Hausdorff dimension of harmonic measures in the plane

by

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Let $\Omega=\mathbf{C}^*/E$ be an open set on the Riemann sphere $\mathbf{C}^*$ and let $\omega(\Omega, Y, z)$ be the harmonic measure of $Y\subset E$ with respect to Ω, evaluated at $z\in\Omega$. For fixed $z\in\Omega$, $\omega(\Omega,\cdot,z)$ is a probability measure supported on $\partial\Omega$. It can be defined by the requirement that

$$\bar{u}(z)=\int u\,d\omega(\Omega,\cdot,z)$$

for all continuous $u:\partial\Omega\to\mathbf{R}$, where $\bar{u}$ is the Perron solution of the Dirichlet problem with boundary values u. We must assume here that E has positive capacity, but not that Ω is regular for the Dirichlet problem.

If z_1 and z_2 belong to the same component of Ω then the corresponding measures $\omega(\Omega,\cdot,z_1)$ and $\omega(\Omega,\cdot,z_2)$ are boundedly absolutely continuous to each other. Accordingly if Ω is a domain, then the possibility that $\omega(\Omega, Y, z)=1$ (or 0) is independent of z, and a set Y such that $\omega(\Omega, Y, z)=1$ is said to support harmonic measure. Recently there has been interest in the metric properties of such sets. Several types of results are known [B, C2, K-W, Mak1, Mak2, Man, Ø, Po, Pr]; here we are concerned with a conjecture of Øksendal [Ø] to the effect that there is always a support set with Hausdorff dimension $\leqslant 1$. This has been proved in certain scale invariant cases by Carleson [C2], Manning [Man], Przytycki [Pr] and for general simply connected domains by Makarov [Mak1]. Our first result is a proof for arbitrary domains.

THEOREM 1. *Let $\Omega\subset\mathbf{C}^*$ be a domain whose complement has positive capacity. Then there is a set $F\subset\partial\Omega$ with Hausdorff dimension $\leqslant 1$, such that $\omega(\Omega,F,z)=1$ for $z\in\Omega$.*

Makarov actually proved more in the simply connected case, namely that if $\varrho:[0,\infty)\to[0,\infty)$ is an increasing function with $\lim_{t\to 0}\varrho(t)/t=0$, then there is a support set with Hausdorff h_ϱ-measure zero. Subsequently Pommerenke [Po] observed that there is even a support set with σ-finite one dimensional Hausdorff measure. These sharp results can also be extended to general domains by our method, but that requires additional technical work and will appear elsewhere.

The proof of Theorem 1 is based on two observations, one coming from Makarov's work and the other from Carleson's. The starting point is that Markarov's proof works for somewhat more general domains than just simply connected ones. To fix ideas let us assume Ω is an exterior domain, i.e. $\Omega=\mathbf{C}^*/E$ where $E\subset\mathbf{C}$ is compact. Let $\delta(z)=\operatorname{dist}(z,E)$. We say Ω satisfies the *capacity density condition* if there is a fixed constant $\eta>0$ such that

(CDC) if $z\in\Omega$ with $\delta(z)$ sufficiently small and if $D=D(z,2\delta(z))$ then

$$\omega(\Omega\cap D, E\cap D, z)\geq\eta.$$

In other words Brownian motion started near $\partial\Omega$ has a definite probability of exiting Ω before moving twice the minimum possible distance. This type of "thick boundary" condition has been useful in various contexts, e.g. [Anc], [B-C], [J-M]. It is satisfied by any simply connected domain and Makarov's proof of Theorem 1 works with minor changes (replace the Riemann map by the universal covering map) on non-simply connected domains satisfying CDC.

It may seem strange that a thick boundary condition would be useful in proving Theorem 1, which after all is trivial if the dimension of the boundary is ≤ 1. However, consider the following situation. Let $z_1,\ldots,z_{N^2}$ be the vertices of a subdivision of the unit square into squares of side $1/N$. For each z_j, let $D_j=D(z_j,r_j)$ be a small disc centered at z_j, and let $\Omega=\mathbf{C}^*/\bigcup D_j$. By appropriate choices of the r_j we can obtain a situation where each D_j has the same harmonic measure, $\omega(\Omega,D_j,\infty)=N^{-2}$. For Theorem 1 to be true in (a suitable inverted case of) this situation, the requirement that each D_j has the same harmonic measure must force $\Sigma\, r_j$ to be bounded independently of N. (Indeed, a simple argument with Wiener's test shows that r_j must be very much smaller than N^{-2} whenever z_j has distance less than $\frac{1}{4}$ to the boundary of the unit square.) This example was pointed out by L. Carleson and by A. O'Farrell.

