Whittaker vectors and the Goodman-Wallach operators

by

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To the memory of my grandmother

§0. Introduction

0.1. Background

The theory of Whittaker functions (vectors) for reductive groups is significant for not only number theory but also representation theory of real semisimple Lie groups. Whittaker functions for reductive algebraic groups have been studied mainly from the viewpoint of number theory, as in the Hecke theory of automorphic forms ([Jc], [JL], [Shl], [Shd], etc.). However, recently the relation between Whittaker functions and some micro-local properties of representations of real semisimple Lie groups (or reductive groups over finite fields) has come to be recognized ([Ko], [Ha2], [Ly], [GW], [Ka1, 2, 3], [Mœ], [Y2], [Ma2]). I expect the study of Whittaker functions to lead us to a better understanding of such deep micro-local structures in representations of real semisimple Lie groups.

This inquiry assumes the following notation and operating definitions. Let G be a connected real semisimple linear Lie group. We fix an Iwasawa decomposition $G = KA_mN_m$ and a minimal parabolic subgroup P_m with the Langlands decomposition $P_m=M_mA_mN_m$. Let $\mathfrak{g}_0$ be the (real) Lie algebra of G and let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{a}_m$, $\mathfrak{n}_m$, $\mathfrak{p}_m$, and $\mathfrak{m}_m$ be the complexified Lie algebras of G, K, A_m, N_m, P_m, and M_m respectively. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. We denote by Σ_m^+ the positive system of the restricted root system corresponding $(\mathfrak{n}_m,\mathfrak{a}_m)$.

We fix a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ such that $\mathfrak{p}\supseteq\mathfrak{p}_m$ and a Levi decomposition $\mathfrak{p}=\mathfrak{l}+\mathfrak{n}$ such that $\mathfrak{l}\supseteq\mathfrak{m}_m+\mathfrak{a}_m$.

Let $\mathfrak{t}_m$ be a Cartan subalgebra of $\mathfrak{m}_m$ and let $\mathfrak{h}=\mathfrak{t}_m+\mathfrak{a}_m$. We denote by Δ the root system with respect to $(\mathfrak{g}, \mathfrak{h})$. We fix a positive root system Δ^+ compatible with Σ_m^+ and denote the simple root system of Δ^+ by Π. Let $\mathfrak{a}$ be the center of $\mathfrak{l}$ and define

$$S=\{\alpha\in\Pi \mid \alpha|_{\mathfrak{a}}=0\}.$$

We denote by (R) the following condition on $\mathfrak{p}$.

(R) $\mathfrak{p} \cap \mathfrak{g}_0$ *is a real form of* $\mathfrak{p}$. *Namely, there exists a parabolic subgroup* P *of* G *whose complexified Lie algebra coincides with* $\mathfrak{p}$.

It is easy to see that $\mathfrak{p}$ may not satisfy this condition (cf. 5.4, Example).

We denote by $\mathscr{A}(G)$ the space of real analytic function on G. For $X, Y\in\mathfrak{g}_0$ and $f\in\mathscr{A}(G)$, we put

$$f(g: X+iY)=\frac{d}{dt}(f(g\exp(tX))+if(g\exp(tY)))|_{t=0},$$
$$f(X+iY: g)=\frac{d}{dt}(f(\exp(tX)g)+if(\exp(tY)))|_{t=0},$$

Let ψ be a character (namely, a one-dimensional representation) on $\mathfrak{n}$. If a function f on G satisfies $f(g:X)=-\psi(X)f(g)$ for all $g\in G$ and $X\in\mathfrak{n}$, then f is called a (real analytic ψ-) Whittaker function on G. We denote by $\mathscr{A}(G, \mathfrak{n}; \psi)$ the space of Whittaker functions. We can introduce a left $U(\mathfrak{g})$-module structure on $\mathscr{A}(G, \mathfrak{n}; \psi)$ by

$$X.f(g)=f(-X: g),$$

for all $f\in\mathscr{A}(G, \mathfrak{n}; \psi)$, $X\in\mathfrak{n}$, and $g\in G$.

If $\mathfrak{p}$ satisfies (R), then $\mathscr{A}(G, \mathfrak{n}; \psi)$ coincides with the following space of an induced representation.

$$\mathscr{A}(G/N; \psi)=\{f\in\mathscr{A}(G) \mid \forall g\in G,\ \forall n\in N, f(gn)=\psi(n)^{-1}f(g)\}.$$

Here, we denote by N the nilradical of P and by the same letter ψ the character on N corresponding to the character ψ on the Lie algebra $\mathfrak{n}$. If ψ is contained in the Richardson orbit with respect to $\mathfrak{p}$, then we call ψ admissible. Here, we regard ψ as an element of $\mathfrak{g}$ via the Killing form of $\mathfrak{g}$.

Let V be an arbitrary left $U(\mathfrak{g})$-module. According to [Ha2], if there exists an embedding (namely, an injective $U(\mathfrak{g})$-homomorphism) $\iota: V\hookrightarrow\mathscr{A}(G, \mathfrak{n}; \psi)$, then we say V has a (algebraic) Whittaker model.

Remark. Here, the usage of the word "model" is different from the original usage of Gelfand-Graev.

Then we can ask:

PROBLEM 0. *When does V have a Whittaker model?*

In order to apply algebraic methods, such as the homological algebra, we introduce the notion of Whittaker vectors. For a left $U(\mathfrak{g})$-module V, we denote by V^* the dual vector space over the complex field. Then V has a natural right $U(\mathfrak{g})$-module structure. We define the space of (dual) Whittaker vectors as follows.

$$\mathsf{Wh}_{\mathfrak{n},\psi}(V) = \{v \in V \mid \forall X \in \mathfrak{n}, X \cdot v = \psi(X)\, v\},$$
$$\mathsf{Wh}^*_{\mathfrak{n},\psi}(V) = \{v \in V^* \mid \forall X \in \mathfrak{n}, v \cdot X = \psi(X)\, v\},$$

For a right $U(\mathfrak{g})$-module M, we also define $\mathsf{Wh}_{\mathfrak{n},\psi}(M)$ and $\mathsf{Wh}^*_{\mathfrak{n},\psi}(M)$ in the same way.

Let V be an irreducible left $U(\mathfrak{g})$-module. For $F \in \mathrm{Hom}_{U(\mathfrak{g})}(V, \mathscr{A}(G, \mathfrak{n}, \psi))$, we define $G(F) \in V^*$ by

$$[\Gamma(F)](v) = [F(v)](e) \quad (v \in V).$$

Here, e is the identity element of G. Immediately, we see $\Gamma(F) \in \mathsf{Wh}^*_{\mathfrak{n},\psi}(V)$ and Γ is injective. Put

$$\mathsf{Wh}^*_{\mathfrak{n},\psi}(V) = \mathrm{Image}(\Gamma).$$

We call an element of $\mathsf{Wh}^G_{\mathfrak{n},\psi}(V)$ a global Whittaker vector. Clearly, the following is equivalent to Problem 0.

PROBLEM 1. *When is* $\mathsf{Wh}^G_{\mathfrak{n},\psi}(V) \neq 0$?

We can also ask:

PROBLEM 2. *When is* $\mathsf{Wh}^*_{\mathfrak{n},\psi}(V) \neq 0$?

PROBLEM 3. *When is* $\mathsf{Wh}^G_{\mathfrak{n},\psi}(V) = \mathsf{Wh}^*_{\mathfrak{n},\psi}(V)$?

PROBLEM 4. *When is* $\mathsf{Wh}^G_{\mathfrak{n},\psi}(V)$ *(or* $\mathsf{Wh}^*_{\mathfrak{n},\psi}(V)$*) finite-dimensional?*

PROBLEM 5. *Determine* $\dim \mathsf{Wh}^G_{\mathfrak{n},\psi}(V)$ *and* $\dim \mathsf{Wh}^*_{\mathfrak{n},\psi}(V)$.

First result with respect to these problems are ascribed to Kostant [Ko]. He has proved that if $\mathfrak{n}$ is the nilradical of some Borel subalegebra of $\mathfrak{g}$ and ψ is admissible, then $\mathrm{Wh}^*_{\mathfrak{n},\psi}(V)\neq 0$ implies the annihilator of V is a minimal primitive ideal. We assume G is quasi-split. Kostant has also proved (the case of $Sl(n,\mathbf{R})$ is ascribed to Casselman and Zuckerman) if G is quasisplit, V is a Harish-Chandra module (cf. 1.4), and the annihilator of V in $U(\mathfrak{g})$ is minimal, then $\mathrm{Wh}^*_{\mathfrak{n},\psi}(V)\neq 0$. He also gave a solution to Problem 5. (Theorem K [Ko]).

At the almost same time, Hashizume considered Whittaker models for Harish-Chandra modules with highest weight vectors ([Ha2]) and introduced Whittaker models with respect to more general class of nilpotent subgroups. Recently, Yamashita ([Y2] Part 2) studied Whittaker models of highest weight modules precisely.

In his thesis [Ly], Lynch generalized important properties of Whittaker vectors, which had been shown by Kostant, to the case that ψ is an arbitrary admissible character.

In [GW], Goodman and Wallach gave the solution of Problem 3 for the case G is quasisplit, $\mathfrak{n}=\mathfrak{n}_m$, and V is a Harish-Chandra module, using some differential operators of infinite order.

From these results, it is suspected that the solution of Problem 1 or 2 is described in terms of some micro-local conceptions, such as the associated variety (for example see [V2], [Ma2]) or the wave front set (cf. [KV], [Ho], [BV]). In fact, Kawanaka (cf. [Ka1, 2, 3]) has shown corresponding results for the generalized Gelfand-Graev representations of reductive algebraic groups over finite fields.

Let ψ be a character of $\mathfrak{n}$. Using the Killing form we regard ψ as an element of the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$.

In [Ma2], it is proved that $\mathrm{Wh}^*_{\mathfrak{n},\psi}(V)\neq 0$ implies that the associated variety of the annihilator of V in $U(\mathfrak{g})$ contains ψ.

0.2. Main results

First, we assume that V is a finitely generated left $U(\mathfrak{n})$-module. Then we easily see the Gelfand-Kirillov dimension $\mathrm{Dim}(V)$ (cf. 1.2) is not more than $\dim \mathfrak{n}$. Put $d=\dim \mathfrak{n}$, and let $c_d(V)$ be the multiplicity of V (cf. 1.2).

Then, using the vanishing theorem of Kostant-Lynch (cf. 2.1), we can get the following solution of Problem 5 (D. A. Vogan gave the author a crucial suggestion (cf. the remark after the proof of Theorem 2.2.1)).

THEOREM A. (Theorem 2.2.1.) *Let V be a left $U(\mathfrak{g})$-module which is finitely generated as an $U(\mathfrak{n})$-module. Let ψ be an admissible character on $\mathfrak{n}$. Then*

$$\dim \mathrm{Wh}^*_{\mathfrak{n},\psi}(V) = c_d(V).$$

As a corollary of this result, we can generalize a result of Kostant ([Ko] Theorem K) to an arbitrary real (not necessary quai-split) semisimple Lie group.

COROLLARY B. (Corollary 2.2.2.) *Let G be an arbitrary semisimple Lie group and let ψ be an admissible character on the nilradical $\mathfrak{n}_m$ of the complexified Lie algebra of a minimal parabolic subgroup of G. Let M be a Harish-Chandra module. Then*

$$\dim \mathrm{Wh}^*_{\mathfrak{n}_m,\psi}(M) = \begin{cases} 0 & \text{if}\ \mathrm{Dim}(M) < d, \\ c_d(M) & \text{if}\ d = \dim \mathfrak{n}_m = \mathrm{Dim}(M). \end{cases}$$

For Problem 3, we will show the following generalization of a result of Goodman and Wallach.

THEOREM C. (Theorem 6.2.1.) *Let M be an irreducible Harish-Chandra module and let ψ be an admissible character on $\mathfrak{n}_m$. Then*

$$\mathrm{Wh}^G_{\mathfrak{n}_m,\psi}(M) = \mathrm{Wh}^*_{\mathfrak{n}_m,\psi}(M).$$

This theorem is proved by the same method as [GW] from Theorem E below.

An aim of this paper is to construct many global Whittaker vectors on an irreducible Harish-Chandra module. Especially, in order to prove Theorem C, we should construct sufficiently many global Whittaker vectors.

Hereafter, we do not assume $\mathfrak{n}=\mathfrak{n}_m$ any more. Let $\psi:\mathfrak{n}\to\mathbf{C}$ be an admissible character.

Now we introduce some notations.

Define

$$\mathsf{P}^{++}_S = \left\{\lambda \in \mathfrak{h}^* \mid \forall\, \alpha \in S, 2\frac{\langle \alpha,\lambda\rangle}{\langle \alpha,\alpha\rangle} \in \mathbf{N}\right\}.$$

For $\lambda \in \mathsf{P}^{++}_S$, we denote by $L(\mathfrak{p},\lambda)$ the highest weight left $U(\mathfrak{g})$-module with a highest weight λ.

Let V be an irreducible Harish-Chandra module and we assume there is some nonsingular pairing between $L(\mathfrak{p},\lambda)$ and V compatible with the $\mathfrak{g}$-actions. The existence of a pairing between V and some highest weight module implies the existence of an embedding of V into some principal series representation.

Let ψ be a non-trivial character on $\mathfrak{n}$. If we found some non-trivial $\omega \in \mathrm{Wh}_{\mathfrak{n},-\psi}(L(\mathfrak{p},\lambda))$, we could easily construct a non-trivial global Whittaker vector on V. Unfortunately, always $\mathrm{Wh}_{\mathfrak{n},-\psi}(L(\mathfrak{p},\lambda))=0$ holds. The idea of Goodman and Wallach [GW] is to consider some completions of $L(\mathfrak{p},\lambda)$ instead of $L(\mathfrak{p},\lambda)$ itself.

First, we consider the formal completion $\hat{L}(\mathfrak{p},\lambda)$ (cf. 3.2).

Then we have:

THEOREM D. (Corollary 3.4.6.) *For $\lambda \in \mathsf{P}_S^{++}$ and an admissible character ψ,*

$$\dim \mathrm{Wh}_{\mathfrak{n},-\psi}(\hat{L}(\mathfrak{p},\lambda)) = c_d(L(\mathfrak{p},\lambda)).$$

Here, $d=\dim \mathfrak{n}$.

This result is a generalization of that of Kostant [Ko] (for irreducible Verma modules) and Lynch [Ly] (for irreducible generalized Verma modules).

We also prove an conjecture of Lynch concerning the dimension of the space of Whittaker vectors in the formal completion of (reducible) generalized Verma modules (Theorem 3.4.7).

The formal completion is too large for our purpose. Hence, according to Goodman and Wallach, we introduce the Gevrey completions $L^{\varkappa}(\mathfrak{p},\lambda)$ for $1 \leq \varkappa$ (cf. 4.2).

We prove:

THEOREM E. (Theorem 4.2.1.) *For an arbitrary character ψ, $\lambda \in \mathsf{P}_S^{++}$, and $1 \leq \varkappa < 2$,*

$$\mathrm{Wh}_{\mathfrak{n},-\psi}(\hat{L}(\mathfrak{p},\lambda)) \subseteq L^{\varkappa}(\mathfrak{p},\lambda).$$

First, Goodman and Wallach [GW] have proved this result for the case that $\mathfrak{n}$ is the nilradical of some Borel subalgebra. (Hence G should be quasi-split.)

Wallach also announced in his lecture at Katata 1986 (and personal discussion 1987), that Goodman and Wallach had proved a corresponding result for the case $\mathfrak{n}$ is a 2-step nilpotent Lie algebra.

Fix $\lambda \in \mathsf{P}_S^{++}$.

Using Theorem D and Theorem E, we can prove:

THEOREM F. (Theorem 5.5.1.) *Let V be an irreducible Harish-Chandra module which has a non-singular pairing with $L(\mathfrak{p},\lambda)$.*

For an arbitrary character ψ on $\mathfrak{n}$, there exists some discrete subset D of $\mathbf{C}$ *such that $0 \notin D$ and for all $z \in \mathbf{C}-D$ there exists some injective map:*

$$\mathsf{Wh}_{\mathfrak{n},-z\psi}(\hat{L}(\mathfrak{p},\lambda)) \hookrightarrow \mathsf{Wh}^G_{\mathfrak{n},z\psi}(V).$$

Moreover, if ψ is admissible and $\mathrm{Dim}(V)=\dim \mathfrak{n}$, *then* $\mathsf{Wh}^G_{\mathfrak{n},z\psi}(V) \neq 0$ *for all $z \notin D$.*

If the condition (R) holds, we have a stronger result:

THEOREM G. (Theorem 5.5.2.) *We assume the condition* (R) *holds. Let V be an irreducible Harish-Chandra module which have a non-singular pairing with $L(\mathfrak{p},\lambda)$.*

For an arbitrary character ψ on $\mathfrak{n}$, there exists some injective map:

$$\mathsf{Wh}_{\mathfrak{n},-\psi}(\hat{L}(\mathfrak{p},\lambda)) \hookrightarrow \mathsf{Wh}^G_{\mathfrak{n},\psi}(V).$$

Moreover, if ψ is admissible and $\mathrm{Dim}(V)=\dim \mathfrak{n}$, *then* $\mathsf{Wh}^G_{\mathfrak{n},\psi}(V) \neq 0$.

0.3. A working hypothesis

Let V be a left $U(\mathfrak{g})$-module and let ψ be an arbitrary character on $\mathfrak{n}$.

In [Ma2], it is proved that $\mathsf{Wh}^*_{\mathfrak{n},\psi}(V) \neq 0$ implies that the associated variety of the annihilator of V in $U(\mathfrak{g})$ contains ψ. Here, using the Killing form, we regard ψ as an element of the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$.

I suspect that "under some good condition" the converse of the above result holds. Specifically, we consider the following situation, which is a special case of Kawanaka's generalized Gelfand-Graev representations ([Ka1,2,3], [Y1]). Let $\mathcal{O}$ be an even nilpotent orbit of $\mathfrak{g}$ and let $u \in \mathcal{O}$. We assume u is contained in $\mathfrak{n}_m \cap \mathfrak{g}_0$ and there exists a Lie algebra homomorphism

$$\phi : sl(2,\mathbf{C}) \to \mathfrak{g}$$

such that

$$\phi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = u,$$

$$\phi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) \in \bar{\mathfrak{n}}_m \cap \mathfrak{g}_0,$$

and we also assume

$$H=\phi\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right)$$

is contained in the center of $\mathfrak{l}_m\cap\mathfrak{g}_0$ and all the eigenvalues of $\mathrm{ad}(H)|\mathfrak{n}_m$ is non-negative.

Put $\mathfrak{g}(k)=\{X\in\mathfrak{g}|\ \mathrm{ad}(H)X=kX\}$ for all even integer k. Since u is even,

$$\mathfrak{p}_u=\sum_{k\in\mathbf{N}}\mathfrak{g}(2k)$$

is a parabolic subalgebra of $\mathfrak{g}$ such that $\mathfrak{p}_u\supseteq\mathfrak{p}_m$. We assume $\mathfrak{p}_u$ satisfies the condition (R). Let $\mathfrak{n}_u$ be the nilradical of $\mathfrak{p}_u$ and let $\mathfrak{a}_u$ be the center of $\mathfrak{g}(0)$. Clearly, $\mathfrak{a}_u\subseteq\mathfrak{h}$. Put

$$S_u=\{\alpha\in\Pi\,|\,\alpha|_{\mathfrak{a}_u}=0\}.$$

It is known that $\mathfrak{p}_u$ is admissible, namely there exists some admissible character on $\mathfrak{n}_u$. (Cf. [SS].)

The following conjecture could be regarded an algebraic version of Kawanaka's conjecture ([Ka3] (2.5.2)).

Conjecture H. *Let V be an irreducible Harish-Chandra module such that* $\mathrm{Dim}(V)\leq\dim\mathfrak{n}_u$ *and let ψ be an admissible character on $\mathfrak{n}_u$. We denote by I the annihilator of V in $U(\mathfrak{g})$. Under the above condition, the followings are equivalent.*

(H1) *The characteristic variety of I coincides with the closure of $\mathcal{O}$.*

(H2) $\mathrm{Wh}^*_{\mathfrak{n}_u,\psi}(V)\neq 0$.

(H3) $\mathrm{Wh}^G_{\mathfrak{n}_u,\psi}(V)\neq 0$.

We remark that clearly (H3) implies (H2), and we see, from [Ma2] Theorem 2, (H2) always implies (H1). Corollary B and Theorem C means Conjecture H holds when $\mathfrak{p}_u=\mathfrak{p}_m$.

Theorem G gives a sufficient condition for (H3) in terms of "minimal" embeddings into principal series.

Hence we can easily see the following working hypothesis implies the above conjecture.

