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On the topology of spaces of holomorphic maps

by

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1. Introduction

Let X and Y be two complex manifolds and form the two spaces $\mathrm{Hol}(X, Y)$ and $\mathrm{Map}(X, Y)$ of respectively holomorphic and continuous maps $X\to Y$, equipped with the compact-open topology.

We will study the inclusion of $\mathrm{Hol}(X, Y)$ into $\mathrm{Map}(X, Y)$ in the case, where X is a Riemann surface and Y is a generalized flag manifold or a loop group.

Let $\mathrm{Hol}_n^*(X, Y)$ and $\mathrm{Map}_n^*(X, Y)$ denote the spaces of based maps of degree n. In [14] G. Segal shows that the inclusion of $\mathrm{Hol}_n^*(X, \mathbf{CP}^m)$ into $\mathrm{Map}_n^*(X, \mathbf{CP}^m)$ is a homology equivalence up to dimension $(n-2g)(2m-1)$, where g is the genus of X. Segal conjectured that a similar statement holds, if $\mathbf{CP}^m$ is replaced by a flag manifold or a Grassmannian, and this was confirmed by M. A. Guest, [7], and F. C. Kirwan, [9].

If G is a compact Lie group, the loop group ΩG has many properties similar to a Grassmannian, see [12]. So it is natural to try to extend Segal's result to the inclusion of $\mathrm{Hol}_n^*(X, \Omega G)$ into $\mathrm{Map}_n^*(X, \Omega G)$, and this is indeed the purpose of this work.

Let $\mathcal{V}_n(X\times\mathbf{CP}^1, X\vee\mathbf{CP}^1, G_\mathbf{C})$ be the space of based isomorphism classes of holomorphic $G_\mathbf{C}$-bundles over $X\times\mathbf{CP}^1$, trivial over the axis $X\vee\mathbf{CP}^1$ and with characteristic class n. In [1] M. F. Atiyah describes how there is an imbedding of $\mathrm{Hol}_n^*(X, \Omega G)$ into $\mathcal{V}_n(X\times\mathbf{CP}^1, X\vee\mathbf{CP}^1, G_\mathbf{C})$.

The main result (Theorem 7.8) is that

$$\lim_{n\to\infty} H_*(\mathcal{V}_n(X\times\mathbf{CP}^1, X\vee\mathbf{CP}^1, G_\mathbf{C})) = H_*(\mathrm{Map}_0^*(X, \Omega G)).$$

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If $X=\mathbf{CP}^1$, then $\mathrm{Hol}_n^*(\mathbf{CP}^1,\Omega G)\hookrightarrow\mathcal{V}_n(\mathbf{CP}^1\times\mathbf{CP}^1,\mathbf{CP}^1\vee\mathbf{CP}^1,G_{\mathbf{C}})$ is a homotopy equivalence and as the methods work equally well for a generalized flag manifold,

$$\lim_{n\to\infty} H_*(\mathrm{Hol}_n^*(\mathbf{CP}^1, Y)) = H_*(\mathrm{Map}_0^*(\mathbf{CP}^1, Y))$$

with Y a generalized flag manifold or a loop group. The degree n might be a multi-index $n=(n_1,\dots,n_r)$ and then $n\to\infty$ means $n_i\to\infty$ for all $i=1,\dots,r$.

Segal's results on projective spaces are stronger. In particular, in each dimension q, the limit $\lim_{n\to\infty} H_q(\mathrm{Hol}_n^*(X,\mathbf{CP}^1))$ is obtained after a finite number of steps. If this result on projective spaces could be proved in the framework of this paper, then the analogous result for loop groups would probably hold.

There is one result in this direction. The induced map on π_0 is an injection if $X=\mathbf{CP}^1$. This gives yet an other proof of the connectivity of certain moduli spaces in algebraic geometry, see [3] and its references.

On the other hand, the method of this paper has the virtue of treating the different target spaces at the same time. The papers [14], [7] and [9] start by proving the result for maps into $\mathbf{CP}^1$, and then use induction to extend the result to the other target spaces.

If D is the open unit disk in $\mathbf{C}$, then the inclusion $\mathrm{Hol}(D, Y)\hookrightarrow\mathrm{Map}(D, Y)$ is a homotopy equivalence. As a surface X can be made by gluing disks together, one could hope to prove that the inclusion $\mathrm{Hol}(X, Y)\hookrightarrow\mathrm{Map}(X, Y)$ is a homotopy equivalence by an induction argument. It would be easy, if the restriction map $\mathrm{Hol}(X, Y)\to\mathrm{Hol}(X', Y)$ was a fibration for a pair $X'\subseteq X$. Unfortunately this is not the case, so we have to be more clever.

A based holomorphic map $X\to\mathbf{CP}^1$ is uniquely determined by its zeros and poles and Segal uses this fact to replace the study of holomorphic maps with the study of configurations of zeros and poles. We will use that a based holomorphic map $X\to\mathbf{CP}^1$ is uniquely determined by its principal parts, and replace the study of holomorphic maps with the study of configurations of principal parts.

As the diffeomorphism group does not act on such configurations, we have to enlarge the space. The 'configuration' space we consider consists of pairs of a complex structure on the underlying real manifold M and a configuration of principal parts in this complex structure. Now the diffeomorphism group acts on the space, but it is no longer a true configuration space, since a global quantity, namely the complex structure, is introduced.

In Sections 2, 3 and 4 the necessary features of complex structures on two dimensional manifolds, flag manifolds and loop groups are described. Most of the material is standard, cf., [6], [12] and [15], so there will be statements without proof or specific references. The main results are Lemma 2.8 and its generalizations Lemma 3.2 and Lemma 4.2.

In Section 5 we introduce the space $\mathcal{M}(M, Y)$ of pairs (f, J), where J is a complex structure on M and f is a J-meromorphic map $M \to Y$. If $\bar{D}$ is the closed unit disk, then we show that $\mathcal{M}(\bar{D}, Y)$ is weakly homotopy equivalent to $\mathrm{Map}(\bar{D}, Y)$.

In Section 6, a principal part of a holomorphic map into Y is defined and we define the space $\mathcal{P}(M, Y)$ of pairs (ξ, J), where J is a complex structure on M and ξ is a configuration of principal parts in this structure. There is a natural map $\mathcal{M}(M, Y) \to \mathcal{P}(M, Y)$, and if $\partial M \neq \varnothing$, then the map is surjective and a weak homotopy equivalence. The most important property of the space $\mathcal{P}(M, Y)$ is that, under certain conditions on an inclusion $M_1 \subseteq M_2$, the restriction map from $\mathcal{P}(M_2, Y)$ to $\mathcal{P}(M_1, Y)$ is a quasifibration. It enables us to get the desired result for a union $M_1 \cup M_2$, if it is known for M_1, M_2 and $M_1 \cap M_2$.

This is used in Section 7, where the results are proved. Starting with the result for $\bar{D}$, we follow the inductive methods of [10]. As long as M is not closed, the relevant restriction maps are quasifibrations, and $\mathcal{M}(M, Y)$ is weak homotopy equivalent to $\mathrm{Map}(M, Y)$. When the manifolds is closed, it is necessary to introduce a stabilized space $\hat{\mathcal{P}}$.

By adding a principal part near infinity, we get a map $\mathcal{P} \to \mathcal{P}$, which increases the degree and $\hat{\mathcal{P}}$ is the telescope of the sequence $\mathcal{P} \to \mathcal{P} \to \mathcal{P} \to \dots$. Now the relevant restriction maps become homology fibrations and we can conclude that $\hat{\mathcal{P}}$ and $\mathrm{Map}^*(M, Y)$ have the same homology type. The next step is to show that if $\mathcal{P}_J$ is the space of configurations of principal parts in a fixed complex structure J, then the inclusion $\mathcal{P}_J \hookrightarrow \mathcal{P}$ is a homotopy equivalence. Finally we show that $\mathcal{P}_{J,n}$ can be identified with $\mathcal{V}_n(X \times \mathbf{CP}^1, X \vee \mathbf{CP}^1, G_{\mathbf{C}})$, where X is M equipped with the complex structure J.

2. Complex structures on two dimensional manifolds

Let M be a compact, connected, oriented two dimensional C^∞-manifold possibly with boundary and corners. Choose a volume form Ω and let F be the subbundle of $\mathrm{End}(TM)$ consisting of endomorphisms A with $A^2 = -1$ and $\Omega(v, Av) \geq 0$ for all $v \in TM$. As the dimension of M is two, the space $\mathcal{C}(M)$ of complex structures on M is the space of smooth sections J in F, equipped with the C^∞-topology. The bundle F has contractible fibers, so $\mathcal{C}(M)$ is contractible. If $J \in \mathcal{C}(M)$, then M_J denotes M equipped with the complex structure J. As the complex structure can vary we will speak of J-holomorphic and J-harmonic functions, maps, forms etc.

Let Diff(M) be the diffeomorphism group of M equipped with the C^∞-topology, then by the results of [4] and [5] we have

LEMMA 2.1. *If M is the sphere or the closed unit disk and J_0 is the standard complex structure, then there exists a continuous map $J\mapsto\phi_J$ from $\mathscr{C}(M)$ to* Diff(M), *such that $\phi_{J_0}=\mathrm{id}$ and $\phi_J\colon M_J\to M_{J_0}$ is holomorphic.*

The volume form Ω together with a complex structure J, determine a unique metric $(\cdot,\cdot)_J$ on M, and if J^* is the adjoint of J, then $-J^*$ is the Hodge star operator for $(\cdot,\cdot)_J$ acting on one-forms. We also let $-J^*$ denote the Hodge star operator acting on zero- and two-forms, i.e., $J^*f=-f\Omega$ and $J^*f\Omega=-f$.

The metric $(\cdot,\cdot)_J$ on M induces a Hermitian metric on the bundle $\Lambda^iM_{\mathbf{C}}$ of complex valued i-forms on M. We also denote this metric by $(\cdot,\cdot)_J$, and in terms of J it can be expressed as $(\phi,\psi)_J\Omega=\phi\wedge-\overline{J^*\psi}$. The space of smooth sections in $\Lambda^iM_{\mathbf{C}}$ is denoted $\Omega^iM_{\mathbf{C}}$, and it has an inner product defined by

$$\langle\phi,\psi\rangle_{J,0}=\int_M(\phi,\psi)_J\Omega=\int_M\phi\wedge-\overline{J^*\psi}.$$

The complex structure J induces a splitting $\Lambda^1M_{\mathbf{C}}=\Lambda^{1,0}M\oplus\Lambda^{0,1}M$ of the complex one-forms into $(1,0)$-forms and $(0,1)$-forms, and a corresponding splitting of the exterior differential $d=\partial_J+\bar\partial_J$. The adjoint operators with respect to $\langle\cdot,\cdot\rangle_{J,0}$ have the following expression $d_J^*=-J^*dJ^*$, $\partial_J^*=-J^*\bar\partial_JJ^*$ and $\bar\partial_J^*=-J^*\partial_JJ^*$.

We inductively define Sobolev inner products on $\Omega^iM_{\mathbf{C}}$ by

$$\langle\phi,\psi\rangle_{J,k}=\langle\phi,\psi\rangle_J+\langle d\phi,d\psi\rangle_{J,k-1}+\langle d_J^*\phi,d_J^*\psi\rangle_{J,k-1},$$

and it is easily seen that J^* is an isometry with respect to these inner products. The corresponding Sobolev norms are defined by $\|\psi\|_{J,k}=\sqrt{\langle\psi,\psi\rangle_{J,k}}$, and for $k\in\mathbf{N}$ we define an operator norm $\|\cdot\|_{J,k}$ on $\mathrm{End}(\Omega^1M_{\mathbf{C}})$ by

$$\|T\|_{J,k}=\sup\{\|T\alpha\|_{J,l}\mid\|\alpha\|_{J,l}\leq 1\text{ and }l\leq k\}.$$

If $k=0$, then we will omit it, i.e., $\langle\cdot,\cdot\rangle_J=\langle\cdot,\cdot\rangle_{J,0}$ and $\|\cdot\|_J=\|\cdot\|_{J,0}$. Let $\alpha,\beta\in\Lambda^1M_{\mathbf{C}}$ and $J,J'\in\mathscr{C}(M)$, then $\alpha\wedge J'^*\beta=\alpha\wedge-J^*J^*J'^*\beta$, so

$$\langle\alpha,\beta\rangle_{J'}=-\langle\alpha,J^*J'^*\beta\rangle_J \tag{2.2}$$

and hence

$$\begin{aligned}|\langle\alpha,\beta\rangle_{J'}-\langle\alpha,\beta\rangle_J| &= |\langle\alpha,(1+J^*J'^*)\beta\rangle_J|\leq\|\alpha\|_J\|(1+J^*J'^*)\beta\|_J\\ &\leq\|1+J^*J'^*\|_J\|\alpha\|_J\|\beta\|_J=\|J^*(J'^*-J^*)\|_J\|\alpha\|_J\|\beta\|_J\\ &\leq\|J^*\|_J\|J'^*-J^*\|_J\|\alpha\|_J\|\beta\|_J=\|J'^*-J^*\|_J\|\alpha\|_J\|\beta\|_J,\end{aligned}$$

especially $|\|\alpha\|_{J'}^2-\|\alpha\|_J^2|\leq\|J'^*-J^*\|_J\|\alpha\|_J^2$. This inequality generalizes by induction to

LEMMA 2.3. *If* $\|J'^*-J^*\|_k\leq 1$, *then*

$$|\|f\|_{J',k}^2-\|f\|_{J,k}^2|\leq 4^{k-1}\|J'^*-J^*\|_{J,k-1}\|f\|_{J,k}^2,\quad \textit{all } f\in\Omega^0M_{\mathbf{C}},\quad \textit{and}$$

$$|\|\alpha\|_{J',k}^2-\|\alpha\|_{J,k}^2|\leq 4^{k}\|J'^*-J^*\|_{J,k}\|\alpha\|_{J,k}^2,\quad \textit{all } \alpha\in\Omega^1M_{\mathbf{C}}.$$

We can now show

PROPOSITION 2.4. *Let* $f_n,f\in\Omega^0M_{\mathbf{C}}$ *and let* $J_n,J\in\mathscr{C}(M)$. *Suppose that* $\int_M f_n\Omega=\int_M f\Omega=0$ *for all* $n\in\mathbf{N}$, $J_n\to J$ *and* $\bar\partial_{J_n}f_n\to\bar\partial_J f$ *in the* C^∞*-topology. Then* $f_n\to f$ *in the* C^∞*-topology.*

Proof. Let λ be the first positive eigenvalue for the Laplacian $\Delta_J=d_J^*d=2\bar\partial_J^*\bar\partial_J$ acting on functions, then

$$\begin{aligned}\|f_n-f\|_{J,k}^2 &\leq 4\left(1+\frac{2}{\lambda}\right)\|\bar\partial_J(f_n-f)\|_{J,k-1}^2\\ &\leq 4\left(1+\frac{2}{\lambda}\right)(\|\bar\partial_J f-\bar\partial_{J_n}f_n\|_{J,k-1}+\|(\bar\partial_{J_n}-\bar\partial_J)f_n\|_{J,k-1})^2.\end{aligned}$$

As $\bar\partial_{J_n}\to\bar\partial_J$, we only need to show that $\|f_n\|_{J,k}$ is bounded. We may assume that $\|J_n^*-J^*\|_{J,k-1}\leq 1$ and then by Lemma 2.3

$$\begin{aligned}\|f_n\|_{J,k}^2 &\leq(1+4^{k-1})\|f_n\|_{J_n,k}^2\leq(1+4^{k-1})4\left(1+\frac{2}{\lambda}\right)\|\bar\partial_{J_n}f_n\|_{J_n,k-1}^2\\ &\leq 4\left(1+\frac{2}{\lambda}\right)(1+4^{k-1})^2\|\bar\partial_{J_n}f_n\|_{J,k-1}^2,\end{aligned}$$

which is bounded, because $\bar\partial_{J_n}f_n\to\bar\partial_J f$. □

The J-harmonic one-forms are characterized by being closed and orthogonal to the exact one-forms with respect to $\langle\,,\rangle_J$. We fix a basis $(\alpha_1(J),\dots,\alpha_{2g}(J))$ for the J-

harmonic one-forms by demanding that $\int_{c_i} \alpha_j(J)=\delta_{i,j}$ for $i,j=1,2,\ldots,2g$, where $(c_1, c_2, \ldots, c_{2g})$ is a fixed canonical homology basis, see [6, p. 54].

PROPOSITION 2.5. *Let $(\alpha_1(J), \ldots, \alpha_{2g}(J))$ be a basis for the J-harmonic one-forms as above. If $J_n \to J$ in the C^∞-topology, then $\alpha_i(J_n) \to \alpha_i(J)$ in the C^∞-topology, for all $i=1,2,\ldots,2g$.*

Proof. Let $i \in \{1,2,\ldots,2g\}$ be given. To ease the notation put $\alpha_n=\alpha_i(J_n)$ and $\alpha=\alpha_i(J)$. As α and α_n represent the same cohomology class, $\alpha_n=\alpha+d\phi_n$, where $d\phi_n$ is uniquely determined by $\langle d\phi_n, d\psi\rangle_{J_n}=-\langle \alpha, d\psi\rangle_{J_n}$ for all $d\psi$. We shall show that $d\phi_n \to 0$. As $J_n \to J$, Lemma 2.3 implies that it is enough to show that $\|d\phi_n\|_{J_n,k} \to 0$ for all k.

First consider the case $k=0$. We may assume that $\|J_n^*-J^*\|_J \leq 1$, and then, using equation (2.2) and the fact that $\alpha \perp d\phi$ with respect to $\langle \cdot, \cdot \rangle_J$, we get

$$\begin{aligned}\|d\phi_n\|_{J_n}^2 &= -\langle \alpha, d\phi_n\rangle_{J_n} = \langle \alpha, (1+J^*J_n^*)\, d\phi_n\rangle_J \\ &\leq \|\alpha\|_J \, \|1+J^*J_n^*\|_J \, \|d\phi_n\|_J \leq \|\alpha\|_J \, \|1+J^*J_n^*\|_J \, 2\|d\phi_n\|_J,\end{aligned}$$

and hence $\|d\phi_n\|_{J_n} \leq 2\|\alpha\|_J \, \|1+J^*J_n^*\|$, which tends to zero.

If $k>0$, we put $L_n=\ldots d_{J_n}^* \, dd_{J_n}^*$ (k terms). The adjoint with respect to $\langle\,,\,\rangle_{J_n}$ is $L_n^*=dd_{J_n}^* d \ldots$. Similarly we put $L=\ldots d_J^* dd_J^*$ and $L^*=dd_J^* d \ldots$.

