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Quasiconformal maps of cylindrical domains

by

JUSSI VÄISÄLÄ

University of Helsinki
Helsinki, Finland

1. Introduction

1.1. We shall consider domains $D\subset\mathbf{R}^3$ which are of the form $G\times\mathbf{R}^1$ where G is a domain in the plane $\mathbf{R}^2$. The main problem considered in this paper is: When is $G\times\mathbf{R}^1$ quasiconformally equivalent to the round ball B^3? It is well known that this is true if G is the disk B^2. Indeed, the sharp lower bound $q_0=K_O(B^2\times\mathbf{R}^1)$ for the outer dilatation $K_O(f)$ for quasiconformal maps $f\colon B^2\times\mathbf{R}^1\to B^3$ is explicitly known:

$$q_0=\frac{1}{2}\int_0^{\pi/2}(\sin t)^{-1/2}\,dt=1.31102\ldots;$$

see [GV, Theorem 8.1]. We shall show that there is a quasiconformal map $f\colon G\times\mathbf{R}^1\to B^3$ if and only if G satisfies the *internal chord-arc condition*, which is recalled in Section 4 of this paper. It implies that the boundary of G is rectifiable.

We also show that if G is bounded then $K_O(f)\geq q_0$, and the equality is possible only if G is a round disk. For unbounded domains the corresponding lower bound is trivially one, which is attained when G is a half plane.

It is of some interest to note that although the result deals solely with quasiconformality, its proof will involve two other classes of maps: the locally bilipschitz maps and the quasisymmetric maps, the latter notion considered in a suitable metric of the product space $\partial^*G\times\mathbf{R}^1$ where ∂^*G is the prime and end boundary of G.

The main result is proved in Section 5 and the dilatation estimate in Section 6. Before that we give preliminary results on John domains, quasisymmetric maps, prime ends and chord-arc conditions. The following auxiliary results may have independent interest: Theorem 2.9 gives a useful condition for a weakly quasisymmetric map to be quasisymmetric. Theorem 2.20 gives a sufficient condition for a quasiconformal map to be quasisymmetric in the internal metric. In Lemma 6.7 we give a dilatation estimate for the boundary map of a quasiconformal map at a point of differentiability.

1.2. *Notation.* Our notation is fairly standard. Thus open balls and spheres in a metric space are written as $B(x,r)$ and $S(x,r)$. In $\mathbf{R}^n$ we may use superscripts as $B^n(x,r)$ and $S^{n-1}(x,r)$. We abbreviate

$$B^n(0,r)=B^n(r)=B(r),\quad B^n(0,1)=B^n,$$

$$S^{n-1}(0,r)=S^{n-1}(r)=S(r),\quad S^{n-1}(0,1)=S^{n-1}.$$

We let H^n denote the upper half space $x_n>0$ of $\mathbf{R}^n$.

A path in $\mathbf{R}^n$ is a continuous map $\alpha\colon\Delta\to\mathbf{R}^n$ of an interval $\Delta\subset\mathbf{R}^1$. The locus of α is $|\alpha|=\alpha\Delta$. If α is a path or an arc, its length is written as $l(\alpha)$. We let $[a,b]$ denote the closed line segment with end points $a,b\in\mathbf{R}^n$. If E is an arc and if $a,b\in E$, $E[a,b]$ will denote the closed subarc of E between a and b. The diameter of a set A in a metric space (X,d) is $d(A)$, the distance between sets $A,B\subset X$ is $d(A,B)$. All closures and boundaries of sets in $\mathbf{R}^n$ are taken in the extended space $\dot{\mathbf{R}}^n=\mathbf{R}^n\cup\{\infty\}$. By a neighborhood we mean an open neighborhood. The complement of a set A is $\complement A$.

2. John domains

2.1. *Definition.* John domains were first considered by John [Jo, p. 402]; the term is due to Martio and Sarvas [MS]. There are plenty of different characterizations of John domains; see [Vä$_5$, 2.17–2.22] and [NV]. We shall adopt the definition based on diameter cigars.

Let $E\subset\mathbf{R}^n$ be an arc with end points a,b. For $x\in E$ we set

$$\delta(x)=\min(d(E[a,x]),d(E[x,b])).$$

For $c\geq 1$ the open set

$$\operatorname{cig}_d(E,c)=\bigcup\{B(x,\delta(x)/c)\colon x\in E\}$$

is called a (diameter) *c-cigar* joining a and b. The terminology differs slightly from that in [Vä$_5$]. In particular, no turning condition is given on the core E of the cigar.

We say that a domain $D\subset\mathbf{R}^n$ is a *c-John* domain if each pair of points in D can be joined by a c-cigar in D.

2.2. *The carrot property.* It is more customary to base the definition of a John domain on carrots than on cigars. We next discuss the relation between these concepts and also give a relative version of the carrot property.

Let again E be an arc in $\mathbf{R}^n$ with end points a, b, and let $c \geq 1$. The set

$$\operatorname{car}_d(E, c) = \bigcup\{B(x, d(E[a, x])/c) : x \in E\} \tag{2.3}$$

is a (diameter) *c-carrot* with vertex a joining a to b. We also allow the possibility that E is an arc in $\dot{\mathbf{R}}^n$ with $b=\infty$; then the union in (2.3) is taken over all $x \in E \setminus \{\infty\}$.

Let $D \subset \mathbf{R}^n$ be a domain. We say that a set $A \subset D$ has the *c-carrot property* in D with *center* $x_0 \in \bar{D}$ if each $x_1 \in A$ can be joined to x_0 by a c-carrot in D. Observe that there are two essentially different possibilities: either $x_0 \in D$ or $x_0 = \infty \in \partial D$. In the first case, excluding the trivial case $D = \mathbf{R}^n$, D is bounded: $D \subset B(x_0, cd(x_0, \partial D))$.

According to the customary definition, a domain $D \neq \mathbf{R}^n$ is a c-John domain if it has the c-carrot property in D with some center $x_0 \in D$. Such domains are always bounded. Our definition 2.1 gives plenty of unbounded John domains. For example, a half space is a 1-John domain. The following lemma summarizes the relations between the cigar and carrot definitions of John domains:

2.4. LEMMA. (a) *If D is a bounded c-John domain, then D has the c_1-carrot property in D with some center $x_0 \in D$ and with $c_1 = c_1(c)$.*

(b) *If a domain $D \subset \mathbf{R}^n$ has the c-carrot property in D with center $x_0 \in D$, then D is a c-John domain.*

(c) *If D is an unbounded c-John domain, then D has the $3c$-carrot property with center ∞.*

Proof. We can obtain (a) and (b) by an easy modification of the proof of the corresponding statements for distance cigars and carrots [Väs, 2.21].

To prove (c), assume that D is an unbounded c-John domain, and let $a \in D$. Choose a sequence of points $x_j \in D$ with $|x_j - a| = 3j$. Join a to x_j by a c-cigar $\operatorname{cig}_d(E_j, c)$ in D. Let b_j be the first point of E_j in $S(a, j)$ and set $F_j = E_j[a, b_j]$. Then

$$d(F_j) \leq 2j \leq |b_j - x_j| \leq d(E_j[b_j, x_j]).$$

Hence

$$\operatorname{car}_d(F_j, c) \subset \operatorname{cig}_d(E_j, c) \subset D. \tag{2.5}$$

For $k=1,\dots,j$, let b_{jk} be the first point of F_j in $S(a,k)$. By the compactness of $S(a,k)$ and by the diagonal process, we find an infinite subset N_1 of the set N of positive integers such that for each $k\in N$, $b_{jk}\to y_k\in S(a,k)$ as $j\to\infty$ in $N_1\cap[k,\infty)$. For every $k\in N$ we can then choose $j(k)\geq k+1$ such that for $u_k=b_{j(k),k}$ and $v_k=b_{j(k),k+1}$ we have

$$|u_k-y_k|<1/6c,\quad |v_k-y_{k+1}|<1/6c. \tag{2.6}$$

Set $A_k=F_{j(k)}[u_k,v_k]$ for $k\geq 2$ and $A_1=F_{j(1)}[a,v_1]$. Assuming $D\neq\mathbf{R}^n$ it is easy to see that the arcs $A_1,[v_1,u_2],A_2,[v_2,u_3],A_3,\dots$ contain a path from a to ∞. Leaving out some loops we obtain an arc E joining a to ∞. We show that $\operatorname{car}_d(E,c)\subset D$.

Let $x\in E$ and write $\delta(x)=d(E([a,x])$. We must show that $B(x,\delta(x)/3c)\subset D$. Let first $x\in A_k$ for some k. The case $k=1$ is clear. Assume that $k\geq 2$ and set $\delta_k(x)=d(F_{j(k)}[a,x])$. Then

$$\delta(x)\leq d(A_k[u_k,x])+2j(k)\leq\delta_k(x)+2\delta_k(x)=3\delta_k(x).$$

By (2.5) this implies

$$B(x,\delta(x)/3c)\subset B(x,\delta_k(x)/c)\subset D.$$

Next assume that $x\in[v_k,u_{k+1}]$. Now $\delta(x)\leq 2j(k)\leq 2\delta_k(v_k)$. Since (2.6) gives $|x-v_k|\leq |u_k-v_k|\leq 1/3c$, we obtain

$$|x-v_k|+\delta(x)/3c\leq 1/3c+2\delta_k(v_k)/3c\leq\delta_k(v_k)/c.$$

By (2.5) this yields

$$B(x,\delta(x)/3c)\subset B(v_k,\delta_k(v_k)/c)\subset D. \qquad \square$$

2.7. *Remark.* A more thorough analysis on various cigar and carrot conditions will be given in [NV], where we also consider domains containing the point at infinity.