Working Hypothesis I. *The condition* (H1) *implies that V has a $\mathfrak{g}$-invariant pairing with $L(\mathfrak{p}_u,\lambda)$ for some $\lambda\in\mathrm{P}^{++}_{S_u}$.*

At present, this hypothesis is mere wishful thinking. However, even if it were false, counter examples to "Working Hypothesis I" would, I believe, be interesting.

I fancy that the deep analysis of the structure of principal series representations by Casian and Collingwood ([CC1], [CC2], [CC3]) enable us to say something about the above hypothesis. In fact, they establish an algorithm to compute "minimal embeddings" which are distinguished by the weight filtration (cf. [CC1]). For example, for $G=Sp(2,\mathbf{R})$ (real rank two), if V has an integral infinitesimal character, the above working hypothesis is true ([CC1], [CC3]).

It is interesting, I think, to re-interpret the results of Casian and Collingwood in more geometrical terms. "Working Hypothesis I" is a candidate of the beginning of such reinterpretation.

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I dedicate this paper to my grandmother Tame Mimura, who had encouraged me and passed away in Tokyo when I was writing this manuscript at MIT.

§ 1. Notations and preliminaries

1.1. Notation

In this article, we use the following notations.

As usual we denote the complex number field, the real number field, the rational number field, the ring of integers, and the set of non-negative integers by $\mathbf{C}$, $\mathbf{R}$, $\mathbf{Q}$, $\mathbf{Z}$, and $\mathbf{N}$ respectively.

For a complex vector space V, we denote by V^* the dual vector space. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, and Δ the root system with respect to $(\mathfrak{g}, \mathfrak{h})$. We fix some positive root system Δ^+ and let Π be the set of simple roots. Put

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \forall H \in \mathfrak{h}, [H, X] = \alpha(H)X\},$$

$$\mathfrak{u} = \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha,$$

$$\bar{\mathfrak{u}} = \sum_{-\alpha \in \Delta^+} \mathfrak{g}_\alpha.$$

Let $\langle\, , \rangle$ be the Killing form of $\mathfrak{g}$.

Next we fix notations for a parabolic subalgebra. Hereafter, through this article, we fix a subset S of Π and the following notations for the parabolic subalgebra determined by S. Let $\tilde{S}$ be the set of the elements of Δ which are written by linear combinations of elements of S over $\mathbf{Z}$. Put

$$\mathfrak{a} = \{H \in \mathfrak{h} \mid \forall \alpha \in S, \alpha(H) = 0\},$$

$$\mathfrak{l} = \mathfrak{h} + \sum_{\alpha \in \tilde{S}} \mathfrak{g}_\alpha,$$

$$\mathfrak{n} = \sum_{\alpha \in \Delta^+ - \tilde{S}} \mathfrak{g}_\alpha,$$

$$\bar{\mathfrak{n}} = \sum_{-\alpha \in \Delta^+ - \tilde{S}} \mathfrak{g}_\alpha,$$

$$\mathfrak{m} = \{X \in \mathfrak{l} \mid \forall H \in \mathfrak{a}, \langle X, Y \rangle = 0\},$$

$$\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n} = \mathfrak{l} + \mathfrak{n},$$

$$\bar{\mathfrak{p}} = \mathfrak{m} + \mathfrak{a} + \bar{\mathfrak{n}} = \mathfrak{l} + \bar{\mathfrak{n}},$$

$$\mathfrak{t} = \mathfrak{h} \cap \mathfrak{m}.$$

Hence $\mathfrak{h}$ is the direct sum of $\mathfrak{t}$ and $\mathfrak{a}$, which are othogonal with respect to the Killing form. For each $\lambda \in \mathfrak{h}^*$, we denote the restriction of λ to $\mathfrak{t}$ (resp. $\mathfrak{a}$) by $\lambda_\mathfrak{t}$ (resp.$\lambda_\mathfrak{a}$). Using the Killing form, we can regard $\mathfrak{t}^*$ and $\mathfrak{a}^*$ as subspaces of $\mathfrak{h}^*$. Then we immediately have $\lambda = \lambda_\mathfrak{t} + \lambda_\mathfrak{a}$.

For all $\lambda \in \mathrm{h}^*$, we define $H_\lambda \in \mathfrak{h}$ by

$$\forall H \in \mathfrak{h}, \quad \langle H, H_\lambda \rangle = \lambda(H).$$

Put

$$\Sigma^+ = \{\beta \in \mathfrak{a}^* \mid \beta \neq 0 \text{ and } \exists \alpha \in \Delta^+, \alpha|_\mathfrak{a} = \beta\}.$$

For $\alpha \in \Sigma^+$, put

$$\mathfrak{n}_\alpha = \{X \in \mathfrak{n} \mid \forall H \in \mathfrak{a}, [H, X] = \alpha(H)X\},$$

$$\bar{\mathfrak{n}}_{-\alpha} = \{X \in \bar{\mathfrak{n}} \mid \forall H \in \mathfrak{a}, [H, X] = -\alpha(H)X\},$$

We define $\Phi \subseteq \Sigma^+$ by

$$\Phi = \{\beta \in \mathfrak{a}^* \mid \beta \neq 0 \text{ and } \exists \alpha \in \Pi,\ \alpha|_{\mathfrak{a}} = \beta\}.$$

We denote by H_0 the element of $\mathfrak{a}$ which satisfies:

$$\beta(H_0) = 1 \quad \text{for all} \quad \beta \in \Phi.$$

Put

$$\mathfrak{g}(i) = \{X \in \mathfrak{g} \mid \mathrm{ad}(H_0)X = iH\}.$$

These define a $\mathbf{Z}$-graded structure on $\mathfrak{g}$.
Put

$$Q_{\mathfrak{a}}^+ = \{\alpha_1 + \ldots + \alpha_l \mid l \in \mathbf{N},\ \alpha_i \in \Sigma^+\ (1 \leq i \leq l)\} \cup \{0\}.$$

For $\mu \in Q_{\mathfrak{a}}^+$, we define

$$U(\mathfrak{n})_\mu = \{P \in U(\mathfrak{n}) \mid \forall H \in \mathfrak{a},\ HP - PH = \mu(H)P\},$$

$$U(\bar{\mathfrak{n}})_{-\mu} = \{P \in U(\bar{\mathfrak{n}}) \mid \forall H \in \mathfrak{a},\ HP - PH = -\mu(H)P\}.$$

Let $G_{\mathbf{C}}$ be the simply-connected connected complex algebraic group corresponding to $\mathfrak{g}$ and let $P_{\mathbf{C}}$ (resp. $\bar{P}_{\mathbf{C}}$) be the parabolic subgroup corresponding to $\mathfrak{p}$ (resp. $\bar{\mathfrak{p}}$).

1.2. Gelfand-Kirillov dimension and multiplicities

We recall two important invariants for finitely generated $U(\mathfrak{g})$-modules, namely Gelfand-Killirov dimension and multiplicity (Bernstein degree). For details, see [V1].

Let $\mathfrak{g}_1$ be an arbitrary Lie algebra over $\mathbf{C}$. Let M be a finitely generated $U(\mathfrak{g}_1)$-module and $v_1, \ldots, v_h$ its generators. Fix a non-negative integer n. Let $U_n(\mathfrak{g}_1)$ be the space of the elements in $U(\mathfrak{g}_1)$ which are written by a products of at most n elements of $\mathfrak{g}_1$. Put $M_n = \Sigma_{1 \leq i \leq h} U_n(\mathfrak{g}_1)v_i$. Then, there exists some polynomial $\chi(x)$ in one variable over $\mathbf{Q}$ such that $\dim_{\mathbf{C}} M_n = \chi(n)$ for sufficiently large n. The Gelfand-

Killirov dimension $\operatorname{Dim} M$ is the degree of $\chi(x)$. Let d be any integer such that $d \geq \operatorname{Dim} M$. Then the multiplicity $c_d(M)$ of M is defined by

$$c_d(M) = \begin{cases} \text{the coefficient of } d!\chi(x) \text{ at } x^{\operatorname{Dim} M} & \text{if } d = \operatorname{Dim} M \\ 0 & \text{if } d > \operatorname{Dim} M. \end{cases}$$

Multiplicities are always non-negative integers. The definitions of Gelfand-Killirov dimensions and multiplicities do not depend on the choice of generators.

Let M_1, M_2, and M_3 be finitely generated $U(\mathfrak{g}_1)$-modules such that

$$\max_{i=1,2,3} (\operatorname{Dim}(M_i)) \leq d,$$

and there exists a short exact sequence of $U(\mathfrak{g}_1)$-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0.$$

Then we have

$$c_d(M_2) = c_d(M_1) + c_d(M_3).$$

1.3 Whittaker vectors

Let $\psi: \mathfrak{n} \to \mathbf{C}$ be any character. We denote by the same letter the algebra homomorphism $\psi: U(\mathfrak{n}) \to \mathbf{C}$ induced from ψ.

Using the Killing form, we can identify the space of characters on $\mathfrak{n}$ and $\mathfrak{g}(-1)$. Thus, hereafter we regard a character ψ as an element of $\mathfrak{g}(-1)$.

Let M be a left $U(\mathfrak{g})$-module. Then the dual vector space M^* has a natural right $U(\mathfrak{g})$-module structures.

We define the space of (dual) Whittaker vectors (cf. [Ma2]) as follows.

$$\mathrm{Wh}_{\mathfrak{n},\psi}(M) = \{ v \in M \mid \forall X \in \mathfrak{n},\ X \cdot v = \psi(X) v \},$$

$$\mathrm{Wh}^*_{\mathfrak{n},\psi}(M) = \{ v \in M^* \mid \forall X \in \mathfrak{n},\ v \cdot X = \psi(X) v \},$$

For a right $U(\mathfrak{g})$-module M, we also define $\mathrm{Wh}_{\mathfrak{n},\psi}(M)$ and $\mathrm{Wh}^*_{\mathfrak{n},\psi}(M)$ in the same way. Namely,

$$\mathrm{Wh}_{\mathfrak{n},\psi}(M) = \{ v \in M \mid \forall X \in \mathfrak{n},\ v \cdot X = \psi(X) v \},$$

$$\mathrm{Wh}^*_{\mathfrak{n},\psi}(M) = \{ v \in M^* \mid \forall X \in \mathfrak{n},\ X \cdot v = \psi(X) v \}.$$

A $\mathfrak{g}$-module M is called a Whittaker module if there exists a cyclic element of M which is contained in $\mathrm{Wh}_{\mathfrak{n},\psi}(M)$.

Next, according to Lynch [Ly], we introduce the following notions. Let ψ be a character on $\mathfrak{n}$. We call ψ admissible when ψ is contained in the Richardson orbit $\mathcal{O}_{\mathfrak{p}}$ with respect to $\mathfrak{p}$. (Namely, $\mathcal{O}_{\mathfrak{p}}$ is a unique $G_{\mathbf{C}}$-orbit such that $\mathcal{O}_{\mathfrak{p}} \cap \mathfrak{n}$ is open dense in $\mathfrak{n}$.) Here, we regard ψ as an element of $\mathfrak{g}(-1)$.

If there exists an admissible character, we call $\mathfrak{p}$ an admissible parabolic subalgebra. It is known that there exist non-admissible parabolic subalgebras. (Cf. [Ly].) But, for example, the complexification of the minimal parabolic subalgebra of a real form of $\mathfrak{g}$ is admissible.

Let $L_{\mathbf{C}}$ be the complex analytic subgroup of $G_{\mathbf{C}}$ corresponding to $\mathfrak{l}$. Then $L_{\mathbf{C}}$ acts on $\mathfrak{g}(-1)$ by the adjoint action. If $\mathfrak{p}$ is admissible, then $\psi \in \mathfrak{g}(-1)$ is admissible if and only if ψ is contained the open $L_{\mathbf{C}}$-orbit in $\mathfrak{g}(-1)$.

If M is a finitely generated $\mathfrak{n}$-module, then the Gelfand-Kirillov dimension of M is clearly less than or equal to dim $\mathfrak{n}$.

1.4. Harish-Chandra modules and global Whittaker vectors

We fix a real form $\mathfrak{g}_0$ of $\mathfrak{g}$ and a connected real semisimple linear Lie group G whose Lie algebra is $\mathfrak{g}_0$. We also fix an Iwasawa decomposition:

$$G = KA_m N_m.$$

Here, K is a maximal compact subgroup of G, A_m is a maximal real-split torus, and N_m is the nilradical of the minimal parabolic subgroup of G. We denote by $\mathfrak{k}$, $\mathfrak{a}_m$, and $\mathfrak{n}_m$ the complexified Lie algebras of K, A_m, and N_m respectively. Let M_m be the centralizer of A_m in K and let $\mathfrak{m}_m$ be the complexified Lie algebra of M_m. Put

$$P_m = M_m A_m N_m,$$

$$\mathfrak{l}_m = \mathfrak{m}_m + \mathfrak{a}_m,$$

$$\mathfrak{p}_m = \mathfrak{m}_m + \mathfrak{a}_m + \mathfrak{n}_m.$$

We denote by log the inverse of exp: $\mathfrak{a}_m \cap \mathfrak{g}_0 \to A_m$.

Let $\bar{\mathfrak{p}}_m$ be the opposite parabolic of $\mathfrak{p}_m$ and let $\bar{\mathfrak{n}}_m$ be the nilradical of $\bar{\mathfrak{p}}_m$.

We assume $\mathfrak{a}_m \subseteq \mathfrak{h}$. Put $S_m = \{\beta \in \Pi \mid \forall H \in \mathfrak{a}_m, \beta(H) = 0\}$. Put $\varrho_m = \varrho_{S_m}$.

A compatible left $(\mathfrak{g}, K)$-module (for example see [BW]) of finite length is called a Harish-Chandra module.

Next we introduce the notion of global Whittaker vectors. We denote by $\mathscr{A}(G)$ the space of real analytic functions on G. For $X, Y \in \mathfrak{g}_0$ and $f \in \mathscr{A}(G)$, we put

$$f(X+iY:g)=\frac{d}{dt}(f(\exp(tX)(g)+if(\exp(tY)g))|_{t=0},$$
$$f(g:Y+iY)=\frac{d}{dt}(f(g\exp(tX))+if(g\exp(tY)))|_{t=0}.$$

Let ψ be an arbitrary character on $\mathfrak{n}$. Put

$$\mathscr{A}(G,\mathfrak{n};\psi)=\{f\in\mathscr{A}(G)|\,\forall g\in G,\ \forall X\in\mathfrak{n}, f(g:X)=-\psi(X)f(g)\}.$$

$\mathscr{A}(G,\mathfrak{n};\psi)$ has a structure of $U(g)$-module by the left action. Let M be a left $U(\mathfrak{g})$-module and let $\mathrm{Hom}_{U(\mathfrak{g})}(M,\mathscr{A}(G,\mathfrak{n};\psi))$ be the space of $U(\mathfrak{g})$-homomorphisms of M to $\mathscr{A}(G,\mathfrak{n};\psi)$. For $F\in\mathrm{Hom}_{U(\mathfrak{g})}(M,\mathscr{A}(G,\mathfrak{n};\psi))$, we define $\Gamma(F)\in M^*$ by

$$[\Gamma(F)](v)=[F(v)](e)\ \text{ for all }\ v\in M.$$

Here, e is the identity element of G. We immediately see $\Gamma(F)\in\mathsf{Wh}^*_{\mathfrak{n},\psi}(M)$. Since any real analytic function is determined by its Taylor expansion, we can easily see Γ is an injective linear map. Put

$$\mathsf{Wh}^G_{\mathfrak{n},\psi}(M)=\mathrm{Image}(\Gamma).$$

We call an element of $\mathsf{Wh}^G_{\mathfrak{n},\psi}(M)$ a (G-) global Whittaker vector on M.

We remark that if there exists a parabolic subgroup P of G and the complexified Lie algebra of the nilradical N of P coincides with $\mathfrak{n}$, then $\mathscr{A}(G,\mathfrak{n};\psi)$ coincides with the following space of an induced representation.

$$\mathscr{A}(G/N;\psi)=\{f\in\mathscr{A}(G)|\,\forall g\in G,\ \forall n\in N, f(gn)=\psi(n)^{-1}f(g)\}.$$

Here, we denote the character on N induced from the character ψ on $\mathfrak{n}$ by the same letter.

1.5 Generalized Verma modules

Define

$$\mathsf{P}^{++}_S=\left\{\lambda\in\mathfrak{h}^*|\,\forall\alpha\in S,\ 2\frac{\langle\alpha,\lambda\rangle}{\langle\alpha,\alpha\rangle}\in\{0,1,2,\dots\}\right\}.$$

Let σ_λ be an irreducible finite-dimensional $\mathfrak{l}$-representation whose highest weight is $\lambda \in \mathsf{P}_S^{++}$. Let V_λ be the representation space of σ_λ and we fix a non-trivial highest weight vector v_λ of σ_λ.

V_λ^* has a natural right $U(\mathfrak{l})$-module structure. Let v_λ^* be a non-zero λ-weight vector of V_λ^*.

We define a left (resp. right) action of $\mathfrak{n}$ (resp. $\bar{\mathfrak{n}}$) on V_λ (resp. V_λ^*) by $X \cdot v = 0$ (resp. $v \cdot X = 0$) for all $X \in \mathfrak{n}$ and $v \in V_\lambda$ (resp. $X \in \bar{\mathfrak{n}}$ and $v \in V_\lambda^*$). Then we can regard V_λ (resp. V_λ^*) as a left $U(\mathfrak{p})$-module (resp. a right $U(\bar{\mathfrak{p}})$-module).

Let $\lambda \in \mathsf{P}_s^{++}$. We define the generalized Verma modules (Lepowski [Le]) as follows.

$$M(\mathfrak{p}, \lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V_\lambda.$$

$$\bar{M}(\bar{\mathfrak{p}}, \lambda) = V_\lambda^* \otimes_{U(\mathfrak{p})} U(\mathfrak{g}).$$

$M(\mathfrak{p},\lambda)$ (resp. $\bar{M}(\bar{\mathfrak{p}},\lambda)$) is a left (resp. right) $U(\mathfrak{g})$-module.

As left $U(\bar{\mathfrak{n}})$-modules (resp. right $U(\mathfrak{n})$-module) we have $M(\mathfrak{p},\lambda) \cong U(\bar{\mathfrak{n}}) \otimes_{\mathbf{C}} V_\lambda$ (resp. $\bar{M}(\bar{\mathfrak{p}},\lambda) \cong V_\lambda^* \otimes_{\mathbf{C}} U(\mathfrak{n})$).

Let $L(\mathfrak{p},\lambda)$ (resp. $\bar{L}(\bar{\mathfrak{p}},\lambda)$) be a unique irreducible quotient $U(\mathfrak{g})$-module of $M(\mathfrak{p},\lambda)$ (resp. $\bar{M}(\bar{\mathfrak{p}},\lambda)$).

Let $q_\lambda : M(\mathfrak{p},\lambda) \to L(\mathfrak{p},\lambda)$ and $\bar{q}_\lambda : \bar{M}(\bar{\mathfrak{p}},\lambda) \to \bar{L}(\bar{\mathfrak{p}},\lambda)$ be the canonical projections. Let $K(\mathfrak{p},\lambda)$ (resp. $\bar{K}(\bar{\mathfrak{p}},\lambda)$) be the kernel of q_λ (resp. $\bar{q}_\lambda$).

Now we consider the situation in 1.4, namely G is a real semisimple Lie group with the complexified Lie algebra $\mathfrak{g}$ and $\mathfrak{a}_m \subseteq \mathfrak{h}$.

We also assume $S_m \subseteq S$.

Then clearly we see $\mathsf{P}_S^{++} \subseteq \mathsf{P}_{S_m}^{++}$ and $\mathfrak{l}_m \subseteq \mathfrak{l}$. For $\lambda \in V_\lambda$, we put

$$E_\lambda = U(\mathfrak{l}_m) \cdot v_\lambda \subseteq V_\lambda.$$

Then E_λ is an irreducible representation of $\mathfrak{l}_m$ with highest weight λ.

For $\lambda \in \mathsf{P}_s^{++}$, we have

$$M(\mathfrak{p}_m, \lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_m)} E_\lambda,$$

$$\bar{M}(\bar{\mathfrak{p}}_m, \lambda) = E_\lambda^* \otimes_{U(\bar{\mathfrak{p}}_m)} U(\mathfrak{g}).$$

1.6 Functions of Gevrey class and differential operators of infinite order

Now we refer to the (ultra-differentiable) functions of Gevrey class introduced in [Gv].