We did not succeed in modifying Makarov's ideas to cover this type of situation and used instead an idea from [C2], namely, the integration by parts formula

$$\frac{1}{2\pi}\int \frac{\partial G}{\partial n}\log\frac{\partial G}{\partial n}\,ds=\gamma+\sum_{\nabla G(z_j)=0} G(z_j). \tag{0.1}$$

Here Ω is an exterior domain with smooth boundary, G is Green's function with pole at ∞, n is the normal into Ω, and $\gamma=\lim_{z\to\infty} G(z)-\log|z|$ is Robin's constant. The point is that the right side is explicitly bounded from below in terms of the diameter of $\mathbf{C}^*/\Omega$.

If Ω is very non-simply connected, then G will have many critical points and the right side of (0.1) will be large. In [C2], this fact is used to show that for square Cantor sets E, harmonic measure for $\mathbf{C}^*/E$ is supported on a set with dimension *strictly* less than 1. This is in contrast with Makarov's deep result in [Mak1], that if Ω is simply connected then any set with dimension less than 1 has harmonic measure zero. A modification of our method can be used to generalize Carleson's result to "Cantor-like" sets without a scale invariant structure; this will appear elsewhere.

The remainder of this paper is the proof of Theorem 1. The proof we give is due to Lennart Carleson; our original argument was similar in spirit but used a different domain modification procedure and was much more complicated. We are grateful to Prof. Carleson for permission to use his argument and in a broader sense, for everything he has done for this subject.

§ 1. Modification of the domain: Preliminaries

In this section we modify Ω to Ω' so that on Ω' the gradient of Green's function is bounded on a large set. This modification will be based on two different constructions, here called the *disk construction* and the *annulus construction*. Let Q be a dyadic square and assume $E\cap Q$ has positive capacity. We let A be a (small) positive constant and B be a large positive constant; the values of A and B will be fixed later. These values will depend on $\delta>0$ when we prove $\mathrm{Dim}(\omega)<1+\delta$.

Disk construction. Assume first that $l(Q)=1$ and $E\cap Q$ has capacity $C(Q)$. Replace $E\cap Q$ by a disk centered at the center of Q and with radius $r=C(Q)^A$. For arbitrary Q, we change scale to a square of sidelength 1, make the above construction, and scale back.

Annulus construction. Let Q' be the closed square with sidelength $l(Q')=Bl(Q)$, with the same center as Q, and with sides parallel to the coordinate axes. Delete from E any point lying in $Q'\setminus Q$.

After performing one of these constructions we obtain a new set E and a new domain Ω, as well as a new harmonic measure. We keep, however, the same notation.

Our next task is to estimate harmonic measure on the new domain. For the annulus construction this is easy; any part of E which remains has its harmonic measure increased. For the disk construction the effects are less obvious. The following two lemmas give the required estimates.

LEMMA 1.1. *Let F and G be closed sets, and let Q be a square with $l(Q)=1/4$. We suppose $F\subset Q$,* distance$(G, Q)\geqslant 1/2$, *and F has capacity $e^{-\gamma}$. Let Ω be the domain with $\partial\Omega=F\cup G$, and let ω be the harmonic measure of F in Ω. Define L to be the curve surrounding Q at distance $1/4$ to Q. Then if $z\in L$,*

$$\omega(z)\sim\frac{1}{1+K(G)\gamma}$$

where $K(G)$ is a constant depending only on G.

Proof. Let $u(z)$ be the harmonic measure of L for the domain Ω' between G and L. We shall prove that

$$K(G)=D(u)=\int\int_{\Omega}|\nabla u|^2\,dx\,dy.$$

Applying first Green's formula, then Harnack's inequality, and then Green's formula again we obtain (n'=inward normal in Ω')

$$\begin{aligned}-\int_L\frac{\partial\omega}{\partial n'}\,ds&=-\int_L u\frac{\partial\omega}{\partial n'}\,ds\\&=-\int_L\omega\frac{\partial u}{\partial n'}\,ds\\&\sim(-m)\int_L\frac{\partial u}{\partial n'}\,ds\\&=m\int_{\Omega'}|\nabla u|^2\,dx\,dy\\&=mD(u),\end{aligned}\tag{1.1}$$

where m is the mean value of ω over L.

