D(\mathbf{g}))=1$ or (2) $\mathbf{c}\mathbf{g}_p\mathbf{c}^{-1}\in\mathbf{\Lambda}$ *for some* $p\in R$, $\mathbf{c}\in\mathbf{E}^-(\mathbf{g})$. *In this case* $u\in\mathfrak{L}(\mathbf{\Lambda})$, $\mathbf{c}sl_2(u,g)\mathbf{c}^{-1}\subset\mathfrak{L}(\mathbf{\Lambda})$ *and* μ *is algebraic; also* $x\mathbf{\Lambda}=\cup\{x\mathbf{\Lambda}^0\mathbf{z}^i: i=0,\dots,n\}$ *for some integer* $n\geq0$, *where* $x=\pi(\mathbf{x}(\mu))$ *and* $\mathbf{\Lambda}^0$ *denotes the connected component of* $\mathbf{\Lambda}$ *containing* $\mathbf{e}$.

In our proof of Theorem 1 we assume that $\mathbf{u}$ is horocyclic. This contains no loss of generality. Indeed, if $\mathbf{u}$ is $\mathbf{\Gamma}$-horocyclic then $(\mathbf{\Gamma}\backslash\mathbf{G},\mu)$ is composed of a finite number of ergodic components of $\mathbf{N}_{\mathbf{u}^k}$ and the action of $\mathbf{u}^k$ on each of these components coincides with the action of the horocyclic element $\exp ku$.

COROLLARY 1. *Let* $\mathbf{G}$ *be a semisimple Lie group. Then*

(1) *Theorem* 1 *holds for all nontrivial elements* $\mathbf{u}$ *of one-parameter unipotent subgroups of* $\mathbf{G}$;

(2) *if* $\mathbf{G}$ *is connected and* $\mathbf{\Gamma}$ *is a compatible lattice in* $\mathbf{G}$ *then Theorem* 1 *holds for all noncentral unipotent elements of* $\mathbf{G}$.

COROLLARY 2. *Let* $\mathbf{G}$ *be a Lie group and* $\mathbf{\Gamma}$ *a uniform lattice in* $\mathbf{G}$. *Let* $\mathbf{H}$ *be a closed subgroup of* $\mathbf{G}$ *such that* $\mathbf{H}\cap\mathbf{\Gamma}$ *is a lattice in* $\mathbf{H}$. *Suppose that* $\mathbf{H}$ *contains a* $\mathbf{\Gamma}$-*horocyclic element of* $\mathbf{G}$. *Then the Lie algebra* $\mathfrak{H}$ *of* $\mathbf{H}$ *is not trivial and* $sl_2(u,g)\subset\mathfrak{H}$ *for some horocyclic* $u\in\mathfrak{G}$ *and a diagonal* $g\in\mathfrak{G}$.

COROLLARY 3. *Let* $\mathbf{G}$ *be a Lie group and* $\mathbf{\Gamma}$ *a uniform lattice in* $\mathbf{G}$. *Then the action of every* $\mathbf{\Gamma}$-*horocyclic element of* $\mathbf{G}$ *on* $\mathbf{\Gamma}\backslash\mathbf{G}$ *is measure rigid. If, in addion,* $\mathbf{G}$ *is connected, semisimple and* $\mathbf{\Gamma}$ *is compatible with* $\mathbf{G}$ *then the action of every unipotent element of* $\mathbf{G}$ *on* $\mathbf{\Gamma}\backslash\mathbf{G}$ *is measure rigid.*

Let $\mathbf{G}=SL(2,\mathbf{R})$ and $\mathbf{\Gamma},\mu,\mathbf{u}=\exp u\in\mathbf{\Lambda}$ be as in Theorem 1. It is clear that if $\mu(D(\mathbf{g}))>0$ then $\mathbf{g}\notin\mathbf{\Lambda}$. This implies that either $\mathfrak{L}(\mathbf{\Lambda})=\{0\}$ or $\mathfrak{L}(\mathbf{\Lambda})=\{tu,t\in R\}$. This and [R4, Corollary 3] give the following generalization of [D1].

COROLLARY 4. *Let* $\mathbf{G}=SL(2,R)$ *and* $\mathbf{\Gamma},\mu,\mathbf{u}=\exp u\in\mathbf{\Lambda}$ *be as in Theorem* 1. *Suppose that the action of* $\mathbf{u}$ *on* $(\mathbf{\Gamma}\backslash\mathbf{G},\mu)$ *is ergodic. Then either* $\mathbf{\Gamma}$ *is a lattice and* μ *is* $\mathbf{G}$-*invariant or* μ *is supported on a closed orbit of* $\mathbf{u}$ *or of* $\mathbf{u}_t=\exp tu$, $t\in R$. *In particular, every unipotent subgroup of* $\mathbf{G}=SL(2,R)$ *is strictly measure rigid.*

Theorem 1 provides some important ergodic theoretic consequences. Namely, it allows to classify up to an isomorphism all ergodic joinings of two horocyclic translations as well as factors of such translations. More specifically, let $\mathbf{G}_i$, $i=1,2$ be a Lie group, $\mathbf{\Gamma}_i$ a lattice in $\mathbf{G}_i$, ν_i a $\mathbf{G}_i$-invariant Borel probability measure on $\mathbf{\Gamma}_i\backslash\mathbf{G}_i=X_i$, $\mathbf{u}^{(i)}\in\mathbf{G}$, $\mathbf{u}=\mathbf{u}^{(1)}\times\mathbf{u}^{(2)}$. A $\mathbf{u}$-invariant Borel probability measure μ on $X=X_1\times X_2$ is called a joining of $\mathbf{u}^{(1)}$ on (X_1,ν_1) and $\mathbf{u}^{(2)}$ on (X_2,ν_2) if $\mu(A\times X_2)=\nu_1(A)$, $\mu(X_1\times B)=\nu_2(B)$ for all Borel subsets $A\subset X_1$, $B\subset X_2$. The joining $\nu_1\times\nu_2$ will be called the trivial joining. We show (Section 7) that if $\mathbf{G}_1$ and $\mathbf{G}_2$ are connected and a joining μ is algebraic then the groups $\mathbf{\Lambda}_1(\mu)$ and $\mathbf{\Lambda}_2(\mu)$ defined by

$$\mathbf{\Lambda}_1(\mu)=\{\mathbf{h}\in\mathbf{G}_1:(\mathbf{h},\mathbf{e})\in\mathbf{\Lambda}(\mu)\},\quad \mathbf{\Lambda}_2(\mu)=\{\mathbf{h}\in\mathbf{G}_2:(\mathbf{e},\mathbf{h})\in\mathbf{\Lambda}(\mu)\}$$

are closed normal subgroups of $\mathbf{G}_1$ and $\mathbf{G}_2$ respectively. Here $\mathbf{\Lambda}(\mu)\subset\mathbf{G}_1\times\mathbf{G}_2$, $\mu(x(\mu)\mathbf{\Lambda}(\mu))=1$, $x(\mu)\in X=X_1\times X_2$. For $\mathbf{c}\in\mathbf{G}_2$ write $\mathbf{\Gamma}_2^{\mathbf{c}}=\{\gamma\mathbf{\Lambda}_2(\mu):\gamma\in\mathbf{c}^{-1}\mathbf{\Gamma}_2\mathbf{c}\}$ and for $z\in X_1$ let

$$\xi_\mu(z)=\{y\in X_2:(z,y)\in x(\mu)\mathbf{\Lambda}(\mu)\}.$$

The set $\xi_\mu(z)$ is called the z-fiber of μ.

THEOREM 2. *Let* $\mathbf{G}_i$ *be a connected Lie group,* $\boldsymbol{\Gamma}_i$ *a lattice in* $\mathbf{G}_i$ *and* $\mathbf{u}^{(i)}\in\mathbf{G}_i$, $i=1,2$. *Let* μ *be an ergodic algebraic joining of* $\mathbf{u}^{(1)}$ *on* $(X_1=\boldsymbol{\Gamma}_1\backslash\mathbf{G}_1,\nu_1)$ *and* $\mathbf{u}^{(2)}$ *on* $(X_2=\boldsymbol{\Gamma}_2\backslash\mathbf{G}_2,\nu_2)$. *Then there is* $\mathbf{c}\in\mathbf{G}_2$ *and a continuous surjective homomorphism* $\alpha\colon\mathbf{G}_1\to\mathbf{G}_2/\boldsymbol{\Lambda}_2(\mu)$ *with kernel* $\boldsymbol{\Lambda}_1(\mu)$, $\alpha(\mathbf{u}^{(1)})=\mathbf{u}^{(2)}\boldsymbol{\Lambda}_2(\mu)$ *such that*

$$\xi_\mu(\boldsymbol{\Gamma}_1\mathbf{h})=\{\boldsymbol{\Gamma}_2\mathbf{c}\boldsymbol{\beta}_i\alpha(\mathbf{h})\colon i=1,\dots,n\}$$

for all $\mathbf{h}\in\mathbf{G}_1$, *where the intersection* $\boldsymbol{\Gamma}_0=\alpha(\boldsymbol{\Gamma}_1)\cap\boldsymbol{\Gamma}_2^{\mathbf{c}}$ *is of finite index in* $\alpha(\boldsymbol{\Gamma}_1)$ *and in* $\boldsymbol{\Gamma}_2^{\mathbf{c}}$, $n=|\boldsymbol{\Gamma}_0\backslash\alpha(\boldsymbol{\Gamma}_1)|$ *and* $\alpha(\boldsymbol{\Gamma}_1)=\{\boldsymbol{\Gamma}_0\boldsymbol{\beta}_i\colon i=1,\dots,n\}$.

Now suppose that $\mathbf{u}^{(i)}\in\mathbf{G}_i$ is $\boldsymbol{\Gamma}_i$-horocyclic and $\mathbf{g}^{(i)}\in\mathbf{G}_i$ is diagonal for $\mathbf{u}^{(i)}$, $i=1,2$. It is clear that $\mathbf{u}=\mathbf{u}^{(1)}\times\mathbf{u}^{(2)}$ is $\boldsymbol{\Gamma}_1\times\boldsymbol{\Gamma}_2$-horocyclic in $\mathbf{G}_1\times\mathbf{G}_2$ and $\mathbf{g}=\mathbf{g}^{(1)}\times\mathbf{g}^{(2)}$ is diagonal for $\mathbf{u}$. Also the $\mathbf{g}$-orbit of $x=(x_1,x_2)\in X$ diverges in X when $n\to\infty$ if and only if so do the $\mathbf{g}^{(1)}$-orbit of x_1 and the $\mathbf{g}^{(2)}$-orbit of x_2 in X_1 and X_2 respectively. We have $\nu_i(D(\mathbf{g}^{(i)}))=0$, since $\mathbf{g}^{(i)}\in\boldsymbol{\Lambda}(\nu_i)$, $i=1,2$. This implies that if μ is a joining of $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ then $\mu(D(\mathbf{g}))=0$. This implies via Theorem 1 that all ergodic joinings of $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ are algebraic. This gives the following

COROLLARY 5 (The Joinings Theorem). (1) *Let* $\mathbf{G}_i$ *be a connected semisimple Lie group,* $\boldsymbol{\Gamma}_i$ *a lattice in* $\mathbf{G}_i$ *and* $\mathbf{u}^{(i)}$ *a unipotent element of* $\mathbf{G}_i$, $i=1,2$. *Let* μ *be an ergodic joining of* $\mathbf{u}^{(1)}$ *and* $\mathbf{u}^{(2)}$. *Then* μ *is algebraic and the fibers of* μ *are given by Theorem* 2.

(2) *If in addition* $\mathbf{G}_i$ *is simple,* $i=1,2$ *and* μ *is nontrivial then every fiber of* μ *is finite and* $\mathbf{G}_1$ *and* $\mathbf{G}_2$ *are locally isomorphic.*

Corollary 5 generalizes our joinings theorem for $\mathbf{G}_i=SL(2,R)$, $i=1,2$ obtained in [R3]. Some restricted results of this nature were also obtained in [W2]. As in [R3, Corollary 4] we obtain the following

COROLLARY 6 (The Rigidity Theorem). *Let* $\mathbf{G}_i$ *be a connected semisimple Lie group,* $\boldsymbol{\Gamma}_i$ *a lattice in* $\mathbf{G}_i$ *containing no nontrivial normal subgroups of* $\mathbf{G}_i$ *and* $\mathbf{u}^{(i)}$ *a unipotent element of* $\mathbf{G}_i$, $i=1,2$. *Suppose that the action of* $\mathbf{u}^{(1)}$ *on* (X_1,ν_1) *is ergodic and there is a measure preserving map* $\psi\colon(X_1,\nu_1)\to(X_2,\nu_2)$ *such that* $\psi(x\mathbf{u}^{(1)})=\psi(x)\mathbf{u}^{(2)}$ *for* ν_1*-almost every* $x\in X_1$. *Then there is* $\mathbf{c}\in\mathbf{G}_2$ *and a surjective homomorphism* $\alpha\colon\mathbf{G}_1\to\mathbf{G}_2$ *such that* $\alpha(\boldsymbol{\Gamma}_1)\subset\mathbf{c}^{-1}\boldsymbol{\Gamma}_2\mathbf{c}$ *and* $\psi(\boldsymbol{\Gamma}_1\mathbf{h})=\boldsymbol{\Gamma}_2\mathbf{c}\alpha(\mathbf{h})$ *for* ν_1*-almost every* $\boldsymbol{\Gamma}_1\mathbf{h}\in X_1$. *Also* α *is a local isomorphism whenever* ψ *is finite to one or* $\mathbf{G}_1$ *is simple and it is an isomorphism whenever* ψ *is one-to-one or* $\mathbf{G}_1$ *is simple with trivial center.*

This corollary generalizes our rigidity theorem for $SL(2,R)$ in [R1]. It was previously obtained in [W1], [W2] by methods from [R1] and [R3].

Let $\mathbf{G}, \mathbf{\Gamma}, \nu$ and $\mathbf{u} \in \mathbf{G}$ be as above. A **u**-invariant measurable partition ξ of $(\mathbf{\Gamma}\backslash\mathbf{G}, \nu)$ is called a factor of **u**. We denote by $\xi(x)$ the atom of ξ containing $x \in \mathbf{\Gamma}\backslash\mathbf{G}=X$. The factor ξ is called *algebraic* if there is a surjective homomorphism $\alpha: \mathbf{G} \to \mathbf{G}$ such that $\xi(x\mathbf{h})=\xi(x)\alpha(\mathbf{h})$ for all $\mathbf{h} \in \mathbf{G}$ and ν-almost every $x \in X$. It was shown in [R2, 3] that if $\mathbf{G}=SL(2,R)$ then every factor of a unipotent element of **G** is algebraic. In general, algebraicity of factors of unipotent translations is rather an exception. Indeed we showed in [R3] that if **u** is the n-fold cartesian product $\mathbf{u}_1 \times \ldots \times \mathbf{u}_n$ of unipotent elements $\mathbf{u}_i \in \mathbf{G}_i=SL(2,R)$, $i=1,\ldots,n$ acting ergodically on $(\mathbf{\Gamma}\backslash\mathbf{G}, \nu^n)$ with $\mathbf{G}=\mathbf{G}_1 \times \ldots \times \mathbf{G}_n$, $\mathbf{\Gamma}=\mathbf{\Gamma}_1 \times \ldots \times \mathbf{\Gamma}_n$, $\nu^n=\nu_1 \times \ldots \times \nu_n$ then every factor of this action has the form $\mathbf{H}\backslash\mathbf{G}/L$, where **H** is a closed subgroup of **G**, containing $\mathbf{\Gamma}$ and L is a closed group of affine maps on $\mathbf{H}\backslash\mathbf{G}$ centralized by **u**. Recently, Witte [W3] showed using our main theorem from [R5] that this is true for general **G** and **u**.

The ideas and techniques we use to prove Theorem 1 are totally different from the methods used by other authors. In [R4] we introduced a dynamical property of unipotent group actions, called the R-property, which plays a crucial role in our analysis. It is a generalization of the property for unipotent flows which we introduced in [R3] and [W1] (in [R3] it is called the H-property and in [W1] the Ratner property). Also we make an essential use of the ergodic theory of nilpotent group actions developed in [R4] (see also [GE]). All the results from [R4] used in this paper are stated in Section 1, so that the paper can be read independently of [R4]. In Section 2 we discuss some features of horocyclicity of u, used in the proof of the basic lemma in Section 3. In Section 4 we discuss conclusion 1 of the main theorem and devote Sections 5–7 to conclusion 2 of the theorem. In Sections 5–6 we shrink the support of μ to a homogeneous set and in Section 7 we show that μ is, in fact, the Haar measure on that set.

The results of this paper were announced in [R5].

I wish to thank Paul Chernoff for valuable discussions on various aspects of the problem.

0. Notations

Throughout this paper unless otherwise stated **G** denotes a real second countable Lie group, equipped with a left invariant Riemannian metric, $\mathfrak{G}$ the Lie algebra of **G**, **e** the identity element of $\mathbf{G}$, $\mathrm{Ad}_{\mathbf{a}}(v)=v(\mathbf{a})=\mathbf{a}^{-1}v\mathbf{a}$, $\mathrm{ad}_b(v)=[v,b]$, $v, b \in \mathfrak{G}$, $\mathbf{a} \in \mathbf{G}$. If $\mathfrak{H} \subset \mathfrak{G}$ then $\exp \mathfrak{H}=\{\exp v: v \in \mathfrak{H}\}$. If $\mathfrak{H}$ (**H**) is a subalgebra of $\mathfrak{G}$ (a subgroup of **G**) then $\mathfrak{I}(\mathfrak{H})$ (**I(H)**) denotes the normalizer of $\mathfrak{H}$ (of $\mathbf{H}$) in $\mathfrak{G}$ (in $\mathbf{G}$). Also g denotes an R-diagonalizable element of $\mathfrak{G}$, $\mathbf{g}(t)=\exp tg$, $\mathbf{g}(1)=\mathbf{g}$, $\mathfrak{E}_\lambda$ the eigenspace of ad_g with the eigenvalue λ,

$$\mathfrak{E}_{\lambda_1}^{\lambda_2}=\sum_{\lambda_1\leq\lambda\leq\lambda_2}\mathfrak{E}_\lambda,\quad \mathfrak{E}^-=\sum_{\lambda<0}\mathfrak{E}_\lambda,\quad \mathbf{E}_{\lambda_1}^{\lambda_2}=\exp\mathfrak{E}_{\lambda_1}^{\lambda_2},\quad \mathbf{E}^-=\exp\mathfrak{E}^-$$

p_λ the projection onto $\mathfrak{E}_\lambda$ induced by the direct sum decomposition $\mathfrak{G}=\Sigma_\lambda\mathfrak{E}_\lambda$, $\chi(v)=\max\{\lambda: p_\lambda(v)\neq 0\}$ if $v\neq 0$. Also u denotes a horocyclic element for g, if such an element exists, $u\in\mathfrak{E}_{-2}$, $\mathbf{u}(t)=\exp tu$, $t\in R$, $\mathbf{u}=\mathbf{u}(1)$ and u^* the element in $\mathfrak{E}_2$ for which $[u,u^*]=g$. A triple (X,d,μ) will mean that X is a metric space with the metric d and a Borel probability measure μ. We shall always assume that when $\mathbf{G}$ acts on a measure space (X,μ) then (X,μ) is a standard Borel space and the action map $(x,\mathbf{g})\to x\mathbf{g}$, $x\in X$, $\mathbf{g}\in\mathbf{G}$ is Borel measurable.

1. Auxiliary results

We begin with the study of eigenspaces of ad_g with g being an R-diagonalizable element of $\mathfrak{G}$.

PROPOSITION 1.1. *A vector $w\in\mathfrak{G}$ has the form $\mathbf{c}g\mathbf{c}^{-1}$ for some $\mathbf{c}\in\mathbf{E}^-$ if and only if $w=g+v$ for some $v\in\mathfrak{E}^-$.*

Proof. If $w=\mathbf{c}g\mathbf{c}^{-1}$ for some $\mathbf{c}\in\mathbf{E}^-$ then

$$w=g+\sum_{i=1}^{\infty}\mathrm{ad}_c^i(g)/i!=g+v,\quad \mathbf{c}=\exp c,\ c\in\mathfrak{E}^-$$

and $v\in\mathfrak{E}^-$. Now let $w=g+v$ for some $v\in\mathfrak{E}^-$. We have $\exp w=\mathbf{g}\cdot\mathbf{b}$ for some $\mathbf{b}\in\mathbf{E}^-$. Define

$$\mathbf{c}_n=\prod_{k=0}^{n}(\mathbf{g}^{-k}\mathbf{b}^{-1}\mathbf{g}^{k})=\mathbf{b}^{-1}(\mathbf{g}^{-1}\mathbf{b}^{-1}\mathbf{g})\dots(\mathbf{g}^{-n}\mathbf{b}^{-1}\mathbf{g}^{n})\in\mathbf{E}^-. \tag{1.1}$$

It follows from the definition of $\mathbf{E}^-$ that

$$\lim_{n\to\infty}\mathbf{c}_n=\mathbf{c}\in\mathbf{E}^-$$

exists. We have

$$\mathbf{cgc}^{-1} = \lim_{n\to\infty}\left(\left[\prod_{k=0}^{n-1}(\mathbf{g}^{-k}\mathbf{b}^{-1}\mathbf{g}^{k})\right]\cdot\mathbf{g}\cdot\left[\prod_{k=0}^{n}(\mathbf{g}^{-(n-k)}\mathbf{b}\mathbf{g}^{n-k})\right]\right)$$
$$(1.2)\qquad = \mathbf{g}\lim_{n\to\infty}\left(\prod_{k=1}^{n}(\mathbf{g}^{-k}\mathbf{b}^{-1}\mathbf{g}^{k})\right)\cdot\left(\prod_{k=0}^{n}(\mathbf{g}^{-(n-k)}\mathbf{b}\mathbf{g}^{n-k})\right) = \mathbf{gb} = \exp w.$$

Also it follows from (1.1) that

$$(1.3)\qquad \mathbf{c} = \lim_{n\to\infty}[\exp(-nw)\exp ng].$$

This and (1.2) show that

$$\exp(w/k) = \mathbf{c}(k)\exp(g/k)\,\mathbf{c}^{-1}(k) = \exp(\mathbf{c}(k)\,(g/k)\,\mathbf{c}^{-1}(k))$$

where $k\in\mathbf{Z}^+$ and

$$\mathbf{c}(k) = \lim_{n\to\infty}[\exp(-nw/k)\exp(ng/k)] = \mathbf{c}$$

by (1.3). Thus $\exp(w/k)=\exp(\mathbf{c}(g/k)\,\mathbf{c}^{-1})$ for all $k=1,2,\ldots$. This implies that $w=\mathbf{c}g\mathbf{c}^{-1}$. □

PROPOSITION 1.2. *Let* $\mathbf{\Lambda}$ *be a closed Lie subgroup of* $\mathbf{G}$ *(not necessarily connected) with the Lie algebra* $\mathfrak{L}(\mathbf{\Lambda})$. *Suppose that* $\mathbf{cgc}^{-1}\in\mathbf{\Lambda}$ *for some* $\mathbf{c}\in\mathbf{E}^-$. *Then*

(1) $v\in\mathfrak{L}(\mathbf{\Lambda})$ *whenever* $v\in\mathfrak{E}^-$, $\exp v\in\mathbf{\Lambda}$;

(2) $\mathbf{c}p_{\chi(v)}(v)\mathbf{c}^{-1}\in\mathfrak{L}(\mathbf{\Lambda})$ *whenever* $v\in\mathfrak{L}(\mathbf{\Lambda})$.

Proof. We have $\mathbf{cg}^n\mathbf{c}^{-1}=\mathbf{g}^n\mathbf{c}_n\in\mathbf{\Lambda}$, where

$$(1.4)\qquad \mathbf{c}_n = (\mathbf{g}^{-n}\mathbf{c}\mathbf{g}^{n})\,\mathbf{c}^{-1}\to\mathbf{c}^{-1},\quad n\to\infty$$

since $\mathbf{c}\in\mathbf{E}^-$.

(1) We have $\mathbf{c}_n^{-1}\mathbf{g}^{-n}(\exp v)\,\mathbf{g}^n\mathbf{c}_n=\exp\mathrm{Ad}_{\mathbf{g}^n\mathbf{c}_n}(v)\in\mathbf{\Lambda}$ and $\exp\mathrm{Ad}_{\mathbf{g}^n\mathbf{c}_n}(v)\to\mathbf{e}, n\to\infty$ by (1.4), since $v\in\mathfrak{E}^-$. This implies that $\mathrm{Ad}_{\mathbf{g}^n\mathbf{c}_n}(v)\in\mathfrak{L}(\mathbf{\Lambda})$ for all $n\geq n_0$ and hence $v\in\mathfrak{L}(\mathbf{\Lambda})$.

(2) We have $v=v_\lambda+v'$ where $\lambda=\chi(v)$, $v_\lambda=p_\lambda(v)$ and $\chi(v')<\lambda$. Also

$$\mathrm{Ad}_{\mathbf{g}^n\mathbf{c}_n}(v) = \mathbf{c}_n^{-1}(\mathbf{g}^{-n}(v_\lambda+v')\,\mathbf{g}^n)\,\mathbf{c}_n = \mathbf{c}_n^{-1}(e^{\lambda n}v_\lambda+v'')\,\mathbf{c}_n\in\mathfrak{L}(\mathbf{\Lambda})$$

$n=1,2,\ldots$, where $\|v''\|\leq e^{n\alpha}\|v'\|$ for some $\alpha<\lambda$. This implies that

$$\lim_{n\to\infty}(\mathbf{c}_n^{-1}(v_\lambda+e^{-\lambda n}v'')\,\mathbf{c}_n) = \mathbf{c}v_\lambda\,\mathbf{c}^{-1}\in\mathfrak{L}(\mathbf{\Lambda})$$

by (1.4), where $\|e^{-\lambda n}v''\|\leq e^{(\alpha-\lambda)n}\|v'\|\to 0, n\to\infty$. This completes the proof. □

PROPOSITION 1.3. *Let* Λ *be as above. Suppose that* $\exp v \in \Lambda$ *for some* $v \in \mathfrak{E}^-$. *Then* v *normalizes* $\mathfrak{L}(\Lambda)$.

Proof. We have $\mathrm{Ad}_{\exp v}(\mathfrak{L}(\Lambda))=\mathfrak{L}(\Lambda)$ and $\chi([w, v])<\chi(w)$ for all $w \in \mathfrak{G}$ with $[w, v]\neq 0$. Let

$$\{\chi(w): 0 \neq w \in \mathfrak{L}(\Lambda)\} = \{\chi_1, \ldots, \chi_n\}$$

where $\chi_1<\chi_2<\ldots<\chi_n$. Using induction on $k=1, \ldots, n$ we shall show that $[w, v] \in \mathfrak{L}(\Lambda)$ for every $0\neq w \in \mathfrak{L}(\Lambda)$ with $\chi(w)=\chi_k$. Indeed, let $0\neq w \in \mathfrak{L}(\Lambda), \chi(w)=\chi_1$. We have

$$\mathrm{Ad}_{\exp v}(w) = w+[w, v]+w' = w+w_0 \in \mathfrak{L}(\Lambda)$$

where $\chi(w')<\chi([w, v])<\chi(w)=\chi_1$ if $w'\neq 0$. This implies that $w_0=[w, v]+w' \in \mathfrak{L}(\Lambda)$ and hence $w_0=0$ since otherwise we would have $\chi(w_0)<\chi(w)=\chi_1$. This implies that $w'=0$ and $[w, v]=0$ since $\chi(w')<\chi([w, v])$ if $w'\neq 0$. This proves that $[w, v] \in \mathfrak{L}(\Lambda)$. Now assume that $[z, v] \in \mathfrak{L}(\Lambda)$ for all $0\neq z \in \mathfrak{L}(\Lambda)$ with $\chi(z)\leq\chi_k$ and let $0\neq w \in \mathfrak{L}(\Lambda)$, $\chi(w)=\chi_{k+1}$. If $[w, v]=0$ we are done. Otherwise we have

$$\mathrm{Ad}_{\exp v}(w) = w+\sum_{k=1}^{m(v)} c_k\, \mathrm{ad}_v^k(w) \in \mathfrak{L}(\Lambda)$$

where $c_k=1/k!$ and $m(v)=\max\{k>0: \mathrm{ad}_v^k(w)\neq 0\}$. This implies that

$$w_0 = \sum_{k=1}^{m(v)} c_k\, \mathrm{ad}_v^k(w) \in \mathfrak{L}(\Lambda).$$

Also $\chi(w_0)<\chi(w)=\chi_{k+1}$. Therefore

$$w_p = \mathrm{ad}_v^p(w_0) = \sum_{i=1}^{m(v)-p} c_i\, \mathrm{ad}_v^{p+i}(w) \in \mathfrak{L}(\Lambda) \tag{1.5}$$

for all $p>0$. In particular, $w_{m(v)-1}=c_1\, \mathrm{ad}_v^{m(v)}(w) \in \mathfrak{L}(\Lambda)$ and hence $\mathrm{ad}_v^{m(v)}(w) \in \mathfrak{L}(\Lambda)$. Therefore $\mathrm{ad}_v^k(w) \in \mathfrak{L}(\Lambda)$ for all $k\geq 2$ by (1.5). This implies that $\mathrm{ad}_v(w) \in \mathfrak{L}(\Lambda)$ since $w_0 \in \mathfrak{L}(\Lambda)$. This completes the proof. □

Now assume that g is diagonal for a horocyclic element $u, g=[u, u^*], u^* \in \mathfrak{E}_2$, $u \in \mathfrak{E}_{-2}$. Define

$$\hat{\mathfrak{E}}_\lambda = \{v \in \mathfrak{E}_\lambda: [v, u^*] = 0\}, \quad \check{\mathfrak{E}}_\lambda = \{v \in \mathfrak{E}_\lambda: [v, u] = 0\}$$
$$\hat{\omega} = \{\lambda: \hat{\mathfrak{E}}_\lambda \neq \{0\}\}, \quad \check{\omega} = \{\lambda: \check{\mathfrak{E}}_\lambda \neq \{0\}\}. \tag{1.6}$$

THEOREM 1.1. *Suppose that g is diagonal for a horocyclic element u. Then*

(1) *every eigenvalue of* ad_g *is an integer;*

(2) *If* λ *is an eigenvalue of* ad_g *then so is* $-\lambda$*;*

(3) $\mathrm{ad}_{u^*}^{-\lambda}(w)\neq 0$ *for every* $0\neq w\in\mathfrak{E}_\lambda$ *with* $\lambda<0$*;* $\mathrm{ad}_u^{\lambda}(w)\neq 0$ *for every* $0\neq w\in\mathfrak{E}_\lambda$ *with* $\lambda>0$*;*

(4) $\lambda\geq 0$ *for all* $\lambda\in\hat{\omega}$ *and* $\lambda\leq 0$ *for all* $\lambda\in\check{\omega}$*. Also* $\mathrm{ad}_u^{\lambda}(\hat{\mathfrak{E}}_\lambda)=\check{\mathfrak{E}}_\lambda$ *for all* $\lambda\in\hat{w}$ *and* $\mathrm{ad}_{u^*}^{-\lambda}(\check{\mathfrak{E}}_\lambda)=\hat{\mathfrak{E}}_{-\lambda}$ *for all* $\lambda\in\check{\omega}$*;*

(5) $\mathfrak{G}=\Sigma\{\mathrm{ad}_u^k(\hat{\mathfrak{E}}_\lambda):\lambda\in\hat{w},\,0\leq k\leq\lambda\}=\Sigma\{\mathrm{ad}_{u^*}^k(\check{\mathfrak{E}}_\lambda):\lambda\in\check{w},\,0\leq k\leq-\lambda\}$*;*

(6) *If* $\mathfrak{E}=\{v_1,\dots,v_n\}$, $0\neq v_i=\mathrm{ad}_u^{k_i}(w_i)$, $k_i\geq 0$, $w_i\in\hat{\mathfrak{E}}_{\lambda_i}$, $\lambda_i\neq\lambda_j$, $i\neq j$ *then* $\mathfrak{E}$ *is linearly independent.*

Proof. This theorem is well known (see, for instance, [H, pp. 31–34]). The proof we give is similar to the proof of an analogous statement for roots of semisimple Lie algebras (see [J]). Let $0\neq v\in\hat{\mathfrak{E}}_\lambda$ for some $\lambda\in R$. We claim that for each $k=1,2,\dots$

$$[\mathrm{ad}_u^k(v),u^*]=\alpha_{k-1}\,\mathrm{ad}_u^{k-1}(v) \tag{1.7}$$

where

$$\alpha_k=(k+1)(\lambda-k)=\alpha_{k-1}+\lambda-2k,\quad \alpha_0=\lambda. \tag{1.8}$$

We shall prove (1.7) by induction on k. Let $k=1$. We have

$$[\mathrm{ad}_u(v),u^*]=[v,[u,u^*]]+[[v,u^*],u]=[v,g]=\lambda v=\alpha_0 v.$$

Now let (1.7) hold for k. Consider

$$\begin{aligned}[\mathrm{ad}_u^{k+1}(v),u^*]&=[\mathrm{ad}_u^k(v),[u,u^*]]+[[\mathrm{ad}_u^k(v),u^*],u]\\&=(\lambda-2k)\,\mathrm{ad}_u^k(v)+\alpha_{k-1}\,\mathrm{ad}_u^k(v)=\alpha_k\,\mathrm{ad}_u^k(v).\end{aligned}$$

This proves our claim. Now let $m=m(v)\geq 0$ be the largest integer for which $\mathrm{ad}_u^m(v)\neq 0$ and let $\mathfrak{L}(v)$ be the subspace of $\mathfrak{G}$ spanned by $\{\mathrm{ad}_u^k(v):0\leq k\leq m\}$. The space $\mathfrak{L}(v)$ is invariant under ad_u and ad_{u^*} and hence under ad_g, since $g=[u,u^*]$. We have

$$0=\mathrm{Tr}(\mathrm{ad}_g)\text{ on }\mathfrak{L}(v)=\sum_{k=0}^{m}(\lambda-2k)=(m+1)(\lambda-m).$$

This implies that $\lambda=m$ and shows that λ is a nonnegative integer when $\lambda\in\hat{w}$. Hence $\hat{\mathfrak{E}}_\lambda=\{0\}$ for $\lambda<0$.

Now we claim that there exists no $w\in\mathfrak{E}_{\lambda+2}$ with $[w,u]=v$. Indeed, suppose on the contrary that such w exists. Let $n=n(w)\geq 0$ be the largest integer for which $\bar{v}=\mathrm{ad}^n_{u^*}(w)\neq 0$. We have $\bar{v}\in\hat{\mathfrak{G}}_{\lambda+2(n+1)}$. We claim that

$$(1.9)\qquad [\mathrm{ad}^k_{u^*}(w),u]=\beta_k\,\mathrm{ad}^{k-1}_{u^*}(w)$$

for all $k=1,\dots,n$ and some $\beta_k\neq 0$ if $n\geq 1$. In order to prove (1.9) one should repeat the argument in the proof of (1.7), using the relation $[v,u^*]=0$. We have $m(\bar{v})=\lambda+2(n+1)$ and therefore

$$(1.10)\qquad \mathrm{ad}^k_u(\bar{v})\neq 0$$

for all $0\leq k\leq\lambda+2(n+1)$ by the definition of $m(\bar{v})$. This and (1.9) show that $v=r\,\mathrm{ad}^{n+1}_u(\bar{v})$ for some $r\neq 0$. This and (1.10) imply that

$$\mathrm{ad}^{\lambda+1}_u(v)=\mathrm{ad}^{m(v)+1}_u(v)\neq 0$$

which contradicts the definition of $m(v)$. This proves our claim. Using the symmetry between u^* and u we prove in the same way that $\lambda\leq 0$ for all $\lambda\in\check{\omega}$ and

$$(1.11)\qquad \text{there exists no } w\in\mathfrak{G} \text{ with } 0\neq[w,u^*]\in\check{\mathfrak{G}}_\lambda,\lambda\in\check{\omega}.$$

Also our argument shows that $\mathrm{ad}^\lambda_u(\hat{\mathfrak{G}}_\lambda)=\check{\mathfrak{G}}_{-\lambda}$ for all $\lambda\in\hat{\omega}$ and $\mathrm{ad}^{-\lambda}_{u^*}(\check{\mathfrak{G}}_\lambda)=\hat{\mathfrak{G}}_{-\lambda}$ for all $\lambda\in\check{\omega}$. This proves (4).

For $v\in\hat{\mathfrak{G}}_\lambda$, $0<\lambda\in\hat{\omega}$ set $\mathfrak{B}_0(v)=\{tv\colon t\in R\}$ and define

$$\mathfrak{B}_n(v)=\{w\in\mathfrak{G}_{\lambda-2n}\colon[w,u^*]\in\mathfrak{B}_{n-1}(v)\}$$

$n=1,2,\dots$. It follows from (1.8) that $\alpha_k\neq 0$ for all $k=1,\dots,\lambda-1$. This and (1.11) imply that

$$(1.12)\qquad \begin{aligned}&\mathfrak{B}_n(v)=\{0\}\quad\text{for}\quad n>\lambda\\ &\mathfrak{B}_n(v)=\mathrm{ad}^n_u(v)+\sum_{i=1}^{n}\sum_{k=0}^{n-i}\mathrm{ad}^k_u(\hat{\mathfrak{G}}_{\lambda-2i}),\quad n=1,\dots,\lambda.\end{aligned}$$

Now let $w\in\mathfrak{G}_\lambda,\lambda\in R$ and $[w,u^*]\neq 0$. Let $n=n(w)>0$ be the largest integer for which $v=\mathrm{ad}^n_{u^*}(w)\neq 0$. We have $v\in\hat{\mathfrak{G}}_{\lambda+2n}$ and $\lambda+2n$ is an integer. This implies that λ is an integer and proves (1). Also $w\in\mathfrak{B}_n(v)$. This and (1.12) prove the first identity in (5). Also $n\leq\lambda+2n$ by (1.12) and hence $n\geq-\lambda$. This implies that $\mathrm{ad}^{-\lambda}_{u^*}(w)\neq 0$ if $\lambda<0$ and $\mathrm{ad}^{-\lambda}_{u^*}(w)\in\mathfrak{G}_{-\lambda}$. This improves (2) and (3) for $\lambda<0$. The proofs of (2) and (3) for $\lambda>0$ and of the second identity in (5) are similar.

Now let us prove (6). We can assume that $v_i \in \mathfrak{E}_\lambda$ for some $\lambda \in R$ and all $i=1,\dots,n$. Suppose on the contrary that $v=\Sigma_{i=1}^n \alpha_i v_i=0$ for some $\alpha_i \in R$ and $\lambda_j=\max\{\lambda_i: \alpha_i \neq 0\}>0$ if $n>1$. Then $k_j=(\lambda_j-\lambda)/2=\max\{k_i: \alpha_i \neq 0\}$ and therefore $\mathrm{ad}_{u*}^{k_j}(v)= \mathrm{ad}_{u*}^{k_j}(\alpha_j v_j)\neq 0$ by (1.7). This contradicts $v=0$ and proves (6). □

We shall use in this paper the ideas and techniques developed in [R4]. Let us state the results from [R4] needed for our proofs.

Let $\mathfrak{N}$ be a subspace of $\mathfrak{G}$ and $\mathfrak{N}^\perp$ a subspace of $\mathfrak{G}$ complementary to $\mathfrak{N}$. Let $p_{\mathfrak{N}}$ and $p_\perp$ denote the projection onto $\mathfrak{N}$ and $\mathfrak{N}^\perp$ respectively. For $x \in \mathfrak{G}, \mathbf{h} \in \mathbf{G}$ let $x(\mathbf{h})=\mathrm{Ad}_{\mathbf{h}}(x)$.