Let U be an open set of $\mathbf{R}^n$ and let $1 \leq \varkappa$. We denote by $C^\infty(U)$ the space of the functions of class C^∞ on U. We call $\varphi \in C^\infty(U)$ a function of Gevrey class of order $\varkappa$ if for any compact subset K of U there exist some $h>0$ and $C>0$ such that

$$\forall \alpha \in \mathbf{N}^n, \quad \sup_{x \in K} |D^\alpha \varphi(x)| \leq C h^{|\alpha|} (|\alpha|!)^\varkappa.$$

Here, for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n$ we put

$$|\alpha| = \alpha_1 + \ldots + \alpha_n,$$
$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$$

We denote by $\mathscr{G}^\varkappa(U)$ the space of the functions of Gevrey class of order $\varkappa$. Since the definition of Gevrey class is local and invariant under any real analytic coordinate transformation, we can define $\mathscr{G}^\varkappa(X)$ for any real analytic manifold X.

From Pringsheim's result, we have $\mathscr{G}^1(X) = \mathscr{A}(X)$.

Next we consider differential operators which acts $\mathscr{G}^\varkappa(U)$. We assume $c_\alpha \in \mathbf{C}$ $(\alpha \in \mathbf{N}^n)$ satisfies

$$\sum_{\alpha \in \mathbf{N}^n} |c_\alpha| \, t^{|\alpha|} (|\alpha|!)^\varkappa < \infty, \tag{1}$$

for all $t>0$.

Put

$$P = \sum_{\alpha \in \mathbf{N}^n} c_\alpha D^\alpha.$$

LEMMA 1.6.1 (cf. [Km] Theorem 2.12, also see [Ro]). *Under the above assumption, P is a continuous linear endmorphism of $\mathscr{G}^\varkappa(U)$.*

If we introduce the topology on $C^\infty(U)$ as usual, then the inclusion map $\iota : \mathscr{G}^\varkappa(U) \hookrightarrow C^\infty(U)$ is continuous.

1.7 Gevrey vectors

We use the notation of 1.4. Here we refer to the Geverey vectors in Banach representations. (For details, see [Gd1] §1, [GW].)

Let π be a strongly continuous representation of G on a Banach space $H=H(\pi)$. We denote the space of C^∞-vectors for π by H_∞ and we also denote the associated representation of $U(\mathfrak{g})$ by $d\pi$.

We fix some basis $X_1, \ldots, X_l$ of $\mathfrak{g}$.

We define continuous semi-norms ϱ_n on H_∞ by

$$\varrho_0(v) = \|v\|,$$

$$\varrho_m(v) = \max_{1\leq j_k\leq d} \|d\pi(X_{j_1} \ldots X_{j_m})\, v\|.$$

For $\varkappa>0$, we put

$$S_\varkappa(\pi) = \{v \in H_\infty \mid \exists M,\ t>0, \forall n \in \mathbf{N}, \varrho_n(v) \leq Mt^n(n!)^\varkappa\}.$$

An element of $S_\varkappa(\pi)$ is called a Gevrey vector of order $\varkappa$.

Put

$$\|v\|_{\varkappa, t} = \sup_{n\geq 0}\{t^{-n}(n!)^{-\varkappa}\varrho_n(v)\}.$$

We topologize $S_\varkappa(\pi)$ as the inductive limit of the normed spaces

$$S_{\varkappa, t}(\pi) = \{\, v \in H_\infty \mid \|v\|_{\varkappa, t} < \infty\},$$

as $t \to \infty$.

Then the natural inclusion $S_\varkappa(\pi) \hookrightarrow H_\infty$ is continuous.

§ 2. Dimensions of the space of dual Whittaker vectors

2.1. The cohomology vanishing theorem of Kostant-Lynch

In this section 2.1, we only consider left modules. However, the argument in this section is applicable to the right modules.

First we define twisted $\mathfrak{n}$-actions (cf. [Ko], [Ly]). We fix an arbitrary character ψ on $\mathfrak{n}$. Let V be a $\mathfrak{n}$-module. For $v \in V$ and $X \in \mathfrak{n}$, we define

$$X * v = X \cdot v - \psi(X) v.$$

We call the above action the ψ-twisted action. Immediately we see $*$ defines another $\mathfrak{n}$-module structure on V. We call this $\mathfrak{n}$-module the ψ-twisted $\mathfrak{n}$-module of V.

We define an $\mathfrak{n}$-module structure on $\mathbf{C}$ by

$$X\cdot z=-\psi(X)z \quad (X\in\mathfrak{n}, z\in\mathbf{C}).$$

We denote this $\mathfrak{n}$-module by $\mathbf{C}_{-\psi}$. Then, for every $\mathfrak{n}$-module V, the ψ-twisted $\mathfrak{n}$-module is identified with $V\otimes_{\mathbf{C}}\mathbf{C}_{-\psi}$.

For an $\mathfrak{n}$-module V, we put

$$H^0(\mathfrak{n}, V)=\{v\in V \mid \forall X\in\mathfrak{n}, X\cdot v=0\}.$$

The functor $V\rightsquigarrow H_0(\mathfrak{n}, V)$ from the category of $\mathfrak{n}$-modules to the category of $\mathbf{C}$-vector space is left exact, and we can define the ith right derived functor $H^i(\mathfrak{n},\cdot)$. For a $\mathfrak{g}$-module M, clearly we can see

$$\mathrm{Wh}_{\mathfrak{n},\psi}(M)=H^0(\mathfrak{n}, M\otimes\mathbf{C}_{-\psi}).$$

Now we can quote the vanishing theorem, which is first proved by Kostant [Ko] for Borel subalgebras and generalized by Lynch [Ly] to the case of admissible parabolic subalgebras.

THEOREM 2.1.1. ([Ly] Lemma 4.3.) *Let ψ be an admissible character on $\mathfrak{n}$, and let a left $\mathfrak{g}$-module M be a Whittaker module* (*see* 1.3) *with respect to ψ. Then $H^i(\mathfrak{n}, M\otimes\mathbf{C}_{-\psi})=0$ for all $i>0$. We also have the same result for a right $\mathfrak{g}$-module.*

Let V be a $\mathfrak{n}$-module. We define

$$V_\eta=\{n\in V \mid \dim_{\mathbf{C}}(U(\mathfrak{n})*v)<\infty\}.$$

If M is a $\mathfrak{g}$-module, then we can easily see M_η is a $\mathfrak{g}$-submodule of M.

LEMMA 2.1.2. ([Ly] Proposition 4.5.) *Let M be a $\mathfrak{g}$-module such that $\dim_{\mathbf{C}}\mathrm{Wh}_{\mathfrak{n},\psi}(M)<\infty$. Then M_η has a finite composition series and each irreducible constituent is a Whittaker module.*

COROLLARY 2.1.3. (Cf. [Ly] Theorem 4.3.) *Let M be a $\mathfrak{g}$-module such that $\dim_{\mathbf{C}}\mathrm{Wh}_{\mathfrak{n},\psi}(M)<\infty$. Then $H^i(\mathfrak{n}, M_\eta\otimes\mathbf{C}_{-\psi})=0$ for $i>0$.*

2.2. Dual Whittaker vectors of an $\mathfrak{n}$-finitely generated $\mathfrak{g}$-module

Now, we are going to prove the following our first main result.

THEOREM 2.2.1. *Let M be a left $U(\mathfrak{g})$-module which is finitely generated as an $U(\mathfrak{g})$-module. Let ψ be an admissible character on $\mathfrak{n}$. Then*

$$\dim_{\mathbf{C}}(\mathrm{Wh}^*_{\mathfrak{n},\psi}(M)) = c_d(M).$$

Here, we put $d = \dim \mathfrak{n}$. (Since M is finitely generated, $\mathrm{Dim}(M) < d$.)

Proof. Let M be a finitely generated $U(\mathfrak{n})$-module. Since $U(\mathfrak{n})$ has finite global homological dimension, V has a finite projective resolution. On the other hand, every finitely generated projective $U(\mathfrak{n})$-module is stably free ([Qu] Theorem 7, also see [Mc]). From [No] 3.3, Lemma 7, we have the following finite free resolution.

$$0 \leftarrow M \leftarrow U(\mathfrak{n})^{\oplus r_1} \leftarrow \dots \leftarrow U(\mathfrak{n})^{\oplus r_m} \leftarrow 0. \tag{2}$$

Here, $U(\mathfrak{n})^{\oplus r_j}$ means

$$\overbrace{U(\mathfrak{n}) \oplus \dots \oplus U(\mathfrak{n})}^{r_j}.$$

Taking the dual of (1), we have

$$0 \rightarrow M^* \rightarrow (U(\mathfrak{n})^*)^{\oplus r_1} \rightarrow \dots \rightarrow (U(\mathfrak{n})^*)^{\oplus r_m} \rightarrow 0. \tag{3}$$

From the Artin-Rees lemma for $U(\mathfrak{n})$ (cf. [Ko] Lemma 4.5), we easily have

$$0 \rightarrow (M^*)_\eta \rightarrow (U(\mathfrak{n})^*)_\eta^{\oplus r_1} \rightarrow \dots \rightarrow (U(\mathfrak{n})^*)_\eta^{\oplus r_m} \rightarrow 0. \tag{3}$$

We can regard the left $U(\mathfrak{n})$-module "$U(\mathfrak{n})$" as the image of a generalized Verma module of with a lowest weight $U(\mathfrak{g})$ under the forgetful functor. On the other hand,

$$\begin{aligned} \dim H^0(\mathfrak{n}, (U(\mathfrak{n})^*)_\eta \otimes \mathbf{C}_{-\psi}) &= \dim \mathrm{Wh}_{\mathfrak{n},\psi}((U(\mathfrak{n})^*)_\eta) \\ &= \dim \mathrm{Wh}_{\mathfrak{n},\psi}((U(\mathfrak{n})^*) \\ &= \dim \mathrm{Wh}^*_{\mathfrak{n},\psi}(U(\mathfrak{n})) \\ &= 1. \end{aligned} \tag{5}$$

From Corollary 2.1.3, we have

$$H^i(\mathfrak{n}, (U(\mathfrak{g})^*)_\eta \otimes \mathbf{C}_{-\psi}) = 0 \quad \text{for} \quad i > 0. \tag{6}$$

From (2), we have

$$0 \rightarrow \mathrm{Wh}_{\mathfrak{n},\psi}(M^*) \rightarrow \mathrm{Wh}_{\mathfrak{n},\psi}((U(\mathfrak{n})^*)^{\oplus r_1}).$$

Especially, we have $\dim \mathrm{Wh}_{\mathfrak{n},\psi}(M^*)<\infty$. Hence from Corollary 2.1.3,

$$H^i(\mathfrak{n},(M^*)_\eta \otimes \mathbf{C}_{-\psi}) = 0 \quad \text{for} \quad i>0. \tag{7}$$

From (4), (5), and (6), we have

$$\begin{aligned}\dim \mathrm{Wh}^*_{\mathfrak{n},\psi}(M) &= \dim \mathrm{Wh}^*_{\mathfrak{n},\psi}((M^*)_\eta)\\ &= \sum_{i=1}^{m}(-1)^{i+1} r_i.\end{aligned} \tag{8}$$

On the other hand, from (2) we have

$$c_d(M) = \sum_{i=1}^{m}(-1)^{i+1} r_i. \tag{9}$$

(7) and (8) imply the desired result. Q.E.D.

Remark. D. A. Vogan suggested the author that the multiplicities relate to the dimensions of the space of dual Whittaker vectors via free resolutions.

We fix a connected real semisimple Lie group G and its Iwasawa decomposition as in 1.4. Let M be an arbitrary Harish-Candra module. Then M is finitely generated as a $U(\mathfrak{n}_m)$-module ([CO] 2.3), and the multiplicity of M as a $U(\mathfrak{g})$-module coincides with the multiplicity of M as a $U(\mathfrak{n}_m)$-module ([Jo] 5.6). Hence, we have the following generalization of a result of Kostant ([Ko] Theorem K).

COROLLARY 2.2.2. *Let ψ be an admissible character on $\mathfrak{n}_m$ and let M be a Harish-Chandra module (with respect to $(\mathfrak{g}, K)$). Then*

$$\dim \mathrm{Wh}^*_{\mathfrak{n}_m,\psi}(M) = \begin{cases} 0 & \text{if } \mathrm{Dim}(M) < d, \\ c_d(M) & \text{if } d = \dim\, \mathfrak{n}_m = \mathrm{Dim}(M). \end{cases}$$

Here, $c_d(M)$ means the multiplicity of M as a $U(\mathfrak{g})$-module.

Since we can easily see that the multiplicity of $\bar{L}(\bar{\mathfrak{p}},\lambda)$ as a $U(\mathfrak{g})$-module coincides with the multiplicity of $\bar{L}(\bar{\mathfrak{p}},\lambda)$ as a $U(\mathfrak{n})$-module, we have another corollary.

COROLLARY 2.2.3. *Let ψ be an admissible character on $\mathfrak{n}$. Put $d = \dim \mathfrak{n}$. For all $\lambda \in \mathsf{P}_S^{++}$, we have*

$$\dim \mathrm{Wh}^*_{\mathfrak{n},\psi}(\bar{L}(\bar{\mathfrak{p}},\lambda)) = c_d(\bar{L}(\bar{\mathfrak{p}},\lambda)) < \infty.$$

Here, $c_d(\bar{L}(\bar{\mathfrak{p}},\lambda))$ *means the multiplicity of* $\bar{L}(\bar{\mathfrak{p}},\lambda)$ *as a* $U(\mathfrak{g})$*-module.*

Especially, if $\mathrm{Dim}(\bar{L}(\bar{\mathfrak{p}},\lambda)) = d$*, then*

$$0 < \dim \mathsf{Wh}^*_{\mathfrak{n},\psi}(\bar{L}(\bar{\mathfrak{p}},\lambda)).$$

The following result is the special case of the formula (7) in the proof of Theorem 2.2.1.

COROLLARY 2.2.4. *Let* ψ *be an admissible character on* $\mathfrak{n}_m$ *and let* M *be a Harish-Chandra module (with respect to* $(\mathfrak{g},K)$*). Then, for all* $i>0$,

$$H^i(\mathfrak{n}_m,(M^*)_\eta \otimes \mathbf{C}_{-\psi}) = 0.$$

§ 3. Whittaker vectors in the completions of highest weight modules

3.1. Canonical pairings for irreducible highest weight modules

Accordings to [Shp] (also see [Ko], [Ly], [GW]), we introduce a pairing between hightest weight modules.

Let $\langle\langle\ ,\ \rangle\rangle$, be the canonical pairing on $V_\lambda^* \times V_\lambda$, that defines a C-linear map

$$P_\lambda\colon V_\lambda^* \otimes_{\mathbf{C}} V_\lambda \to \mathbf{C}.$$

Let Q_λ be the canonical projection:

$$Q_\lambda\colon \bar{M}(\bar{\mathfrak{p}},\lambda) \otimes_{\mathbf{C}} M(\mathfrak{p},\lambda) \to \bar{M}(\bar{\mathfrak{p}},\lambda) \otimes_{U(\mathfrak{g})} M(\mathfrak{p},\lambda) = V_\lambda^* \otimes_{\mathbf{C}} V_\lambda.$$

The composition $P_\lambda \circ Q_\lambda$ defines a pairing on $\bar{M}(\bar{\mathfrak{p}},\lambda) \times M(\mathfrak{p},\lambda)$, which we denote by the same letter $\langle\langle\ ,\ \rangle\rangle$.

From the definition, we have:

LEMMA 3.1.1. *For all* $P \in U(\mathfrak{g})$, $v^* \in \bar{M}(\bar{\mathfrak{p}},\lambda)$, *and* $v \in M(\mathfrak{p},\lambda)$,

$$\langle\langle v^* \cdot P, v\rangle\rangle = \langle\langle v^*, P\cdot v\rangle\rangle.$$

Then we easily have:

COROLLARY 3.1.2. *For all* $\lambda \in \mathsf{P}_S^{++}$, *the followings hold.*

$$K(\mathfrak{p},\lambda) = \{P \in M(\mathfrak{p},\lambda) \mid \forall Q \in \bar{M}(\bar{\mathfrak{p}},\lambda), \langle\langle Q,P\rangle\rangle = 0\},$$
$$\bar{K}(\bar{\mathfrak{p}},\lambda) = \{Q \in \bar{M}(\bar{\mathfrak{p}},\lambda) \mid \forall P \in \bar{M}(\bar{\mathfrak{p}},\lambda), \langle\langle Q,P\rangle\rangle = 0\}.$$

Especially, $\langle\langle\ ,\ \rangle\rangle$ *induces a non-singular pairing on* $\bar{L}(\bar{\mathfrak{p}},\lambda)\times L(\mathfrak{p},\lambda)$. *(We denote this pairing by the same letter* $\langle\langle\ ,\ \rangle\rangle$.)

Since the irreducibility of $M(\mathfrak{p},\lambda)$ implies that of $\bar{M}(\bar{\mathfrak{p}},\lambda)$, we have:

COROLLARY 3.1.3. *If* $M(\mathfrak{p},\lambda)$ *is irreducible, then* $\langle\langle\ ,\ \rangle\rangle$ *is non-degenerate.*

For $\mu\in\mathrm{Q}_{\mathfrak{a}}^{+}$, put

$$L(\mathfrak{p},\lambda)_{-\mu}=\{P\in L(\mathfrak{p},\lambda)\,|\,\forall H\in\mathfrak{a},H\cdot P=(\lambda(H)-\mu(H))P\},$$

$$\bar{L}(\bar{\mathfrak{p}},\lambda)_{\mu}=\{Q\in\bar{L}(\bar{\mathfrak{p}},\lambda)\,|\,\forall H\in\mathfrak{a},Q\cdot H=(\lambda(H)-\mu(H))P\}.$$

We also define $M(\mathfrak{p},\lambda)_{-\mu}$, $\bar{M}(\bar{\mathfrak{p}},\lambda)_{\mu}$, $K(\mathfrak{p},\lambda)_{-\mu}$, and $\bar{K}(\bar{\mathfrak{p}},\lambda)_{\mu}$ in the same way.

Hence we have

$$L(\mathfrak{p},\lambda)_{-\mu}=q_{\lambda}(M(\mathfrak{p},\lambda)_{-\mu}),$$

$$\bar{L}(\bar{\mathfrak{p}},\lambda)_{\mu}=\bar{q}_{\lambda}(\bar{M}(\bar{\mathfrak{p}},\lambda)_{\mu}).$$

Immediately, we have the following direct sum decomposition of irreducible highest weight modules.

$$L(\mathfrak{p},\lambda)=\bigotimes_{\mu\in\mathrm{Q}_{\mathfrak{a}}^{+}}L(\mathfrak{p},\lambda)_{-\mu},$$

$$\bar{L}(\bar{\mathfrak{p}},\lambda)=\bigotimes_{\mu\in\mathrm{Q}_{\mathfrak{a}}^{+}}\bar{L}(\bar{\mathfrak{p}},\lambda)_{\mu}.$$

Especially, these are finite-dimensional as $\mathbf{C}$-vector spaces.

We easily have:

LEMMA 3.1.4. (1) *Let* $\mu,\nu\in\mathrm{Q}_{\mathfrak{a}}^{+}$ *be distinct. Then the restriction of* $\langle\langle\ ,\ \rangle\rangle$ *to* $\bar{L}(\bar{\mathfrak{p}},\lambda)_{\mu}\times L(\mathfrak{p},\lambda)_{-\nu}$ *is zero.*

(2) *For* $\mu\in\mathrm{Q}_{\mathfrak{a}}^{+}$, *the restriction of* $\langle\langle\ ,\ \rangle\rangle$ *to* $\bar{L}(\bar{\mathfrak{p}},\lambda)_{\mu}\times L(\mathfrak{p},\lambda)_{-\mu}$ *is non-degenerate.*

3.2. Formal completions and algebraic duals

We define the formal completion of $L(\mathfrak{p},\lambda)$ by

$$\hat{L}(\mathfrak{p},\lambda)=\prod_{\mu\in\mathrm{Q}_{\mathfrak{a}}^{+}}L(\mathfrak{p},\lambda)_{-\mu}.$$

We also define $\hat{M}(\mathfrak{p},\lambda)$ and $\hat{K}(\mathfrak{p},\lambda)$ in the same way. From the same argument of the proof of [GW] Lemma 2.2, $\hat{L}(\mathfrak{p},\lambda)$, $\hat{M}(\mathfrak{p},\lambda)$, and $\hat{K}(\mathfrak{p},\lambda)$ coincide with the $\bar{\mathfrak{n}}$-completion (cf. [GW] 2) of $L(\mathfrak{p},\lambda)$, $M(\mathfrak{p},\lambda)$, and $K(\mathfrak{p},\lambda)$ respectively. Especially, these have $\mathfrak{g}$-module structures. The natural embedding $L(\mathfrak{p},\lambda)\hookrightarrow\hat{L}(\mathfrak{p},\lambda)$ is a $U(\mathfrak{g})$-homormorphism.