2.8. *Terminology.* We recall the definition of quasisymmetry [TV]. Let X and Y be metric spaces with distance written as $|a-b|$, let $\eta:[0,\infty)\to[0,\infty)$ be a homeomorphism and $f:X\to Y$ an embedding. If $|a-x|\leq t|b-x|$ implies $|f(a)-f(x)|\leq\eta(t)|f(b)-f(x)|$ for all $a,b,x\in X$ and $t>0$, f is η-quasisymmetric or η-QS. If $H\geq 1$ and if $|a-x|\leq|b-x|$ implies $|f(a)-f(x)|\leq H|f(b)-f(x)|$, f is weakly H-QS. An η-QS map is weakly H-QS with $H=\eta(1)$. The converse is true for certain space. We give in Theorem 2.9 a result in this direction which is related to but more useful than [TV, 2.15].

The main result of this section is Theorem 2.20. It states that under certain conditions, a QC map is QS in the internal metric. This result has also applications in the theory of John disks [NV]. Therefore we give it in a form which is stronger than what is actually needed in this paper.

As in [TV] we say that a metric space X is k-homogeneously totally bounded or k-HTB if $k:[1/2,\infty)\to[1,\infty)$ is an increasing function and if, for each $\alpha\geq 1/2$, every closed ball $\bar{B}(x,r)$ in X can be covered with sets $A_1,\dots,A_s$ such that $s\leq k(\alpha)$ and $d(A_j)<r/\alpha$ for all j. If $t>0$ and if A is a bounded k-HTB set whose points have mutual distances at least t, card $A\leq k(d(A)/t)$.

2.9. Theorem. *Suppose that X and Y are k-HTB metric spaces and that X is pathwise connected. Then every weakly H-QS map $f:X\to Y$ is η-QS with η depending only on H and k.*

Proof. Let $a,b,x\in X$ be distinct points with $|a-x|=t|b-x|$. We must find an estimate

$$|f(a)-f(x)|\leq\eta(t)|f(b)-f(x)| \tag{2.10}$$

where $\eta(t)\to 0$ as $t\to 0$. We know that (2.10) is valid for $t\leq 1$ with $\eta(t)=H$.

Suppose first that $t>1$. Set $r=|b-x|$ and choose an arc γ from x to a. Define inductively successive points $a_0,\dots,a_s$ of γ so that $a_0=x$, a_{j+1} is the last point of γ in $\bar{B}(a_j,r)$, and a_s is the first of these points outside $B(x,|x-a|)$. Then $|a_i-a_j|\geq r$ for $0\leq i<j<s$. Since X is k-HTB, we have

$$s\leq k(|x-a|/r)=k(t).$$

Since f is weakly H-QS, we obtain

$$|f(a_1)-f(x)|\leq H|f(b)-f(x)|,$$

and by induction

$$|f(a_{j+1})-f(a_j)|\leq H|f(a_j)-f(a_{j-1})|\leq H^{j+1}|f(b)-f(x)|$$

for $1\leq j\leq s-1$. This implies

$$|f(a_s)-f(x)|\leq sH^s|f(b)-f(x)|.$$

Since $|a-x|\leq|a_s-x|$, we obtain (2.10) with $\eta(t)=sH^{s+1}$, $s=k(t)$.

Next assume that $t<1$. Set $r=|x-b|$ and choose points $b_j\in S(x,3^{-j}r)$, $j\geq 0$, with $b_0=b$. Let s be the smallest integer with $3^{-s}r\leq|x-a|$. Then

$$s \geq \frac{\ln(1/t)}{\ln 3} = s_0(t). \tag{2.11}$$

If $0 \leq i < j < s$, we have $2|x-b_j| \leq |b_i-b_j|$, which implies that $|a-b_j| \leq |b_i-b_j|$. Hence

$$|f(a)-f(b_j)| \leq H|f(b_i)-f(b_j)|, \quad |f(x)-f(b_j)| \leq H|f(b_i)-f(b_j)|,$$

and thus

$$|f(a)-f(x)| \leq 2H|f(b_i)-f(b_j)|.$$

On the other hand, $|b_j-x| \leq |b-x|$ implies that the points $f(b_0), \ldots, f(b_{s-1})$ lie in the ball $\bar{B}(x, H|f(b)-f(x)|)$. Since Y is k-HTB, we get

$$s \leq k(2H^2|f(b)-f(x)|/|f(a)-f(x)|).$$

Since $s_0(t) \to \infty$ as $t \to 0$, this and (2.11) yield (2.10) with some $\eta(t)$ converging to 0 together with t. □

2.12. *The internal metrics.* Let $D \subset \mathbf{R}^n$ be a domain. For $a, b \in D$ we write

$$\delta_D(a, b) = \inf d(\alpha), \quad \lambda_D(a, b) = \inf l(\alpha),$$

where the infima are taken over all arcs (equivalently paths) joining a and b in D. Then δ_D and λ_D are metrics of D consistent with the usual topology. In this paper we prefer to work with δ_D, whose boundary behavior and some other properties are simpler than that of λ_D. For this reason we also work with diameter cigars and carrots.

We let d denote the euclidean metric. Then $d \leq \delta_D \leq \lambda_D$.

2.13. LEMMA. *Let $D \subset \mathbf{R}^n$ be a domain and let $E \subset D$ be connected. Then the diameters $\delta_D(E)$ and $d(E)$ are equal.*

Proof. Trivially $d(E) \leq \delta_D(E)$. Let $\varepsilon > 0$ and choose a domain D_0 such that $E \subset D_0 \subset D$ and $d(D_0) \leq d(E) + \varepsilon$. Let $a, b \in E$ and choose an arc α joining a and b in D_0. Then

$$\delta_D(a, b) \leq d(\alpha) \leq d(D_0) \leq d(E) + \varepsilon.$$

Hence $\delta_D(E) \leq d(E) + \varepsilon$. Since ε is arbitrary, the lemma follows. □

2.14. LEMMA. *Suppose that $D \subset \mathbf{R}^n$ is a domain, that $A \subset D$ has the c-carrot property in D and that e is a metric of A with $\delta_D \leq e \leq d$. Then (A, e) is k-HTB with $k = k_{c,n}$.*

Proof. Consider a closed ball $\bar{B}_e(x,r)$ in the metric e, where $x\in A$ and $r>0$. Suppose that $x_1,\dots,x_s\in\bar{B}_e(x,t)$ with $e(x_i,x_j)\geq r/2$ for $i\neq j$. It suffices to find an upper bound $s\leq s_0(c,n)$.

Choose carrots $\mathrm{car}_d(E_j,c)\subset D$ joining x_j to the center x_0. Since $\delta_D(x_i,x_j)\geq r/2$, $d(E_j)<r/4$ for at most one j, and we may thus assume that $d(E_j)\geq r/4$ for all j. We can then choose points $y_j\in E_j$ such that the subarcs $F_j=E_j[x_j,y_j]$ satisfy $d(F_j)=r/8$. Then the balls $B_j=B(y_j,r/8c)$ are contained in D. We show that these balls are disjoint. If B_i meets B_j for $i\neq j$, the set $\gamma=F_i\cup[y_i,y_j]\cup F_j$ joins x_i and x_j in D, and hence $\delta_D(x_i,x_j)\leq d(\gamma)$. Since $\delta_D(x_i,x_j)\geq r/2$ and since

$$d(\gamma)\leq d(F_i)+d(F_j)+|y_i-y_j|<r/8+r/8+r/4c\leq r/2,$$

this gives a contradiction. It follows that $|y_i-y_j|\geq r/4c$ for $i\neq j$. On the other hand,

$$|y_i-x|\leq|y_i-x_i|+|x_i-x|\leq d(F_i)+e(x_i,x)\leq r/8+r=9r/8.$$

Since $(\mathbf{R}^n,d)$ is HTB, this gives $s\leq s_0(c,n)$ as desired. □

2.15. *Terminology*. Suppose that C_0 and C_1 are disjoint continua in $\dot{\mathbf{R}}^n$, that $t>0$ and that

$$d(C_0,C_1)\leq t\min(d(C_0),d(C_1)).$$

Then the family $\Gamma=\Delta(C_0,C_1;\dot{\mathbf{R}}^n)$ of all paths joining C_0 and C_1 in $\dot{\mathbf{R}}^n$ satisfies the standard modulus estimate

$$M(\Gamma)\geq\phi_0(t,n)>0, \tag{2.16}$$

where the function $t\mapsto\phi_0(t,n)$ is a decreasing self homeomorphism of the positive real line $(0,\infty)$; see e.g. [GM, 2.6].