PROPOSITION 1.4 [R4, Proposition 1.5]. *Let* $x \in \mathfrak{N}^\perp, y \in \mathfrak{N}$ *and for* $n \in \mathbf{Z}^+$, $k=0,1,\dots,n$ *let* $\mathbf{h}_0(n)=\mathbf{e}, \mathbf{h}_k(n)=\exp[p_{\mathfrak{N}}(x(\mathbf{h}_{k-1}^{-1}(n)))/n]\cdot \mathbf{h}_{k-1}(n)\exp(y/n)$. *Assume that* $\|x(\mathbf{h}_k^{-1}(n))\|\le C$ *for all* $n \in \mathbf{Z}^+, k=0,1,\dots,n$ *and some* $C>0$. *Then*

$$\begin{aligned}\exp(x+y) &= \lim_{n\to\infty}\left[\prod_{i=0}^{n-1}\exp(p_\perp(x(\mathbf{h}_i^{-1}(n)))/n)\cdot \mathbf{h}_n^{(n)}\right]\\ &= \left[\lim_{n_p\to\infty}\left(\prod_{i=0}^{n_p-1}\exp(p_\perp(x(\mathbf{h}_i^{-1}(n_p)))/n_p)\right)\right]\\ &\quad\times\left[\lim_{n_p\to\infty}\left(\prod_{i=n_p-1}^{0}\exp(p_{\mathfrak{N}}(x(\mathbf{h}_i^{-1}(n_p)))/n_p)\right)\right]\exp y\end{aligned}\tag{1.13}$$

where $\{n_p: p=1,2,\dots\}$ *is a subsequence of* $\{1,2,\dots\}$.

We will also need the following fact, which can be found in [J]. Namely, for all sufficiently small $x, y \in \mathfrak{G}$ one has

$$\exp x \exp y = \exp\left(x+y+[x,y]/2+\sum_{n=3}^{\infty} c_n(x,y)\right)\tag{1.14}$$

where each $c_n(x,y)$ is a linear combination with universal coefficients of the commutators of the form $[z_1,[z_2,[\dots,[z_{n-1},z_n],\dots]]]$ with $z_i \in \{x,y\}, i=1,\dots,n$ and the series in (1.14) is norm absolutely convergent.

Now let b be a nilpotent element of $\mathfrak{G}$, $\mathfrak{N}$ a subalgebra of $\mathfrak{G}$, normalized by b and $\mathfrak{N}^\perp$ the orthogonal complement of $\mathfrak{N}$ in $\mathfrak{G}$.

Note 1.1. We shall often use in this paper orthogonal complements of subspaces of $\mathfrak{G}$. In fact, arbitrary complements would suffice, but we take orthogonal complements for convenience. Also we can always redefine the Riemannian metric on $\mathfrak{G}$ in such a way that the arbitrary complements occurring in the argument would be orthogonal in this metric.

Define $\mathfrak{A}_0=\mathfrak{N}$, $\mathfrak{A}_0^\perp=\mathfrak{N}^\perp$ and

$$\mathfrak{A}_n = \{v\in \mathfrak{A}_{n-1}^\perp : \mathrm{ad}_b^n(v)\in\mathfrak{N}\},\quad \mathfrak{A}_{-1}=\mathfrak{G} \tag{1.15}$$

where $\mathfrak{A}_n^\perp$ denotes the orthogonal complement of $\mathfrak{A}_n$ in $\mathfrak{A}_{n-1}^\perp$, $n=1,2,\ldots$. Note that some of the $\mathfrak{A}_n$ might be trivial. We have $\mathfrak{N}^\perp=\Sigma_{i=1}^r\mathfrak{A}_i$ for some $r=r(b)\in\mathbf{Z}^+$. For $v\in\mathfrak{N}^\perp$, $t\in R$ write $b(t)=\exp tb$ and $v=\Sigma_{i=1}^r v_i$, $v_i\in\mathfrak{A}_i$, $i=1,\ldots,r$. We have

$$\begin{aligned} v(\mathbf{b}(t)) = \mathrm{Ad}_{\mathbf{b}(t)}(v) &= \sum_{i=1}^{r}\sum_{k=0}^{r}\frac{t^k}{k!}\mathrm{ad}_b^k(v_i)\\ &= v_{\mathfrak{N}}(b,t)+\hat v(b,t)+v'(b,t)\end{aligned}$$

where

$$\begin{aligned} v_{\mathfrak{N}}(b,t) &= \sum_{i=1}^{r}\sum_{k=i}^{r}\frac{t^k}{k!}\mathrm{ad}_b^k(v_i)\in\mathfrak{N}\\ \hat v(b,t) &= \sum_{i=1}^{r}\frac{t^{i-1}}{(i-1)!}\mathrm{ad}_b^{i-1}(v_i) = \sum_{i=1}^{r}\hat v_i(b,t)\\ v'(b,t) &= \sum_{i=2}^{r}\sum_{k=0}^{i-2}\frac{t^k}{k!}\mathrm{ad}_b^k(v_i).\end{aligned} \tag{1.16}$$

PROPOSITION 1.5 [R4, Corollary 3.1]. *There are $t_0(b)=t_0(b,\mathfrak{N})>2$, $Q(b)=Q(b,\mathfrak{N})>r(b)$ such that if* $\max\{\|p_\perp(v(\mathbf{b}(s)))\|: 0\leqslant s\leqslant t\}\leqslant\theta$ *for some* $t\geqslant t_0(b)$, $\theta>0$ *then*

$$\begin{aligned} &\|\mathrm{ad}_b^k(v_i)\|\leqslant Q(b)\,\theta/t^{i-1} \quad \textit{for all}\quad i=2,\ldots,r;\ 0\leqslant k\leqslant i-2\\ &\|v'(b,s)\|\leqslant Q(b)\,\theta/t \quad \textit{for all}\quad 0\leqslant s\leqslant t\\ &\|v-v_1\|\leqslant Q(b)\,\theta/t.\end{aligned} \tag{1.17}$$

PROPOSITION 1.6 [R4, Proof of Lemma 3.2]. *Given $0<c<1$ there are $t_0(c,b)=t_0(c,b,\mathfrak{N})>1$* and *$0<\omega(c,b)=\omega(c,b,\mathfrak{N})<1$ such that if* $\max\{\|p_\perp(v(\mathbf{b}(s)))\|:0\leqslant s\leqslant t\}=\theta$ *for some $t\geqslant t_0(c,b)$, $\theta>0$ and $v\in\mathfrak{N}^\perp$ then here are $0\leqslant s_0\leqslant t$ and $j\in\{1,\ldots,r\}$ such that*

$$\begin{aligned} &\omega(c,b)\,\theta\leqslant \|p_\perp(v(\mathbf{b}(s_0)))\|\leqslant\theta\\ &\|p_\perp(v(\mathbf{b}(s_0)))-p_\perp(\hat v_j(b,s_0))\|\leqslant c\|p_\perp(\hat v_j(b,s_0))\|.\end{aligned}$$

Now assume that $\mathfrak{N}$ consists of nilpotent elements of $\mathfrak{G}$ and $\mathbf{N}=\exp\mathfrak{N}$ is simply connected. Let $B=\{b_1, \dots, b_q\}$ be a basis in $\mathfrak{N}$ and for $v\in\mathfrak{N}$ let $\alpha_i(v)$ be the b_i-coordinate of v. The basis B is called *triangular* [R4, Definition 2.1] if $\alpha_k[b_i, b_j]=0$ for all $k\leq \max\{i,j\}, i,j\in\{1, \dots, q\}$. The basis B is called *regular* if it is a permutation of a triangular basis. All bases in this paper are assumed to be regular. Let $\varphi: R^q\to\mathbf{N}$ be defined by

$$\varphi(t_1, \dots, t_q) = \exp t_1 b_1 \exp t_2 b_2 \dots \exp t_q b_q.$$

The map φ is a diffeomorphism from R^q onto $\mathbf{N}$ and $\lambda(\mathbf{A})=m(\varphi^{-1}(\mathbf{A})), \mathbf{A}\subset\mathbf{N}$ is a Haar measure on $\mathbf{N}$ [R4, Proposition 2.1], where m denotes a Lebesgue measure on R^q and $\mathbf{A}$ a Borel subset of $\mathbf{N}$. For $s_i\geq 1, i=1, \dots, q$ define

$$\mathbf{F}_B(s_1, \dots, s_q) = \{\varphi(t_1, \dots, t_q): |t_i| \leq s_i, i=1, \dots, q\}.$$

The sequence $\mathbf{F}_B(s_1, \dots, s_q), \min_i s_i\to\infty$ is called *B-regular* [R4, Definition 2.2] if $s_i=\varrho\sigma_i(s)$ for some parameter $\varrho>0$ and some functions $0<\sigma_i(s)\uparrow\infty, s\to\infty$ with

$$\sigma_i(s)\,\sigma_j(s) \leq \sigma_k(s) \quad \text{for all} \quad s\geq 1 \tag{1.18}$$

whenever $\alpha_k([b_i, b_j])\neq 0, i,j,k\in\{1, \dots, q\}$.

Henceforth the symbol $\mathbf{F}_\varrho(s)$ or $\mathbf{F}_\varrho^B(s)$ will mean a B-regular sequence $\mathbf{F}_B(s_1, \dots, s_q)$ with $s_i=\varrho\sigma_i(s), i=1, \dots, q$ for some functions $\sigma_i(s)$ satisfying (1.18). Define

$$\Phi_\varrho(s) = \{v\in\mathfrak{N}: |\alpha_i(v)| \leq \varrho\sigma_i(s), i=1, \dots, q\}. \tag{1.19}$$

It follows from (1.14) and (1.18) that there is $0<\varrho_0<0.1$ such that if $0<\varrho<\varrho_0$ then

$$\mathbf{h}\mathbf{F}_\varrho(s) \subset \mathbf{F}_{3\varrho}(s) \quad \text{for all} \quad \mathbf{h}\in\mathbf{F}_\varrho(s),\ s\geq 1 \tag{1.20}$$

$$\exp v\in\mathbf{F}_\varrho(s) \ \Rightarrow\ v\in\Phi_{2\varrho}(s), \quad v\in\Phi_\varrho(s) \ \Rightarrow\ \exp v\in\mathbf{F}_{2\varrho}(s). \tag{1.21}$$

Define $\tau(s)=\min\{\varrho\sigma_i(s): i=1, \dots, q\}$ and for $v\in\mathfrak{G}$ let

$$\beta(v, s) = \max\{\|p_\perp(v(\mathbf{h}))\|: \mathbf{h}\in\mathbf{F}_\varrho(s)\}$$

$v(\mathbf{h})=\mathrm{Ad}_\mathbf{h}(v)$. Also let $\mathfrak{J}(\mathfrak{N})$ denote the normalizier of $\mathfrak{N}$ in $\mathfrak{G}$, $\mathfrak{N}_\mathfrak{J}^\perp=\mathfrak{J}(\mathfrak{N})\cap\mathfrak{N}^\perp$ and $p_\mathfrak{J}^\perp$ the projection onto $\mathfrak{N}_\mathfrak{J}^\perp$.

Proposition 1.7 (The R-property [R4, Theorem 3.1]). *There exists $t_0=t_0(B)>1$, $L=L(B)>1, 0<\eta=\eta(B)<1$ such that if $v\in\mathfrak{N}^\perp, \tau(t)>t_0$ and $\beta(v, t)\leq\theta$ for some $\theta>0$ then*

$$\|p_\perp(v(\mathbf{h}))-p_{\mathfrak{J}}^\perp(v(\mathbf{h}))\|\leq L\theta/\tau(t)$$

for all $\mathbf{h}\in\mathbf{F}_\varrho(t)$ *and*

$$\|p_\perp(v(\mathbf{h}_0))-p_\perp(v(\mathbf{h}))\|\leq\varepsilon\theta$$

for all $\mathbf{h}\in\mathbf{h}_0\mathbf{F}_{\eta\varepsilon\varrho}(t)$, *all* $\mathbf{h}_0\in\mathbf{F}_\varrho(t)$ *and all* $0<\varepsilon<1$.

A sequence $\mathbf{F}_\varrho(s)$ is said to be *consistent* with $\mathfrak{G}$ if there is a constant $Q=Q(B,\varrho)>1$ such that

$$p_{\mathfrak{N}}(v(\mathbf{h}))\in\Phi_{Q\alpha\theta}(t)$$

for all $\mathbf{h}\in\mathbf{F}_\alpha(t)$ and all $0\leq\alpha\leq\varrho$, whenever $\beta(v,t)\leq\theta$, $v\in\mathfrak{G}$, $\|v\|\leq\theta$, $\tau(t)\geq t_0(B)$.

Proposition 1.8 [R4, Lemma 3.3]. *Suppose that* $\mathbf{F}_\varrho(s)$ *is consistent with* $\mathfrak{G}$. *Then there is* $\mathbf{C}=\mathbf{C}(B,\varrho)\geq 10LQ/\eta$ *such that if* $\beta(v,t)\leq\theta$ *for some* $v\in\mathfrak{G}$, $\|v\|\leq\theta$, $0<\theta<0.1\mathbf{C}^{-1}\varrho$, $\tau(t)\geq\max\{t_0,\mathbf{C}^{-1}\theta\}=t(\theta)$ *then*

$$\exp v(\mathbf{h})=\exp(p_{\mathfrak{J}}^\perp(v(\mathbf{h}))+r(v,\mathbf{h})+\varepsilon(v,\mathbf{h}))\cdot\bar{\mathbf{h}}=\exp(\omega(v,\mathbf{h}))\cdot\bar{\mathbf{h}} \tag{1.22}$$

for all $\mathbf{h}\in\mathbf{F}_\varrho(t)$, *where* $\bar{\mathbf{h}}\in\mathbf{F}_{C\theta}(t)$ *and*

$$r(v,\mathbf{h})\in\mathfrak{N}_{\mathfrak{J}}^\perp,\quad \|r(v,\mathbf{h})\|\leq\mathbf{C}\theta^2,\quad \varepsilon(v,\mathbf{h})\in\mathfrak{N}^\perp,\quad \|\varepsilon(v,\mathbf{h})\|\leq\mathbf{C}\theta/\tau(t).$$

Now let $\mathbf{F}_\varrho(s)$ be consistent with $\mathfrak{G}$ and let $\beta(v,t)\leq\theta$ for some $v\in\mathfrak{G}$, $\|v\|\leq\theta$, $\tau(t)\geq t(\theta)$, $0<\theta<0.1\mathbf{C}^{-1}\varrho$. Let $\varphi(\cdot)=\varphi(v,\cdot):\mathbf{F}_\varrho(t)\to(\exp v)\mathbf{N}$ be defined by

$$\varphi(\mathbf{h})=\mathbf{h}\exp\omega(v,\mathbf{h})=(\exp v)\cdot\mathbf{h}\cdot(\bar{\mathbf{h}})^{-1} \tag{1.23}$$

where $\omega(v,\mathbf{h})$ and $\bar{\mathbf{h}}$ are as in (1.22).

Proposition 1.9 [R4, Corollary 3.5]. *Let* $\mathbf{h}_0,\mathbf{h}_0\mathbf{h}\in\mathbf{F}_\varrho(t)$ *and* $\mathbf{h}\in\mathbf{F}_\delta(t)$ *for some* $0<\delta<\varrho$. *Then*

$$\varphi(\mathbf{h}_0\mathbf{h})=\varphi(\mathbf{h}_0)\mathbf{h}\psi(\mathbf{h}_0,\mathbf{h}) \tag{1.24}$$

where $\psi(\mathbf{h}_0,\mathbf{h})\in\mathbf{F}_{C\theta\delta}(t)\subset\mathbf{F}_{0.1\delta}(t)$.

The following two results are concerned with ergodic actions of $\mathbf{N}$. Let λ denote a Haar measure on $\mathbf{N}$.

PROPOSITION 1.10 [R4, Theorem 2.1]. *Let* $\mathbf{N}$ *act on a probability space* (X, μ) *with* μ *being* $\mathbf{N}$*-invariant. Suppose that the action of* $\mathbf{N}$ *is ergodic. Then given* $A \subset X$, $\mu(A) > 1-\alpha$, $0<\alpha<1$ *and a regular sequence* $\mathbf{F}_\varrho(s)$ *in* $\mathbf{N}$ *there exists* $Q(A) \subset X$, $\mu(Q(A))=1$ *such that if* $x \in Q(A)$ *then*

$$\liminf_{s\to\infty}[\lambda(A \cap x\mathbf{F}_\varrho(s))/\lambda(\mathbf{F}_\varrho(s))] \geqslant 1-\beta(\alpha)$$

for some $0<\beta(\alpha)\to 0$, *when* $\alpha \to 0$, *where* $\lambda(D)$ *for* $D \subset x\mathbf{F}_\varrho(s)$ *is defined to be* $\lambda\{\mathbf{h} \in \mathbf{F}_\varrho(s) : x\mathbf{h} \in D\}$.

Now let $\mathbf{G}$ act on a metric space (X, d, μ) with μ being a Borel probability measure on (X, d). Let

$$\mathbf{\Lambda} = \mathbf{\Lambda}(\mathbf{G}, X, \mu) = \{\mathbf{g} \in \mathbf{G} : \text{the action of } \mathbf{g} \text{ preserves } \mu\}.$$

We say that the action of $\mathbf{G}$ on (X, d) is *uniform* if given $\varepsilon>0$ there is $c(\varepsilon)>0$ such that $d(x\mathbf{g}_1, x\mathbf{g}_2)<\varepsilon$ for all $x \in X$ whenever $d_{\mathbf{G}}(\mathbf{g}_1, \mathbf{g}_2)<c(\varepsilon)$, $\mathbf{g}_1, \mathbf{g}_2 \in \mathbf{G}$. Let $\mathbf{I}(\mathbf{N})$ denote the normalizer of $\mathbf{N}$ in $\mathbf{G}$.

PROPOSITION 1.11 [R4, Theorem 2.2]. *Let* $\mathbf{G}$ *act uniformly by homeomorphisms on* (X, d, μ) *with* μ *being* $\mathbf{N}$*-invariant. Suppose that the action of* $\mathbf{N}$ *is ergodic. Then given* $\varepsilon>0$ *and a nonempty compact subset* $\mathbf{K} \subset \mathbf{I}(\mathbf{N}) - \mathbf{\Lambda}$ *there exist a compact subset* $Y = Y(\varepsilon, \mathbf{K}) \subset X$ *with* $\mu(Y) > 1-\varepsilon$ *and* $\delta = \delta(\varepsilon, Y) > 0$ *such that* $d(Y, Y\mathbf{k}) > \delta$ *for all* $\mathbf{k} \in \mathbf{K}$.

Note 1.2 [R4, Notes 2.3 and 3.2]. Let $\mathfrak{H}$ be a simply connected subalgebra of $\mathfrak{G}$ spanned by $\mathfrak{N}$ and a nilpotent element $u \in \mathfrak{G}$, normalizing $\mathfrak{N}$. Let $B_{\mathfrak{H}} = \{u, b_1, \ldots, b_q\}$ be a regular basis in $\mathfrak{H}$ with $B = \{b_1, \ldots, b_q\}$ being a regular basis in $\mathfrak{N}$. Let $\mathbf{u} = \mathbf{z}\exp u$, $\mathbf{z} \in \mathbf{Z}(\mathbf{G})$, $\mathbf{u}^k \notin \mathbf{N} = \exp \mathfrak{N}$, $k \in \mathbf{Z}$ and $\mathbf{N}_\mathbf{u} = \{\mathbf{u}^k\mathbf{N} : k \in \mathbf{Z}\}$. For a Borel subset $\mathbf{A} \subset \mathbf{N}_\mathbf{u}$ define $\lambda_\mathbf{u}(\mathbf{A}) = \Sigma_{k\in\mathbf{Z}}\lambda(\mathbf{A} \cap \mathbf{u}^k\mathbf{N})$, where for $\mathbf{D} \subset \mathbf{u}^k\mathbf{N}$ we define $\lambda(\mathbf{D}) = \lambda(\mathbf{u}^{-k}\mathbf{D})$ with λ being a Haar measure on $\mathbf{N}$. Then $\lambda_\mathbf{u}$ is a Haar measure on $\mathbf{N}_\mathbf{u}$. Define

$$\mathbf{F}(n; s_1, \ldots, s_q) = \{\mathbf{u}^k\mathbf{F}_B(s_1, \ldots, s_q) : -n \leqslant k \leqslant n\}$$

and call the sequence $\mathbf{F}(n; s_1, \ldots, s_q)$ regular when $\min_i\{n, s_i\} \to \infty$ if $\mathbf{F}_B(s_1, \ldots, s_q) = \mathbf{F}_\varrho^B(s)$ is regular, $n = \varrho n(s)$, $n(s) \uparrow \infty$, $s \to \infty$ and $n(s)\sigma_i(s) \leqslant \sigma_k(s)$, $s \geqslant 1$ whenever $\alpha_k([u, b_i]) \neq 0$, $i, k \in \{1, \ldots, q\}$ (see (1.17)). Propositions 1.8–1.11 hold also with $\mathbf{N}_\mathbf{u}$ in place of $\mathbf{N}$. One should only substitute λ by $\lambda_\mathbf{u}$ and $\mathbf{F}_\varrho(s)$ by the regular sequence $\mathbf{F}(n; s_1, \ldots, s_q)$ just described.

PROPOSITION 1.12 [R4, Theorem 5]. *Let* $\mathbf{G}$ *act on a metric space* (X, d, μ) *with* μ *being* $\mathbf{N}$*-invariant. Suppose that* $\mathbf{g}^n\mathbf{h}\mathbf{g}^{-n}\to\mathbf{e}, n\to\infty$ *for all* $\mathbf{h}\in\mathbf{N}$ *and some* $\mathbf{g}\in\mathbf{\Lambda}$ *and the action of* $\mathbf{N}$ *is ergodic. Then the action of* $\mathbf{g}$ *is mixing.*

Finally we include the following simple fact used in the proof of the main theorem.

PROPOSITION 1.13. *Let* $\mathbf{H}$ *be a Lie group with a Haar measure* λ *acting on a probability space* (X, μ) *with* μ *being* $\mathbf{H}$*-invariant. Then given* $A\subset X, \mu(A)=1$ *and* $\mathbf{F}\subset\mathbf{H}, \lambda(\mathbf{F})>0$ *there is* $X(A,\mathbf{F})\subset X, \mu(X(A,\mathbf{F}))=1$ *such that if* $x\in X(A,\mathbf{F})$ *then* $x\mathbf{h}\in A$ *for* λ*-almost every* $\mathbf{h}\in\mathbf{F}$.

Proof. It suffices to assume that $\lambda(\mathbf{F})<\infty$. Let f denote the characteristic function of A. We have

$$\int_X f(x\mathbf{h})\,d\mu=\int_X f(x)\,d\mu=1$$

for all $\mathbf{h}\in\mathbf{H}$. This implies that

$$\int_{\mathbf{F}}\left(\int_X f(x\mathbf{h})\,d\mu\right)d\lambda(\mathbf{h})/\lambda(\mathbf{F})=\int_X\left[\left(\int_{\mathbf{F}} f(x\mathbf{h})\,d\lambda(\mathbf{h})\right)\Big/\lambda(\mathbf{F})\right]d\mu=1$$

and therefore $\int_{\mathbf{F}} f(x\mathbf{h})\,d\lambda(\mathbf{h})/\lambda(\mathbf{F})=1$ for μ-almost every $x\in X$. This implies that for such x we have $f(x\mathbf{h})\in\mathbf{F}$ for λ-almost every $\mathbf{h}\in\mathbf{F}$, since $0\leqslant f(x)\leqslant 1$ for all $x\in X$. This completes the proof. □

2. The significance of horocyclicity

Throughout this section we assume that $\mathbf{G}$ is a Lie group, g an R-diagonalizable element of $\mathfrak{G}$, $u\in\mathfrak{G}$ a horocyclic element for $g, \mathbf{u}(s)=\exp su, s\in R, \mathbf{u}=\mathbf{u}(1)$ and $\mathfrak{E}_k$, $k=0,\pm1,\dots,\pm m$ the eigenspace of ad_g with the eigenvalue k (see Theorem 1.1). Note that some of the $\mathfrak{E}_k$ might be trivial. Write $\mathfrak{E}^k=\mathfrak{E}^k_{-m}, k=0,\pm1,\dots,\pm m$, $\mathfrak{E}^{-1}=\mathfrak{E}^-$. We have

$$\mathfrak{G}=\sum_{k=-m}^{m}\mathfrak{E}_k$$

—a direct sum decomposition. Let p_k denote the projection onto $\mathfrak{E}_k$, induced by this decomposition.

We shall use the symmetry arising from horocyclicity of u to obtain the necessary tools used in the proof of the basic lemma.

Let $\mathfrak{N}\subset\mathfrak{E}^{-2}$ be a subalgebra of $\mathfrak{G}$ normalized by u. As above we denote by $p_\perp(v)$ and $p_{\mathfrak{N}}(v)$ the projection of $v\in\mathfrak{G}$ onto $\mathfrak{N}^\perp$ and $\mathfrak{N}$ respectively. Let

$$\mathfrak{A}_i=\mathfrak{A}_i(u)=\{v\in\mathfrak{A}_{i-1}^\perp\colon \mathrm{ad}_u^i(v)\in\mathfrak{N}\},\quad i=1,\ldots,m+1$$

$$\mathfrak{A}_0=\mathfrak{N},\quad \mathfrak{A}_0^\perp=\mathfrak{N}^\perp,\quad \mathfrak{A}_{-1}^\perp=\mathfrak{G}$$

be as in (1.15) for $b=u$, where $\mathfrak{A}_{i-1}^\perp$ denotes the orthogonal complement of $\mathfrak{A}_{i-1}$ in $\mathfrak{A}_{i-2}^\perp$, $i=1,2,\ldots,m+1$ (see Note 1.1).

LEMMA 2.1. *Let* $0\neq v_i\in\mathfrak{A}_i$, $i=0,\ldots,m+1$. *Then*

$$\chi(\mathrm{ad}_u^k(v_i))\leqslant 2i-2k-2 \tag{2.1}$$

for all $k\geqslant 0$ *with* $\mathrm{ad}_u^k(v_i)\neq 0$.

Proof. It follows from the definition of $\mathfrak{A}_i$ that

$$\mathrm{ad}_u^k(v_i)\neq 0$$

for all $0\leqslant k\leqslant i-1, i=1,\ldots,m+1$. Let us show that

$$\chi(v_i)\leqslant 2i-2$$

for all $1\leqslant i\leqslant m+1$. This would imply (2.1), since $\chi(u)=-2$. Suppose on the contrary that

$$\chi(v_i)>2i-2$$

Then

$$\chi(\mathrm{ad}_u^{i-1}(v_i))>0.$$

This implies that

$$\mathrm{ad}_u^i(v_i))\neq 0$$

by Theorem 1.1, since u is horocyclic for g. Therefore

$$\chi(\mathrm{ad}_u^i(v_i))>-2$$

and

$$\mathrm{ad}_u^i(v_i)\notin\mathfrak{N}.$$

This contradicts the definition of $\mathfrak{A}_i$. □

It follows from [R4, Proposition 1.4] that $\mathbf{N}=\exp\mathfrak{N}$ is a simply connected unipotent subgroup of $\mathbf{G}$. Let $B^{(k)}$ be a maximal set of vectors in $\mathfrak{N}$ such that $\chi(b)=k$ for all $b\in B^{(k)}$ and $\{p_k(b): b\in B^{(k)}\}$ forms a linearly independent set in $\mathfrak{E}_k, k=-2,\dots,-m$. The set $B=\bigcup_{k=-m}^{-2}B^{(k)}=\{b_1,\dots,b_q\}$ is a regular basis in $\mathfrak{N}$. Let $\mathbf{F}_\varrho(s)$, $0<\varrho<0.1\varrho_0$ be the B-regular sequence in $\mathbf{N}$ defined with

$$\sigma_i(s)=s^k \quad \text{if} \quad \chi(b_i)=-2k \quad \text{or} \quad -(2k+1), \quad i=1,\dots,q,\ k=1,\dots,m \tag{2.2}$$

and let $\Phi_\varrho(s)$ be as in (1.19).

From now on we assume that $u\in\mathfrak{N}$ and $u=b_1\in B$.

Note 2.1. The argument below works for the case $u\notin\mathfrak{N}$ as well. One should only substitute $\mathbf{N}$ by $\mathbf{N}_\mathbf{u}$ (see Note 1.2) and $\mathbf{F}_\varrho(s)$ by the regular sequence in $\mathbf{N}_\mathbf{u}$ described in Note 1.2.

LEMMA 2.2. *Let $t_0=t_0(u)$ be as in Proposition 1.5 for $b=u$ and let $v\in\mathfrak{G}$, $\|v\|\leq\theta$ for some $0<\theta<1$. Suppose that*

$$\max\{\|p_\perp(v(\mathbf{u}(s)))\|: 0\leq s\leq t\}\leq\theta$$

for some $t\geq t_0$. Then

$$\begin{aligned} p_\perp(v(\mathbf{u}(s))) &= v^0(u,s)+\sum_{j=1}^m z_j(v,s)\\ p_{\mathfrak{N}}(v(\mathbf{u}(s))) &\in \Phi_{2Q\theta}(s)\end{aligned} \tag{2.3}$$

for all $0\leq s\leq t$, where $Q=Q(u)$ is as in (1.17) and

$$v^0(u,s)\in\mathfrak{E}^0\cap\mathfrak{N}^\perp,\quad \|v^0(u,s)\|\leq Q\theta$$

$$\|z_j(v,s)\|\leq Q\theta/t^j,\quad 0<\chi(z_j(v,s))\leq 2j,\quad j=1,\dots,m.$$

Proof. We have

$$v=p_\perp(v)+p_{\mathfrak{N}}(v)=v_\perp+v_{\mathfrak{N}}$$

$$p_{\mathfrak{N}}(v(\mathbf{u}(s)))=p_{\mathfrak{N}}(v_\perp(\mathbf{u}(s)))+v_{\mathfrak{N}}(\mathbf{u}(s)).$$

It is clear that $v_{\mathfrak{N}}(\mathbf{u}(s))\in\Phi_\theta(s)$, since $\|v_{\mathfrak{N}}\|\leq\theta$. This says that we can assume $v\in\mathfrak{N}^\perp$. We have

$$p_\perp(v(\mathbf{u}(s)))=p_\perp(\hat{v}(u,s))+p_\perp(v'(u,s))$$

where

$$\hat{v}(u,s)=\sum_{i=1}^{m+1}\frac{s^{i-1}}{(i-1)!}\,\mathrm{ad}_u^{i-1}(v_i)=\sum_{i=1}^{m+1}\hat{v}_i(u,s)$$

$$v'(u,s)=\sum_{i=2}^{m+1}\sum_{k=0}^{i-2}\frac{s^k}{k!}\,\mathrm{ad}_u^k(v_i)=\sum_{i=2}^{m+1}\sum_{k=0}^{i-2}v'_{i,k}(u,s)$$

for all $0\leqslant s\leqslant t$ by (1.16), where v_i denotes the projection of v onto $\mathfrak{A}_i$. It follows from (1.17) and Lemma 2.1 that

$$\chi(\hat{v}_i(u,s))\leqslant 0,\quad i=1,\ldots,m+1$$

$$\chi(v'_{i,k}(u,s))\leqslant 2i-2k-2$$

$$\|v'_{i,k}(u,s)\|\leqslant Q\theta/t^{i-k-1},\quad i=2,\ldots,m+1,\ k=0,\ldots,i-2$$

for all $0\leqslant s\leqslant t$. This implies that

$$p_\perp(v'(u,s))=\sum_{j=1}^{m}z_j(v,s)$$

$$\|z_j(v,s)\|\leqslant Q\theta/t^j,\quad \chi(z_j(v,s))\leqslant 2j,\quad j=1,\ldots,m$$

for all $0\leqslant s\leqslant t$. Also

$$\chi(v^0)\leqslant 0,\quad \|v^0\|\leqslant Q\theta$$

where

$$v^0=v^0(u,s)=p_\perp(\hat{v}(u,s)).$$

We have

$$\|p_{\mathfrak{N}}(v(\mathbf{u}(s)))-v_{\mathfrak{N}}(u,s)\|\leqslant Q\theta$$

where

$$v_{\mathfrak{N}}(u,s)=\sum_{i=1}^{m+1}\sum_{k=i}^{m+1}\frac{s^k}{k!}\,\mathrm{ad}_u^k(v_i)$$

by (1.16). It follows from Lemma 2.1 that

$$\chi(\mathrm{ad}_u^k(v_i))\leqslant 2i-2k-2 \tag{2.4}$$

if $\mathrm{ad}_u^k(v_i) \neq 0$. Also

$$\left\| \frac{s^k}{k!} \mathrm{ad}_u^k(v_i) \right\| \leq Q\theta s^k/t^{i-1} \leq Q\theta s^{k-i+1}$$

for all $0 \leq s \leq t$ by (1.17). This, (2.2) and (2.4) imply (2.3). □

LEMMA 2.3. *Suppose that* $v \in \mathfrak{N}^\perp$ *and*

$$v = v^0 + \sum_{j=1}^{m} z_j$$

where

$$v^0 \in \mathfrak{E}^0 \cap \mathfrak{N}^\perp, \quad \|v^0\| \leq \varepsilon, \quad \|z_j\| \leq \gamma/t^j, \quad z_j \in \mathfrak{E}^{2j} \cap \mathfrak{N}^\perp, \quad t \geq 1.$$

Then

$$\begin{gathered} p_\perp(v(\mathbf{h})) = f(v, \mathbf{h}) + \sum_{j=1}^{m} z_{\mathbf{h},j} \\ p_{\mathfrak{N}}(v(\mathbf{h})) \in \Phi_{\bar{Q}\varrho \max\{\varepsilon, \gamma\}}(t) \end{gathered} \tag{2.5}$$

for all $\mathbf{h} \in \mathbf{F}_\varrho(t)$ *where* $\bar{Q} > 0$ *is a constant and*

$$f(v, \mathbf{h}) = v^0 + v^0_{\mathbf{h}} + \xi_{\mathbf{h}} \in \mathfrak{E}^0 \cap \mathfrak{N}^\perp$$

$$v^0_{\mathbf{h}} \in \mathfrak{E}^0 \cap \mathfrak{N}^\perp, \quad \xi_{\mathbf{h}} \in \mathfrak{E}^{-2} \cap \mathfrak{N}^\perp, \quad \|v^0_{\mathbf{h}}\| \leq \bar{Q}\gamma\varrho$$

$$z_{\mathbf{h},j} \in \mathfrak{E}^{2j} \cap \mathfrak{N}^\perp, \quad \|z_{\mathbf{h},j}\| \leq \bar{Q}\gamma/t^j, \quad j = 1, \ldots, m.$$

Proof. Recall that

$$v(\mathbf{h}) = \mathrm{Ad}_{\mathbf{h}}(v).$$

We have

$$v(\mathbf{h}) = v^0(\mathbf{h}) + \sum_{j=1}^{m} z_j(\mathbf{h}).$$

It is clear that

$$v^0(\mathbf{h}) = v^0 + \xi'_{\mathbf{h}} + \alpha'_{\mathbf{h}} \tag{2.6}$$

where

$$\xi'_{\mathbf{h}} \in \mathfrak{G}^{-2} \cap \mathfrak{N}^{\perp}, \quad \alpha'_{\mathbf{h}} \in \Phi_{Q_1 \varepsilon \varrho}(t)$$

for some $Q_1 > 0$ since $\chi(v^0) \leq 0$. Now let us assume for simplicity that

$$\mathbf{h} = \exp(\varrho t^l b), \quad b \in B$$

$$\chi(b) = -2l \quad \text{or} \quad -(2l+1), \quad l \in \mathbf{Z}^+.$$

We have

$$z_j(\mathbf{h}) - z_j = \sum_{k=1}^{m(\mathbf{h})} \frac{\varrho^k t^{kl}}{k!} \operatorname{ad}_b^k(z_j) = \sum_{k=1}^{m(\mathbf{h})} p_{k,j}$$

$$= \sum_{kl<j} p_{k,j} + p_{j/l,j} + \sum_{kl>j} p_{k,j} = \sum_{kl<j} p_{k,j} + v^0_{\mathbf{h},j} + \xi''_{\mathbf{h},j} + \alpha''_{\mathbf{h},j}$$

where

$$p_{k,j} \in \mathfrak{G}^{2j-2kl}, \quad \|p_{k,j}\| \leq Q_2 \varrho \gamma / t^{j-kl}$$

for some $Q_2 > 0, k = 1, \ldots, m(\mathbf{h}), j = 1, \ldots, m$ and therefore

$$v^0_{\mathbf{h},j} \in \mathfrak{G}^0, \quad \|v^0_{\mathbf{h},j}\| \leq Q_2 \varrho \gamma$$

$$\xi''_{\mathbf{h},j} \in \mathfrak{G}^{-2} \cap \mathfrak{N}^{\perp}, \quad \alpha''_{\mathbf{h},j} \in \Phi_{Q_2 \varrho \gamma}(t).$$

This and (2.6) imply (2.5) if we set

$$\xi_{\mathbf{h}} = \xi'_{\mathbf{h}} + \sum_{j=1}^{m} \xi''_{\mathbf{h},j}, \quad v^0_{\mathbf{h}} = p_{\perp}\left(\sum_{j=1}^{m} v^0_{\mathbf{h},j}\right)$$

$$z_{\mathbf{h},j} = p_{\perp}\left(z_j + \sum_{i-kl=j} p_{k,i}\right)$$

and note that

$$\left\| p_{\mathfrak{N}}(v(\mathbf{h})) - \alpha'_{\mathbf{h}} - \sum_{j=1}^{m} \alpha''_{\mathbf{h},j} \right\| \leq Q_3 \varrho \gamma$$

for some $Q_3 > 0$. This completes the proof. □

COROLLARY 2.1. *Suppose that*

$$\max\{\|p_\perp(v(\mathbf{u}(s)))\|: 0 \leq s \leq \varrho t\} \leq \theta \tag{2.7}$$

for some $v \in \mathfrak{G}$, $\|v\| \leq \theta$, $0<\theta<1$ *and some* $t \geq t_0/\varrho$. *Then*

$$p_\perp(v(\mathbf{h})) = v^0(\mathbf{h}) + \xi_\mathbf{h} + z_\mathbf{h}$$

$$p_\mathfrak{N}(v(\mathbf{h})) \in \Phi_{\tilde{Q}\theta\alpha}(t)$$

for all $\mathbf{h} \in \mathbf{F}_\alpha(t)$ *and all* $0 \leq \alpha \leq \varrho$, *where*

$$v^0(\mathbf{h}) \in \mathfrak{E}^0 \cap \mathfrak{N}^\perp, \quad \|v^0(\mathbf{h})\| \leq \tilde{Q}\theta$$

$$\xi_\mathbf{h} \in \mathfrak{E}^{-2} \cap \mathfrak{N}^\perp, \quad z_\mathbf{h} \in \mathfrak{N}^\perp, \quad \|z_\mathbf{h}\| \leq \tilde{Q}\theta/t$$

and $\tilde{Q}=\tilde{Q}(B,\varrho)>2\bar{Q}Q/\varrho^m$ *is a constant.*

Proof. We have using (2.7) and Lemma 2.2

$$v = v^0 + \sum_{j=1}^m z_j + v_\mathfrak{N}$$

where $v^0 \in \mathfrak{E}^0 \cap \mathfrak{N}^\perp$, $\|v^0\| \leq Q\theta$, $v_\mathfrak{N} \in \mathfrak{N}$, $\|v_\mathfrak{N}\| \leq \theta$ and $z_j \in \mathfrak{E}_1^{2j} \cap \mathfrak{N}^\perp$, $\|z_j\| \leq Q\theta/t^j\varrho^j$. This and Lemma 2.3 imply the corollary. □

Corollary 2.1 shows that $\mathbf{F}_\varrho(s)$ is consistent with $\mathfrak{G}$. This and Proposition 1.8 imply the following

COROLLARY 2.2. *There exists* $\mathbf{C}=\mathbf{C}(B,\varrho) \geq 10L(B)\tilde{Q}(B)/\eta(B)$ *such that* (1.22) *holds for* $\mathbf{F}_\varrho(s)$ *and* (1.24) *holds for* φ *where* φ *is as in* (1.23).