We have $\hat{L}(\mathfrak{p},\lambda)=\hat{M}(\mathfrak{p},\lambda)/\hat{K}(\mathfrak{p},\lambda)$.

We can extend $\langle\langle\ ,\ \rangle\rangle$ to $\bar{L}(\bar{\mathfrak{p}},\lambda)\times\hat{L}(\mathfrak{p},\lambda)$ in the obvious way. If $Q\in\hat{L}(\mathfrak{p},\lambda)$ satisfies $\langle\langle P,Q\rangle\rangle=0$ for all $P\in\bar{L}(\bar{\mathfrak{p}},\lambda)$, then we easily have $Q=0$.

Next result is obvious.

LEMMA 3.2.1. *Let V_i $(i\in\mathbf{N})$ be a family of finite dimensional complex vector spaces. Put $V=\oplus_{i\in\mathbf{N}}V_i$. Then the algebraic dual V^* coincides with $\Pi_{j\in\mathbf{N}}(V_j)^*$. Here we define $f(v)=0$ for all $f\in(V_j)^*$ and $v\in V_i$ such that $j\neq i$.*

From Lemma 3.1.4 and Lemma 3.2.1, we immediately have

PROPOSITION 3.2.2. *The algebraic dual $\bar{L}(\bar{\mathfrak{p}},\lambda)^*$ of $\bar{L}(\bar{\mathfrak{p}},\lambda)$ is isomorphic to $\hat{L}(\mathfrak{p},\lambda)$ as a $\mathfrak{g}$-module via the canonical pairing $\langle\langle\ ,\ \rangle\rangle$.*

Namely, for all $P\in U(\mathfrak{g})$, $v^*\in\bar{L}(\bar{\mathfrak{p}},\lambda)$, and $v\in\hat{L}(\mathfrak{p},\lambda)$,

$$\langle\langle v^*\cdot P,v\rangle\rangle=\langle\langle v^*,P\cdot v\rangle\rangle.$$

Remark. Though $\hat{M}(\mathfrak{p},\lambda)$ and $\bar{M}(\bar{\mathfrak{p}},\lambda)^*$ are isomorphic as $\mathfrak{m}+\mathfrak{a}$-modules, they are not isomorphic as $\mathfrak{g}$-modules if $M(\mathfrak{p},\lambda)\neq L(\mathfrak{p},\lambda)$.

3.3. Dual Whittaker vectors on an irreducible highest weight module

Let $\psi:\mathfrak{n}\to\mathbf{C}$ be any character.

From Proposition 3.2.2 and Theorem 2.2.3, we have

PROPOSITION 3.3.1.

$$\mathrm{Wh}_{\mathfrak{n},\psi}(\hat{L}(\mathfrak{p},\lambda))=\mathrm{Wh}^*_{\mathfrak{n},\psi}(\bar{L}(\bar{\mathfrak{p}},\lambda)).$$

Especially, if ψ is admissible, then we have

$$\dim_{\mathbf{C}}(\mathrm{Wh}_{\mathfrak{n},\psi}(\hat{L}(\mathfrak{p},\lambda)))=c_d(\bar{L}(\bar{\mathfrak{p}},\lambda)).$$

Here, $d=\dim\mathfrak{n}$ and $c_d(\bar{L}(\bar{\mathfrak{p}},\lambda))$ is the multiplicity of $\bar{L}(\bar{\mathfrak{p}},\lambda)$.

Remark. we can easily see that for $\bar{L}(\bar{\mathfrak{p}},\lambda)$, the Gelfand-Kirillov dimension and multiplicities as a $\mathfrak{g}$-module coincide with those as an $\mathfrak{n}$-module.

COROLLARY 3.3.2. *If ψ is admissible,*

$$\mathrm{Wh}_{\mathfrak{n},\psi}(\hat{L}(\mathfrak{p},\lambda)) \neq 0 \quad \textit{if and only if} \quad \mathrm{Dim}(\bar{L}(\bar{\mathfrak{p}},\lambda))=\dim \mathfrak{n}.$$

We write any element ω of $\mathrm{Wh}_{\mathfrak{n},\psi}(\hat{L}(\mathfrak{p},\lambda))$ as the following formal sum.

$$w = \sum_{\mu \in Q_{\alpha}^{+}} w_{-\mu},$$

where $w_{-\mu} \in L(\mathfrak{p},\lambda)_{-\mu}$.

Next we consider the case that $\bar{L}(\bar{\mathfrak{p}},\lambda) = \bar{M}(\bar{\mathfrak{p}},\lambda)$. For $v \in V_{\lambda}$, we define $v \otimes \psi \in \mathrm{Wh}^{*}_{\mathfrak{n},\psi}(\bar{M}(\bar{\mathfrak{p}},\lambda))$ by

$$(v \otimes \psi)(v^{*} \otimes_{\mathbf{C}} P) = \langle\langle v^{*}, v \rangle\rangle \psi(P),$$

where $v^{*} \in V^{*}_{\lambda}$ and $P \in U(\mathfrak{n})$. We denote the corresponding element in $\mathrm{Wh}_{\mathfrak{n},\psi}(\hat{M}(\mathfrak{p},\lambda))$ by $\psi_{v}(\lambda)$. Then we can immediately see

$$\psi_{v}(\lambda)_{0} = 1 \otimes_{\mathbf{C}} v \in \mathbf{C} \otimes_{\mathbf{C}} V_{\lambda}.$$

We easily get:

PROPOSITION 3.3.3. ([Ly] Chapter 5.) *We assume $\bar{M}(\bar{\mathfrak{p}},\lambda)$ is irreducible. Then*

$$\mathrm{Wh}_{\mathfrak{n},\psi}(\hat{M}(\mathfrak{p},\lambda)) = \{\psi_{v}(\lambda) \mid v \in V_{\lambda}\}.$$

3.4. A conjecture of Lynch

Fix $\lambda \in \mathbf{P}_{S}^{++}$ and an admissible character $\psi : \mathfrak{n} \to \mathbf{C}$. From Lemma 2.1.2, and Proposition 3.3.1, we have:

LEMMA 3.4.1. *For all $i > 0$, we have*

$$H^{i}(\mathfrak{n}, \hat{L}(\mathfrak{p},\lambda)_{\eta} \otimes \mathbf{C}_{-\psi}) = 0.$$

The following result is proved just the same way as [HS] Lemma 2.37.

PROPOSITION 3.4.2. *For every finitely generated $U(\mathfrak{n})$-module V, the inclusion $(V^{*})_{\eta} \hookrightarrow V^{*}$ induces isomorphisms*

$$H^{i}(\mathfrak{n}, (V^{*})_{\eta} \otimes \mathbf{C}_{-\psi}) \cong H^{i}(\mathfrak{n}, V^{*} \otimes \mathbf{C}_{-\psi}).$$

From Lemma 3.4.1, Proposition 3.4.2, and Proposition 3.2.2, we have:

PROPOSITION 3.4.3. *For all $i>0$, we have*

$$H^i(\mathfrak{n}, \hat{L}(\mathfrak{p}, \lambda) \otimes \mathbf{C}_{-\psi}) = 0.$$

Now we prove:

LEMMA 3.4.4. *There exists an increasing sequence of sub-$U(\mathfrak{g})$-modules of $\hat{M}(\mathfrak{p}, \lambda)$ (namely a filtration)*

$$0 = \hat{M}_0 \subseteq \hat{M}_1 \subseteq \dots \subseteq \hat{M}_l = \hat{M}(\mathfrak{p}, \lambda)$$

such that for each $1 \leq i \leq l$ there exists some λ_i such that

$$\hat{M}_i / \hat{M}_{i-1} \cong \hat{L}(\mathfrak{p}, \lambda_i).$$

Proof. It is well-known (and can be easily seen) that there exists a filtration of $M(\mathfrak{p}, \lambda)$

$$0 = M_0 \subseteq M_1 \dots \subseteq M_l = M(\mathfrak{p}, \lambda)$$

such that for each $1 \leq i \leq l$ there exists some λ_i such that

$$M_i / M_{i-1} \cong L(\mathfrak{p}, \lambda_i).$$

Let $\hat{M}_i$ be the $\bar{\mathfrak{n}}$-completion of M_i (cf. [GW] 2). From [GW] Proposition 2.1, (2), we can easily see $\{\hat{M}_i | 0 \leq i \leq l\}$ satisfies the desired conditions. Q.E.D.

LEMMA 3.4.5. *Put $d = \dim \mathfrak{n}$. Then*

$$c_d(\bar{L}(\bar{\mathfrak{p}}, \lambda)) = c_d(L(\mathfrak{p}, \lambda)).$$

Proof. Let τ be the complexified Cartan involution with respect to the normal real form of $\mathfrak{g}$ corresponding to the decomposition

$$\mathfrak{g} = \bar{\mathfrak{u}} + \mathfrak{h} + \mathfrak{u}.$$

Then we have $\tau(\bar{\mathfrak{n}}) = \mathfrak{n}$. For $v \in \bar{L}(\bar{\mathfrak{p}}, \lambda)$, if we define a left $U(\mathfrak{g})$-action "$*$" by

$$X * v = v \cdot (-\tau(X)),$$

then as a left $U(\mathfrak{g})$-module immediately we see

$$(\bar{L}(\bar{\mathfrak{p}}, \lambda), *) \cong L(\mathfrak{p}, \lambda).$$

Since we can easily see

$$U_m(\mathfrak{g})*v_\lambda^* = v_\lambda^* \cdot U_m(\mathfrak{g})$$

for all $m \geq 0$, we have the desired conclusion. Q.E.D.

From Proposition 3.3.1, we have:

COROLLARY 3.4.6. *For every* $\lambda \in \mathsf{P}_S^{++}$ *and every admissible character* ψ,

$$\dim \mathsf{Wh}_{\mathfrak{n},\psi}(\hat{L}(\mathfrak{p},\lambda)) = c_d(L(\mathfrak{p},\lambda)).$$

Here, $d = \dim \mathfrak{n}$.

Now we prove the following result which is conjectured by T. E. Lynch.

THEOREM 3.4.7. *For every* $\lambda \in \mathsf{P}_S^{++}$ *and an admissible character* ψ,

$$\dim \mathsf{Wh}_{\mathfrak{n},\psi}(\hat{M}(\mathfrak{p},\lambda)) = \dim V_\lambda.$$

Proof. Proposition 3.4.3 and Lemma 3.4.4 implies:

$$\mathsf{Wh}_{\mathfrak{n},\psi}(\hat{M}(\mathfrak{p},\lambda)) = \sum_{i=1}^{l} \dim \mathsf{Wh}_{\mathfrak{n},\psi}(\hat{L}(\mathfrak{p},\lambda)).$$

On the other hand, from Corollary 3.4.6, and 1.2, we have

$$\sum_{i=1}^{l} \dim \mathsf{Wh}_{\mathfrak{n},\psi}(\hat{L}(\mathfrak{p},\lambda_i)) = \sum_{i=1}^{l} c_d(L(\mathfrak{p},\lambda_i))$$

$$= c_d(M(\mathfrak{p},\lambda))$$

$$= \dim V_\lambda.$$

Here, we put $d = \dim \mathfrak{n}$. Hence we have the desired result. Q.E.D.

§4. Whittaker vectors in Gevrey completions

4.1. Unitary structures in root spaces

For $\alpha \in \Phi$, we put

$$\Delta^+(\alpha) = \{\beta \in \Delta \mid \beta|_{\mathfrak{a}} = \alpha\}.$$

Considering the normal real form $\mathfrak{g}_1$ of $\mathfrak{g}$ with respect to the decomposition $\mathfrak{g}=\bar{\mathfrak{u}}+\mathfrak{h}+\mathfrak{u}$, we see there exists an involution τ of $\mathfrak{g}$ such that $\tau|_{\mathfrak{h}}=-\mathrm{id}_{\mathfrak{h}}$ and $\tau(\bar{\mathfrak{u}})=\mathfrak{u}$. Let $\sigma\colon \mathfrak{g}\to\mathfrak{g}$ be the complex conjugation with respect to $\mathfrak{g}_1$. Put $\theta=\sigma\circ\tau$ $(=\tau\circ\sigma)$. Then, if we regard $\mathfrak{g}$ as a real Lie algebra, θ is a Cartan involution of $\mathfrak{g}$.

We define an Hermitian product on $\mathfrak{g}$ by

$$(X, Y) = -\langle X, \theta(Y)\rangle .$$

Fix $\alpha\in\Sigma^+$. For simplicity, we denote the restriction of $(\ ,\ )$ to $\mathfrak{n}_\alpha$ by the same letter.

For each $\beta\in\Delta^+$, we choose a non-zero element X_β of $\mathfrak{g}_\beta\cap\mathfrak{g}_1$ such that $(X_\beta, X_\beta)=1$. Put $\bar{X}_{-\beta}=-\theta(X_\beta)$. Then $\bar{X}_{-\beta}\in\mathfrak{g}_{-\beta}$ and for $\beta,\gamma\in\Delta^+$ we have

$$\langle X_\beta, \bar{X}_{-\gamma}\rangle = \begin{cases} 1 & \text{if } \beta=\gamma, \\ 0 & \text{otherwise.}\end{cases}$$

Then $\{\bar{X}_{-\beta}\mid\beta\in\Delta^+(\alpha)\}$ is an orthonormal basis of $\bar{\mathfrak{n}}_{-\alpha}$. Let $M_{\mathbf{C}}$ be the connected simply-connected complex Lie group corresponding to $\mathfrak{m}$. Since $\mathfrak{m}$ is invariant under θ, we can consider the compact real form M of $M_{\mathbf{C}}$ corresponding to the Cartan involution $\theta|_{\mathfrak{m}}$. Since M is simply-connected and connected, we can define an action σ_λ of M on V_λ which is compatible with the action of Lie algebra $\mathfrak{m}$. We also consider the adjoint action $\mathrm{Ad}(m)$ $(m\in M)$ on $\bar{\mathfrak{n}}_{-\alpha}$.

LEMMA 4.1.1. *For each* $\alpha\in\Sigma^+$, $(\mathrm{Ad}, \bar{\mathfrak{n}}_{-\alpha}, (\ ,\ ))$ *is a unitary representation of* M.

Proof. For $m\in M$, X, $Y\in\bar{\mathfrak{n}}_{-\alpha}$, we have

$$\begin{aligned}(\mathrm{Ad}(m)X, \mathrm{Ad}(m)Y) &= -\langle \mathrm{Ad}(m)X, \theta(\mathrm{Ad}(m)Y)\rangle \\ &= -\langle \mathrm{Ad}(m)X, \mathrm{Ad}(\theta(m))\,\theta(Y)\rangle \\ &= -\langle \mathrm{Ad}(m)X, \mathrm{Ad}(m)\,\theta(Y)\rangle \\ &= -\langle X, \theta(Y)\rangle \\ &= (X, Y).\end{aligned}$$

Q.E.D.

4.2. Gevrey completions

Now we introduce Gevrey completions of generalized Verma modules. First we define a family of seminorms on generalized Verma modules which are essentially introduced in [GW] (also see [Ra], [Gd2]) for Verma modules. Hereafter we fix a positive real number $\varkappa$.

Let $S(\bar{\mathfrak{n}})$ be the symmetic algebra of $\bar{\mathfrak{n}}$ and let

$$\tau\colon S(\bar{\mathfrak{n}}) \to U(\bar{\mathfrak{n}})$$

be the symmetrization map. We fix a numeration $\{\beta_1, \dots, \beta_h\}$ of $\Delta^+ - \bar{S}$. Here, $h = \dim \bar{\mathfrak{n}}$. Put $\bar{X}_i = \bar{X}_{\beta_i}$.

For $\mathbf{I} = (i_1, \dots, i_h) \in \mathbf{N}^h$, put

$$\bar{X}(\mathbf{I}) = \tau(\bar{X}_1^{i_1} \dots \bar{X}_h^{i_h}),$$

$$\beta(\mathbf{I}) = \sum_{k=1}^{h} i_k \beta_k,$$

$$|\mathbf{I}| = \sum_{k=1}^{h} i_k.$$

We fix some $\lambda \in \mathrm{P}_S^{++}$ and put $d = \dim V_\lambda$. Since M is compact, there exists some positive definite Hermitian inner product $(\ ,\ )_\lambda$ on V_λ which unitarizes the action of M on V_λ. We can assume $(v_\lambda, v_\lambda)_\lambda = 1$. We fix an orthonormal basis $v_1, \dots, v_d$ of V_λ such that $v_1 = v_\lambda$.

Then we can write each $P \in \hat{M}(\mathfrak{p}, \lambda)$ uniquely as follows.

$$P = \sum_{j=1}^{d} \sum_{\mathbf{I} \in \mathbf{N}^h} P(j, \mathbf{I}) \bar{X}(\mathbf{I}) \otimes v_j \quad \text{(formal sum)}.$$

Here $P(j, \mathbf{I}) \in \mathbf{C}$ for all $1 \leq j \leq d$ and $\mathbf{I} \in \mathbf{N}_h$.

For $\varkappa \geq 1$ and $t > 0$, put

$$\|P\|_{\varkappa, t} = \sum_{j=1}^{d} \sum_{\mathbf{I} \in \mathbf{N}^h} |P(j, \mathbf{I})|\, t^{|\mathbf{I}|} (|\mathbf{I}|!)^{\varkappa},$$

$$M^{\varkappa}(\mathfrak{p}, \lambda) = \{P \in \hat{M}(\mathfrak{p}, \lambda) \mid \forall t > 0, \|P\|_{\varkappa, t} < \infty\}.$$

Next we define Gevrey completions of irreducible heighest weight modules.

Let $\hat{q}_\lambda : \hat{M}(\mathfrak{p}, \lambda) \to \hat{L}(\mathfrak{p}, \lambda)$ be the natural projection. We define

$$L^{\varkappa}(\mathfrak{p}, \lambda) = \hat{q}_\lambda(M^{\varkappa}(\mathfrak{p}, \lambda)).$$

Now we can state one of the main results of this paper.

THEOREM 4.2.1. *For all* $1 \leq \varkappa < 2$ *and* $\lambda \in \mathsf{P}_S^{++}$, *we have*

$$\mathsf{Wh}_{\mathfrak{n},\psi}(\hat{L}(\mathfrak{p},\lambda)) \subseteq L^{\varkappa}(\mathfrak{p},\lambda).$$

4.3. Recursion formula

In this section we fix an arbitrary character ψ on $\mathfrak{n}$ and $w \in \mathsf{Wh}_{\mathfrak{n},\psi}(\hat{L}(\mathfrak{p},\lambda))$.

Put $n = \dim \mathfrak{a}$ and $m = \dim \mathfrak{t}$. Let $H_1, \ldots, H_n$ (resp. $T_1, \ldots, T_m$) be an orthonormal basis of $\mathfrak{a}$ (resp. $\mathfrak{t}$).

We denote by Ω the Casimir element in $U(\mathfrak{g})$. Namely

$$\Omega = \sum_{i=1}^{n} H_i^2 + \sum_{i=1}^{m} T_j^2 + \sum_{\beta \in \Delta^+} (\bar{X}_{-\beta} X_\beta + X_\beta \bar{X}_{-\beta}).$$

Put

$$\Omega_M = \sum_{j=1}^{m} T_i^2 + \sum_{\beta \in \Delta^+ \cap \bar{S}} (\bar{X}_{-\beta} X_\beta + X_\beta \bar{X}_{-\beta}).$$

Then, up to scalar factor Ω_M coincides with the Casimir element for $\mathfrak{m}$.

Put

$$\varrho = \frac{1}{2} \sum_{\beta \in \Delta^+} \beta \in \mathfrak{h}^*,$$

$$\varrho_M = \frac{1}{2} \sum_{\beta \in \Delta^+ \cap \bar{S}} \beta \in \mathfrak{t}^* \subseteq \mathfrak{h}^*.$$

$$\varrho_S = \frac{1}{2} \sum_{\beta \in \Delta^+ - \bar{S}} \beta \in \mathfrak{a}^* \subseteq \mathfrak{h}^*.$$

We fix $\lambda \in \mathsf{P}_S^{++}$ and $\mu \in \mathsf{Q}_{\mathfrak{a}}^+$.