As $\|d\phi_n\|_{J_n,k}^2=\|d\phi_n\|_{J_n,k-1}^2+\|L_n \, d\phi_n\|_{J_n}^2$, an induction argument gives that we only need to consider the last term.

$$\begin{aligned}\|L_n \, d\phi_n\|_{J_n}^2 &= \langle d\phi_n, L_n^* L_n \, d\phi_n\rangle_{J_n} = -\langle \alpha, L_n^* L_n \, d\phi_n\rangle_{J_n} + \langle \alpha, L^* L d\phi_n\rangle_J \\ &= \langle L_n \alpha, J^*J_n^* L_n \, d\phi_n\rangle_J + \langle L\alpha, L d\phi_n\rangle_J \\ &= \langle (L_n - L)\, \alpha, J^*J_n^* L_n \, d\phi_n\rangle_J + \langle L\alpha, (J^*J_n^* L_n + L)\, d\phi_n\rangle_J \\ &\leq \|(L-L_n)\, \alpha\|_J \, \|J^*J_n^* L_n \, d\phi_n\|_J + \|L\alpha\|_J \, \|(J^*J_n^* L_n + L)\, d\phi_n\|_J.\end{aligned}$$

As $L_n \to L$ and $J^*J_n^* \to -1$, we only need to show that $\|d\phi_n\|_{J,k}$ is bounded, or by Lemma 2.3 that $\|d\phi_n\|_{J_n,k}$ is bounded. The case $k=0$ is already shown, and if $k>0$, then as above we only need to consider $\|L_n \, d\phi_n\|_{J_n}$. We have

$$\begin{aligned}\|L_n \, d\phi_n\|_{J_n}^2 &= \langle d\phi_n, L_n^* L_n \, d\phi_n\rangle_{J_n} = -\langle \alpha, L_n^* L_n \, d\phi_n\rangle_{J_n} \\ &= -\langle L_n \alpha, L_n \, d\phi_n\rangle_{J_n} \leq \|L_n \alpha\|_{J_n} \, \|L_n \, d\phi_n\|_{J_n},\end{aligned}$$

so if $\|J_n^* - J^*\|_J \leq 1$, then $\|L_n\, d\phi_n\|_{J_n} \leq \|L_n\, \alpha\|_{J_n} \leq \|\alpha\|_{J_n, k} \leq (1+4^k)\, \|\alpha\|_{J,k}$, and the proof is complete. □

We get a basis $(\omega_1(J), \omega_2(J), \ldots, \omega_g(J))$ for the J-holomorphic differentials by putting $\omega_j(J) = a_j(J) - iJ^* a_j(J)$ for $j = 1, 2, \ldots, g$, see [6, Proposition III.2.7.]. Hence the holomorphic differentials depend continuously on the complex structure. Similarly we have

Proposition 2.6. *The Weierstrass points depend continuously on the complex structure.*

Proof. It is a local question, so consider the Weierstrass points in some disk $D' \subseteq M$. Choose, continuously depending on J, a J-holomorphic homeomorphism $\phi_J: D \to D'$. Let $(\omega_1(J), \omega_2(J), \ldots, \omega_g(J))$ be a basis for the J-holomorphic differentials as above and define holomorphic functions $f_{J,j}: D \to \mathbf{C}$ by letting $f_{J,j}\, dz = \phi_J^*(\omega_j(J))$. These functions depend continuously on J as does the matrix

$$[\omega_1(J), \omega_2(J), \ldots, \omega_g(J)] = \begin{pmatrix} f_{J,1} & f_{J,2} & \cdots & f_{J,g} \\ f'_{J,1} & f'_{J,2} & \cdots & f'_{J,g} \\ \vdots & \vdots & & \vdots \\ f^{(g-1)}_{J,1} & f^{(g-1)}_{J,2} & \cdots & f^{(g-1)}_{J,g} \end{pmatrix}.$$

Now we only have to observe that the J-Weierstrass points in D' is the image by ϕ_J of the zeros of $\det[\omega_1(J), \omega_2(J), \ldots, \omega_g(J)]$, see [6]. □

With the same notation as above, assume that $\phi_J(0)$ is the same point p for all $J \in \mathscr{C}(M)$ and that p is a non-Weierstrass point in the complex structure J_0. For J in a neighbourhood of J_0, $\det[\omega_1(J), \omega_2(J), \ldots, \omega_g(J)] \neq 0$. So the inverse matrix

$$[\omega_1(J), \omega_2(J), \ldots, \omega_g(J)]\,(0)^{-1}$$

exists, and it depends continuously on J. If

$$(\xi_1(J), \xi_2(J), \ldots, \xi_g(J)) = (\omega_1(J), \omega_2(J), \ldots, \omega_g(J))\, [\omega_1(J), \omega_2(J), \ldots, \omega_g(J)]\,(0)^{-1},$$

then $(\xi_1(J), \xi_2(J), \ldots, \xi_g(J))$ is a basis for the J-holomorphic differentials adapted to the point p, and we have shown

Lemma 2.7. *If p is a non J_0-Weierstrass point, then for J in a neighbourhood of J_0, we can find a basis, continuously dependent on J, for the J-holomorphic differentials adapted to the point p.*

If U is a domain in $\mathbf{C}$ and $f\colon U\to\mathbf{CP}^1$ is a meromorphic function with a finite number of poles, then we can write $f=p/q+h$, where p and q are polynomials and $h\colon U\to\mathbf{C}$ is holomorphic. The following lemma is a generalization of this result.

LEMMA 2.8. *Let M be a closed surface and let D_0, D_1 and D_2 be open disks in M such that $\bar{D}_1\cap\bar{D}_2=\emptyset$ and $\bar{D}_0\subseteq D_1$. Put $T=D_1\setminus\bar{D}_0$, let J be a complex structure on M and let $Q\in D_2$ be a non J-Weierstrass point.*

Any J-holomorphic function $f\colon T\to\mathbf{C}$ can be written uniquely as a sum $f=F_1|_T+F_2|_T$ where $F_1\colon D_1\to\mathbf{C}$ and $F_2\colon M\setminus(\bar{D}_0\cup\{Q\})\to\mathbf{C}$ are J-holomorphic functions such that if z is a J-parameter vanishing at Q, then $F_2(z)=\Sigma_{n=-g}^{\infty}d_n z^n$ with $d_0=0$.

Furthermore, if z depends continuously on J (which we may assume), then F_1 and F_2 depend continuously on f and J in the compact-open topology, as long as Q is a non-Weierstrass point.

Proof. Uniqueness is clear. To prove the existence, we first consider the case $f(w)=\Sigma_{n=-N}^{N}c_n w^n$, where w is a parameter on D_1, vanishing at $P\in D_0$. There exists a meromorphic function F_2 on M, which at P has the same principal part as f, has no poles outside $\{P,Q\}$ and at Q has the expression $F_2(z)=\Sigma_{n=-g}^{\infty}d_n z^n$. Indeed, a configuration of principal parts (a Mittag-Leffler distribution of meromorphic functions) comes from a globally defined meromorphic function if and only if the induced cohomology class in $H^1(M,\mathcal{O})$ is zero. As we allow an extra pole of up to order g at Q, the induced class in $H^1(M,\mathcal{O}(g\cdot[Q]))$ must be zero, and by Serre duality, $H^1(M,\mathcal{O}(g\cdot[Q]))=H^0(M,\mathcal{O}(K_M-g\cdot[Q]))=0$ since Q is a non-Weierstrass point. We may of course assume that $d_0=0$. If we put $F_1=f-F_2|_T$ then F_1 extends to a holomorphic function $D_1\to\mathbf{C}$.

The next step is to show that F_1 and F_2 depend continuously on f and J. For that purpose we will determine the principal part of F_2 at Q.

Let c_1 be a circle in T around P and let c_2 be a circle in D_2 around Q. Let $(\xi_1,\xi_2,\dots,\xi_g)$ be a basis for the J-holomorphic differentials adapted to the point Q, i.e., $\xi_j=(z^{j-1}+(\text{order}\geq g))\,dz$. The principal part of F_2 at Q is $f'=\Sigma_{n=-g}^{-1}d_n z^n$, and the coefficients $d_{-1},d_{-2},\dots,d_{-g}$ can be determined by

$$d_{-k}=\int_{c_2}f'\xi_k=\int_{c_2}F_2\xi_k=\pm\int_{c_1}F_2\xi_k=\pm\int_{c_1}f\xi_k.$$

If we choose z to depend continuously on J, then $(\xi_1,\xi_2,\dots,\xi_g)$ and the numbers $d_{-1},d_{-2},\dots,d_{-g}$ depend continuously on J. Hence if we consider f' as a function $D_2\setminus\{Q\}\to\mathbf{C}$, then f' depends continuously on J and f. If $\tilde{D}_1\subseteq D_1$ and $\tilde{D}_2\subseteq D_2$ are closed disks containing $\tilde{D}_0$ and Q respectively in their interior, then we can extend $f|_{\tilde{D}_1\setminus D_0}$ and $f'|_{\tilde{D}_2\setminus\{Q\}}$ to one smooth function $G: M\setminus(D_0\cup\{Q\})\to \mathbf{C}$ such that G depends continuously on f and f' and hence on f and J.

We define a differential α on M by letting $\alpha=-\bar{\partial}_J G$ outside $\tilde{D}_1\cup\tilde{D}_2$ and zero on $\tilde{D}_1\cup\tilde{D}_2$. Consider the equation $\bar{\partial}_J u=\alpha$ on M. As

(1) F_2-G is a solution on $M\setminus(\tilde{D}_1\cup\tilde{D}_2)$,

(2) $(F_2-G)|_T=F_2|_T-f=F_1|_T$ extends J-holomorphically to D_1 and

(3) $(F_2-G)|_{D_2\setminus\{Q\}}=F_2|_{D_2\setminus\{Q\}}-f'$ extends J-holomorphically to D_2,

there exists a solution with $u(Q)=0$. By Proposition 2.4 the solution depends continuously on J and α, and hence on J and f. This implies that F_1 and F_2 depend continously on f and J.

By continuity the map $f\mapsto(F_1,F_2)$ extends to the space of all functions f. □

3. Flag manifolds

Let $\mathbf{k}=(k_1,k_2,\ldots,k_r)$ be an ordered set of positive integers and put $n=\Sigma\, k_i$. The *(generalized) flag manifold* $Fl_{\mathbf{k}}$ is the space of subspaces $(E_1,E_2,\ldots,E_r)$ of $\mathbf{C}^n$, such that $\dim(E_i)=k_1+k_2+\ldots+k_i$ and $E_1\subseteq E_2\subseteq\ldots\subseteq E_r=\mathbf{C}^n$.

A flag $(E_1,E_2,\ldots,E_r)$ in $Fl_{\mathbf{k}}$ can be represented by a $(n\times n)$-matrix (a_{ij}) in $Gl_n(\mathbf{C})$, such that E_i is the span of the first $k_1+k_2+\ldots+k_i$ columns. A generic flag can uniquely be represented by an $n\times n$-matrix of the form

$$A=\begin{pmatrix} E & 0 & \ldots & 0\\ A_{2,1} & E_2 & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ A_{r,1} & \ldots & A_{r,r-1} & E_r\end{pmatrix},$$

where E_i is the identity $(k_i\times k_i)$-matrix and $A_{i,j}$ is an arbitrary $(k_i\times k_j)$-matrix. The subspace of these flags is called the *affine part* of $Fl_{\mathbf{k}}$ and is denoted $(Fl_{\mathbf{k}})_a$. Furthermore, such matrices form a subgroup $N_{\mathbf{k}}$ of $Gl_n(\mathbf{C})$, which acts on $Fl_{\mathbf{k}}$ from the left and acts transitively and freely on $(Fl_{\mathbf{k}})_a$.

The complement of $(Fl_{\mathbf{k}})_a$ is called the *infinite part* and is denoted $(Fl_{\mathbf{k}})_\infty$. It is a subvariety of $Fl_{\mathbf{k}}$ given by the equation $\Pi_{l=1}^{r-1}\det(a_{ij})_{i,j\leq k_1+\ldots+k_l}=0$.

Unless $r=2$ and we are considering a Grassmanian, $(Fl_{\mathbf{k}})_\infty$ is reducible with irreducible components $Y_1,Y_2,\ldots,Y_{r-1}$, where Y_l is given by the equation

$$\det(a_{ij})_{i,j\leq k_1+\ldots+k_l}=0.$$

If U is an open subset of a Riemann surface and $f\colon U\to Fl_{\mathbf{k}}$ is a holomorphic map with $f(U)\cap(Fl_{\mathbf{k}})_a\neq\emptyset$, then we can consider f as a meromorphic map into $(Fl_{\mathbf{k}})_a$. The set of *poles* is $f^{-1}((Fl_{\mathbf{k}})_\infty)$, which is a discrete subset of U.

For $l=1,2,\dots,r-1$ we put $N_l=\{A\mid i-j\neq l\Rightarrow A_{i,j}=0\}$ and let π_l denote the projection $N_{\mathbf{k}}=N_1\oplus\dots\oplus N_{r-1}\to N_l$. The composition in $N_{\mathbf{k}}$ is given by $(AB)_{i,j}=A_{i,j}+\Sigma_{l=j+1}^{i-1}A_{i,l}B_{l,j}+B_{i,j}$, and if $A\in N_l\oplus\dots\oplus N_r$, then

$$\pi_l(AB)=\pi_l(BA)=\pi_l(A+B-I)=\pi_l(A)+\pi_l(B)-I. \tag{3.1}$$

On an open Riemann surface, any Mittag-Leffler distribution comes from a globally defined meromorphic function. If $\mathbf{CP}^1$ is replaced by a flag manifold $Fl_{\mathbf{k}}$ this generalizes to:

LEMMA 3.2. *Let $\bar{M}$ be a compact surface with $\partial M\neq\emptyset$, and let $\bar{D}_1$, $\bar{D}_2$ be disjoint closed disks in $M=\bar{M}\setminus\partial M$. Put $c_i=\partial D_i$ and let $J\in\mathscr{C}(\bar{M})$. If, for $i=1,2$, $f_i\colon\bar{D}_i\to Fl_{\mathbf{k}}$ is J-holomorphic with $f_i(c_i)\subseteq(Fl_{\mathbf{k}})_a$, then there exist J-holomorphic maps $f\colon\bar{M}\to Fl_{\mathbf{k}}$ and $g_i\colon\bar{D}_i\to N_{\mathbf{k}}$ such that $f_i=g_if|_{\bar{D}_i}$ and the poles of f is contained in $D_1\cup D_2$.*

Furthermore, for small variations of J, the map f can be chosen such that it depends continuously on f_1, f_2 and J.

Proof. First choose a closed surface $\tilde{M}$ with $\bar{M}\subseteq\tilde{M}$ and a continuous extension map $\mathscr{C}(\bar{M})\to\mathscr{C}(\tilde{M})$. Then any complex structure J on $\bar{M}$ can be considered as a complex structure on $\tilde{M}$. Next, choose a point Q in $\tilde{M}\setminus\bar{M}$, which is a non-Weierstrass point in the given complex structure.

We can find open disks D_1' and D_2', such that $\bar{D}_i'\subseteq D_i$ and $f_i^{-1}((Fl_{\mathbf{k}})_\infty)\subseteq D_i'$. Let $T_i=D_i\setminus D_i'$ and consider $f_i|_{T_i}$ as a map $T_i\to N_{\mathbf{k}}$. If the composition in $N_{\mathbf{k}}$ was addition, then Lemma 2.8 would give the result. Instead an induction argument using (3.1) and lemma 2.8 works. □

Remark 3.3. If $\bar{M}\subseteq S^2$, then we do not need the assumption $\partial M\neq\emptyset$, i.e., the lemma holds for $\bar{M}=S^2$.

4. Loop groups

Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$ and consider the space of based loops in G, i.e., the space of smooth maps $\gamma\colon S^1\to G$ with $\gamma(1)=1$. It is an infinite dimensional Lie group, and we let the *loop group* ΩG be the identity component.(1) The Lie Algebra of ΩG is $\Omega\mathfrak{g}$, i.e., the space of smooth maps $\gamma: S^1 \mapsto \mathfrak{g}$ with $\gamma(1)=0$.

The complexification of G is denoted $G_{\mathbf{C}}$ and has Lie algebra $\mathfrak{g}_{\mathbf{C}}=\mathfrak{g}\otimes_{\mathbf{R}}\mathbf{C}$. We let $LG_{\mathbf{C}}$ denote the identity component of all loops in $G_{\mathbf{C}}$. It too is an infinite dimensional Lie group, and we may consider ΩG as a subgroup of $LG_{\mathbf{C}}$. Let $L^+G_{\mathbf{C}}$ denote the subgroup of loops $\gamma\in LG_{\mathbf{C}}$, which are the boundary value of a holomorphic map $D\to G_{\mathbf{C}}$, where D is the open unit disk in $\mathbf{C}$, and let $L^-G_{\mathbf{C}}$ denote the subgroup of loops $\gamma\in LG_{\mathbf{C}}$, which are the boundary value of a holomorphic map $D_\infty\to F_{\mathbf{C}}$, where $D_\infty=\mathbf{CP}^1\setminus\bar{D}$. The Lie groups $LG_{\mathbf{C}}$, $L^+G_{\mathbf{C}}$ and $L^-G_{\mathbf{C}}$ have the Lie algebras $L\mathfrak{g}_{\mathbf{C}}$, $L^+\mathfrak{g}_{\mathbf{C}}$ and $L^-\mathfrak{g}_{\mathbf{C}}$.

The multiplication map $\Omega G\times L^+G_{\mathbf{C}}\to LG_{\mathbf{C}}$ is a diffeomorphism, see [12, chapter 8], so the loop group is also a homogeneous space of $LG_{\mathbf{C}}$. The description $\Omega G\cong LG_{\mathbf{C}}/L^+G_{\mathbf{C}}$ makes ΩG into a complex manifold, but not into a complex Lie group. The multiplication in ΩG is not holomorphic, but left multiplication by a fixed element is holomorphic.