It is clear that $D(u)\leqslant$constant and that we may assume γ is large. Let $G(z,\varrho)$ be

Green's function (with pole at ϱ) for $\hat{\Omega}=\Omega\setminus\Omega'$. ($\hat{\Omega}$ is the part of Ω inside L.) Let v be the harmonic measure of F in $\hat{\Omega}$. Then

$$v(z)=\frac{1}{\gamma'}\int_F G(z,\varrho)\,d\mu(\varrho)$$

where μ is a positive measure of mass $\|\mu\|=1$, and where $\gamma\sim\gamma'$. By Green's formula (n=inward normal in $\hat{\Omega}$),

$$\begin{aligned}\frac{1}{2\pi}\int_L \frac{\partial\omega}{\partial n'}\,ds &= \frac{1}{2\pi}\int_F \frac{\partial\omega}{\partial n}\,ds\\ &= \frac{1}{2\pi}\int_F v\frac{\partial\omega}{\partial n}\,ds\\ &= \frac{1}{2\pi}\int_{\partial\hat{\Omega}} \omega\frac{\partial v}{\partial n}\,ds\\ &= \frac{-1}{\gamma'}+\frac{1}{2\pi}\int_L \omega\frac{\partial v}{\partial n}\,ds\\ &= \frac{-1}{\gamma'}(1-\omega(z_0))\end{aligned}\tag{1.2}$$

for some $z_0\in L$. The last follows from the mean value theorem because $\partial v/\partial n>0$ on L and

$$\frac{1}{2\pi}\int_L \frac{\partial v}{\partial n}\,ds=\frac{1}{\gamma'}.$$

Combining (1.1) and (1.2) we see that

$$mD(u)\sim\frac{1}{\gamma}(1-\omega(z_0)),$$

and the lemma now follows from Harnack's inequality.

LEMMA 1.2. *Let F_1 and Ω_1 and F_2 and Ω_2 be as in Lemma 1.1 with G outside Q satisfying $G=\partial\Omega_j\setminus F_j$, $j=1,2$. Let $e^{-\gamma_1}$ and $e^{-\gamma_2}$ be the capacities of F_1 respectively F_2, and assume that Q satisfies the* (B) *annulus condition. Define L to be the circle with center equal to center of Q and radius $\frac{1}{4}B$. Then given any $\varepsilon>0$ there is a value $B(\varepsilon)$ so that if $(1+\varepsilon)\gamma_1\leq\gamma_2$, $B\geq B(\varepsilon)$, and if $\omega_1(z)$ and $\omega_2(z)$ are the harmonic measures of any $Y\subset G$,*

$$\omega_1(z) \leqslant \omega_2(z), \quad z \in L,$$

and hence outside L.

Proof. Let $R=\frac{1}{4}B$ and shrink the picture in the scale $1/R$ so that L becomes the unit circle. Let $D_j=U\setminus F_j$, $j=1,2$ where U is the unit disk. Let $g_j(z,\varrho)$ be Green's function for D_j. The lemma follows from the estimate

$$\frac{\partial g_1}{\partial n}(z,\varrho) < \frac{\partial g_2}{\partial n}(z,\varrho), \quad |z|=1, \quad |\varrho|=1/2. \tag{1.3}$$

For assume that (1.3) holds but that

$$\frac{\omega_1(z_0)}{\omega_2(z_0)} = \operatorname*{Max}_{|z|=1/2} \frac{\omega_1(z)}{\omega_2(z)} = \lambda > 1,$$

where $|z_0|=1/2$. By the maximum principle ($\lambda\omega_2-\omega_1>0$ on Y),

$$\lambda\omega_2-\omega_1>0 \quad \text{on} \quad |z|=1.$$

However, taking $\varrho=z_0$ in (1.3), Green's theorem yields ($\omega_j=0$ on F_j),

$$\begin{aligned}
0 &= \lambda\omega_2(z_0)-\omega_1(z_0) \\
&= \frac{1}{2\pi}\int_{|z|=1} \frac{\partial g_2}{\partial n}(z,z_0)\,\lambda\omega_2(z)\,ds(z) - \frac{1}{2\pi}\int_{|z|=1} \frac{\partial g_1}{\partial n}(z,z_0)\,\omega_1(z)\,ds(z) \\
&> \frac{1}{2\pi}\int \frac{\partial g_2}{\partial n}(z,z_0)\,(\lambda\omega_2(z)-\omega_1(z))\,ds(z) \\
&> 0,
\end{aligned}$$

which is a contradiction.