Then, we have

$$\begin{aligned}\Omega &= \sum_{i=1}^{n} H_i^2 + \sum_{j=1}^{m} T_j^2 + H_{2\varrho} + 2 \sum_{\beta \in \Delta^+} \bar{X}_{-\beta} X_\beta \\ &= \Omega_M + \sum_{i=1}^{n} H_i^2 + H_{2\varrho_S} + 2 \sum_{\beta \in \Delta^+ - \bar{S}} \bar{X}_{-\beta} X_\beta.\end{aligned}$$

LEMMA 4.3.1. *For all* $v \in \hat{L}(\mathfrak{p}, \lambda)$,

$$\Omega \cdot v = (\langle \lambda, \lambda \rangle + \langle \lambda, 2\varrho \rangle) v.$$

Proof. Let u be any element of $\bar{L}(\bar{\mathfrak{p}}, \lambda)$. We have only to show the statement of the lemma for u. Since Ω is contained in the center of $U(\mathfrak{g})$, the lemma follows from the following formula, which we can easily deduce.

$$(v_\lambda^* \otimes 1) \cdot \Omega = (\langle \lambda, \lambda \rangle + \langle \lambda, 2\varrho \rangle)(v_\lambda^* \otimes 1). \qquad \text{Q.E.D.}$$

From the lemma, we have

$$\begin{aligned}(\langle \lambda, \lambda \rangle + \langle \lambda, 2\varrho \rangle)\, w_{-\mu} &= \Omega w_{-\mu} \\ &= \Omega_M w_{-\mu} + \left(\sum_{i=1}^{n} H_i^2 + H_{2\varrho_S} \right) w_{-\mu} + 2 \sum_{\beta \in \Delta^+ - S} \bar{X}_{-\beta} X_\beta w_{-\mu}.\end{aligned}$$

Here

$$w = \sum_{\mu \in \mathrm{Q}_\mathfrak{a}^+} w_{-\mu} \quad \text{(formal sum)},$$

$$w_{-\mu} \in L(\mathfrak{p}, \lambda)_{-\mu} \quad (\mu \in \mathrm{Q}_\mathfrak{a}^+).$$

Hence we have:

LEMMA 4.3.2. (Recursion formula.)

$$(\langle \lambda, \lambda + 2\varrho \rangle - \langle \lambda_\mathfrak{a} - \mu, \lambda_\mathfrak{a} - \mu + 2\varrho_S \rangle - \Omega_M) \cdot w_{-\mu} = 2 \sum_{\alpha \in \Phi} \sum_{\beta \in \Delta^+(\alpha)} (\psi(X_\beta) \bar{X}_{-\beta} \cdot w_{-\mu+\alpha}).$$

Here, if $\mu - \alpha \notin \mathrm{Q}_\mathfrak{a}^+$, *then we define* $w_{-\mu+\alpha} = 0$.

We define $T_\lambda(\mu) \in U(\mathfrak{m})$ by

$$T_\lambda(\mu) = \langle \lambda, \lambda + 2\varrho \rangle - \langle \lambda_\mathfrak{a} - \mu, \lambda_\mathfrak{a} - \mu + 2\varrho_S \rangle - \Omega_M.$$

4.4. Some unitary representations of M

Let r be a positive integer. For $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \Phi^r$, put

$$W(\boldsymbol{\alpha}) = W(\alpha_1, \ldots, \alpha_r) = \bar{\mathfrak{n}}_{-\alpha_r} \otimes_{\mathbf{C}} \cdots \otimes_{\mathbf{C}} \bar{\mathfrak{n}}_{-\alpha_1} \otimes_{\mathbf{C}} V_\lambda.$$

For $\boldsymbol{\alpha} \in \Phi^r$ we put

$$\boldsymbol{\alpha}(i) = \alpha_1 + \ldots + \alpha_i \quad (1 \leqslant i \leqslant r).$$

We define a positive definite Hermitian product $(\ ,\ )_{\boldsymbol{\alpha}}$ on $W(\alpha_1, \ldots, \alpha_r)$ such that

$$\{\bar{X}_{-\beta_r} \otimes \ldots \otimes \bar{X}_{-\beta_1} \otimes v_j | \beta_i \in \Delta^+(\alpha_i)\ (1 \leqslant i \leqslant r),\ 1 \leqslant j \leqslant d\}$$

is an orthonormal basis of $W(\alpha_1, \ldots, \alpha_r)$. We can immediately see $W(\alpha_1, \ldots, \alpha_r)$ is a unitary representation of M with respect to $(\ ,\ )_{\boldsymbol{\alpha}}$. Put

$$\|x\|_{\boldsymbol{\alpha}} = (x, x)_{\boldsymbol{\alpha}} \quad (x \in W(\boldsymbol{\alpha})).$$

Let $\alpha_{r+1} \in \Phi$. We regard $\bar{X}_{-\beta}$ ($\beta \in \Delta^+(\alpha_{r+1})$) as an operator

$$\bar{X}_{-\beta}: W(\alpha_1, \ldots, \alpha_r) \to W(\alpha_1, \ldots, \alpha_{r+1})$$
$$P \otimes \bar{X}_{-\beta} \rightsquigarrow P.$$

Then the operator norm of this operator "$\bar{X}_{-\beta}$" is less than or equal to 1. Then immediately we have:

LEMMA 4.4.1. *There exists a positive constant C_1 which does not depend on $\alpha_1, \ldots, \alpha_{r+1}$ such that*

$$\left\| \sum_{\beta \in \Delta^+(\alpha_{r+1})} \psi(X_\beta) \bar{X}_{-\beta} \right\| \leqslant C_1.$$

Next we consider the action of Ω_M on $W(\alpha_1, \ldots, \alpha_r)$. Since Ω_M is contained in the center of $U(\mathfrak{m})$ and $W(\alpha_1, \ldots, \alpha_r)$ is completely reducible, we have:

LEMMA 4.4.2. *Ω_M acts on $W(\alpha_1, \ldots, \alpha_r)$ as a diagonalizable linear operator and moreover distinct eigenspaces are orthogonal to each other. The eigenvalues of Ω_M are all non-negative.*

4.5. An estimate of $\|T_\lambda(\boldsymbol{\mu})^{-1}\|$

Let $\mathfrak{g}_1$ be the normal real form of $\mathfrak{g}$ defined in 4.1. and let σ be the complex conjugation with respect to $\mathfrak{g}_1$. We denote the induced conjugation on α^*, $\mathfrak{h}^*$, and so on by the same letter σ. For $\xi \in \mathfrak{a}^*$, put

$$\Re\xi = \frac{\xi+\sigma(\xi)}{2},$$

$$\Im\xi = \frac{\xi+\sigma(\xi)}{2i}.$$

Put

$$\Upsilon(\lambda) = \left\{\mu \in Q_{\mathfrak{a}}^+ \,\middle|\, \frac{\langle \lambda_t, \lambda_t+2\varrho_M\rangle + 2\langle \mu, \Re\lambda_{\mathfrak{a}}-\varrho_S\rangle}{\langle \mu,\mu\rangle} \geqslant \frac{1}{2}\right\}.$$

LEMMA 4.5.1. $\Upsilon(\lambda)$ *is finite.*

Proof. Put

$$\mathfrak{a}_0^* = \{\xi \in \mathfrak{a}^* \mid \xi(\mathfrak{g}_0 \cap \mathfrak{a}) \subseteq \mathbf{R}\}.$$

The lemma is deduced from the fact that

$$\left\{\xi \in \mathfrak{a}_0^* \,\middle|\, \frac{\langle \lambda_t, \lambda_t+2\varrho_M\rangle + 2\langle \xi, \Re\lambda_{\mathfrak{a}}-\varrho_S\rangle}{\langle \xi,\xi\rangle} \geqslant \frac{1}{2}\right\}$$

is compact set of $\mathfrak{a}_0^* \cong \mathbf{R}^n$. Q.E.D.

Hereafter we fix a non-negative integer $s(\lambda)$ such that

$$\{\mu \in Q_{\mathfrak{a}}^+ \mid |\mu| \leqslant s(\lambda)\} \supseteq \Upsilon(\lambda).$$

Then we have:

PROPOSITION 4.5.2. *Let r be a positive integer. We fix $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_r)\in\Phi^r$. Put $\mu=\boldsymbol{\alpha}(r)=\alpha_1+\ldots+\alpha_r$. If $r>s(\lambda)$, then $T_\lambda(\mu)$ acts on $W(\alpha_1,\ldots,\alpha_r)$ as an inversible linear operator.*

Moreover we have

$$\|T_\lambda(\mu)^{-1}\| < 2\langle \mu,\mu\rangle^{-1}.$$

Here $\|T_\lambda(\mu)^{-1}\|$ is the norm as a operator on the Hilbert space $W(\alpha_1,\ldots,\alpha_r)$.

Proof. We have

$$\langle \lambda, \lambda+2\varrho\rangle - \langle \lambda_{\mathfrak{a}}-\mu, \lambda_{\mathfrak{a}}-\mu+2\varrho_S\rangle$$

$$= \langle \lambda_t, \lambda_t + 2\varrho_M \rangle + \langle \lambda_a, \lambda_a + 2\varrho_S \rangle - \langle \mu, \mu \rangle$$
$$+ \langle \mu, \lambda_a + 2\varrho_S \rangle + \langle \lambda_a, \mu \rangle - \lambda_a, \lambda_a + 2\varrho_S \rangle$$
$$= \langle \lambda_t, \lambda_t + 2\varrho_M \rangle + \langle \mu, 2\lambda_a + 2\varrho_S \rangle - \langle \mu, \mu \rangle .$$

Hence

$$T_\lambda(\mu) = \langle \lambda_t, \lambda_t + 2\varrho_M \rangle + \langle \mu, 2\lambda_a + 2\varrho_S \rangle - \langle \mu, \mu \rangle - \Omega_M .$$

Let η be an eigenvalue of $T_\lambda(\mu)$ as a linear operator on $W(\alpha_1, \ldots, \alpha_r)$.

From Lemma 4.4.2, we have

$$\Re \eta \leqslant \langle \lambda_t, \lambda_t + 2\varrho_M \rangle + \langle \mu, 2\Re\lambda_a + 2\varrho_S \rangle - \langle \mu, \mu \rangle .$$

Since $\mu \notin \mathrm{Y}(\lambda)$, from Lemma 3.5.1 we have

$$\Re \eta < -\tfrac{1}{2}\langle \mu, \mu \rangle .$$

The proposition is easily deduced from this fact. Q.E.D.

Next we introduce a new positive definite inner product $\{\ ,\ \}$ on $\mathfrak{a}_0^*$ as follows. For $\alpha_1, \alpha_2 \in \Phi$,

$$\{\alpha_1, \alpha_2\} = \begin{cases} 0 & \text{if} \quad \alpha_1 \neq \alpha_2, \\ 1 & \text{if} \quad \alpha_1 = \alpha_2 \end{cases} .$$

Then there exist positive constants C_2 and c_2 such that

$$C_2^{-1}\{x, x\} \leqslant \langle x, x \rangle \leqslant c_2\{x, x\},$$

for all $x \in \mathfrak{a}_0^*$.

Immediately we have:

COROLLARY 4.5.3. *We fix $(\alpha_1, \ldots, \alpha_r) \in \Phi^r$. Put $\mu = \alpha_1 + \ldots + \alpha_r$. Then we have*

$$\|T_\lambda(\mu)^{-1}\| < 2C_2\{\mu, \mu\}^{-1}.$$

4.6. Proof of Theorem 4.2.1

We use the notations of sections 3.1–3.5. Hereafter we fix $\lambda \in \mathrm{P}_S^{++}$, an arbitrary character ψ on $\mathfrak{n}$, and $w \in \mathrm{Wh}_{\mathfrak{n},\psi}(\bar{L}(\mathfrak{p}, \lambda))$. Then we have the following formal expression.

$$w = \sum_{\mu \in Q_{\mathfrak{a}}^{+}} w_{-\mu},$$

where $w_{-\mu} \in L(\mathfrak{p}, \lambda)_{-\mu}$.

Let H_0 be the element of $\mathfrak{a}$ such that $\alpha(H_0) = 1$ for all $\alpha \in \Phi$ (cf. 1.1). Put $|\nu| = \nu(H_0)$ for $\nu \in Q_{\mathfrak{a}}^{+}$. For $\nu \in Q_{\mathfrak{a}}^{+}$, we define.

$$I(\nu) = \{(\alpha_1, \ldots, \alpha_{|\nu|}) \mid \alpha_i \in \Phi \ (1 \leq i \leq |\nu|), \nu = \alpha_1 + \ldots + \alpha_{|\nu|}\}.$$

Let $\mu \in Q_{\mathfrak{a}}^{+}$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{|\mu|}) \in I(\mu)$. Put $\boldsymbol{\alpha}(r) = \alpha_1 + \ldots + \alpha_r$ for $1 \leq r \leq |\mu|$. We define a $\mathfrak{l}$-homomorphism $\mathbf{p}_{\boldsymbol{\alpha}, r}$ from $W(\alpha_1, \ldots, \alpha_r)$ to $M(\mathfrak{p}, \lambda)_{-\boldsymbol{\alpha}(r)}$ as follows.

$$\mathbf{p}_{\boldsymbol{\alpha}, r}(X_1 \otimes \ldots \otimes X_r \otimes v) = X_1 X_2 \ldots X_r \cdot v.$$

Here $X_i \in \bar{\mathfrak{n}}_{-\alpha_i}$ $(1 \leq i \leq r)$ and $v \in V$.

Let $\mathbf{q}_\mu$ be the natural projection from $M(\mathfrak{p}, \lambda)_{-\mu}$ to $L(\mathfrak{p}, \lambda)_{-\mu}$.

For $\mu \in Q_{\mathfrak{a}}^{+}$, we define

$$W(\mu) = \bigoplus_{\boldsymbol{\alpha} \in I(\mu)} W(\boldsymbol{\alpha}),$$

$$\mathbf{p}_\mu = \sum_{\boldsymbol{\alpha} \in I(\mu)} \mathbf{p}_{\boldsymbol{\alpha}, |\mu|}.$$

$W(\mu)$ has a unitary structure induced from $(W(\boldsymbol{\alpha}), \|\ \|_{\boldsymbol{\alpha}})$. We denote $\|\ \|_\mu$ the norm of the unitary structure on $W(\mu)$. We can easily see $\mathbf{p}_\mu : W(\mu) \to M(\mathfrak{p}, \lambda)_{-\mu}$ is a surjective $\mathfrak{l}$-homomorphism. Put $\mathbf{r}_\mu = \mathbf{q}_\mu \circ \mathbf{p}_\mu$.

For μ such that $|\mu| \leq s(\lambda)$, we hereafter fix $\tilde{w}_{-\mu} \in W(\mu)$ such that $\mathbf{r}_\mu(\tilde{w}_{-\mu}) = w_{-\mu}$. Since $\mathbf{r}_\mu$ intertwine the action of $T_\lambda(\mu)$, Proposition 4.5.2 implies:

LEMMA 4.6.1. *Let $\mu \in Q_{\mathfrak{a}}^{+}$ satisfy $|\mu| > s(\lambda)$. Then $T_\lambda(\mu)$ is inversible on $L(\mathfrak{p}, \lambda)_{-\mu}$ and $M(\mathfrak{p}, \lambda)_{-\mu}$.*

Now we introduce some notations. Let $\mu \in Q_{\mathfrak{a}}^{+}$ satisfy $|\mu| > s(\lambda)$. Put

$$D(\mu) = \{\nu \in Q_{\mathfrak{a}}^{+} \mid |\nu| = s(\lambda), \mu - \nu \in Q_{\mathfrak{a}}^{+}\}.$$

For $\alpha \in \Phi$, put

$$Y_\alpha = \sum_{\beta \in \Delta^{+}(\alpha)} \psi(X_\beta) \bar{X}_{-\beta}.$$

We denote by $\tilde{Y}_\alpha$ the lifting of Y_α to a linear map of $W(\alpha_1, \dots, \alpha_l)$ to $W(\alpha_1, \dots, \alpha_l, \alpha)$ or of $W(\xi)$ to $W(\xi+\alpha)$. (Cf. 4.4.) Here $\alpha_1, \dots, \alpha_l \in \Phi$ and $\xi \in \mathrm{Q}_\alpha^+$.

Then Lemma 4.6.1 and Lemma 4.3.2 imply:

LEMMA 4.6.2. *Let $\mu \in \mathrm{Q}_\alpha^+$ satisfy $|\mu|-s(\lambda)=r>0$. Then*

$$w_{-\mu} = 2^r \sum_{\nu \in D(\mu)} \sum_{\boldsymbol{\alpha} \in I(\mu-\nu)} T_\lambda(\boldsymbol{\alpha}(r)+\nu)^{-1} Y_{\alpha_r} T_\lambda(\boldsymbol{\alpha}(r-1)+\nu)^{-1} Y_{\alpha_{r-1}} \cdots T_\lambda(\boldsymbol{\alpha}(1)+\nu)^{-1} Y_{\alpha_1} w_{-\nu}.$$

Here $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r)$ and $\boldsymbol{\alpha}(i) = \alpha_1 + \dots + \alpha_i$. Hence $\mu = \boldsymbol{\alpha}(r)+\nu$.

For $\mu \in \mathrm{Q}_\alpha^+$ such that $|\mu|-s(\lambda)=r>0$, we define an element of $W(\mu)$ by

$$\tilde{w}_{-\mu} = 2^r \sum_{\nu \in D(\mu)} \sum_{\boldsymbol{\alpha} \in I(\mu-\nu)} T_\lambda(\boldsymbol{\alpha}(r)+\nu)^{-1} \tilde{Y}_{\alpha_r} T_\lambda(\boldsymbol{\alpha}(r-1)+\nu)^{-1} \tilde{Y}_{\alpha_{r-1}} \cdots T_\lambda(\boldsymbol{\alpha}(1)+\nu) \tilde{Y}_{\alpha_1} \tilde{w}_{-\nu}.$$

Clearly we have

$$\mathbf{r}_\mu(\tilde{w}_{-\mu}) = w_{-\mu}.$$

Next we are going to estimate $\|\tilde{w}_{-\mu}\|_\mu$. First we introduce some positive constants which only depends on ψ and λ. Put

$$C_3 = \max\{\|\tilde{w}_{-\nu}\|_\nu \mid \nu \in \mathrm{Q}_\alpha^+, |\nu| = s(\lambda)\},$$
$$D_\lambda = \{\nu \in \mathrm{Q}_\alpha^+ \mid |\nu| = s(\lambda)\},$$
$$C_4 = \max_{\nu \in D_\lambda}\left(\sum_{\boldsymbol{\alpha} \in I(\nu)}\left(\prod_{1 \leq j \leq s(\lambda)} \{\boldsymbol{\alpha}(j), \boldsymbol{\alpha}(j)\}\right)\right).$$

Fix $\mu \in \mathrm{Q}_\alpha^+$ such that $|\mu|-s(\lambda)=r>0$. From the definition, we have

$$\begin{aligned} \|\tilde{w}_{-\mu}\|_\mu &\leq C_3(4C_1C_2)^r \sum_{\nu \in D(\mu)} \sum_{\boldsymbol{\alpha} \in I(\mu-\nu)} \left(\prod_{1 \leq i \leq r} \{\boldsymbol{\alpha}(i)+\nu, \boldsymbol{\alpha}(i)+\nu\}^{-1}\right) \\ &\leq C_3C_4(4C_1C_2)^r \sum_{\boldsymbol{\alpha} \in I(\mu)} \left(\prod_{1 \leq i \leq |\mu|} \{\boldsymbol{\alpha}(i), \boldsymbol{\alpha}(i)\}^{-1}\right). \end{aligned} \tag{1}$$

We fix a numeration $\{\gamma_1, \dots, \gamma_n\}$ of Φ. Then every $\mu \in \mathrm{Q}_\alpha^+$ is represented as follows.

$$\mu = \sum_{i=1}^n m_i \gamma_i,$$

where m_i $(1 \leqslant i \leqslant n)$ are non-negative integers. From the formula in [GW] (4.9), we have

$$\sum_{\boldsymbol{\alpha} \in I(\mu)} \left(\prod_{1 \leqslant i \leqslant |\mu|} \{\boldsymbol{\alpha}(i), \boldsymbol{\alpha}(i)\}^{-1} \right) = \prod_{i=1}^{n} (m_i!)^{-2}. \tag{2}$$

We easily have

$$\frac{|\mu|!}{m_1! \dots m_n!} \leqslant n^{|\mu|}. \tag{3}$$

From (1), (2), and (3), we have:

LEMMA 4.6.3. *There exists some positive constant C_5 such that for all $\mu \in \mathrm{Q}_\mathfrak{a}^+$*

$$\|\tilde{w}_{-\mu}\|_\mu \leqslant C_5^{|\mu|+1}(|\mu|!)^{-2}.$$

Put $\bar{w}_{-\mu} = \mathbf{p}_\mu(\tilde{w}_{-\mu})$ for $\mu \in \mathrm{Q}_\mathfrak{a}^+$. Next we are going to estimate $\|\bar{w}_{-\mu}\|_{\varkappa, t}$ for $1 \leqslant \varkappa < 2$ and $t > 0$.