If $L_1^-G_{\mathbf{C}}=\{\gamma\in L^-G_{\mathbf{C}}|\ \gamma(\infty)=1\}$, then the multiplication map $L_1^-G_{\mathbf{C}}\times L^+G_{\mathbf{C}}\to LG_{\mathbf{C}}$ is a diffeomorphism onto a dense open subset of $LG_{\mathbf{C}}$, see [12, chapter 8], so $L_1^-G_{\mathbf{C}}$ can be considered as an open dense subset of ΩG. Moreover, the inclusion $L_1^-G_{\mathbf{C}}\hookrightarrow\Omega G$ is holomorphic, and the multiplication in $L_1^-G_{\mathbf{C}}$ extends to a holomorphic left action of $L_1^-G_{\mathbf{C}}$ on ΩG. The Lie algebra of $L_1^-G_{\mathbf{C}}$ is $L_0^-\mathfrak{g}_{\mathbf{C}}=\{\gamma\in L^-\mathfrak{g}_{\mathbf{C}}|\ \gamma(\infty)=0\}$, so ΩG is a complex manifold modeled on $L_0^-\mathfrak{g}_{\mathbf{C}}$

The loop group ΩG can be considered as a kind of infinite dimensional Grassmannian, see [12], and as such $L_1^-G_{\mathbf{C}}$ is the affine part of ΩG. The complement is called the infinite part and is denoted $(\Omega G)_\infty$.

This is very similar to the situation in the preceding section. The loop group ΩG corresponds to the flag manifold $Fl_{\mathbf{k}}$, and $L_1^-G_{\mathbf{C}}$ corresponds to the group $N_{\mathbf{k}}\cong(Fl_{\mathbf{k}})_a$. There is one difference between the groups $N_{\mathbf{k}}$ and $L_1^-G_{\mathbf{C}}$, namely the exponential map. It is an isomorphism in the case of $N_{\mathbf{k}}$, but this may not be so in the case of $L_1^-G_{\mathbf{C}}$. Hence as a complex manifold $L_1^-G_{\mathbf{C}}$ need not be a vector space, but it is contractible by the homomorphisms $\gamma\mapsto\gamma_t$, $t\in[0,1]$, where $\gamma_t(z)=\gamma(t^{-1}z)$.

We will need the description of elements in ΩG as holomorphic bundles over $\mathbf{CP}^1$, see [12, section 8.10]. The idea is simple. A loop $\gamma\in\Omega G$ is used to glue the trivial $G_{\mathbf{C}}$-bundle over $\bar{D}$ and $\bar{D}_\infty$ together and thus obtain a $G_{\mathbf{C}}$-bundle over $\mathbf{CP}^1$. To be precise, an

(1) Normally all components are considered, but as we later will consider based maps into ΩG, we will only need the identity component.

element of ΩG is the same as an isomorphism class of pairs (P, τ), where P is a holomorphic principal $G_{\mathbf{C}}$-bundle on $\mathbf{CP}^1$ and τ is a trivialization of P over $\bar{D}_\infty$, i.e., a smooth section of $P|_{\bar{D}_\infty}$, which is holomorphic over D_∞. The elements of $L_1^- G_{\mathbf{C}} \subseteq \Omega G$ correspond to pairs (P, τ), where P is the trivial bundle, and the action of $L_1^- G_{\mathbf{C}}$ on ΩG corresponds to the map $(\gamma, (P, \tau)) \mapsto (P, \gamma\tau)$. Holomorphic maps into ΩG are described by

PROPOSITION 4.1. *If X is a complex manifold, then a holomorphic map from X to ΩG is the same thing as an isomorphism class of pairs (P, τ), where P is a holomorphic principal $G_{\mathbf{C}}$-bundle on $X \times \mathbf{CP}^1$ and τ is a trivialization of P over $X \times \bar{D}_\infty$.*

If U is an open subset of a Riemann surface X and $f: U \to \Omega G$ is holomorphic with $f(U) \cap L_1^- G_{\mathbf{C}} \neq \emptyset$, then f can be considered as a meromorphic map into $L_1^- G_{\mathbf{C}} = (\Omega G)_a$. The set of *poles* is $f^{-1}((\Omega G)_\infty)$, which is a discrete subset of U. If we use Proposition 4.1 and identify f with a pair (P, τ), where P is a holomorphic $G_{\mathbf{C}}$-bundle over $U \times \mathbf{CP}^1$, then a point $\alpha \in U$ is a pole if and only if the line $\{\alpha\} \times \mathbf{CP}^1$ is a *jumping line*, i.e., if and only if the bundle $P|_{\{\alpha\} \times \mathbf{CP}^1}$ is non-trivial.

We end the chapter on loop groups with the equivalent of Lemma 3.2.

LEMMA 4.2. *Let $\bar{M}$ be a compact surface with non-empty boundary, and let D_1, D_2 be disjoint closed disks in $M = \bar{M} \setminus \partial M$. Put $c_i = \partial D_i$ and let $J \in \mathscr{C}(\bar{M})$. If, for $i = 1, 2$, $f_i: \bar{D}_i \to \Omega G$ is J-holomorphic with $f_i(c_i) \subseteq L_1^- G_{\mathbf{C}}$, then there exist J-holomorphic maps $f: \bar{M} \to \Omega G$ and $g_i: \bar{D}_i \to L_1^- G_{\mathbf{C}}$ such that $f_i = g_i f|_{\bar{D}_i}$ and the set of poles of f is contained in $D_1 \cup D_2$.*

Furthermore, for small variations of f_1, f_2 and J, the map f can be chosen such that it depends continuously on f_1, f_2 and J.

Proof. The two maps $f_1: \bar{D}_1 \to \Omega G$ and $f_2: \bar{D}_2 \to \Omega G$ correspond to two pairs (P_i, τ_i), where P_i is a J-holomorphic $G_{\mathbf{C}}$-bundle over $\bar{D}_i \times \mathbf{CP}^1$ and τ_i is a trivialization of P_i over $\bar{D}_i \times \bar{D}_\infty$. The bundle P_i is trivial outside the jumping lines $f^{-1}((\Omega G)_\infty) \times \mathbf{CP}^1$, so by gluing $P_1 \cup P_2$ to the trivial bundle over $(\bar{M} \times \mathbf{CP}^1) \setminus \{\text{jumping lines}\}$, we get a J-holomorphic $G_{\mathbf{C}}$-bundle P over $\bar{M} \times \mathbf{CP}^1$.

As $\partial M \neq \emptyset$, there exists a trivialization τ of P over $\bar{M} \times \bar{D}_\infty$. The pair (P, τ) corresponds to a J-holomorphic map $f: \bar{M} \to \Omega G$, and the difference between the trivializations $\tau|_{\bar{D}_i \times \bar{D}_\infty}$ and τ_i is a J-holomorphic map $g_i: \bar{D}_i \times \bar{D}_\infty \mapsto G_{\mathbf{C}}$. We can choose τ such that $g_i(x, \infty) = 1$ for all $x \in \bar{D}_i$, so g_i is a J-holomorphic map $\bar{D}_i \to L_1^- G_{\mathbf{C}}$. The maps f, g_1 and g_2 have all the required properties, but we still have to show that this process can be made continuously.

Let $y_0=(f_1^0,f_2^0,J^0)$ be given. Put $U=D_1\cup D_2$ and choose an open subset $V\subseteq\bar{M}$, such that $U\cup V=\bar{M}$ and $f_i^0(\bar{V}\cap\bar{D}_i)\subseteq(\Omega G)_a$ for $i=1,2$. Finally, choose a neighbourhood W of y_0 in the space of triples (f_1,f_2,J) with $f_i\in\mathrm{Map}(\bar{D}_i,\Omega G)$ and $J\in\mathscr{C}(\bar{M})$ such that $f_i(\bar{V}\cap\bar{D}_i)\subseteq(\Omega G)_a$ and f_i is J-holomorphic.

The evaluation map $F\colon W\times\bar{U}\to\Omega G$, given by $F(f_1,f_2,J,x)=f_i(x)$ if $x\in\bar{D}_i$, defines a pair (P_U,τ_U), where P_U is a $G_{\mathbf{C}}$-bundle over $W\times\bar{U}\times\mathbf{CP}^1$, and τ_U is a trivialization of P_U over $W\times\bar{U}\times\bar{D}_\infty$. The bundle P_U is J-holomorphic, when restricted to $\{(f_1,f_2,J)\}\times U\times\mathbf{CP}^1$, and the trivialization τ_U is J-holomorphic, when restricted to $\{(f_1,f_2,J)\}\times U\times D_\infty$. Furthermore, P_U can be trivialized over $W\times(\bar{U}\cap\bar{V})\times\mathbf{CP}^1$, and the trivialization can be chosen such that it is J-holomorphic, when restricted to $\{(f_1,f_2,J)\}\times(U\cap V)\times\mathbf{CP}^1$.

By gluing P_U to the trivial bundle over $W\times\bar{V}\times\mathbf{CP}^1$, we get a $G_{\mathbf{C}}$-bundle P over $W\times\bar{M}\times\mathbf{CP}^1$, which is J-holomorphic, when restricted to $\{(f_1,f_2,J)\}\times M\times\mathbf{CP}^1$ and is trivial over $W\times\bar{V}\times\mathbf{CP}^1$. We only need to find a trivialization τ of P over $W\times\bar{M}\times\bar{D}_\infty$, which is J-holomorphic, when restricted to $\{(f_1,f_2,J)\}\times M\times D_\infty$, and is equal to τ_U on $W\times\bar{U}\times\{\infty\}$.

If $x\in\bar{U}\cap\bar{V}$, then $F(y,x)\in(\Omega G)_a\cong L_1^-G_{\mathbf{C}}$, and the transition function from the trivialization over $W\times\bar{U}\times\bar{D}_\infty$ to the trivilization over $W\times\bar{V}\times\bar{D}_\infty$ is exactly $F|_{W\times(\bar{U}\cap\bar{V})}$ considered as a map $W\times(\bar{U}\cap\bar{V})\times\bar{D}_\infty\to G_{\mathbf{C}}$.

Let $t\colon\bar{M}\to[0,1]$ be a smooth map, such that $t(\bar{M}\setminus U)=0$ and $t(\bar{U}\setminus V)=1$. We define $\psi_V\colon W\times\bar{V}\to L_1^-G_{\mathbf{C}}$ by letting $\psi_V(y,x)(z)=1$ if $x\in\bar{V}\setminus U$ and $\psi_V(y,x)(z)=F(y,x)(t(x)z)$ if $x\in\bar{U}\cap\bar{V}$, and $\psi_U\colon W\times\bar{U}\to L_1^-G_{\mathbf{C}}$ by $\psi_U=1$ on $\bar{U}\setminus V$ and $\psi_U=F^{-1}\psi_V$ on $\bar{U}\cap\bar{V}$.

The map ψ_V defines an isomorphism of the trivial bundle over $W\times\bar{V}\times\bar{D}_\infty$, and ψ_U defines an isomorphism of the trivial bundle over $W\times\bar{U}\times\bar{D}_\infty$. As $\psi_U=F\psi_V$, when restricted to $W\times(\bar{U}\cap\bar{V})\times\bar{D}_\infty$, we get a trivialization ϕ of P over $W\times\bar{M}\times\bar{D}_\infty$. The trivialization ϕ is holomorphic when restricted to $\{(f_1,f_2,J)\}\times M\times\{\infty\}$, and is equal to τ_U, when restricted to $W\times\bar{U}\times\{\infty\}$.

For any map $\psi\colon W\times\bar{M}\to L_1^-G_{\mathbf{C}}$, the product $\psi\phi$ is a new trivialization of P over $W\times\bar{M}\times\bar{D}_\infty$. We want to find a ψ, such that $\psi\phi$ is J-holomorphic, when restricted to $\{(f_1,f_2,J)\}\times M\times D_\infty$. As P is J^0-holomorphically trivial over $\{y_0\}\times M\times D_\infty$, we can find $\psi\colon\bar{M}\to L_1^-G_{\mathbf{C}}$, such that $\psi\phi$ is J^0-holomorphic, when restricted to $\{y_0\}\times M\times D_\infty$. To ease notation, we assume that ϕ is already J^0-holomorphic, when restricted to $\{y_0\}\times M\times D_\infty$. This corresponds to assuming that ψ_U and ψ_V are J^0-holomorphic, when restricted to respectively $\{y_0\}\times U\times D_\infty$ and $\{y_0\}\times V\times D_\infty$.

We shall find a map $\psi\colon W\times\bar{M}\to L_1^-G_{\mathbf{C}}$ such tat $\psi\psi_U$ and $\psi\psi_V$ are J-holomorphic when restricted to respectively $\{(f_1,f_2,J)\}\times U$ and $\{(f_1,f_2,J)\}\times V$. Let $\Omega^1(\bar{M}, L_0^-\mathfrak{g}_C)$ be the space of one-forms on $\bar{M}$ with values in $L_0^-\mathfrak{g}_C$, and define h: $W\to\Omega^1(\bar{M}, L_0^-\mathfrak{g}_C)$ by

$$h(f_1,f_2,J)=\begin{cases}-(\bar{\partial}_J\psi_U)\,\psi_U^{-1} & \text{on } U\\ -(\bar{\partial}_J\psi_V)\,\psi_V^{-1} & \text{on } V.\end{cases}$$

This is well-defined, because the difference between ψ_U and ψ_V is J-holomorphic.

Our task is to find ψ, such that $\psi^{-1}\bar{\partial}_J\psi=h$. It we put

$$\Omega^{0,1}(W\times\bar{M}, L_0^-\mathfrak{g}_C)=\{(f_1,f_2,J,h)\in W\times\Omega^1(\bar{M}, L_0^-\mathfrak{g}_C)|\ h\in\Omega_J^{0,1}(\bar{M}, L_0^-\mathfrak{g}_C)\},$$

then $(y, h(y))\in\Omega^{0,1}(W\times\bar{M}, L_0^-\mathfrak{g}_C)$ all $y\in W$, and as ϕ is J^0-holomorphic, when restricted to $\{y_0\}\times M\times D_\infty$, we have $h(y_0)=0$. Now consider the map

$$H: W\times C^\infty(\bar{M}, L_1^-G_C)\to\Omega^{0,1}(W\times\bar{M}, L_0^-\mathfrak{g}_C)$$
$$(f_1,f_2,J,\psi)\mapsto(f_1,f_2,J,\psi^{-1}\bar{\partial}_J\psi).$$

We shall show that H has a right inverse, and to do that we use the Nash-Moser inverse function theorem, see [8]. The first step is to find the differential of H.

The tangent space at (y,ψ) of $W\times C^\infty(\bar{M}, L_1^-G_C)$ is $T_yW\times C^\infty(\bar{M}, L_0^-\mathfrak{g}_C)$, and the tangent space at (y,h) of $\Omega^{0,1}(W\times\bar{M}, L_0^-\mathfrak{g}_C)$ is $T_yW\times\Omega_J^{0,1}(\bar{M}, L_0^-\mathfrak{g}_C)$. Let $y=(f_1,f_2,J)\in W$, $A\in C^\infty(\bar{M}, L_0^-\mathfrak{g}_C)$ and $B=(B_1,B_2,K)\in T_yW$, where $B_j=T_{f_j}\mathrm{Map}(\bar{D}_j,\Omega G)$ and $K\in T_J\mathscr{C}(M)$. Then

$$DH(y,\psi)(B,A)=\left(B,\psi^{-1}\frac{i}{2}K\,d\psi+\psi^{-1}(\bar{\partial}_jA)\,\psi\right).$$

By [8, III, Theorem 1.1.3] it is enough to show that DH has a smooth tame family of right inverses. So we shall be able to solve the equation

$$\bar{\partial}_JA=\psi h\psi^{-1}-\frac{i}{2}K\,d\psi\,\psi^{-1} \tag{4.3}$$

where $h\in\Omega_J^{0,1}(\bar{M}, L_0^-\mathfrak{g}_C)$, such that the solution A is a smooth tame function of J,ψ,K and h.

If $\partial M=\varnothing$ and the righthand side of (4.3) lies in the images of $\bar{\partial}_J$ then this is possible, see [8, II, Theorem 3.3.3]. We have $\partial M\neq\varnothing$, so we will close M and extend our data in a suitable way.

Let $\tilde{M}$ be a closed surface containing $\bar{M}$. By [13] there exists a smooth tame map

$(J, \psi, K, h) \mapsto (\tilde{J}, \tilde{\psi}, \tilde{K}, \tilde{h})$ which extends the data from $\bar{M}$ to $\tilde{M}$. Next we modify $\tilde{h}$ to $\tilde{h}$ such that $R = \tilde{\psi}\tilde{h}\tilde{\psi}^{-1} - (i/2)\tilde{K}\, d\tilde{\psi}\, \tilde{\psi}^{-1}$ lies in the images of $\bar{\partial}_{\tilde{J}}$.

The image of $\bar{\partial}_{\tilde{J}}$ is the forms $\alpha \in \Omega_{\tilde{J}}^{0,1}(\tilde{M}, L_0^- \mathfrak{g}_{\mathbf{C}})$ such that $\int_{\tilde{M}} \alpha \wedge \omega = 0$ for all $\tilde{J}$-holomorphic forms ω. In § 2 we constructed a basis $(\omega_1(\tilde{J}), \ldots, \omega_g(\tilde{J}))$ for the $\tilde{J}$-holomorphic differentials and by examining the proof of Proposition 2.5 we see that the map $\tilde{J} \mapsto \omega_j(\tilde{J})$ is smooth and tame.

Choose forms $f_1, \ldots, f_g \in \Omega_{\tilde{J}}^{0,1} \tilde{M}$ such that $f_j|_{\bar{M}} = 0$ and the matrix

$$(a_{i,j}(\tilde{J}))_{i,j=1,\ldots,g} = \left(\int_{\tilde{M}} \pi_{\tilde{J}}^{0,1} f_i \wedge \omega_j(\tilde{J}) \right)_{i,j=1,\ldots,g},$$

where $\pi_{\tilde{J}}^{0,1}$ is the projection onto $\Omega_{\tilde{J}}^{0,1}\tilde{M}$, is regular for $\tilde{J} = \tilde{J}_0$. Then the same is true for $\tilde{J}$ in a neighbourhood of $\tilde{J}_0$ and by making W smaller we may assume that it is true for all $\tilde{J}$. Let $(b_{i,j}(\tilde{J}))_{i,j=1,\ldots,g}$ be the inverse matrix and put $f_i(\tilde{J}) = \Sigma_{j=1}^{g} b_{i,j}(\tilde{J})\, \pi_{\tilde{J}}^{0,1} f_i$. Then $\int_{\tilde{M}} f_i(\tilde{J}) \wedge \omega_j(\tilde{J}) = \delta_{i,j}$, and if we put

$$\tilde{h} = \tilde{h} - \sum_{i=1}^{g} f_i(\tilde{J}) \otimes \int_{\tilde{M}} \left(\tilde{h} - \frac{i}{2} \tilde{\psi}^{-1} \tilde{K}\, d\tilde{\psi} \right) \wedge \omega_i(\tilde{J}),$$

then R lies in the images of $\bar{\partial}_{\tilde{J}}$. As mentioned above, $\bar{\partial}_{\tilde{J}}$ now has a smooth tame family of right inverses. As the restriction from $\tilde{M}$ to $\bar{M}$ obviously is smooth and tame, the differential DH has a smooth tame family of right inverses, and the proof is complete. □

5. Spaces of holomorphic maps

In the following Y denotes either a flag manifold $Fl_{\mathbf{k}}$ or a loop group ΩG. It is a complex manifold and even a complex projective variety. We let Y_a denote the affine part of Y and let $Y_\infty = Y \setminus Y_a$ denote the infinite part of Y. The affine part is isomorphic to a contractible complex Lie group N, and the composition $N \times N \to N$ extends to a holomorphic left action $N \times Y \to Y$ of N on Y. The infinite part is the union $Y_\infty = Y_1 \cup \ldots \cup Y_r$ of finitely many irreducible algebraic varieties $Y_1, \ldots, Y_r$.