We say that a pair of disjoint continua C_0, C_1 in a domain $D\subset\dot{\mathbf{R}}^n$ is *t-standard* in D, $t>0$, if

$$\delta_D(C_0,C_1)\leq t\min(d(C_0),d(C_1)).$$

Let $\phi\colon(0,\infty)\to(0,\infty)$ be a decreasing homeomorphism. A domain $D\subset\dot{\mathbf{R}}^n$ is called *ϕ-broad* if for each $t>0$ and each t-standard pair (C_0,C_1) in D, the path family $\Gamma=\Delta(C_0,C_1;D)$ satisfies the inequality

$$M(\Gamma)\geq\phi(t). \tag{2.17}$$

In this paper we need only the case where D is a half space, for which (2.17) is well known to be true for $\phi(t)=\phi_0(t,n)/2$. More generally, if D is a c-QED domain in the sense of [GM], D is ϕ-broad with $\phi=\phi_0(t,n)/c$. The domains $B^p\times\mathbf{R}^{n-p}$ are not broad for $1\leq p\leq n-1$. The reader interested only in this paper can skip Lemma 2.18 and read the proof of Theorem 2.20 assuming that the domain D is a half space, in which case $\delta_D=d$.

2.18. LEMMA. *Let $D\subset\mathbf{R}^n$ be a ϕ-broad domain and let e be a metric of D with $d\leq e\leq\delta_D$. Then (D,e) is k-HTB with $k=k_{\phi,n}$.*

Proof. We consider again a closed ball $\bar{B}_e(x,r)$ and points $x_1,\dots,x_s\in\bar{B}_e(x,r)$ with $|x_i-x_j|\geq r/2$ for $i\neq j$. We must show that $s\leq s_0(\phi,n)$.

Choose a positive number $q=q(\phi,n)<1/16$ such that

$$2\omega_{n-1}\left(\ln\frac{1-4q}{8q}\right)^{1-n}\leq\phi(1), \tag{2.19}$$

where ω_{n-1} is the area of S^{n-1}. Join x_i to x by an arc $E_i\subset D$. Since $E_i\cup E_j$ joins x_i and x_j in D, we have

$$r/2\leq e(x_i,x_j)\leq\delta_D(x_i,x_j)\leq d(E_i)+d(E_j)$$

for $i\neq j$. Hence $d(E_i)<r/4$ for at most one i, and we may thus assume that $d(E_i)\geq r/4$ for all i. Choose subarcs α_i and β_i of E_i such that x_i is an end point of α_i and

$$d(\alpha_i)=d(\beta_i)=\delta_D(\alpha_i,\beta_i)=qr.$$

Since $q<1/16$, this is possible. Then (α_i,β_i) is a 1-standard pair in D, and hence $M(\Gamma_i)\geq\phi(1)$ for $\Gamma_i=\Delta(\alpha_i,\beta_i;D)$. Setting $B_i=B_e(x_i,r/4)$, $a_i=m(B_i)$ and $\Gamma_i^*=\{\gamma\in\Gamma_i\colon|\gamma|\subset B_i\}$, we have

$$M(\Gamma_i^*)\leq a_i(qr)^{-n}.$$

We next estimate $M(\Gamma_i\setminus\Gamma_i^*)$. If $\gamma\in\Gamma_i\setminus\Gamma_i^*$, there is $y\in|\gamma|$ with $e(y,x_i)\geq r/4$. Since $\alpha_i\cup|\gamma|$ joins x_i and y_i in D, we have

$$\delta_D(y,x_i)\leq d(\alpha_i)+d(|\gamma|)=qr+d(|\gamma|).$$

Since $e\leq\delta_D$, this yields $d(|\gamma|)\geq r/4-qr$. Hence γ meets $\complement B(x_i,r/8-qr/2)$. On the other hand, $d(\alpha_i)=qr$ implies that $\alpha_i\subset\bar{B}(x_i,qr)$. Hence γ also meets $\bar{B}(x_i,qr)$, and we obtain

$$M(\Gamma_i\setminus\Gamma_i^*)\leq\omega_{n-1}\left(\ln\frac{r/8-qr/2}{qr}\right)^{1-n}\leq\phi(1)/2$$

by (2.19). Consequently,

$$\phi(1)\le M(\Gamma_i)\le M(\Gamma_i^*)+M(\Gamma_i\setminus\Gamma_i^*)\le a_i(qr)^{-n}+\phi(1)/2,$$

and hence $a_i\ge q^n r^n\phi(1)/2$.

Since $e(x_i,x_j)\ge r/2$, the balls B_i are disjoint. They are contained in the ball $\bar{B}_e(x,5r/4)\subset\bar{B}(x,2r)$, and hence

$$\Omega_n 2^n r^n\ge\sum_{i=1}^{s}a_i\ge sq^n r^n\phi(1)/2,$$

where Ω_n is the volume of B^n. This gives the desired bound

$$s\le 2^{n+1}\Omega_n/q^n\phi(1)=s_0(\phi,n).$$ □

2.20. THEOREM. *Suppose that $f\colon D\to D'$ is a K-QC map between domains $D,D'\subset\mathbf{R}^n$, where D is ϕ-broad. Suppose also that $A\subset D$ is a pathwise connected set and that fA has the c_1-carrot property in D' with center $y_0\in\bar{D}$. If $y_0\neq\infty$ and hence $y_0\in D'$, we assume that $d(A)\le c_2d(f^{-1}(y_0),\partial D)$. If $y_0=\infty$, we assume that f extends to a homeomorphism $D\cup\{\infty\}\to D'\cup\{\infty\}$.*

Then $f|A$ is η-QS in the metrics δ_D and $\delta_{D'}$ with η depending only on the data $v=(c_1,c_2,K,\phi,n)$.

Proof. For brevity we write $\delta=\delta_D$ and $\delta'=\delta_{D'}$. From Lemmas 2.14 and 2.18 it follows that there is $k=k_v$ such that (fA,δ') and (A,δ) are k-HTB. By Theorem 2.9 it suffices to show that $f|A$ is weakly H-QS with $H=H(v)$. Let a,b,x be distinct points in A with $\delta(a,x)\le\delta(b,x)=r$. Set

$$a'=f(a),\quad b'=f(b),\quad x'=f(x),\quad \alpha=\delta'(a',x'),\quad \beta=\delta'(b',x').$$

We must find H such that $\alpha\le H\beta$. We may assume that $D\neq\mathbf{R}^n\neq D'$. We set

$$\tau=d(x,\partial D),\quad \tau'=d(x',\partial D'),$$

and let $M_j\ge1$, $q_j\le1$ denote positive constants depending only on v.

We shall consider 5 cases. The first two are auxiliary cases. The cases 3, 4, 5 cover the whole situation. If $y_0=\infty$, Case 3 is the general case.

Case 1. $\tau\ge2r$. Now a and b lie in the ball $B=B(x,3r/2)$, and $\delta(a,x)=|a-x|$, $\delta(b,x)=|b-x|=r$. By [Vä$_3$, 2.4] $f|B$ is η-QS in the euclidean metric with $\eta=\eta_{K,n}$. By [TV, 2.11] fB is of M_1-bounded turning, $M_1=2\eta(1)$. This implies

$$\frac{\alpha}{\beta} \leq \frac{M_1|a'-x'|}{|b'-x'|} \leq M_1\eta\left(\frac{|a-x|}{|b-x|}\right) \leq M_1\eta(1).$$

Case 2. $B(x',\alpha)\subset D'$. Now $\alpha=|a'-x'|$. We may assume that $\beta<\alpha$ and thus $\beta=|b'-x'|$. Let R' be the ring $B(x',\alpha)\setminus\bar{B}(x',\beta)$. The components of the complement of $R=f^{-1}R'$ are $C_0=f^{-1}\bar{B}(x',\beta)$ and $C_1=\complement f^{-1}B(x',\alpha)$. The continuum C_0 is bounded and contains x and b while C_1 is unbounded and contains a and $\complement D$. If $[x,b]$ meets $\complement D$, then $d(C_0,C_1)\leq|b-x|\leq d(C_0)$. But this is also true if $[x,b]\subset D$, because then

$$d(C_0,C_1)\leq|a-x|\leq\delta(a,x)\leq\delta(b,x)=|b-x|\leq d(C_0).$$

Let Γ_R be the path family associated with the ring R. Then the Teichmüller estimate [Vä$_2$, 11.9] gives $M(\Gamma_R)\geq q_1$. Hence

$$q_1\leq KM(\Gamma_{R'})=K\omega_{n-1}\left(\ln\frac{\alpha}{\beta}\right)^{1-n},$$

which gives the desired bound $\alpha\leq H_1\beta$ with $H_1=H_1(v)$.

Case 3. $\delta(x,x_0)\geq 2r$ where $x_0=f^{-1}(y_0)$. If $y_0=\infty$, then $x_0=\infty$, and this is the general case. Join x' and b' by an arc $C_0\subset D'$ with $d(C_0)<2\beta$. Join a' to y_0 by a carrot $\operatorname{car}_d(E,c)\subset D'$. For $y\in E\setminus\{a',y_0\}$ set $\sigma(y)=d(E[a',y])$. Then $B(y,\sigma(y)/c_1)\subset D'$. We consider two subcases.