First we introduce some notations. Put

$$\bar{\Phi} = \bigcup_{\alpha \in \Phi} \Delta^+(\alpha).$$

For $\mu = \mathrm{Q}_\mathfrak{a}^+$, we put

$$J(\mu) = \{(\beta_1, \dots, \beta_{|\beta|}) | \beta_i \in \bar{\Phi}\, (1 \leqslant i \leqslant |\mu|),\ \mu = (\beta_1 + \dots + \beta_{|\mu|})_\mathfrak{a}\}.$$

If we put $C_6 = \operatorname{card} \bar{\Phi}$, then we have

$$\operatorname{card} J(\mu) \leqslant C_6^{|\mu|}. \tag{4}$$

For any positive integer r and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_r) \in \bar{\Phi}^r$, put

$$\bar{X}_{\boldsymbol{\beta}} = \bar{X}_{\beta_1} \dots \bar{X}_{\beta_r} \in U(\bar{\mathbf{n}}).$$

We choose an orthonomal basis $v_1, \dots, v_d$ of V_λ as in 3.2. Then Lemma 4.6.3 can be rewritten as follows.

LEMMA 4.6.4. *For each $\mu \in \mathrm{Q}_\mathfrak{a}^+$, $\bar{w}_{-\mu}$ has the following expression.*

$$\bar{w}_{-\mu} = \sum_{j=1}^{d} \sum_{\boldsymbol{\beta} \in J(\mu)} Q(j, \boldsymbol{\beta}) \bar{X}_{\boldsymbol{\beta}} v_j,$$

where $Q(j, \boldsymbol{\beta}) \in \mathbf{C}$ $(1 \leq j \leq d; \boldsymbol{\beta} \in J(\mu))$ *satisfy*

$$\sum_{j=1}^{d} \sum_{\boldsymbol{\beta} \in J(\mu)} |Q(j, \boldsymbol{\beta})|^2 \leq (C_5^{|\mu|+1}(|\mu|!)^{-2})^2.$$

Especially, for $1 \leq j \leq d$ *and* $\boldsymbol{\beta} \in J(\mu)$, *we have*

$$|Q(j, \boldsymbol{\beta})| \leq C_5^{|\mu|+1}(|\mu|!)^{-2}.$$

We quote:

LEMMA 4.6.5. ([Gd2] (2.3), (2.4).) *Put* $h = \dim \bar{\mathfrak{n}}$. *For* $\mu \in Q_\alpha^+$ *and* $\boldsymbol{\beta} \in J(\mu)$, *we have the following expression.*

$$\frac{1}{|\mu|!} \bar{X}_{\boldsymbol{\beta}} = \sum_{\mathbf{I} \in \mathbf{N}^h} \frac{1}{|\mathbf{I}|!} c_{\mathbf{I}}^{\boldsymbol{\beta}} \bar{X}(\mathbf{I}),$$

where $c_{\mathbf{I}}^{\boldsymbol{\beta}} \in \mathbf{C}$ *satisfies the following conditions* (A), (B).

(A) $c_{\mathbf{I}}^{\boldsymbol{\beta}} = 0$ *if* $|\mathbf{I}| > |\mu|$.

(B) *There exists some constant* C_7 *which only depends on the structure of* $\bar{\mathfrak{n}}$ *and satisfies*

$$|c_{\mathbf{I}}^{\boldsymbol{\beta}}| \leq C_7^{|\mu|}.$$

Put

$$K(\mu) = \{\mathbf{I} \in \mathbf{N}^h \mid |\mathbf{I}| \leq |\mu|\}.$$

Then there exist some constant C_8 such that

$$\operatorname{card} K(\mu) \leq C_8^{|\mu|}.$$

From Lemma 4.6.4 and Lemma 4.6.5, we have

$$\bar{w}_{-\mu} = |\mu|! \sum_{j=1}^{d} \sum_{\mathbf{I} \in K(\mu)} \sum_{\boldsymbol{\beta} \in J(\mu)} \frac{1}{|\mathbf{I}|!} Q(j, \boldsymbol{\beta})\, c_{\mathbf{I}}^{\boldsymbol{\beta}} \bar{X}(\mathbf{I}) v_j$$

We fix $1 \leq \varkappa < 2$ and $t > 0$. Then we have

$$\begin{aligned} \|\bar{w}_{-\mu}\|_{\varkappa, t} &\leq |\mu|! \sum_{j=1}^{d} \sum_{\boldsymbol{\beta} \in J(\mu)} \sum_{\mathbf{I} \in K(\mu)} |Q(j, \boldsymbol{\beta})|\, |c_{\mathbf{I}}^{\boldsymbol{\beta}}| t^{|\mathbf{I}|} (|\mathbf{I}|!)^{\varkappa - 1} \\ &\leq dC_5(C_5 C_6 C_7 t)^{|\mu|}(|\mu|!)^{-1} \sum_{|\mathbf{I}| \leq |\mu|} (|\mathbf{I}|!)^{\varkappa - 1} \qquad (5) \\ &\leq A(Bt)^{|\mu|}(|\mu|!)^{\varkappa - 2}. \end{aligned}$$

where $A = dC_5$ and $B = C_5 C_6 C_7 C_8$.

Put

$$\bar{w} \in \sum_{\mu \in \mathsf{Q}_\mathfrak{a}^+} \bar{w}_{-\mu}.$$

Now we have

$$\begin{aligned} \|\bar{w}\|_{\varkappa, t} &\leq \sum_{\mu \in \mathsf{Q}_\mathfrak{a}^+} \|\bar{w}_{-\mu}\|_{\varkappa, t} \\ &\leq A \sum_{\mu \in \mathsf{Q}_\mathfrak{a}^+} ((Bt)^{|\mu|}(|\mu|!)^{\varkappa-2} \qquad (6) \\ &< \infty. \end{aligned}$$

This implies $\bar{w} \in M^\varkappa(\mathfrak{p}, \lambda)$. Since $\hat{q}(\bar{w}) = w$, we get the desired conclusion. Q.E.D.

4.7. Whittaker vectors in the Geverey completion of a generalized Verma module

The following results is deduced from just the same argument of Theorem 4.2.1.

THEOREM 4.7.1. *For all* $1 \leq \varkappa < 2$ *and* $\lambda \in \mathsf{P}_S^{++}$, *we have*

$$\mathsf{Wh}_{\mathfrak{n}, \psi}(\hat{M}(\mathfrak{p}, \lambda)) \subseteq M^\varkappa(\mathfrak{p}, \lambda).$$

For the later use, we slightly generalize the above result.

Fix $\lambda \in \mathsf{P}_S^{++}$. We define

$$O(\lambda) = \{\xi \in \mathfrak{a}^* | \bar{M}(\bar{\mathfrak{p}}, \lambda + \xi) \text{ is irreducible}\}.$$

As a finite dimensional $\mathfrak{m}$-module, we identify V_λ and $V_{\lambda+\xi}$ $(\xi \in \mathfrak{a}^*)$.

From Proposition 3.3.3, we have:

LEMMA 4.7.2. *For all* $\xi \in O(\lambda)$,

$$\mathsf{Wh}_{\mathfrak{n}, \psi}(\hat{M}(\mathfrak{p}, \lambda + \xi)) = \{\psi_v(\lambda + \xi) | v \in V_\lambda\}.$$

The following result is easily deduced from the proof of Theorem 4.2.1.

LEMMA 4.7.3. *Let T be a compact subset of $O(\lambda)$. We fix $v \in V_\lambda$. Then, for all* $1 \leq \varkappa < 2$ *and* $t > 0$, *we have*

$$\sup_{\xi \in T} \|\psi_v(\lambda + \xi)\|_{\varkappa, t} < \infty$$

Put $\Phi^{\infty}=\bigcup_{r\in\mathbf{N}}\Phi^{r}$, where we put $\Phi^{0}=\{\phi\}$. For all $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_r)\in\Phi^{\infty}$, put

$$|\boldsymbol{\alpha}|=\alpha_1+\dots+\alpha_r.$$

Fix $\lambda\in\mathsf{P}_S^{++}$ and $\xi\in\mathfrak{a}^*$. We write

$$\psi_v(\lambda+\xi)=\sum_{\mu\in\mathsf{Q}_{\mathfrak{a}}^{+}}\psi_v(\lambda+\xi)_{-\mu},$$

where $\psi_v(\lambda+\xi)_{-\mu}\in M(\mathfrak{p},\lambda)_{-\mu}$.

PROPOSITION 4.7.4. *Fix $\lambda\in\mathsf{P}_S^{++}$ and fix $\xi\in\mathfrak{a}^*$ such that $\langle\mu,\xi\rangle\neq 0$ for all $\mu\in\mathsf{Q}_{\mathfrak{a}}^{+}$. Then there exists some $\varepsilon>0$ such that $z\rightsquigarrow\psi_v(\lambda+z\xi)_{-\mu}$ is holomorphic on $\{z\in\mathbf{C}\,|\,0<|z|<\varepsilon\}$ for all $\mu\in\mathsf{Q}_{\mathfrak{a}}^{+}$.*

Proof. From the proof of Proposition 4.5.2, we have

$$T_{\lambda+z\xi}(\mu)=\langle\lambda_t,\lambda_t+2\varrho_M\rangle+\langle\mu,2\lambda_{\mathfrak{a}}+2\varrho_S\rangle+\langle\mu,2z\xi\rangle-\langle\mu,\mu\rangle-\Omega_M.$$

$T_{\lambda+z\xi}(\mu)$ is inversible for $|\mu|>>0$, $0<|z|<<1$. On the other hands, there exists some discrete subset U of $\mathbf{R}$ such that all the eigenvalues of Ω_M on $W(\boldsymbol{\alpha})$ contained in U for all $\boldsymbol{\alpha}\in\Phi^{\infty}$. Therefore, $T_{\lambda+z\xi}(\alpha_1+\dots+\alpha_r)$ is inversible on $W(\boldsymbol{\alpha})$ for all $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_r)\in\Phi^{\infty}$. Hence, we get the desired conclusion from Lemma 4.3.2. Q.E.D.

§.5. The Goodman-Wallach operators

5.1. Principal series

Hereafter throughout §5, we fix some $S\subseteq\Pi$ such that $S_m\subseteq S$ and use the notations of 1.4. and 1.5. Namely, G is a connected real semisimple Lie group with finite center, $\mathfrak{g}$ is the complexified Lie algebra of G, $E_\lambda\subseteq V_\lambda$, ... etc.

Let M_0 be the connected component of M_m containing the identity element. Put $P_0=M_0A_mN_m$. Then P_0 is the connected component of P_m containing the identity element.

Hereafter we denote by $*$ either "m" or "0". Hence (M_*,P_*) is either (M_m,P_m) or (M_0,P_0). Let σ be a finite dimensional irreducible unitaty representation of M_* and let E_σ be the representation space of σ.

For any real analytic Lie group X, we denote by $\mathcal{M}(X)$ the space of measurable functions with respect to the (left) Haar measure. Let $\mathscr{F}$ be $\mathscr{G}^{\varkappa}$ $(1\leq\varkappa)$, C^{∞}, $\mathscr{A}$ or $\mathscr{M}$. Let $\mathscr{F}(G;E_\sigma)$ (resp. $\mathscr{F}(K;E_\sigma)$) be the space of E_σ-valued functions on G (resp. K) belonging to the class $\mathscr{F}$.

Let $\mathscr{X}=\{X_1,\dots,X_n\}$ be a basis of a real vector space $\mathfrak{k}\cap\mathfrak{g}_0$. For $t>0$, $\varkappa>1$, and $f\in C^{\infty}(K;E_\sigma)$, put

$$\|f\|^K_{k,t}=\inf\{C>0\,|\,\forall n\in\mathbf{N},\forall X_{i_1},\dots,X_{i_n}\in\mathscr{X},\sup_{k\in K}\|f(X_{i_1}\dots X_{i_n}:k)\|_{E_\sigma}\leq Ct^n(|n|!)^{\varkappa}\}.$$

Here, $\|\ \|_{E_\sigma}$ is the norm of E_σ. Put

$$\mathscr{G}^{\varkappa}(K;E_\sigma)_t=\{f\in C^{100}(K;E_\sigma)|\,\|f\|^K_{\varkappa,t}<\infty\}.$$

From [GW] Corollary 1.2, we have

$$\mathscr{G}^{\varkappa}(K;E_\sigma)=\bigcup_{t>0}\mathscr{G}^{\varkappa}(K;E_\sigma)_t.$$

Moreover, the topology of $\mathscr{G}^{\varkappa}(K;E_\sigma)$ coincides with the direct limit of those of the normed spaces $\mathscr{G}^{\varkappa}(K;E_\sigma)_t$.

Let $\mathbf{H}_*(\sigma)$ (resp. $\mathscr{G}^{\varkappa}_*(\sigma)$) be the space of E_σ-valued square-integrable functions (resp. ultradifferential functions of Gevrey class of order $\varkappa$) f on K which satisfies $f(km)=\sigma(m)^{-1}f(k)$ for all $k\in K$ and $m\in M_*$. Since $\mathscr{G}^{\varkappa}_*(\sigma)$ is a closed subspace of $\mathscr{G}^{\varkappa}(K;E_\sigma)$, we introduce the subspace topology on $\mathscr{G}^{\varkappa}_*(\sigma)$.

Let $\nu\in\mathfrak{a}^*_m$. We define

$$\begin{aligned}\mathscr{F}(G/P_*;L_{\sigma,\nu})=\{f\in\mathscr{F}(G;E_\sigma)|\,&\forall g\in G,\ \forall m\in M*,\ \forall a\in A_m,\ \forall n\in N_m,\\ &f(gman)=e^{(\nu-\varrho_m)(\log a)}\sigma(m)^{-1}f(g)\}.\end{aligned}$$

We regard $\mathscr{F}(G/P_*;L_{\sigma,\nu})$ as a G-module by the left action. Namely, put

$$\pi^*_{\sigma,\nu}(g_1)f(g)=f(g_1^{-1}g),$$

for all $f\in\mathscr{F}(G/P_*;L_{\sigma,\nu})$ and $g_1,g\in G$.

For $f\in\mathbf{H}_\sigma$ (or $f\in\mathscr{G}^{\varkappa}_*(\sigma)$), we define $e_\nu(f)\in\mathscr{M}(G/P_*;L_{\sigma,\nu})$ by

$$e_\nu(f)(kan)=e^{(\nu-\varrho_m)(\log a)}f(k),$$

for $k\in K$, $a\in A_m$, $n\in N_m$. Put

$$\mathbf{H}_*(\sigma,\nu)=\{e_\nu(f)\in\mathscr{M}(G/P_*;L_{\sigma,\nu})|f\in\mathbf{H}_*(\sigma)\}.$$

We can immediately see

$$e_\nu(\mathscr{G}^\varkappa_*(\sigma))=\mathscr{G}^\varkappa(G/P_*;L_{\sigma,\nu}).$$

Since $\mathbf{H}_*(\sigma,\nu)$ is isomorphic to $\mathbf{H}_*(\sigma)$ as vector spaces, we can easily see that $(\pi^*_{\sigma,\nu}, \mathbf{H}_*(\sigma,\nu))$ is a strongly continuous representation of G on a Hilbert space.

Clearly, we can see $\mathbf{H}_0(\sigma,\nu)$ is a direct sum of a finite number of principal series representations in the usual sense. Namely, we have

$$\mathbf{H}_0(\sigma,\nu)=\bigoplus_{\tau\in\hat{M}_m}[\sigma:\tau|_{M_0}]\mathbf{H}_m(\tau,\nu).$$

Here, $\hat{M}_m$ is the set of equivalence class of finite-dimensional irreducible representations of M_m and $[\sigma:\tau|_{M_0}]$ is the multiplicities.

From the same argument as the proof of [GW] Lemma 5.1, we see the space $S_\varkappa(\pi^*_{\sigma,\nu})$ of Gevrey vectors of order $1\leq\varkappa$ in $(\pi^*_{\sigma,\nu}, \mathbf{H}_*(\sigma,\nu))$ coincides with $\mathscr{G}^\varkappa(G/P_*;L_{\sigma,\nu})$, and is also isomorphic to $\mathscr{G}^\varkappa_*(\sigma)$ as topological vector spaces. We simply denote $\|f|_K\|^K_{\varkappa,t}$ by $\|f\|^K_{\varkappa,t}$ for all $f\in\mathscr{G}^\varkappa_*(G/P_*;L_{\sigma,\nu})$ and $t>0$.

We denote by $\mathscr{A}_K(G/P_*;L_{\sigma,\nu})$ (resp. $\mathbf{H}_*(\sigma)_K$) the space of K-finite elements of $\mathbf{H}_*(\sigma,\nu)$ (resp. $\mathbf{H}_*(\sigma)$).

$\mathscr{A}_K(G/P_*;L_{\sigma,\nu})$ has a structure of a $(\mathfrak{g},K)$-module induced from the G-module structure of $\mathbf{H}_*(\sigma,\nu)$. $\mathscr{A}_K(G/P_*;L_{\sigma,\nu})$ is a Harish-Chandra module. We can easily see $\mathscr{A}_K(G/P_*;L_{\sigma,\nu})$ is isomorphic to $\mathbf{H}_*(\sigma)_K$ as a K-module via e_ν.

Hereafter throughout § 5, we fix some $S\subseteq\Pi$ such that $S_m\subseteq S$.

Let $\lambda\in\mathbf{P}^{++}_S$. We denote by σ_λ the finite dimensional unitary representation of M_0 with the highest weight λ_t with respect to $(\mathfrak{t},\mathfrak{u}\cap\mathfrak{m}_m)$. Then we can identify E_{σ_λ} with E_λ.

Let σ^*_λ be a contragradient representation of σ_λ. Namely, $E_{\sigma^*_\lambda}=E^*_\lambda$. Put $\lambda'=\lambda_{\mathfrak{a}_m}$.

Define

$$\mathscr{G}^\varkappa(G/P_0,\lambda)=\mathscr{G}^\varkappa(G/P_0;L_{\sigma^*_\lambda,\lambda'+\varrho_M}),$$

$$\mathscr{A}_K(G/P_0,\lambda)=\mathscr{A}^\varkappa(G/P_0;L_{\sigma^*_\lambda,\lambda'+\varrho_M}),$$

Since $\mathscr{G}^{\varkappa}(G/P_0, \lambda)$ is the space of the Gevrey vectors in $H(\sigma_\lambda^*, \lambda_{\mathfrak{a}_m})$, hereafter we regard $\mathscr{G}^{\varkappa}(G/P_0, \lambda)$ as a topological vector space (cf. 1.6).

5.2. Embeddings into principal series

Let V be an irreducible Harish-Chandra module. We fix $\lambda \in \mathsf{P}_S^{++}$.

Put

$$\operatorname{Emb}(V;\lambda) = \operatorname{Hom}_{\mathfrak{g},K}(V, \mathscr{A}_K(G/P_0, \lambda)).$$

We remark that the highest weight vector v_λ of V_λ is also a highest weight vector of E_λ. For $\iota \in \operatorname{Emb}(V;\lambda)$, we define $\delta_\iota \in V^*$ by

$$\delta_\iota(v) = [[\iota(v)](e)](v_\lambda) \quad (v \in V).$$

Here, e is the identity element of G, and we remark that $[\iota(v)(e)] \in E_\lambda^*$.

We define ${}^t X = -X$ for $X \in \mathfrak{g}$. Then we can extend $X \rightsquigarrow {}^t X$ to the anti-automorphism of $U(\mathfrak{g})$.

Now we consider V^* as a left $U(\mathfrak{g})$-module, namely for $v^* \in V^*$, $v \in V$, and $Y \in U(\mathfrak{g})$, we define

$$[Y \cdot v^*](v) = v^*({}^t Y \cdot v).$$

For $\iota \in \operatorname{Emb}(V; \lambda)$, we can easily see

LEMMA 5.2.1. *The map*

$$\Xi_\iota : M(\mathfrak{p}_m, \lambda) \ni Y \otimes v_\lambda \rightsquigarrow Y \cdot \delta_\iota \in V^* \quad (Y \in U(\mathfrak{g}))$$

is a well-defined $U(\mathfrak{g})$-homomorphism.

Put

$$\operatorname{Emb}^\circ(V;\lambda) = \{\iota \in \operatorname{Emb}(V;\lambda) \mid \operatorname{Image}(\Xi_\iota) \text{ is irreducible}\}.$$

Let $\mathscr{A}_K^\circ(G/P_0, \lambda)$ (resp. $\mathscr{G}_\circ^{\varkappa}(G/P_0, \lambda)$) be the space of the elements f of $\mathscr{A}_K(G/P_0, \lambda)$ (resp. $\mathscr{G}^{\varkappa}(G/P_0, \lambda)$) which satisfy

$$\sum_{i=1}^{l} f(g : Y_i)(e_i) = 0 \quad (g \in G),$$

for $Y_1, \dots, Y_l \in U(\bar{\mathfrak{n}}_m)$ and $e_1, \dots, e_l \in E_\lambda$ such that $\Sigma_{i=1}^{l} Y_i \oplus e_i \in K(\mathfrak{p}_m, \lambda)$. Clearly,

$$\mathscr{A}_K^\circ(G/P_0, \lambda) \subseteq \mathscr{G}_o^\kappa(G/P_0, \lambda).$$

Put

$$\mathrm{Emb}_S(V) = \bigcup_{\lambda \in P_S^{++}} \mathrm{Emb}^\circ(V; \lambda).$$

We call an element of $\mathrm{Emb}_S(V)$ a minimal S-embedding.