If X is a Riemann surface and $f: X \to Y$ is a holomorphic map, which does not map into Y_∞, then the set of poles, $f^{-1}(Y_\infty)$, is a discrete subset of X. To each point $\alpha \in X$ and $i = 1, \ldots, r$ the ith order $\mathrm{ord}_{i,\alpha} f$ of f at α is defined as the order of contact between $f(U)$ and Y_i at $f(\alpha)$, where U is a neighbourhood of α, such that $f^{-1}(Y_\infty) \cap U \subseteq \{\alpha\}$. The total order, $\mathrm{ord}_\alpha f$, of f at α is of the sum the ith orders, and α is a pole if and only if $\mathrm{ord}_\alpha f > 0$. The ith degree of f is $\deg_i f = \Sigma_\alpha \mathrm{ord}_{i,\alpha} f$, and the total degree is

$\deg f=\deg_1 f+\ldots+\deg_r f=\Sigma_a \operatorname{ord}_a f$. If X is closed, the degrees are finite, and the r-tuple $(\deg_1 f, \ldots, \deg_r f)$ determines which component of $\operatorname{Map}(X, Y)$, f lies in.

Let $\bar{M}$ be a compact two-dimensional manifold, possibly with boundary and corners, and put $M=\bar{M}\setminus\partial M$. Equip the space $\operatorname{Map}(M, Y)$ of continuous maps from M to Y with the compact-open topology.

If $f\in \operatorname{Hol}_J(M, Y)=\{f\in \operatorname{Map}(M, Y) \mid f \text{ is } J\text{-holomorphic}\}$ and $f(M)\cap Y_a\neq\emptyset$, then we call f a J-meromorphic map and we have the concepts of poles, orders and degrees of f.

We let $\mathcal{M}_n(\bar{M})$ be the space of pairs (f, J) in $\operatorname{Map}(M, Y)\times\mathscr{C}(\bar{M})$ such that f is J-meromorphic with $\deg f=n$, and if M' is any subset of M, then we let $\mathcal{M}_n(\bar{M}, M')$ be the space of pairs (f, J) in $\mathcal{M}_n(\bar{M})$ such that the poles of f is outside M'. We put $\mathcal{M}_{\leq n}(\bar{M}, M')=\bigcup_{k=0}^n \mathcal{M}_k(\bar{M}, M')$ and $\mathcal{M}(\bar{M}, M')=\lim_{n\to\infty}\mathcal{M}_{\leq n}(\bar{M}, M')$.

If the complex structure is fixed, then we have the spaces $\mathcal{M}_{J,n}(M, M')$ and $\mathcal{M}_{J,\leq n}(M, M')$ consisting of J-meromorphic maps with the right degree. We put $\mathcal{M}_J(M, M')=\lim_{n\to\infty}\mathcal{M}_{J,\leq n}(\bar{M}, M')$ and if $M'=\emptyset$, then we omit it, i.e., $\mathcal{M}(\bar{M})=\mathcal{M}(\bar{M}, \emptyset)$, etc.

The restriction of the projection $\operatorname{Map}(M, Y)\times\mathscr{C}(\bar{M})\to\operatorname{Map}(M, Y)$ to $\mathcal{M}(\bar{M})$ fits into the commutative diagram

$$\begin{array}{ccc} \mathcal{M}_J(M) & \longrightarrow & \mathcal{M}(\bar{M}) \\ \downarrow & & \downarrow \\ \operatorname{Hol}_J(M, Y) & \longrightarrow & \operatorname{Map}(M, Y) \end{array}$$

In this section we consider the case $\bar{M}=\bar{D}=\{z\in\mathbf{C} \mid |z|\leq 1\}$ and show that the maps in the diagram are homotopy equivalences.

LEMMA 5.1. *Let J_0 be any complex structure on $\bar{D}$. There exists a map ψ from $\mathcal{M}(\bar{D})$ to $\mathcal{M}_{J_0}(D)$, such that $\psi(f, J_0)=f$, and the map $\mathcal{M}(\bar{D})\to\mathcal{M}_{J_0}(D)\times\mathscr{C}(\bar{D})$ given by $(f, J)\mapsto(\psi(f, J), J)$ is a homeomorphism.*

Proof. Let $\phi_J\colon D_J\to D_{J_0}$ be the map from Lemma 2.1 and define ψ by

$$\psi(f, J)=f\circ\phi_J^{-1}. \qquad \square$$

LEMMA 5.2. *The inclusion $\operatorname{Hol}_J(D, Y)\hookrightarrow\operatorname{Map}(D, Y)$ is a homotopy equivalence.*

Proof. Let J_0 be the standard complex structure on $\bar{D}$ and let $\phi\colon \bar{D}_{J_0}\to\bar{D}_J$ be a holomorphic homeomorphism with $\phi(0)=0$. Define for $t\in[0, 1]$, $\psi_t\colon\bar{D}\to\bar{D}$ by $\psi_t(z)=\phi(t\phi^{-1}(z))$. Then ψ_t is J-holomorphic for all $t\in[0, 1]$, $\psi_0=0$ and $\psi_1=\mathrm{id}$. We define a homotopy inverse $F\colon \mathrm{Map}(D, Y) \to \mathrm{Hol}_J(D, Y)$ to the inclusion by $F(f)(z)=f(0)$, and only have to observe that F is homotopic to the identity on both $\mathrm{Hol}_J(M, Y)$ and $\mathrm{Map}(D, Y)$ by the homotopy $(t,f)\mapsto f\circ\psi_t$.

LEMMA 5.3. *The map* $\mathcal{M}_J(D)\to \mathrm{Hol}_J(D, Y)\setminus \mathrm{Hol}_J(D, Y_\infty)$ *is a homotopy equivalence.*

Proof. Let $\psi_t\colon \bar{D}\to\bar{D}$ be the map defined in the proof above. We define a homotopy inverse to the map in the Lemma by $f\mapsto f\circ\psi_{1/2}$. □

Let $\widetilde{\mathrm{Hol}}_J(\bar{D}, Y)$ be the space of $f\in \mathrm{Map}(\bar{D}, Y)$ such that $f|_D$ is J-holomorphic and $f(\bar{D})$ is contained in a chart, and let $\mathfrak{n}$ be the Lie algebra of N. Then we have

LEMMA 5.4. $\widetilde{\mathrm{Hol}}_J(\bar{D}, Y)$ *is a complex manifold modelled on* $\mathrm{Hol}_J(\bar{D}, \mathfrak{n})$.

LEMMA 5.5. *The inclusion* $\mathrm{Hol}_J(D, Y)\setminus \mathrm{Hol}_J(D, Y_\infty)\hookrightarrow \mathrm{Hol}_J(D, Y)$ *is a homotopy equivalence.*

Proof. Choose a metric on Y and a $k>0$, such that any subset of Y with diameter less than k is contained in a chart. Let ψ_t be the J-holomorphic map defined in the proof of Lemma 5.2. For $f\in \mathrm{Hol}_J(D, Y)$, we let $t(f)$ be the maximal $t\in[0, 1/2]$ such that $\mathrm{diam}(f\circ\psi_t(\bar{D}))\leq k$. The number $t(f)$ depends continuously on f, so we can define a map ϕ from $\mathrm{Hol}_J(D, Y)$ to $\widetilde{\mathrm{Hol}}_J(\bar{D}, Y)$ by $\phi(f)=f\circ\psi_{t(f)}$. This is a homotopy inverse to the restriction $r\colon \widetilde{\mathrm{Hol}}_J(\bar{D}, Y)\to \mathrm{Hol}_J(D, Y)$, because

$$r\circ\phi(f)=f\circ\psi_{t(f)}\sim f\circ\psi_1=f \quad\text{and}\quad \phi\circ r(f)=f\circ\psi_{t(f)}\sim f\circ\psi_1=f$$

by obvious homotopies.

Moreover, the subspaces $\mathrm{Hol}_J(D, Y)\setminus \mathrm{Hol}_J(D, Y_\infty)$ and $\mathrm{Hol}_J(D, Y_\infty)$ are preserved by the homotopies. So it is enough to show that the inclusion

$$\widetilde{\mathrm{Hol}}_J(\bar{D}, Y)\setminus \mathrm{Hol}_J(\bar{D}, Y_\infty)\hookrightarrow \widetilde{\mathrm{Hol}}_J(\bar{D}, Y)$$

is a homotopy equivalence.

This is the case because $\widetilde{\mathrm{Hol}}_J(\bar{D}, Y)$ is a manifold and $\widetilde{\mathrm{Hol}}_J(\bar{D}, Y)\cap \mathrm{Hol}_J(\bar{D}, Y_\infty)$ has infinite codimension in the sense of the following lemma. □

LEMMA 5.6. *If* $f\in \widetilde{\mathrm{Hol}}_J(\bar{D}, Y)\cap \mathrm{Hol}_J(\bar{D}, Y_\infty)$, *then there exist a neighbourhood* W *of* 0 *in* $\mathrm{Hol}_J(\bar{D}, \mathbf{C})$ *and an imbedding* $i\colon W\hookrightarrow \widetilde{\mathrm{Hol}}_J(\bar{D}, Y)$, *such that*

(1) $i^{-1}(\mathrm{Hol}_J(\bar{D}, Y_\infty))=\{0\}$, *and*

(2) *every smooth curve* γ *in* $\mathrm{Hol}_J(\bar{D}, Y_\infty)$ *with* $\gamma(0)=f$ *has* $\gamma'(0)\notin di(T_0 W\setminus\{0\})$.

Proof. We can consider f as a map $\bar{D}\to\mathfrak{n}$ and as Y_∞ has complex codimension one in Y, there exists a $g\in\mathrm{Hol}_J(\bar{D},\mathfrak{n})$, such that $g(0)$ is not tangent to Y_∞ at $f(0)$ ($\mathfrak{n}$ is a vector space so it makes sense to consider $g(0)$ as a tangentvector at any point). We can choose an $\varepsilon>0$, such that for a $z\in D$ with $|z|<\varepsilon$, $g(z)$ is not tangent to Y_∞ at $f(z)$.

If W is a sufficiently small neighbourhood of 0 in $\mathrm{Hol}_J(\bar{D},\mathbf{C})$, then we have an imbedding $i\colon W\hookrightarrow\widetilde{\mathrm{Hol}}_J(\bar{D},Y)\colon h\mapsto f+hg$. We see that $di_0(h)=hg$, and if $h(z)g(z)$ is tangent to Y_∞ at a point $f(z)$ with $|z|<\varepsilon$, then we must have $h(z)=0$. If γ is a smooth curve in $\mathrm{Hol}_J(\bar{D},Y_\infty)$ with $\gamma(0)=f$ and $\gamma'(0)=hg$ for a $h\in\mathrm{Hol}_J(\bar{D},\mathbf{C})$, then $h(z)g(z)$ is tangent to Y_∞ at $f(z)$ for all z. Hence $h(z)=0$ for all z with $|z|<\varepsilon$, and as h is holomorphic, h is identically zero.

So condition (2) of the lemma is satisfied and if W is sufficiently small, condition (1) is satisfied too. □

We finally state

LEMMA 5.7. *Let $\bar{D}$ be the closed unit disk in $\mathbf{C}$ and let J be any complex structure on $\bar{D}$. Then the maps in the commutative diagram*

$$\begin{array}{ccc} \mathcal{M}_J(D) & \longrightarrow & \mathcal{M}(\bar{D}) \\ \downarrow & & \downarrow \\ \mathrm{Hol}_J(D,Y) & \longrightarrow & \mathrm{Map}(D,Y) \end{array}$$

are homotopy equivalences.

Proof. The two horizontal maps are homotopy equivalences by Lemma 5.1 and Lemma 5.2, and the lefthand vertical map is a homotopy equivalence by Lemma 5.3 and Lemma 5.5. But then the last map is a homotopy equivalence too. □

6. Spaces of principal parts

Let J be a complex structure on $\bar{M}$ and let $\mathcal{O}_J$ and $\mathcal{M}_J$ denote the sheaves of respectively J-holomorphic and J-meromorphic maps into N. I.e., for an open subset $U\subseteq M$, we let $\mathcal{O}_J(U)=\mathrm{Hol}_J(U,N)$ and $\mathcal{M}_J(U)=\mathrm{Hol}_J(U,Y)\setminus\mathrm{Hol}_J(U,Y_\infty)$.

The action of N on Y induces an action of $\mathcal{O}_J(U)$ on $\mathcal{M}_J(U)$, which clearly preserves poles and their orders. So we can define the quotient sheaf $\mathcal{P}_J=\mathcal{M}_J/\mathcal{O}_J$ called the *sheaf of J-principal parts*. A *configuration of J-principal parts* is a global section of $\mathcal{P}_J$.

As noted above, a pole, the order of a point and the degree of a configuration of principal parts are well defined concepts. We are only interested in finite configurations, so we let $\mathcal{P}_J(M)$ be the set of global section ξ of $\mathcal{P}_J$ with $\deg \xi < \infty$. We furthermore let $\mathcal{P}_{J,\leq n}(M)$ and $\mathcal{P}_{J,n}(M)$ denote the set of $\xi \in \mathcal{P}_J(M)$ with respectively $\deg \xi \leq n$ and $\deg \xi = n$.

If M' and M both are subsets of a surface $\bar{M}$, then we let $\mathcal{P}_J(M, M')$ be the space of $\xi \in \mathcal{P}_J(M)$ with $\xi|_{M \cap M'} = 0$ and similar for $\mathcal{P}_{J,\leq n}(M, M')$ and $\mathcal{P}_{J,n}(M, M')$.

Finally the complex structure varies, and we get the space $\mathcal{P}(\bar{M})$ consisting of pairs (ξ, J) where $J \in \mathscr{C}(\bar{M})$ and $\xi \in \mathcal{P}_J(M)$, and the spaces $\mathcal{P}_{\leq n}(\bar{M})$, $\mathcal{P}_n(\bar{M})$, $\mathcal{P}(M, M')$, $\mathcal{P}_{\leq n}(M, M')$ and $\mathcal{P}_n(M, M')$ whose definition should be obvious.

Let $\mathscr{A}(\bar{M}, M')$ be the quotient of the free Abelian monoid, generated by points of $\bar{M} \setminus M'$ by the relation, which identifies points on ∂M with zero, see [14, p. 45], and define the *pole map* $\mathcal{P}(\bar{M}, M') \to \mathscr{A}(\bar{M}, M')$ by $(\xi, J) \mapsto \Sigma_{\alpha \in M} \operatorname{ord}_\alpha \xi \cdot \alpha$.

A J-holomorphic map $f: M \to Y$ with $f(M) \cap Y_a \neq \emptyset$ and $\deg f < \infty$ defines a configuration $[f]$ of J-principal parts with $\deg_\alpha[f] = \deg_\alpha f$ all $\alpha \in M$, i.e., we have a map $\mathcal{M}(\bar{M}, M') \to \mathcal{P}(\bar{M}, M'): (f, J) \mapsto ([f], J)$, which preserves the degree.

LEMMA 6.1. *Let $f, f' \in \mathcal{M}_J(M)$. then $[f] = [f']$ if and only if there exists a J-holomorphic map $g: M \to N$, such that $f' = gf$.*

Proof. The 'if' part is clear, so assume $[f] = [f']$. Let $\alpha_1, \ldots, \alpha_n$ be the poles of f and f' and put $V = M \setminus \{\alpha_1, \ldots, \alpha_n\}$. There exist neighbourhoods U_i of α_i and J-holomorphic maps $g_i: U_i \to N$, such that $f'|_{U_i} = g_i f|_{U_i}$ for all $i = 1, \ldots, n$. On V we can consider f and f' as maps into N. So on $V \cap U_i$ we must have $g_i|_{V \cap U_i} = f'|_{V \cap U_i} f|_{V \cap U_i}^{-1}$ and hence $g: M \to N$ can be defined by $g(x) = g_i(x)$ if $x \in U_i$ and $g(x) = f'(x) f^{-1}(x)$ if $x \in V$. □

The Lemma says that the fiber at $([f]), J)$ of the map $\mathcal{M}(\bar{M}) \to \mathcal{P}(\bar{M})$ is $\mathcal{O}_J(M)$.

In the case of $Y = \Omega G$, Proposition 4.1 implies that $\mathcal{P}_J(M)$ is the set of holomorphic $G_{\mathbf{C}}$-bundles on $M_J \times \mathbf{CP}^1$ with only finitely many jumping lines.

Before we equip $\mathcal{P}(\bar{M})$ with a topology, we will study the action of $\mathcal{O}_J(M)$ on $\mathcal{M}_J(M)$ a little closer.

LEMMA 6.2. $\operatorname{Hol}_J(M, N)$ *acts freely on* $\mathcal{M}_J(M)$.