Subcase 3a. There is $y\in E$ with $|y-x'|<\sigma(y)/2c_1$. Now $\tau'>\sigma(y)/2c_1$. Let $\theta=\theta_K^n$: $[0,1)\to[0,\infty)$ be the well-known distortion function for QC maps [Vä, 18.1], and set $t_0=\theta^{-1}(1/2)$. If $|b'-x'|\leq t_0\tau'$, then $|b-x|\leq\tau/2$, which implies $r\leq\tau/2$, and we have Case 1. Assume that $|b'-x'|>t_0\tau'$. Now $[x',y]\cup E[y,a']$ joins x' and a' in D', and hence

$$\alpha\leq|x'-y|+\sigma(y)\leq\sigma(y)/2c_1+\sigma(y)<2\sigma(y).$$

Since

$$\beta\geq|b'-x'|\geq t_0\tau'>t_0\sigma(y)/2c_1,$$

we obtain $\alpha/\beta\leq 4c_1/t_0$.

Subcase 3b. $|y-x'|\geq\sigma(y)/2c_1$ for all $y\in E$. Now $C_0\cap E=\emptyset$. Consider the path families $\Gamma'=\Delta(C_0,E;D')$ and $\Gamma=f^{-1}\Gamma'$. By Lemma 2.13 we obtain

$$d(f^{-1}C_0)=\delta(f^{-1}C_0)\geq\delta(x,b)=r,$$

$$d(f^{-1}E)=\delta(f^{-1}E)\geq\delta(x_0,a)\geq\delta(x_0,x)-\delta(a,x)\geq 2r-r=r,$$

$$\delta(f^{-1}C_0,f^{-1}E)\leq\delta(a,x)\leq r.$$

Hence the pair $(f^{-1}C_0, f^{-1}E)$ is 1-standard in D. Since D is ϕ-broad, we have $M(\Gamma)\geq\phi(1)$.

We next show that the number

$$\delta_0 = \inf\left\{\frac{d(|\gamma|)}{d(C_0)}: \gamma\in\Gamma'\right\}$$

is bounded by a constant M_2. We may assume that $\delta_0>2$. Then each $\gamma\in\Gamma'$ meets the spheres $S(x', d(C_0))$ and $S(x', \delta_0 d(C_0)/2)$, which implies

$$M(\Gamma')\leq\omega_{n-1}\left(\ln\frac{\delta_0}{2}\right)^{1-n}.$$

Since $M(\Gamma')\geq M(\Gamma)/K\geq\phi(1)/K$, this yields $\delta_0\leq M_2$.

Since $d(C_0)<2\beta$, there is $\gamma\in\Gamma'$, with $d(|\gamma|)\leq 2M_2\beta$. Let $y\in|\gamma|\cap E$. Then

$$\sigma(y)/2c_1\leq|y-x'|\leq d(|\gamma|)+d(C_0)<2M_2\beta+2\beta,$$

and hence

$$\alpha\leq d(C_0)+d(|\gamma|)+\sigma(y)<2\beta+2M_2\beta+4c_1(M_2+1)\beta=M_3\beta.$$

This completes the proof of Case 3.

In Cases 4 and 5 we assume that $y_0\neq\infty$. Then $y_0\in D', x_0\in D$ and $d(A)\leq c_2 d(x_0, \partial D)$. Using an auxiliary similarity we may assume that $d(y_0, \partial D')=1$. For every $y\in fA$ there is a carrot $\mathrm{car}_d(E, c)$ joining y to y_0. Then $B(y_0, d(E)/c_1)\subset D'$, which implies $d(E)/c_3\leq 1$ and thus $\delta'(y, y_0)\leq c_1$. Consequently, we have always

$$\alpha\leq 2c_1. \tag{2.21}$$

Case 4. $|x'-y_0|<1/2$. As usual, we let $L(x,f,r)$ and $l(x,f,r)$ denote the supremum and infimum of $|f(z)-f(x)|$ over $z\in S(x,r)\cap D$. Writing $r_0=l(x', f^{-1}, 1/2)$ we have $\bar{B}(x, r_0)\subset D$. If $r\leq r_0$, then $\delta(a,x)\leq r$ implies $a\in\bar{B}(x, r_0)$, and hence $a'\in\bar{B}(x', 1/2)\subset D$, which yields $\alpha=|a'-x'|\leq 1/2$, and we are thus in Case 2. Choosing $a_0, b_0\in S(x, r_0)$ with $|f(a_0)-x'|=L(x,f,r_0)$ and $|f(b_0)-x'|=l(x,f,r_0)$ we have Case 2 also for the triple (x, a_0, b_0). Hence

$$1/2=L(x,f,r_0)\leq H_1 l(x,f,r_0)$$

with $H_1=H_1(v)$. If $r>r_0$, then

$$\beta \geqslant l(x,f,r) > l(x,f,r_0) \geqslant 1/2H_1,$$

and hence (2.21) gives $\alpha/\beta \leqslant 4c_1H_1$.

Case 5. $\delta(x,x_0) \leqslant 2r$ and $|x'-y_0| \geqslant 1/2$. We may assume that $\beta < 1/8$, since otherwise (2.21) gives $\alpha/\beta \leqslant 16c_1$. Join x' and b' by an arc $C_0 \subset D'$ with $d(C_0) < 2\beta$, and set $C_1 = \bar{B}(y_0, 1/4)$. We consider the path families $\Gamma' = \Delta(C_0, C_1; D')$ and $\Gamma = f^{-1}\Gamma'$. Since $C_0 \subset \bar{B}(x', 2\beta)$ and $d(x', C_1) \geqslant 1/4$, we have

$$M(\Gamma') \leqslant \omega_{n-1}\left(\ln\frac{1}{8\beta}\right)^{1-n}. \tag{2.22}$$

Since in view of Lemma 2.13,

$$d(f^{-1}C_0) = \delta(f^{-1}C_0) \geqslant \delta(b,x) = r,$$

$$d(f^{-1}C_1) \geqslant l(y_0, f^{-1}, 1/4) \geqslant \theta^{-1}(1/4)\, d(x_0, \partial D) \geqslant \theta^{-1}(1/4)\, \delta(A)/c_2$$
$$\geqslant \theta^{-1}(1/4)\, \delta(b,x)/c_2 = r/M_4,$$

$$\delta(f^{-1}C_0, f^{-1}C_1) \leqslant \delta(x,x_0) \leqslant 2r,$$

the pair $(f^{-1}C_0, f^{-1}C_1)$ is $2M_4$-standard in D. Since D is ϕ-broad, we obtain $M(\Gamma) \geqslant \phi(2M_4)$. Since $M(\Gamma) \leqslant KM(\Gamma')$, this and (2.22) yield $\beta \geqslant q_3$. By (2.21) we have $\alpha/\beta \leqslant 2c_1/q_3$. □

2.23. *Finite connectedness.* We recall that a domain $D \subset \dot{\mathbf{R}}^n$ is finitely connected at a boundary point b if b has arbitrarily small neighborhoods U such that $U \cap D$ has only a finite number of components. Equivalently [Vä$_2$, 17.7], each neighborhood U contains a neighborhood V such that V meets only a finite number of components of $U \cap D$. If this number of components is one, D is locally connected at b.

A somewhat stronger form of the following result will be proved in [NV, 2.18]:

2.24. Lemma. *A John domain $D \subset \mathbf{R}^n$ is finitely connected on the boundary. An unbounded John domain is locally connected at* ∞.

3. Prime ends

3.1. Suppose that $f\colon B^n \to D$ is a QC map. Then f can be extended to a homeomorphism $f^*\colon \bar{B}^n \to D^*$ where D^* is the prime end compactification of D, obtained by adding the set

∂^*D of prime ends to D. This idea of Carathéodory has been extended from the plane to higher dimensions by Zorich [Zo] and by Näkki [Nä]. We present a simple self-contained version, which is valid in the special case where D is finitely connected on the boundary; see Section 2.23.

Suppose that the domain $D\subset\mathbf{R}^n$ is finitely connected on the boundary. An endcut of D is a path $\alpha:[a,b)\to D$ such that $\alpha(t)\to z\in\partial D$ as $t\to b$. We write $z=h(\alpha)$. A subendcut of α is a restriction to a subinterval $[a_1,b]$. If U is a neighborhood of $h(\alpha)$, there is a unique component $A(U,\alpha)$ of $U\cap D$ containing a subendcut of α. Two endcuts α and β are equivalent, written $\alpha\sim\beta$, if $h(\alpha)=h(\beta)$ and if $A(U,\alpha)=A(U,\beta)$ for every neighborhood U of $h(\alpha)$. The equivalence class $[\alpha]$ of α is a *prime end* of D, and their collection ∂^*D is the prime end boundary of D. We write $D^*=D\cup\partial^*D$. There is a natural impression map $i_D:D^*\to\bar{D}$, defined by $i_D([\alpha])=h(\alpha)$ for $[\alpha]\in\partial^*D$ and by $i_D|D=\text{id}$. If D is locally connected at a point $b\in\partial D$, $i_D^{-1}(b)$ consists of a single point, which is often identified with b. In particular, if ∂D is homeomorphic to S^{n-1}, we can identify $\partial^*D=\partial D$.

Suppose that $f:B^n\to D$ is QC. By [Vä$_1$, 17.10, 17.14] f has a continuous extension $\bar{f}:\bar{B}^n\to\bar{D}$. Every point-inverse $\bar{f}^{-1}(y)$ is totally disconnected. Indeed, if $E\subset\bar{f}^{-1}(y)$ is a nondegenerate continuum, the family of all endcuts α of B^n with $h(\alpha)\in E$ has infinite modulus while its image is of modulus zero.