Now let $\mathfrak{H}$ be a subalgebra of $\mathfrak{E}^- = \mathfrak{E}^{-1}$ and $\mathfrak{H}^\perp$ the orthogonal complement of $\mathfrak{H}$ in $\mathfrak{G}$. Let $\bar{B}=\{\bar{b}_1, \ldots, \bar{b}_n\}$ be a regular basis in $\mathfrak{H}$ and $\bar{F}_\varrho(s)$ be the $\bar{B}$-regular sequence in $\mathbf{H}=\exp\mathfrak{H}$ defined with

$$\bar{\sigma}_i(s) = s^{-\chi(\bar{b}_i)}, \quad i = 1, \ldots, n. \tag{2.8}$$

Let $\mathfrak{I}_0(\mathfrak{H}) = \mathfrak{I}(\mathfrak{H}) \cap \mathfrak{E}^0$, $\mathfrak{I}_0^\perp(\mathfrak{H}) = \mathfrak{I}_0(\mathfrak{H}) \cap \mathfrak{H}^\perp$. Note that $\mathrm{ad}_b^{2m+1}(v)=0$ for all $b \in \mathfrak{H}$, $v \in \mathfrak{G}$. Now let

$$\max\{\|p_{\mathfrak{H}^\perp}(\bar{v}(\mathbf{h}))\|: \mathbf{h} \in \bar{\mathbf{F}}_p(t)\} \leq \theta \tag{2.9}$$

for some $t \geqslant t_0(\bar{B})/\varrho$, $0<\theta<1$ and some $\bar{v}=v+z\in\mathfrak{G}$ with $v\in\mathfrak{E}^0$, $\|v\|\leqslant\theta$, $\|z\|\leqslant\theta/t^{2m+1}$. Then

$$\|p_{\mathfrak{H}^\perp}(z(\mathbf{h}))\| \leqslant \tilde{Q}\theta/t, \quad p_{\mathfrak{H}}(\bar{v}(\mathbf{h}))\in\Phi_{\tilde{Q}\theta\alpha}(t) \tag{2.10}$$

for some $\tilde{Q}=\tilde{Q}(\bar{B})>1$, all $\mathbf{h}\in\bar{\mathbf{F}}_\alpha(t)$ and all $0\leqslant\alpha\leqslant\varrho$. This implies that

$$\|p_{\mathfrak{H}^\perp}(v(\mathbf{h}))\| \leqslant 2\theta \tag{2.11}$$

for all $\mathbf{h}\in\bar{\mathbf{F}}_\varrho(t)$, if $t_0(\bar{B})$ is sufficiently large. Also $p_{\mathfrak{H}^\perp}(v(\mathbf{h}))\in\mathfrak{E}^0\cap\mathfrak{H}^\perp$ and $\mathfrak{E}^0$ is consistent with $\bar{\mathbf{F}}_\varrho(s)$. We have using the R-property

$$\begin{aligned} &\|p_{\mathfrak{H}^\perp}(v(\mathbf{h}))-p_{\mathfrak{Z}_0^\perp(\mathfrak{H})}(v(\mathbf{h}))\| \leqslant 2L(\bar{B})\,\theta/t\varrho \\ &\|p_{\mathfrak{H}^\perp}(\bar{v}(\mathbf{h}))-p_{\mathfrak{Z}_0^\perp(\mathfrak{H})}(v(\mathbf{h}))\| \leqslant 3L(\bar{B})\,\theta/t\varrho \end{aligned} \tag{2.12}$$

for all $\mathbf{h}\in\bar{\mathbf{F}}_\varrho(t)$. Using this, (2.10), (2.11) and Proposition 1.8 we get the following

PROPOSITION 2.1. *There exists* $\mathbf{C}=\mathbf{C}(\varrho,\bar{B})\geqslant 10L(\bar{B})\,\tilde{Q}(\bar{B})/\eta(\bar{B})$ *such that if* (2.9) *holds for some* $0<\theta<0.1\mathbf{C}^{-1}\varrho$, $t\geqslant\max\{t_0(\bar{B})\,\varrho^{-1}, \mathbf{C}/\theta\varrho\}=\bar{t}_0(\varrho,\theta)$ *and* $\bar{v}=v+z\in\mathfrak{G}$, $v\in\mathfrak{E}^0$, $\|v\|\leqslant\theta$, $\|z\|\leqslant\theta/t^{2m+1}$ *then*

$$\exp\bar{v}(\mathbf{h}) = \exp(p_{\mathfrak{Z}_0^\perp(\mathfrak{H})}(v(\mathbf{h}))+r(v,\mathbf{h})+\varepsilon(\bar{v},\mathbf{h}))\cdot\bar{\mathbf{h}} = (\exp\bar{\omega}(\bar{v},\mathbf{h}))\,\bar{\mathbf{h}} \tag{2.13}$$

for all $\mathbf{h}\in\bar{\mathbf{F}}_\varrho(t)$, *where* $r(v,\mathbf{h})\in\mathfrak{Z}_0^\perp(\mathfrak{H})$, $\|r(v,\mathbf{h})\|\leqslant\mathbf{C}\theta^2$, $\varepsilon(\bar{v},\mathbf{h})\in\mathfrak{H}^\perp$, $\|\varepsilon(\bar{v},\mathbf{h})\|\leqslant\mathbf{C}\theta/t$, $\bar{\mathbf{h}}\in\bar{\mathbf{F}}_{\mathbf{C}\theta}(t)$.

Now let (2.9) hold for some $t\geqslant\bar{t}(\varrho,\theta)$ and let $\bar{\varphi}:\bar{\mathbf{F}}_\varrho(t)\to(\exp\bar{v})\mathbf{H}$ be defined by $\bar{\varphi}(\mathbf{h})=\mathbf{h}\exp\bar{\omega}(\bar{v},\mathbf{h})$. Using Proposition 1.9 we obtain that

$$\bar{\varphi}(\mathbf{h}_0\,\mathbf{h}) = \bar{\varphi}(\mathbf{h}_0)\cdot\mathbf{h}\cdot\bar{\psi}(\mathbf{h}_0,\mathbf{h}) \tag{2.14}$$

for all $\mathbf{h}_0, \mathbf{h}_0\mathbf{h}\in\bar{\mathbf{F}}_\varrho(t), \mathbf{h}\in\bar{\mathbf{F}}_\delta(t)$, where $\bar{\psi}(\mathbf{h}_0,\mathbf{h})\in\bar{\mathbf{F}}_{\mathbf{C}\theta\delta}(t)$.

3. The basic lemma

In this section we assume that $\mathfrak{G}$, g, $\mathbf{g}=\exp g$ and u are as in Section 2 and use the notations of that section. Also we choose for convenience a Riemannian metric on $\mathfrak{G}$ in which the subspace $\mathfrak{E}_k$, $k=0,\pm1,\dots,\pm m$ are mutually orthogonal (see Note 1.1). Recall that some of the $\mathfrak{E}_k$ might be trivial.

We assume that $\mathbf{G}$ acts uniformly by homeomorphisms on a metric space (X, d) with a Borel probability measure μ. By [R4, Proposition 1.1] the group $\mathbf{\Lambda}=\mathbf{\Lambda}(\mathbf{G}, X, \mu)$ is a closed Lie subgroup of $\mathbf{G}$. We denote by $\mathfrak{L}(\mathbf{\Lambda})$ the Lie algebra of $\mathbf{\Lambda}$, which might be trivial. Define

$$(3.1) \qquad \mathfrak{N}=\mathfrak{L}(\mathbf{\Lambda})\cap\mathfrak{E}^{-2}, \quad \mathfrak{H}=\mathfrak{L}(\mathbf{\Lambda})\cap\mathfrak{E}^{-}$$

$\mathfrak{E}^{-}=\mathfrak{E}^{-1}$. We assume that $\mathbf{u}=\exp u\in\mathbf{\Lambda}$. Then u normalizes $\mathfrak{L}(\mathbf{\Lambda})$, $\mathfrak{N}$ and $\mathfrak{H}$ by Proposition 1.3.

We shall construct a special decomposition of $\mathfrak{G}$ induced by $\mathfrak{L}(\mathbf{\Lambda})$, which plays a crucial role in the proof of the main theorem.

Let $\mathfrak{A}_1=\mathfrak{A}_1(u)=\{v\in\mathfrak{N}^{\perp}: \mathrm{ad}_u(v)\in\mathfrak{N}\}$ and let $\mathfrak{P}=\mathfrak{A}_1+\mathfrak{N}$. It follows from Lemma 2.1 that $\mathfrak{P}\subset\mathfrak{E}^0$. Note that $\mathfrak{E}_k\subset\mathfrak{N}^{\perp}$ for all $k>-2$ by our choice of the Riemannian metric on $\mathfrak{G}$.

We say that a set Q of vectors in $\mathfrak{G}$ is a k-set, $k=0, \pm 1, \ldots, \pm m$ if $\chi(v)=k$, $\|v\|=1$ for all $v\in Q$ and $\{p_k(v): v\in Q\}$ is linearly independent in $\mathfrak{E}_k$. For $n\in\mathbf{Z}$ define

$$\nu(n)=n/2 \text{ if } n \text{ is even}; \quad \nu(n)=(n+1)/2 \text{ if } n \text{ is odd}$$

and set $\nu(v)=\nu(\chi(v))$. It follows from Theorem 1.1 that if $\chi(v)>0$ then $\mathrm{ad}_u^{\nu(v)}(v)\neq 0$ and

$$(3.2) \qquad \begin{aligned} &\chi(\mathrm{ad}_u^{\nu(v)}(v))=0 \quad \text{if } \chi(v) \text{ is even} \\ &\chi(\mathrm{ad}_u^{\nu(v)}(v))=-1 \quad \text{if } \chi(v) \text{ is odd.} \end{aligned}$$

Let $\mathfrak{P}^{\perp}$ be the orthogonal complement of $\mathfrak{A}_1$ in $\mathfrak{N}^{\perp}$ and let $W^{(k)}, k=1, \ldots, m$ be a maximal k-set in $\mathfrak{P}^{\perp}$ such that

$$(3.3) \qquad \mathrm{ad}_u^{\nu(k)}(v)\in\mathfrak{L}(\mathbf{\Lambda})$$

for all $v\in W^{(k)}, k=1, \ldots, m$. Note that some (or all) of the $W^{(k)}$ might be empty.

Let $\mathfrak{D}_1=\mathfrak{E}^0\cap\mathfrak{P}^{\perp}$. Let $\mathfrak{E}_k^{(1)}$ denote the orthogonal complement in $\mathfrak{E}_k$ of the subspace spanned by $\{p_k(w): w\in W^{(k)}\}$ if $W^{(k)}\neq\varnothing$ and set $\mathfrak{E}_k^{(1)}=\mathfrak{E}_k$ if $W^{(k)}=\varnothing, k=1,2, \ldots, m$. Let $\mathfrak{V}_1$ be the subspace of $\mathfrak{N}^{\perp}$ spanned by $\mathfrak{D}_1$ and $\mathfrak{E}_k^{(1)}, k=1, \ldots, m$. It follows from the definition of $W^{(k)}$ that

$$(3.4) \qquad \mathrm{ad}_u^{\nu(v)}(v)\notin\mathfrak{L}(\mathbf{\Lambda})$$

for all $0\neq v\in\mathfrak{V}_1$ with $\chi(v)>0$. Also $\mathfrak{G}=\mathfrak{W}+\mathfrak{V}_1+\mathfrak{P}$, where $\mathfrak{W}$ is the subspace of $\mathfrak{P}^{\perp}$ spanned by $W^{(k)}$, $k=1, \ldots, m$.

Now we shall construct by induction a sequence $\mathfrak{B}_2, \ldots, \mathfrak{B}_{m+1}$ of subspaces of $\tilde{\mathfrak{B}}_1$ by the following procedure. Set

$$\mathfrak{B}_2 = \{v \in \tilde{\mathfrak{B}}_1 \colon \mathrm{ad}_u^2(v) \in \mathfrak{N}\}$$

and assume that $\mathfrak{B}_2, \ldots, \mathfrak{B}_n$ have been constructed. For $k=1, \ldots, m$ let $Y_n^{(k)}$ be a maximal k-set in $\mathfrak{B}_n$ and let $\mathfrak{D}_n = \mathfrak{E}^0 \cap \mathfrak{B}_n$. Note that for some (or all) k the set $Y_n^{(k)}$ might be empty. Let $\mathfrak{E}_k^{(n)}$ be the orthogonal complement in $\mathfrak{E}_k^{(n-1)}$ of the subspace of $\mathfrak{E}_k^{(n-1)}$ spanned by $\{p_k(v) \colon v \in Y_n^{(k)}\}$ if $Y_n^{(k)} \neq \emptyset$ and $\mathfrak{E}_k^{(n)} = \mathfrak{E}_k^{(n-1)}$ if $Y_n^{(k)} = \emptyset$. Also let $\tilde{\mathfrak{D}}_n$ be the orthogonal complement of $\mathfrak{D}_n$ in $\tilde{\mathfrak{D}}_{n-1}$ and $\tilde{\mathfrak{B}}_n$ the subspace spanned by $\mathfrak{E}_k^{(n)}$ and $\tilde{\mathfrak{D}}_n$, $k=1, \ldots, m$. Define

$$\mathfrak{B}_{n+1} = \{v \in \tilde{\mathfrak{B}}_n \colon \mathrm{ad}_u^{n+1}(v) \in \mathfrak{N}\}.$$

This completes our construction. Let

$$\mathfrak{B} = \sum_{i=2}^{m+1} \mathfrak{B}_i, \quad \mathfrak{D} = \sum_{i=2}^{m+1} \mathfrak{D}_i = \mathfrak{E}^0 \cap \mathfrak{P}^{\perp}$$

and let $\mathfrak{Y}$ be the subspace of $\mathfrak{B}$ spanned by $Y = \bigcup_{n=2}^{m+1} \bigcup_{k=1}^{m} Y_n^{(k)}$. We have

$$\mathfrak{B} = \mathfrak{Y} + \mathfrak{D}, \quad \mathfrak{G} = \mathfrak{W} + \mathfrak{B} + \mathfrak{P}. \tag{3.5}$$

Next we shall decompose $\mathfrak{P}$. We have $\mathfrak{H} \subset \mathfrak{P}$. Let

$$\tilde{\mathfrak{J}}(\mathfrak{H}) = \mathfrak{J}(\mathfrak{H}) \cap \mathfrak{P}.$$

We have $\tilde{\mathfrak{J}}(\mathfrak{H}) \subset \mathfrak{J}_0(\mathfrak{H}) = \mathfrak{E}^0 \cap \mathfrak{J}(\mathfrak{H}) \subset \mathfrak{J}(\mathfrak{N})$. Also $\tilde{\mathfrak{J}}(\mathfrak{H}) = \mathfrak{J}_0(\mathfrak{H})$ if $u \in \mathfrak{N}$. Let $\mathfrak{Z}$ be the orthogonal complement of $\tilde{\mathfrak{J}}(\mathfrak{H})$ in $\mathfrak{P}$ and let $\mathfrak{F} = \mathfrak{D} + \mathfrak{Z}$. We have

$$\mathfrak{G} = \mathfrak{W} + \mathfrak{Y} + \mathfrak{F} + \tilde{\mathfrak{J}}(\mathfrak{H}). \tag{3.6}$$

Now let $u^* \in \mathfrak{E}_2$ be such that $[u, u^*] = g$ and let $\tilde{\mathfrak{E}}_2 \subset \mathfrak{E}_2$ be spanned by $\{p_2(w), w \in W^{(2)}\}$. We can assume without loss of generality that $u^* \in \mathfrak{E}_2^{(1)}$ if $u^* \notin \tilde{\mathfrak{E}}_2$. Then $u^* \in \mathfrak{B}_2$ if $u \in \mathfrak{N}$. In this case we assume that $u^* \in Y_2^{(2)}$. Define

$$\bar{\Psi} = \bigcup_{i=2}^{m+1} \bigcup_{k > i-1} Y_i^{(k)} = \{\psi_1, \ldots, \psi_l\} \subset Y$$

$$Y - \Psi = \{c_1, \ldots, c_r\}.$$

It follows from the definition of Ψ that if $u^* \notin \mathfrak{E}_2$ and $u \in \mathfrak{N}$ then $u^* \in \Psi$ (since in this case $u^* \in Y_2^{(2)} \subset \Psi$) and hence $\Psi \neq \emptyset$. In case $u^* \notin \mathfrak{E}_2$ and $u \notin \mathfrak{N}$ the set Ψ might or might not be empty. Thus if $u^* \notin \mathfrak{E}_2$ then

$$\text{either } \Psi \neq \emptyset \text{ or } u \notin \mathfrak{N}. \tag{3.7}$$

We shall show in subsequent sections that when $\mathbf{G}$ acts by right translations on $(X = \boldsymbol{\Gamma} \backslash \mathbf{G}, \mu)$ and the action of $\mathbf{N_u} = \{\mathbf{u}^k \exp \mathfrak{N} : k \in \mathbf{Z}\}$ is ergodic then the following holds. If $\Psi \neq \emptyset$ or $u \notin \mathfrak{N}$ then $\mu(D(\mathbf{g})) = 1$. Otherwise there is $x \in X$ such that $\mu(x \exp(\mathfrak{W} + \mathfrak{J}(\mathfrak{H}))) = 1$. The following basic lemma makes a first step towards this goal. Namely, it shows that there is a set $A \subset X$ of positive μ-measure such that if $x, y \in A$ are sufficiently close and $y = x \exp \bar{v}$ for some $\bar{v} = w + y + f + i$, $w \in \mathfrak{W}$, $y \in \mathfrak{Y}$, $f \in \mathfrak{F}$, $i \in \mathfrak{J}(\mathfrak{H})$ with $\|w\|$ small then $\|y\|$ and $\|f\|$ ought to be small. The order of magnitude is also important. In order to state this lemma define

$$W = \bigcup_{k=1}^{m} W^{(k)} = \{w_1, \ldots, w_p\}$$

$$\mathfrak{W}(\alpha, t) = \left\{ w \in \mathfrak{W} : w = \sum_{i=1}^{p} \alpha_i w_i,\ |\alpha_i| \leq \alpha t^{-\nu(w_i)},\ i = 1, \ldots, p \right\} \tag{3.8}$$

where $\alpha > 0$ is small and $t > 1$ is large. Also for $v \in \mathfrak{G}$ let $w(v), y(v), f(v)$ denote the projection of v onto $\mathfrak{W}$, $\mathfrak{Y}$ and $\mathfrak{F}$ respectively. The set $\Psi \cup (Y - \Psi) = \{\psi_1, \ldots, \psi_l, c_1, \ldots, c_r\}$ is a basis in $\mathfrak{Y}$. Let $c_i(v)$ and $\psi_j(v)$ denote the c_i-coordinate and the ψ_j-coordinate of $y(v)$ respectively, $i = 1, \ldots, r; j = 1, \ldots, l$.

LEMMA 3.1 (Basic). *Suppose that the action of* $\mathbf{N_u}$ *on* (X, d, μ) *is ergodic. Then there are constants* $0 < \Theta < 1$, $0 < a < 1$, $\mathbf{C} > 1$ *with the following property. Given* $0 < \theta \leq \Theta$, $\varepsilon > 0$ *there are* $t(\varepsilon, \theta) > 1$, $0 < \gamma = \gamma(\varepsilon, \theta) < 1$ *and a compact* $A = A(\varepsilon, \theta) \subset X$, $\mu(A) > 1 - \varepsilon$ *such that if* $x, y \in A$, $y = x \exp i_1 \exp v \exp i_2$, $i_1, i_2 \in \mathfrak{J}(\mathfrak{H})$, $v \in \mathfrak{G}$, $\|i_1\|, \|i_2\|, \|v\| \leq \gamma$ *and* $w(v) \in \mathfrak{W}(a\theta, t)$ *for some* $t \geq t(\varepsilon, \theta)$ *then*

$$|c_i(v)| \leq \mathbf{C}\theta t^{-\chi(c_i)}, \quad i = 1, \ldots, r$$

$$|\psi_j(v)| \leq \mathbf{C}\theta t^{-\nu(\psi_j)}, \quad j = 1, \ldots, l$$

$$\|f(v)\| \leq \mathbf{C}\theta t^{-\beta}$$

for some $\beta > 0$. (*In fact,* β *can be taken to be* $1/2(m+1)$.)

To prove Lemma 3.1 we use Proposition 1.11 and the R-property for $\mathbf{N_u}$ and $\mathbf{H_u}$. From now till the end of this section we assume that $u \in \mathfrak{N}$ and note that our argument below works for the case $u \notin \mathfrak{N}$ as well. One should only substitute $\mathbf{N}$ by $\mathbf{N_u}$ and $\mathbf{F}_\varrho(s)$ by the regular sequence in $\mathbf{N_u}$ described in Note 1.2. We begin with the study of the decomposition (3.5).

LEMMA 3.2. *If* $0 \neq v \in \mathfrak{V}_i$, $i=2, \dots, m+1$ *then*

$$\mathrm{ad}_u^{i-1}(v) \notin \mathfrak{L}(\mathbf{\Lambda}). \tag{3.9}$$

Proof. We have

$$\mathrm{ad}_u^{i-1}(v) \notin \mathfrak{N} \tag{3.10}$$

for all $0 \neq v \in \mathfrak{V}_i$ by the definition of $\mathfrak{V}_i$. First let us prove (3.9) for $0 \neq v \in \mathfrak{V}_i$ with $\chi(v) \leq 0$. It follows from (3.10) that $\mathrm{ad}_u^{i-1}(v) \neq 0$ and $\chi(\mathrm{ad}_u^{i-1}(v)) \leq -2(i-1) \leq -2$, since $i \geq 2$. This implies that $\mathrm{ad}_u^{i-1}(v) \in \mathfrak{E}^{-2}$ and therefore $\mathrm{ad}_u^{i-1}(v) \notin \mathfrak{L}(\mathbf{\Lambda})$ by (3.10) and the definition of $\mathfrak{N}$.

Now let $\chi(v)>0$, $0 \neq v \in \mathfrak{V}_i$. Then $\nu(v) \leq i-1$, since $\mathrm{ad}_u^{\nu(v)}(v) \notin \mathfrak{N}$ by (3.4), but $\mathrm{ad}_u^i(v) \in \mathfrak{N}$ by the definition of $\mathfrak{V}_i$. If $\nu(v)=i-1$ we are done by (3.4). Let $\nu(v) \leq i-2$. Then $\chi(v) \leq 2(i-2)$ and therefore $\chi(\mathrm{ad}_u^{i-1}(v)) \leq 2(i-2)-2(i-1)=-2$. This and (3.10) imply that $\mathrm{ad}_u^{i-1}(v) \notin \mathfrak{L}(\mathbf{\Lambda})$. This completes the proof. □

Henceforth $p_\perp(v)$ denotes the projection of v onto $\mathfrak{N}^\perp$.

LEMMA 3.3. *There exists* $0<c=c(u, \mathbf{\Lambda}) \leq 0.1$, $0<\omega=\omega(c, u)<1$ *and* $\bar{t}=\bar{t}(c, u)>1$ *such that if* $v \in \mathfrak{V}$ *and*

$$\max\{\|p_\perp(v(\mathbf{u}(s)))\|: 0 \leq s \leq t\} = \theta$$

for some $0<\theta<1$, $t \geq \bar{t}$ *then there is* $0 \leq s_0 = s_0(c, \theta, v, t) \leq t$ *such that*

$$\|p_\perp(v(\mathbf{u}(s_0)))\| \geq \omega\theta$$

$$p_\perp(v(\mathbf{u}(s_0)))+r \notin \mathfrak{L}(\mathbf{\Lambda})$$

for all $r \in \mathfrak{G}$ *with* $\|r\| \leq 10c\|p_\perp(v(\mathbf{u}(s_0)))\|$.

Proof. We shall use Proposition 1.6. Write

$$\mathfrak{Z}_i = \mathrm{ad}_u^{i-1}(\mathfrak{V}_i)$$

$i=2,\dots,m+1$. We have using Lemma 3.2

$$\mathfrak{Z}_i\cap\mathfrak{L}(\mathbf{\Lambda})=\{0\},\quad p_\perp(\mathfrak{Z}_i)\cap\mathfrak{L}(\mathbf{\Lambda})=\{0\},\quad i=2,\dots,m+1.$$

Let $0<c=c(u,\mathbf{\Lambda})\leq 0.1$ be so small that if $w\in\bigcup_{i=2}^{m+1}p_\perp(\mathfrak{Z}_i)$ then

$$w+r\notin\mathfrak{L}(\mathbf{\Lambda}) \tag{3.11}$$

for all $r\in\mathfrak{G}$ with $\|r\|\leq 20c\|w\|$. Now let $\bar{t}=t_0(c,u,\mathfrak{N})>1$, $0<\omega=\omega(c,u,\mathfrak{N})<1$ be as in Proposition 1.6 and let $t\geq\bar{t}$. It follows then from Proposition 1.6 that there is $0\leq s_0\leq t$ and $j\in\{2,\dots,m+1\}$ such that

$$\begin{gathered}\|p_\perp(v(\mathbf{u}(s_0)))\|\geq\omega\theta\\ \|p_\perp(v(\mathbf{u}(s_0)))-p_\perp(\hat{v}_j(u,s_0))\|\leq c\|p_\perp(\hat{v}_j(u,s_0))\|\end{gathered}\tag{3.12}$$

where $\hat{v}_j(u,s_0)$ is proportional to $\mathrm{ad}_u^{j-1}(v_j)$ and v_j denotes the projection of v onto $\mathfrak{V}_j$. We have $p_\perp(\hat{v}_j(u,s_0))\in p_\perp(\mathfrak{Z}_j)$. This, (3.11) and (3.12) imply the lemma. □

It follows from the definition of $\mathfrak{N}$ and $\mathfrak{H}$ that

$$(\mathfrak{N}^\perp\cap\mathfrak{E}^{-2})\cap\mathfrak{L}(\mathbf{\Lambda})=\{0\},\quad(\mathfrak{H}^\perp\cap\mathfrak{E}^{-})\cap\mathfrak{L}(\mathbf{\Lambda})=\{0\}.$$

We shall assume that $0<c<0.1$ in Lemma 3.3 is so small that if $\xi\in\mathfrak{N}^\perp\cap\mathfrak{E}^{-2}$, $\xi'\in\mathfrak{H}^\perp\cap\mathfrak{E}^-$ then

$$\xi+r\notin\mathfrak{L}(\mathbf{\Lambda}),\quad\xi'+r'\notin\mathfrak{L}(\mathbf{\Lambda}) \tag{3.13}$$

for all $r,r'\in\mathfrak{G}$ with $\|r\|\leq 10c\|\xi\|$, $\|r'\|\leq 10c\|\xi'\|$.

Now let $B=\{b_1,\dots,b_q\}$ be a regular basis in $\mathfrak{N}$, $u=b_1$ and $\mathbf{F}_\varrho(s)$ the B-regular sequence in $\mathbf{N}=\exp\mathfrak{N}$ defined in (2.2) with some fixed $0<\varrho<0.1\varrho_0$. Now we shall specify the choice of θ. Let $\mathbf{C}=\max\{\mathbf{C}(B,\varrho),\mathbf{C}(\bar{B},\varrho)\}$, $t_0=\max\{t_0(B),t_0(\bar{B})\}$, $\eta=\min\{\eta(B),\eta(\bar{B})\}$ (see Corollary 2.2, Proposition 2.1 and the R-property) where $\bar{B}$ is a regular basis in $\mathfrak{H}$ and $\bar{\mathbf{F}}_\varrho(s)$ the $\bar{B}$-regular sequence in $\mathbf{H}=\exp\mathfrak{H}$ defined in (2.8). Let $0<\theta_1<1$ be so small that if $v\in\mathfrak{G}$, $v\notin\mathfrak{L}(\mathbf{\Lambda})$ and $\|v\|\leq 2\theta_1$, then $\exp v\notin\mathbf{\Lambda}$. Let

$$\Theta=0.01\min\{\theta_1,\varrho c\omega\mathbf{C}^{-2}\},\quad\alpha=0.1c\mathbf{C}^{-1} \tag{3.14}$$

where c and ω are as in Lemma 3.3 and let $0<\theta<\Theta$ be fixed. Define

$$\bar{\theta}=0.1c\theta\mathbf{C}^{-1},\quad\theta_0=\omega\bar{\theta},\quad a(\theta)=\alpha\theta.$$

We shall denote by $\mathfrak{K}$ the closure of the set

$$\{v\in\mathfrak{I}(\mathfrak{N}): 0.1\theta_0\leq\|v\|\leq 2\theta,\ v+r\notin\mathfrak{L}(\mathbf{\Lambda}) \text{ for all } r\in\mathfrak{G} \text{ with } \|r\|\leq c\|v\|\}$$

and define $\mathbf{K}=\exp\mathfrak{K}$. The set $\mathbf{K}$ is a compact subset of $\mathbf{I}(\mathbf{N})-\mathbf{\Lambda}$, which might be empty.

Now let $w\in\mathfrak{W}(a,t)$. Write $\nu_i=\nu(w_i)$ and recall that $\nu_i\geq 1$ for all $i=1,\dots,p$. We have

$$\begin{aligned} w(\mathbf{u}(s)) &= \sum_{i=1}^{p}\sum_{j=0}^{m}\frac{s^j}{j!}\operatorname{ad}_u^j(\alpha_i w_i)\\ &= \sum_{i=1}^{p}\sum_{j=0}^{\nu_i-1}\frac{s^j}{j!}\operatorname{ad}_u^j(\alpha_i w_i)+\sum_{i=1}^{p}\frac{s^{\nu_i}}{\nu_i!}\operatorname{ad}_u^{\nu_i}(\alpha_i w_i)\\ &\quad+\sum_{i=1}^{p}\sum_{j=\nu_i+1}^{m}\frac{s^j}{j!}\operatorname{ad}_u^j(\alpha_i w_i) = w'(s)+\hat{w}(s)+w_{\mathfrak{N}}(s)\end{aligned}$$

where $\hat{w}(s)\in\mathfrak{L}(\mathbf{\Lambda})$, $w_{\mathfrak{N}}(s)\in\mathfrak{N}$ and

$$\|\hat{w}(s)\|\leq \mathbf{C}a,\quad \|w'(s)\|\leq \mathbf{C}at^{-1} \tag{3.15}$$

for all $0\leq s\leq t$ by (3.3), (3.8) and (1.17). We have $\hat{w}(s)\in\mathfrak{L}(\mathbf{\Lambda})$ and $\chi(\hat{w}(s))\leq 0$ by (3.2). Therefore $\hat{w}(s)\in\mathfrak{I}(\mathfrak{N})$ and $p_\perp(\hat{w}(s))=p_{\mathfrak{I}}^\perp(\hat{w}(s))$, where $p_{\mathfrak{I}}^\perp$ denotes the projection onto $\mathfrak{N}_{\mathfrak{I}}^\perp=\mathfrak{I}(\mathfrak{N})\cap\mathfrak{N}^\perp$. This and (3.15) imply that if $w\in\mathfrak{W}(a,t)$ then

$$\begin{aligned}&\|p_\perp(w(\mathbf{u}(s)))-p_{\mathfrak{I}}^\perp(w(\mathbf{u}(s)))\|\leq \mathbf{C}at^{-1}\\ &\|p_{\mathfrak{I}}^\perp(w(\mathbf{u}(s)))\|\leq \mathbf{C}a(1+t^{-1})\end{aligned} \tag{3.16}$$

for all $0\leq s\leq t$.

For $\bar{v}\in\mathfrak{G}$ write $\bar{v}=w+v+p$, where $w\in\mathfrak{W}$, $v\in\mathfrak{V}$, $p\in\mathfrak{P}$ (see (3.5)) and let

$$\beta(\bar{v},t)=\max\{\|p_\perp(\bar{v}(\mathbf{h}))\|: \mathbf{h}\in\mathbf{F}_\varrho(t)\}.$$

LEMMA 3.4. *Given $0<\delta<0.1c\theta_0$, $T>1$ there exist $t(\delta,T)>T$ and $0<\gamma=\gamma(u,\delta,T)<\delta$ such that if $\bar{v}\in\mathfrak{G}$, $\bar{v}=v+w+p$, $\|\bar{v}\|\leq\gamma$ and*

$$w\in\mathfrak{W}(a(\theta),t),\quad \beta(\bar{v},t)=\theta \tag{3.17}$$

for some $t\geq t(\delta,T)$ then there exists $\mathbf{h}_t\in\mathbf{F}_\varrho(t)$ such that

$$\exp\bar{v}(\mathbf{h})=\exp(k(\bar{v},\mathbf{h})+z(\bar{v},\mathbf{h}))\cdot\bar{\mathbf{h}}=\exp\omega(\bar{v},\mathbf{h})\cdot\bar{\mathbf{h}} \tag{3.18}$$

for all $\mathbf{h}\in\mathbf{h}_t\mathbf{F}_{\eta\delta\varrho}(t)$, *where*

$$\tilde{\mathbf{h}}\in\mathbf{F}_{C\theta}(t),\quad k(\bar{v},\mathbf{h})\in\mathfrak{K},\quad \|z(\bar{v},\mathbf{h})\|\leq 4\delta,\quad z(\bar{v},\mathbf{h})\in\mathfrak{N}^{\perp}. \tag{3.19}$$

Proof. Let $0<\gamma<0.1\delta$ be so small that if $\bar{v}\in\mathfrak{G}$, $\|\bar{v}\|\leq\gamma$ and $\beta(\bar{v},t')=\theta_0/2$ for some $t'>1$ then

$$t'\geq\max\{T,10t_0\bar{t}m\mathbf{C}/\varrho\delta\theta_0\}=t(\delta,T) \tag{3.20}$$

where $\bar{t}$ is as in Lemma 3.3. Now let $\|\bar{v}\|\leq\gamma$ and let (3.17) hold for some $t\geq t(\delta,T)$. In particular, we have

$$\max\{\|p_{\perp}(\bar{v}(\mathbf{u}(s)))\|:0\leq s\leq t\varrho\}\leq\theta.$$

We have using (3.16)

$$\begin{gathered}p_{\perp}(\bar{v}(\mathbf{u}(s)))=p_{\perp}(v(\mathbf{u}(s)))+p_{\perp}(w(\mathbf{u}(s)))+p_{\perp}(p)\\ \|p_{\perp}(w(\mathbf{u}(s)))-p_{\mathfrak{Z}}^{\perp}(w(\mathbf{u}(s)))\|\leq\mathbf{C}a(\theta)\varrho^{-1}t^{-1}\leq 0.01\delta\\ \|p_{\mathfrak{Z}}^{\perp}(w(\mathbf{u}(s)))\|\leq 0.1c\theta_0(1+\delta)\end{gathered} \tag{3.21}$$

for all $0\leq s\leq\varrho t$. Suppose first that

$$\max\{\|p_{\perp}(v(\mathbf{u}(s)))\|:0\leq s\leq\varrho t\}\geq\bar{\theta}. \tag{3.22}$$

By Lemma 3.3 there exists $s_0\in[t(\delta,T),t]$ such that

$$\|p_{\perp}(v(\mathbf{u}(s_0)))\|\geq\theta_0,\quad p_{\perp}(v(\mathbf{u}(s_0)))+r\notin\mathfrak{L}(\mathbf{\Lambda}) \tag{3.22}$$

for all $r\in\mathfrak{G}$ with $\|r\|\leq 10c\|p_{\perp}(v(\mathbf{u}(s_0)))\|$. Write $\tilde{\theta}=\|p_{\perp}(v(\mathbf{u}(s_0)))\|$. We have using the R-property

$$\begin{gathered}\|p_{\perp}(\bar{v}(\mathbf{u}(s_0)))-p_{\mathfrak{Z}}^{\perp}(\bar{v}(\mathbf{u}(s_0)))\|\leq\mathbf{C}\theta/t\varrho\leq 0.1\delta\\ p_{\mathfrak{Z}}^{\perp}(\bar{v}(\mathbf{u}(s_0)))=p_{\mathfrak{Z}}^{\perp}(v(\mathbf{u}(s_0)))+p_{\mathfrak{Z}}^{\perp}(w(\mathbf{u}(s_0)))+p_{\mathfrak{Z}}^{\perp}(p).\end{gathered} \tag{3.24}$$

This and (3.21) imply

$$\begin{gathered}\|p_{\perp}(v(\mathbf{u}(s_0)))-p_{\mathfrak{Z}}^{\perp}(v(\mathbf{u}(s_0)))\|\leq 0.3\delta\\ \|p_{\mathfrak{Z}}^{\perp}(\bar{v}(\mathbf{u}(s_0)))-p_{\mathfrak{Z}}^{\perp}(v(\mathbf{u}(s_0)))-p_{\mathfrak{Z}}^{\perp}(w(\mathbf{u}(s_0)))\|\leq 0.1\delta.\end{gathered} \tag{3.25}$$

Therefore

$$0.9\tilde{\theta} \leq \|p_{\mathfrak{Z}}^{\perp}(v(\mathbf{u}(s_0)))\| \leq 1.1\tilde{\theta} \leq 1.1\theta \tag{3.26}$$

since $\delta \leq 0.1c\theta_0 \leq 0.1\tilde{\theta}$. We have using the R-property

$$\|p_{\mathfrak{Z}}^{\perp}(\bar{v}(\mathbf{h})) - p_{\mathfrak{Z}}^{\perp}(\bar{v}(\mathbf{u}(s_0)))\| \leq \delta\theta \leq 0.1\delta \tag{3.27}$$

for all $\mathbf{h} \in \mathbf{u}(s_0)\mathbf{F}_{\eta\delta\varrho}(t)$. Now we use (1.22) and Corollary 2.2 to write

$$\exp \bar{v}(\mathbf{h}) = \exp(p_{\mathfrak{Z}}^{\perp}(\bar{v}(\mathbf{h})) + r(\bar{v}, \mathbf{h}) + \varepsilon(\bar{v}, \mathbf{h})) \cdot \bar{\mathbf{h}} \tag{3.28}$$

for all $\mathbf{h} \in \mathbf{F}_{\varrho}(t)$, where

$$\begin{gathered} \bar{\mathbf{h}} \in \mathbf{F}_{C\theta}(t), \quad r(\bar{v}, \mathbf{h}) \in \mathfrak{N}_{\mathfrak{Z}}^{\perp}, \quad \|r(\bar{v}, \mathbf{h})\| \leq \mathbf{C}\theta^2 \leq 0.1c\tilde{\theta} \\ \varepsilon(\bar{v}, \mathbf{h}) \in \mathfrak{N}^{\perp}, \quad \|\varepsilon(\bar{v}, \mathbf{h})\| \leq \mathbf{C}\theta/t\varrho \leq 0.1\delta \end{gathered} \tag{3.29}$$

by our choice of θ. Now set

$$k(\bar{v}, \mathbf{h}) = p_{\mathfrak{Z}}^{\perp}(v(\mathbf{u}(s_0))) + p_{\mathfrak{Z}}^{\perp}(w(\mathbf{u}(s_0))) + r(\bar{v}, \mathbf{h}) \in \mathfrak{Z}(\mathfrak{N}).$$

We have using (3.21), (3.26) and (3.29)

$$0.7\theta_0 \leq 0.7\tilde{\theta} \leq \|k(\bar{v}, \mathbf{h})\| \leq 1.2\tilde{\theta} \leq 1.2\theta$$

$$\|p_{\mathfrak{Z}}^{\perp}(\bar{v}(\mathbf{h})) + r(\bar{v}, \mathbf{h}) - k(\bar{v}, \mathbf{h})\| \leq 0.6\delta$$

for all $\mathbf{h} \in \mathbf{u}(s_0)\mathbf{F}_{\eta\delta\varrho}(t)$ by (3.27). Finally, it follows from (3.21), (3.23) and (3.29) that

$$k(\bar{v}, \mathbf{h}) + r \notin \mathfrak{L}(\mathbf{\Lambda})$$

for all $r \in \mathfrak{G}$ with $\|r\| \leq c\|k(\bar{v}, \mathbf{h})\|$. This shows that $k(\bar{v}, \mathbf{h}) \in \mathfrak{K}$ and (3.18), (3.19) hold for all $\mathbf{h} \in \mathbf{u}(s_0)\mathbf{F}_{\eta\delta\varrho}(t)$. This proves our lemma for the case (3.22), if we set $\mathbf{h}_t = \mathbf{u}(s_0)$.