Proof. Let $g \in \operatorname{Hol}_J(M, N)$ and $f \in \mathcal{M}_J(M)$ and assume that $gf = f$. As N acts freely on Y_a, $g(x) = 1$ for $x \in f^{-1}(Y_a)$, but $f^{-1}(Y_a)$ is dense in M, and thus $g = 1$. □

LEMMA 6.3. *Let $U \subseteq M$, let $a_1, \dots, a_m \in U \setminus \partial U$ and put $V = U \setminus \{a_1, \dots, a_m\}$. Let $J_n \in \mathscr{C}(\bar{M})$ and let g_n be a map $U \to N$, such that g_n is J_n-holomorphic. If $J_n \to J \in \mathscr{C}(\bar{M})$ and $g_n|_V \to g$, where $g: V \to N$ is J-holomorphic, then g extends to a J-holomorphic map $g: U \to N$, and $g_n \to g$.*

Proof. Let $a \in U \setminus V$ and choose a disk D_a in $(V \setminus \partial U) \cup \{a\}$ around a. Choose, continuously depending on $J' \in \mathscr{C}(M)$, a J'-holomorphic homeomorphism $\phi_{J'}: D_a \to D$, such that $\phi_{J'}(a) = 0$. Let $c = \{z \in \mathbf{C} |\, |z| = \frac{2}{3}\}$. We can imbed N as a closed subset of a complex topological vector space E. In the case of a loop group, E is not a Banach space, but there do exist norms $\|\cdot\|_m$ on E, and a sequence in E converges if and only if it converges in all these norms. If $x \in D_a \setminus \{a\}$, then $g(x) = \sum_{k=-\infty}^{\infty} a_k \phi_J(x)^k$ with

$$a_k = \frac{1}{2\pi i} \int_c \frac{g \circ \phi_J^{-1}(z)}{z^{k+1}} dz \in E.$$

As $g_n \to g$ and $\phi_{J_n}^{-1} \to \phi_J^{-1}$ uniformly on c, we have $a_n = 0$ if $n < 0$. Thus g extends to a J-holomorphic map $g: V \cup \{a\} \to N$. Let $K = \phi_J^{-1}(\{z \in \mathbf{C} |\, |z| \leq \frac{1}{3}\})$. It is a compact neighbourhood of a, and $\operatorname{dist}(\phi_J(K), c) = \frac{1}{3}$, hence $\operatorname{dist}(\phi_{J_n}(K), c) > \frac{1}{4}$, if n is sufficiently large. For such an n, an $x_0 \in K$ and a norm $\|\cdot\|$ as above

$$\|g_n(x_0) - g(x_0)\| = \left\| \frac{1}{2\pi i} \int_c \frac{g_n \circ \phi_{J_n}^{-1}(z)}{z - \phi_{J_n}(x_0)} dz - \frac{1}{2\pi i} \int_c \frac{g \circ \phi_J^{-1}(z)}{z - \phi_J(x_0)} \right\|$$

$$\leq \frac{1}{2\pi} \int_c \left\| \frac{(z - \phi_J(x_0))(g_n \circ \phi_{J_n}^{-1}(z) - g \circ \phi_J^{-1}(z)) + (\phi_{J_n}(x_0) - \phi_J(x_0))\, g \circ \phi_J^{-1}(z)}{(z - \phi_{J_n}(x_0))(z - \phi_J(x_0))} \right\| dz$$

$$\leq 3 \int_c (\|g_n \circ \phi_{J_n}^{-1}(z) - g \circ \phi_J^{-1}(z)\| + \|\phi_{J_n}(x_0) - \phi_J(x_0)\|\, \|g \circ \phi_J^{-1}(z)\|)\, dz.$$

As $g_n \circ \phi_{J_n}^{-1}(z) \to g \circ \phi_J^{-1}(z)$ uniformly on c, $\phi_{J_n} \to \phi_J$ uniformly on K and $\|g \circ \phi_J^{-1}(z)\|$ is bounded on c, we have $\|g_n - g\| \to 0$ uniformly on K. Hence $g_n \to g$ uniformly on compact subsets of $V \cup \{a\}$. Finally induction on the number of points in $U \setminus V$ finishes the proof. □

LEMMA 6.4. *Let J_n be a sequence of complex structures on $\bar{M}$, let $g_n \in \mathcal{O}_{J_n}(M)$ and let $f_n \in \mathcal{M}_{J_n}(M)$. If $J_n \to J \in \mathscr{C}(\bar{M})$, $f_n \to f \in \mathcal{M}_J(M)$ and $g_n f_n \to \tilde{f} \in \mathcal{M}_J(M)$, then there exists a $g \in \mathcal{O}_J(M)$, such that $g_n \to g$ and $\tilde{f} = gf$.*

Proof. Put $V = f^{-1}(Y_a) \cap \tilde{f}^{-1}(Y_a)$. Then $M \setminus V$ is finite, we can consider $f|_V$ and $\tilde{f}|_V$ as maps into N. Define $g: V \to N$ by $g = \tilde{f}|_V f|_V^{-1}$. Let K be a compact subset of V. As Y_a is open and $f(K)\subseteq Y_a$, we have that $f_n(K)\subseteq Y_a\cong N$ if n is sufficiently large. Then $g_n|_K=g_n|_K f_n|_K f_n|_K^{-1}\to \tilde{f}|_K f|_K^{-1}=g|_K$. By Lemma 6.3, g extends to a J-holomorphic map $g:M\to N$ and $g_n\to g$, which in turn implies that $g_nf_n\to gf$, and thus $\tilde{f}=gf$. □

COROLLARY 6.5. *$\mathcal{O}_J(M)$ acts properly on $\mathcal{M}_J(M)$.*

There is obviously the following generalization of Lemma 3.2 and Lemma 4.2.

LEMMA 6.6. *Let $\bar{M}$ be a two-dimensional compact connected manifold with non-empty boundary and let $\bar{D}_1,\dots,\bar{D}_n$ be disjoint closed disks in M. Suppose we have J-holomorphic maps $f_i:\bar{D}_i\to Y$ with $f_i(\partial D_i)\subseteq Y_a$, then there exist J-holomorphic maps $f:\bar{M}\to Y$ and $g_i:\bar{D}_i\to N$ such that $f_i=g_i f|_{\bar{D}_i}$ and the poles of f is contained in $D_1\cup\dots\cup D_n$.*

Furthermore, for small variations of $f_1,\dots,f_n$ and J, the choices can be made, such that f and $g_1,\dots,g_n$ depend continuously on $f_1,\dots,f_n$ and J.

COROLLARY 6.7. *If $\bar{M}$ is a compact connected surface with $\partial M\neq\emptyset$, then the map $\mathcal{M}(\bar{M})\to\mathcal{P}(\bar{M})$ is surjective, and as sets $\mathcal{P}_J(M)=\mathcal{M}_J(M)/\mathcal{O}_J(M)$.*

We are now ready to define the topology on $\mathcal{P}(\bar{M})$ in the case, where M has a boundary. For a compact subset K of M, we let $\mathcal{M}(K)$ denote the space of pairs $(f,J)\in\mathrm{Map}(K,Y)\times\mathscr{C}(\bar{M})$, where f extends to an element of $\mathcal{M}_J(U)$ for some neighbourhood U of K. We define an equivalence relation $\sim$ on $\mathcal{M}(K)$ by letting $(f_1,J_1)\sim(f_2,J_2)$, if $J_1=J_2$ and there exist a neighbourhood U of K and a map $g\in\mathrm{Hol}_{J_1}(U,N)$, such that $f_1=g|_K f_2$. Equip $\mathcal{M}(K)/\sim$ with the quotient topology. Put the weakest topology on $\mathcal{P}_{\leqslant n}(\bar{M})$, which makes the restriction map $\mathcal{P}_{\leqslant n}(\bar{M})\to\mathcal{M}(K)/\sim$ continuous for all compact subsets K of M. Finally let $\mathcal{P}(\bar{M})=\lim_{n\to\infty}\mathcal{P}_{\leqslant n}(\bar{M})$. If $\partial M\neq\emptyset$, then the maps $\mathcal{M}(\bar{M})\to\mathcal{P}(\bar{M})$ and $\mathcal{P}(\bar{M})\to\mathcal{A}(\bar{M})$ are continuous.

If $\bar{D}_1,\dots,\bar{D}_k$ are disjoint disks in M, and, for $i=1,\dots,k$, $f_i:D_i\to Y$ is a J-holomorphic map with $f_i(D_i)\cap Y_a\neq\emptyset$ and $\deg f_i<\infty$, then we get a configuration of J-principal parts in M denoted $[f_1]\cup\dots\cup[f_k]$, and no matter what the boundary of M is, every configuration of J-principal parts is of this form.

Equip $\bar{D}$ with the standard complex structure and let H_n denote the space of J-holomorphic maps $f:\bar{D}\to Y$ such that $\deg f|_D=n$. Choose, for $i=1,\dots,k$ and $J\in\mathscr{C}(\bar{M})$, J-holomorphic imbeddings $\phi_{iJ}:\bar{D}\to M$ which depend continuously on J, such that $\phi_{iJ}(\bar{D})\cap\phi_{jJ}(\bar{D})=\emptyset$, if $i\neq j$. If $n=n_1+\dots+n_k$, then there is a map

$$H_{n_1}\times\dots\times H_{n_k}\times\mathscr{C}(\bar{M})\to\mathcal{P}_n(\bar{M})$$

defined by

$$(f_1,\dots,f_k,J)\mapsto([f_1\circ\phi_{1J}^{-1}]\cup\dots\cup[f_k\circ\phi_{kJ}^{-1}],J).$$

Two sets of maps $(f_1,\dots,f_k)$ and $(f'_1,\dots,f'_k)$ give the same configuration if and only if there for each $i=1,\dots,k$ exists a map $g_i\in\mathrm{Hol}(\check{D},N)$, such that $f'_i=g_if_i$. If we put $H_n/\sim=H_n/\mathrm{Hol}(\check{D},N)$, then Lemma 6.6 implies

LEMMA 6.8. *If $\partial M\neq\emptyset$, then the map above induces a local homeomorphism*

$$(H_{n_1}/\sim)\times\dots\times(H_{n_k}/\sim)\times\mathscr{C}(\check{M})\hookrightarrow\mathscr{P}_n(\check{M}),$$

and every element of $\mathscr{P}_n(\check{M})$ has a neighbourhood, which is the image of such a homeomorphism.

In particular, the transition functions between spaces of the form

$$(H_{n_1}/\sim)\times\dots\times(H_{n_k}/\sim)\times\mathscr{C}(\check{M})$$

are homeomorphism. This is even the case if $\partial M=\emptyset$, because we can always remove a disk from M without disturbing a given configuration of principal parts. So if $\partial M=\emptyset$, the topology on $\mathscr{P}(M)$ can be defined by declaring the inclusions

$$(H_{n_1}/\sim)\times\dots\times(H_{n_k}/\sim)\times\mathscr{C}(M)\hookrightarrow\mathscr{P}(M)$$

to be local homeomorphisms. The subspace $\mathscr{P}_n(M)$ is then open and closed in $\mathscr{P}(M)$, and we still have

LEMMA 6.9. *The maps $\mathscr{M}(\check{M})\to\mathscr{P}(\check{M})\to\mathscr{A}(\check{M})$ are continuous.*

Let $\tilde{H}_n=\{f\in H_n|\ f(\check{D})$ is contained in a chart$\}$. Then $\tilde{H}_n$ is an open subset of $\widetilde{\mathrm{Hol}}(\check{D},Y)$ and hence a complex manifold modelled on $\mathrm{Hol}(\check{D},\mathfrak{n})$, see Lemma 5.4. The following result is obvious.

LEMMA 6.10. *The restriction of the action $F\colon\mathrm{Hol}(\check{D},N)\times\tilde{H}_n\to H_n$ to $F^{-1}(\tilde{H}_n)$ is holomorphic.*

As a corollary we have

LEMMA 6.11. *$\tilde{H}_n/\sim$ is a manifold, and the projection $\tilde{H}_n\to\tilde{H}_n/\sim$ has local sections.*

Proof. $\widetilde{\mathrm{Hol}}(\bar{D}, Y)$ acts freely and properly on H_n, and a neighbourhood of the identity acts smoothly on $\tilde{H}_n$. □

In Lemma 6.8 we may clearly replace H_n with $\tilde{H}_n$, i.e., we have

LEMMA 6.12. *The maps*

$$(\tilde{H}_{n_1}/\sim)\times\ldots\times(\tilde{H}_{n_k}/\sim)\times\mathscr{C}(\bar{M})\to\mathscr{P}_n(\bar{M})$$

are local homeomorphisms and cover $\mathscr{P}_n(\bar{M})$.

As the fiber of $\mathscr{M}(\bar{M})\to\mathscr{P}(\bar{M})$ is $\mathcal{O}_J(M)$, which is contractible, it is not surprising that the map is a weak homotopy equivalence, but before we can prove it, we need to show that it is a quasifibration.

LEMMA 6.13. *If* $\partial M\neq\varnothing$, *then the map* $\pi\colon\mathscr{M}_n(\bar{M})\to\mathscr{P}_n(\bar{M})$ *is a quasifibration over any open subset of* $\mathscr{P}_n(\bar{M})$.

Proof. By [2, Satz 2.2], it is enough to show that π is a quasifibration over arbitrarily small open subsets. Locally we have a commutative diagram

$$\begin{array}{ccc}\tilde{H}_{n_1}\times\ldots\times\tilde{H}_{n_k}\times\mathscr{C}(\bar{M}) & \longrightarrow & \mathscr{M}_n(\bar{M})\\ \downarrow & & \downarrow\pi\\ (\tilde{H}_{n_1}/\sim)\times\ldots\times(\tilde{H}_{n_k}/\sim)\times\mathscr{C}(\bar{M}) & \longrightarrow & \mathscr{P}_n(\bar{M})\end{array}$$

As there are local sections of $\tilde{H}_{n_i}\to\tilde{H}_{n_i}/\sim$, there are local sections of π. Let $\sigma\colon W\to\mathscr{M}_n(\bar{M})$ be a section of π over an open subset $W\subseteq\mathscr{P}_n(\bar{M})$. We only need to show that $\pi|_{\pi^{-1}(W)}\colon\pi^{-1}(W)\to W$ is a quasifibration. Let $\tilde{W}$ be the set of triples $(g,\xi,J)\in\mathrm{Map}(M,N)\times W$ such that g is J-holomorphic, and consider the map $(g,\xi,J)\mapsto g\sigma(\xi,J)$ from $\tilde{W}$ to $\pi^{-1}(W)$. It is a homeomorphism, so we only have to show that the projection $\tilde{W}\to W$ is a quasifibration. This is trivial, as a contraction of N induces a fiber preserving deformation of $\tilde{W}$ onto $\{0\}\times W$. □

We can now show

LEMMA 6.14. *If* $\partial M\neq\varnothing$, *then the map* $\pi\colon\mathscr{M}(\bar{M})\to\mathscr{P}(\bar{M})$ *is a quasifibration.*

Proof. As $\mathscr{P}(\bar{M})=\lim_{n\to\infty}\mathscr{P}_{\leq n}(\bar{M})$, it is by [2, Satz 2.15] enough to show that π is a quasifibration, when restricted to $\mathscr{M}_{\leq n}(\bar{M})$. This we do by induction on n. Assume that the restriction to $\mathcal{M}_{\leq n-1}(\bar{M})$ is a quasifibration. Choose a neighbourhood $B(\varepsilon)$ of ∂M in $\bar{M}$, homeomorphic to $\partial M\times[0,\varepsilon)$ and let W be the set of pairs $(\xi,J)\in\mathcal{P}_{\leq n}(\bar{M})$ with $\deg\xi|_{M\setminus B(\varepsilon)}\leq n-1$. Then W is a neighbourhood of $\mathcal{P}_{\leq n-1}(\bar{M})$ in $\mathcal{P}_{\leq n}(\bar{M})$, and it is enough to show that π is a quasifibration, when restricted to $\pi^{-1}(W)$, $\mathcal{M}_n(\bar{M})$ and $\mathcal{M}_n(\bar{M})\cap\pi^{-1}(W)$ respectively. By Lemma 6.13, the last two restrictions are quasifibrations, so we need only consider $\pi|_{\pi^{-1}(W)}:\pi^{-1}(W)\to W$. As the fibers of π are contractible, it is by [2, Hilfsatz 2.10] enough to find a deformation $\psi_t\colon W\to W$, $t\in[0,1]$, such that

(1) $\psi_0=\mathrm{id}$,

(2) $\psi_t(\mathcal{P}_{\leq n-1}(\bar{M}))\subseteq\mathcal{P}_{\leq n-1}(\bar{M})$ for all t,

(3) $\psi_1(W)=\mathcal{P}_{\leq n-1}(\bar{M})$ and

(4) ψ_t lifts to a deformation of $\pi^{-1}(W)$.

Choose a vector field on $\bar{M}$, such that the corresponding flow ϕ_t preserves $\bar{M}\setminus B(\varepsilon)$ and has $\phi_1(\bar{M})\subseteq(\bar{M})\setminus B(\varepsilon)$. We put $\tilde{\psi}_t((f,J))=(f\circ\phi_t,\phi_t(J))$. This defines a deformation $\tilde{\psi}_t$ of $\pi^{-1}(W)$, which clearly descends to the wanted deformation ψ_t of W. □

We have already noted that the fibers of $\mathcal{M}(\bar{M})\to\mathcal{P}(\bar{M})$ are contractible, so we get

LEMMA 6.15. *If $\partial M\neq\emptyset$, then the map $\mathcal{M}(\bar{M})\to\mathcal{P}(\bar{M})$ is a weak homotopy equivalence.*

Two configurations ξ_1 and ξ_2 of J-principal parts without common poles give rise to a new configuration $\xi_1\cup\xi_2$ of J-principal parts called the *union* or the *sum* of ξ_1 and ξ_2.

LEMMA 6.16. *Addition of principal parts is a continuous map:*

$$\{((\xi_1,J),(\xi_2,J))\in\mathcal{P}(\bar{M})\times\mathcal{P}(\bar{M})|\ \mathrm{pole}\,\xi_1\cap\mathrm{pole}\,\xi_2=\emptyset\}\to\mathcal{P}(\bar{M}).$$

Proof. Let $((\xi_{1n},J_n),(\xi_{2n},J_n))\to((\xi_1,J),(\xi_2,J))$ be a convergent sequence in the space above. Let $\alpha_1,\dots,\alpha_{k_1}$ be the poles of ξ_1 and let $\alpha_{k_1+1},\dots,\alpha_k$ be the poles of ξ_2. Choose disjoint closed disks $\bar{D}_1,\dots,\bar{D}_k$ in M, with $\alpha_i\in D_i$ all $i=1,\dots,k$. Let, for $j=1,2$, $\bar{\xi}_{jn}$ be the part of ξ_{jn}, which lies in $D_1\cup\dots\cup D_k$. Then $(\bar{\xi}_{jn},J_n)\to(\xi_j,J)$ and, for n sufficiently large, $\deg\bar{\xi}_{jn}=\deg\xi_j=n_j$. We obviously have that $(\bar{\xi}_{1n}\cup\bar{\xi}_{2n},J_n)\to(\xi_1\cup\xi_2,J)$, and if K is any compact subset of M, then $\bar{\xi}_{jn}|_K=\xi_{jn}|_K$, if n is large. Hence $(\xi_{1n}\cup\xi_{2n},J_n)\to(\xi_1\cup\xi_2,J)$. □

LEMMA 6.17. *The fiber of the pole map $\mathcal{P}_1(\bar{M})\to\mathcal{A}_1(M)$, restricted to configurations with one simple pole, has r connected components, one for each irreducible component Y_i of $Y_\infty=Y_1\cup\dots\cup Y_r$.*

Proof. Let $\alpha\in M=\mathcal{A}_1(M)$ be given. Choose for $J\in\mathcal{C}(\bar{M})$, a J-holomorphic imbedding $\phi_J:\bar{D}\to M$, such that $\phi_J(0)=\alpha$ and ϕ_J depends continuously on J. The fiber over α of the pole map is homeomorphic to

$$\{([f],J)\in(\bar{H}_1/\sim)\times\mathcal{C}(\bar{M})\,|\,f(0)\in Y_\infty\}.$$

As $\mathcal{C}(\bar{M})$ is contractible, it is enough to consider the space

$$\{[f]\in\bar{H}_1/\sim|\,f(0)\in Y_\infty\}.$$

Let $[f]$ be an element of this space. Then $f(\bar{D})\cap Y_\infty=\{f(0)\}$, and the order of contact is one. Thus $f(0)$ is a simple point of Y_∞, and as the sets $Y_i\cap Y_j$ consist of singular points for $i\neq j$, the fiber has at least r connected components.