We have:

LEMMA 5.2.2. *For all* $\iota \in \mathrm{Emb}^\circ(V, \lambda)$, *we have*

$$\iota(V) \subseteq \mathscr{A}_K^\circ(G/P_0, \lambda).$$

Proof. Let $Y_1, \dots, Y_l \in U(\bar{\mathfrak{n}}_m)$ and $e_1, \dots, e_l \in E_\lambda$ satisfy $\Sigma_{i=1}^{l} Y_i \oplus e_i \in K(\mathfrak{p}_m, \lambda)$.

Then there exist $Z_1, \dots, Z_l \in U(\mathfrak{m}_m)$ which satisfy $e_i = Z_i \cdot v_\lambda$. Let Q be an arbitrary element of $U(\mathfrak{g})$.

Since $\iota \in \mathrm{Emb}^\circ(V, \lambda)$, we have

$$\sum_{i=1}^{l} Q Y_i Z_i \cdot \delta_\iota(v) = 0.$$

for all $v \in V$. If we put $f = \iota(v)$, then we have

$$\begin{aligned} 0 &= \sum_{i=1}^{l} f(Q Y_i Z_i : e)(v_\lambda) \\ &= \sum_{i=1}^{l} [{}^t Z_i \cdot f(e : Q Y_i)](v_\lambda) \\ &= \sum_{i=1}^{l} [f(e : Q Y_i)](Z_i \cdot v_\lambda) \\ &= \sum_{i=1}^{l} [f(e : Q Y_i)](e_i) \end{aligned}$$

Since $Q \in U(\mathfrak{g})$ is arbitrary and the function

$$G \in g \rightsquigarrow \sum_{i=1}^{l} [f(g : Y_i)](e_i)$$

is a real analytic function on G, we have the desired conclusion.

5.3. The definition of the Goodman-Wallach Operators

Fix an arbitrary character $\psi:\mathfrak{n}\to\mathbf{C}$, $1\leq\varkappa<2$, and $\lambda\in\mathrm{P}_S^{++}$. Let w be an element of $L^\varkappa(\mathfrak{p},\lambda)$. From the definition of $L^\varkappa(\mathfrak{p},\lambda)$, we can choose $\bar{w}\in M^\varkappa(\mathfrak{p},\lambda)$ such that $q_\lambda(w)$.

Then we can write as follows (cf. 4.2).

$$\bar{w}=\sum_{j=1}^{d}\sum_{\mathbf{I}\in\mathbf{N}^h} W(j,\mathbf{I})\bar{X}(\mathbf{I})\otimes v_j.$$

Here $W(j,\mathbf{I})\in\mathbf{C}$ for all $1\leq j\leq d$ and $\mathbf{I}\in\mathbf{N}^h$, and $\bar{X}(\mathbf{I})$ and $v_1,\dots,v_d$ are defined in 4.2. We fix $Z_i\in U(\mathfrak{m})$ such that $v_i=Z_i\cdot v_\lambda$ for all $1\leq i\leq d$.

We define $\delta_i\in\mathscr{G}^\varkappa(G/P_0,\lambda)^*$ $(1\leq i\leq d)$ by

$$\delta_i(f)=[f(Z_i:e)](v_\lambda)\quad(f\in\mathscr{G}^\varkappa(G/P_0,\lambda)).$$

$\mathscr{G}^\varkappa(G/P_0,\lambda)\cong\mathscr{G}_0^\varkappa(\sigma_\lambda^*)$ as topological vector spaces and the natural embedding

$$\mathscr{G}_0^\varkappa(\sigma_\lambda^*)\hookrightarrow C^\infty(K;E_{\sigma_\lambda^*})$$

is continuous. Hence we can see δ_i is continuous linear functional on $\mathscr{G}^\varkappa(G/P_0,\lambda)$ since δ_i clearly defines a continuous linear functional on $C^\infty(K;E_{\sigma_\lambda^*})$. Namely, we have:

LEMMA 5.3.1. *Fix arbitrary $t>0$ and some $1\leq i\leq d$. Then, there exists some positive number $|\delta_i|_{\varkappa,t}$ such that for every $f\in\mathscr{G}^\varkappa(G/P_0,\lambda)$ which satisfies $\|f\|_{\varkappa,t}^K<\infty$, the following estimate holds.*

$$|\delta_i(f)|\leq|\delta_i|_{\varkappa,t}\|f\|_{\varkappa,t}^K.$$

Put $|\delta|_{\varkappa,t}=\max_{1\leq i\leq d}|\delta_i|_{\varkappa,t}$.

For $f\in\mathscr{G}^\varkappa(G/P_0,\lambda)$ we define

$$\omega_w(f)=\sum_{j=1}^{d}\sum_{\mathbf{I}\in\mathbf{N}^h} W(j,\mathbf{I})[f(\bar{X}(\mathbf{I})Z_j:e)](v_\lambda).$$

From Theorem 4.2.1, and [GW] (2.2), we see ω_w is a continuous linear functional on $\mathscr{G}^\varkappa(G/P_0,\lambda)$.

Namely, we have:

PROPOSITION 5.3.2 *Fix arbitrary $t>s>0$. Then, for all $f\in\mathscr{G}^{\varkappa}(G/P_0,\lambda)$ which satisfy $\|f\|^K_{\varkappa,s}<\infty$, the following estimate holds.*

$$|\omega_w(f)|\leq\|\bar{w}\|_{\varkappa,\varepsilon s}|\delta|_{\varkappa,t}\|f\|^K_{\varkappa,s},$$

where $\varepsilon=t^{\varkappa}(t-s)^{-\varkappa}$ and $\bar{w}$ is an element of $M^{\varkappa}(\mathfrak{p},\lambda)$ such that $q_\lambda(\bar{w})=w$.

Moreover we easily see the definition of ω_w does not depend on the choice of $\bar{w}$ if we restrict ω_w to $\mathscr{G}^{\varkappa}_{\circ}(G/P_0,\lambda)$.

Hereafter we assume $w\in\mathrm{Wh}_{\mathfrak{n},-\psi}(\hat{L}(\mathfrak{p},\lambda))\subseteq L^{\varkappa}(\mathfrak{p},\lambda)$.

We denote the restriction of ω_w to $\mathscr{A}^{\circ}_K(G/P_0,\lambda)$ by the same letter. We have:

PROPOSITION 5.3.3. *For all $f\in\mathscr{G}^{\varkappa}_{\circ}(G/P_0,\lambda)$, we have*

$$\omega_w(X\cdot f)=\psi(X)\omega_w(f)\quad(X\in\mathfrak{n}).$$

Especially $\omega_w\in\mathrm{Wh}^*_{\mathfrak{n},\psi}(\mathscr{A}^{\circ}_K(G/P_0,\lambda))$.

Proof. We define $\bar{w}\in M^{\varkappa}(\mathfrak{p},\lambda)$ as in 4.6. This proposition is directly deduced from the following fact.

$$X\cdot\bar{w}=-\psi(X)\bar{w}\quad\mathrm{mod}\,\hat{K}(\mathfrak{p},\lambda)\cap M^{\varkappa}(\mathfrak{p},\lambda).\qquad\text{Q.E.D.}$$

Remark. The Whittaker vector ω_w is introduced by Goodman and Wallach in [GW] when G is a real quasi-split semi-simple Lie group.

Put

$$\pi_\lambda(g)f(g_1)=f(g^{-1}g_1),$$
$$[\Omega_w(f)](g)=\omega_w(\pi_\lambda(g^1)f),$$

for $f\in\mathscr{G}^{\varkappa}_{\circ}(G/P_0,\lambda)$ and $g,g_1\in G$.

From Theorem 4.2.1, Lemma 1.6.1, [Gd1] (2.6), we can easily see $\Omega_w(f)\in\mathscr{G}^{\varkappa}(G)$, since $f\in\mathscr{G}^{\varkappa}(G)$.

We put

$$\mathscr{G}^{\varkappa}(G,\mathfrak{n};\psi)=\{f\in\mathscr{G}^{\varkappa}(G)|\,\forall g\in G,\forall X\in\mathfrak{n},f(g:X)=-\psi(X)f(g)\}.$$

Especially $\mathscr{G}^1(G,\mathfrak{n};\psi)=\mathscr{A}(G,\mathfrak{n};\psi)$. Using the left action we can regard $\mathscr{G}^{\varkappa}(G,\mathfrak{n};\psi)$ as a left G-module. From Proposition 5.3.1, we can easily see $\Omega_w(f)\in\mathscr{G}^{\varkappa}(G,\mathfrak{n};\psi)$.

We call the G-homomorphism

$$\Omega_w : \mathscr{G}^x(G/P_0, \lambda) \to \mathscr{G}^x(G, \mathfrak{n}, \psi)$$

the Goodman-Wallach Operator (attached to $w \in \mathrm{Wh}_{\mathfrak{n}, -\psi}(\hat{M}(\mathfrak{p}, \lambda))$).

5.4. The injectivity of the Goodman-Wallach Operators

We fix $\lambda \in \mathrm{P}_S^{++}$ and $w \in \mathrm{Wh}_{\mathfrak{n}, -\psi}(\hat{L}(\mathfrak{p}, \lambda))$.

Then w is uniquely written as

$$w = \sum_{\mu \in \mathrm{Q}_\mathfrak{a}^+} w_{-\mu}$$

where $w_{-\mu} \in L(\mathfrak{p}, \lambda)_{-\mu}$. We define $\bar{w}_{-\mu}$ such that $q_\mu(\bar{w}_{-\mu}) = w_{-\mu}$, as 4.6, for all $\mu \in \mathrm{Q}_\mathfrak{a}^+$.

First we prove:

PROPOSITION 5.4.1. *$w \neq 0$ if and only if $w_0 \neq 0$.*

Proof. We assume $w_0 = 0$. Then, via the canonical pairing of 3.1, w defines $w^* \in \mathrm{Wh}^*_{\mathfrak{n}, -\psi}(\bar{L}(\bar{\mathfrak{p}}, \lambda))$ such that $w^*(1 \otimes V_\lambda^*) = 0$. However, since $\bar{L}(\bar{\mathfrak{p}}, \lambda)$ is generated by $(1 \otimes V_\lambda^*$ as $U(\mathfrak{n})$-module, we have $w^* = 0$. This means $w = 0$. Q.E.D.

For $z \in \mathbf{C}$ we put

$$\bar{w}(z) = \sum_{\mu \in \mathrm{Q}_\mathfrak{a}^+} z^{|\mu|} \bar{w}_{-\mu},$$

$$w(z) = q_\lambda(\bar{w}(z)).$$

We can easily see $w(z) \in \mathrm{Wh}_{\mathfrak{n}, -z\psi}(\hat{L}(\mathfrak{p}, \lambda))$.

Then we have:

LEMMA 5.4.2. *For all $\mathscr{A}_K^\circ(G/P_0, \lambda)$, $\omega_{w(z)}(f)$ is an entire holomorphic function in z.*

Proof. Fix $f \in \mathscr{A}_K^\circ(G/P_0, \lambda)$ and $t > 0$ such that $\|f\|_{x,t}^K < \infty$.

We have

$$\omega_{w(z)}(f) = \sum_{\mu \in \mathrm{Q}_\mathfrak{a}^+} z^{|\mu|} \omega_{w_{-\mu}}(f). \tag{1}$$

From 4.6 (5), there exist some $A, B>0$ such that

$$\|\bar{w}_{-\mu}\|_{\varkappa,t} \leqslant A(Bt)^{|\mu|}(|\mu|!)^{\varkappa-2},$$

for all $t>0$ and $\mu \in Q_\alpha^+$. Hence, from Proposition 5.3.2 we have

$$\begin{aligned} |\omega_{w_{-\mu}}(f)| &\leqslant \|w_{-\mu}\|_{\varkappa,\, 2^{k+1}t} |\delta|_{\varkappa, 2t} \|f\|^K_{\varkappa, t} \\ &\leqslant \sum_{\mu \in Q_\alpha^+} A|\delta|_{\varkappa, 2t}\|f\|^K_{\varkappa, t}(|\mu|!)^{\varkappa-2}(B2^{\varkappa-2}t|z|)^{|\mu|}. \end{aligned} \tag{2}$$

Hence the right hand side of (1) converges uniformly on $|z|<R$ for all $R>0$. Q.E.D.

Next we prove:

THEOREM 5.4.3. *We assume $w \in \mathrm{Wh}_{\mathfrak{n},-\psi}(\hat{L}(\mathfrak{p},\lambda))$ is non-zero. Then there exists some discrete subset D of $\mathbf{C}$ such that $0 \notin D$ and for all $z \in \mathbf{C}-D$,*

$$\Omega_{w(z)} : \mathscr{A}^\circ_K(G/P_0, \lambda) \to \mathscr{A}(G, \mathfrak{n}; z\psi)$$

is injective.

Proof. Put

$$U = \{z \in \mathbf{C} \mid \Omega_{w(z)} : \mathscr{A}^\circ_K(G/P_0, \lambda) \to \mathscr{A}(G, \mathfrak{n}; z\psi) \text{ is not injective}\}.$$

First we show $0 \notin U$. There exist some $Z \in U(\mathfrak{m})$ such that $Z \otimes v_\lambda = w_0$. We define $\delta_Z \in \mathscr{A}_K(G/P_0, \lambda)^*$ by

$$\delta_Z(f) = [f(e:Z)](v_\lambda) \quad (f \in \mathscr{A}_K(G/P_0, \lambda)).$$

Clearly, $\delta_Z = \omega_{w(0)}$.

Since V_λ is an irreducible $U(\mathfrak{m})$-module, there exists some $Y \in U(\mathfrak{m})$ such that $YZ \cdot v_\lambda = v_\lambda$. We can easily see

$$[{}^tY^tQ \cdot \Omega_{w_0}(f)](e) = [f(e)](Q \cdot v_\lambda),$$

for all $Q \in U(\mathfrak{m}_m)$. Hence $\Omega_{w(0)} : \mathscr{A}^\circ_\varkappa(G/P_0, \lambda) \to \mathscr{A}(G, \mathfrak{n}; \psi)$ is injective.

Next we assume U has a limit point. Since kernel of $\Omega_{w(z)}$ is $(\mathfrak{g}, K)$-submodule of $\mathscr{A}^\circ_K(G/P_0, \lambda)$ and $\mathscr{A}^\circ_K(G/P_0, \lambda)$ has finite length, there exists some non-trivial $(\mathfrak{g}, K)$-submodule of $\mathscr{A}^\circ_K(G/P_0, \lambda)$ such that

$$W = \{z \in \mathbf{C} \mid V \supseteq \mathrm{Ker}(\Omega_{w(z)})\}$$

also has a limit point. Choose a non-trivial element $h \in V$. Then we can immediately see that there exists some $k \in K$ such that $[h(k:Z)](v_\lambda) \neq 0$. Since V is K-invariant, there exists some $f \in V$ such that $[f(e:Z)](V_\lambda) \neq 0$, namely $\omega_{w(0)}(f) \neq 0$. On the other hand, we have

$$\omega_{w(z)}(f) = 0 \quad (z \in W).$$

Since W has a limit point, Lemma 5.4.2 implies $\omega_{w(z)}(f) = 0$ for all $z \in \mathbf{C}$. This is a contradiction. Q.E.D.

Next we consider the following condition on S.

(R) $\mathfrak{p} \cap \mathfrak{g}_0$ *is a real form of* $\mathfrak{p}$.

Example 1. If $G = SU(n, 1)$ $(n > 1)$ and $\mathfrak{p}$ is a maximal parabolic subalgebra of $\mathfrak{g}$ which contains the complexification of the minimal parabolic subalgebra for G, then the condition (R) does not hold.

Example 2. If G is real-split, then the condition (R) always holds for all $S \supseteq S_m$.

Under the assumption (R), we can prove a stronger result than Theorem 5.4.3. Hereafter, we assume (R) holds.

Put $\mathfrak{a}_\# = \mathfrak{a}_m \cap \mathfrak{a} \cap \mathfrak{g}_0$. Let $A_\#$ (resp. N) be the analytic subgroup of G corresponding to $\mathfrak{a}_\#$ (resp. $\mathfrak{n} \cap \mathfrak{g}_0$ and let $M_\#$ be the centerizer of $A_\#$ in G. Let P be the normalizer of $\mathfrak{p} \cap \mathfrak{g}_0$ in G. Then P has a Langlands decomposition $P = M_\# A_\# N$.

Let $n = \dim \mathfrak{a}_\#$. Let Σ_m^+ the restricted root system with respect to $(\mathfrak{n}_m, \mathfrak{a}_m)$ and let Π_m be the set of simple roots of Σ_m^+. Put

$$\begin{aligned}\Pi_S &= \{\alpha \in \Pi_m \mid \alpha|_{\mathfrak{a}_\#} \neq 0\} \\ &= \{\alpha_1, \ldots, \alpha_n\}.\end{aligned}$$

We denote the restriction of α_i $(1 \leq i \leq n)$ to $\mathfrak{a}_\#$ by the same letters.

Then, we can easily see Π_S forms a basis of the dual vector space of a real vector space $\mathfrak{a}_\#$. Let $\{H_1, \ldots, H_n\}$ be the dual basis of $\mathfrak{a}_\#$.

Put $\mathbf{R}_+^n = \{(x_1, \ldots, x_n) \in \mathbf{R}^n \mid x_i > 0 \ (1 \leq i \leq n)\}$. For $x = (x_1, \ldots, x_n) \in \mathbf{R}_+^n$, we define

$$a_x = \exp\left(-\sum_{i=1}^n \log(x_i) H_i\right) \in A_\#.$$

Then we see $x \rightsquigarrow a_x$ is an isomorphism of $\mathbf{R}^n_+$ to $A_\#$.

For $\lambda \in \mathfrak{h}^*$ and $x=(x_1, \dots, x_n) \in \mathbf{R}^n_+$, we define

$$t^\lambda = \exp\left(\lambda\left(\sum_{i=1}^n \log(x_i) H_i\right)\right).$$

Fix $1<\varkappa<2$. Let $w \in \mathrm{Wh}_{\mathfrak{n},-\psi}(\hat{L}(\mathfrak{p},\lambda))$. According to the proof of Theorem 4.2.1, there exists some $\bar{w}_{-\mu} \in M(\mathfrak{p},\lambda)_{-\mu}$ ($\mu \in \mathrm{Q}^+_\mathfrak{a}$) such that

$$\hat{q}_\lambda\left(\sum_{\mu \in \mathrm{Q}^+_\mathfrak{a}} \bar{w}_{-\mu}\right) = w,$$

and there exist some $A, B>0$ such that

$$\|\bar{w}_{-\mu}\|_{\varkappa, t} \leq A(Bt)^{|\mu|}(|\mu|!)^{\varkappa-2},$$

for all $t>0$ and $\mu \in \mathrm{Q}^+_a$.

Put $\bar{w}_{-\mu}=\Sigma_{i=1}^d W^{(i)}_{-\mu} \otimes Z_i v_\lambda$, where $W^{(i)}_{-\mu} \in U(\bar{\mathfrak{n}})_{-\mu}$ and $Z_i \in U(\mathfrak{m})$ for $(1 \leq i \leq d)$.

We consider $\omega_w(\pi_\lambda(a_x^{-1} f)$ for $f \in \mathscr{G}^\varkappa_\circ(G/P_0, \lambda)$ and $x \in \mathbf{R}^n_+$. Put $w_{-\mu}=\mathbf{q}_\mu(\bar{w}_\mu)$. We have

$$\begin{aligned} \omega_{w_{-\mu}}(\pi_\lambda a_x^{-1})f) &= \sum_{i=1}^d [f(a_x\colon W^{(i)}_{-\mu} Z_i)](v_\lambda) \\ &= \sum_{i=1}^d [f(x^\mu W^{(i)} Z_i\colon a_x)](v_\lambda) \\ &= x^{\mu-\lambda}\, \omega_{w_{-\mu}}(f). \end{aligned}$$

For $x=(x_1, \dots, x_n) \in \mathbf{R}^n_+$, put

$$|x| = |x_1| + \dots + |x_n|.$$

Then, we get:

THEOREM 5.4.4. *We assume the condition* (R). *Let* $1 \leq \varkappa < 2$ *and let* $\lambda \in \mathrm{P}^{++}_{-S}$. *We fix* $w \in \mathrm{Wh}_{\mathfrak{n},-\psi}(\hat{L}(\mathfrak{p},\lambda))$. *We choose* $\bar{w}_{-\mu}$ *for* $\mu \in \mathrm{Q}^+_\mathfrak{a}$ *as above.*

Then for all $\mathscr{G}^\varkappa_\circ(G/P_0, \lambda)$ *and* $x \in \mathbf{R}^n_+$ *we have*

$$x^\lambda \omega_w(\pi(a_x f) = \sum_{\mu \in \mathrm{Q}^+_\mathfrak{a}} \omega_{w_{-\mu}}(f)\, x^\mu. \tag{4}$$

Moreover for all $R>0$, the above series is unformly convergent for $|x|<R$.