Now assume that

$$\max\{\|p_{\perp}(\bar{v}(\mathbf{u}(s)))\| : 0 \leq s \leq \varrho t\} < \bar{\theta}. \tag{3.30}$$

Then there exists $\mathbf{h}_t \in \mathbf{F}_{\varrho}(t)$ such that

$$\|p_{\perp}(\bar{v}(\mathbf{h}_t))\| = \theta. \tag{3.31}$$

We have using Corollary 2.1 and the R-property

$$
(3.32)\qquad \begin{aligned} &\|p_\perp(\bar v(\mathbf{h}_t)) - p_{\mathfrak{Z}}^\perp(\bar v(\mathbf{h}_t))\| \leq \mathbf{C}\theta/t\varrho \leq 0.1\delta \\ &p_\perp(\bar v(\mathbf{h}_t)) = v^0(\mathbf{h}_t) + \xi_{\mathbf{h}_t} + z_{\mathbf{h}_t} \end{aligned}
$$

where $\|v^0(\mathbf{h}_t)\| \leq \mathbf{C}\bar\theta \leq 0.1c\theta$, $\xi_{\mathbf{h}_t} \in \mathfrak{N}^\perp \cap \mathfrak{E}^{-2}$, $\|z_{\mathbf{h}_t}\| \leq \mathbf{C}\bar\theta/t\varrho \leq 0.1\delta$. This implies

$$0.9\theta \leq \|\xi_{\mathbf{h}_t}\| \leq 1.1\theta$$

$$\|p_{\mathfrak{Z}}^\perp(\bar v(\mathbf{h}_t)) - \xi_{\mathbf{h}_t}\| \leq 0.5c\theta.$$

This gives via (3.13)

$$(3.33)\qquad p_{\mathfrak{Z}}^\perp(\bar v(\mathbf{h}_t)) + r \notin \mathfrak{L}(\mathbf{\Lambda})$$

for all $r \in \mathfrak{G}$ with $\|r\| \leq 5c\|p_{\mathfrak{Z}}^\perp(\bar v(\mathbf{h}_t))\|$. As above we use the R-property to get

$$(3.34)\qquad \|p_{\mathfrak{Z}}^\perp(\bar v(\mathbf{h})) - p_{\mathfrak{Z}}^\perp(\bar v(\mathbf{h}_t))\| \leq \delta\theta$$

for all $\mathbf{h} \in \mathbf{h}_t \mathbf{F}_{\eta\delta\varrho}(t)$. Now set for $\mathbf{h} \in \mathbf{h}_t \mathbf{F}_{\eta\delta\varrho}(t)$

$$k(\bar v, \mathbf{h}) = p_{\mathfrak{Z}}^\perp(\bar v(\mathbf{h}_t)) + r(\bar v, \mathbf{h}) \in \mathfrak{Z}(\mathfrak{N})$$

where $r(\bar v, \mathbf{h})$ is as in (3.28). We have using (3.31), (3.32), (3.29) and (3.34)

$$0.7\theta \leq \|k(\bar v, \mathbf{h})\| \leq 1.3\theta$$

$$\|p_{\mathfrak{Z}}^\perp(\bar v(\mathbf{h})) + r(\bar v, \mathbf{h}) - k(\bar v, \mathbf{h})\| \leq 0.3\delta$$

for all $\mathbf{h} \in \mathbf{h}_t \mathbf{F}_{\eta\delta\varrho}(t)$. Finally, it follows from (3.29) and (3.33) that

$$k(\bar v, \mathbf{h}) + r \notin \mathfrak{L}(\mathbf{\Lambda})$$

for all $r \in \mathfrak{G}$ with $\|r\| \leq c\|k(\bar v, \mathbf{h})\|$. This shows that $k(\bar v, \mathbf{h}) \in \mathfrak{K}$ and (3.18), (3.19) hold for every $\mathbf{h} \in \mathbf{h}_t \mathbf{F}_{\eta\delta\varrho}(t)$. This completes the proof. □

Now we shall prove a similar lemma for $\mathbf{H}$, using (3.13) and Proposition 2.1. Let $\bar B$ and $\bar{\mathbf{F}}_\varrho(s)$ be as in Proposition 2.1. For $\bar v \in \mathfrak{G}$ write

$$\bar\beta(\bar v, s) = \max\{\|p_{\mathfrak{H}^\perp}(\bar v(\mathbf{h}))\| : \mathbf{h} \in \bar{\mathbf{F}}_\varrho(s)\}.$$

LEMMA 3.5. *For a given* $0<\delta<0.1c\theta$, $T>1$ *let* $t(\delta, T)$ *be as in* (3.19). *Suppose that* $\bar\beta(\bar v, t)=\theta$ *for some* $t\geq t(\delta, T)$ *and some* $\bar v\in\mathfrak{G}$, $\bar v=z+v$, $v\in\mathfrak{E}^0$, $\|v\|\leq\delta$ *with* $\|z\|\leq\theta/t^{2m+1}$. *Then there exists* $\mathbf{h}_t\in\bar{\mathbf{F}}_\varrho(t)$ *such that*

$$(3.35)\qquad \exp(\bar v(\mathbf{h}))=\exp(k(\bar v,\mathbf{h})+z(\bar v,\mathbf{h}))\cdot\bar{\mathbf{h}}=\exp\bar\omega(\bar v,\mathbf{h})\cdot\bar{\mathbf{h}}$$

for all $\mathbf{h}\in\mathbf{h}_t\bar{\mathbf{F}}_{\eta\delta\varrho}(t)$, *where* $k(\bar v,\mathbf{h})\in\mathfrak{K}$, $\bar{\mathbf{h}}\in\bar{\mathbf{F}}_{\mathbf{C}\theta}(t)$, $z(\bar v,\mathbf{h})\in\mathfrak{N}^\perp$, $\|z(\bar v,\mathbf{h})\|\leq4\delta$.

Proof. Let $\mathbf{h}_t\in\bar{\mathbf{F}}_\varrho(t)$ be such that

$$(3.36)\qquad \|p_{\mathfrak{H}^\perp}(\bar v(\mathbf{h}_t))\|=\theta.$$

We have using (2.12)

$$(3.37)\qquad \begin{aligned}&\|p_{\mathfrak{H}^\perp}(\bar v(\mathbf{h}_t))-p_{\mathfrak{J}_0^\perp(\mathfrak{H})}(v(\mathbf{h}_t))\|\leq\mathbf{C}\theta/t\leq0.1\delta\\ &\|p_{\mathfrak{H}^\perp}(\bar v(\mathbf{h}_t))-p_{\mathfrak{H}^\perp}(v(\mathbf{h}_t))\|=\|p_{\mathfrak{H}^\perp}(z(\mathbf{h}_t))\|\leq\mathbf{C}\theta/t\leq0.1\delta.\end{aligned}$$

It is clear that

$$p_{\mathfrak{H}^\perp}(v(\mathbf{h}_t))=p_{\mathfrak{H}^\perp}(v)+\xi_{\mathbf{h}_t}$$

for some $\xi_{\mathbf{h}_t}\in\mathfrak{E}^-\cap\mathfrak{H}^\perp$, since $v\in\mathfrak{E}^0$. This implies that

$$\|p_{\mathfrak{H}^\perp}(\bar v(\mathbf{h}_t))-\xi_{\mathbf{h}_t}\|\leq\delta+0.1\delta\leq0.2c\theta.$$

This implies via (3.13) and (3.37) that

$$(3.38)\qquad p_{\mathfrak{J}_0^\perp(\mathfrak{H})}(v(\mathbf{h}_t))+r\notin\mathfrak{L}(\mathbf{\Lambda})$$

for all $r\in\mathfrak{G}$ with $\|r\|\leq4c\|p_{\mathfrak{J}_0^\perp(\mathfrak{H})}(v(\mathbf{h}_t))\|$. We have using Proposition 2.1

$$(3.39)\qquad \exp\bar v(\mathbf{h})=\exp(p_{\mathfrak{J}_0^\perp(\mathfrak{H})}(v(\mathbf{h}))+r(v,\mathbf{h})+\varepsilon(\bar v,\mathbf{h}))\cdot\bar{\mathbf{h}}$$

for all $\mathbf{h}\in\bar{\mathbf{F}}_\varrho(t)$, where $\bar{\mathbf{h}}\in\bar{\mathbf{F}}_{\mathbf{C}\theta}(t)$, $r(v,\mathbf{h})\in\mathfrak{J}_0^\perp(\mathfrak{H})\subset\mathfrak{J}(\mathfrak{N})$, $\|r(v,\mathbf{h})\|\leq\mathbf{C}\theta^2\leq0.1c\theta$, $\varepsilon(\bar v,\mathbf{h})\in\mathfrak{H}^\perp$, $\|\varepsilon(\bar v,\mathbf{h})\|\leq\mathbf{C}\theta/t\varrho\leq0.1\delta$. We have using the R-property and the definition of $t(\delta, T)$

$$(3.40)\qquad \|p_{\mathfrak{J}_0^\perp(\mathfrak{H})}(v(\mathbf{h}))-p_{\mathfrak{J}_0^\perp(\mathfrak{H})}(v(\mathbf{h}_t))\|\leq2\delta\theta$$

for all $\mathbf{h}\in\mathbf{h}_t\mathbf{F}_{\eta\delta\varrho}(t)$. Define

$$k(\bar v,\mathbf{h})=p_{\mathfrak{J}_0^\perp(\mathfrak{H})}(v(\mathbf{h}_t))+r(v,\mathbf{h})\in\mathfrak{J}(\mathfrak{N}).$$

We have using (3.38)

$$k(\bar{v}, \mathbf{h})+r \notin \mathfrak{L}(\mathbf{A})$$

for all $r \in \mathfrak{G}$ with $\|r\| \leq c\|k(\bar{v}, \mathbf{h})\|$. This, (3.36) and (3.37) say that $k(\bar{v}, \mathbf{h}) \in \mathfrak{K}$. Also

$$\|p_{\mathfrak{Z}_0^\perp(\mathfrak{H})}(v(\mathbf{h}))+r(v, \mathbf{h})-k(\bar{v}, \mathbf{h})\| \leq 2\delta\theta$$

for all $\mathbf{h} \in \mathbf{h}_t \bar{\mathbf{F}}_{\eta\delta\varrho}(t)$ by (3.40). This and (3.39) complete the proof of the lemma. □

COROLLARY 3.1. *If* $\mathfrak{V}+\mathfrak{J} \neq \{0\}$ *then* $\mathfrak{K} \neq \emptyset$.

Proof. Let $0 \neq v+j \in \mathfrak{V}+\mathfrak{J}$. If $v \neq 0$ then $\|p_\perp(v(\mathbf{u}(s)))\| \to \infty$, when $s \to \infty$. Then (3.17) holds for some $t \geq t(\delta, T)$ and $\bar{v}=v$, if $\|v\|$ is sufficiently small. If $v=0$ then $j \neq 0$ and $j \notin \mathfrak{Z}(\mathfrak{H})$, $j \in \mathfrak{E}^0$ and hence $\bar{\beta}(j, t) \to \infty$, when $t \to \infty$. Then $\bar{\beta}(j, t)=\theta$ for some $t \geq t(\delta, T)$ if $\|j\|$ is sufficiently small. It follows then from (3.18) and (3.35) that $\mathfrak{K} \neq \emptyset$. □

Now let $\mathbf{x}=\exp v$ for some $v \in \mathfrak{N}^\perp$ (some $v \in \mathfrak{H}^\perp$). If $\|v\|$ is sufficiently small then there is a neighborhood $\mathbf{O_N(e)} \subset \mathbf{N}$ ($\mathbf{O_H(e)} \subset \mathbf{H}$) and a diffeomorphism $\varphi: \mathbf{O_N(e)} \to \mathbf{xN}$ ($\bar{\varphi}: \mathbf{O_H(e)} \to \mathbf{xH}$) such that $\varphi(\mathbf{y})=\mathbf{y} \exp v_{\mathbf{y}}$ ($\bar{\varphi}(\mathbf{z})=\mathbf{z} \exp \bar{v}_{\mathbf{z}}$) for every $\mathbf{y} \in \mathbf{O_N(e)}$ ($\mathbf{z} \in \mathbf{O_H(e)}$) and some $v_{\mathbf{y}} \in \mathfrak{N}^\perp$ ($\bar{v}_{\mathbf{z}} \in \mathfrak{H}^\perp$). We choose Θ in (3.14) so small that if $\|v\| \leq 5\Theta$ then

$$|[\lambda(\mathbf{B})/\lambda(\varphi(\mathbf{B}))]-1| \leq 0.01$$

(3.41)

$$|[\lambda(\mathbf{A})/\lambda(\bar{\varphi}(\mathbf{A}))]-1| \leq 0.01$$

for every Borel subset $\mathbf{B} \subset \mathbf{O_N(e)}$ ($\mathbf{A} \subset \mathbf{O_H(e)}$), where $\lambda(\varphi(\mathbf{B}))$ ($\lambda(\bar{\varphi}(\mathbf{A}))$) is defined to be $\lambda(\mathbf{x}^{-1}\varphi(\mathbf{B}))$ ($\lambda(\mathbf{x}^{-1}\bar{\varphi}(\mathbf{A}))$) with λ being a Haar measure on $\mathbf{N}$ (on $\mathbf{H}$).

Now let $v \in \mathfrak{G}$, $\exp v=\mathbf{x}$ and $\beta(v, t) \leq \theta$ ($\bar{\beta}(v, t) \leq \theta$) for some $t \geq t(\delta, T)$. Let $\varphi: \mathbf{F}_\varrho(t) \to \mathbf{xN}$ ($\bar{\varphi}: \bar{\mathbf{F}}_p(t) \to \mathbf{xH}$) be as in Corollary 2.2 (as in (2.14)). Namely,

$$\varphi(\mathbf{h})=\mathbf{h} \exp \omega(v, \mathbf{h})=\mathbf{xh}(\bar{\mathbf{h}}^{-1}), \quad \mathbf{h} \in \mathbf{F}_\varrho(t)$$

$$\bar{\varphi}(\mathbf{h})=\mathbf{h} \exp \bar{\omega}(v, \mathbf{h})=\mathbf{xh}(\bar{\mathbf{h}}')^{-1}, \quad \mathbf{h} \in \bar{\mathbf{F}}_\varrho(t)$$

where $\omega(v, \mathbf{h})$ and $\bar{\omega}(v, \mathbf{h})$ are as in (3.18), (3.35) and

$$\bar{\mathbf{h}} \in \mathbf{F}_{C\theta}(t) \subset \mathbf{F}_{0.1\varrho}(t), \quad \bar{\mathbf{h}}' \in \bar{\mathbf{F}}_{C\theta}(t) \subset \bar{\mathbf{F}}_{0.1\varrho}(t).$$

This, (1.24), (2.14) and Corollary 2.2 imply that $\varphi, \bar\varphi$ are injective and

$$(3.42)\qquad \begin{aligned} &\varphi(\mathbf{F}_\varrho(t)) \subset \mathbf{x}\mathbf{F}_{2\varrho}(t), \quad \bar\varphi(\bar{\mathbf{F}}_\varrho(t)) \subset \mathbf{x}\bar{\mathbf{F}}_{2\varrho}(t) \\ &\varphi(\mathbf{h}\mathbf{F}_\alpha(t)) \subset \varphi(\mathbf{h})\,\mathbf{F}_{2\alpha}(t), \quad \bar\varphi(\mathbf{h}\bar{\mathbf{F}}_\alpha(t)) \subset \bar\varphi(\mathbf{h})\,\bar{\mathbf{F}}_{2\alpha}(t) \end{aligned}$$

for all $\mathbf{h}\in\mathbf{F}_\varrho(t)$ ($\mathbf{h}\in\bar{\mathbf{F}}_\varrho(t)$), all $0<\alpha<1$ with $\mathbf{h}\mathbf{F}_\alpha(t)\subset\mathbf{F}_\varrho(t)$ ($\mathbf{h}\bar{\mathbf{F}}_\alpha(t)\subset\bar{\mathbf{F}}_\varrho(t)$). It follows also from (3.41) (via the fact that the Haar measure λ and the Riemannian metric on $\mathbf{G}$ are left invariant) that

$$(3.43)\qquad \begin{aligned} &|[\lambda(\mathbf{B})/\lambda(\varphi(\mathbf{B}))]-1| \leqslant 0.01 \\ &|[\lambda(\mathbf{A})/\lambda(\bar\varphi(\mathbf{A}))]-1| \leqslant 0.01 \end{aligned}$$

for all Borel subsets $\mathbf{B}\subset\mathbf{F}_\varrho(t)$ ($\mathbf{A}\subset\bar{\mathbf{F}}_\varrho(t)$), where $\lambda(\varphi(\mathbf{B}))$ and $\lambda(\bar\varphi(\mathbf{A}))$ are as above. These relations will be used in the proof of Lemma 3.6 below. To state this lemma let us go back to the decomposition $\mathfrak{G}=\mathfrak{W}+\mathfrak{V}+\mathfrak{J}+\mathfrak{I}_0(\mathfrak{H})$. For $\bar v\in\mathfrak{G}$ write $\bar v=w+v+j+i$ and for $v\in\mathfrak{V}$ write $v=\sum_{n=2}^{m+1} v_n$, $v_n\in\mathfrak{V}_n$.

LEMMA 3.6. *Suppose that the action of* $\mathbf{N}$ *on* (X,d,μ) *is ergodic. Then given* $\varepsilon>0$ *there are* $t(\varepsilon)>1$, $0<\gamma=\gamma(\varepsilon)<1$ *and a compact* $A=A(\varepsilon)\subset X, \mu(A)>1-\varepsilon$ *such that if* $x,y\in A, y=x\exp i_1\exp\bar v\exp i_2$ *for some* $i_1,i_2\in\mathfrak{I}_0(\mathfrak{H})$, $\bar v\in\mathfrak{G}$, $\|i_1\|,\|i_2\|,\|\bar v\|\leqslant\gamma$, $\bar v=w+v+j+i$ *and* $w\in\mathfrak{W}(a(\theta),t)$ *for some* $t\geqslant t(\varepsilon)$ *then*

$$\|j\|\leqslant \mathbf{C}\theta t^{-1/2(m+1)}, \quad \|v_n\|\leqslant \mathbf{C}\theta t^{n-1}$$

for all $n=2,\dots,m+1$.

Proof. If $\mathfrak{V}+\mathfrak{J}=\{0\}$ then the lemma is obvious. Suppose that $\mathfrak{V}+\mathfrak{J}\neq\{0\}$. Then $\mathbf{K}\neq\emptyset$ by Corollary 3.1. Recall that $\mathbf{K}\subset\mathbf{I}(\mathbf{N})-\mathbf{\Lambda}$.

Let $0<r<1$ and $0<\zeta(a)<1$ for $0<a<1$ be defined by

$$r=\min\{\lambda(\mathbf{F}_\varrho(s))/\lambda(\mathbf{F}_{2\varrho}(s)),\ \lambda(\bar{\mathbf{F}}_\varrho(s)/\lambda(\bar{\mathbf{F}}_{2\varrho}(s))\}$$

$$\zeta(a)=\min\{\lambda(\mathbf{F}_{\eta a\varrho/4}(s))/\lambda(\mathbf{F}_{2\varrho}(s)),\ \lambda(\bar{\mathbf{F}}_{\eta a\varrho/4}(s))/\lambda(\bar{\mathbf{F}}_{2\varrho}(s))\}.$$

Note that r and $\zeta(a)$ do not depend on ϱ and s by [R4, Proposition 2.1].

Let $0<\alpha_1<1$ be so small that $10\beta(\alpha_1)<0.01r$, where $0<\beta(\alpha)<\alpha$ is as in Proposition 1.10. Let $Y=Y(\alpha_1,\mathbf{K})\subset X$, $\mu(Y)>1-\alpha_1$ and $\delta(\alpha_1,Y)>0$ be as in Proposition 1.11. We have

$$d(Y, Y\mathbf{k}) > \delta(\alpha_1, Y) \tag{3.44}$$

for all $\mathbf{k} \in \mathbf{K}$.

Let $0<\bar\delta=\bar\delta(\alpha_1, Y)<1$ be so small that if $d_G(\mathbf{e}, \mathbf{z})<\bar\delta$ then $d(x, x\mathbf{z})<0.1\delta(\alpha_1, Y)$ for all $x \in X$. Let $\delta=0.1 \min\{\bar\delta, c\theta_0\}$ where c and θ_0 are as in (3.14). Let $0<\alpha_2<0.01\zeta(\delta)$ be so small that $10\beta(\alpha_2)<0.01r\zeta(\delta)$. By Proposition 1.10 there is $A_1 \subset X, \mu(A_1)>1-\alpha_2$ and $\tau_1>1$ such that if $x \in A_1, t>\tau_1$ then

$$\begin{aligned}
&\lambda(Y \cap x\mathbf{F}_{\eta\delta\varrho/2}(t))/\lambda(\mathbf{F}_{\eta\delta\varrho/2}(t)) > 1-\beta(\alpha_1)\\
&\lambda(Y \cap x\mathbf{F}_{\eta\delta\varrho}(t))/\lambda(\mathbf{F}_{\eta\delta\varrho}(t)) > 1-\beta(\alpha_1)\\
&\lambda(Y \cap x\bar{\mathbf{F}}_{\eta\delta\varrho/2}(t))/\lambda(\bar{\mathbf{F}}_{\eta\delta\varrho/2}(t)) > 1-\beta(\alpha_1)\\
&\lambda(Y \cap x\bar{\mathbf{F}}_{\eta\delta\varrho}(t))/\lambda(\bar{\mathbf{F}}_{\eta\delta\varrho}(t)) > 1-\beta(\alpha_1) > 1-0.001r.
\end{aligned} \tag{3.45}$$

Using again Proposition 1.10 we get a compact $A \subset X, \mu(A)>1-\varepsilon$ and $\tau_2>1$ such that if $x \in A, t>\tau_2$ then

$$\begin{aligned}
&\lambda(A_1 \cap x\mathbf{F}_{\varrho}(t))/\lambda(\mathbf{F}_{\varrho}(t)) > 1-\beta(\alpha_2) > 1-0.001r\zeta(\delta)\\
&\lambda(A_1 \cap x\mathbf{F}_{2\varrho}(t))/\lambda(\mathbf{F}_{2\varrho}(t)) > 1-\beta(\alpha_2)\\
&\lambda(A_1 \cap x\bar{\mathbf{F}}_{\varrho}(t))/\lambda(\bar{\mathbf{F}}_{\varrho}(t)) > 1-\beta(\alpha_2).\\
&\lambda(A_1 \cap x\bar{\mathbf{F}}_{2\varrho}(t))/\lambda(\bar{\mathbf{F}}_{2\varrho}(t)) > 1-\beta(\alpha_2).
\end{aligned} \tag{3.46}$$

Define

$$T = T(\varepsilon) = \max\{\tau_1, \tau_2\}, \quad t(\varepsilon) = [t(\delta, T)]^{2(m+1)}, \quad \gamma = \gamma(\varepsilon) = \gamma(\delta, T)$$

where $t(\delta, T)$ and $\gamma(\delta, T)$ are as in Lemma 3.4.

Now suppose that $x, y \in A$, $y=x \exp i_1 \exp \bar v \exp i_2$ for some $i_1, i_2 \in \mathfrak{J}_0(\mathfrak{H})$, $\bar v \in \mathfrak{G}$, $\|i_1\|, \|i_2\|, \|\bar v\| \leq \gamma$ and

$$\bar v = w+v+j+i, \quad w \in \mathfrak{W}(a(\theta), t) \tag{3.47}$$

for some $t \geq t(\varepsilon)$. Suppose that $v \neq 0$. We claim that

$$\max\{\|p_\perp(v(\mathbf{u}(s)))\|: 0 \leq s \leq \varrho t\} < 2\theta. \tag{3.48}$$

Indeed, suppose on the contrary that (3.48) does not hold. Then there exists $\bar{t}$: $T\leqslant t(\delta, T)\leqslant\bar{t}\leqslant t$ such that

$$\beta(\bar{v}, \bar{t}) = \theta$$

by (3.16) and (3.20), where $\beta(\bar{v}, \bar{t})$ is as in Lemma 3.4. Set $\mathbf{i}_1=\exp i_1, \mathbf{i}_2=\exp(-i_2)$, $\bar{x}=x\mathbf{i}_1, \bar{y}=y\mathbf{i}_2, \mathbf{h}(\mathbf{i}_1)=\mathbf{i}_1\mathbf{h}\mathbf{i}_1^{-1}, \mathbf{h}\in\mathbf{N}$. We have $\bar{y}=\bar{x}\exp\bar{v}$ and

$$x\mathbf{h}(\mathbf{i}_1)\,\mathbf{i}_1\exp(\omega(\bar{v}, \mathbf{h}))\,\mathbf{i}_2 = \bar{x}\varphi(\mathbf{h})\,\mathbf{i}_2 = \bar{y}\mathbf{h}\tilde{\mathbf{h}}\mathbf{i}_2 = y\mathbf{h}(\mathbf{i}_2^{-1})\,\tilde{\mathbf{h}}(\mathbf{i}_2^{-1}) \tag{3.49}$$

for all $\mathbf{h}(\mathbf{i}_1)\in\mathbf{F}_\varrho(t)$, where $\varphi(\mathbf{h})$ and $\tilde{\mathbf{h}}\in\mathbf{F}_{0.1\varrho}(t)$ are as in (3.42). Here

$$\mathbf{h}(\mathbf{i}_2^{-1})\in\mathbf{F}_{\varrho+\gamma}(t)\subset\mathbf{F}_{1.1\varrho}(t), \quad \tilde{\mathbf{h}}(\mathbf{i}_2^{-1})\in\mathbf{F}_{0.1\varrho+\gamma}(t)\subset\mathbf{F}_{0.2\varrho}(t).$$

It follows then from (3.42), (3.43), (3.45), (3.46), (3.49) and the definition of A that there are $\mathbf{h}_t\in\mathbf{F}_\varrho(t)$ and $\mathbf{h}\in\mathbf{h}_t\mathbf{F}_{\eta\delta\varrho}(t)$ such that

$$x\mathbf{h}_t(\mathbf{i}_1)\in A_1, \quad \bar{x}\varphi(\mathbf{h}_t)\,\mathbf{i}_2\in A_1$$

$$x\mathbf{h}(\mathbf{i}_1)\in Y, \quad \bar{x}\varphi(\mathbf{h})\,\mathbf{i}_2\in Y$$

$$\bar{x}\varphi(\mathbf{h})\,\mathbf{i}_1 = x\mathbf{h}(\mathbf{i}_1)\cdot\mathbf{i}_1\cdot\mathbf{k}(\bar{v}, \mathbf{h})\cdot\mathbf{z}(\bar{v}, \mathbf{h})\cdot\mathbf{i}_2 = x\mathbf{h}(\mathbf{i}_1)\,\mathbf{k}(\bar{v}, \mathbf{h})\cdot\boldsymbol{\delta}(\bar{v}, \mathbf{h})$$

for some $\mathbf{k}=\mathbf{k}(\bar{v}, \mathbf{h})\in\mathbf{K}$ and $d_G(\mathbf{e}, \boldsymbol{\delta}(\bar{v}, \mathbf{h}))\leqslant 4\delta<\bar{\delta}$. This implies that

$$d(Y, Y\mathbf{k}) < \delta(\alpha_1, Y)$$

in contradiction with (3.44). This proves (3.48). It follows then from (1.17) that

$$\|v_n\| \leqslant 2Q\theta/(t\varrho)^{n-1} < \mathbf{C}\theta/t^{n-1}, \quad n = 2, \ldots, m+1.$$

We have $\bar{v}=z+p$, where $z=w+v, p=j+i$ and $\|z\|\leqslant\theta l^{-(2m+1)}$, where $l=t^{1/2(m+1)}>t(\delta, T)$. Arguing as above and using Lemma 3.5 we show that $\bar{\beta}(\bar{v}, l)<\theta$ and $\bar{\beta}(p, l)\leqslant 2\theta$. Now we use the R-property to get

$$\|j\| = \|p-i\| \leqslant 2L\theta/l\varrho \leqslant \mathbf{C}\theta/l \leqslant \mathbf{C}\theta t^{-1/2(m+1)}.$$

This completes the proof of the lemma. □

Proof of Lemma 3.1. Let $\mathbf{C}>1$, $0<\Theta<1$, $0<a<1$ be as in (3.14) and (3.41) and let $0<\theta\leqslant\Theta$, $\varepsilon>0$ be given. Let $t(\varepsilon)=t(\varepsilon, \theta)$, $\gamma=\gamma(\varepsilon)=\gamma(\varepsilon, \theta)$ and $A=A(\varepsilon)=A(\varepsilon, \theta)$ be as in

Lemma 3.6. Let $x, y \in A$, $y = x \exp i_1 \exp \bar{v} \exp i_2$, $i_1, i_2 \in \mathfrak{I}_0(\mathfrak{H})$, $\|i_1\|, \|i_2\|, \|\bar{v}\| \leq \gamma$, $\bar{v} = w + v + j + i$ and $w \in W(a(\theta), t)$ for some $t \geq t(\varepsilon)$. Then

$$\|j\| \leq \mathbf{C}\theta t^{-1/2(m+1)}, \quad \|v_n\| \leq \mathbf{C}\theta / t^{n-1}, \quad n = 2, \ldots, m+1 \tag{3.50}$$

by Lemma 3.6. We have $\mathfrak{B} = \mathfrak{Y} + \mathfrak{D} = \Sigma_{n=2}^{m+1} \mathfrak{B}_n$, $v_n = y(v_n) + d(v_n)$, where $y(v_n) \in \mathfrak{Y} \cap \mathfrak{B}_n$, $d(v_n) \in \mathfrak{D} \cap \mathfrak{B}_n$. It follows from (3.50) that

$$\|d(v_n)\| \leq \bar{\mathbf{C}}\theta / t^{n-1} \leq \bar{\mathbf{C}}\theta / t^{-1}, \quad \|y(v_n)\| \leq \bar{\mathbf{C}}\theta / t^{n-1} \tag{3.51}$$

for some $\bar{\mathbf{C}} > 1$ and all $n = 2, \ldots, m+1$. Let

$$\Psi_n = \{\psi \in \Psi : \psi \in \mathfrak{B}_n\}, \quad C_n = \{c \in Y - \Psi : c \in \mathfrak{B}_n\}, \quad n = 2, \ldots, m+1.$$

We have

$$y = y(v_n) = \sum_{\psi \in \Psi_n} \psi(y)\, \psi + \sum_{c \in C_n} \mathfrak{c}(y)\, c.$$

This and (3.51) show that

$$|\psi(y)| \leq \mathbf{C}'\theta / t^{n-1} \leq \mathbf{C}'\theta / t^{\nu(\psi)}, \quad \psi \in \Psi_n$$

$$|\mathfrak{c}(y)| \leq \mathbf{C}'\theta / t^{n-1} \leq \mathbf{C}'\theta / t^{\chi(c)}, \quad c \in C_n$$

for some $\mathbf{C}' > 1$, since $\nu(v) \leq n-1$ for all $v \in \mathfrak{B}_n$ with $\chi(v) > 0$ by (3.4) and $\chi(c) \leq n-1$ for all $c \in C_n$ by the definition of $Y - \Psi$. This, (3.50) and (3.51) complete the proof, since $f(\bar{v}) = j + \Sigma_{n=2}^{m+1} d(v_n)$. □

4. Divergence of g-orbits

In this section we assume that $\mathbf{G}$ acts by right translations on $(X = \mathbf{\Gamma} \backslash \mathbf{G}, d, \mu)$ with $\mathbf{\Gamma}$ being a discrete subgroup of $\mathbf{G}$ and use the notations of Section 3. Thus $\mathbf{\Lambda} = \mathbf{\Lambda}(\mathbf{G}, \mathbf{\Gamma}, \mu)$, $\mathfrak{N} = \mathfrak{L}(\mathbf{\Lambda}) \cap \mathfrak{E}^{-2}$, $\mathbf{N} = \exp \mathfrak{N}$, u is horocyclic for g, $\mathbf{u} = \exp u$, $\mathbf{N_u} = \{\mathbf{u}^k \mathbf{N} : k \in \mathbf{Z}\}$. Let $\mathfrak{O}_\delta(\mathfrak{G})$ denote the δ-ball at O in $\mathfrak{G}$, $\mathbf{O}_\delta(\mathbf{G}) = \exp \mathfrak{O}_\delta(\mathfrak{G})$ and let $\pi : \mathbf{G} \to X$ be the covering projection $\pi(\mathbf{h}) = \mathbf{\Gamma h}$. For $x \in X$ define

$$\Delta(x) = 0.5 \max\{\delta > 0 : \pi \text{ is one-to-one on } \mathbf{xO}_\delta(\mathbf{G}), \mathbf{x} \in \pi^{-1}\{x\}\}. \tag{4.1}$$

Now let $\mathbf{g}_p = \exp pg$, $p \in R$, $\mathbf{g} = \mathbf{g}_1$. Recall (see the introduction) that the $\mathbf{g}_p$-orbit of $x \in X$ is said to diverge when $n \to \infty$ if $\Delta(x\mathbf{g}_p^n) \to 0$, when $n \to \infty$. Let $D(\mathbf{g}_p) = \{x \in X : \text{the } \mathbf{g}_p\text{-orbit of } x \text{ diverges when } n \to \infty\}$. It is clear that $D(\mathbf{g}_p) = D(\mathbf{g}_q)$ for all $p, q \in R^+$.

Let Ψ be as in (3.7) for the decomposition $\mathfrak{G}=\mathfrak{W}+\mathfrak{Y}+\mathfrak{F}+\tilde{\mathfrak{Z}}(\mathfrak{H})$ induced by $\mathfrak{L}(\mathbf{\Lambda})$. In this section we shall prove the following theorem.

THEOREM 4.1. *Suppose that the action of* $\mathbf{N_u}$ *on* (X,μ) *is ergodic and either* $\Psi\neq\emptyset$ *or* $u\notin\mathfrak{N}$. *Then* $\mu(D(\mathbf{g}))=1$.

Let us show that conclusion 1 of the main theorem follows from Theorem 4.1. Indeed, it is contained in the following corollary.

COROLLARY 4.1. *Suppose that the action of* $\mathbf{N_u}$ *on* (X,μ) *is ergodic and* $\mathbf{c}\mathbf{g}_p\mathbf{c}^{-1}\notin\mathbf{\Lambda}$ *for some* $p>0$ *and all* $\mathbf{c}\in\mathbf{E}^-$. *Then* $\mu(D(\mathbf{g}))=1$.

Proof. It follows from Proposition 1.1 that $-pg+v\notin\mathfrak{L}(\mathbf{\Lambda})$ for any $v\in\mathfrak{E}^-$, since $\mathbf{c}\mathbf{g}_p\mathbf{c}^{-1}\notin\mathbf{\Lambda}$ for all $\mathbf{c}\in\mathbf{E}^-$. This implies that if $\bar{v}=pu^*+v'$ with $\chi(v')\leq 1$ then $\bar{v}\notin\mathfrak{W}$, since $\nu(\bar{v})=1$ and $\mathrm{ad}_u^{\nu(\bar{v})}(\bar{v})=-pg+v$, where $v=\mathrm{ad}_u(v')\in\mathfrak{E}^-$. This implies that $u^*\notin\tilde{\mathfrak{E}}_2$—the subspace of $\mathfrak{E}_2$ spanned by $\{p_2(w): w\in W_2^{(2)}\}$. Then either $\Psi\neq\emptyset$ or $u\notin\mathfrak{N}$ by (3.7) and $\mu(D(\mathbf{g}))=1$ by Theorem 4.1. □

We shall prove Theorem 4.1 assuming as before that $u\in\mathfrak{N}$ and note that for the case $u\notin\mathfrak{N}$ one should only substitute $\mathfrak{Z}_0(\mathfrak{H})$ by $\tilde{\mathfrak{Z}}(\mathfrak{H})$ and $\mathbf{H}$ by $\mathbf{H_u}$ in the argument below.

The proof of Theorem 4.1 is based on Lemma 4.1 below which says roughly speaking that if $A\subset X$ is as in Lemma 3.1 then A can be covered by small boxes of the form $x\exp\Pi$ where Π is a parallelopiped in $\mathfrak{G}$ with faces parallel to $\mathfrak{W}$, $\mathfrak{Y}$, $\mathfrak{F}$ and $\mathfrak{Z}_0(\mathfrak{H})$ adjusted to the rate of expansion by $\mathrm{Ad}_{\mathbf{g}}$ and the number of these boxes is much smaller than the reciprocal of the area of the (W,Ψ)-face of Π. We begin with the description of these boxes.

Define $\Phi=\Psi\cup W$ and for a subset $D\subset\Phi$ define

$$\nu(D)=\sum_{v\in D}\nu(v),\quad \chi(D)=\sum_{v\in D}\chi(v).$$

Also define

$$M=\chi(\Phi)-\nu(\Psi). \tag{4.2}$$

In view of Lemma 3.1 we introduce the following notations. For $v\in\mathfrak{G}$ let $\xi(v)$ be $\chi(v)$ or $\nu(v)$. For $t\geq 1$, $\sigma>0$ define

$$\mathfrak{V}^{\xi}(t,\sigma)=\{v\in\mathfrak{Y}+\mathfrak{F}\colon |\mathfrak{c}_i(v)|\leq\sigma t^{-\xi(c_i)},\ i=1,\dots,r,\ |\psi_j(v)|\leq\sigma t^{-\xi(\psi_j)},$$

$$(4.3)\qquad j=1,\dots,l,\ \|f(v)\|\leq\sigma t^{-\beta}\}$$

$$\mathfrak{W}^{\xi}(t,\sigma)=\{w\in\mathfrak{W}\colon |\alpha_i(w)|\leq\sigma t^{-\xi(w_i)},\ i=1,\dots,p\}$$

$$\mathfrak{S}^{\xi}(t,\sigma)=\mathfrak{W}^{\xi}(t,\sigma)+\mathfrak{V}^{\xi}(t,\sigma).$$

Now let $\mathbf{C}>1$, $0<\Theta<1$, $0<a<1$ be as in Lemma 3.1 and let $0<\theta<\Theta$ be chosen later. Define

$$(4.4)\qquad \begin{aligned}&\mathfrak{V}^{\xi}(t)=\mathfrak{V}^{\xi}(t,\mathbf{C}\theta),\quad \mathfrak{W}^{\xi}(t)=\mathfrak{W}^{\xi}(t,a(\theta)/\bar{a})\\ &\mathfrak{S}^{\xi}(t)=\mathfrak{W}^{\xi}(t)+\mathfrak{V}^{\xi}(t)\end{aligned}$$

where $a(\theta)=a\theta$ and we set for technical reasons $\bar{a}=10^{m^4}$. We have

$$(4.5)\qquad \begin{aligned}&\chi([v,w])\leq\chi(v)+\chi(w)\\ &\nu([v,w])\leq\nu(v)+\nu(w)\end{aligned}$$

for all v, w with $\chi(v),\chi(w)\geq 0$. This implies via (1.14) that if $z, v\in 4\mathfrak{S}^{\xi}(t)$ then

$$\exp(z+v)=\exp z\exp(v+v'+i')$$

for some $v'\in\mathfrak{S}^{\xi}(t,K_1\mathbf{C}^2\theta^2)$, $i'\in\mathfrak{I}_0(\mathfrak{H})$, $\|i'\|\leq K_1\mathbf{C}^2\theta^2t^{-2\beta}$ and some $K_1\geq 1$. Using the fact that $\mathfrak{I}_0(\mathfrak{H})\subset\mathfrak{E}^0$ we get via (1.14)

$$(4.6)\qquad \exp(z+v)=\exp z\exp(v+\hat{v})\exp i$$

for some $\hat{v}\in\mathfrak{S}^{\xi}(t,K\mathbf{C}^2\theta^2)$, $i\in\mathfrak{I}_0(\mathfrak{H})$, $\|i\|\leq K\mathbf{C}^2\theta^2t^{-2\beta}$, if θ is sufficiently small, where $K\geq K_1$. Now set $b=4$ and choose $0<\theta<\Theta$ so small that if we define

$$(4.7)\qquad \hat{\theta}=100\bar{a}K\mathbf{C}^2\theta b^m/a$$

then

$$(4.8)\qquad (1+\hat{\theta})^{\chi(\Phi)}<3/2.$$

We have for $\hat{v}$ in (4.6)

$$(4.9)\qquad \hat{v}\in\mathfrak{S}^{\xi}(t,\hat{\theta}a(\theta)\,b^{-m}/\bar{a})\subset\hat{\theta}\mathfrak{S}^{\xi}(bt).$$

For $\mathfrak{U}\subset\mathfrak{W}+\mathfrak{Y}+\mathfrak{F}$ and $\gamma>0$ write

$$(4.10)\qquad \begin{aligned}&\mathbf{B}(\mathfrak{U},\gamma)=\{\exp v\exp i\colon v\in\mathfrak{U},\ i\in\mathfrak{J}_0(\mathfrak{H}),\ \|i\|\leqslant\gamma\}\\ &\mathbf{B}^{\xi}(\gamma,t)=\mathbf{B}(\mathfrak{S}^{\xi}(t),\gamma).\end{aligned}$$

The set $x\mathbf{B}^{\xi}(\gamma,t)$, $x\in X$ will be called a (ξ,t)-box at x.