On the other hand, the set of singular points in Y_∞ is a proper subvariety of Y_∞ and has at least complex codimension one. Hence the set Y_i^s of points in Y_i, which are simple in Y_∞ is connected. Around each point $y\in Y_i^s$, exist local coordinates (u,v) on Y, such that Y_i is given by the equation $u=0$. In these coordinates, f is given by a pair of maps $f(z)=(u(z),v(z))$ with $u(z)=\Sigma_{n=1}^\infty u_n z^n$, $u_1\neq 0$. We put $f_t(z)=(u_t(z),v_t(z))$ with $u_t(z)=z\Sigma_{n=1}^\infty u_n(tz)^{n-1}$ and $v_t(z)=v(tz)$. This gives us a curve f_t from $f=f_1$ to f_0. The map f_t has only one simple pole at 0 for all t, and $f_0(z)=(u_1z,v(0))$. By covering a curve in Y_i^s from $f_0(0)=(0,v(0))$ to a base point $y_i\in Y_i^s$ with a finite number of local coordinates, f_0 can be deformed such that the new f_0 has $f_0(0)=y_i$ and in local coordinates $f_0(z)=(u_1z,0)$. Finally we just have to deform u_1 into a base point. □

Higher order poles can be split continuously in the following sense.

LEMMA 6.18. *Given a J-principal part ξ at $\alpha\in M$ and a neighbourhood U of α. Then ξ can be deformed continuously into a configuration of principal parts in U, all with simple poles.*

Proof. We use induction on the order $\mathrm{ord}_\alpha\xi$ of the principal part. If $\mathrm{ord}_\alpha\xi=1$, there is nothing to show. So we need only to show that we continuously can split a principal part of order $m\geq 2$ into a configuration of two or more principal parts in U, which then necessarily have strictly lower orders.

We may assume that $U=D$, $\alpha=0$ and $f:D\to Y$ is a representative for ξ, which maps D into a chart. If $f(0)\in Y_\infty$ is a simple point, then there exist local coordinates (u,v) on Y, such that Y_∞ is given by the equation $u=0$. The map f is given by a pair of maps $f(z)=(u(z),v(z))$. Put $v_t=v$ and $u_t(z)=tz+u(z)$. Then $f_t(z)=(u_t(z),v_t(z))$ defines a curve f_t starting at $f=f_0$. For $t\neq 0$, f_t has a simple pole at 0 and hence some other pole in the vicinity of 0. If $f(0)$ is a singular point on Y_∞, then it is obviously enough to find a curve f_t with $f_0=f$, such that $f_t(0)$ is a simple point on Y_∞ for $t\neq 0$. Let u be a local coordinate on Y around $f(0)$, such that f is given by $f(z)=u(z)$, with $u(0)=0$. The singular points have at least complex codimension one in Y_∞, so there exists a curve $\tilde{u}(t)$ such that $\tilde{u}(0)=0$, which corresponds to the singular point $f(0)$, and $\tilde{u}(t)$ corresponds to simple point on Y_∞ for $t\neq 0$. We define the curve f_t by $f_t(z)=\tilde{u}(t)+u(z)$. □

Remark 6.19. If Y_∞ is irreducible, then the last two results show that the space $\mathcal{P}(\bar{M})$ is connected.

If $\bar{M}'$ is another compact surface and $\bar{M}'\subseteq\bar{M}$, then the restriction from $\bar{M}$ to $\bar{M}'$ is a continuous map $r:\mathcal{P}(\bar{M})\to\mathcal{P}(\bar{M}')$ and the fiber $r^{-1}(\xi',J')$ is homeomorphic to $\{(\xi,J)\in\mathcal{P}(\bar{M},M')|\ J|_{\bar{M}'}=J'\}$ by the map $(\xi,J)\mapsto(\xi\cup\xi',J)$. We will show that r is a quasifibration under certain conditions.

We say that $\bar{M}'\subseteq\bar{M}$ is nicely imbedded, if $\partial M'\cap M$ only has finitely many connected components $\partial_1,\dots,\partial_k$, and the closure $\bar{\partial}_i$ of each of these has the topology of a line, intersects ∂M transversally and has a neighbourhood $B_i(\varepsilon)$ in $\bar{M}$ homeomorphic to $\bar{\partial}_i\times(-\varepsilon,\varepsilon)$, such that $B_i(\varepsilon)\cap B_j(\varepsilon)=\emptyset$, if $i\neq j$. We put $B(\varepsilon)=B_1(\varepsilon)\cup\dots\cup B_k(\varepsilon)$. Then $B(\varepsilon)$ is a neighbourhood of $\overline{\partial M'\cap M}$ homeomorphic to $\overline{\partial M'\cap M}\times(-\varepsilon,\varepsilon)$.

LEMMA 6.20. *Let $\bar{M}'\subseteq\bar{M}$ be nicely imbedded and let $r:\mathcal{P}(\bar{M})\to\mathcal{P}(\bar{M}')$ be the restriction map. If $W\subseteq\mathcal{P}_n(\bar{M}')$ is open, then $r|_{r^{-1}(W)}:r^{-1}(W)\to W$ is a quasifibration.*

Proof. It is enough to show that r has the following weak form of the homotopy lifting property:

Let P be compact, and let $h:P\to r^{-1}(W)$ and $\bar{H}:P\times[0,1]\to W$ be maps, such that $\bar{H}(x,t)=r\circ h(x)$ for all $x\in P$ and $t\in[0,1/2]$. Then there exists a lift of $\bar{H}$, i.e., a map $H:P\times[0,1]\to r^{-1}(W)$, such that $r\circ H=\bar{H}$ and $H(x,0)=h(x)$ for all x.

Let h and $\bar{H}$ be as above. We can write $\bar{H}(x,t)=(\xi'(x,t),J'(x,t))$, and then $h(x)=(\xi'(x,0)\cup\xi(x),J(x))$, where the poles of $\xi(x)$ are contained in $M\setminus M'$. It is tempting to put $H(x,t)=(\xi'(x,t)\cup\xi(x)$, an extension of $J'(x,t))$, but $\xi(x)$ need to be holomorphic with respect to the extension of $J'(x,t)$. Let us for the moment assume that the poles of $\xi(x)$ are contained in an open set V with $\bar{V}\cap\bar{M}'=\emptyset$. Then we can choose the extension $J(x,t)$ of $J'(x,t)$ such that $J(x,t)|_{\bar{V}}=J(x)|_{\bar{V}}$ and all is well. The strategy is now first (while t goes from 0 to 1/2) to push $\xi(x)$ away from $\partial M'$ and then use the construction above. The details are as follows.

Choose an open set U, such that the poles of $\xi'(x,t)$ are contained in U for all

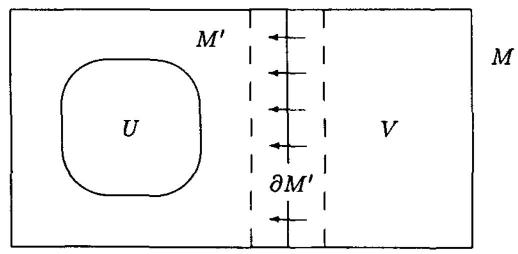

Fig. 6.1

$(x, t) \in P \times [0,1]$, and $\bar{U} \subseteq M'$. Choose for each $x \in P$ a vector field $v(x)$ on $\bar{M}$ which is $J(x)$-holomorphic in a neighbourhood $B(\varepsilon)$ of $\overline{\partial M' \cap M}$, with $\overline{B(\varepsilon)} \cap \bar{U} = \emptyset$. Let $t \mapsto \phi(x, t)$ be the flow restricted to $\overline{M \setminus M'}$, and put $V = M \setminus \overline{B(\varepsilon) \cup M'}$. We can choose $v(x)$ such that ϕ is continuous in $(x, t), M \setminus M' \subseteq \phi(x, 1)(V)$ for all $x \in P$ and such that $\phi(x, t)$ is $J(x)$-holomorphic in a neighbourhood of $\overline{\partial M' \cap M}$, see Figure 6.1.

As $\phi(x, t)$ is $J(x)$-holomorphic near $\overline{\partial M' \cap M}$, we can choose a continuous map $J: P \times [0, 1] \to \mathscr{C}(\bar{M})$, such that

(1) $J(x, t)|_{\bar{M}'} = J'(x, t)$, for all $t \in [0, 1]$,

(2) $J(x, t)|_{\bar{M} \setminus \bar{M}'} = \phi(x, 2t)(J(x))|_{\bar{M} \setminus \bar{M}'}$, for $t \in [0, 1/2]$ and

(3) $J(x, t)|_{\bar{V}} = \phi(x, 1)(J(x))|_{\bar{V}}$, for $t \in [1/2, 1]$.

As the poles of $\xi(x) \circ \phi(x, 1)$ lie in V we can regard $\xi(x) \circ \phi(x, 1)$ as a configuration of $J(x, t)$-principal parts for $t \in [1/2, 1]$. Hence it is possible to define the homotopy $H: P \times [0, 1] \to r^{-1}(W)$ by

$$H(x, t) = \begin{cases} (\xi'(x, t) \cup \xi(x) \circ \phi(x, 2t), J(x, t)), & \text{for } 0 \leq t \leq 1/2 \\ (\xi'(x, t) \cup \xi(x) \circ \phi(x, 1), J(x, t)), & \text{for } 1/2 \leq t \leq 1. \end{cases}$$

Obviously $r \circ H = \bar{H}$ and $H(x, 0) = h(x)$ all x. □

We can now show

Proposition 6.21. *Let $\bar{M}' \subseteq \bar{M}$ be nicely imbedded and assume that every component of $\partial M'$ intersects ∂M. Then the restriction map $r: \mathscr{P}(\bar{M}) \to \mathscr{P}(\bar{M}')$ is a quasifibration.*

Proof. As $\mathscr{P}(\bar{M}') = \lim_{n \to \infty} \mathscr{P}_{\leq n}(\bar{M}')$, it is enough to show that r is a quasifibration over $\mathscr{P}_{\leq n}(\bar{M}')$, which we do by induction on n. By Lemma 6.20, r is a quasifibration over $\mathscr{P}_{\leq 0}(\bar{M}') = \mathscr{P}_0(M')$, so the start of the induction is secured. Assume that r is a quasifibration over $\mathscr{P}_{\leq n-1}(\bar{M}')$.

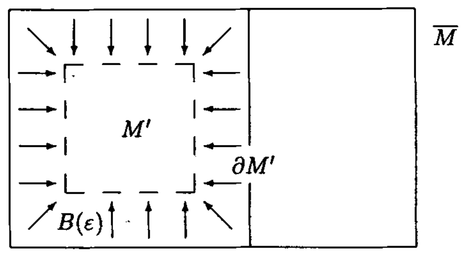

Fig. 6.2

Let $B'(\varepsilon)$ be a neighbourhood of $\partial M'$ in $\bar{M}'$, homeomorphic to $\partial M'\times[0,\varepsilon)$, and let W be the set of pairs $(\xi,J)\in\mathscr{P}_{\leq n}(\bar{M}')$ with $\xi|_{M'\setminus B(\varepsilon)}\leq n-1$. It is a neighbourhood of $\mathscr{P}_{\leq n-1}(\bar{M}')$ in $\mathscr{P}_{\leq n}(\bar{M}')$, and by Lemma 6.20, r is a quasifibration over $\mathscr{P}_n(\bar{M}')$ and $W\cap\mathscr{P}_n(\bar{M}')$. Thus, it is enough to show that r is a quasifibration over W, see [2, Satz 2.2].

As in [2] and [10] we only have to contract W onto $\mathscr{P}_{\leq n-1}(\bar{M}')$ and show that the contraction lifts to a deformation of $r^{-1}(W)$, which is a weak homotopy equivalence on the fibers. Choose a vector field on $\bar{M}$, such that the induced flow ϕ_t satisfies

(1) $\phi_t(M')\subseteq M'$ for all t,

(2) $\phi_t(M'\setminus B'(\varepsilon))\subseteq M'\setminus B'(\varepsilon)$ for all t and

(3) $\phi_1(M')\subseteq M'\setminus B'(\varepsilon)$.

See Figure 6.2.

We define deformations d_t of W and D_t of $r^{-1}(W)$ by

$$d_t(\xi',J')=(\xi'\circ\phi_t,\phi_t(J'))\quad\text{and}\quad D_t(\xi,J)=(\xi\circ\phi_t,\phi_t(J)).$$

As $r\circ D_t=d_t\circ r$, $d_t(\mathscr{P}_{\leq n-1}(\bar{M}'))\subseteq\mathscr{P}_{\leq n-1}(\bar{M}')$ and $d_1(W)\subseteq\mathscr{P}_{\leq n-1}(\bar{M}')$, we only have to show that $D_1|_{r^{-1}((\xi',J'))}: r^{-1}(\xi',J')\to r^{-1}(d_1(\xi',J'))$ is a weak homotopy equivalence, see [2, Satz 2.10]. The fiber $r^{-1}(\xi',J')$ is homeomorphic to the space F_0 of pairs $(\xi,J)\in\mathscr{P}(\bar{M},M')$ with $J|_{\bar{M}'}=J'$ and $r^{-1}(d_1(\xi',J'))$ is homeomorphic to the space F_1 of pairs $(\xi,J)\in\mathscr{P}(\bar{M},M')$ with $J|_{\bar{M}'}=\phi_1(J')$. If we consider D_1 as a map $F_0\to F_1$, then $D_1(\xi,J)=((\tilde{\xi}\cup\xi)\circ\phi_1,\phi_1(J))$, where $\tilde{\xi}$ is a (possibly empty) configuration of principal parts in $B'(\varepsilon)\cap M'$, which by the flow ϕ_t is moved to $M\setminus M'$. The configuration $\xi\circ\phi_1$ is pushed away from $\partial M'$, and it is possible to move $\tilde{\xi}$ along $\partial M'$ to ∂M. Hence D_1 is homotopy equivalent to the map $D: F_0\to F_1$, given by $D(\xi,J)=(\xi\circ\phi_1,\phi_1(J))$. We want to find a homotopy inverse $\hat{D}: F_1\to F_0$.

We cannot use D^{-1} as it would move principal parts in $M\setminus M'$ into M'. Instead we will first move the principal parts away from $\partial M'$, and then use D^{-1}, but this process must not change the complex structure in $\bar{M}'$. There are no principal parts in $\bar{M}'$ so we need only worry about the complex structure in a neighbourhood of $\partial M' \cap M$. So we will move the principal parts by a flow which is holomorphic in a neighbourhood of $\partial M' \cap M$.

Let $B(\varepsilon)$ be a neighbourhood of $\overline{\partial M' \cap M}$ in $\bar{M}$, which is homeomorphic to $\overline{\partial M' \cap M} \times (-\varepsilon, \varepsilon)$ and let $s \mapsto \psi(t, s)$ be the flow of a vector field on $\bar{M}$, such that

(1) $\psi(t, s)$ depends continuously on (t, s),

(2) $\psi(t, s)$ is $\phi_t(J')$-holomorphic on $B(\varepsilon) \cap \bar{M}'$ for all (t, s),

(3) $M \setminus (B(\varepsilon) \cup M') \subseteq \psi(t, s)(M \setminus (B(\varepsilon) \cup M'))$ for all (t, s),

(4) $M \setminus M' \subseteq \psi(t, 1)(M \setminus (B(\varepsilon) \cup M'))$ for all t,

(5) there exists an $n \in \mathbf{N}$ such that

(i) $\phi_{1/n}(M \setminus M') \subseteq \psi(t, s) \circ \phi_{1/n}(M \setminus M')$ for all (t, s),

(ii) $\phi_{1/n}(M \setminus M') \subseteq \psi(t, 1)(M \setminus M')$ for all t.

If $n = 1$, then we could move the principal parts away from $\partial M'$ by ψ, and then use ϕ to move them back and at the same time change the complex structure on $\bar{M}'$, from $\phi_1(J')$ to J', i.e., use D^{-1}. For an arbitrarily n we do the same, but in several steps. The details are as follows. Put

$$\theta = \phi_{1/n}^{-1} \circ \psi\left(\frac{1}{n}, 1\right) \circ \phi_{1/n}^{-1} \circ \psi\left(\frac{2}{n}, 1\right) \circ \ldots \circ \phi_{1/n}^{-1} \circ \psi(1, 1),$$

and define for a $J \in \mathscr{C}(\bar{M})$ with $J|_{\bar{M}'} = \phi_1(J')$, a complex structure $h(J)$ on $\bar{M}$ by $h(J)|_{\bar{M}'} = J'$ and $h(J)|_{\bar{M} \setminus M'} = \theta(J)$. Now $\hat{D}: F_1 \to F_0$ is defined by $\hat{D}(\xi, J) = (\xi \circ \theta, h(J))$.

We shall show that $D \circ \hat{D}$ and $\hat{D} \circ D$ are homotopic to the identity. First we consider $\hat{D} \circ D$ and define $\theta_t: \bar{M} \to \bar{M}$ for $k/n \leq t \leq (k+1)/n$ by

$$\theta_t = \phi_{k/n} \circ \psi\left(\frac{n-k}{n}, nt-k\right) \circ \phi_{1/n}^{-1} \circ \psi\left(\frac{n-k+1}{n}, 1\right) \circ \ldots \circ \phi_{1/n}^{-1} \circ \psi(1, 1).$$

For a $J \in \mathscr{C}(\bar{M})$ with $J|_{\bar{M}'} = J'$, we define $h_t(J) \in \mathscr{C}(\bar{M})$ by $h_t(J) = J'$ on M' and $h_t(J) = \theta_t(J)$ on $\bar{M} \setminus M'$. Finally $H_t: F_0 \to F_0$ is defined by $H_t(\xi, J) = (\xi \circ \theta_t, h_t(J))$. Clearly $H_0 = \mathrm{id}$ and $H_1 = \hat{D} \circ D$.