If α is an endcut of B^n, $f\alpha$ is an endcut of D. We show that $h(\alpha)=h(\beta)$ if and only if $f\alpha\sim f\beta$. If $h(\alpha)=h(\beta)=b$ and if U is a neighborhood of $h(f\alpha)=h(f\beta)=\bar{f}(b)$, then there is $r>0$ such that $f[B^n\cap B^n(b,r)]$ is contained in a component of $U\cap D$ and contains subendcuts of both $f\alpha$ and $f\beta$. Hence $f\alpha\sim f\beta$. Conversely, let $f\alpha\sim f\beta$ and suppose that $h(\alpha)\neq h(\beta)$. Then $h(f\alpha)=h(f\beta)=y$. Since $\bar{f}^{-1}(y)$ is totally disconnected, there is a compact set $F\subset\bar{B}^n\setminus\bar{f}^{-1}(y)$ separating $h(\alpha)$ and $h(\beta)$ in $\bar{B}^n$. We may assume that $|\alpha|\cap F=\emptyset=|\beta|\cap F$. Choose a connected neighborhood U of y such that $U\cap\bar{f}F=\emptyset$. Since $f[F\cap D]$ separates $f\alpha$ and $f\beta$ in D, $A(U,f\alpha)\neq A(U,f\beta)$, a contradiction.

If β is an endcut of D, then $\partial B^n\cap\operatorname{cl}f^{-1}|\beta|$ is a connected set in $\bar{f}^{-1}(h(\beta))$, hence a point. Thus $f^{-1}\beta$ is an endcut of B^n. It follows that f has a unique bijective extension $f^*:\bar{B}^n\to D^*$ satisfying $f^*([\alpha])=[f\alpha]$ and hence $i_Df^*=\bar{f}$.

If $\alpha:[a,b)\to D$ is an endcut of D, we say that α joins $\alpha(a)$ and $[\alpha]=u\in\partial^*D$. Similarly, an open path α in D joins elements $u,v\in\partial^*D$ if α has subpaths representing u and v. We can then extend the definition of the internal distance $\delta_D(a,b)$ (see Section 2.12) to all a,b in $Q=D^*\setminus i_D^{-1}(\infty)$. It is easy to see that δ_D is a metric of Q. We show that δ_D is consistent with the topology of Q, that is, f^* defines a homeomorphism $f^{*-1}Q\to Q$ in δ_D.

Let $u\in Q\cap\partial^*D$, set $b=f^{*-1}(u)$, and let $\varepsilon>0$. Since $\bar{f}$ is continuous, there is $U=\bar{B}^n\cap B^n(b,r)$ such that $\bar{f}U\subset B^n(i_D(u),\varepsilon/2)$. Then $\delta_D(\bar{f}(x),u)<\varepsilon$ for all $x\in U$. Hence f^* is continuous at b in δ_D. Next let U be as above, and choose a compact set $F\subset\bar{B}^n\setminus\bar{f}^{-1}(i_D(u))$ separating b and $S(b,r)\cap\bar{B}^n$ in $\bar{B}^n$. Then $d(\bar{f}F,i_D(u))=q>0$. Since $\delta_D(y,u)<q$ implies $f^{*-1}(y)\in U$, f^{*-1} is continuous at u in δ_D.

3.2. *Cylindrical domains.* Let G be a domain in $\mathbf{R}^n$ and let $D=G\times\mathbf{R}^1\subset\mathbf{R}^{n+1}$. We assume that G is finitely connected on the boundary. Clearly D has also this property. We shall derive a relation between the prime ends of G and D.

Suppose first that G is bounded. If α is an endcut of G and if $t\in\mathbf{R}^1$, then $\alpha_t(s)=(\alpha(s),t)$ defines an endcut α_t in D with $h(\alpha_t)=(h(\alpha),t)$. We obtain a natural injective map $j\colon G^*\times\mathbf{R}^1\to D^*$ with $j|G\times\mathbf{R}^1=\mathrm{id}$ and $j([\alpha],t)=[\alpha_t]$. Moreover, the image $\mathrm{im}\,j$ is the set $Q_D=D^*\setminus i_D^{-1}(\infty)$.

The metric δ_G of G^* and the euclidean metric of $\mathbf{R}^1$ define the product metric

$$\varrho((x,t),(x',t'))=\delta_G(x,x')+|t-t'| \tag{3.3}$$

in $G^*\times\mathbf{R}^1$. We show that j satisfies the bilipschitz condition

$$\varrho(z,z')/2\leq\delta_D(j(z),j(z'))\leq 2\varrho(z,z') \tag{3.4}$$

for all $z=(x,t)$, $z'=(x',t')$ in $G^*\times\mathbf{R}^1$.

Let $P_1\colon D\to G$ and $P_2\colon D\to\mathbf{R}^1$ be the natural projections. If α joins $j(z)$ and $j(z')$ in D, then $P_1\alpha$ joins x and x' in G, and hence $\delta_G(x,x')\leq d(P_1|\alpha|)\leq d(|\alpha|)$, which implies $\delta_G(x,x')\leq\delta_D(j(z),j(z'))$. Furthermore, $|t-t'|\leq d(P_2|\alpha|)\leq d|\alpha|$, and hence $|t-t'|\leq\delta_D(j(z),j(z'))$. The first inequality of (3.4) follows.

Next assume that β joins x and x' in G. Then $j(z)$ and $j(z')$ can be joined by a path α consisting of subpaths of β_t and $\beta_{t'}$ and of a vertical line segment of length $|t-t'|$. We have

$$\delta_D(j(z),j(z'))\leq d(|\alpha|)\leq 2d(|\beta|)+|t-t'|,$$

which yields the second inequality of (3.4). The set $D^*\setminus Q_D=i_D^{-1}(\infty)$ clearly consists of two elements, represented by endcuts $\alpha_1,\alpha_2\colon[0,\infty)\to D$, defined by $\alpha_1(t)=(x_0,t)$, $\alpha_2(t)=(x_0,-t)$ where $x_0\in G$ is arbitrary. We set $[\alpha_1]=+\infty$, $[\alpha_2]=-\infty$. Then D^* can be identified with $(G^*\times\mathbf{R}^1)\cup\{-\infty,+\infty\}$.

Next let G be unbounded. Write $Q_G=G^*\setminus i_G^{-1}(\infty)$. As above, we obtain a natural bilipschitz map $j\colon Q_G\times\mathbf{R}^1\to D^*$. Now D is locally connected at ∞, and $D^*\setminus\mathrm{im}\,j$ consists of the single point $\infty=i_D^{-1}(\infty)$. We can thus identify $D^*=(Q_G\times\mathbf{R}^1)\cup\{\infty\}$.

4. Chord-arc conditions

4.1. *The* CA *condition.* We first recall the ordinary chord-arc condition. Let (X, d) be a metric space, let $\dot{X}=X\cup\{\infty\}$ be its one-point extension, and let $A\subset\dot{X}$ be a Jordan curve (topological circle). Suppose that A is locally rectifiable, that is, every compact subarc of $A\setminus\{\infty\}$is rectifiable. If $a, b\in A\setminus\{\infty\}$, we let $\sigma(a, b)$ denote the length of the shorter component of $A\setminus\{a, b\}$. If $c\geq 1$ and if

$$\sigma(a, b)\leq cd(a, b)$$

for all finite $a, b\in A$, we say that A is a *c-chord-arc* curve, or briefly, A is c-CA. Equivalently, A is CA if and only if A is a bilipschitz image of S^1 or $\dot{\mathbf{R}}^1$.

4.2. *The* ICA *condition.* We next recall the internal chord-arc condition from [Vä$_7$]. Let D be a simply connected proper subdomain of $\mathbf{R}^2$ and let D be finitely connected on the boundary. Then the prime end boundary ∂^*D of D is Jordan curve. If α is as subarc of ∂^*D, the impression $i_D|\alpha$ is a path in $\dot{\mathbf{R}}^2$ and has a well-defined length $l(\alpha)$, possibly infinite, called the length of α. Suppose that $i_D(u)=\infty$ for at most one $u\in\partial^*D$, then also written as ∞, and that $l(\alpha)<\infty$ for every compact subarc of $\partial^*D\setminus\{\infty\}$. Then ∂^*D is said to be locally rectifiable. If $u, v\in\partial^*D\setminus\{\infty\}$, we let $\sigma_D(u, v)$ denote the length of the shorter component of $\partial^*D\setminus\{u, v\}$. Let $\delta_D(u, v)$ be the internal distance as in Section 3.1. If $c\geq 1$ and if

$$\sigma_D(u, v)\leq c\delta_D(u, v) \tag{4.3}$$

for all $u, v\in\partial^*D\setminus\{\infty\}$, we say that D satisfies the *internal c-chord-arc* condition, or briefly, D is c-ICA.

In [Vä$_7$] we used a slightly different definition where δ_D was replaced by the metric λ_D (see Section 2.8). Since $\delta_D\leq\lambda_D$, (4.3) implies the c-ICA condition of [Vä$_7$]. As noted in [Vä$_7$, 2.6], the converse is also true, up to the constants. However, the converse is not needed in this paper. The ICA condition has also been considered in [La] and in [Po].