For a given $0<\varepsilon<0.1$ let $A=A(\varepsilon,\theta)$, $t(\varepsilon)=t(\varepsilon,\theta)$, $\gamma(\varepsilon)=\gamma(\varepsilon,\theta)\leqslant\theta$ be as in Lemma 3.1. Also for $0<\gamma<\gamma(\varepsilon)$ let

$$(4.11)\qquad \tau(\gamma)=\max\{t(\varepsilon),(100\gamma)^{1/\beta}\}.$$

LEMMA 4.1. *Given $0<\gamma<\gamma(\varepsilon)$, $\tau>\tau(\gamma)$ there exists $n(\gamma,\tau)>1$ such that for every integer $n\geqslant n(\gamma,\tau)$ there are $a_1(n),\dots,a_{M(n)}(n)\in X$ such that*

$$A\subset\bigcup_{i=1}^{M(n)}a_i(n)\,\mathbf{B}(a\mathfrak{S}^{\chi}(t_n),\gamma)$$

where $t_n=\tau b^n/(1+\hat{\theta})^n$, $M(n)=Lb^{nM}$, M is as in (4.2) *and $L=L(\gamma,\tau)\geqslant1$, $a\geqslant1$ are constants.*

The proof of this lemma uses Lemma 3.1 and is given in an appendix at the end of this section.

In order to derive Theorem 4.1 from Lemma 4.1 we need to make an observation. Let $\bar{\mathbf{G}}$ be a Lie subgroup of $\mathbf{G}$ with the Lie algebra $\bar{\mathfrak{G}}$ and let $\mathfrak{G}=\mathfrak{L}_1+\mathfrak{L}_2+\bar{\mathfrak{G}}$ be the direct sum of $\bar{\mathfrak{G}}$ and some subspaces $\mathfrak{L}_1,\mathfrak{L}_2\subset\mathfrak{G}$. For $\mathfrak{U}_i\subset\mathfrak{L}_i$, $i=1,2$ and $\mathbf{D}\subset\bar{\mathbf{G}}$ write

$$\mathbf{B}(\mathfrak{U}_1,\mathfrak{U}_2)=\{\exp v_1\exp v_2\colon v_1\in\mathfrak{U}_1,\ v_2\in\mathfrak{U}_2\}$$

$$\mathbf{B}(\mathfrak{U}_1,\mathfrak{U}_2)\cdot\mathbf{D}=\{\mathbf{bd}\colon\mathbf{b}\in\mathbf{B}(\mathfrak{U}_1,\mathfrak{U}_2),\mathbf{d}\in\mathbf{D}\}.$$

Now let $\mathbf{v}$ be a $\bar{\mathbf{G}}$-invariant Borel probability measure on X. Suppose that

$$(4.12)\qquad \begin{aligned}&y\mathbf{B}(\mathfrak{U}_1,\mathfrak{U}_2)\,\mathbf{O}_\delta(\bar{\mathbf{G}})\subset x\mathbf{O}_{\Delta(x)}(\mathbf{G})\\ &\mathbf{v}(y\mathbf{B}(\mathfrak{U}_1,\mathfrak{U}_2)\,\mathbf{O}_\delta(\bar{\mathbf{G}}))>0\end{aligned}$$

for some $\mathfrak{U}_i\subset\mathfrak{O}_\delta(\mathfrak{L}_i)$, $i=1,2$, $0<\delta<0.1\Delta(x)$, $x,y\in X$. It is a fact that (4.12) implies that

$$(4.13)\qquad \frac{\mathbf{v}(y\mathbf{B}(\mathfrak{U}_1,\mathfrak{U}_2)\cdot\mathbf{D}_1)}{\mathbf{v}(y\mathbf{B}(\mathfrak{U}_1,\mathfrak{U}_2)\cdot\mathbf{D}_2)}=\frac{\lambda(\mathbf{D}_1)}{\lambda(\mathbf{D}_2)}$$

for all Borel subsets $\mathbf{D}_1, \mathbf{D}_2 \subset \mathbf{O}_\delta(\bar{\mathbf{G}})$, $\lambda(\mathbf{D}_2)>0$, where λ denotes a right invariant Haar measure on $\bar{\mathbf{G}}$. Expression (4.13) will be used in the proof of Theorem 4.1. Now let

$$H(\Phi) = \{\mathrm{ad}_u^{\chi(v)}(v)/\|\mathrm{ad}_u^{\chi(v)}(v)\| : v \in \Phi\}.$$

It follows from Theorem 1.1 that $\mathrm{ad}_u^{\chi(v)}(v) \neq 0$ for all $0 \neq v$ with $\chi(v)>0$ and

$$\chi(\mathrm{ad}_u^{\chi(v)}(v)) = -\chi(v). \tag{4.14}$$

This implies via the definition of W and Ψ that $H(\Phi) \subset \mathfrak{H}$ (recall that we assumed $u \in \mathfrak{H}$). Also $|H(\Phi)|=|\Phi|$ and $H(\Phi)$ is linearly independent in $\mathfrak{H}$ by Theorem 1.1. Let $H=\{h_1, \ldots, h_s\}$ be a basis of unit vectors in $\mathfrak{H}$, containing $H(\Phi)$. (One can show that, in fact, $H(\Phi)$ is a basis in $\mathfrak{H}$.) We have $\chi(H) \leqslant -\chi(\Phi)$ by (4.14).

For $v \in \mathfrak{G}$, $t \in R$ and a subspace $\mathfrak{L} \subset \mathfrak{G}$ define

$$v(t) = \mathbf{g}_{-t} v \mathbf{g}_t, \quad \mathfrak{L}(t) = \mathbf{g}_{-t} \mathfrak{L} \mathbf{g}_t.$$

We have

$$\begin{aligned} &\chi(v(t)) = \chi(v), \quad v \in \mathfrak{G},\ t \in R \\ &\|v(t)\| \leqslant e^{\chi(v)t} \|v\|, \quad v \in \mathfrak{G},\ t \geqslant 0. \end{aligned} \tag{4.15}$$

Define

$$H(t) = \{h(t)/\|h(t)\| : h \in H\} = \{h_1(t), \ldots, h_s(t)\}$$

$$\mathbf{H}(t) = \exp \mathfrak{H}(t).$$

The set $H(t)$ is a basis of unit vectors in $\mathfrak{H}(t)$. For $v \in \mathfrak{H}(t)$ let $h_i(t, v)$ be the $h_i(t)$-coordinate of v. Define

$$\mathfrak{U}_\delta(t) = \{v \in \mathfrak{H}(t) : |h_i(v, t)| \leqslant \delta\}, \quad \mathbf{U}_\delta(t) = \exp \mathfrak{U}_\delta(t).$$

We have using (4.15)

$$\mathbf{g}_{-t} \mathfrak{U}_\delta(0) \mathbf{g}_t \subset \{v \in \mathfrak{H}(t) : |h_i(t, v)| \leqslant \delta e^{t\chi(h_i)}, \quad i = 1, \ldots, s\}.$$

This implies that

$$\lambda(\mathbf{g}_{-t} \mathbf{U}_\delta(0) \mathbf{g}_t)/\lambda(\mathbf{U}_\delta(t)) \leqslant \bar{C} e^{t\chi(H)} \leqslant \bar{C} e^{-t\chi(\Phi)} \tag{4.16}$$

for all $t \geqslant 0$, all $0<\delta<1$ and some $\bar{C}>0$, where λ denotes a Haar measure on $\mathbf{H}(t)$.

Now let $\mathfrak{H}^c$ be a subspace of $\mathfrak{Z}_0(\mathfrak{H})$ complementary to $\mathfrak{H}$. We can rephrase Lemma 4.1 as follows. Given $0<\gamma<\gamma(\varepsilon)$, $\tau>\tau(\gamma)$ there exists $n(\gamma,\tau)>1$ such that for every $n\geq n(\gamma,\tau)$ there are $a_1(n),\dots,a_{M(n)}(n)\in X$ such that

$$A\subset \bigcup_{i=1}^{M(n)} a_i(n)\,\mathbf{B}(a\mathfrak{S}^\chi(t_n),\ \mathfrak{O}_\gamma(\mathfrak{H}^c))\cdot \mathbf{U}_\gamma(0)$$

where $t_n, M(n)$ and $a>0$ are as in Lemma 4.1. Write

$$\sigma=\ln(b/(1+\hat{\theta}))>0.$$

We have

$$\mathbf{g}_{-n\sigma}\,\mathbf{B}(a\mathfrak{S}^\chi(t_n),\ \mathfrak{O}_\gamma(\mathfrak{H}^c))\,\mathbf{g}_{n\sigma}\subset \mathbf{O}_{2\max\{\gamma,\,a\tau^{-1}\}}(\mathbf{G}) \tag{4.17}$$

for all $n\in\mathbf{Z}^+$ by the definition of t_n, $\mathfrak{S}^\chi(t_n)$ and the fact that $\mathfrak{O}_\gamma(\mathfrak{H}^c)\subset\mathfrak{Z}_0(\mathfrak{H})\subset\mathfrak{E}^0$.

Proof of Theorem 4.1. We shall show that if $\Psi\neq\varnothing$ then $\mu(D(\mathbf{g}_\sigma))=1$. For $0<\delta<1$ define

$$K(\delta)=\{x\in X\colon \Delta(x)\geq 10\delta\}$$

$$A(\delta)=\{x\in A\colon x\mathbf{g}_{\sigma n}\in K(\delta)\text{ for infinitely many } n\in\mathbf{Z}^+\}.$$

We claim that $\mu(A(\delta))=0$ for all $0<\delta<1$. Indeed, we have

$$A(\delta)=\bigcap_{k=0}^{\infty}\bigcup_{n=k}^{\infty}A_n(\delta) \tag{4.18}$$

where

$$A_n(\delta)=\{x\in A\colon x\mathbf{g}_{\sigma n}\in K(\delta)\}.$$

Let $0<\gamma<\gamma(\varepsilon)$, $\tau>\tau(\gamma)$ be so small that

$$\max\{\gamma,a\tau^{-1}\}\leq\delta. \tag{4.19}$$

For $n\geq n(\gamma,\tau)$ let

$$A_{n,i}=A_{n,i}(\delta)=A_n(\delta)\cap\left[a_i(n)\,\mathbf{B}(a\mathfrak{S}^\chi(t_n),\ \mathfrak{O}_\gamma(\mathfrak{H}^c))\,\mathbf{U}_\gamma(0)\right]$$

$i=1,\dots,M(n)$. We have

$$A_n(\delta)=\bigcup_{i=1}^{M(n)}A_{n,i}. \tag{4.20}$$

Suppose that $\mu(A_{n,i})>0$ and let $x\in A_{n,i}$. Write $x_n=x\mathbf{g}_{\sigma n}\in K(\delta)$. We have using (4.17)

$$\begin{aligned} A_{n,i}\mathbf{g}_{\sigma n} &\subset (a_i(n)\,\mathbf{g}_{\sigma n})(\mathbf{g}_{-\sigma n}\mathbf{B}(a\mathfrak{S}^{\chi}(t_n),\ \mathfrak{O}_\gamma(\mathfrak{H}^c))\,\mathbf{g}_{\sigma n}(\mathbf{g}_{-\sigma n}\mathbf{U}_\gamma(0)\,\mathbf{g}_{\sigma n}) \\ &\subset (a_i(n)\,\mathbf{g}_{\sigma n})(\mathbf{g}_{-\sigma n}\mathbf{B}(a\mathfrak{S}^{\chi}(t_n),\ \mathfrak{O}_\gamma(\mathfrak{H}^c))\,\mathbf{g}_{\sigma n})\cdot\mathbf{U}_\gamma(n\sigma)\subset x_n\,\mathbf{O}_{\Delta(x_n)}(\mathbf{G}) \end{aligned} \tag{4.21}$$

by (4.19) and our definition of $K(\delta)$. Now let μ_n be the Borel probability measure on X defined by

$$\mu_n(E)=\mu(E\mathbf{g}_{-n\sigma})$$

for every Borel subset $E\subset X$. It is clear that μ_n is invariant under the action of $\mathbf{H}(n\sigma)=\mathbf{g}_{-n\sigma}\mathbf{H}\mathbf{g}_{n\sigma}$. It follows now from (4.13), (4.16), (4.21) and the definition of μ_n that

$$\begin{aligned} \mu(A_{n,i}) = \mu_n(A_{n,i}\,\mathbf{g}_{n\sigma}) &\leq \lambda(\mathbf{g}_{-n\sigma}\,\mathbf{U}_\gamma(0)\,\mathbf{g}_{n\sigma})/\lambda(\mathbf{U}_\gamma(n\sigma)) \\ &\leq \bar{C}e^{-\chi(\Phi)\,n\sigma} = \bar{C}(b(1+\hat{\theta})^{-1})^{-n\chi(\Phi)} \end{aligned}$$

for all $n\geq n(\gamma,\tau)$. This and (4.20) imply that

$$\begin{aligned} \mu(A_n(\delta)) &\leq \bar{C}(b(1+\hat{\theta})^{-1})^{-n\chi(\Phi)}M(n) \\ &\leq \bar{C}L(\gamma,\tau)\,(b(1+\hat{\theta})^{-1})^{-n\chi(\Phi)}b^{n(\chi(\Phi)-\nu(\Psi))} \\ &= \bar{L}(1+\hat{\theta})^{n\chi(\Phi)}b^{-n\nu(\Psi)} \leq \bar{L}(3/8)^n \end{aligned}$$

for all $n\geq n(\gamma,\tau)$ by (4.8), the definition of $b=4$ and the fact that $\nu(\Psi)\geq 1$, since $\Psi\neq\varnothing$. This implies that the series $\Sigma\,\mu(A_n(\delta))$ converges and therefore $\mu(A(\delta))=0$ by (4.18). We have

$$A\cap D(\mathbf{g}_\sigma)=A-\bigcup_{k=1}^{\infty}A(1/k)$$

and hence

$$\mu(A\cap D(\mathbf{g}_\sigma))=\mu(A)>0.$$

This implies that $\mu(D(\mathbf{g}_\sigma))=1$, since $x\mathbf{N}\subset D(\mathbf{g}_\sigma)$ for all $x\in D(\mathbf{g}_\sigma)$ and the action of $\mathbf{N}$ is ergodic. This completes the proof. □

Appendix 4.1

Here we shall prove Lemma 4.1. Let $0<\gamma<\gamma(\varepsilon)$ and $\tau>\tau(\gamma)$ be given. Let $x\in A$ and $B=x\mathbf{B}(\mathfrak{S}^{\nu}(\tau),0.1\gamma)$. Define

$$(4.22)\qquad \mathfrak{Q} = \mathfrak{Q}(B) = \{v \in \mathfrak{S}^{\nu}(\tau) : x \exp v \exp i \in A \text{ for some } i \in \mathfrak{J}_0(\mathfrak{H}),\ \|i\| \leq 0.1\gamma\}.$$

Write $t_n = \tau b^n/(1+\hat{\theta})^n$, $n=1,2,\ldots$, $\gamma_{-1}=\gamma_0=0$, $\gamma_n=\Sigma_{k=0}^{n-1} t_k^{-\beta}$, $\beta=1/2(m+1)$. We have $\gamma_n<0.1\gamma$ for all n, since $\tau \geq \tau(\gamma)$.

For $\mathfrak{S} \subset \mathfrak{G}$, $z \in \mathfrak{S}$ define $\mathfrak{S}(z)=\mathfrak{S}-z$. It is clear that

$$(4.23)\qquad \mathfrak{S}^{\xi}(z, t) = \mathfrak{S}^{\xi}(t) - z \subset 2\mathfrak{S}^{\xi}(t)$$

for all $z \in \mathfrak{S}^{\xi}(t)$, $t \geq 1$.

LEMMA 4.2. *For every* $n \in \mathbf{Z}^+$ *there are* $\mathbf{y}_1(n), \ldots, \mathbf{y}_{s(n)}(n) \in \mathbf{B}(\mathfrak{Q}, \gamma_{n-1})$ *such that*

$$\exp \mathfrak{Q} \subset \bigcup_{i=1}^{s(n)} \mathbf{y}_i(n)\, \mathbf{B}(2\mathfrak{S}^{\nu}(t_n), \gamma_n)$$

and $s(n) \leq b^{n\nu(W)}$.

Proof. In view of (4.23) it is enough to show that for every $n \in \mathbf{Z}^+$ there are $\mathbf{y}_1(n), \ldots, \mathbf{y}_{s(n)}(n) \in \mathbf{B}(\mathfrak{Q}, \gamma_{n-1})$ and $z_1(n), \ldots, z_{s(n)}(n) \in \mathfrak{W}^{\nu}(t_n)$ such that

$$(4.24)\qquad \exp \mathfrak{Q} \subset \bigcup_{i=1}^{s(n)} \mathbf{y}_i(n)\, \mathbf{B}(\mathfrak{S}^{\nu}(z_i(n), t_n), \gamma_n)$$

where $s(n) \leq b^{n\nu(W)}$. We shall prove this by induction on n. For $n=0$ set $s(0)=1$, $\mathbf{y}_1(0)=\mathbf{e}$, $z_1(0)=0$. Suppose that (4.24) holds for n. In order to prove it for $n+1$ it is enough to show that if $\mathbf{y} \in \mathbf{B}(\mathfrak{Q}, \gamma_{n-1})$, $z \in \mathfrak{W}^{\nu}(t_n)$ and

$$\mathfrak{Q}_n = \{v \in \mathfrak{S}^{\nu}(z, t_n) : \mathbf{y} \exp v \exp i \in \exp \mathfrak{Q} \text{ for some } i \in \mathfrak{J}_0(\mathfrak{H}),\ \|i\| \leq \gamma_n\}$$

then there are s: $1 \leq s \leq b^{\nu(W)} = b(W)$, $\mathbf{q}_1, \ldots, \mathbf{q}_s \in \exp \mathfrak{Q}_n$ and $z_1, \ldots, z_s \in \mathfrak{W}^{\nu}(t_{n+1})$ such that

$$(4.25)\qquad \exp \mathfrak{Q}_n \subset \bigcup_{i=1}^{s} \mathbf{q}_i \mathbf{B}(\mathfrak{S}^{\nu}(z_i, t_{n+1}), t_n^{-\beta}).$$

We have

$$\mathfrak{S}^{\nu}(z, t_n) = \mathfrak{W}^{\nu}(z, t_n) + \mathfrak{V}^{\nu}(t_n).$$

Let $u_1, \ldots, u_{b(W)} \in \mathfrak{W}^{\nu}(z, t_n)$ be such that

$$(4.26)\qquad \mathfrak{W}^{\nu}(z, t_n) = \bigcup_{i=1}^{b(W)} (u_i + \mathfrak{W}^{\nu}(bt_n)).$$

(Note that $\mathfrak{W}^\nu(z, t_n)$ is a parallelopiped with sides parallel to w_i of length $a(\theta)\, t_n^{-\nu(w_i)}/\bar{a}$. In (4.26) we partition $\mathfrak{W}^\nu(z, t_n)$ into parallelopipeds whose w_i-sides have the length $a(\theta)\,(bt_n)^{-\nu(w_i)}/\bar{a}$. There are exactly $b(W)$ of such parallelopipeds.) We have

$$\mathfrak{S}^\nu(z, t_n) = \bigcup_{i=1}^{b(W)} (u_i + \mathfrak{W}^\nu(bt_n) + \mathfrak{V}^\nu(t_n)) = \bigcup_{i=1}^{b(W)} (u_i + \mathfrak{T}(t_n)).$$

Let $z_i' \in \mathfrak{T}(t_n)$ be such that $u_i + z_i' = q_i \in \mathfrak{Q}_n$ and set $\mathbf{q}_i = \exp q_i$. We have

$$q_i + \mathfrak{T}(z_i', t_n) = u_i + z_i' + \mathfrak{T}(z_i', t_n) = u_i + \mathfrak{T}(t_n).$$

Also if $v \in \mathfrak{T}(z_i', t_n)$ then

$$v \in 2\mathfrak{W}^\nu(bt_n) + 2\mathfrak{V}^\nu(t_n) \subset 2\mathfrak{S}^\nu(t_n).$$

It follows then from (4.9) that

$$\exp(q_i + v) = \mathbf{q}_i \exp \bar{v} \exp \bar{\imath}$$

where

$$\bar{v} \in \mathfrak{T}(z_i', t_n/(1+\hat{\theta})), \quad \bar{\imath} \in \mathfrak{I}_0(\mathfrak{H}), \ \|\bar{\imath}\| \leq t_n^{-\beta}.$$

This implies that

$$\exp(u_i + \mathfrak{T}(t_n)) \subset \mathbf{q}_i\, \mathbf{B}(\mathfrak{T}(z_i', t_n/(1+\hat{\theta})), t_n^{-\beta}).$$

Now suppose that

$$\exp(q_i + v) \in \exp \mathfrak{Q}_n.$$

It follows then from (4.27), Lemma 3.1 and the definition of $\mathfrak{Q}$ that

$$\bar{v} \in \mathfrak{W}^\nu(z_i, t_{n+1}) + \mathfrak{V}^\nu(t_{n+1}) = \mathfrak{S}^\nu(z_i, t_{n+1})$$

for some $z_i \in \mathfrak{W}^\nu(t_{n+1})$. This proves (4.25) and completes the proof of the lemma. □

In order to prove Lemma 4.1 we need to partition further the sets

$$\mathbf{y}_i(n)\, \mathbf{B}(2\mathfrak{S}^\nu(t_n), \gamma_n)$$

from Lemma 4.2.

For $v\in\mathfrak{G}$ with $\chi(v)>0$ and $\alpha>0$ define

$$\alpha(v)=\min\{\alpha,\chi(v)-\nu(v)\}.$$

Note that if $\chi(v)=1$ then $\nu(v)=1$ and therefore $\alpha(v)=0$. Let $\mathfrak{S}^{\nu+\alpha}(t)$ be as in (4.4) with $\xi(v)=(\nu+\alpha)(v)$ and let

$$j(\Phi)=\max\{\chi(v)-\nu(v):v\in\Phi\}$$

$$r(\beta)=j(\Phi)/\beta\in\mathbf{Z}^{+}.$$

For each $r\in\mathbf{Z}^{+}$ with $1\leq r\leq r(\beta)$ and $1\leq k\leq m$ define

$$\mathfrak{S}_k^{\nu+r\beta}(t)=\{v\in\mathfrak{S}^{\nu+(r-1)\beta}:|a_i(v)|\leq a(\theta)\,t^{-(\nu+r\beta)(w_i)}/\bar{a},\ |c_p(v)|\leq\mathbf{C}\theta t^{-(\nu+r\beta)(c_p)},$$

$$|\psi_j(v)|\leq\mathbf{C}\theta t^{-(\nu+r\beta)(\psi_j)}\text{ for all }w_i,c_p,\psi_j\text{ with }\chi(w_i),\chi(c_p),\chi(\psi_j)\leq k\}.$$

Also define

$$\mathfrak{M}_k^{\nu+r\beta}(t)=\{v\in\mathfrak{S}_k^{\nu+r\beta}:|c_i(v)|\leq\mathbf{C}\theta t^{-\chi(c_i)},\ i=1,\dots,r\}$$

$$\mathfrak{M}^{\nu+r\beta}(t)=\bigcap_{k=1}^{m}\mathfrak{M}_k^{\nu+r\beta}(t).$$

Now let $w\in d\mathfrak{S}_k^{\nu+r\beta}(t)$, $v\in d\mathfrak{S}_{k+1}^{\nu+r\beta}(t)$ from some $1\leq d\leq 2^{m(r(\beta)+1)}$. Arguing as in (4.6) and using (4.7) we get

$$\exp(w+v)=\exp w\exp\bar{v}\exp i \tag{4.28}$$

for some $\bar{v}\in 2d\mathfrak{S}_{k+1}^{\nu+r\beta}$, $i\in\mathfrak{J}_0(\mathfrak{H})$, $\|i\|\leq t^{-2\beta}$.

It follows from Lemma 4.2 and Lemma 3.1 that

$$\exp\mathfrak{Q}\subset\bigcup_{i=1}^{s(n)}\mathbf{y}_i(n)\,\mathbf{B}(2\mathfrak{M}^{\nu}(t_n),\gamma_n)$$

for all $n\in\mathbf{Z}^{+}$. Now let $n\in\mathbf{Z}^{+}$ be fixed and let $\mathbf{y}=\mathbf{y}_i(n)$ for some $i\in\{1,\dots,s(n)\}$. Let

$$\mathfrak{Q}_n=\{v\in 2\mathfrak{M}^{\nu}(t_n):\mathbf{y}\exp v\exp i\in\exp\mathfrak{Q}\text{ for some }i\in\mathfrak{J}_0(\mathfrak{H}),\ \|i\|\leq\gamma_n\}.$$

Let $\mathbf{q}\in\mathbf{B}(\mathfrak{Q}_n,0.01\gamma)$ and let

$$\mathfrak{Q}_{k,r}(\mathbf{q})=\{v\in d\mathfrak{M}_k^{\nu+r\beta}(t_n):\mathbf{q}\exp v\exp i\in\exp\mathfrak{Q}_n\text{ for some }i\in\mathfrak{J}_0(\mathfrak{H}),\ \|i\|\leq 0.1\gamma\}$$

where $1\leq k\leq m-1$, $0\leq r\leq r(\beta)$, $1\leq d\leq 2^{m(r(\beta)+1)}$.

LEMMA 4.3. *There are* $\mathbf{q}_1, \ldots, \mathbf{q}_s \in \exp \mathfrak{Q}_{k,r}(\mathbf{q})$ *such that*

$$\exp \mathfrak{Q}_{k,r}(\mathbf{q}) \subset \bigcup_{i=1}^{s} \mathbf{q}_i \mathbf{B}(2d\mathfrak{M}_{k+1}^{\nu+r\beta}(t_n), t_n^{-\beta})$$

where $s \leq (t_n^\beta+1)^{|\zeta_{k+1}|}$, $\zeta_k = \{v \in \Phi : \chi(v) = k\}$ *and* $|\zeta_k|$ *denotes the cardinal number of* ζ_k.

Proof. The proof is similar to that of Lemma 4.2. Let $p(k)=(t_n^\beta+1)^{|\zeta_{k+1}|}$ and let $m_1, \ldots, m_{p(k)} \in d\mathfrak{M}_k^{\nu+r\beta}(t_n)$ be such that

$$d\mathfrak{M}_k^{\nu+r\beta}(t_n) \subset \bigcup_{i=1}^{p(k)} (m_i + d\mathfrak{M}_{k+1}^{\nu+r\beta}(t_n)) = \bigcup_{i=1}^{p(k)} (m_i + \mathfrak{T}_{k+1}(t_n)). \tag{4.29}$$

Let $z_i \in \mathfrak{T}_{k+1}(t_n)$ be such that $q_i = m_i + z_i \in \mathfrak{Q}_{k,r}(\mathbf{q})$. Set $\mathbf{q}_i = \exp q_i$. We have

$$m_i + \mathfrak{T}_{k+1}(t_n) = q_i + \mathfrak{T}_{k+1}(z_i, t_n).$$

Let $v \in \mathfrak{T}_{k+1}(z_i, t_n) \subset 2d\mathfrak{M}_{k+1}^{\nu+r\beta}(t_n)$. We have using (4.28)

$$\exp(q_i + v) = \mathbf{q}_i \exp \bar{v} \exp i$$

for some $\bar{v} \in 2d\mathfrak{S}_{k+1}^{\nu+r\beta}(t_n)$, $i \in \mathfrak{J}_0(\mathfrak{H})$, $\|i\| \leq t_n^{-2\beta}$. It follows now from the definition of $\mathfrak{Q}_{k,r}(\mathbf{q})$ that if $q_i + v \in \mathfrak{Q}_{k,r}(\mathbf{q})$ then

$$\bar{v} \in 2d\mathfrak{M}_{k+1}^{\nu+r\beta}(t_n).$$

This and (4.29) complete the proof of the lemma. □

For $0 \leq r \leq r(\beta)$ define

$$\eta(r) = \{v \in \Phi : 0 \leq \chi(v) - \nu(v) \leq r\beta\}$$

$$k(r) = \min\{\chi(v) : v \in \Phi - \eta(r)\} \geq 2.$$

Let now $n(\gamma, \tau) > 1$ be so large that

$$t_n^{-\beta} m r(\beta) \leq 0.01\gamma \tag{4.30}$$

for all $n \geq n(\gamma, \tau)$.

COROLLARY 4.2. *Let* $0 \leq r \leq r(\beta) - 1$, $n \geq n(\gamma, \tau)$, $\mathbf{q} \in B(\mathfrak{Q}_n, mrt_n^{-\beta})$,

$\mathfrak{Q}_r(\mathbf{q}) = \{v \in d\mathfrak{M}^{\nu+r\beta}(t_n) : \mathbf{q} \exp v \exp i \in \exp \mathfrak{Q}_n$ *for some* $i \in \mathfrak{J}_0(\mathfrak{H})$, $\|i\| \leq mrt_n^{-\beta}\}$, *where* $d=2^{mr+1}$. *Then there are* $\mathbf{p}_1^{(r)},\dots,\mathbf{p}_{\tau(r)}^{(r)}\in\mathbf{B}(\mathfrak{Q}_r(\mathbf{q}),mt_n^{-\beta})$ *such that*

$$\exp\mathfrak{Q}_r(\mathbf{q})\subset\bigcup_{i=1}^{\tau(r)}\mathbf{p}_i\mathbf{B}(2^m d\mathfrak{M}^{\nu+(r+1)\beta}(t_n),mt_n^{-\beta})$$

where $\tau(r)\leq(t_n^\beta+1)^{\sum_{i=k(r)}^m|\zeta_i|}$.

Proof. We have $\mathfrak{M}^{\nu+r\beta}(t_n)=\mathfrak{M}_{k(r)-1}^{\nu+(r+1)\beta}(t_n)$. This and Lemma 4.3 imply the corollary. □

It follows from the definition of $\nu(v)$ that $\chi(v)-\nu(v)=k$ whenever $\chi(v)=2k$ or $2k+1$. This shows that $k(r)=2k$ for all $(k-1)/\beta\leq r<k/\beta$, $1\leq k\leq j(\Phi)$. This implies that

$$\sum_{r=0}^{r(\beta)}\sum_{i=k(r)}^{m}|\zeta_i|=\frac{1}{\beta}\sum_{j=0}^{j(\Phi)}j|\xi_j|=(\chi(\Phi)-\nu(\Phi))/\beta \tag{4.31}$$

where $\xi_j=\{v\in\Phi:\chi(v)-\nu(v)=j\}$.

COROLLARY 4.3. *Let* $n\geq n(\gamma,\tau)$. *Then there are* $\mathbf{b}_1(n),\dots,\mathbf{b}_{\delta(n)}(n)\in\mathbf{B}(\mathfrak{Q}_n,mr(\beta)\,t_n^{-\beta})$ *such that*

$$\exp\mathfrak{Q}_n\subset\bigcup_{i=1}^{\delta(n)}\mathbf{b}_i(n)\,\mathbf{B}(2^{mr(\beta)+1}\mathfrak{S}^\chi(t_n),mr(\beta)\,t_n^{-\beta}) \tag{4.32}$$

where $\delta(n)\leq(t_n^\beta+1)^{(\chi(\Phi)-\nu(\Phi))/\beta}$.

Proof. For $r=0$ let $\mathbf{q}=\mathbf{e}\in\mathbf{B}(\mathfrak{Q}_n,0)$. We have $\mathfrak{Q}_0(\mathbf{q})=\mathfrak{Q}_n$. Applying Corollary 4.1 we get

$$\exp\mathfrak{Q}_n\subset\bigcup_{i=1}^{\tau(0)}\mathbf{p}_i(0)\,\mathbf{B}(2^{m+1}\mathfrak{M}^{\nu+\beta}(t_n),mt_n^{-\beta})$$

where $\mathbf{p}_1(0),\dots,\mathbf{p}_{\tau(0)}\in\mathbf{B}(\mathfrak{Q}_n,mt_n^{-\beta})$ and $\tau(0)\leq(t_n^\beta+1)^{\sum_{i=k(0)}^m|\zeta_i|}$. Note that $\mathfrak{S}^\chi(t_n)=\mathfrak{M}^{\nu+r(\beta)\beta}(t_n)$. This and (4.31) show that we need $r(\beta)$ successive applications of Corollary 4.2 to get (4.32). This completes the proof. □

Proof of Lemma 4.1. Let $\mathscr{E}=\{x_1\mathbf{B}(\mathfrak{S}^\nu(\tau),0.1\gamma),\dots,x_{L_1}\mathbf{B}(\mathfrak{S}^\nu(\tau),0.1\gamma)\}$ be a cover of A by (ν,τ)-boxes at $x_1,\dots,x_{L_1}\in A$. Let $n(\gamma,\tau)$ be as in (4.30), $n\geq n(\gamma,\tau)$ and let $\mathfrak{Q}^{(i)}=\mathfrak{Q}(x_i\mathbf{B}(\mathfrak{S}^\nu(\tau),0.1\gamma)$ be as in (4.22), $i=1,\dots,L_1$. It follows from Lemma 4.2 and Corollary 4.3 that there are $a_1(n),\dots,a_{M(n)}(n)\in\bigcup_{i=1}^{L_1}x_i\mathbf{B}(\mathfrak{Q}^{(i)},0.2\gamma)$ such that

$$A\subset\bigcup_{j=1}^{M(n)}a_j(n)\,\mathbf{B}(2^{mr(\beta)+1}\mathfrak{S}^\chi(t_n),0.3\gamma)$$

where

$$M(n) \leqslant L_1\, s(n)\, (t_n^\beta + 1)^{(\chi(\Phi)-\nu(\Phi))/\beta} \leqslant Ls(n)\, b^{n(\chi(\Phi)-\nu(\Phi))} \leqslant Lb^{n(\chi(\Phi)-\nu(\Phi)+\nu(W))} = Lb^{nM}$$

for some $L=L(\gamma,\tau)>1$. This completes the proof of the lemma if we set $a=2^{mr(\beta)+1}$. □

5. Conclusion 2 of the Main Theorem

From now till the end of the paper we assume that $\mathbf{G}$ and $(X=\mathbf{\Gamma}\backslash\mathbf{G}, d, \mu)$ are as in Section 4, $\mathbf{N}=\exp\mathfrak{N}$, $\mathfrak{N}=\mathfrak{L}(\mathbf{\Lambda})\cap\mathfrak{E}^{-2}$ and $\mathbf{c}\mathbf{g}_p\mathbf{c}^{-1}\in\mathbf{\Lambda}$ for some $\mathbf{c}\in\mathbf{E}^-$, $p\in R$. It follows from Proposition 1.2 that $u\in\mathfrak{N}$ and $\mathbf{c}u\mathbf{c}^{-1}\in\mathfrak{L}(\mathbf{\Lambda})$, since $\mathbf{u}=\exp u\in\mathbf{\Lambda}$. Also we assume that the action of $\mathbf{N}$ is ergodic.

Let $\mathbf{\Lambda}_c=\mathbf{c}^{-1}\mathbf{\Lambda}\mathbf{c}$ and let μ_c be the Borel probability measure on X defined by $\mu_c(E)=\mu(E\mathbf{c}^{-1})$ for every Borel subset $E\subset X$. It is clear that $\mathbf{\Lambda}_c=\mathbf{\Lambda}(\mathbf{G},\mathbf{\Gamma},\mu_c)$. Also $\mathbf{g}_p\in\mathbf{\Lambda}_c$, $u\in\mathfrak{L}(\mathbf{\Lambda}_c)$, $\mathfrak{N}_c=\mathbf{c}^{-1}\mathfrak{N}\mathbf{c}=\mathfrak{L}(\mathbf{\Lambda}_c)\cap\mathfrak{E}^{-2}$ and the action of $\exp\mathfrak{N}_c$ on (X,μ_c) is ergodic. It is clear that μ_c is algebraic if and only if so is μ. This says that we can simply assume without loss of generality that $\mathbf{g}_p\in\mathbf{\Lambda}$, $u\in\mathfrak{L}(\mathbf{\Lambda})$ and prove that then $sl_2(u,g)\subset\mathfrak{L}(\mathbf{\Lambda})$ and μ is algebraic.

We have $\mu(D(\mathbf{g}))=0$, since $\mathbf{g}_p\in\mathbf{\Lambda}$. This implies via Theorem 4.1 that $\Psi=\varnothing$. Using (3.7) and $u\in\mathfrak{N}$ we conclude that $u^*+v\in\mathfrak{W}$ for some v with $\chi(v)<2$. Therefore $\mathrm{ad}_u(u^*+v)=-g+v'\in\mathfrak{L}(\mathbf{\Lambda})$ for some v' with $\chi(v')<0$ by the definition of $\mathfrak{M}$. This implies that $g\in\mathfrak{L}(\mathbf{\Lambda})$ by Proposition 1.2. We shall show later that $u^*\in\mathfrak{L}(\mathbf{\Lambda})$, too.

Let us note that the set Ψ in (3.7) depends on the choices of bases and complementary spaces occurring in the construction of $\mathfrak{W}$ and $\mathfrak{V}$ (see Note 1.1), while the fact $\Psi\neq\varnothing$ or $\Psi=\varnothing$ and Theorem 4.1 do not depend on those choices.