The proof that $D \circ \hat{D}$ is homotopic to the identity is similar. □

7. The results

In this section we will show the topology of the space of holomorphic maps resembles the topology of the space of continuous maps. First a non-closed surface is considered.

PROPOSITION 7.1. *Let $\bar{M}$ be a compact surface, and assume that every component of $\bar{M}$ has non empty boundary. Then the map $\mathcal{M}(\bar{M})\to \mathrm{Map}(M, Y)$ is a weak homotopy equivalence.*

Proof. The surface $\bar{M}$ can be made by gluing disks together, and we use induction on the number of disks. The start of the induction is secured by Lemma 5.7. So assume $\bar{M}=\bar{M}_1\cup\bar{M}_2$, and that the proposition is true for $\bar{M}_1$, $\bar{M}_2$ and $\bar{M}_1\cap\bar{M}_2$. We may assume that the inclusions $\bar{M}_1\cap\bar{M}_2\subseteq\bar{M}_1$ and $\bar{M}_2\subseteq\bar{M}$ satify the conditions of Proposition 6.21. Consider the diagram

$$\begin{array}{ccccccccc} \mathcal{P}(\bar{M}) \to & \mathcal{P}(\bar{M}_1) & & \mathcal{M}(\bar{M}) \to & \mathcal{M}(\bar{M}_1) & & \mathrm{Map}(M, Y) \to & \mathrm{Map}(M_1, Y) \\ \downarrow & \downarrow & \leftleftarrows & \downarrow & \downarrow & \to & \downarrow & \downarrow \\ \mathcal{P}(\bar{M}_2) \to & \mathcal{P}(\bar{M}_1\cap\bar{M}_2) & & \mathcal{M}(\bar{M}_2) \to & \mathcal{M}(\bar{M}_1\cap\bar{M}_2) & & \mathrm{Map}(M_2, Y) \to & \mathrm{Map}(M_1\cap M_2, Y) \end{array}$$

where the maps in the squares are restrictions. The maps between the squares are weak homotopy equivalences, except possibly, the map $\mathcal{M}(\bar{M})\to\mathrm{Map}(M, Y)$. The right-hand square is homotopy cartesian, and if the middle square is weak homotopy cartesian, the proof is complete. The left-hand square is weak homotopy cartesian, because the vertical maps are quasifibrations, but then the middle square is weak homotopy cartesian too. □

It is unfortunately impossible to apply the proof of Proposition 7.1 in the case, where $\partial M=\emptyset$, because the relevant restrictions are not quasifibrations. Indeed, in the proof of Proposition 6.21, it was crucial to be able to push a configuration ξ to the boundary ∂M. In order to overcome this difficulty, a new stabilized space is introduced.

Let M be a closed surface. Choose open subsets $M_1, M_2\subseteq M$, such that $\bar{M}_1$ and $\bar{M}_2$ are manifolds with boundaries, and $\bar{M}_1$ and $M\setminus M_2$ are closed disks with $M\setminus M_2\subseteq M_1$. Then $M=M_1\cup M_2$, and $M_1\cap M_2$ is an annulus, see Figure 7.1.

Choose a sequence of disks $D_1, D_2, \ldots$ in M_1 such that $\bar{D}_{k+1}\subseteq D_k$ all k, and $\bar{D}_\infty=\bigcap D_k$ is a disk with $\partial M_2\cap D_\infty\neq\emptyset$. Choose for all k, a point $a_k\in D_k\setminus\bar{D}_{k+1}$, such that $\overline{\{a_k|k\in\mathbb{N}\}}\cap\bar{M}_2=\emptyset$, see Figure 7.2.

Now choose continuously depending on $J\in\mathscr{C}(\bar{M}_1)$, a J-holomorphic imbedding $\phi_{Jk}: D\to D_k\setminus(\bar{D}_{k+1}\cup\bar{M}_2)$, such that $\phi_{Jk}(0)=a_k$. *If* $Y_1, \ldots, Y_r$ are the irreducible components of Y_∞, then for each $i=1, \ldots, r$, we choose a holomorphic map $f_i: D\to Y$, such that 0 is the only pole and $\mathrm{ord}_{j,0}f_i=\delta_{ij}$. We define a J-principal part ξ_{Jk} at a_k, with $\mathrm{ord}_j\xi_{Jk}=\delta_{ij}$ where $k\equiv i \pmod r$, by $\xi_{Jk}=[f_i\circ\phi_{Jk}^{-1}]$. Define imbeddings $\mathcal{P}(M, \bar{D}_k)\hookrightarrow\mathcal{P}(M, \bar{D}_{k+1})$ by

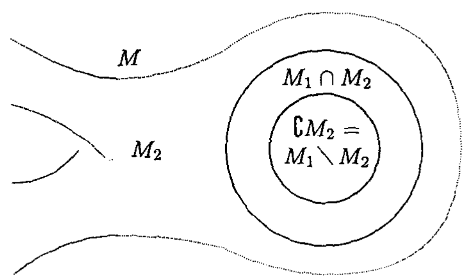

Fig. 7.1

$(\xi, J)\mapsto(\xi\cup\xi_{J|_{\bar{M}_1}k}, J)$ and $\mathcal{P}(M_1, \bar{D}_k)\hookrightarrow\mathcal{P}(M_1, \bar{D}_{k+1})$ by $(\xi, J)\mapsto(\xi\cup\xi_{Jk}, J)$, and form the telescopes $\hat{\mathcal{P}}(M, \bar{D}_\infty)$ of the sequence $\mathcal{P}(M, \bar{D}_1)\hookrightarrow\mathcal{P}(M, \bar{D}_2)\hookrightarrow\ldots$ and $\hat{\mathcal{P}}(\bar{M}_1, \bar{D}_\infty)$ of the sequence $\mathcal{P}(\bar{M}_1, \bar{D}_1)\hookrightarrow\mathcal{P}(\bar{M}_1, \bar{D}_2)\hookrightarrow\ldots$, then we have

PROPOSITION 7.2. *There is a homology cartesian commutative diagram:*

$$\begin{array}{ccc} \hat{\mathcal{P}}(M, \bar{D}_\infty) & \longrightarrow & \hat{\mathcal{P}}(\bar{M}_1, \bar{D}_\infty) \\ \downarrow r & & \downarrow r \\ \mathcal{P}(\bar{M}_2, \bar{D}_\infty) & \longrightarrow & \mathcal{P}(\bar{M}_1\cap\bar{M}_2, \bar{D}_\infty). \end{array}$$

Proof. If we let a homology fibration be as in [11], then it is enough to show that the restriction maps r are homology fibrations. Let $\bar{M}'$ denote either M or $\bar{M}_1$ and put $\bar{M}_2'=\bar{M}_2\cap\bar{M}'$. Let (ξ_0, J_0) belong to $\mathcal{P}(\bar{M}_2', \bar{D}_\infty)$, let $\beta_1, \ldots, \beta_k$ be the poles of ξ_0 and let $\nu_1, \ldots, \nu_k$ be their orders. Let $B(\varepsilon)$ be a neighbourhood of $\partial M_2'$ in M, which is homeomorphic to $\partial M_2'\times(-\varepsilon, \varepsilon)$, and choose $\varepsilon>0$ such that $\alpha_i, \beta_j\notin\overline{B(2\varepsilon)}$ and

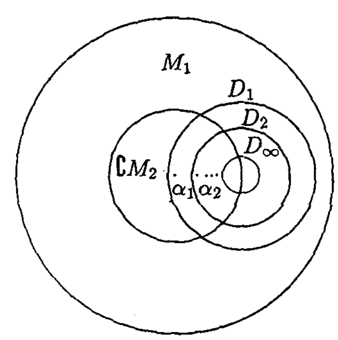

Fig. 7.2

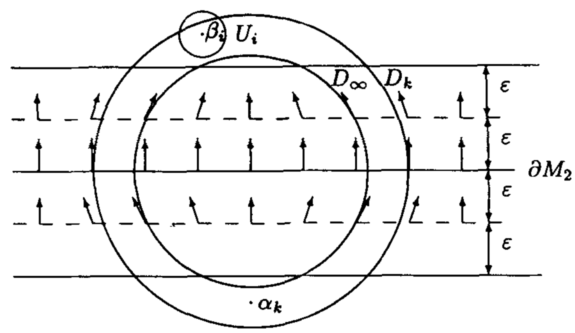

Fig. 7.3

$D_\infty \cap (M_2 \setminus B(2\varepsilon)) \neq \varnothing$, see Figure 7.3. Choose for $i=1,\dots,k$ an open disk U_i around β_i, such that $\bar{U}_i \subseteq M_2' \setminus (\overline{B(2\varepsilon)} \cup \bar{D}_\infty)$ and $\bar{U}_i \cap \bar{U}_j = \varnothing$ if $i \neq j$.
The set

$$\{(\xi, J) \in \mathcal{P}(\bar{M}_2', \bar{D}_\infty) |\ \deg \xi|_{U_i} = \nu_i \text{ and } \operatorname{pole}(\xi) \subseteq U_1 \cup \dots \cup U_k \cup B(\varepsilon)\}$$

is a neighbourhood of (ξ_0, J_0) in $\mathcal{P}(\bar{M}', \bar{D}_\infty)$. So by Lemma 6.12, (ξ_0, J_0) has a neighbourhood homeomorphic to

$$(\tilde{H}_{\nu_1}/\sim) \times \dots \times (\tilde{H}_{\nu_k}/\sim) \times \{(\xi, J) \in \mathcal{P}(\bar{M}_2', \bar{D}_\infty) |\ \operatorname{pole}(\xi) \subseteq B(\varepsilon)\}.$$

As $\tilde{H}_{\nu_1}/\sim$ is a manifold (ξ_0, J_0) has a neighbourhood W in $\mathcal{P}(\bar{M}_2', \bar{D}_\infty)$, homeomorphic to $B_1 \times \dots \times B_k \times B$, where B_i is an open contractible subset of $\tilde{H}_{\nu_i}/\sim$, and $B = \{(\xi, J) \in \mathcal{P}(\bar{M}_2', \bar{D}_\infty) |\ \operatorname{pole}(\xi) \subseteq B(\varepsilon)\}$. From the diagram

$$\begin{array}{ccc} r^{-1}(W) & \xrightarrow{\ \simeq\ } & B_1 \times \dots \times B_k \times r^{-1}(B) \\ \Big\downarrow {\scriptstyle r|_{r^{-1}(W)}} & & \Big\downarrow {\scriptstyle \mathrm{id} \times r|_{r^{-1}(B)}} \\ W & \xrightarrow{\ \simeq\ } & B_1 \times \dots \times B_k \times B, \end{array}$$

it is seen that we only have to show that B is contractible, and that the inclusions of the fibers of r in $r^{-1}(B)$ are homology equivalences.

Choose a vector field on M, which vanishes outside $B(2\varepsilon)$, is tangent to ∂D_i for all i, is transversal to $\partial M_2'$ and points into M_2', see Figure 7.3.

Let ϕ_t be the flow on M, induced by this vector field. We may assume that

(1) $\phi_t = \mathrm{id}$ outside $B(2\varepsilon)$ for all t,

(2) $\phi_t(D_k) = D_k$ for all t,

(3) $\phi_t(M_2' \cup B(\varepsilon)) \subseteq M_2' \cup B(\varepsilon)$ for all t, and

(4) $\phi_1(M_2' \cup B(\varepsilon)) \subseteq M_2' \setminus \overline{B(\varepsilon)}$.

The flow ϕ_t induces a deformation h_t of B, given by $h_t(\xi, J) = (\xi \circ \phi_t, \phi_t(J))$. As $h_1(B) \cong \mathscr{C}(\bar{M}_2')$, B is contractible. Let $(\xi', J') \in B$. We shall show that the inclusion $r^{-1}(\xi', J') \hookrightarrow r^{-1}(B)$ is a homology equivalence. Define a deformation H_t of $r^{-1}(B)$ by $H_t(\xi, J, s) = (\xi \circ \phi_t, \phi_t(J), s)$. Then $H_1(r^{-1}(B))$ is the space of triples $(\xi, J, s) \in \hat{\mathscr{P}}(\bar{M}', \bar{D}_\infty)$ with pole$(\xi) \subseteq M \setminus \overline{M_2 \cup B(\varepsilon)}$. Let F_1 be the space of triples $(\xi, J, s) \in H_1(r^{-1}(B))$ with $J|_{\bar{M}_2'} = \phi_1|_{\bar{M}_2'}(J')$, and consider the diagram

$$\begin{array}{ccc} r^{-1}(\xi', J') & \to & r^{-1}(B) \\ \downarrow{\scriptstyle H_1} & & \downarrow{\scriptstyle H_1} \\ F_1 & \to & H_1(r^{-1}(B)). \end{array}$$

We will show that the two vertical maps and the lower horizontal map are homology equivalences, and hence that the top horizontal map is a homology equivalence.

First consider $H_1: r^{-1}(B) \to H_1(r^{-1}(B))$. If $i: H_1(r^{-1}(B)) \to r^{-1}(B)$ is the inclusion, then $i \circ H_1 = H_1 \sim H_0 = \text{id}$, and as $H_t(H_1(r^{-1}(B))) \subseteq H_1(r^{-1}(B))$ for all t, we also have $H_1 \circ i = H_1|_{H_1(r^{-1}(B))} \sim H_0|_{H_1(r^{-1}(B))} = \text{id}$. Next consider the inclusion $F_1 \hookrightarrow H_1(r^{-1}(B))$. Choose a deformation D_t of $\mathscr{C}(\bar{M}')$ such that

(1) $D_0 = \text{id}$,

(2) $D_t(J)|_{M \setminus (M_2 \cup B(\varepsilon))} = J|_{M \setminus (M_2 \cup B(\varepsilon))}$ for all J,

(3) $D_t(J)|_{\bar{M}_2'} = \phi_1|_{\bar{M}_2'}(J')$ if $J|_{\bar{M}_2'} = J'$ for all t, and

(4) $D_1(J)|_{\bar{M}_2'} = \phi_1|_{\bar{M}_2'}(J')$ for all J.

Define a deformation $\tilde{D}_t$ of $H_1(r^{-1}(B))$ by $\tilde{D}_t(\xi, J, s) = (\xi, D_t(J), s)$. This deformation contracts $H_1(r^{-1}(B))$ onto F_1, hence the inclusion $F_1 \hookrightarrow H_1(r^{-1}(B))$ is a homotopy equivalence.

Only the map $H_1: r^{-1}(\xi', J') \to F_1$ remains. Let F_0 the space of triples $(\xi, J, t) \in \hat{\mathscr{P}}(\bar{M}', \bar{D}_\infty)$ with pole$(\xi) \subseteq M \setminus \bar{M}_2$ and $J|_{\bar{M}_2'} = J'$. This space is homeomorphic to $r^{-1}(\xi', J')$ by the map $F_0 \to r^{-1}(\xi', J')$, which maps (ξ, J, t) to $(\xi \cup \xi', J, t)$. By this identification, H_1 corresponds to the map $H: F_0 \to F_1$ by

$$(\xi, J, t) \mapsto (\xi \circ \phi_1 \cup \xi' \circ \phi_1, \phi_1(J), t).$$

By Lemma 6.18 and Lemma 6.17, we can split $\xi' \circ \phi_1$ into simple principal parts, move these principal parts along $\phi_1^{-1}(\partial M_2)$ to the points α_k, and finally deform them into the standard form ξ_{kJ}.

The spaces F_0 and F_1 are the telescopes of the sequences $F_0^1 \to F_0^2 \to \dots$ and $F_1^1 \to F_1^2 \to \dots$ respectively, where F_0^n is the space of pairs $(\xi, J) \in \mathcal{P}(\bar{M}', \bar{D}_n)$ with $\text{pole}(\xi) \subseteq M \setminus \bar{M}_2$ and $J|_{\bar{M}_2'} = J'$ and F_1^n is the space of pairs $(\xi, J) \in \mathcal{P}(\bar{M}', \bar{D}_n)$ with $\text{pole}(\xi) \subseteq M \setminus \overline{M_2 \cup B(\varepsilon)}$ and $J|_{\bar{M}_2'} = \phi_1(J')|_{\bar{M}_2'}$. We define $\tilde{H}: F_0^n \to F_1^n$ by

$$\tilde{H}(\xi, J) = (\xi \circ \phi_1, \phi_1(J)).$$

It is enough to show that $\tilde{H}$ is a homotopy equivalence, and this can be proved by the same method as in the proof of Proposition 6.21. □

We can now show that $\hat{\mathcal{P}}(M, \bar{D}_\infty)$ and $\text{Map}(M, \bar{D}_\infty; Y, Y_a)$, which is the space of maps $f: M \to Y$ with $f(\bar{D}_\infty) \subseteq Y_a$, have the same homology type. Let H_1 be the homotopy theoretical fiber product of $\hat{\mathcal{P}}(\bar{M}_1, \bar{D}_\infty)$ and $\mathcal{P}(\bar{M}_2, \bar{D}_\infty)$, let H_2 be the homotopy theoretical fiber product of $\mathcal{M}(\bar{M}_1, \bar{D}_1)$ and $\mathcal{M}(\bar{M}_2, \bar{D}_1)$ and let H_3 be the homotopy theoretical fiber product of $\text{Map}(M_1, \bar{D}_\infty; Y, Y_a)$ and $\text{Map}(\bar{M}_2, \bar{D}_\infty; Y, Y_a)$

The inclusion $\text{Map}(M, \bar{D}_\infty; Y, Y_a) \hookrightarrow H_3$ is a homotopy equivalence, and by Proposition 7.2, the inclusion $\hat{\mathcal{P}}(M, \bar{D}_\infty) \hookrightarrow H_1$ is a homology equivalence. All in all we have

THEOREM 7.3. *In the commutative diagram*

$$\begin{array}{ccccc} \hat{\mathcal{P}}(M, \bar{D}_\infty) & \longleftarrow & \mathcal{M}(M, \bar{D}_1) & \longrightarrow & \text{Map}(M, \bar{D}_\infty; Y, Y_a) \\ \downarrow & & \downarrow & & \downarrow \\ H_1 & \longleftarrow & H_2 & \longrightarrow & H_3, \end{array}$$

the bottom horizontal maps are weak homotopy equivalences, the left-hand vertical map is a homology equivalence and the right-hand vertical map is a homotopy equivalence.