We next show that the ICA condition is a special case of the general CA condition:

4.4. Lemma. *Let D be a simply connected proper subdomain of* $\mathbf{R}^2$ *and let D be finitely connected on the boundary. Then D is c-*ICA *if and only if ∂^*D is c-*CA *in the metric δ_D.*

Proof. Suppose that $i_D(u)=\infty$ for at most one $u\in\partial^*D$, also written as ∞. It suffices to show that if $\alpha\subset\partial^*D\setminus\{\infty\}$ is a compact arc, then its length $l_\delta(\alpha)$ in the metric δ_D is equal to $l(\alpha)$. Let $u, v \in \partial^* D \setminus \{\infty\}$. If a path β joins u and v in D, then $|i_D(u) - i_D(v)| \leq d(|\beta|)$. Hence $|i_D(u) - i_D(v)| \leq \delta_D(u, v)$. It follows that $l(\alpha) \leq l_\delta(\alpha)$.

Conversely, if $\gamma \subset \partial^* D \setminus \{\infty\}$ is an arc with end points u, v and if $\varepsilon > 0$, then there is a path β joining u and v in $i_D \gamma + B^n(\varepsilon)$. Since

$$d(|\beta|) \leq d(i_D \gamma) + 2\varepsilon \leq l(\gamma) + 2\varepsilon,$$

we have $\delta_D(u, v) \leq l(\gamma)$. If the points $u_0, \ldots, u_k$ divide α to subarcs $\alpha_1, \ldots, \alpha_k$, we obtain

$$\sum_{j=1}^{k} \delta_D(u_j, u_{j-1}) \leq \sum_{j=1}^{k} l(\alpha_j) = l(\alpha),$$

and thus $l_\delta(\alpha) \leq l(\alpha)$. □

5. The main theorem

5.1. *Terminology.* A homeomorphism $f: D \to D'$ between domains in $\mathbf{R}^n$ is of *L-bounded length distortion*, abbreviated L-BLD, if

$$l(\alpha)/L \leq l(f\alpha) \leq L l(\alpha)$$

for every path α in D, or equivalently, each point in D has a neighborhood in which f is L-bilipschitz. More general discrete open BLD maps are considered in [MV]. An L-BLD homeomorphism is K-QC with $K = L^{n-1}$. Compared with QC maps, the BLD maps have a pleasant behavior in cartesian products; the product of two L-BLD maps is again L-BLD.

Suppose that for each $c \geq 1$ there are given conditions $A(c)$ and $B(c)$. We say that A and B are *equivalent up to constants* if for each $c \geq 1$ there is $c_1 \geq 1$ such that $A(c) \Rightarrow B(c_1)$ and $B(c) \Rightarrow A(c_1)$. The parameter can also be written as K or L.

We next give the main result of this paper:

5.2. THEOREM. *Let G be a simply connected proper subdomain of $\mathbf{R}^2$. Then the following conditions are equivalent up to constants:*

(1) *There is a K-QC map $B^3 \to G \times \mathbf{R}^1$.*

(2) *G is finitely connected on the boundary and c-ICA.*

(3) *There is an L-BLD homeomorphism $G_0 \to G$ where G_0 is either a round disk or a half plane.*

(4) *There is an L-BLD homeomorphism $G_0 \times \mathbf{R}^1 \to G \times \mathbf{R}^1$, where G_0 is as in* (3).

Proof. The implication (2)⇒(3) was proved in [Vä$_7$, 3.4, 3.7, 3.8, 3.11]. In fact, it was proved in the seemingly stronger form in which the ICA condition was given in the metric λ_D instead of δ_D. The unbounded case had been proved earlier by Latfullin [La], which was unfortunately overlooked in [Vä$_7$].

If $f\colon G_0 \to G$ is the L-BLD homeomorphism given by (3), then $f\times \mathrm{id}\colon G_0\times \mathbf{R}^1 \to G\times \mathbf{R}^1$ is L-BLD, and hence (4) is true. Since $G_0\times\mathbf{R}^1$ is QC homeomorphic to B^3, (4) clearly implies (1). It remains to show that (1) implies (2).

Replacing B^3 by its Möbius image H^3 we assume that there is a K-QC map $f\colon H^3\to G\times\mathbf{R}^1 = D$. We first show that G is a c_1-John domain. Here and later, we let $c_1, c_2, \dots$ and $q_1, q_2, \dots$ denote constants depending only on K with $c_j \geq 1$ and $0<q_j<1$. Let $x_0\in\mathbf{R}^2$, let $r>0$, and suppose that $B^2(x_0, r)\setminus G$ has two components E_1, E_2 meeting $B^2(x_0, r/c)$. By [NV,4.5] it suffices to find an upper bound $c\leq c_2$. Now $E_1\times\{0\}$ and $E_2\times\{0\}$ are contained in different components of $B^3((x_0,0),r)\setminus D$. Thus [GV, Theorem 6.1] gives an estimate $c\leq e^{MK}$ where M is a universal constant. Hence G is a c_1-John domain. From 2.24 it follows that G is finitely connected on the boundary. We divide the rest of the proof into two cases:

Case 1. G is bounded. We extend f to a homeomorphism $f^*\colon \bar{H}^3\to D^*$; see Section 3.1. We identify $D^* = (G^*\times\mathbf{R}^1)\cup\{-\infty, +\infty\}$ as in Section 3.2. Performing an auxiliary Möbius transformation of $\mathbf{R}^3$ we may assume that $f^*(0) = -\infty$ and $f^*(\infty) = +\infty$.

We may assume that G has the c_1-carrot property in G with center $x_0\in G$; see Lemma 2.4(a). We may normalize $d(x_0, \partial G) = 1$. Then

$$\delta_G(x, x_0) < c_1 \tag{5.3}$$

for all $x\in G$. For $r>0$ set $S_+(r) = H^3\cap S^2(r)$. The projection of $fS_+(r)$ into the x_3-axis is an interval or a point. Let $r_0\leq r_1$ be its end points. An easy modification of the proof of [GV, Lemma 8.1] gives the estimate

$$r_1 - r_0 < c_3 = (Km(G)/\psi(1))^{1/2} \tag{5.4}$$

where $\psi(1)$ is the same universal constant as in [GV].

Choose positive numbers $r<r'$ such that $r_1 = 0$, $r_0' = 1$. We may assume that $r=1$. We want to apply Theorem 2.20 to the map $f\colon H^3\to D$ with $A = H^3\cap(B^3(r')\setminus\bar{B}^3(r))$. Choose $z_0\in H^3\cap S^2$ such that $y_0 = f(z_0)$ is of the form (x_0, t_0); then $r_0\leq t_0\leq 0$. We show that fA has the c_4-carrot property in D with center y_0 and with $c_4 = 1 + c_1 + 2c_3$. Assume that $y_1 = (x_1, t_1)\in fA\subset G\times[r_0, r_1']$. Join x_1 to x_0 by a 2-dimensional carrot $\mathrm{car}_d(E_0, c)\subset G$. Set

$$E_1 = E_0\times\{t_1\}, \quad E_2 = \{x_0\}\times[t_0, t_1], \quad E = E_1\cup E_2.$$

Then E is an arc joining y_1 to y_0. We show that $\mathrm{car}_d(E, c_4) \subset D$.

Let $y \in E$ and set $\delta(y) = d(E[y_1, y])$. We must verify that $B^3(y, \delta(y)/c_4) \subset D$. If $y \in E_1$, we can write $y = (x, t_1)$ with $B^2(x, \delta(y)/c_1) \subset G$. Hence

$$B^3(y, \delta(y)/c_4) \subset B^3(y, \delta(y)/c_1) \subset G \times \mathbf{R}^1 = D.$$

If $y \in E_2$, we write $y = (x_0, t)$. Then (5.4) implies

$$\delta(y) \leqslant \delta(E) \leqslant \delta(E_1) + r_1' - r_0 \leqslant \delta(E_1) + 1 + 2c_3.$$

Since $d(x_0, \partial G) = 1$, we have $\delta(E_1) \leqslant c_1$, and hence $\delta(y) \leqslant c_4$. Thus

$$B^2(y, \delta(y)/c_4) \subset B^3(y, 1) \subset B^2(x_0, 1) \times \mathbf{R}^1 \subset D.$$

It follows that fA has the c_4-carrot property in D.

We still need an upper bound for $d(A)/d(z_0, \partial H^3)$. Write $s = d(z_0, \partial H^3)$ and observe that $d(A) = 2r'$. We thus have to find an estimate

$$r' \leqslant c_5 s. \tag{5.5}$$

Let Γ be the family of paths joining $S_+(1)$ and $S_+(r')$ in A. Then

$$M(\Gamma) = 2\pi(\ln r')^{-2}.$$

Since $B(x_0, 1) \subset G \subset B(x_0, c_1)$, [Vä$_1$, 7.2] and (5.4) easily give the estimates

$$\frac{\pi}{(1+2c_3)^2} \leqslant M(f\Gamma) \leqslant \pi c_1^2.$$

Since f is K-QC, we obtain

$$1 + q_1 \leqslant r' \leqslant c_6.$$

Hence (5.5) reduces to

$$s \geqslant q_2. \tag{5.6}$$

We may assume that $s < q_1$. Let C_0 be the vertical segment of length s joining z_0 to ∂H^3, let $C_1 = S_+(r')$, and let $\Gamma_1 = \Delta(C_0, C_1; H^3)$. Then

$$M(\Gamma_1) \leqslant 4\pi\left(\ln \frac{q_1}{s}\right)^{-2}. \tag{5.7}$$

Set $r''=1-q_1$. Arguing as above with path families we get the estimate $r_0''>-c_7$. Then fC_0 lies between the planes $x_3=-c_7$ and $x_3=0$. Moreover, fC_1 lies between the planes $x_3=1$ and $x_3=r_1'<1+c_3$. Let Z be the cylinder $B^2(x_0,1)\times(-c_7,1+c_3)\subset D$. Then there are continua $C_0'\subset Z\cap fC_0$ and $C_1'\subset Z\cap fC_1$ with diameters at least 1/2. As a quasiball Z is c_8-QED, and hence [GM, 2.6] gives an estimate

$$M(f\Gamma_1)\geq M(\Delta(C_0',C_1';Z))\geq M(\Delta(C_0',C_1';\mathbf{R}^n))/c_8\geq q_3.$$

Since $M(f\Gamma_1)\leq KM(\Gamma_1)$, this and (5.7) yield (5.6).