In fact, in the case $\mathbf{g}\in\mathbf{\Lambda}$ there is a natural way to construct $\mathfrak{W}$ and $\mathfrak{V}$, provided by Theorem 1.1 and Proposition 1.2. More specifically, it follows from Proposition 1.2 that $\mathfrak{Z}(\mathfrak{H})$ and every nontrivial $\mathfrak{E}_p^q\cap\mathfrak{L}(\mathbf{\Lambda})$, $-m\leqslant p\leqslant q\leqslant m$ all have bases, consisting of eigenvectors of ad_g. Here m is the maximal eigenvalue of ad_g. Let

$$\hat{\mathfrak{E}}_\lambda = \{v\in\mathfrak{E}_\lambda : [v,u^*]=0\}, \quad \hat{\omega}=\{\lambda : \hat{\mathfrak{E}}_\lambda \neq \{0\}\}$$

be as in Theorem 1.1, $\lambda\geqslant 0$ for $\lambda\in\hat{\omega}$. It is clear that if $w\in\hat{\mathfrak{E}}_\lambda, \lambda>0$ then

$$\mathrm{ad}_u^{\nu(\lambda)}(w) = \mathrm{ad}_u^{\nu(z)}(z) \tag{5.1}$$

for all $z\in\{\mathrm{ad}_u^k(w) : 0\leqslant k\leqslant\lambda/2\}$. For $0<\lambda\in\hat{\omega}$ define

$$\hat{\mathfrak{W}}_\lambda = \{v \in \hat{\mathfrak{E}}_\lambda : \mathrm{ad}_u^{\nu(\lambda)}(v) \in \mathfrak{L}(\mathbf{\Lambda})\}$$

and let $\hat{\mathfrak{Y}}_\lambda$ be a subspace of $\hat{\mathfrak{E}}_\lambda$ complementary to $\hat{\mathfrak{W}}_\lambda$. Let $\mathfrak{W}$ and $\mathfrak{Y}$ be the subspaces spanned by $\{\mathrm{ad}_u^k(w)\colon w \in \hat{\mathfrak{W}}_\lambda,\ 0<\lambda \in \hat{\omega},\ 0 \leqslant k < \lambda/2\}$ and $\{\mathrm{ad}_u^k(y)\colon y \in \hat{\mathfrak{Y}}_\lambda,\ 0<\lambda \in \hat{\omega},\ 0 \leqslant k<\lambda/2\}$ respectively. It follows from Theorem 1.1 that $\mathrm{ad}_u^{\lambda+1}(v)=0$ for all $v \in \hat{\mathfrak{E}}_\lambda$. This implies that $\mathrm{ad}_u^\lambda(y) \notin \mathfrak{N}$ for all $0 \neq y \in \hat{\mathfrak{Y}}_\lambda$, since otherwise we would have $y \in \Psi$ and $\Psi \neq \varnothing$. This implies that

$$\mathrm{ad}_u^\lambda(y) \notin \mathfrak{L}(\mathbf{\Lambda}), \quad y \in \hat{\mathfrak{Y}}_\lambda \tag{5.2}$$

by the definition of $\hat{\mathfrak{Y}}_\lambda$.

Proposition 5.1. (1) *Let $v \in \mathfrak{E}_\lambda$ for some $\lambda>0$. Then $v \in \mathfrak{W}$ if and only if* $\mathrm{ad}_u^k(v) \in \mathfrak{L}(\mathbf{\Lambda})$ *for some $\nu(\lambda) \leqslant k \leqslant \lambda$;*

(2) *The subalgebra $\mathfrak{H} = \mathfrak{E}^- \cap \mathfrak{L}(\mathbf{\Lambda})$ is spanned by*

$$\{\mathrm{ad}_u^k(w)\colon w \in \hat{\mathfrak{W}}_\lambda, 0<\lambda \in \hat{\omega},\ \lambda/2<k \leqslant \lambda\}.$$

Proof. If $v \in \mathfrak{W}$ then $\mathrm{ad}_u^k(v) \in \mathfrak{L}(\mathbf{\Lambda})$ for all $k \geqslant \nu(\lambda)$ by (5.1) and the definition of $\mathfrak{W}$. Now let $v \neq 0$ and $\mathrm{ad}_u^k(v) \in \mathfrak{L}(\mathbf{\Lambda})$ for some $\nu(\lambda) \leqslant k \leqslant \lambda$. It follows from Theorem 1.1 that there are non-zero $v_1, \ldots, v_n \in \mathfrak{E}_\lambda$ such that $v=v_1+\ldots+v_n$ and $v_i=\mathrm{ad}_u^{k_i}(v_i')$ for some $v_i' \in \hat{\mathfrak{E}}_{\lambda_i}$, $0<\lambda_i \in \hat{\omega}$, $k_i=(\lambda_i-\lambda)/2$, $\lambda_1>\lambda_2>\ldots>\lambda_n$. We have $\mathrm{ad}_u^{\lambda_1}(v_1')=\mathrm{ad}_u^{\lambda+k_1}(v_1)=\mathrm{ad}_u^{\lambda+k_1}(v) \in \mathfrak{L}(\mathbf{\Lambda})$. Therefore $v_1' \in \hat{\mathfrak{W}}_{\lambda_1}$ by (5.2), $v_1 \in \mathfrak{W}$ and $\mathrm{ad}_u^k(v_1) \in \mathfrak{L}(\mathbf{\Lambda})$, since $k \geqslant \nu(\lambda)$. This implies that $\mathrm{ad}_u^k(v-v_1) \in \mathfrak{L}(\mathbf{\Lambda})$. Applying this argument n times we get $v_i \in \mathfrak{W}$ for all $i=1,\ldots,n$. The proof of (2) is similar □

It follows from Proposition 1.2 that if $v \in \mathfrak{G}$, $\chi(v)>0$ and $\mathrm{ad}_u^{\nu(v)}(v) \in \mathfrak{L}(\mathbf{\Lambda})$ then $\mathrm{ad}_u^{\nu(v)}(p_{\chi(v)}(v)) \in \mathfrak{L}(\mathbf{\Lambda})$ and therefore $p_{\chi(v)}(v) \in \mathfrak{W}$ by Proposition 5.1. This shows that $\mathfrak{W}$ is indeed as required in Section 3. Now we define bases in $\mathfrak{W}, \mathfrak{Y}$ and $\mathfrak{H}$ by

$$W = \{\mathrm{ad}_u^k(w)/\|\mathrm{ad}_u^k(w)\|\colon w \in \hat{W}_\lambda,\ 0<\lambda \in \hat{\omega},\ 0 \leqslant k<\lambda/2\} = \{w_1, \ldots, w_p\}$$

$$Y = \{\mathrm{ad}_u^k(y)/\|\mathrm{ad}_u^k(y)\|\colon y \in \hat{Y}_\lambda,\ 0<\lambda \in \hat{\omega},\ 0 \leqslant k<\lambda/2\} = \{c_1, \ldots, c_r\} \tag{5.3}$$

$$H = \{\mathrm{ad}_u^k(w)/\|\mathrm{ad}_u^k(w)\|\colon w \in \hat{W}_\lambda,\ 0<\lambda \in \hat{\omega},\ \lambda/2<k \leqslant \lambda\} = \{h_1, \ldots, h_p\}$$

where $\hat{W}_\lambda$, $\hat{Y}_\lambda$ are bases of unit vectors in $\hat{\mathfrak{W}}_\lambda$ and $\hat{\mathfrak{Y}}_\lambda$ respectively. We have $Y=Y-\Psi$ since $\Psi=\varnothing$. We now define $\mathfrak{D}$ to be the subspace spanned by $\{\mathrm{ad}_u^k(y)\colon y \in \hat{\mathfrak{Y}}_\lambda,\ 0<\lambda \in \hat{\omega},\ \lambda/2 \leqslant k<\lambda-1\}$. Thus $\mathfrak{G}=\mathfrak{W}+\mathfrak{Y}+\mathfrak{D}+\mathfrak{P}$, where $\mathfrak{P}=\{v \in \mathfrak{G}\colon \mathrm{ad}_u(v) \in \mathfrak{N}\}$. It is clear that $\mathrm{ad}_u^\lambda(v) \in \mathfrak{P}$ for all $v \in \hat{\mathfrak{E}}_\lambda$.

It follows from Proposition 1.2 that $\mathfrak{P}$ and $\mathfrak{I}_0(\mathfrak{H})=\mathfrak{I}(\mathfrak{H})\cap\mathfrak{E}^0\subset\mathfrak{P}$ are spanned by eigenvectors of ad_g. Let $\bar{\mathfrak{Y}}_\lambda=\{y\in\hat{\mathfrak{Y}}_\lambda: \mathrm{ad}_u^\lambda(y)\in\mathfrak{I}_0(\mathfrak{H})\}$ and let $\mathfrak{Z}^-$ be the subspace of $\mathfrak{I}_0(\mathfrak{H})$ spanned by $\{\mathrm{ad}_u^\lambda(y): y\in\bar{\mathfrak{Y}}_\lambda,\ 0<\lambda\in\hat{\omega}\}$. We have $\mathfrak{Z}^-+\mathfrak{H}=\mathfrak{E}^-\cap\mathfrak{I}_0(\mathfrak{H})$. Also let $\mathfrak{Z}_0=\mathfrak{E}_0\cap\mathfrak{I}_0(\mathfrak{H})$. We have

$$\mathfrak{I}_0(\mathfrak{H})=\mathfrak{H}+\mathfrak{Z}^-+\mathfrak{Z}_0.$$

Finally let $\mathfrak{I}$ be a subspace of $\mathfrak{P}$ complementary to $\mathfrak{I}_0(\mathfrak{H})$. Summarizing we have

$$\mathfrak{G}=\mathfrak{W}+\mathfrak{Y}+\mathfrak{F}+\mathfrak{Z}+\mathfrak{H} \tag{5.4}$$

where $\mathfrak{F}=\mathfrak{D}+\mathfrak{I}$, $\mathfrak{Z}=\mathfrak{Z}^-+\mathfrak{Z}_0$.

Now let $\mathfrak{E}$ be any of the subspaces in (5.4) with the chosen above basis $\{e_1,\dots,e_n\}$ of eigenvectors of ad_g. For $\gamma>0$ define

$$\mathfrak{U}_\gamma(\mathfrak{E})=\left\{v\in\mathfrak{E}: v=\sum_{i=1}^n e_i(v)\,e_i,\ |e_i(v)|<\gamma\right\}$$

and let a compact $A=A(\theta,\varepsilon)\subset X$, $\mu(A)>1-\varepsilon$, $0<\varepsilon<0.1$ be as in Lemma 3.1 for some $0<\theta<\Theta$, specified later. Set

$$\Delta=\Delta(A)=\min\{\Delta(x): x\in A\}>0$$

where $\Delta(x)$ is defined in (4.1). In this section we shall prove the following

THEOREM 5.1. *Given $0<\delta<\Delta$ there exists a subset $X(\delta)\subset X$, $\mu(X(\delta))>0$ such that if $x\in X(\delta)$ then*

$$\mu(x\exp\mathfrak{U}_\delta(\mathfrak{W}+\mathfrak{Z}_0)\exp\mathfrak{U}_\delta(\mathfrak{H}))>0.$$

The proof of this theorem begins with the following lemma.

LEMMA 5.1. *Given $0<\delta<\Delta$ there exists $X_1(\delta)\subset X$, $\mu(X_1(\delta))>0$ such that if $x\in X_1(\delta)$ then*

$$\mu(x\exp\mathfrak{U}_\delta(\mathfrak{W}+\mathfrak{Y}+\mathfrak{Z}_0)\exp\mathfrak{U}_\delta(\mathfrak{H}))>0.$$

The proof of this lemma is based on Lemma 5.2 below analogous to Lemma 4.1. Let $\mathfrak{S}^\chi(t)$ and $\tau(\gamma)$ be as in (4.4) and (4.11) respectively. Recall that $\Psi=\emptyset$ in (4.3).

LEMMA 5.2. *Given $0<\gamma<\gamma(\varepsilon)$ and $\tau>\tau(\gamma)$ there exists $n(\gamma,\tau)>1$ such that for each integer $n\geqslant n(\gamma,\tau)$ there are $a_1(n),\dots,a_{L(n)}(n)\in X$ such that*

$$(5.5)\qquad A\subset \bigcup_{i=1}^{L(n)} a_i(n)\,[\exp(a\mathfrak{S}^\chi(\tau b^n)+\mathfrak{U}_\gamma(\mathfrak{Z}))]\,\mathbf{U}_\gamma(\mathbf{H})$$

where $\mathbf{U}_\gamma(\mathbf{H})=\exp\mathfrak{U}_\gamma(\mathfrak{H})$, $L(n)\leq Lb^{n\chi(W)}$ *and* $L=L(\gamma,\tau)>1$, $a>1$, $b>1$ *are constants.*

The proof of Lemma 5.2 uses Lemma 3.1 and is given in an appendix at the end of this section. Write

$$\mathfrak{R}(n)=a\mathfrak{S}^\chi(\tau b^n)+\mathfrak{U}_\gamma(\mathfrak{Z}),\quad B_i(n)=a_i(n)\,(\exp\mathfrak{R}(n))\,\mathbf{U}_\gamma(\mathbf{H}).$$

It follows from (4.5) that if $x\in B_i(n)$, $i=1,\dots,L(n)$ then

$$(5.6)\qquad B_i(n)\subset x[\exp(3\mathfrak{R}(n))]\,\mathbf{U}_{3\gamma}(\mathbf{H}).$$

Now let $\mathfrak{W}^\chi(t)$ be as in (4.4) and let

$$\mathfrak{Y}^\chi(t)=\{v\in\mathfrak{Y}:|c_i(v)|\leq \mathbf{C}\theta t^{\chi(c_i)},\ i=1,\dots,r\}.$$

Set $\sigma=\ln b$. It follows from the definition of $\mathfrak{S}^\chi(t)$ and the fact $\mathfrak{F}\subset\mathfrak{E}^0$, $\mathfrak{Z}^-\subset\mathfrak{E}^-$ that

$$(5.7)\qquad \mathbf{g}_\sigma^{-n}\,\mathfrak{R}(n)\,\mathbf{g}_\sigma^n=a\mathfrak{W}^\chi(\tau)+a\mathfrak{Y}^\chi(\tau)+\mathfrak{U}_\gamma(\mathfrak{Z}_0)+\mathfrak{O}_n(\mathfrak{F}+\mathfrak{Z}^-)$$

for all $n\geq 1$, $0<\gamma<1$, where

$$(5.8)\qquad \mathfrak{O}_n(\mathfrak{F}+\mathfrak{Z}^-)\subset\mathfrak{U}_{b^{-n\beta}}(\mathfrak{F}+\mathfrak{Z}^-).$$

Proof of Lemma 5.1. Let $0<\delta<\Delta$ be given and let $0<\gamma<\gamma(\varepsilon)$ and $\tau>\tau(\gamma)$ be such that

$$\max\{3\gamma,3a\tau^{-1}\}\leq 0.1\delta.$$

Let $n(\gamma,\tau)$ be as in Lemma 5.2 and let $n_0\geq n(\gamma,\tau)$ be so big that $b^{-n\beta}\leq 0.1\delta$ for all $n\geq n_0$. For $n\geq 1$ we have

$$B_i(n)\,\mathbf{g}_\sigma^n=(a_i(n)\,\mathbf{g}_\sigma^n)\exp(\mathbf{g}_\sigma^{-n}\,\mathfrak{R}(n)\,\mathbf{g}_\sigma^n)\,(\mathbf{g}_\sigma^{-n}\,\mathbf{U}_\gamma(\mathbf{H})\,\mathbf{g}_\sigma^n)\subset(a_i(n)\,\mathbf{g}_\sigma^n)\exp(\mathbf{g}_\sigma^{-n}\,\mathfrak{R}(n)\,\mathbf{g}_\sigma^n)\,\mathbf{U}_\gamma(\mathbf{H})$$
$$=\Omega_i(n).$$

Now let $n\geq n_0$ and $A_n=A\cap A\mathbf{g}_\sigma^{-n}$. We have $\mu(A_n)>1-2\varepsilon$, since $g\in\mathfrak{L}(\mathbf{\Lambda})$. Define

$$\bar{J}_n=\{i:\mu(A_n\cap B_i(n))>0\}.$$

It follows from (5.6) and (5.7) that

$$\Omega_i(n) \subset x_i \mathbf{O}_{\Delta(x_i)}(\mathbf{G})$$

for all $i \in \bar{J}_n$ and some $x_i \in A$, where $\mathbf{O}_{\Delta(x)}(\mathbf{G})$ is as in (4.1). This implies via (4.13) and (4.16) that

$$\mu(B_i(n)) = \mu(B_i(n)\,\mathbf{g}_\sigma^n) \leqslant D\mu(\Omega_i(n))\, b^{n\chi(H)} = D\mu(\Omega_i(n))\, b^{-n\chi(W)} \tag{5.9}$$

for all $i \in \bar{J}_n$, $n \geqslant n_0$ and some $D>1$, where $\chi(H) = \Sigma_{i=1}^p \chi(h_i) = -\Sigma_{i=1}^p \chi(w_i) = -\chi(W)$ (see (5.3)). Define

$$J_n = \{i \in \bar{J}_n : \mu(\Omega_i(n)) \geqslant (1-2\varepsilon)/2DL\}$$

where $L = L(\gamma, \tau) > 1$ is as in (5.5). We have

$$A_n - \bigcup_{i \notin J_n} (A_n \cap B_i(n)) \subset \bigcup_{i \in J_n} B_i(n).$$

This implies via (5.9) and the definition of J_n that

$$\begin{aligned} \mu\Big(\bigcup_{i \in J_n} B_i(n)\Big) &\geqslant \mu(A_n) - \mu\Big(\bigcup_{i \notin J_n} (A_n \cap B_i(n))\Big) \\ &\geqslant \mu(A_n) - \sum_{i \notin J_n} \mu(B_i(n)) \geqslant (1-2\varepsilon)/2 > 1/4 \end{aligned}$$

since $L(n) \leqslant L b^{n\chi(W)}$. Now let

$$X_n = X_n(\delta) = \Big(\bigcup_{i \in J_n} B_i(n)\Big)\,\mathbf{g}_\sigma^n.$$

We have $\mu(X_n) \geqslant 1/4$ for all $n \geqslant n_0$. If $x \in X_n$ then $x\mathbf{g}_\sigma^{-n} \in B_i(n)$ for some $i \in J_n$ and

$$\Omega_i(n) \subset x \exp(\mathbf{g}_\sigma^{-n}(3\mathfrak{R}(n))\,\mathbf{g}_\sigma^n) \cdot \mathbf{U}_{3\gamma}(\mathbf{H}) \subset x \exp[\mathfrak{U}_\delta(\mathfrak{W} + \mathfrak{Y} + \mathfrak{Z}_0) + 3\mathfrak{L}_n(\mathfrak{F} + \mathfrak{Z}^-)] \cdot \mathbf{U}_\delta(\mathbf{H})$$

by (5.6) and (5.7). This implies that

$$\mu(x \exp[\mathfrak{U}_\delta(\mathfrak{W} + \mathfrak{Y} + \mathfrak{Z}_0) + 3\mathfrak{L}_n(\mathfrak{F} + \mathfrak{Z}^-)] \cdot \mathbf{U}_\delta(\mathbf{H})) > (1-2\varepsilon)/2DL \tag{5.10}$$

for all $x \in X_n$, $n \geqslant n_0$. Now let

$$X_1 = X_1(\delta) = \bigcap_{i=n_0}^{\infty} \bigcup_{n=i}^{\infty} X_n = \{x \in X : x \in X_n \text{ for infinitely many } n \geqslant n_0\}.$$

We have $\mu(X_1)\geq 1/4$ and

$$\mu(x\exp[\mathfrak{U}_\delta(\mathfrak{W}+\mathfrak{Y}+\mathfrak{Z}_0)]\,\mathbf{U}_\delta(\mathbf{H}))>0$$

for all $x\in X_1$ by (5.8) and (5.10). This completes the proof of the lemma. □

Next we shall eliminate $\mathfrak{Y}$. To do so let us look closer at the subspaces $\mathfrak{W}$ and $\mathfrak{H}$. Let

$$K=\{\mathrm{ad}_u^k(w)\colon w\in\hat{\mathfrak{W}}_\lambda,\ 0<\lambda\in\hat{\omega},\ 0\leq k\leq\lambda\}$$

and let $\mathfrak{K}$ be the subspace of $\mathfrak{G}$ spanned by K. For $v\in K$ there is a unique $\lambda(v)\in\hat{\omega}$, $z(v)\in\hat{\mathfrak{W}}_\lambda$ and $0\leq k(v)\leq\lambda$ such that $v=\mathrm{ad}_u^{k(v)}(z(v))$. Also $\mathfrak{H}=\mathfrak{K}\cap\mathfrak{E}^-$. We have

$$(5.11)\qquad [v,u]\in\mathfrak{K},\quad [v,u^*]\in\mathfrak{K}$$

whenever $v\in\mathfrak{K}$ (see (1.7)).

PROPOSITION 5.2. *Suppose that* $v\in\mathfrak{G}$, $w\in K$, $k(w)<\lambda(w)$ *and* (1) $[v,[w,u]]\in\mathfrak{K}$, (2) $[[v,u^*],[w,u]]\in\mathfrak{K}$. *Then* $[v,w]\in\mathfrak{K}$.

Proof. We have

$$k_1=[[v,[w,u]],u^*]\in\mathfrak{K}$$

by (5.11) and (1). We have using (1.8)

$$k_1=[v,[[w,u],u^*]]+k_2=(k(w)+1)(\lambda(w)-k(w))\,[v,w]+k_2.$$

where $k_2\in\mathfrak{K}$ by (2). This implies that $[v,w]\in\mathfrak{K}$, since $k(w)<\lambda(w)$. □

PROPOSITION 5.3. *Let* v,w *be eigenvectors of* ad_g, $v\in\mathfrak{H}$ *and* $w\in\mathfrak{K}$. *Then*

(1) $[v,w]\in\mathfrak{H}$ *if* $\chi(w)\leq 0$;
(2) $[v,w]\in\mathfrak{H}$ *if* $\chi(w)>0$ *and* $\chi(v)<-\chi(w)$;
(3) $[v,w]\in\mathfrak{K}+\check{\mathfrak{E}}_0$ *if* $\chi(v)=-\chi(w)$;
(4) $[v,w]\in\mathfrak{W}$ *if* $-\chi(w)<\chi(v)\leq -1$.

Proof. Let z be an eigenvector of ad_g. It follows from Theorem 1.1 that

$$(5.12)\qquad \begin{aligned} &\text{if } \chi(z)=0 \text{ and } [z,u]\in\mathfrak{H} \quad \text{then } z\in\mathfrak{K}+\check{\mathfrak{E}}_0\\ &\text{if } \chi(z)>0 \text{ and } [z,u]\in\mathfrak{K}+\check{\mathfrak{E}}_0,\quad \text{then } [z,u]\in\mathfrak{K} \text{ and } z\in\mathfrak{W}.\end{aligned}$$

(1) If $\chi(w)\leq 0$ then $w\in\mathfrak{L}(\Lambda)$ and hence $[v,w]\in\mathfrak{H}$, since $\chi([v,w])<0$.

(2) We shall prove (2) by induction on $\chi(w)$ assuming with no loss of generality that $w\in K$. Then $k(w)<\lambda(w)$ since $\chi(w)>0$. Let $\chi(w)=1$ and $\chi(v)<-1$. We have $[w,u]\in\mathfrak{H}$, $\chi([v,u^*])\leq 0$. It follows then from (1) that $[v,[w,u]]\in\mathfrak{H}$ and $[[v,u^*],[w,u]]\in\mathfrak{H}$. This implies $[v,w]\in\mathfrak{H}$ by Proposition 5.2. Now suppose that (2) holds for all w with $\chi(w)\in\{0,1,\dots,n-1\}$, $n>1$. Let $\chi(w)=n$ and $\chi(v)<-n$. We have $\chi([w,u])=n-2$, $\chi(v)<-(n-2)$, $\chi([v,u^*])<-(n-2)$. This implies by the inductive hypothesis that $[v,[w,u]]\in\mathfrak{H}$, $[[v,u^*],[w,u]]\in\mathfrak{H}$. Therefore $[v,w]\in\mathfrak{H}$ by Proposition 5.2. This proves (2).

(3) Let $\chi(v)=-\chi(w)$. We have $\chi([v,u])<-\chi(w)$ and $\chi(v)<-\chi([w,u])$. This implies via (1) and (2) that $[[v,w]],u]\in\mathfrak{H}$ and hence $[v,w]\in\mathfrak{K}+\check{\mathfrak{E}}_0$ by (5.12).

(4) We shall prove (4) by induction on $\chi(w)$. Let $\chi(w)=2$ and $\chi(v)=-1$. We have $\chi([w,u])=0$, $\chi([v,u])=-3<-\chi(w)$. This implies by (1) and (2) that $[[v,w],u]\in\mathfrak{H}$ and hence $[v,w]\in\mathfrak{W}$ by (5.12), since $\chi([v,w])>0$. Now assume that (4) holds for all w with $\chi(w)\in\{1,2,\dots,n-1\}$, $n>2$. Thus

$$(5.13)\qquad [v,w]\in\mathfrak{K}+\check{\mathfrak{E}}_0\quad\text{for all}\quad v\in\mathfrak{H},\ \chi(w)\in\{1,2,\dots,n-1\},\ n>2.$$

Now let $\chi(w)=n$ and $-n<\chi(v)\leq -1$. To show that $[v,w]\in\mathfrak{W}$ we use induction on $\chi(v)$. Indeed, let $\chi(v)=-n+1$. We have $\chi([v,u])=-n-1<-\chi(w)$. Therefore $[[v,u],w]\in\mathfrak{H}$ by (2). Also $\chi([w,u])=n-2\in\{1,2,\dots,n-1\}$ and hence $[v,[w,u]]\in\mathfrak{H}$ by (5.13). This implies that $[[v,w],u]\in\mathfrak{H}$ and hence $[v,w]\in\mathfrak{W}$ by (5.12). Now assume that

$$(5.14)\qquad [v,w]\in\mathfrak{K}+\check{\mathfrak{E}}_0\quad\text{for all } v \text{ with}\quad -n\leq\chi(v)\leq -n+k$$

where $1\leq k<n-1$. Let $\chi(v)=-n+k+1$. We have $-n\leq\chi([v,u])=-n+k-1<-n+k$. Therefore $[[v,u],w]\in\mathfrak{K}+\check{\mathfrak{E}}_0$ by (5.14). Also $[v,[w,u]]\in\mathfrak{K}+\check{\mathfrak{E}}_0$ by (5.13). Therefore $[v,w]\in\mathfrak{W}$ by (5.12), since $\chi([v,w])>0$. This completes the proof of (4) and of the proposition. □

Corollary 5.1. *The space $\mathfrak{W}$ is a Lie subalgebra of $\mathfrak{G}$, normalized by $\mathfrak{Z}_0$.*

Proof. Let $w\in\mathfrak{W}\cap\mathfrak{E}_p$, $v\in\mathfrak{W}\cap\mathfrak{E}_q$, $p,q>0$. *For* $k,l\geq 0$ *write*

$$z_{k,l}=\left[\mathrm{ad}_u^k(w),\mathrm{ad}_u^l(v)\right].$$

We have

$$\mathrm{ad}_u^{p+q}([w,v])=z=\sum_{k,l:\,k+l=p+q}a_{k,l}z_{k,l}$$

for some $a_{k,l}\in R$. If both $\chi(\mathrm{ad}_u^k(w))=p-2k\leqslant 0$ and $\chi(\mathrm{ad}_u^l(v))=q-2l\leqslant 0$ then $z_{k,l}\in\mathfrak{L}(\mathbf{\Lambda})$ by the definition of $\mathfrak{W}$. Now let $p-2k>0$. Then $q-2l=-(p+q)-(p-2k)<-(p-2k)$ and therefore $z_{k,l}\in\mathfrak{H}\subset\mathfrak{L}(\mathbf{\Lambda})$ by Proposition 5.3. Thus $z_{k,l}\in\mathfrak{L}(\mathbf{\Lambda})$ for all $k,l\colon k+l=p+q$. This implies that $z\in\mathfrak{L}(\mathbf{\Lambda})$ and therefore $[w,v]\in\mathfrak{W}$ by Proposition 5.1. This proves that $\mathfrak{W}$ is a subalgebra of $\mathfrak{G}$. Now let $i\in\mathfrak{Z}_0$ and $\alpha_{k,l}=[\mathrm{ad}_u^k(w),\mathrm{ad}_u^l(i)]$, $k,l\geqslant 0$, $k+l=p$. If $k=p, l=0$ then $\mathrm{ad}_u^p(w)\in\mathfrak{H}$ and $\alpha_{k,l}\in\mathfrak{H}$, since $i\in\mathfrak{Z}_0\subset\mathfrak{J}_0(\mathfrak{H})$. If $-p<p-2k\leqslant 0$ then $l>0$ and $\mathrm{ad}_u^k(w),\mathrm{ad}_u^l(i)\in\mathfrak{L}(\mathbf{\Lambda})$ and hence $\alpha_{k,l}\in\mathfrak{L}(\mathbf{\Lambda})$. If $p-2k>0$ then $-2l=-p-(p-2k)<-(p-2k)$ and hence $\alpha_{k,l}\in\mathfrak{L}(\mathbf{\Lambda})$ by Proposition 5.3. This proves that $\mathrm{ad}_u^p([w,i])\in\mathfrak{L}(\mathbf{\Lambda})$ and hence $[w,i]\in\mathfrak{W}$ by Proposition 5.1. This completes the proof. □

Now we shall prove Theorem 5.1. First let us note that if $\delta>0$ is sufficiently small then the map $(v,h)\to\exp v\exp h$ from $\mathfrak{U}_\delta(\mathfrak{W}+\mathfrak{Y}+\mathfrak{F}+\mathfrak{Z})\times\mathfrak{U}_\delta(\mathfrak{H})$ onto a neighborhood of $\mathbf{e}$ is a diffeomorphism. If $\mathbf{y}=\exp v\exp h$ we write $v=v(\mathbf{y}), h=h(\mathbf{y})$. Also $v(\mathbf{y})=w(\mathbf{y})+c(\mathbf{y})+f(\mathbf{y})+z(\mathbf{y})$ where $w(\mathbf{y})\in\mathfrak{W}$, $c(\mathbf{y})\in\mathfrak{Y}$, $f(\mathbf{y})\in\mathfrak{F}$, $z(\mathbf{y})\in\mathfrak{Z}$. Let $c_\lambda(\mathbf{y})$ $(w_\lambda(\mathbf{y}))$ denote the projection of $c(\mathbf{y})$ $(w(\mathbf{y}))$ onto $\mathfrak{Y}\cap\mathfrak{E}_\lambda$ $(\mathfrak{W}\cap\mathfrak{E}_\lambda)$, $\lambda=1,\dots,m$. We will need the following quantities

$$\begin{aligned}d&=\max\{\|[v,w]\|\colon v,w\in\mathfrak{G},\ \|v\|=\|w\|=1\}\\ \eta&=\min\{\|[v,u]\|\colon v\in\mathfrak{E}_\lambda,\ \lambda>0,\ \|v\|=1\}>0.\end{aligned}\tag{5.15}$$

Recall that $[v,u]\neq 0$ for all $0\neq v\in\mathfrak{E}_\lambda$ with $\lambda>0$ by Theorem 1.1.

Proof of Theorem 5.1. Let $0<\delta<\Delta$ be given and let $0<\delta_0<\delta$ be so small that

$$10m\delta_0 d^m\leqslant 0.1\eta.$$

Let $X(\delta_0)\subset X$, $\mu(X(\delta_0))>0$ be as in Lemma 5.1 and let

$$\begin{aligned}Y&=\{(x,x\mathbf{y})\colon x\in X(\delta_0),\ \mathbf{y}\in\mathbf{B}(\delta_0)\}\\ \mathbf{B}(\delta_0)&=\exp\mathfrak{U}_{\delta_0}(\mathfrak{W}+\mathfrak{Y}+\mathfrak{Z}_0)\exp\mathfrak{U}_{\delta_0}(\mathfrak{H}).\end{aligned}\tag{5.16}$$

For $\lambda=1,\dots,m$ and $\alpha>0$ define

$$Y(\alpha,\lambda)=\{(x,x\mathbf{y})\in Y\colon\|c_\lambda(\mathbf{y})\|\geqslant\alpha\}.$$

It is enough to prove that

$$\bar{\mu}(Y(\alpha,\lambda))=0$$

for all $\alpha>0$ and $\lambda=1,\dots,m$, where $\bar\mu=\mu\times\mu$. First let us prove that

$$\bar\mu(Y(\alpha,1))=0$$

for all $\alpha>0$. Indeed, suppose on the contrary that $\bar\mu(Y(\alpha,1))>0$ for some $\alpha>0$. Since the action of $\mathbf{u}_t\times\mathbf{u}_t$ on $X\times X$ preserves $\bar\mu$, there is an arbitrary small $0<t<\alpha/2$ and $(x,x\mathbf{y})\in Y(\alpha,1)$ such that $(x\mathbf{u}_t, x\mathbf{y}\mathbf{u}_t)\in Y$. We have

$$\mathbf{u}_{-t}\mathbf{y}\mathbf{u}_t=\exp(\mathrm{Ad}_{u_t}(v(\mathbf{y})))\exp(\mathbf{u}_{-t}h(\mathbf{y})\,\mathbf{u}_t)=\exp(v(\mathbf{y},t))\cdot\mathbf{h}(\mathbf{y},t) \tag{5.17}$$

for some $v(\mathbf{y},t)\in\mathfrak{U}_{\delta_0}(\mathfrak{W}+\mathfrak{Y}+\mathfrak{Z}_0)$, $\mathbf{h}(y,t)\in\mathbf{U}_{\delta_0}(\mathbf{H})$, since $\mathbf{u}_{-t}\mathbf{y}\mathbf{u}_t\in\mathbf{B}(\delta_0)$, if t is sufficiently small. In particular,

$$f(v(\mathbf{y},t))=0=z^-(v(\mathbf{y},t)). \tag{5.18}$$

We shall get a contradiction to (5.18). We have

$$v(\mathbf{y})=w(\mathbf{y})+c(\mathbf{y})+z_0(\mathbf{y})$$

$$\mathrm{Ad}_{\mathbf{u}_t}(v(\mathbf{y}))=v(\mathbf{y})+f_{-1}(t)+O(t)+\hat h(t)=\hat v(\mathbf{y},t)+\hat h(\mathbf{y},t)$$

where

$$O(t)=O(t,\mathbf{y})\in\sum_{\lambda\neq-1}\mathfrak{E}_\lambda,\quad \|O(t)\|\leqslant 2\delta_0 mdt\leqslant 0.1\eta t$$

$$\hat h(t)=\hat h(\mathbf{y},t)=\sum_{\lambda<0}\hat h_\lambda(t),\quad \hat h_\lambda(t)\in\mathfrak{H},\quad \|\hat h_\lambda(t)\|\leqslant 2\delta_0 dt^{\nu(-\lambda)}\leqslant 0.1\eta t \tag{5.19}$$

and $f_{-1}(t)=f_{-1}(t,\mathbf{y})$ denotes the projection of $\mathrm{Ad}_{\mathbf{u}_t}(c(\mathbf{y}))$ onto $\mathfrak{E}_{-1}$. We have

$$f_{-1}(t)=t[c_1(\mathbf{y}),u]+O_1(t)\in\mathfrak{F}+\mathfrak{Z}^-\quad\text{(see the definition of }\mathfrak{F}\text{ and }\mathfrak{Z}^-)$$

$$\|O_1(t)\|\leqslant 2\delta_0 dt^2\leqslant 0{,}1\eta\alpha t \tag{5.20}$$

$$\|f_{-1}(t)\|\geqslant\|c_1(\mathbf{y})\|\eta t-0.1\eta\alpha t\geqslant 0.9\|c_1(\mathbf{y})\|\eta t$$

since $\|c_1(\mathbf{y})\|\geqslant\alpha$. Note that for $\hat h_{-1}(t)$ and $\hat h_{-2}(t)$ in (5.19) we have

$$\hat h_{-1}(t)=t[w_1(\mathbf{y}),u]+O(t^2),\quad \hat h_{-2}(t)=t[z_0(\mathbf{y}),u]+O(t^2) \tag{5.21}$$

where $\|O(t^2)\|\leqslant 0.1\eta t^2\leqslant 0.1\eta\alpha t$. Here $[z_0(\mathbf{y}),u]\in\mathfrak{H}$, since $z_0(\mathbf{y})\in\mathfrak{Z}_0\subset\mathfrak{I}_0(\mathfrak{H})$.

Now we shall use Proposition 1.5 to represent $\mathrm{Ad}_{\mathbf{u}_t}(\hat v(\mathbf{y},t)+\hat h(\mathbf{y},t))$ as the product

$\exp(v(\mathbf{y},t))\tilde{\mathbf{h}}(\mathbf{y},t)$ for some $v(\mathbf{y},t)\in\mathfrak{W}+\mathfrak{Y}+\mathfrak{F}+\mathfrak{Z}=\mathfrak{H}^{\perp}$ and some $\tilde{\mathbf{h}}(\mathbf{y},t)\in\mathbf{H}$. Here $\hat{v}(\mathbf{y},t)\in\mathfrak{H}^{\perp}$. Let $\mathbf{h}=\exp h$ with $h=\Sigma_{\lambda<0}h_\lambda$ being as in (5.19). In view of Proposition 1.5 we have to look at $\mathrm{Ad}_{\mathbf{h}}(\hat{v}(\mathbf{y},t))$. Let p_{-1} denote the projection of $\mathrm{Ad}_{\mathbf{h}}(\hat{v}(\mathbf{y},t))$ onto $\mathfrak{E}_{-1}$. We have

$$p_{-1}=f_{-1}(t)+[c_1(\mathbf{y}),h_{-2}]+[z_0(\mathbf{y}),h_{-1}]+[w_1(\mathbf{y}),h_{-2}]+O(t^2)$$
$$=f_{-1}(t)+[c_1(\mathbf{y}),h_{-2}]+\tilde{h}+O(t^2)$$

where $\tilde{h}\in\mathfrak{H}$ by Proposition 5.3 and

$$\|[c_1(\mathbf{y}),h_{-2}]\|\leqslant 0.1\eta t\|c_1(\mathbf{y})\|\leqslant 0.2\|f_{-1}(t)\|$$

$$\|O(t^2)\|\leqslant 0.1\eta t^2\leqslant 0.1\eta\alpha t\leqslant 0.1\|c_1(\mathbf{y})\|\eta t\leqslant 0.2\|f_{-1}(t)\|$$

by (5.20). This implies that for each $\mathbf{h}\in\mathbf{H}$ satisfying (5.19) the projection of $\mathrm{Ad}_{\mathbf{h}}(\hat{v}(\mathbf{y},t))$ onto $(\mathfrak{D}+\mathfrak{Z}^-)\cap\mathfrak{E}_{-1}$ is $f_{-1}(t)+f_{-1}(t,\mathbf{h})$, where $\|f_{-1}(t,\mathbf{h})\|\leqslant 0.4\|f_{-1}(t)\|$. This implies via (1.14) and Proposition 1.5 that if δ_0 is sufficiently small then

$$\max\{f(v(\mathbf{y},t)),z^-(v(\mathbf{y},t))\}>0$$

contradicting (5.18). This proves that $\bar{\mu}(Y(\alpha,1))=0$ for all $\alpha>0$.