Proof. We have only to show the uniform convergence. This is showed by the same argument of the proof of Lemma 5.4.2, using (3). Q.E.D.

Since $\Omega_{w(0)}$ is injective (cf. the proof of Theorem 5.4.3), we have:

COROLLARY 5.4.5. *We assume that* (R) *holds. Then for all non-zero* $w \in \mathsf{Wh}_{\mathfrak{n},-\psi}(\hat{L}(\mathfrak{p},\lambda))$,

$$\Omega_w: \mathscr{G}_\circ^\times(G/P_0,\lambda) \to \mathscr{G}^\times(G/N;\psi)$$

is injective.

5.5. Existence of a global Whittaker vector

Let V be an irreducible Harish-Chandra module and $\lambda \in \mathbf{P}_S^{++}$. We assume $\mathrm{Emb}^\circ(V;\lambda) \neq 0$. Fix a non-trivial $\iota \in \mathrm{Emb}^\circ(V;\lambda)$ and an arbitrary character $\psi: \mathfrak{n} \to \mathbf{C}$.

For $w \in \mathsf{Wh}_{\mathfrak{n},-\psi}(\hat{L}(\mathfrak{p},\lambda))$, we define $\Psi_{\iota,\psi}(w) \in \mathsf{Wh}_{\mathfrak{n},\psi}^G(V)$ by

$$[\Psi_{\iota,\psi}(w)](v) = \omega_w(\iota(v)) \quad (v \in V).$$

From [V1], we have $\mathrm{Dim}(V)=\mathrm{Dim}(L(\mathfrak{p},\lambda))$ when $\mathrm{Emb}^\circ(V;\lambda) \neq 0$.

Hence, from the Theorem 5.4.3 and Corollary 2.2.3, we immediately have:

THEOREM 5.5.1. *There exists some discrete subset D of $\mathbf{C}$ such that $0 \notin D$ and for all $z \in \mathbf{C}-D$* the map

$$\Psi_{\iota,z\psi}: \mathsf{Wh}_{\mathfrak{n},-z\psi}(\hat{L}(\mathfrak{p},\lambda)) \to \mathsf{Wh}_{\mathfrak{n},z\psi}^G(V)$$

is injective.

Moreover, if ψ is admissible, $\mathrm{Dim}(V)=\dim \mathfrak{n}$, *and* $\mathrm{Emb}_S(V) \neq 0$, *then* $\mathsf{Wh}_{\mathfrak{n},z\psi}^G(V) \neq 0$.

If the condition (R) holds, from Corrollary 5.4.5, we have a stronger result:

THEOREM 5.5.2. *We assume the condition* (R) *holds. Then the map*

$$\Psi_{\iota,\psi}: \mathsf{Wh}_{\mathfrak{n},-\psi}(\hat{L}(\mathfrak{p},\lambda)) \to \mathsf{Wh}_{\mathfrak{n},\psi}^G(V)$$

is injective.

Moreover, if $\mathrm{Dim}(V)=\dim\mathfrak{n}$ *and* ψ *is admissible, then* $\mathrm{Emb}_S(V)\neq 0$ *implies* $\mathrm{Wh}^G_{\mathfrak{n},\psi}(V)\neq 0$.

§ 6. Whittaker vectors attached to an admissible character on the nilradical of a minimal parabolic subgroup

6.1. Whittaker vectors on principal series

In § 6, we use the notation in § 1 and § 5 freely. We also assume hereafter in § 6 $S=S_m$. Hence we have $\mathfrak{p}=\mathfrak{p}_m$, $\mathfrak{n}=\mathfrak{n}_m$, $\mathfrak{l}=\mathfrak{l}_m$, ... etc. However $\mathfrak{m}=\mathfrak{m}_m$ and $\mathfrak{a}=\mathfrak{a}_m$ may not hold.

Let W be the little Weyl group with respect to $(\mathfrak{g},\mathfrak{a}_m)$. Put $\mathfrak{t}_m=\mathfrak{h}\cap\mathfrak{m}_m$. Hence we have $\mathfrak{h}=\mathfrak{t}_m\oplus\mathfrak{a}_m$ and $\mathfrak{t}_m$ and $\mathfrak{a}_m$ are orthogonal with respect to the Killing form. Using Killing form, we can regard $\mathfrak{t}^*_m$ and $\mathfrak{a}^*_m$ as subspaces of $\mathfrak{h}^*$. Since $\mathfrak{t}\subseteq\mathfrak{t}_m$, and the restriction of each element of $\mathfrak{a}^*_m$ to $\mathfrak{t}_m$ is zero, we have

$$\mathfrak{a}^*_m\subseteq \mathsf{P}^{++}_{S_m}.$$

First, we consider the Whittaker vectors on spherical principal series with unique quotients.

Let id_m be the trivial representation of M_m.

Let $\mathscr{F}$ be $\mathscr{G}^\kappa$, $\mathscr{A}_K$, or C^∞. For $\nu\in\mathfrak{a}^*_m$, we put

$$\mathscr{F}(G/P_m;L_\nu)=\mathscr{F}(G/P_m;L_{\mathrm{id}_m,\nu}).$$

Then we have natural embeddings:

$$\mathscr{A}_K(G/P_m;L_\nu)\hookrightarrow\mathscr{A}_K(G/P_0,\nu-\varrho_m),$$

$$\mathscr{G}^\kappa(G/P_m;L_\nu)\hookrightarrow\mathscr{G}_\kappa(G/P_0,\nu-\varrho_m),$$

Put

$$(a^*_m)_-=\{H\in\mathfrak{a}^*_m|\,\forall\,\alpha\in\Pi_m,\Re(\alpha(H))>0\}.$$

We quote:

THEOREM 6.1.1. ([Ly] Theorem 6.2.2, Corollary 6.2, also see [Ko] Theorem 5.2.1, Lemma 5.2.) *We assume* $\nu\in(\mathfrak{a}^*_m)_-$ *and* $\psi\colon U(\mathfrak{n}_m)\to\mathbf{C}$ *is an admissible character. Then,*

$\mathscr{A}_K(G/P_m; L_\nu)$ is a free $U(\mathfrak{n}_m)$-module with card W *generators. Hence we have*

$$\dim \mathrm{Wh}^*_{\mathfrak{n}_m,\psi}(\mathscr{A}_K(G/P_m; L_\nu)) = \mathrm{card}\, W.$$

Remark. The dimensions of dual Whittaker vectors on non-spherical principal series are also known ([Ko] Theorem I, [Ly] Theorem 6.4). Namely,

$$\dim \mathrm{Wh}_{\mathfrak{n}_m,\psi}(\mathscr{A}_K(G/P_m; L_{\sigma,\nu})) = \mathrm{card}\, W \dim V_\sigma.$$

For $f \in C^\infty(K/M_m)$, we define $e_\nu(f) \in C^\infty(G/P_m; L_\nu)$ by

$$e_\nu(f)(kan) = e^{(\nu-\varrho_m)(\log a)} f(kM_m).$$

For each $w \in W$, we fix a representative $w^* \in G$. We put $\bar{\mathfrak{n}}_w = \mathrm{Ad}(w^*)\mathfrak{n} \cap \bar{\mathfrak{n}}$. Let $\bar{N}_w = \bar{N} \cap w^{*-1} N w^*$ be the corresponding analytic subgroup of $\bar{N}$.

We define for $f \in C^\infty(G/P_m; L_\nu)$ and $w \in W$,

$$\delta_{\nu,w}(f) = \int_{\bar{N}_w} f(w^* \bar{n}_w)\, d\bar{n}_w.$$

We denote the restriction of $\delta_{\nu,w}$ to $\mathscr{G}^\varkappa(G/P_m; L_\nu)$ by the same letter.

We also define for $f \in C^\infty(K/M_m)$,

$$\delta_w(\nu, f) = \delta_{\nu,w}(e_\nu(f)).$$

Put

$$(\mathfrak{a}_m^*)_{--} = \left(\nu \in (\mathfrak{a}_m^*)_- \mid \forall \alpha \in \Sigma_m^+, 2\frac{\langle \alpha, \nu\rangle}{\langle \alpha, \alpha\rangle} \notin \mathbf{Z}\right).$$

THEOREM 6.1.2. ([He], [Sch], cf. [Wa].)

(1) *For all $\nu \in (\mathfrak{a}_m^*)_{--}$ and $w \in W$, $\delta_{\nu,w}$ is a continuous linear functional on $C^\infty(G/P_m; L_\nu)$. Hence $\delta_{\nu,w}$ is also continuous on $\mathscr{G}^\varkappa(G/P_0; L_\nu)$ ($\varkappa \geq 1$).*

Especially, for each $t>0$ there exists some positive number $|\delta_{\nu,w}|_{\varkappa,t}$ such that for every $f \in \mathscr{G}^\varkappa(G/P_m; L_\nu)$ which satisfies $\|f\|^K_{\varkappa,t} < \infty$, the following estimate holds

$$|\delta_{\nu,w}(f)| \leq |\delta_{\nu,w}|_{\varkappa,t} \|f\|^K_{\varkappa,t}.$$

Moreover if T is a compact subset of $(\mathfrak{a}_m^*)_{--}$, then we can choose $|\delta_{\nu,w}|_{\varkappa,t}$ for $\nu \in T$ such that

$$\sup_{\nu \in T} |\delta_{\nu,w}|_{\varkappa,t} < \infty.$$

(2) *For all* $f\in C^\infty(K/M)$, $\nu\rightsquigarrow\delta_w(\nu,f)$ *is a holomorphic function on* $(\mathfrak{a}_m^*)_{--}$.

We fix $\nu\in(\mathfrak{a}_m^*)_-$. Then we can easily see there exist some $\xi\in\mathfrak{a}_m^*$ which satisfies the assumption of Proposition 4.7.4. (Even if $\Sigma^+\neq\Sigma_m^+$, we can easily see this fact.) We may aslo assume $\nu+z\xi\in(\mathfrak{a}_m^*)_{--}$ for all $0<|z|<<1$. We fix some non-zero $v\in V_{\mathrm{id}_m}$. Let ε be a sufficiently small positive integer. For $z\in\mathbf{C}$ such that $0<|z|<\varepsilon$, we consider $(-\psi_v(w(\nu+z\xi)-\varrho_m)\in\mathrm{Wh}_{\mathfrak{n}_m,-\psi}(\hat{M}(\mathfrak{p},w(\lambda+z\xi)-\varrho_m))$ defined in 3.3.

There exist unique $P^w_{-\mu}(z)\in U(\bar{\mathfrak{n}})_{-\mu}$ for all $\mu\in Q_{\mathfrak{a}}^+$ and $w\in W$ such that

$$(-\psi)_v(w(\lambda+z\xi)-\varrho_m)=\sum_{\mu\in Q_{\mathfrak{a}}^+}P^w_{-\mu}(z)\otimes v,$$

and $z\rightsquigarrow P^w_{-\mu}(z)$ is holomorphic on $0<|z|<\varepsilon$ for all $\mu\in Q_{\mathfrak{a}}^+$ and $w\in W$.

For $f\in\mathscr{G}^\varkappa(G/P_m;L_{\nu+z\xi})$, we define

$$w_{w,z}(f)=\delta_{w,\nu+z\xi}\left(\sum_{\mu\in Q_{\mathfrak{a}}^+}d\pi_{\nu+z\xi}({}^t(P^w_{-\mu}(z)))f)\right).$$

Here, $d\pi_{\nu+z\xi}$ is the differential representation of $\mathscr{G}^\varkappa(G/P_m;L_{\nu+z\xi})$.

From Proposition 4.7.4, Theorem 6.1.2, and [GW] (2.2), we have:

LEMMA 6.1.3. *Let* $1\leq\varkappa<2$. *Fix* $(\mathfrak{a}_m^*)_-$. *We choose* ξ *as above and let* ε *be a sufficiently small positive number. Then,* $\omega_{w,z}$ *satisfies: Put* $V_z=\mathscr{G}^\varkappa(G/P_m;L_{\nu+z\xi})$.

(1) $\omega_{w,z}$ *is a continuous linear functional on* V_z *and contained in* $\mathrm{Wh}^*_{\mathfrak{n}_m,z\psi}(V_z)$ *for all* $0<|z|<\varepsilon$. *Especially, for each* $t>0$ *there exists some positive number* $|\delta_{\nu,w}|_{\varkappa,t}$ *such that for every* $f\in\mathscr{G}^\varkappa(G/P_m;L_\nu)$ *which satisfies* $\|f\|^K_{\varkappa,t}<\infty$, *the following estimate holds*

$$\omega_{w,z}(f)|\leq\|(-\psi)_v(\nu+z\xi)\|_{\varkappa,2^\varkappa t}|\delta_{\nu+z\xi,w}|_{\varkappa,2t}\|f\|^K_{\varkappa,t}.$$

(2) *For all* $f\in\mathscr{G}^\varkappa(K/M)$, $z\rightsquigarrow\omega_{w,z}(e_{\nu+z\xi}(f))$ *is holomorphic on* $0<|z|<\varepsilon$.

Put $r=\operatorname{card}W$. Fix $\lambda\in(\mathfrak{a}^*)_-$ and choose $\xi\in\mathfrak{a}_m^*$ as above. From Theorem 6.1.1, for all z such that $|z|<\varepsilon$, $\dim\mathrm{Wh}_{\mathfrak{n}_m,\psi}(\mathscr{A}_K(G/P_m;L_{\nu+z\xi}))=r$.

Let $\mathscr{A}_K(K/M_m)$ be the space of K-finite functions on K/M_m.

Now, we can prove the following result in just the same way as [GW] Lemma 5.11.

LEMMA 6.1.4. *For all z such that $|z|<\varepsilon$, we can define a basis*

$$y_z^{(1)}, \ldots, y_z^{(r)}$$

of $\mathrm{Wh}^*_{\mathfrak{n}_m,\psi}(\mathscr{A}_K(G/P_m;L_{\nu+z\xi}))$ *such that for all* $f\in\mathscr{A}_K(K/M_m)$ *and* $1\leq i\leq r$, *the map*

$$z \rightsquigarrow y_z^{(i)}(e_{\nu+z\xi}(f))$$

is holomorphic on $|z|<\varepsilon$.

We have:

PROPOSITION 6.1.5. *Fix* $1\leq\varkappa<2$ *and fix an arbitrary* $\nu\in(\mathfrak{a}_m^*)_-$. *Then, every element of* $\mathrm{Wh}^*_{\mathfrak{n}_m,\psi}(\mathscr{A}_K(G/P_m;L_\nu))$ *can be extended to a continuous linear functional on* $\mathscr{G}^\varkappa(G/P_m;L_\nu)$. *Especially,*

$$\mathrm{Wh}^*_{\mathfrak{n}_m,\psi}(\mathscr{A}_K(G/P_m;L_\nu))=\mathrm{Wh}^G_{\mathfrak{n}_m,\psi}(\mathscr{A}_K(G/P_m;L_\nu)).$$

Proof. Now we can apply the same method as the proof of [GW] Lemma 5.12. Clearly, we have only to extend $y_0^{(i)}$ $(1\leq i\leq r)$ to a continuous linear functional on $\mathscr{G}^\varkappa(G/P_m;L_\nu)$.

We fix $t>0$, $1\leq i\leq r$, and $f\in\mathscr{A}_K(K/M_m)$ which satisfies $\|f\|^K_{\varkappa,t}<\infty$. From Lemma 6.1.3 and Lemma 6.1.4, we can write

$$y_z^{(i)}(e_{\nu+z\xi}(f))=\sum_{w\in W} d_{i,w}(z)\omega_{w,z}(e_{\nu+z\xi}(f)),$$

for all $0<|z|<\varepsilon$. Here, $d_{i,w}(z)$ $(w\in W)$ is homorphic functions defined on $0<|z|<\varepsilon$.

From Theorem 6.1.1, and [GW] (2.2), we have

$$|y_z^{(i)}(e_{\nu+z\xi}(f))|\leq\sum_{w\in W}|d_{i,w}(z)|\,\|(-\psi)_\upsilon(w(\nu+z\xi)-\varrho_m)\|_{\varkappa,2^\varkappa t}|\delta_{\nu,w}|_{\varkappa,t}\|f\|^K_{\varkappa,t}.$$

From Lemma 4.7.3, and Theorem 6.1.4, we have

$$\sup_{|z|=\varepsilon/2}\sum_{w\in W}|d_{i,w}(z)|\,\|(-\psi)_\upsilon(w(\nu+z\xi)-\varrho_m)\|_{\varkappa,2^\varkappa t}|\delta_{\nu+z\xi,w}|_{\varkappa,t}<\infty.$$

We denote by M the above constant.

Hence by the maximum principle,

$$|y_0^{(i)}(e_\nu(f))| \leq \max_{|z|=\varepsilon/2} |y_z^{(i)}(e_{\nu+z\xi}(f))| \leq M\|f\|_{x,t}.$$

Since $e_\nu(A_K(K/M_m))=\mathscr{A}_K(G/P_m; L_\nu)$ is dense in $\mathscr{G}^x(G/P_m; L_\nu)$, we have the desired result. Q.E.D.

Now that we have proved Proposition 6.1.5, we can prove, using Corollary 2.1.3 and Corollary 2.2.4, the following theorem in just the same way as the quasi-split case, namely [GW] Theorem 5.2.

THEOREM 6.1.6. *We assume ψ is an admissible character on $\mathfrak{n}_m$. Let σ be an irreducible finite dimensional representation of M_m and $\nu \in \mathfrak{a}_m^*$. If*

$$y \in \mathrm{Wh}^*_{\mathfrak{n}_m,\psi}(\mathscr{A}_K(G/P_m; L_{\sigma,\nu})),$$

then w extends to a continuous functional on $\mathscr{G}^x(G/P_m; L_{\sigma,\nu})$.

Especially,

$$\mathrm{Wh}^*_{\mathfrak{n}_m,\psi}(\mathscr{A}_K(G/P_m; L_{\sigma,\nu})) = \mathrm{Wh}^G_{\mathfrak{n}_m,\psi}(\mathscr{A}_K(G/P_m; L_{\sigma,\nu})).$$

6.2 Global Whittaker vectors

As a corollary of Theorem 6.1.6, we have one of the main results of this paper.

THEOREM 6.2.1. *Let V be an irreducible Harish-Chandra module, and let ψ: $U(\mathfrak{n}_m) \to \mathbf{C}$ be an admissible character. Then,*

$$\mathrm{Wh}^G_{\mathfrak{n}_m,\psi}(V) = \mathrm{Wh}^*_{\mathfrak{n}_m,\psi}(V).$$

Remark. $\dim \mathrm{Wh}^*_{\mathfrak{n}_m,\psi}(V)$ is given in Corollary 2.2.2.

Proof. From Casselman's embedding theorem, there exists some irreducible representation σ of M_m and $\nu \in \mathfrak{a}_m^*$ such that there exists some embedding

$$\iota\colon V \hookrightarrow \mathscr{A}_K(G/P_m; L_{\sigma,\nu}).$$

For simplicity, put $M=\mathscr{A}_K(G/P_m; L_{\sigma,\nu})$. Hence we have an exact sequence

$$0 \to (M/V)^* \to M^* \to V^* \to 0.$$

From [KO] Lemma 4.5, the following is exact.

$$0 \to (M/V)^*_\eta \to M^*_\eta \to V^*_\eta \to 0.$$

From Corollary 2.2.4, we have an exact sequence

$$0 \to \mathsf{Wh}^*_{\mathfrak{n}_m, \psi}(M/V) \to \mathsf{Wh}^*_{\mathfrak{n}_m, \psi}(M) \to \mathsf{Wh}^*_{\mathfrak{n}_m, \psi}(V) \to 0.$$

This means every $\mathsf{Wh}^*_{\mathfrak{n}_m, \psi}(V)$ extends to an element of $\mathsf{Wh}^*_{\mathfrak{n}_m, \psi}(M)$. Hence, by Theorem 6.1.6, we have the desired result. Q.E.D.

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