We have now verified all hypotheses of Theorem 2.20. Thus $f|A$ is η_1-QS with respect to the euclidean metric of A and the metric δ_D of fA. Here and later, we let $\eta_1,\eta_2,\ldots$ denote homeomorphisms $\eta_j\colon[0,\infty)\to[0,\infty)$ depending only on K. Let F be the closure of fA in D^*. Then $f^{*-1}|F$ is η_2-QS in the metric δ_D. By Section 3.2, the metric δ_D is 2-bilipschitz equivalent to the product metric ϱ of $G^*\times\mathbf{R}^1$. Hence the restriction $f_1\colon\partial^*G\times[0,1]\to\mathbf{R}^2$ of f^{*-1} is η_3-QS in ϱ. From [Vä$_5$, 5.6] and from (5.3) it follows that the Jordan curve ∂^*G is c_9-CA in δ_G. By Lemma 4.4 this means that G is c_9-ICA.

Case 2. G is unbounded. We again extend f to a homeomorphism $f^*\colon\bar{H}^3\to D^*$. We use the identification $D^*=(Q_G\times\mathbf{R}^1)\cup\{\infty\}$, $Q_G=G^*\setminus i_G^{-1}(\infty)$, explained in Section 3.2. Since G is a John domain, we can write $i_G^{-1}(\infty)=\infty$ and thus $Q_G=G^*\setminus\{\infty\}$; see Lemma 2.24. We may assume that $f^*(\infty)=\infty$. We want to apply Theorem 2.20 to the map $f\colon H^3\to D$ with $A=H^3$. Suppose that $y=(x,t)\in D$. Since G is c_1-John, there is a 2-dimensional carrot $\mathrm{car}_d(E,2c_1)$ joining x to ∞ in G; see Lemma 2.4. Then $\mathrm{car}_d(E\times\{t\},2c_1)$ joins y to ∞ in D. Hence D has the $2c_1$-carrot property in D. We can thus apply Theorem 2.20 and conclude that f is η_4-QS in δ_D. As in Case 1, this implies that $f_2=f^{*-1}|(\partial^*G\setminus\{\infty\})\times\mathbf{R}^1$ is η_5-QS in the product metric ϱ. From [Vä$_6$, 5.4] it follows that ∂^*G is c_{10}-CA in δ_D, and hence G is c_{10}-ICA. □

6. Dilatation estimates

6.1. *Terminology*. We recall that the outer dilatation $K_O(f)$ of a homeomorphism $f\colon D\to D'$ between domains in $\mathbf{R}^n$ is the infimum of all $K\geq1$ such that

$$M(\Gamma)\leq KM(f\Gamma)$$

for every path family Γ in D. The inner dilatation of f is $K_I(f)=K_O(f^{-1})$. If D is homeomorphic to B^n, the outer coefficient of quasiconformality $K_O(D)$ is the infimum (in fact, minimum) of the numbers $K_O(f)$ over all homeomorphisms $f\colon D\to B^n$. Thus $K_O(D)<\infty$ if and only if D is QC equivalent to a ball.

The exact value of $K_O(D)$ is known for only very few domains D. One of these is the round cylinder $D=B^2\times\mathbf{R}^1$ for which

$$K_O(D)=q_0=\frac{1}{2}\int_0^{\pi/2}(\sin t)^{-1/2}\,dt=\int_0^1(1-t^4)^{-1/2}\,dt=1.31102\ldots. \qquad (6.2)$$

See [GV, Theorem 8.1] and observe that [GV] writes $K_O(f)^2$ for our $K_O(f)$.

In this section we estimate $K_O(D)$ for domains of the form $D=G\times\mathbf{R}^1$, $G\subset\mathbf{R}^2$. Trivially $K_O(D)\geq 1$ for all D. If $K_O(D)=1$, D must be a Möbius image of B^3. This happens precisely when G is a half plane.

We next consider the case where G is bounded. If $K_O(D)<\infty$, Theorem 5.2 implies that $l(\partial^*G)<\infty$. The number

$$b(G)=\frac{l(\partial^*G)^2}{4\pi m(G)}$$

is called the *isoperimetric constant* of G. If $l(\partial^*G)=\infty$ or if $l(\partial^*G)$ is not defined, we set $b(G)=\infty$. By the isoperimetric inequality, we have always $b(G)\geq 1$, and $b(G)=1$ if and only if G is a round disk.

We shall prove the following generalization of the round case mentioned above:

6.3. THEOREM. *For every bounded simply connected domain $G\subset\mathbf{R}^2$ we have*

$$K_O(G\times\mathbf{R}^1)\geq q_0\,b(G)^{1/4},$$

where q_0 is the constant in (6.2). Hence $K_O(G\times\mathbf{R}^1)\geq q_0$ for all bounded G, and $K_O(G\times\mathbf{R}^1)=q_0$ if and only if G is a round disk.

Proof. We try to rewrite the proof of the round case [GV, Theorem 8.1] in the more general setting. Set $D=G\times\mathbf{R}^1$, and let $f\colon H^3\to D$ be a QC map with $K_O(f^{-1})=K_I(f)=K$. By Theorem 5.2, G is finitely connected on the boundary, $l(\partial^*G)<\infty$, and G is c-ICA for some $c=c(K)$. We must show that $K\geq q_0\,b(G)^{1/4}$. Let again

$$f^*\colon \bar{H}^3\to D^*=(G^*\times\mathbf{R}^1)\cup\{-\infty,+\infty\}$$

be the homeomorphic extension of f; see Section 3. We may assume that $f^*(0)=-\infty$, $f^*(\infty)=+\infty$. Let $a<b$ be real numbers, let Γ' be the family of all vertical segments $\{y\}\times(a,b)$, $y\in\partial^*G$, and let $\Gamma=f^{*-1}\Gamma'$. We shall prove in Section 6.5 the inequality

$$M_2(\Gamma)\geq\frac{l(\partial^*G)}{K(b-a)}. \qquad (6.4)$$

We remark that if G has a smooth boundary, then (6.4) follows easily from [GV, Theorem 4.3] which implies that the induced boundary map of $H^2\setminus\{0\}$ onto $\partial G\times\mathbf{R}^1$ is K-QC.

We show how the theorem follows from (6.4). For every positive number r let again r_0 and r_1 denote the infimum and supremum of $P_3(f(x))$ over $x\in S_+(r)=S^2(r)\cap H^3$. Let $0<r<s<\infty$, let Z be the positive x_3-axis, and set

$$R=B^3(s)\setminus\bar{B}^3(r),\quad R_0=R\cap\mathbf{R}^2,\quad A=R\cap H^3,\quad E=R\cap Z.$$

Consider the path families

$$\Gamma_1=\Delta(E,R_0;A),\quad \Gamma_2=\Delta(S^2(r),S^2(s);R_0).$$

For $a=r_0, b=s_1$, each member of the family Γ of (6.4) has a subpath in Γ_2; hence

$$M_2(\Gamma_2)\geqslant\frac{l(\partial^*G)}{K(s_1-r_0)}.$$

Applying [Vä$_1$, Theorem 3.4] as in [GV, Lemma 3.7] we obtain the estimate

$$M_2(f\Gamma_1)\geqslant\frac{\pi^{3/2}(s_0-r_1)}{2m(G)^{1/2}}.$$

Since $M(f\Gamma_1)\leqslant KM(\Gamma_1)$, these inequalities yield

$$K^2M(\Gamma_1)\,M_2(\Gamma_2)\geqslant\pi^2b(G)^{1/2}\frac{s_0-r_1}{s_1-r_0}.$$

On the other hand, we have

$$M(\Gamma_1)=\frac{\pi}{2q_0^2}\ln\frac{s}{r},\quad M_2(\Gamma_2)=2\pi\left(\ln\frac{s}{r}\right)^{-1};$$

see [GV, Lemma 3.8]. Hence

$$K^2\geqslant q_0^2\,b(G)^{1/2}\frac{s_0-r_1}{s_1-r_0}.$$

As $s\to\infty$, this and (5.4) give $K\geqslant q_0\,b(G)^{1/4}$ as desired.

6.5. *Proof of* (6.4). We shall use an elaboration of the argument in [GV, p. 30]. Fix $v_0\in\partial^*G$ and write

$$D_2=(\partial^*G\setminus\{v_0\})\times(a,b),\quad D_1=f^{*-1}D_2.$$

For $\lambda=l(\partial^*G)$ let $\phi:\partial^*G\setminus\{v_0\}\to(0,\lambda)$ be a length-preserving map, that is, $l(\phi^{-1}(0,t))=t$ for $0<t<\lambda$. Let D_3 be the rectangle $(0,\lambda)\times(a,b)\subset\mathbf{R}^2$ and let $g:D_2\to D_3$ be the homeomorphism $\phi\times\text{id}$.