Now let us show that $\bar{\mu}(Y(\alpha,2))=0$. We can assume without loss of generality that $c_1(\mathbf{y})=0$ for all $(x,x\mathbf{y})\in Y$. Suppose that $\bar{\mu}(Y(\alpha,2))>0$ for some $\alpha>0$. Arguing as above we get $0<t<\alpha/2$ and $(x,x\mathbf{y})\in Y(\alpha,2)$ such that $(x\mathbf{u}_t,x\mathbf{y}\mathbf{u}_t)\in Y$. Thus

$$\mathbf{u}_{-t}\mathbf{y}\mathbf{u}_t=\exp(v(\mathbf{y},t))\cdot\mathbf{h}(\mathbf{y},t)$$

where $v(\mathbf{y},t)$, $\mathbf{h}(\mathbf{y},t)$ are as in (5.17) if t is sufficiently small. In particular,

$$f(v(\mathbf{y},t))=0. \tag{5.22}$$

We have

$$\mathrm{Ad}_{\mathbf{u}_t}(v(\mathbf{y}))=w(\mathbf{y})+c(\mathbf{y})+e_0(t)+O(t)+\hat{h}(t)=\hat{v}(\mathbf{y},t)+\hat{h}(\mathbf{y},t)$$

where $O(t)\in\Sigma_{\lambda\neq 0}\mathfrak{E}_\lambda$, $\hat{h}(t)$ satisfy (5.19), (5.21) and $e_0(t)$ denotes the projection of $\mathrm{Ad}_{\mathbf{u}_t}(v(\mathbf{y}))$ onto $\mathfrak{E}_0$. We have

$$e_0(t)=t[c_2(\mathbf{y}),u]+(z_0(\mathbf{y})+t[w_2(\mathbf{y}),u])+O_1(t)=f_0(t)+k_0(t)$$

$$\|O_1(t)\|\leqslant 0.1\eta\alpha t,\quad k_0(t)\in\mathfrak{I}(\mathfrak{H}),\quad f_0(t)\in\mathfrak{D}\quad\text{and} \tag{5.23}$$

$$\|f_0(t)\|\geqslant\|c_2(\mathbf{y})\|\eta t-0.1\eta\alpha t\geqslant 0.9\|c_2(\mathbf{y})\|\eta t.$$

Now let $\mathbf{h}\in\mathbf{H}$, $\mathbf{h}=\exp h$ with h satisfying (5.19) and let p_0 be the projection of $\mathrm{Ad}_{\mathbf{h}}(\hat{v}(\mathbf{y},t))$ onto $\mathfrak{E}_0$. We have

$$(5.24)\qquad \begin{aligned} p_0 &= f_0(t)+k_0(t)+[c_2(\mathbf{y}),h_{-2}]+[w_2(\mathbf{y}),h_{-2}]+[w_1(\mathbf{y}),h_{-1}]+O(t^2)\\ &= f_0(t)+[c_2(\mathbf{y}),h_{-2}]+k(t)+O(t^2)\end{aligned}$$

where as above

$$(5.25)\qquad \|O(t^2)\|\leqslant 0.1\eta\alpha t,\quad \|[c_2(\mathbf{y}),h_{-2}]\|\leqslant 0.1\eta t\|c_2(\mathbf{y})\|\leqslant 0.2\|f_0(t)\|$$

and $k(t)\in\mathfrak{Z}+\mathfrak{Z}(\mathfrak{H})$ by Proposition 5.3. Note that we have used the fact that $c_1(\mathbf{y})=0$ in (5.24). Expressions (5.23) and (5.25) imply via Proposition 1.5 that $f(v(\mathbf{y},t))\neq 0$ contradicting (5.22). This proves that $\bar{\mu}(Y(\alpha,2))=0$ for all $\alpha>0$.

To prove that $\bar{\mu}(Y,\alpha,\lambda))=0$ for all $\lambda=1,\ldots,m$ we use induction on λ. Suppose that $\mu(Y(\alpha,\lambda))=0$ for all $\alpha>0$, $\lambda=1,\ldots,n$, $2\leqslant n<m$. We can assume that

$$(5.26)\qquad c_i(\mathbf{y})=0$$

for all $i=1,\ldots,n$ and all $(x,x\mathbf{y})\in Y$. Suppose on the contrary that $\bar{\mu}(Y(\alpha,n+1))>0$ for some $\alpha>0$. As above let $0<t<\alpha/2$ and $(x,x\mathbf{y})\in Y(\alpha,n+1)$ be such that $(x\mathbf{u}_t,x\mathbf{y}\mathbf{u}_t)\in Y$. We have for $v(\mathbf{y},t)$ as in (5.17)

$$c_{n-1}(v(\mathbf{y},t))=0$$

by (5.26), since $n-1\geqslant 1$. We have

$$\mathrm{Ad}_{\mathbf{u}_t}(v(\mathbf{y}))=v(\mathbf{y})+e_{n-1}(t)+O(t)+\hat{h}(t)=\hat{v}(\mathbf{y},t)+\hat{h}(\mathbf{y},t)$$

where $O(t)\in\Sigma_{\lambda\neq n-1}\mathfrak{E}_\lambda$, $\hat{h}(t)$ satisfy (5.19), (5.21) and e_{n-1} denotes the projection of $\mathrm{Ad}_{\mathbf{u}_t}(v(\mathbf{y}))-w_{n-1}(\mathbf{y})$ onto $\mathfrak{E}_{n-1}$. We have

$$e_{n-1}(t)=t[c_{n+1}(\mathbf{y}),u]+t[w_{n+1}(\mathbf{y}),u]+O(t^2)=c_{n-1}(t)+\hat{w}_{n-1}(t)$$

where as above

$$\|O(t^2)\|\leqslant 0.1\eta\alpha t,\quad \hat{w}_{n-1}(t)\in\mathfrak{W}\cap\mathfrak{E}_{n-1},\quad c_{n-1}(t)\in\mathfrak{Y}\cap\mathfrak{E}_{n-1}$$

$$\|c_{n-1}(t)\|\geqslant\|c_{n+1}(\mathbf{y})\|\eta t-0.1\eta\alpha t\geqslant 0.9\|c_{n+1}(\mathbf{y})\|\eta t.$$

Now let $\mathbf{h}\in\mathbf{H}$ be as in (5.19) and let p_{n-1} denote the projection of $\mathrm{Ad}_{\mathbf{h}}(\hat{v}(\mathbf{y},t))$ onto $\mathfrak{E}_{n-1}$.

We have

$$p_{n-1} = w_{n-1}(\mathbf{y}) + c_{n-1}(t) + \hat{w}_{n-1}(t) + [c_{n+1}(\mathbf{y}), h_{-2}] + [w_{n+1}(\mathbf{y}), h_{-2}] + [w_n(\mathbf{y}), h_{-1}] + O(t^2) = c_{n-1}(t) + [c_{n+1}(\mathbf{y}), h_{-2}] + \tilde{w} + O(t^2)$$

where $\|O(t^2)\| \leq 0.1\eta\alpha t$ and

$$\|[c_{n+1}(\mathbf{y}), h_{-2}]\| \leq 0.1\eta \|c_{n+1}(\mathbf{y})\| t \leq 0.2\|c_{n-1}(t)\|$$

and $\tilde{w} \in \mathfrak{W}$ by Proposition 5.3, since $n \geq 2$. Thus for each $\mathbf{h} \in \mathbf{H}$, satisfying (5.19) the projection of $\mathrm{Ad}_{\mathbf{h}}(\hat{v}(\mathbf{y}, t))$ onto $\mathfrak{Y} \cap \mathfrak{E}_{n-1}$ is $c_{n-1}(t) + c_{n-1}(t, \mathbf{h})$, where $\|c_{n-1}(t, \mathbf{h})\| \leq 0.2\|c_{n-1}(t)\|$. This implies via (1.13) and (1.14) that

$$c_{n-1}(v(\mathbf{y}, t)) \neq 0$$

contradicting (5.26). This completes the proof of the theorem. □

Appendix 5.1

Here we shall prove Lemma 5.2. The proof is somewhat similar to that of Lemma 4.1.

For $v \in \mathfrak{G}$ let $\xi(v) = \chi(v)(1-\alpha) + \alpha$, where $\alpha = 1/4(m+1)^2 = \beta/2(m+1)$ and $\beta = 1/2(m+1)$ is as in Lemma 5.1. Note that $\xi(v) = 1$ if $\chi(v) = 1$ and $\max\{\nu(v), \chi(v) - \beta\} < \xi(v)$ if $1 < \chi(v) \leq m$. Write

$$\xi(W) = \sum_{i=1}^{p} \xi(w_i).$$

We set for technical simplicity $b = 2^{4(m+1)^2}$. With this choice of b the number $b^{\xi(v)}$ is an integer for all $v \in \mathfrak{G}$. Now let $\mathfrak{W}^{\xi}(t)$, $\mathfrak{V}^{\xi}(t)$ and $\mathfrak{S}^{\xi}(t)$ be as in (4.4), where θ is chosen so small that for $\hat{\theta}$ defined in (4.7) we have $\hat{\theta} < 1$. Recall that $\Psi = \varnothing$. It follows from (4.5), (4.9) and (1.14) that if $z, v \in 4\mathfrak{S}^{\xi}(t)$, $t \geq 1$ then

$$\exp(z+v) = \exp z \exp(v+\hat{v}) \exp i \tag{5.27}$$

for some $\hat{v} \in t^{-\alpha}\hat{\theta}\mathfrak{S}^{\xi}(bt) \subset t^{-\alpha}\mathfrak{S}^{\xi}(bt)$ and some $i \in \mathfrak{J}_0(\mathfrak{H})$, $\|i\| \leq t^{-2\beta}$.

Now let $0 < \gamma < \gamma(\varepsilon)$ and $\tau \geq \tau(\gamma)$ be given. Let $x \in A$ and $B = x\mathbf{B}(\mathfrak{S}^{\xi}(\tau), 0.1\gamma)$, where $\mathbf{B}(\mathfrak{U}, \gamma)$, $\mathfrak{U} \subset \mathfrak{W} + \mathfrak{V} + \mathfrak{J}$ is as in (4.10). Define

$$\mathfrak{Q} = \mathfrak{Q}(B) = \{v \in \mathfrak{S}^{\xi}(\tau) : x \exp v \exp i \in A \text{ for some } i \in \mathfrak{J}_0(\mathfrak{H}),\ \|i\| \leq 0.1\gamma\}. \tag{5.28}$$

Write $t_n=\tau b^n$, $n=1,2,\dots$, $\gamma_{-1}=\gamma_0=0$, $\gamma_n=\Sigma_{k=0}^{n-1} t_k^{-\beta}$. We have $\gamma_n<0.1\gamma$ for all n.

LEMMA 5.3. *For every $n\in\mathbf{Z}^+$ there are $\mathbf{y}_1(n),\dots,\mathbf{y}_{s(n)}(n)\in\mathbf{B}(\mathfrak{Q},\gamma_{n-1})$ such that*

$$\exp\mathfrak{Q}\subset\bigcup_{i=1}^{s(n)}\mathbf{y}_i(n)\,\mathbf{B}(4\mathfrak{S}^\xi(t_n),\gamma_n)$$

where $s(n)\leq b^{n\xi(W)}$.

Proof. We have $\pi_n=\Pi_{k=0}^{n-1}(1+t_k^{-\alpha})\leq 2$ for all n by our choice of τ and b. Also $\mathfrak{S}^\xi(z,t)=\mathfrak{S}^\xi(t)-z\subset 2\mathfrak{S}^\xi(t)$ for all $z\in\mathfrak{S}^\xi(t)$. This shows that it is enough to prove that for every $n\in\mathbf{Z}^+$ there are $\mathbf{y}_1(n),\dots,\mathbf{y}_{s(n)}(n)\in\mathbf{B}(\mathfrak{Q},\gamma_{n-1})$ and $z_1(n),\dots,z_{s(n)}(n)\in\mathfrak{W}_n^\xi(t_n)$ such that

$$\exp\mathfrak{Q}\subset\bigcup_{i=1}^{s(n)}\mathbf{y}_i(n)\,\mathbf{B}(\mathfrak{S}_n^\xi(z_i(n),t_n),\gamma_n) \tag{5.29}$$

where $s(n)\leq b^{n\xi(W)}$ and $\mathfrak{W}_n^\xi(t_n)=\pi_n\mathfrak{W}^\xi(t_n)$, $\mathfrak{S}_n^\xi(t_n)=\pi_n\mathfrak{S}^\xi(t_n)$. We shall prove this by induction on n. For $n=0$ set $s(0)=1,\mathbf{y}_1(0)=\mathbf{e},z_1(0)=0$. Assume that (5.29) holds for n. In order to prove it for $n+1$ it is enough to show that if $\mathbf{y}\in\mathbf{B}(\mathfrak{Q},\gamma_{n-1})$, $z\in\mathfrak{W}_n^\xi(t_n)$ and

$$\mathfrak{Q}_n=\{v\in\mathfrak{S}_n^\xi(z,t_n):\mathbf{y}\exp v\exp i\in\exp\mathfrak{Q}\ \text{ for some } i\in\mathfrak{I}_0(\mathfrak{H}),\ \|i\|\leq\gamma_n\}$$

then there are s: $1\leq s\leq b^{\xi(W)}=b(\xi,W)$, $\mathbf{q}_1,\dots,\mathbf{q}_s\in\exp\mathfrak{Q}_n$ and $z_1,\dots,z_n\in\mathfrak{W}_{n+1}^\xi(t_{n+1})$ such that

$$\exp\mathfrak{Q}_n\subset\bigcup_{i=1}^{s}\mathbf{q}_i\,\mathbf{B}(\mathfrak{S}_{n+1}^\xi(z_i,t_{n+1}),t_n^{-\beta}). \tag{5.30}$$

We have

$$\mathfrak{S}_n^\xi(z,t_n)=\mathfrak{W}_n^\xi(z,t_n)+\mathfrak{V}_n^\xi(t_n).$$

Let $u_1,\dots,u_{b(\xi,W)}\in\mathfrak{W}_n^\xi(z,t_n)$ be such that

$$\mathfrak{W}_n^\xi(z,t_n)=\bigcup_{i=1}^{b(\xi,W)}(u_i+\mathfrak{W}_n^\xi(bt_n)).$$

We have

$$\mathfrak{S}_n^\xi(z,t_n)=\bigcup_{i=1}^{b(\xi,W)}(u_i+\mathfrak{W}_n^\xi(bt_n)+\mathfrak{V}_n^\xi(t_n))=\bigcup_{i=1}^{b(\xi,W)}(u_i+\mathfrak{T}_n(t_n)).$$

Let $z_i' \in \mathfrak{T}_n(t_n)$ be such that $u_i + z_i' = q_i \in \mathfrak{Q}_n$ and set $\mathbf{q}_i = \exp q_i$. We have

$$q_i + \mathfrak{T}_n(z_i', t_n) = u_i + z_i' + \mathfrak{T}_n(z_i', t_n) = u_i + \mathfrak{T}_n(t_n).$$

Also if $v \in \mathfrak{T}_n(z_i', t_n)$ then

$$v \in 4\mathfrak{W}^\xi(bt_n) + 4\mathfrak{V}^\xi(t_n) \subset 4\mathfrak{S}^\xi(t_n).$$

It follows then from (5.27) that

$$\exp(q_i + v) = \mathbf{q}_i \exp \bar{v} \exp \bar{i}$$

where $\bar{v} \in (1 + t_n^{-\alpha}) \mathfrak{T}_n(t_n) - z_i'$, $\bar{i} \in \mathfrak{J}_0(\mathfrak{H})$, $\|\bar{i}\| \leq t_n^{-\beta}$. This implies that

$$\exp(u_i + \mathfrak{T}(t_n)) \subset \mathbf{q}_i \mathbf{B}((1 + t_n^{-\alpha}) \mathfrak{T}_n(t_n) - z_i', t_n^{-\beta}).$$

Now suppose that

$$\exp(q_i + v) \in \exp \mathfrak{Q}_n.$$

It follows then from Lemma 3.1 and the definition of $\mathfrak{Q}$ that

$$\bar{v} \in \mathfrak{W}_{n+1}^\xi(z_i, t_{n+1}) + \mathfrak{V}_{n+1}^\xi(t_{n+1}) = \mathfrak{S}_{n+1}^\xi(z_i, t_{n+1})$$

for some $z_i \in \mathfrak{W}_{n+1}^\xi(t_{n+1})$, since $\xi(v) \geq \nu(v)$ if $\chi(v) \geq 1$. This proves (5.30) and completes the proof of the lemma. □

For $1 \leq k \leq m$ define

$$\mathfrak{S}_k^\xi(t) = \{v \in \mathfrak{S}^\xi(t) \colon |\alpha_i(v)| \leq a(\theta)\, t^{-\chi(w_i)}/\bar{a},\ |c_p(v)| \leq \mathbf{C}\theta t^{-\chi(c_p)} \text{ for all } i, p \text{ with } \chi(w_i), \chi(c_p) \leq k\}$$

$$\mathfrak{M}_k^\xi(t) = \{v \in \mathfrak{S}_k^\xi(t) \colon |c_p(v)| \leq \mathbf{C}\theta t^{-\chi(c_p)},\ p = 1, \ldots, r\}.$$

We have $\mathfrak{S}^\xi(t) = \mathfrak{S}_1^\xi(t)$ since $\xi(v) = 1$ if $\chi(v) = 1$. Also

$$\mathfrak{S}^\chi(t) = \mathfrak{S}_m^\xi(t) = \mathfrak{M}_m^\xi(t). \tag{5.31}$$

Now let $w \in d\mathfrak{S}_k^\xi(t)$, $v \in d\mathfrak{S}_{k+1}^\xi(t)$ for some $1 \leq d \leq 2^{m+2}$. Arguing as in (5.27) we get

$$\exp(w + v) = \exp w \exp \bar{v} \exp i$$

for some $\bar{v} \in 2d\mathfrak{S}_{k+1}^\xi(t)$, $i \in \mathfrak{J}_0(\mathfrak{H})$, $\|i\| \leq t^{-2\beta}$, since $\xi(v) > \chi(v) - \beta$ and $\hat{\theta} < 1$ in (4.7).

It follows from Lemma 5.3 and Lemma 3.1 that

$$\exp \mathfrak{Q} \subset \bigcup_{i=1}^{s(n)} \mathbf{y}_i(n)\, \mathbf{B}(4\mathfrak{M}_1^{\xi}(t_n), \gamma_n)$$

for all $n \in \mathbf{Z}^+$. Now let $n \in \mathbf{Z}^+$ be fixed and let $\mathbf{y}=\mathbf{y}_i(n)$ for some $i \in \{1, \ldots, s(n)\}$. Let

$$\mathfrak{Q}_n = \{v \in 4\mathfrak{M}_1^{\xi}(t_n) : \mathbf{y} \exp v \exp i \in \exp \mathfrak{Q} \text{ for some } i \in \mathfrak{I}_0(\mathfrak{H}),\ \|i\| \leqslant \gamma_n\}.$$

For $\mathbf{q} \in \mathbf{B}(\mathfrak{Q}_n, 0.01\gamma)$ and $1 \leqslant k \leqslant m$ let

$$\mathfrak{Q}_k(\mathbf{q}) = \{v \in d_k\, \mathfrak{M}_k^{\xi}(t_n) : \mathbf{q} \exp v \exp i \in \exp \mathfrak{Q}_n \text{ for some } i \in \mathfrak{I}_0(\mathfrak{H}),\ \|i\| \leqslant 0.01\gamma\}$$

where $d_k=2^{k+1}$. The proof of the following lemma is identical with that of Lemma 4.3 in Appendix 4.1.

LEMMA 5.4. *There are* $\mathbf{q}_1, \ldots, \mathbf{q}_{\tau(k+1)} \in \exp \mathfrak{Q}_k(\mathbf{q})$ *such that*

$$\exp \mathfrak{Q}_k(\mathbf{q}) \subset \bigcup_{i=1}^{\tau(k+1)} \mathbf{q}_i\, \mathbf{B}(d_{k+1}\, \mathfrak{M}_{k+1}^{\xi}(t_n), t_n^{-\beta})$$

where $\tau(k+1) \leqslant (t_n+1)^{\chi(\zeta_{k+1})-\xi(\zeta_{k+1})}$ *and* $\zeta_k = \{w \in W : \chi(w) = k\}$.

Now let $n(\gamma, \tau) > 1$ be so large that $t_n^{-\beta} m \leqslant 0.01\gamma$ for all $n \geqslant n(\gamma, \tau)$.

COROLLARY 5.2. *Let* $n \geqslant n(\gamma, \tau)$. *Then there are* $\mathbf{b}_1(n), \ldots, \mathbf{b}_{\delta(n)}(n) \in \mathbf{B}(\mathfrak{Q}_n, mt_n^{-\beta})$ *such that*

$$\exp \mathfrak{Q}_n \subset \bigcup_{i=1}^{\delta(n)} \mathbf{b}_i(n)\, \mathbf{B}(2^{m+2}\mathfrak{S}^{\chi}(t_n), mt_n^{-\beta}) \tag{5.32}$$

and $\delta(n) \leqslant (t_n+1)^{\chi(W)-\xi(W)}$.

Proof. For $k=1$ let $\mathbf{q}=\mathbf{e} \in \mathbf{B}(\mathfrak{Q}_n, 0)$. We have $\mathfrak{Q}_1(\mathbf{q}) = \mathfrak{Q}_n$, $d_1=4$. Applying Lemma 5.4 we get $\mathbf{q}_1, \ldots, \mathbf{q}_{\tau(2)} \in \exp \mathfrak{Q}_1(\mathbf{q})$ with

$$\exp \mathfrak{Q}_n \subset \bigcup_{i=1}^{\tau(2)} \mathbf{q}_i\, \mathbf{B}(d_2\, \mathfrak{M}_2^{\xi}(t_n), t_n^{-\beta})$$

where $\tau(2) \leqslant (t_n+1)^{\chi(\zeta_2)-\xi(\zeta_2)}$. In view of (5.31) it is clear that we need m successive applications of Lemma 5.4 to get (5.32). □

Proof of Lemma. 5.2. Let $\mathscr{C} = \{x_1\, \mathbf{B}(\mathfrak{S}^{\xi}(\tau), 0.1\gamma), \ldots, x_{L_1}\, \mathbf{B}(\mathfrak{S}^{\xi}(\tau), 0.1\gamma)\}$ be a cover of A by (ξ, τ)-boxes at $x_1, \ldots, x_{L_1} \in A$. Let $n(\gamma, \tau)$ be as above, $n \geqslant n(\gamma, \tau)$ and let

$\mathfrak{Q}^{(i)}=\mathfrak{Q}(x_i\mathbf{B}(\mathfrak{S}^{\xi}(\tau),0.1\gamma))$ be as in (5.28), $i=1,\dots,L_1$. It follows from Lemma 5.3 and Corollary 5.2 that there are $a_1(n),\dots,a_{M(n)}(n)\in\bigcup_{i=1}^{L_1}x_i\mathbf{B}(\mathfrak{Q}^{(i)},0.2\gamma)$ such that

$$A\subset\bigcup_{j=1}^{M(n)}a_j(n)\,\mathbf{B}(2^{m+2}\mathfrak{S}^{\chi}(t_n),0.3\gamma)$$

where $M(n)\leq L_1\,s(n)\,(t_n+1)^{\chi(W)-\xi(W)}\leq Lb^{n\chi(W)}$ for some $L=L(\gamma,\tau)>1$. This completes the proof of the lemma if we set $a=2^{m+2}$. □

6. The support of μ

Let $\mathfrak{Z}_0=\mathfrak{E}_0\cap\mathfrak{I}(\mathfrak{H})$ be as in Theorem 5.1. Define $\mathfrak{Z}(\mathbf{\Lambda})=\mathfrak{Z}_0\cap\mathfrak{L}(\mathbf{\Lambda})$. We have $\mathfrak{Z}(\mathbf{\Lambda})+\mathfrak{H}=\mathfrak{I}_0(\mathfrak{H})\cap\mathfrak{L}(\mathbf{\Lambda})$. Set $\mathfrak{L}=\mathfrak{W}+\mathfrak{Z}(\mathbf{\Lambda})+\mathfrak{H}$. In this section we shall prove the following theorem.

THEOREM 6.1. (1) *The subspace* $\mathfrak{L}$ *is a Lie subalgebra of* $\mathfrak{G}$.

(2) *There exists* $x\in X$ *such that* $\mu(x\mathbf{L})=1$, *where* $\mathbf{L}$ *denotes the connected Lie subgroup of* $\mathbf{G}$ *with the Lie algebra* $\mathfrak{L}$.

It is clear that $\mathfrak{L}(\mathbf{\Lambda})\subset\mathfrak{L}$. We shall show later that $\mathfrak{L}=\mathfrak{L}(\mathbf{\Lambda})$ and $\mathbf{L}=\mathbf{\Lambda}^0$. Let $\mathfrak{J}_0$ be a subspace of $\mathfrak{P}\cap\mathfrak{E}_0$ complementary to $\mathfrak{Z}_0$. It follows from Proposition 5.3 that

$$[w_\lambda,h_{-\lambda}]\in(\mathfrak{K}\cap\mathfrak{E}_0)\cup\check{\mathfrak{E}}_0\subset\mathfrak{Z}_0+\mathfrak{J}_0$$

for all $h_{-\lambda}\in\mathfrak{H}\cap\mathfrak{E}_{-\lambda}$, $\lambda>0$ and all $w\in\mathfrak{W}$, where w_λ denotes the projection of w onto $\mathfrak{E}_\lambda$. Define

$$\bar{\mathfrak{W}}=\{w\in\mathfrak{W}\colon[w_\lambda,h_{-\lambda}]\in\mathfrak{Z}_0\text{ for all }h_{-\lambda}\in\mathfrak{H}\cap\mathfrak{E}_{-\lambda},\ \lambda>0\}.$$

PROPOSITION 6.1. *The subspace* $\bar{\mathfrak{W}}\subset\mathfrak{W}$ *is a Lie subalgebra of* $\mathfrak{W}$, *normalized by* $\mathfrak{Z}_0$.

Proof. Let $w\in\bar{\mathfrak{W}}\cap\mathfrak{E}_p$, $v\in\bar{\mathfrak{W}}\cap\mathfrak{E}_q$, $p,q>0$ and let $h\in\mathfrak{H}\cap\mathfrak{E}_{-(p+q)}$. We have

$$[[w,v],h]=[w,[v,h]]+[w,h],v]. \tag{6.1}$$

Also $[v,h]\in\mathfrak{H}\cap\mathfrak{E}_{-p}$, $[w,h]\in\mathfrak{H}\cap\mathfrak{E}_{-q}$ by Proposition 5.3. This implies that $[[w,v],h]\in\mathfrak{Z}_0$ by (6.1), since $w,v\in\bar{\mathfrak{W}}$, and proves that $\bar{\mathfrak{W}}$ is a subalgebra of $\mathfrak{W}$. Now let $i\in\mathfrak{Z}_0$ and $h\in\mathfrak{H}\cap\mathfrak{E}_{-p}$. We have

$$[[w,i],h]=[w,[i,h]]+[w,h],i]\in\mathfrak{Z}_0$$

where $[i, h] \in \mathfrak{H} \cap \mathfrak{E}_{-p}$, since $i \in \mathfrak{Z}_0 \subset \mathfrak{I}(\mathfrak{H})$ and $[w, h] \in \mathfrak{Z}_0$, since $w \in \bar{\mathfrak{W}}$. This proves that $[w, i] \in \bar{\mathfrak{W}}$ and that $\mathfrak{Z}_0$ normalizes $\bar{\mathfrak{W}}$.

Lemma 6.1. *Given* $0<\delta<\Delta$ *there is* $\bar{X}(\delta) \subset X, \mu(\bar{X}(\delta))>0$ *such that if* $x \in \bar{X}(\delta)$ *then*

$$\mu(x \exp \mathfrak{U}_\delta(\bar{\mathfrak{W}}+\mathfrak{Z}_0) \exp \mathfrak{U}_\delta(\mathfrak{H}))>0.$$

Proof. The proof is similar to that of Theorem 5.1. Recall that the basis H of $\mathfrak{H}$ defined in Section 5 consists of eigenvectors of ad_g. For $h \in \mathfrak{H}_{-\lambda}=\mathfrak{H} \cap \mathfrak{E}_{-\lambda}$, $\lambda>0$ let

$$\mathfrak{W}(h)=\{w \in \mathfrak{W} \cap \mathfrak{E}_\lambda : [w, h] \in \mathfrak{Z}_0\}$$

and let $\mathfrak{W}^\perp(h)$ be a subspace of $\mathfrak{W} \cap \mathfrak{E}_\lambda$ complementary to $\mathfrak{W}(h)$. Define

$$\bar{H}=\{h \in H : \mathfrak{W}^\perp(h) \neq \{0\}\}$$

$$\zeta=\min\{\|j_0([w, h])\| : h \in \bar{H},\ w \in \mathfrak{W}^\perp(h),\ \|w\|=1\}>0$$

where $j_0(v)$ denotes the projection of $v \in \mathfrak{J}_0+\mathfrak{Z}_0$ onto $\mathfrak{J}_0$.

Now let $0<\delta<\Delta$ be given and let $0<\delta_0<\delta$ be so small that

$$2mp\delta_0 d \leq 0.1\zeta$$

where d is as in (5.15) and $p=\operatorname{card} H$. Let $X(\delta_0) \subset X, \mu(X(\delta_0))>0$ be as in Theorem 5.1 and let

$$Y=\{(x, x\mathbf{y}) : x \in X(\delta_0), \mathbf{y} \in \mathbf{B}(\delta_0)\}$$

$$\mathbf{B}(\delta_0)=\exp \mathfrak{U}_{\delta_0}(\mathfrak{W}+\mathfrak{Z}_0) \exp \mathfrak{U}_{\delta_0}(\mathfrak{H}).$$

For $\alpha>0$, $\lambda=1, \ldots, m$ define

$$Y(\alpha, \lambda)=\{(x, x\mathbf{y}) \in Y : \max\{\|w_\perp(h, \mathbf{y})\| : h \in \bar{H} \cap \mathfrak{E}_{-\lambda}\} \geq \alpha\}$$

where $w_\perp(h, \mathbf{y})$ denotes the projection of $w_\lambda(\mathbf{y})$ onto $\mathfrak{W}^\perp(h)$. It suffices to prove that

$$\bar{\mu}(Y(\alpha, \lambda))=0 \tag{6.2}$$

for all $\alpha>0$ and all $\lambda=1, \ldots, m$ where $\bar{\mu}=\mu \times \mu$.

First let us prove (6.2) for $\lambda=1$. Suppose on the contrary that $\bar{\mu}(Y(\alpha, 1))>0$ for some $\alpha>0$. Then there is $(x, x\mathbf{y}) \in Y(\alpha, 1)$ and an arbitrary small $0<t(h)<\alpha/2$, $h \in \bar{H}_{-1}=\bar{H} \cap \mathfrak{E}_{-1}$ such that

(6.3) $$(x\exp(t(h)\,h), xy\exp(t(h)\,h))\in Y$$

since the action of $\exp th\times\exp th$, $h\in H$ on $X\times X$ preserves $\bar{\mu}$. Let $\bar{h}=\bar{h}(\mathbf{y})\in\bar{H}_{-1}$ be such that

(6.4) $$\|w_{\perp}(\bar{h},\mathbf{y})\|=\max\{\|w_{\perp}(h,\mathbf{y})\|: h\in\bar{H}_{-1}\}\geq\alpha.$$

Write

$$t=t(\bar{h}),\quad \mathbf{h}_t=\exp t\bar{h}.$$

We have

$$\mathbf{h}_{-t}\mathbf{y}\mathbf{h}_t=\exp(\mathrm{Ad}_{\mathbf{h}_t}(v(\mathbf{y})))\,(\mathbf{h}_{-t}\mathbf{h}(\mathbf{y})\,\mathbf{h}_t)=\exp(v(\mathbf{y},t))\,\mathbf{h}(\mathbf{y},t)$$

where $v(\mathbf{y},t)\in\mathfrak{U}_{\delta_0}(\mathfrak{W}+\mathfrak{Z}_0)$, $\mathbf{h}(\mathbf{y},t)\in\mathbf{U}_{\delta_0}(\mathbf{H})$ by (6.3), since $\mathbf{h}_{-t}\mathbf{y}\mathbf{h}_t\in\mathbf{B}(\delta_0)$ if t is sufficiently small. In particular,

(6.5) $$j_0(v(\mathbf{y},t))=0.$$

We have

$$\mathrm{Ad}_{\mathbf{h}_t}(v(\mathbf{y}))=w(\mathbf{y})+e_0(t)+O(t)+\hat{h}(t)=\hat{v}(\mathbf{y},t)+\hat{h}(\mathbf{y},t)$$

where $O(t)\in\Sigma_{\lambda\neq 0}\mathfrak{E}_{\lambda}$, $\|O(t)\|\leq 2m\delta_0\,dt\leq 0.1\zeta t$,

(6.6) $$\hat{h}(t)=\hat{h}(\mathbf{y},t)=\sum_{\lambda>0}\hat{h}_{-\lambda}(t),\quad \hat{h}(t)\in\mathfrak{H}\cap\mathfrak{E}_{-\lambda}$$
$$\|\hat{h}_{-\lambda}(t)\|\leq 2\delta_0\,dt^{\lambda}\leq 0.1\zeta t^{\lambda}$$

and $e_0(t)$ denotes the projection of $\mathrm{Ad}_{\mathbf{h}_t}(v(\mathbf{y}))$ onto $\mathfrak{E}_0$. We have

(6.7) $$e_0(t)=z_0(\mathbf{y})+t[w_1(\mathbf{y}),\bar{h}]+O(t^2)=i_0(t)+j_0(t)$$

where $\|O(t^2)\|\leq 2\delta_0\,mdt^2\leq 0.1\zeta\alpha t$, $i_0(t)\in\mathfrak{Z}_0$, $j_0(t)\in\mathfrak{I}_0$ and

(6.8) $$\|j_0(t)\|\geq\|w_{\perp}(\bar{h},\mathbf{y})\|\zeta t-0.1\zeta\alpha t\geq 0.9\zeta t\|w_{\perp}(\bar{h},\mathbf{y})\|$$

by (6.4). Now let $\mathbf{h}=\exp h\in\mathbf{H}$ with h being as in (6.6) and let p_0 denote the projection of $\mathrm{Ad}_{\mathbf{h}}(\hat{v}(\mathbf{y},t))$ onto $\mathfrak{E}_0$. We have

(6.9) $$p_0=i_0(t)+j_0(t)+[w_1(\mathbf{y}),h_{-1}]+O(t^2)$$

where $\|O(t^2)\| \leqslant 0.1\zeta\alpha t$ and

$$\|j_0([w_1(\mathbf{y}), h_{-1}]\| \leqslant 2p\delta_0 dt\|w_\perp(\bar{h}, \mathbf{y})\| \leqslant 0.1\zeta t\|w_\perp(\bar{h}, \mathbf{y})\| \leqslant 0.2\|j_0(t)\|$$

by (6.4) and (6.8). Thus for each $\mathbf{h}$, satisfying (6.6) we have $j_0(\mathrm{Ad}_{\mathbf{h}}(\hat{v}(\mathbf{y}, t))) = j_0(t) + j_0(t, \mathbf{h})$, where $\|j_0(t, \mathbf{h})\| \leqslant 0.2\|j_0(t)\|$. This implies via (1.13) and (1.14) that $j_0(v(\mathbf{y}, t)) \neq 0$ contradicting (6.5). This proves that $\bar{\mu}(Y(\alpha, 1)) = 0$ for all $\alpha > 0$.

To prove that $\bar{\mu}(Y(\alpha, \lambda)) = 0$ for all $\alpha > 0$ and all $\lambda = 1, \ldots, m$ we use induction on λ. Suppose that $\bar{\mu}(Y(\alpha, \lambda)) = 0$ for all $\alpha > 0$ and all $\lambda = 1, \ldots, n$. We can therefore assume that

$$[w_\lambda(\mathbf{y}), h_{-\lambda}] \in \mathfrak{Z}_0 \tag{6.10}$$

for all $(x, x\mathbf{y}) \in Y$, $h_{-\lambda} \in \mathfrak{H} \cap \mathfrak{E}_{-\lambda}$, $\lambda = 1, \ldots, n$. Now suppose that $\bar{\mu}(Y(\alpha, n+1)) > 0$ for some $\alpha > 0$. As above there exists $(x, x\mathbf{y}) \in Y(\alpha, n+1)$ and sufficiently small $0 < t(h) < \alpha/2$, $h \in \bar{H}_{-(n+1)}$ such that

$$(x \exp(t(h)\, h), x\mathbf{y} \exp(t(h)\, h)) \in Y.$$

Let $\bar{h} = \bar{h}(\mathbf{y}) \in \bar{H}_{-(n+1)}$ be such that

$$\|w_\perp(\bar{h}, \mathbf{y})\| = \max\{\|w_\perp(h, \mathbf{y})\| : h \in \bar{H}_{-(n+1)}\} \geqslant \alpha$$

and let $t = t(\bar{h})$, $\mathbf{h}_t = \exp t\bar{h}$. We have $\mathbf{h}_{-t}\mathbf{y}\mathbf{h}_t = \exp v(\mathbf{y}, t) \cdot \mathbf{h}(\mathbf{y}, t)$ where as above

$$j_0(v(\mathbf{y}, t)) = 0. \tag{6.11}$$

We can now repeat the above argument writing

$$e_0(t) = z_0(\mathbf{y}) + t[w_{n+1}(\mathbf{y}), \bar{h}] + O(t^2) = i_0(t) + j_0(t)$$

in (6.7) and

$$p_0 = i_0(t) + j_0(t) + [w_{n+1}(\mathbf{y}), h_{-n+1}] + \bar{i}_0(t) + O(t^2)$$

in (6.9), where $O(t^2)$ is as in (6.9),

$$\bar{i}_0(t) = \sum_{\lambda=1}^{n} [w_\lambda(\mathbf{y}), h_{-\lambda}] \in \mathfrak{Z}_0$$

by (6.10) and

$$\|j_0([w_{n+1}(\mathbf{y}), h_{-(n+1)}])\| \leqslant 0.1\zeta t\|w_\perp(\bar{h}, \mathbf{y})\| \leqslant 0.2\|j_0(t)\|.$$

This implies that $j_0(v(\mathbf{y},t))\neq 0$, contradicting (6.11). This completes the proof of the lemma. □

LEMMA 6.2. *Let* $\tilde{\mathfrak{W}}$ *be a subalgebra of* $\mathfrak{W}$ *possessing the following property: given* $0<\delta<1$ *there is* $x=x(\delta)\in X$ *such that*

$$\mu(x\exp\mathfrak{U}_\delta(\tilde{\mathfrak{W}})\exp\mathfrak{U}_\delta(\mathfrak{Z}_0)\exp\mathfrak{U}_\delta(\mathfrak{H}))>0.$$

Then $\tilde{\mathfrak{W}}=\mathfrak{W}$.