Proof. We only have to show that

$$\mathcal{M}(\bar{M}_1, \bar{D}_1) \to \hat{\mathcal{P}}(\bar{M}_1, \bar{D}_\infty) \quad \text{and} \quad \mathcal{M}(\bar{M}_1, \bar{D}_1) \to \text{Map}(M_1, \bar{D}_\infty; Y, Y_a)$$

are equivalences, but this is trivial, as all three spaces are contractible. □

The same conclusion holds, if the complex structure is fixed, but before we can show that, some terminology is needed.

Imbed M in $\mathbf{R}^3$, and choose a tubular neighbourhood U of M. The imbedding and U can be chosen, such that any subset of M with diameter less than 10, is contained in a disk in M and has its convex hull contained in U. Let $a_1, \dots, a_n \in M$ be points with weights $\nu_1, \ldots, \nu_n$. If $\operatorname{diam}(\{\alpha_1, \ldots, \alpha_n\}) \leq 10$, then the ordinary center of mass lies in U and can be projected down to a point on M, which we will call the *center of mass*, and which depends continuously on the configuration $(\alpha_1^{\nu_1}, \ldots, \alpha_n^{\nu_n})$ of points in M.

Choose a point $x_\infty \in M$, and put $M' = M \setminus \{x_\infty\}$. Blow the metric up at x_∞, such that any subset M' with diameter less than 10 is contained in a disk in M', and any configuration of points in $M \setminus \{x\}$ with diameter less than 10 has a well defined center of mass.

Let $r \in \mathbf{R}_+$ and $\xi \in \mathscr{A}_{\leq n}(M')$. If $\operatorname{diam}(\xi) \leq r \cdot 4^{\deg \xi - n}$, then ξ is called *r-small*.

Lemma 7.4. *If ξ_1 and ξ_2 are r-small and $\xi_1 \cap \xi_2 \neq \emptyset$, then $\xi_1 \cup \xi_2$ is r-small.*

Proof. If $\xi_1 \subseteq \xi_2$ or $\xi_2 \subseteq \xi_1$ there is nothing to show, so we may assume that $\deg(\xi_1 \cup \xi_2) \geq \max\{\deg \xi_1, \deg \xi_2\} + 1$. Then

$$\begin{aligned} \operatorname{diam}(\xi_1 \cup \xi_2) &\leq \operatorname{diam} \xi_1 + \operatorname{diam} \xi_2 \leq r \cdot 4^{\deg \xi_2 - n} + r \cdot 4^{\deg \xi_2 - n} \\ &\leq 2r \cdot 4^{\max\{\deg \xi_1, \deg \xi_2\} - n} \leq r \cdot 4^{\deg(\xi_1 \cup \xi_2) - n}, \end{aligned}$$

i.e., $\xi_1 \cup \xi_2$ is r-small. □

Two configurations ξ_1 and ξ_2 are called *r-independent*, if any r-small subconfiguration of $\xi_1 \cup \xi_2$ is contained in either ξ_1 or ξ_2.

Lemma 7.5. *If ξ is not r-small, then we can write $\xi = \xi_1 \cup \xi_2$ with $\xi_1 \cap \xi_2 = \emptyset$ and $\xi_1, \xi_2 \neq \emptyset$, such that any proper 2r-small subconfiguration is contained in either ξ_1 or ξ_2.*

Remark. Then the configuration ξ_1 and ξ_2 are r-independent, but they need not be $2r$-independent, because ξ may be $2r$-small.

Proof. Choose $x, y \in \xi$, such that $\operatorname{dist}(x, y) = \operatorname{diam} \xi \geq r^{\deg \xi + n}$. Let ξ_1 be a maximal $2r$-small proper subconfiguration of ξ containing x, and let $\xi_2 = \xi \setminus \xi_1$. Then $\operatorname{diam} \xi_1 < 2r \cdot 4^{\deg \xi - 1 - n}$, and hence

$$\operatorname{dist}(y, \xi) \geq \operatorname{dist}(x, y) - \operatorname{diam} \xi_1 > r \cdot 4^{\deg \xi + n} - 2r \cdot 4^{\deg \xi - 1 - n} = 2r \cdot 4^{\deg \xi - 1 - n}.$$

Assume $\xi' \subseteq \xi$ is $2r$-small, $\xi' \cap \xi_1 \neq \emptyset$ and $\xi' \cap \xi_2 \neq \emptyset$. We shall show that $\xi' = \xi$. As $\xi' \cap \xi_1 \neq \emptyset$, Lemma 7.4 implies that $\xi' \cup \xi_1$ is $2r$-small, and as ξ_1 is maximal, we must have $\xi' \cup \xi_1 = \xi$. Especially $y \in \xi'$, and hence

$$\operatorname{diam} \xi' \geq \operatorname{dist}(y, \xi_1) > 2r \cdot 4^{\deg \xi - 1 - n}.$$

As ξ' is $2r$-small, we have $\deg \xi' > \deg \xi - 1$ and thus $\xi' = \xi$. □

We can now show

LEMMA 7.6. *Let M be a closed surface with base point x_∞ and let J be any complex structure on M. Then the inclusion $\mathscr{P}_J(M, \{x_\infty\}) \hookrightarrow \mathscr{P}(M, \{x_\infty\})$ is a homotopy equivalence.*

Proof. It is clearly enough to show that the inclusion of $\mathscr{P}_{J,n}(M, \{x_\infty\})$ into $\mathscr{P}_n(M, \{x_\infty\})$ is a homotopy equivalence for all n.

We want to define a map $\mathscr{P}_{\leq n}(M, \{x_\infty\}) \times \mathscr{C}(M) \to \mathscr{P}_{\leq n}(M, \{x_\infty\})$ of the form $(\xi, J, J') \mapsto (\psi(\xi, J, J'), J')$, which preserves degree and satisfies

(1) $\psi(\xi, J, J) = \xi$ and

(2) $\psi(\xi_1 \cup \xi_2, J, J') = \psi(\xi_1, J, J') \cup \psi(\xi_2, J, J')$ if pole(ξ_1) and pole(ξ_2) are 2-independent, considered as elements of $\mathscr{A}_{\leq n}(M')$, where (2) only is needed for an induction argument. The map ψ turns J-principal parts into J'-principal parts. We define ψ inductively, but first we choose a vector field v on M, which only vanishes at x_∞.

If $(\xi, J, J') \in \mathscr{P}_1(M, \{x_\infty\}) \times \mathscr{C}(M)$ and $\alpha \in M'$, then we let D_α be the disk in M with center α and radius one. Let $\phi_{J',J}: D_{\alpha J'} \to D_{\alpha J}$ be the unique holomorphic homeomorphism such that $\phi_{J',J}(\alpha) = \alpha$ and $d\phi_{J',J}(v(\alpha)) = c \cdot v(\alpha)$ with $c > 0$. Define ψ by $\psi(\xi, J, J') = \xi \circ \phi$. As ϕ depends continuously on α, J and J', the map ψ depends continuously on (ξ, J) and J'. Condition (2) is empty in this case, and as $\phi = \mathrm{id}$, if $J = J'$, condition (1) is satisfied.

Assume ψ is defined on $\mathscr{P}_{\leq(k-1)}(M, \{x_\infty\}) \times \mathscr{C}(M)$ with $2 \leq k \leq n$, and put

$$\begin{aligned}\tilde{\mathscr{P}} &= \{(\xi, J) \in \mathscr{P}_{\leq k}(M, \{x_\infty\}) \mid \text{pole}(\xi) \text{ is 1-small} \Rightarrow \deg \xi \leq k-1\} \\ &= \mathscr{P}_{\leq(k-1)}(M, \{x_\infty\}) \cup \{(\xi, J) \in \mathscr{P}_k(M, \{x_\infty\}) \mid \text{diam pole}(\xi) > 4^{k-n}\}.\end{aligned}$$

If $\deg \xi = k$, and diam pole$(\xi) > 4^{k-n}$, then we write $\xi = \xi_1 \cup \xi_2$ according to Lemma 7.5. Define $\tilde{\psi}$ on $\tilde{\mathscr{P}} \times \mathscr{C}(M)$ by

$$\tilde{\psi}(\xi, J, J') = \begin{cases} \psi(\xi, J, J'), & \text{if } \deg \xi \leq k-1, \\ \psi(\xi_1, J, J') \cup \psi(\xi_2, J, J'), & \text{if } \deg \xi = k. \end{cases}$$

As ψ satisfies condition (2), $\tilde{\psi}$ is well-defined, and clearly $\tilde{\psi}$ is continuous and satisfies condition (1) and (2).

We now let

$$\bar{\mathcal{P}}=\{(\xi,J)\in\mathcal{P}_{\leq k}(M,\{x_\infty\})\mid \text{pole}(\xi)\text{ is 2-small}\Rightarrow \deg\xi\leq k-1\}$$
$$=\mathcal{P}_{\leq(k-1)}(M,\{x_\infty\})\cup\{(\xi,J)\in\mathcal{P}_k(M,\{x_\infty\})\mid \text{diam pole}(\xi)>2\cdot 4^{k-n}\}\subseteq\bar{\mathcal{P}}.$$

If $\deg\xi=k$, and diam pole$(\xi)\leq 2\cdot 4^{k-n}$, i.e., if $\xi\notin\bar{\mathcal{P}}$, then we let α be the center of mass of pole(ξ) and put $D_\alpha=\{x\in M\mid \text{dist}(x,\alpha)<5\}$. As D_α is a disk in M containing pole(ξ), we can define $\phi_{J',J}\colon D_{\alpha J'}\to D_{\alpha J}$ as above. Choose a homotopy

$$H\colon \mathcal{C}(M)\times\mathcal{C}(M)\times[0,1]\to\mathcal{C}(M),$$

such that $H(J,J',0)=J$, $H(J,J',1)=J'$ and $H(J,J,t)=J$. Put $t(\xi)=4^{n-k}\cdot\text{diam pole}(\xi)-1$ and define ψ on $\mathcal{P}_{\leq k}(M,\{x_\infty\})$ by

$$\psi(\xi,J,J')=\begin{cases}\tilde{\psi}(\xi,J,J'), & \text{if }(\xi,J)\in\bar{\mathcal{P}},\\ \tilde{\psi}(\xi,J,H(J,J',t(\xi)))\circ\phi_{J',H(J,J',t(\xi))}, & \text{if }\deg\xi=k\text{ and }0\leq t(\xi)\leq 1,\\ \xi\circ\phi_{J',J}, & \text{if }\deg\xi=k\text{ and }t(\xi)\leq 0.\end{cases}$$

It is easily checked that ψ is well-defined, continuous and satisfies condition (1) and condition (2).

We can now define a homotopy inverse $\theta\colon\mathcal{P}_n(M,\{x_\infty\})\to\mathcal{P}_{J'}(M,\{x_\infty\})$ to the inclusion $\xi\mapsto(\xi,J')$, by $\theta(\xi,J)=\psi(\xi,J,J')$. □

Put $\mathcal{P}^*(M)=\mathcal{P}(M,\{x_\infty\})$ and $\mathcal{P}^*_J(M)=\mathcal{P}_J(M,\{x_\infty\})$. Let D' be any disk in M containing x_∞. By choosing a vector field, which pushes principal parts away from x_∞, we see that the inclusion $\mathcal{P}(M,\bar{D}')\hookrightarrow\mathcal{P}^*(M)$ is a homotopy equivalence. We put

$$\hat{\mathcal{P}}_0(M,\bar{D}_\infty)=\text{Tel}(\mathcal{P}_0(M,\bar{D}_1)\hookrightarrow\mathcal{P}_0(M,\bar{D}_2)\hookrightarrow\ldots)\subseteq\hat{\mathcal{P}}(M,\bar{D}_\infty).$$

If $x_\infty\in D_\infty$, then by the remarks above and Lemma 7.6:

$$H_*(\hat{\mathcal{P}}_0(M,\bar{D}_\infty))=\lim_{n\to\infty}H_*(\mathcal{P}_n(M,\bar{D}_{n+1}))=\lim_{n\to\infty}H_*(\mathcal{P}^*_n(M))$$
$$=\lim_{n\to\infty}H_*(\mathcal{P}^*_{J,n}(M)),$$

for all $J\in\mathcal{C}(M)$. Similarly we let $\text{Map}^*_0(M,Y)$ be the space of maps $f\colon M\to Y$, such that $f(x_\infty)=1\in N\cong Y_a$ and $\deg_1 f=\ldots=\deg_k f=0$ and let $\text{Map}_0(M,\bar{D}_\infty;Y,Y_a)$ be the space of maps $f\colon M\to Y$, such that $f(\bar{D}_\infty)\subseteq Y_a$ and $\deg_1 f=\ldots=\deg_k f=0$. As $\mathrm{Map}_0^*(M, Y)$ is homotopy equivalent to $\mathrm{Map}_0(M, \bar{D}_\infty; Y, Y_a)$, we have

$$H_*(\mathrm{Map}_0^*(M, Y)) = H_*(\hat{\mathscr{P}}(M, \bar{D}_\infty)) = \lim_{n\to\infty} H_*(\mathscr{P}^*_{J,n}(M)),$$

for all $J\in\mathscr{C}(M)$.

Fix $J\in\mathscr{C}(M)$, let X denote the Riemann surface M_J and put $\mathscr{P}^*_n(X)=\mathscr{P}^*_{J,n}(M)$. If G is a compact Lie group, then we let $\mathscr{V}_n(X\times\mathbf{CP}^1, X\vee\mathbf{CP}^1, G_{\mathbf{C}})$ denote the space of based isomorphism classes of holomorphic $G_{\mathbf{C}}$-bundles over $X\times\mathbf{CP}^1$, trivial over $X\vee\mathbf{CP}^1$, based at (x_∞, ∞) and with characteristic class n, see [1].

Proposition 7.7. *If $Y=\Omega G$, then*

$$\mathscr{P}^*_n(X) = \mathscr{V}_n(X\times\mathbf{CP}^1, X\vee\mathbf{CP}^1, G_{\mathbf{C}}).$$

Proof. If $X'=X\setminus\{x_\infty\}$, then a configuration of principal parts in X without a pole at x_∞ can be represented by a holomorphic map $f\colon X'\to\Omega G$, which by Proposition 4.5 is the same as an isomorphism class of a pair (P', τ), where P' is a holomorphic $G_{\mathbf{C}}$-bundle on $X'\times\mathbf{CP}^1$, and τ is a trivialization of P' over $X'\times\bar{D}_\infty$. The different choices of f correspond to different trivializations τ, but they all agree on $X'\times\{\infty\}$, i.e., a configuration of principal parts gives a pair (P', τ'), where τ' is a trivialization of P' over $X'\times\{\infty\}$. We can find a neighbourhood U of x_∞, such that P' is trivial over $(X'\cap U)\times\mathbf{CP}^1$ and τ' determines the trivialization uniquely. By gluing P' to the trivial bundle over $U\times\mathbf{CP}^1$, we get a bundle P over $X\times\mathbf{CP}^1$, and τ' extends uniquely to a trivialization over $X\vee\mathbf{CP}^1$. Thus we obtain an element of $\mathscr{V}_n(X\times\mathbf{CP}^1, X\vee\mathbf{CP}^1, G_{\mathbf{C}})$.

Assume on the other hand that we have a bundle P over $X\times\mathbf{CP}^1$, which is trivial over $X\vee\mathbf{CP}^1$. Then the restriction to $X'\times\bar{D}_\infty$ is trivial, and by extending the trivialization over $X'\times\{\infty\}$ to $X'\times\bar{D}_\infty$, the transition functions to sets of the form $U\times\bar{D}$, give us a holomorphic map $f\colon X'\to\Omega G$. Different choices of the trivialization correspond to a multiplication of f with a map $g\colon X'\to L_1^- G_{\mathbf{C}}$, i.e., we get a well-defined configuration of principal parts in X' and hence an element of $\mathscr{P}^*_n(X)$. □

From [1] we have that $\mathscr{V}_n(\mathbf{CP}^1\times\mathbf{CP}^1, \mathbf{CP}^1\vee\mathbf{CP}^1, G_{\mathbf{C}})$ and $\mathrm{Hol}^*_n(\mathbf{CP}^1, \Omega G)$ are diffeomorphic, and by Remark 3.3, $\mathscr{P}^*_n(\mathbf{CP}^1)=\mathrm{Hol}^*_n(\mathbf{CP}^1, Y)$, if Y is a generalized flag manifold. All in all we have

Theorem 7.8. *Let X be Riemann surface and Y a generalized flag manifold or a loop group. If $X=\mathbf{CP}^1$, then*

$$H_*(\mathrm{Map}_0^*(\mathbf{CP}^1, Y)) = \lim_{n\to\infty} H_*(\mathrm{Hol}_n^*(\mathbf{CP}^1, Y)),$$

and if $Y=\Omega G$, *then*

$$H_*(\mathrm{Map}_0^*(X, \Omega G)) = \lim_{n\to\infty} H_*(\mathcal{V}_n(X\times \mathbf{CP}^1, X\vee \mathbf{CP}^1, G_{\mathbf{C}})).$$

The connected components of $\mathrm{Map}^*(\mathbf{CP}^1, Y)$ are the spaces $\mathrm{Map}_{\mathbf{k}}^*(\mathbf{CP}^1, Y)$ of based maps $\mathbf{CP}^1\to Y$ with multidegree $\mathbf{k}\in \mathbf{Z}^r$. By Lemma 5.7 and Lemma 6.18, the connected components of $\mathrm{Hol}^*(\mathbf{CP}^1, Y)\cong\mathcal{P}^*(\mathbf{CP}^1)$ are the spaces

$$\mathrm{Hol}_{\mathbf{k}}^*(\mathbf{CP}^1, Y) = \mathrm{Hol}^*(\mathbf{CP}^1, Y)\cap \mathrm{Map}_{\mathbf{k}}^*(\mathbf{CP}^1, Y),$$

with $\mathbf{k}=(k_1, \dots, k_r)$ and $k_i\geqslant 0$ for $i=1, \dots, r$. Hence we have

THEOREM 7.9. *If Y is a generalized flag manifold or a loop group, then the inclusion* $\mathrm{Hol}^*(\mathbf{CP}^1, Y)\hookrightarrow \mathrm{Map}^*(\mathbf{CP}^1, Y)$ *induces an injection*

$$\pi_0(\mathrm{Hol}^*(\mathbf{CP}^1, Y) \hookrightarrow \pi_0(\mathrm{Map}^*(\mathbf{CP}^1, Y)).$$

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