Since G is c-ICA, ϕ is locally c-bilipschitz in δ_G. Hence g is locally $2c$-bilipschitz if D_2 is considered with the product metric ϱ (see (3.2)) and D_3 with the euclidean metric. Let $f_1:D_1\to D_2$ be the homeomorphism defined by f^*. The proof of Theorem 5.2 shows that f_1 is η-QS in ϱ with $\eta=\eta_K$. Hence $h=gf_1:D_1\to D_3$ is locally η_1-QS with $\eta_1=4c^2\eta$, and thus h is K_1-QC with $K_1=4c^2\eta(1)$. We shall show that h is, in fact, K-QC. This will imply (6.4), since the family Γ_3 of all vertical segments $\{t\}\times(a,b)$, $0<t<\lambda$, has modulus $\lambda/(b-a)$ and since Γ consists of $h^{-1}\Gamma_3$ and the single arc $f_1^{-1}[\{v_0\}\times(a,b)]$.

As before, we let $i_G:G^*\to\bar{G}$ and $i_D:D^*\to\bar{D}$ denote the impression maps. The path $\psi=i_G\phi^{-1}:(0,\lambda)\to\mathbf{R}^2$ is parametrized by the arc length, and thus $|\psi'(t)|=1$ a.e. It follows that the derivative of the map $i_Dg^{-1}=\psi\times\text{id}$: $D_3\to\mathbf{R}^3$ is a linear isometry $A(z)$: $\mathbf{R}^2\to\mathbf{R}^3$ for almost every $z\in D_3$.

Fix $x_0\in D_1$ such that h is differentiable at x_0 with a nonzero jacobian and such that $A(z_0)$ exists and is a linear isometry for $z_0=h(x_0)$. Then the linear map $T=A(z_0)\,h'(x_0)$ is the derivative of the map $f_0=i_Df_1:D_1\to\mathbf{R}^3$ at x_0. Observe that f_0 is a restriction of the continuous extension $\bar{f}:\bar{H}^3\to\bar{D}$ of f. From Lemma 6.7 below it follows that $|T|\le Kl(T)$. Hence $|h'(x_0)|\le Kl(h'(x_0))$. Since this is true for almost every $x_0\in D_1$, h is K-QC. □

6.6. *Derivative at a boundary point.* At the end of Section 6.5 we needed the estimate $|T|\le Kl(T)$, where $l(T)$ is the minimum of $|Tx|$ over $x\in S^{n-2}$ and T is the derivative of the boundary map induced by the QC map f. We prove the corresponding result for an arbitrary dimension, since it may be useful also elsewhere. Suppose that $2\le p\le n$ and that $T:\mathbf{R}^p\to\mathbf{R}^n$ is a linear map. Let J_pT be the p-measure TB^p. If $J_pT>0$, the inner and outer dilatations of T are

$$H_I(T)=\frac{J_pT}{l(T)^p},\quad H_O(T)=\frac{|T|^p}{J_pT}.$$

If $p=2$, we have $H_I(T)=H_O(T)=|T|/l(T)$.

If $f:D\to D'$ is a QC map, we have

$$K_I(f)=\operatorname*{ess\,sup}_{x\in D}H_I(f'(x)),\quad K_O(f)=\operatorname*{ess\,sup}_{x\in D}H_O(f'(x)).$$

If $f:H^n\to H^n$ is QC, it induces a boundary map $g:\dot{\mathbf{R}}^{n-1}\to\dot{\mathbf{R}}^{n-1}$, for which $K_I(g)\le K_I(f)$, $K_O(g)\le K_O(f)$. This was proved in [GV, Lemma 4.6] for $n=3$ and in [Ge$_1$, Corollary,

p. 95] for all $n\geq 3$. We generalize the result to the case where the image domain is arbitrary:

6.7. LEMMA. *Suppose that* $f:\bar{B}^n(r)\cap\bar{H}^n$ *is a continuous map, that* $f|B^n(r)\cap H^n$ *is* QC *and that* $g=f|B^{n-1}(r)$ *is differentiable at the origin with* $J_{n-1}g'(0)>0$. *Then* $H_I(g'(0))\leq K_I(f)$ *and* $H_O(g'(0))\leq K_O(f)$.

Proof. We shall use normal families and the recent surprising local maximum principle of Gehring to reduce the lemma to the case $f: H^n\to H^n$ mentioned above. We may assume that $r=1$, that $f(0)=0$ and that $f'(0)$ maps $\mathbf{R}^{n-1}$ onto itself. Write $D=B^n\cap H^n$ and $T=f'(0)$. Consider the maps $f_j: j\bar{D}\to\mathbf{R}^n$ defined by $f_j(x)=jf(x/j)$. For every $x\in\mathbf{R}^n_+=\bar{H}^n\setminus\{\infty\}$, $f_j(x)$ is defined for large j. We show that the family (f_j) is equicontinuous in $\mathbf{R}^n_+$ in the spherical metric.

Since each $f_j|jD$ omits 0 and ∞, the equicontinuity in H^n follows from the general equicontinuity properties of QC maps [Vä$_2$, 19.3]. Let $x_0\in\mathbf{R}^{n-1}$ be a boundary point. Let $0<r<1$ and let $j>|x_0|+2$. Let $a_j(r)$ and $b_j(r)$ denote the supremum of $|f_j(x)-f_j(x_0)|$ over $x\in H^n\cap\bar{B}(x_0,r)$ and $x\in\mathbf{R}^{n-1}\cap\bar{B}(x_0,2r)$, respectively. By the local maximum principle of Gehring [Ge$_2$, 2.1, 2.10], we have

$$a_j(r)\leq cb_j(r) \tag{6.8}$$

where c depends only on $K(f)$ and n. Since $T=g'(0)$, we can write

$$g(x)=Tx+|x|h(x),\quad |h(x)|\leq\varepsilon(|x|)$$

for some homeomorphism $\varepsilon:[0,\infty)\to[0,\infty)$. If $x\in\mathbf{R}^{n-1}\cap\bar{B}(x_0,2r)$, we have

$$\begin{aligned}|f_j(x)-f_j(x_0|&=|T(x-x_0)+|x|h(x/j)-|x_0|h(x_0/j)|\\&\leq 2|T|r+2(|x_0|+2)\,\varepsilon((|x_0|+2)/j)\\&=2|T|r+\delta(j)\end{aligned}$$

where $\delta(j)\to 0$ as $j\to\infty$. By (6.8) this yields

$$a_j(r)\leq 2c|T|r+c\delta(j),$$

which implies the equicontinuity of (f_j) at x_0.

By Ascoli's theorem, (f_j) has a subsequence converging to a map $F:\mathbf{R}^n_+\to\dot{\mathbf{R}}^n$, uniformly in the spherical metric in compact sets. Since each $f_j|jD$ has the same dilatations as f, $F|H^n$ is either constant or a QC map with $K_I(F)\leq K_I(f)$, $K_O(F)\leq K_O(f)$.

On the other hand, $F|\mathbf{R}^{n-1}$ is the linear map T onto $\mathbf{R}^{n-1}$. It follows that $F|H^n$ is a QC map onto H^n or onto the lower half space. By the aforementioned result of [Ge$_1$], we have $H_I(T)=K_I(T)\leqslant K_I(F)\leqslant K_I(f)$, and similarly for the outer dilatation. □

6.9. *An upper bound.* We next study the sharpness of the bound in Theorem 6.3. For $t\geqslant 1$ let $\varkappa(t)$ be the infimum of the numbers $K_O(G\times\mathbf{R}^1)$ over all bounded domains $G\subset\mathbf{R}^2$ with $b(G)\geqslant t$. Then Theorem 6.3 gives the inequality

$$\varkappa(t)\geqslant q_0 t^{1/4}. \tag{6.10}$$

For $t=1$ this holds as an equality. For $t>1$ we presumably have a strict inequality, since the estimate for $M(f\Gamma_1)$ in the proof of Theorem 6.3 is not necessarily sharp.

To get an upper bound for $\varkappa(t)$ we construct an explicit example. For $s\geqslant 1$ let $g\colon\mathbf{R}^3\to\mathbf{R}^3$ be the linear map $g(x)=(sx_1,x_2,x_3)$. Let G_s be the ellipse gB^2, and let $D_s=G_s\times\mathbf{R}^1=g[B^2\times\mathbf{R}^1]$. Then $K_I(g)=s$ and hence $K_O(D_s)\leqslant sK_O(B^2\times\mathbf{R}^1)=q_0 s$. Setting $\beta(s)=b(G_s)$ we thus have

$$\varkappa(t)\leqslant q_0\beta^{-1}(t). \tag{6.11}$$

For $t=1$ (6.10) and (6.11) give the equality $\varkappa(1)=q_0$. The estimate $l(\partial G_s)>4s$ gives the inequality

$$\varkappa(t)<\pi^2 q_0 t/4. \tag{6.12}$$

It seems reasonable to conjecture that $\varkappa(t)/t$ tends to a finite limit as $t\to\infty$.

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