Proof. For $k=1,\dots,m$ let $\tilde{W}^{(k)}$ be a maximal set of unit vectors in $\tilde{\mathfrak{W}}$ such that $\chi(v)=k$ for all $v\in\tilde{W}^{(k)}$ and $\{p_k(v): v\in\tilde{W}^{(k)}\}$ is linearly independent in $\mathfrak{E}_k$, $k=1,\dots,m$. The set $\tilde{W}=\bigcup_{k=1}^m\tilde{W}^{(k)}=\{\tilde{w}_1,\dots,\tilde{w}_s\}$ is a basis in $\tilde{\mathfrak{W}}$. For $0<\delta<1$ and $t>1$ define

$$\tilde{\mathfrak{W}}^\chi(\delta,t)=\left\{w\in\tilde{\mathfrak{W}}: w=\sum_{i=1}^s w_i\tilde{w}_i, |w_i|\leqslant\delta t^{-\chi(\tilde{w}_i)}\right\}.$$

As in the proof of Lemma 5.2 (see Appendix 5.1) we show that for all sufficiently small $\delta>0$ and every integer $n>1$ there are $\mathbf{a}_1(n),\dots,\mathbf{a}_{L(n)}(n)\in\exp\mathfrak{U}_{a\delta}(\tilde{\mathfrak{W}})$ such that

$$\exp\mathfrak{U}_\delta(\tilde{\mathfrak{W}})\subset\bigcup_{i=1}^{L(n)}\mathbf{a}_i(n)\exp(a\tilde{\mathfrak{W}}^\chi(\delta,b^n))$$

where $L(n)\leqslant Lb^{n\chi(\tilde{W})}$, $b, a, L=L(\delta)>1$ are constants and

$$\chi(\tilde{W})=\sum_{i=1}^s\chi(\tilde{w}_i).$$

Define

$$\mathbf{B}_n(\delta)=\exp(a\tilde{\mathfrak{W}}^\chi(\delta,b^n))\exp\mathfrak{U}_\delta(\mathfrak{Z}_0)\exp\mathfrak{U}_\delta(\mathfrak{H}).$$

We have

$$\mathbf{B}(\delta)=\exp\mathfrak{U}_\delta(\tilde{\mathfrak{W}})\exp\mathfrak{U}_\delta(\mathfrak{Z}_0)\exp\mathfrak{U}_\delta(\mathfrak{H})\subset\bigcup_{i=1}^{L(n)}\mathbf{a}_i(n)\,\mathbf{B}_n(\delta).$$

Now suppose on the contrary that $\tilde{\mathfrak{W}}$ is a proper subspace of $\mathfrak{W}$. Then

$$\chi(\tilde{W})\leqslant\chi(W)-1. \tag{6.12}$$

Let K be a compact subset of X with $\mu(K)>0.9$ and let $\delta>0$ be so small that $10a\delta\leqslant\Delta(K)=\min\{\Delta(x): x\in K\}$ (see (4.1)). Let $x=x(\delta)\in X$ be such that

$$\mu(x\mathbf{B}(\delta))=\alpha>0.$$

Set $\sigma=\ln b$. Since the action of $\mathbf{g}_\sigma$ on (X,μ) is ergodic (see Proposition 1.12) there is $\bar{B}\subset x\mathbf{B}(\delta)$, $\mu(\bar{B})>0.9\alpha$ and $n_0>1$ such that if $y\in\bar{B}$ and $n\geqslant n_0$ then the relative frequency of K on the orbit interval $\{y, y\mathbf{g}_\sigma, \dots, y\mathbf{g}_\sigma^n\}$ is at least 0.8. We claim that for every $n\geqslant n_0$ there is $n\leqslant k(n)\leqslant 2n$ and a subset $D(k(n))\subset\bar{B}$ such that

$$\begin{gathered}\mu(D(k(n)))\geqslant 0.3\mu(\bar{B})\geqslant 0.1\alpha\\ D(k(n))\,\mathbf{g}_\sigma^{k(n)}\subset K.\end{gathered}\tag{6.13}$$

To prove the claim we apply a standard argument based on the Fubini theorem. More specifically, for every $n\geqslant n_0$ and every $y\in\bar{B}$ the relative frequency of K on $\{y\mathbf{g}_\sigma^n, \dots, y\mathbf{g}_\sigma^{2n}\}$ is at least 0.3. Let

$$\bar{K}=\{(y,k): y\in\bar{B},\ k\in\{n,\dots,2n\},\ y\mathbf{g}_\sigma^k\in K\}\subset\bar{B}\times I_n$$

where $I_n=\{n,\dots,2n\}$. Let ν be the probability measure on $\bar{B}\times I_n$, which is the product of $\mu/\mu(\bar{B})$ and the normalized counting measure on I_n. We have $\nu(\bar{K})\geqslant 0.3$. This implies via the Fubini theorem that there exists $k=k(n)\in I_n$ such that

$$(\mu/\mu(\bar{B}))\,(\bar{K}\cap(\bar{B}\times\{k\}))>0.3.$$

Define

$$D(k)=\{y\in\bar{B}: (y,k)\in\bar{K}\}.$$

It is clear that $D(k)$ satisfies (6.13). We have $k=k(n)\to\infty$, when $n\to\infty$. Also

$$D(k)=\bigcup_{i=1}^{L(k)} D(k)\cap(x\mathbf{a}_i(k)\,\mathbf{B}_k(\delta))=\bigcup_{i=1}^{L(k)} D_i(k).\tag{6.14}$$

Let $J=\{i: \mu(D_i(k))>0\}$. We have

$$\begin{aligned}D_i(k)\,\mathbf{g}_\sigma^k&\subset(x\mathbf{a}_i(k)\,\mathbf{g}_\sigma^k)\exp(\mathbf{g}_\sigma^{-k}(a\tilde{\mathfrak{W}}^\chi(\delta,b^k))\,\mathbf{g}_\sigma^k)\exp\mathfrak{U}_\delta(\mathfrak{Z}_0)\cdot\mathbf{U}_\delta(k,\mathbf{H})\\ &\subset(x\mathbf{a}_i(k)\,\mathbf{g}_\sigma^k)\exp(\mathbf{g}_\sigma^{-k}(a\tilde{\mathfrak{W}}^\chi(\delta,b^k))\,\mathbf{g}_\sigma^k)\exp\mathfrak{U}_\delta(\mathfrak{Z}_0)\cdot\mathbf{U}_\delta(\mathbf{H})\subset z_i\,\mathbf{O}_{\Delta(z_i)}(\mathbf{G})\end{aligned}$$

for some $z_i\in K$ and all $i\in J$ by the definition of δ. Here

$$\mathbf{U}_\delta(k, \mathbf{H}) = \exp\{h \in \mathfrak{H}: |h_i(h)| \leq \delta b^{k\chi(h_i)}, i=1, \ldots, p\}$$

and $H=\{h_1, \ldots, h_p\}$ is the basis of eigenvectors in $\mathfrak{H}$ used above. This implies via (4.13) and (4.16) that

$$\mu(D_i(k)) = \mu(D_i(k)\,\mathbf{g}_\sigma^n) \leq Cb^{k\chi(H)}$$

for all $i \in J$ and some $C>1$, where $\chi(H)=-\chi(W)$. This and (6.14) give

$$\mu(D(k)) \leq CL(k)b^{-k\chi(W)} \leq CLb^{k(\chi(\bar{W})-\chi(W))} \leq CLb^{-k}$$

by (6.12). Thus $\mu(D(k)) \to 0$ when $k=k(n) \to \infty$. This contradicts (6.13) and completes the proof of the lemma. □

It follows from Proposition 6.1 and Lemmas 6.1 and 6.2 that $\bar{\mathfrak{W}}=\mathfrak{W}$. Thus we get the following

COROLLARY 6.1. $[w_\lambda, h_{-\lambda}] \in \mathfrak{Z}_0$ *for all* $w \in \mathfrak{W}$, $h_{-\lambda} \in \mathfrak{H} \cap \mathfrak{E}_{-\lambda}$, $\lambda>0$.

Next we shall substitute $\mathfrak{Z}_0$ by $\mathfrak{Z}(\mathbf{\Lambda})$ in Lemma 6.1. To do so we shall use Proposition 1.11 and the H-regular sequence $\mathbf{F}_\varrho(s)$ in $\mathbf{H}$ defined with

$$\sigma_i(s) = s^{-\chi(h_i)}, \quad i=1, \ldots, p.$$

It follows from Corollary 6.1 that if $w \in \mathfrak{W}$, $z \in \mathfrak{Z}_0$ and $\mathbf{h} \in \mathbf{H}$ then

$$\exp \mathrm{Ad}_{\mathbf{h}}(w+z) = \exp(z+\bar{w}(w,z,\mathbf{h})+\bar{z}(w,z,\mathbf{h})) \cdot \bar{\mathbf{h}}(w,z,\mathbf{h}) \tag{6.15}$$

where

$$\bar{w}=\bar{w}(w,z,\mathbf{h}) \in \mathfrak{W}, \quad \bar{z}=\bar{z}(w,z,\mathbf{h}) \in \mathfrak{Z}_0 \quad \text{and} \quad \bar{\mathbf{h}}=\bar{\mathbf{h}}(w,z,\mathbf{h}) \in \mathbf{H}.$$

For $x \in X$, $y=x\exp(w+z)$ define $\varphi=\varphi_{x,y}: x\mathbf{H} \to y\mathbf{H}$ by

$$\varphi_{x,y}(x\mathbf{h}) = y\mathbf{h}(\bar{\mathbf{h}})^{-1} = x\mathbf{h}\exp(z+\bar{w}+\bar{z}).$$

Let $0<\theta<1$ be so small that if $\|w\|, \|z\| \leq \theta$ then

$$\begin{gathered} |[\lambda(B)/\lambda(\varphi(B))]-1| \leq 0.1 \\ \varphi(x\mathbf{F}_\varrho(1)) \subset y\mathbf{F}_{2\varrho}(1) \end{gathered} \tag{6.16}$$

for all $0<\varrho<1$ and all Borel subsets $B \subset x\mathbf{F}_\varrho(1)$, where as before

$$\lambda(B)=\lambda\{\mathbf{h}\in\mathbf{F}_\varrho(1):x\mathbf{h}\in B\}$$

with λ being a Haar measure on $\mathbf{H}$. Also we will need

$$r=\lambda(\mathbf{F}_\varrho(s))/\lambda(\mathbf{F}_{2\varrho}(s)))=2^{-p},\quad \varrho,s>0.$$

LEMMA 6.3. *Given* $0<\delta<\Delta$ *there is* $\hat{X}(\delta)\subset X$, $\mu(\hat{X}(\delta))>0$ *such that if* $x\in\hat{X}(\delta)$ *then*

$$\mu(x\exp\mathfrak{U}_\delta(\mathfrak{W}+\mathfrak{Z}(\mathbf{\Lambda}))\exp\mathfrak{U}_\delta(\mathfrak{H}))>0. \tag{6.17}$$

Proof. Let $\mathbf{K}_i$, $i=1,2,\ldots$ be nonempty compact subsets of $\exp\mathfrak{Z}_0$ such that

$$\bigcup_{i=1}^{\infty}\mathbf{K}_i=\exp[\mathfrak{U}_\theta(\mathfrak{Z}_0)-\mathfrak{Z}(\mathbf{\Lambda})].$$

Now let $0<\alpha<0.1r$ be so small that $\beta(\alpha)\leqslant0.1$, where $\beta(\alpha)$ is as in Proposition 1.10. It follows from Proposition 1.11 that there are $Y_i\subset X$, $\mu(Y_i)>1-\alpha$ and $\delta_i>0$ such that

$$d(Y_i,Y_i\mathbf{k})\geqslant\delta_i \tag{6.18}$$

for all $\mathbf{k}\in\mathbf{K}_i$, $i=1,2,\ldots$.

Now let $0<\delta<0.1\min\{\theta,\Delta\}$ be given and let $0<\varrho_i<1$ be so small that if $w\in\mathfrak{W}$, $z\in\mathfrak{Z}_0$, $\|w\|,\|z\|\leqslant\delta$ then

$$\|\bar{z}(w,z,\mathbf{h})\|\leqslant0.1\delta_i \tag{6.19}$$

for all $\mathbf{h}\in\mathbf{F}_{\varrho_i}(1)$, where $\bar{z}(w,z,\mathbf{h})$ is as in (6.15). Using Proposition 1.10 we get $X_i\subset X$, $\mu(X_i)>0.9$ and $t_i>1$ such that if $x\in X_i$, $t>t_i$ then

$$\lambda(Y_i\cap x\mathbf{F}_{\varrho_i}(t))/\lambda(\mathbf{F}_{\varrho_i}(t))\geqslant1-0.2r$$

$$\lambda(Y_i\cap x\mathbf{F}_{2\varrho_i}(t))/\lambda(\mathbf{F}_{2\varrho_i}(t))\geqslant1-0.2r.$$

Since the action of $\mathbf{g}=\mathbf{g}_{-1}$ is ergodic on (X,μ) (see Proposition 1.12) there is $\hat{X}\subset X$, $\mu(\hat{X})=1$ such that if $x\in\hat{X}$ then the relative frequency of X_i on $\{x,x\mathbf{g},\ldots,x\mathbf{g}^n\}$ tends to $\mu(X_i)$ when $n\to\infty$ for all $i=1,2,\ldots$. Now let $X(\delta)\subset X,\mu(X(\delta))>0$ be as in Theorem 5.1 and let $\hat{X}(\delta)=X(\delta)\cap\hat{X},\mu(\hat{X}(\delta))>0$. Let us show that if $x\in\hat{X}(\delta)$ then (6.17) holds for x. Define

$$\bar{X}=\{x\in X:\lambda(\hat{X}\cap x\mathbf{F}_1(1))/\lambda(\mathbf{F}_1(1))=1\}$$

$$\bar{B}(\delta)=(x\exp\mathfrak{U}_\delta(\mathfrak{W}+\mathfrak{Z}_0)\exp\mathfrak{U}_\delta(\mathfrak{H}))\cap\bar{X},\quad x\in\bar{X}(\delta).$$

Here $\mu(\bar{X})=1$ by Proposition 1.13 and $\mu(\bar{B}(\delta))>0$. We claim that

$$\bar{B}(\delta)\subset x\exp\mathfrak{U}_\delta(\mathfrak{W}+\mathfrak{Z}(\mathbf{\Lambda}))\exp\mathfrak{U}_\delta(\mathfrak{H}). \tag{6.20}$$

Indeed, let $xy\in\bar{B}(\delta)$, $\mathbf{y}=(\exp v(\mathbf{y}))\cdot\mathbf{h}(\mathbf{y})=\bar{\mathbf{y}}\mathbf{h}(\mathbf{y})$ for some $v(\mathbf{y})=w(\mathbf{y})+z(\mathbf{y})$, $\mathbf{h}=\mathbf{h}(\mathbf{y})\in\mathbf{U}_\delta(\mathbf{H})$, $w=w(\mathbf{y})\in\mathfrak{U}_\delta(\mathfrak{W})$, $z=z(\mathbf{y})\in\mathfrak{U}_\delta(\mathfrak{Z}_0)$. We have to show that

$$z=z(\mathbf{y})\in\mathfrak{Z}(\mathbf{\Lambda}). \tag{6.21}$$

Suppose on the contrary that $z\in\mathfrak{Z}_0-\mathfrak{Z}(\mathbf{\Lambda})$. Then $\exp z\in\mathbf{K}_i$ for some $i=1,2,\ldots$. Since $xy\in\bar{X}$ there is $a\in x\bar{\mathbf{y}}\mathbf{F}_{0.01\varrho_i}(1)\cap\hat{X}$. Now let $\tau_i>1$ be so big that

$$\delta e^{-\tau_i}<0.1\delta_i,\quad e^{\tau_i}>t_i.$$

It follows from the definition of $\hat{X}$ that there is $\tau\geq\tau_i$ such that

$$a_\tau=a\mathbf{g}^\tau\in X_i,\quad x_\tau=x\mathbf{g}^\tau\in X_i.$$

Set $t=e^\tau$, $\bar{y}_\tau=x\bar{\mathbf{y}}\mathbf{g}^\tau$, $a_\tau\in\bar{y}_\tau\mathbf{F}_{0.01\varrho_i}(t)$. We have from the definition of X_i

$$\lambda(Y_i\cap a_\tau\mathbf{F}_{2\varrho_i}(t))/\lambda(\mathbf{F}_{2\varrho_i}(t))\geq 1-0.2r$$

$$\lambda(Y_i\cap x_\tau\mathbf{F}_{\varrho_i}(t))/\lambda(\mathbf{F}_{\varrho_i}(t))\geq 1-0.2r\geq 0.8.$$

This implies via (6.16) that

$$\begin{gathered}\lambda(Y_i\cap\bar{y}_\tau\mathbf{F}_{2\varrho_i}(t))/\lambda(\mathbf{F}_{2\varrho_i}(t))\geq 1-0.3r\\ \lambda(\varphi(Y_i\cap x_\tau\mathbf{F}_{\varrho_i}(t))\cap\bar{y}_\tau\mathbf{F}_{2\varrho_i}(t))/\lambda(\mathbf{F}_{2\varrho_i}(t))\geq 0.7r\end{gathered} \tag{6.22}$$

where $\varphi=\varphi_{x_\tau,\bar{y}_\tau}$, $\bar{y}_\tau=x_\tau\exp(z+w_\tau)$, $w_\tau=\mathbf{g}^{-\tau}w\mathbf{g}^\tau$, $\|w_\tau\|\leq\delta e^{-\tau}\leq 0.1\delta_i$. It follows from (6.22) that there is $c\in Y_i$ such that $\varphi(c)\in Y_i$. Also $c=x_\tau\mathbf{h}_\tau$ for some $\mathbf{h}_\tau\in\mathbf{F}_{\varrho_i}(t)$, $\mathbf{h}_\tau=\mathbf{g}^{-\tau}\mathbf{h}\mathbf{g}^\tau$ for some $\mathbf{h}\in\mathbf{F}_{\varrho_i}(1)$. We have from the definition of φ

$$\varphi(c)=c\exp(z+\bar{w}_\tau+\bar{z})$$

where $\bar{w}_\tau=\mathbf{g}^{-\tau}\bar{w}\mathbf{g}^\tau$, $\|\bar{w}_\tau\|\leq\delta e^{-\tau}\leq 0.1\delta_i$ and $\bar{w}=\bar{w}(z,w,\mathbf{h})$, $\bar{z}=\bar{z}(z,w,\mathbf{h})$ are as in (6.15), $\|\bar{z}\|\leq 0.1\delta_i$ by (6.19). This implies that

$$d(\varphi(c),c\exp z)\leq 0.2\delta_i$$

which contradicts (6.18), since $\exp z\in\mathbf{K}_i$, $c,\varphi(c)\in Y$. This proves (6.20) and (6.21) and completes the proof of the lemma. □

Now let

$$\tilde{\mathfrak{W}} = \{w \in \mathfrak{W}: [w_\lambda, h_{-\lambda}] \in \mathfrak{Z}(\mathbf{\Lambda}) \text{ for all } h_{-\lambda} \in \mathfrak{H} \cap \mathfrak{E}_{-\lambda},\ \lambda > 0\}.$$

Repeating the argument in the proof of Proposition 6.1 and Lemma 6.1 we get the following

LEMMA 6.4. (1) *The space* $\tilde{\mathfrak{W}}$ *is a Lie subalgebra of* $\mathfrak{W}$, *normalized by* $\mathfrak{Z}(\mathbf{\Lambda})$;
(2) *Given* $0<\delta<\Delta$ *there is* $\tilde{X}(\delta) \subset X$, $\mu(\tilde{X}(\delta))>0$ *such that if* $x \in \tilde{X}(\delta)$ *then*

$$\mu(x \exp \mathfrak{U}_\delta(\tilde{\mathfrak{W}} + \mathfrak{Z}(\mathbf{\Lambda})) \exp \mathfrak{U}_\delta(\mathfrak{H})) > 0.$$

Proof of Theorem 6.1. (1) It follows from Lemmas 6.4 and 6.2 that $\tilde{\mathfrak{W}} = \mathfrak{W}$. This and Proposition 5.3 imply that $\mathfrak{L} = \mathfrak{W} + \mathfrak{Z}(\mathbf{\Lambda}) + \mathfrak{H}$ is a Lie subalgebra of $\mathfrak{G}$.

(2) Lemma 6.4 asserts that there is $x \in X$ such that $\mu(x \exp \mathfrak{U}_\delta(\mathfrak{L}))>0$. This implies that $\mu(x\mathbf{L})=1$, since $z\mathbf{H} \subset x\mathbf{L}$ for every $z \in x \exp \mathfrak{U}_\delta(\mathfrak{L})$ and the action of $\mathbf{H}$ on (X, μ) is ergodic. This completes the proof. □

7. Algebraicity of μ

Let $\mathbf{L} \subset \mathbf{G}$, $x \in X$, $\mu(x\mathbf{L})=1$ be as in Theorem 6.1. Let ν be the Riemannian volume on $x\mathbf{L}$ induced by a left invariant Riemannian metric on $\mathbf{L}$ via the local diffeomorphism $\mathbf{l} \to x\mathbf{l}$, $\mathbf{l} \in \mathbf{L}$. It follows from the definition of $\mathbf{L}$ that ν is invariant under the action of $\mathbf{g}$ on $x\mathbf{L}$. Also it is invariant under the action of $\mathbf{H}$ and $\mathbf{W}$ since $\mathbf{H}$ and $\mathbf{W}$ consist of unipotent elements. Here $\mathbf{W}$ denotes the Lie subgroup of $\mathbf{L}$ with the Lie algebra $\mathfrak{W}$. We shall show in this section that ν is finite and coincides with μ up to a factor. This would imply that $\mathbf{xLx}^{-1} \cap \mathbf{\Gamma}$ is a lattice in $\mathbf{xLx}^{-1}$, $\mathbf{x} \in \pi^{-1}\{x\}$ and μ is $\mathbf{L}$-invariant. Write

$$\mathfrak{Z}(\mathbf{\Lambda}) + \mathfrak{H} = \mathfrak{Q}, \quad Q_\delta(x) = x \exp \mathfrak{U}_\delta(\mathfrak{Q})$$

$$W_\delta(x) = x \exp \mathfrak{U}_\delta(\mathfrak{W}).$$

If $0<\delta<0.1\Delta(x)$ is sufficiently small then for each $y \in Q_\delta(x)$ and each $z \in W_\delta(x)$ the intersection $W_{10\delta}(y) \cap Q_{10\delta}(z)$ consists of exactly one point $p = p(y, z)$. Define

$$W(y) = W(p) = \{p(y, z): z \in W_\delta(x)\}$$

$$Q(z) = Q(p) = \{p(y, z): y \in Q_\delta(x)\}$$

$$B_\delta(x) = \bigcup_{y \in Q_\delta(x)} W(y).$$

We have

$$B_\delta(x) = \bigcup_{q \in W(p)} Q(q) = \bigcup_{s \in Q(p)} W(s)$$

for all $p \in B_\delta(x)$. We can assume without loss of generality that $\mu(B_{\delta/2}(x))>0$.

PROPOSITION 7.1. *There exists $B \subset B_\delta(x)$ with $\nu(B)=\nu(B_\delta(x))$ such that if $z \in B$ then $z\mathbf{g}^{-n} \in B_\delta(x)$ for infinitely many $n \in \mathbf{Z}^+$, where $\mathbf{g}=\mathbf{g}_1$.*

Proof. Since the action of $\mathbf{g}$ on (X,μ) is ergodic (Proposition 1.12) there is $C \subset B_\delta(x)$, $\mu(C)=\mu(B_\delta(x))$ such that if $y \in C$ then $y\mathbf{g}^{-n} \in B_{\delta/2}(x)$ for infinitely many $n \in \mathbf{Z}^+$. Since $\mathfrak{Q} \subset \mathfrak{L}(\mathbf{\Lambda})$ there is $\tilde{C} \subset B_\delta(x)$, $\mu(\tilde{C})=\mu(B_\delta(x))$ such that if $z \in \tilde{C}$ then

$$\lambda(C \cap Q(z))/\lambda(Q(z)) = 1$$

by Proposition 1.13 where λ denotes a $\mathbf{Q}$-invariant measure in $z\mathbf{Q}$, with $\mathbf{Q}$ being the Lie subgroup of $\mathbf{L}$ with the Lie algebra $\mathfrak{Q}$. Pick $z_0 \in \tilde{C}$ and define

$$B = \bigcup_{y \in C \cap Q(z_0)} W(y).$$

It is clear that $\nu(B)=\nu(B_\delta(x))$. Now let $z \in B$. Then $z \in W(y)$ for some $y \in C$. We have $d(y\mathbf{g}^{-n}, z\mathbf{g}^{-n}) \to 0$, when $n \to \infty$ and $y\mathbf{g}^{-n} \in B_{\delta/2}(x)$ for infinitely many $n \in \mathbf{Z}^+$. This implies that $z\mathbf{g}^{-n} \in B_\delta(x)$ for infinitely many $n \in \mathbf{Z}^+$. □

For $z \in B$ let $n(z)=\min\{n \geq 1 : z\mathbf{g}^{-n} \in B_\delta(x)\}$ and let $\varphi : B \to B_\delta(x)$ be defined by $\varphi(z)=z\mathbf{g}^{-n(z)}$. It is clear that the map φ preserves ν and therefore $\nu(\varphi(B))=\nu(B)=\nu(B_\delta(x))$. This implies that we can assume without loss of generality that $\varphi(B)=B$. Define

$$\Omega = \{y \in x\mathbf{L} : y = z\mathbf{g}^{-k} \text{ for some } z \in B,\ 0 \leq k < n(z)\}.$$

The action of $\mathbf{g}$ on (Ω, ν) is measure preserving. Let $\bar{\nu}$ be the Borel measure on $x\mathbf{L}$ defined by $\bar{\nu}(D)=\nu(D \cap \Omega)$ for every Borel subset $D \subset x\mathbf{L}$.

LEMMA 7.1. (1) $\nu(\Omega)<\infty$; (2) $\mu=\bar{\nu}/\nu(\Omega)$.

Proof. Let f be a continuous function on $x\mathbf{L}$ with compact support and let $f_\mu=\int f\,d\mu$. Since the action of $\mathbf{g}$ on $(x\mathbf{L},\mu)$ is ergodic, there is a subset $C_f \subset B_\delta(x)$, $\mu(C_f)=\mu(B_\delta(x))$ such that if $y \in C_f$ then

$$S_{n,f}(y) = \sum_{i=0}^{n-1} f(y\mathbf{g}^{-i})/n \to f_\mu, \quad n \to \infty. \tag{7.1}$$

Let $\tilde{C}_f \subset B_\delta(x)$, $\mu(\tilde{C}_f)=\mu(B_\delta(x))$ be such that if $z \in \tilde{C}_f$ then

$$\lambda(C_f \cap Q(z))/\lambda(Q(z)) = 1.$$

Pick $\tilde{z} \in \tilde{C}_f$ and define

$$B_f = B \cap \bigcup_{y \in C_f \cap Q(\tilde{z})} W(y)$$

$$\Omega_f = \{\omega \in \Omega : \omega = z\mathbf{g}^{-k} \text{ for some } z \in B_f, 0 \leq k < n(z)\}.$$

We have $\nu(\Omega_f)=\nu(\Omega)$. Now let $z \in B_f$. Then $z \in W(y)$ for some $y \in C_f$. We have

$$d(z\mathbf{g}^{-n}, y\mathbf{g}^{-n}) \to 0, \quad n \to \infty.$$

This and (7.1) imply that

$$S_{n,f}(z) \to f_\mu, \quad n \to \infty$$

for all $z \in B_f$, since f is uniformly continuous. Also

(7.2) $$S_{n,f}(\omega) \to f_\mu, \quad n \to \infty$$

for all $\omega \in \Omega_f$. Now let f be nonnegative with $f_\mu \neq 0$. It follows then from the Fatou's lemma that

$$f_\mu \nu(\Omega) = \int_{\Omega_f} f_\mu \, d\nu \leq \lim_{n\to\infty} \int_{\Omega_f} S_{n,f} \, d\nu = \int_\Omega f \, d\nu < \infty.$$

This proves that $\nu(\Omega)<\infty$. Now we use (7.2) and the Dominated Convergence Theorem to get

$$f_{\bar{\nu}} = \int_\Omega f \, d\nu = \int_\Omega S_{n,f} \, d\nu \to \int_\Omega f_\mu \, d\nu = \nu(\Omega) f_\mu$$

for every continuous function f on $x\mathbf{L}$ with compact support. This proves that $\mu=\bar{\nu}/\nu(\Omega)$. □

Proof of the Main Theorem. In view of Lemma 7.1 it remains to prove that $\nu=\bar{\nu}$. To do so it suffices to show that for every $p \in x\mathbf{L}$

$$\nu(O_{0.1\delta}(p)-\Omega)=0$$

where $O_\gamma(p)=p\mathbf{O}_\gamma(\mathbf{e})$ and $\mathbf{O}_\gamma(\mathbf{e})$ denotes the ball of radius γ in $\mathbf{L}$ centered at $\mathbf{e}$. Let $B\subset B_\delta(x)$ be as above and let $\bar{B}\subset B$, $\nu(\bar{B})=\nu(B)$ be such that if $y\in\bar{B}$ then

$$\lambda(B\cap W(y))/\lambda(W(y))=1$$

where λ denotes a $\mathbf{W}$-invariant measure on $y\mathbf{W}$. Define

$$\bar{\Omega}=\{\omega\in\Omega:\omega\mathbf{g}^{-n}\in\bar{B}\cap B_{\delta/2}(x) \text{ for infinitely many } n\in\mathbf{Z}^+\}.$$

We have $\nu(\bar{\Omega})=\nu(\Omega)$, since $\mu=\hat{\nu}=\bar{\nu}/\nu(\Omega)$ and the action of $\mathbf{g}$ on (Ω,μ) is ergodic. If $\omega\in\bar{\Omega}$ then $W_{10\delta}(\omega)\,\mathbf{g}^{-n}\subset W(y)$ for some $n\in\mathbf{Z}^+$ and some $y\in\bar{B}$. This implies by the definition of $\bar{B}$ that

$$\lambda(W_{10\delta}(\omega)\,\mathbf{g}^{-n}\cap B)/\lambda(W_{10\delta}(\omega)\,\mathbf{g}^{-n})=1$$

and hence

$$\lambda(W_{10\delta}(\omega)\cap\Omega)/\lambda(W_{10\delta}(\omega))=1$$

for all $\omega\in\bar{\Omega}$, since Ω is $\mathbf{g}$-invariant. Now let

$$\hat{\Omega}=\{\omega\in\Omega:\lambda(\bar{\Omega}\cap Q_{10\delta}(\omega))/\lambda(Q_{10\delta}(\omega))=1\}.$$

We have

$$\nu(\hat{\Omega})=\nu(\Omega) \tag{7.3}$$

by Proposition 1.13, since $\hat{\nu}=\mu$ is $\mathbf{Q}$-invariant. It follows now from the definition of $\hat{\Omega}$ that if $\omega\in\hat{\Omega}$ then

$$\nu(B_\delta(\omega)\cap\Omega)=\nu(B_\delta(\omega)). \tag{7.4}$$

Here $B_\delta(\omega)$ is defined to be $\pi(\tilde{\omega}\mathbf{B}_\delta(\mathbf{e}))$, where $\tilde{\omega}\in\pi^{-1}\{\omega\}$, π denotes the projection $\pi(\mathbf{h})=\mathbf{\Gamma h}$, $\mathbf{h}\in\mathbf{G}$ and $\mathbf{B}_\delta(\mathbf{e})$ is defined by $\pi(\tilde{\mathbf{x}}\mathbf{B}_\delta(\mathbf{e}))=B_\delta(x)$, $\tilde{\mathbf{x}}\in\pi^{-1}\{x\}$. It follows now from (7.3) and (7.4) that

$$\nu(B_\delta(\omega)\cap\hat{\Omega})=\nu(B_\delta(\omega))$$

for all $\omega\in\hat{\Omega}$. Now let $p\in x\mathbf{L}$. Then we can find $x=\omega_1,\dots,\omega_n$ such that $\omega_i\in B_\delta(\omega_{i-1})\cap\hat{\Omega}$, $i=2,\dots,n$ and $O_{0.1\delta}(p)\subset B_\delta(\omega_n)$. This implies via (7.4) that

$$\nu(O_{0.1\delta}(p)-\Omega)=0$$

and proves that $\nu=\bar{\nu}$.

We have just proved that $\mathbf{L}\subset\mathbf{\Lambda}$. This implies that $\mathfrak{L}=\mathfrak{L}(\mathbf{\Lambda})$, since $\mathfrak{L}(\mathbf{\Lambda})\subset\mathfrak{L}$ by the definition of $\mathfrak{L}$. Therefore $\mathbf{L}=\mathbf{\Lambda}^0$. Now let us show that $x\mathbf{L}=x\mathbf{\Lambda}$. Indeed, suppose on the contrary that there is $\boldsymbol{\alpha}\in\mathbf{\Lambda}$ such that $x\boldsymbol{\alpha}\notin x\mathbf{L}$. We have $\boldsymbol{\alpha}^{-1}\mathbf{L}\boldsymbol{\alpha}=\mathbf{L}$ and hence $x\mathbf{L}\boldsymbol{\alpha}=x\boldsymbol{\alpha}\mathbf{L}$, $x\mathbf{L}\boldsymbol{\alpha}\cap x\mathbf{L}=\varnothing$. This implies that $\mu(x\mathbf{L}\boldsymbol{\alpha})=0$. This gives a contradiction since the action of $\boldsymbol{\alpha}$ preserves μ and $\mu(x\mathbf{L})=1$. This completes the proof of the main theorem. □

Proof of Theorem 2. Let μ be an ergodic algebraic joining of $\mathbf{u}^{(1)}$ on $(X_1=\mathbf{\Gamma}_1\backslash\mathbf{G}_1,\nu_1)$ and $\mathbf{u}^{(2)}$ on $(X_2=\mathbf{\Gamma}_2\backslash\mathbf{G}_2,\nu_2)$. We have $\mu(x\mathbf{\Lambda})=1$ for some $x=(x_1,x_2)\in X=X_1\times X_2$, where $\mathbf{\Lambda}=\mathbf{\Lambda}(\mu)\subset\mathbf{G}_1\times\mathbf{G}_2$. Let $\mathbf{H}=\{\mathbf{z}_1\in\mathbf{G}_1:(\mathbf{z}_1,\mathbf{z}_2)\in\mathbf{\Lambda}$ for some $\mathbf{z}_2\in\mathbf{G}_2\}$ and let $\mathbf{H}(\delta)=\mathbf{H}\cap\mathbf{O}_\delta(\mathbf{G}_1)$. We have

$$\mu(x_1\,\mathbf{O}_\delta(\mathbf{G}_1)\times X_2)=\mu(x_1\,\mathbf{H}(\delta)\times X_2)$$

if $\delta>0$ is sufficiently small. This implies that

$$\nu_1(x_1\,\mathbf{O}_\delta(\mathbf{G}_1))=\nu_1(x_1\,\mathbf{H}(\delta))$$

since μ is a joining. This implies that $\mathbf{O}_\delta(\mathbf{G}_1)=\mathbf{H}(\delta)$ since $\mathbf{H}$ is a subgroup of $\mathbf{G}_1$ and ν_1 a Haar measure on $\mathbf{G}_1$. Therefore $\mathbf{G}_1\subset\mathbf{H}$, since $\mathbf{G}_1$ is connected. This implies that for each $\mathbf{h}\in\mathbf{G}_1$ there is $\tilde{\mathbf{h}}\in\mathbf{G}_2$ with $(\mathbf{h},\tilde{\mathbf{h}})\in\mathbf{\Lambda}$. Applying the same argument to $\mathbf{G}_2$ we get that for each $\tilde{\mathbf{h}}\in\mathbf{G}_2$ there is $\mathbf{h}\in\mathbf{G}_1$ with $(\mathbf{h},\tilde{\mathbf{h}})\in\mathbf{\Lambda}$. Then $(\mathbf{h},\tilde{\mathbf{h}}\mathbf{\Lambda}_2)\subset\mathbf{\Lambda}$, where

$$\mathbf{\Lambda}_2=\mathbf{\Lambda}_2(\mu)=\{\mathbf{z}\in\mathbf{G}_2:(\mathbf{e},\mathbf{z})\in\mathbf{\Lambda}\}$$

and hence

$$(\mathbf{e},\tilde{\mathbf{h}}\mathbf{\Lambda}_2\tilde{\mathbf{h}}^{-1})\subset\mathbf{\Lambda},\quad \tilde{\mathbf{h}}\mathbf{\Lambda}_2\tilde{\mathbf{h}}^{-1}\subset\mathbf{\Lambda}_2$$

by the definition of $\mathbf{\Lambda}_2$. This proves that $\mathbf{\Lambda}_2$ is a normal subgroup of $\mathbf{G}_2$. Similarly, we show that $\mathbf{\Lambda}_1$ is a normal subgroup of $\mathbf{G}_1$. We have

$$(\mathbf{h},\tilde{\mathbf{h}}\mathbf{\Lambda}_2)=(\{\mathbf{h}\}\times\mathbf{G}_2)\cap\mathbf{\Lambda}.$$

Define $\alpha(\mathbf{h})=\tilde{\mathbf{h}}\mathbf{\Lambda}_2$. We have $\alpha(\mathbf{u}^{(1)})=\mathbf{u}^{(2)}\mathbf{\Lambda}_2$, since $\mathbf{u}=(\mathbf{u}^{(1)},\mathbf{u}^{(2)})\in\mathbf{\Lambda}$. Also α is a continuous, surjective homomorphism from $\mathbf{G}_1$ onto $\mathbf{G}_2/\mathbf{\Lambda}_2$. We have

$$\xi(\mathbf{\Gamma}_1\mathbf{h})=\xi(\mathbf{\Gamma}_1)\,\alpha(\mathbf{h}),\quad \mathbf{h}\in\mathbf{G}_1$$

(7.5)

$$\xi(\mathbf{\Gamma}_1)=\bigcup_{\gamma\in\mathbf{\Gamma}_1}A(\gamma)$$

where $A(\gamma)=\mathbf{\Gamma}_2\mathbf{c}\alpha(\gamma)$ for some $\mathbf{c}\in\mathbf{G}_2$ and $A(\gamma_1)=A(\gamma_2)$ if $A(\gamma_1)\cap A(\gamma_2)\neq\varnothing$. Thus

$$\xi(\mathbf{\Gamma}_1)=\bigcup_{i=1}^{\infty} A_i$$

where $A_i=A(\gamma_i)$ for some $\gamma_i\in\mathbf{\Gamma}_1$ and $A_i\cap A_j=\varnothing$ if $i\neq j$. Also for $y=\mathbf{\Gamma}_1\mathbf{h}$ we have

$$\xi(y)=\xi(\mathbf{\Gamma}_1\mathbf{h})=\bigcup_{i=1}^{\infty} A_i\,\alpha(\mathbf{h})=\bigcup_{i=1}^{\infty} A_i(y). \tag{7.6}$$

Now let μ_y be the probability measure on X_2 such that

$$\mu(C)=\int_{X_1}\mu_y(C_y)\,d\nu_1(y)$$

for every measurable $C\subset X_1\times X_2$, where $C_y=\{z\in X_2\colon (y,z)\in C\}, y\in X_1$. We have

$$\mu_y(A)=\mu_{y\mathbf{u}^{(1)}}(A\mathbf{u}^{(2)}) \tag{7.7}$$

for all measurable $A\subset X_2$, since μ is $\mathbf{u}=(\mathbf{u}^{(1)},\mathbf{u}^{(2)})$-invariant. Define

$$f(y)=\max\{\mu_y(A_i(y))\colon i=1,2,\dots\}.$$

It follows from (7.7) that f is constant on orbits of $\mathbf{u}^{(1)}$ and hence $f(y)=\beta>0$ for ν_1-almost every $y\in X_1$, since the action of $\mathbf{u}^{(1)}$ on (X_1,ν_1) is ergodic. Now let

$$C=\{(y,z)\in X\colon z\in A_i(y) \text{ with } \mu_y(A_i(y))=\beta\}.$$

The set C is $\mathbf{u}$-invariant and $\mu(C)>0$. Therefore $\mu(C)=1$, since the action of $\mathbf{u}$ on (X,μ) is ergodic. This implies that $\mu_y(A_i(y))=\mu_y(A_j(y))=\beta$ for all $i,j=1,2,\dots$ and ν_1-almost every $y\in X_1$. This proves that there exists $n\geqslant 1$ such that

$$\xi(y)=\bigcup_{i=1}^{n} A_i(y)$$

for all $y\in X_1$, $A_i(y)\cap A_j(y)=\varnothing$, $i\neq j$, since the union in (7.6) is a disjoint union. We have

$$\xi(\mathbf{\Gamma}_1)=\bigcup_{i=1}^{n} A_i$$

where $A_i=\mathbf{\Gamma}_2\mathbf{c}\alpha(\gamma_i)$ for some $\gamma_i\in\mathbf{\Gamma}_1$, $i=1,\dots,n$. This and (7.5) imply that for each $\gamma\in\mathbf{\Gamma}_1$ there is γ_i such that

$$\alpha(\gamma)\,\alpha(\gamma_i^{-1})\in(\mathbf{c}^{-1}\mathbf{\Gamma}_2\mathbf{c})\,\mathbf{\Lambda}_2=\mathbf{\Gamma}_2^{\mathbf{c}}. \tag{7.8}$$

Let $\mathbf{\Gamma}_0=\alpha(\mathbf{\Gamma}_1)\cap\mathbf{\Gamma}_2^c$. Expression (7.8) shows that for every $\alpha(\gamma)\in\alpha(\mathbf{\Gamma}_1)$ there is $\alpha(\gamma_i)$ with $\alpha(\gamma)\,\alpha(\gamma_i^{-1})\in\mathbf{\Gamma}_0$. Therefore $\alpha(\mathbf{\Gamma}_1)=\{\mathbf{\Gamma}_0\,\alpha(\gamma_i): i=1,\dots,n\}$. Also $\mathbf{\Gamma}_0\,\alpha(\gamma_i)\neq\mathbf{\Gamma}_0\,\alpha(\gamma_j)$ if $i\neq j$, since $\mathbf{\Gamma}_2\,c\alpha(\gamma_i)\neq\mathbf{\Gamma}_2\,c\alpha(\gamma_j)$. This shows that $n=|\Gamma_0\backslash\alpha(\Gamma_1)|$ and completes the proof of the theorem. □

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