To prove (1.3) we first set $u_j(z)$ to be the harmonic measure of F_j in D_j. Then

$$u_j(z) = \frac{1}{\bar{\gamma}_j}\int_{F_j} \log\left|\frac{1-z\bar{\varrho}}{z-\varrho}\right| d\mu_j(\varrho)$$

where μ_j is the appropriate unit mass, and

$$\frac{1}{\bar{\gamma}_j} = \frac{1+O(1/R)}{\gamma_j+\log R}. \tag{1.4}$$

An application of the maximum principle shows that for $|\varrho|=1/2$,

$$g_1(z,\varrho) < \log\left|\frac{1-z\bar{\varrho}}{z-\varrho}\right| - \left(\log 2 - \frac{C_1}{R}\right) u_1(z)$$

and

$$g_2(z,\varrho) > \log\left|\frac{1-z\varrho}{z-\varrho}\right| - \left(\log 2 + \frac{C_2}{R}\right) u_2(z)$$

where C_1, $C_2>0$ are two universal constants. We now fix $|z|=3/4$ and compare these last two inequalities. Since $(1+\varepsilon)\gamma_1 \leq \gamma_2$, it follows from the representation of u_j and (1.4) that

$$g_1(z,\varrho) < g_2(z,\varrho), \quad |\varrho| = 1/2.$$

Since g_1 and g_2 vanish on $|z|=1$, (1.3) follows from the above inequality.

Lemmas 1.1 and 1.2 have the following immediate consequence.

LEMMA 1.3 (Main Lemma). *Let $A=1+\varepsilon$. Suppose $Q\cap E \neq \emptyset$ and suppose Q satisfies the* (B) *annulus condition where $B \geq B(\varepsilon)$. When we perform an* (A) *disk construction, harmonic measure (at ∞) of a set outside of Q increases (Lemma 1.2). The harmonic measure for Q itself decreases by at most a bounded factor (Lemma 1.1).*

For the proof of the theorem we also require some properties of Hausdorff measure. Let $\varphi(r)$ be increasing for $r>0$ with $\varphi(0)=0$. For a set E we then define as in [C1]

$$h_\varphi(E) = \inf \sum_{j=1}^{\infty} \varphi(r_j)$$

where the infimum is taken over all coverings $E \subset \bigcup_{j=1}^{\infty} \{|z-z_j|<r_j\}$ and where *no restrictions* are made on the size of r_j. For the following lemma (compare [C1]) we also require $\varphi(r)/r \leq \varphi(2r) \leq C\varphi(r)$, and $\varphi(r)/r \leq 1$, $r \leq 1$.

LEMMA 1.4. *Let $E \subset \{|z|<1\}$ have capacity $e^{-\gamma}$. Then*

$$h_\varphi(E) \leq \text{const.}\ \varphi(e^{-\gamma}).$$

Proof. There exists (see [C1], page 7) a positive measure μ of total mass $m \geq Ch_\varphi(E)$ such that $\mu(a,r)=\mu(\{|z-a|<r\}) \leq \varphi(r)$ for all $a, r<1$. The logarithmic potential V of μ satisfies

$$V(z)=\int \log\frac{1}{|z-\varrho|}\,d\mu(\varrho)=\int_0^2 \log\frac{1}{r}\,d\mu(z,r)\le\int_0^{m^*}\log\frac{1}{r}\,d\varphi(r),$$

where $\varphi(m^*)=m$. Consequently,

$$V(z)\le m\log\frac{1}{m^*}+\int_0^{m^*}\frac{\varphi(r)}{r}\,dr$$

and therefore

$$\gamma\le\sup_z\frac{1}{m}V(z)\le\log\frac{1}{m^*}+1.$$

The lemma now follows immediately.

Now let Q be a square, $l(Q)=r\le 1/4$, and let $E^*\cap Q$ have capacity $e^{-\gamma}$. When we apply the $(1+\varepsilon)$ disk construction to $E^*\cap Q$ we obtain a disk Δ' of radius r' and capacity $e^{-\gamma'}$. Since $r\le 1/4$, it is easily seen that $\gamma'\le(1+\varepsilon)\gamma$. Setting $\varphi(t)=t^{1+2\varepsilon}$ we see by Lemma 1.4 that

$$\begin{aligned} h_\varphi(E^*\cap Q) &\le C\varphi(e^{-\gamma})\\ &= Ce^{-(1+2\varepsilon)}\\ &\le Ce^{-(1+\varepsilon/2)\gamma'}\\ &= C(r')^{1+\varepsilon/2}. \end{aligned}\tag{1.5}$$

§ 2. Modification of the domain

Let Ω be a domain whose boundary, E, is contained in $\{|z|<1/2\}$. We fix two large integers M and N, and set $\varrho=2^{-N}$. Let $\mathcal{G}_N$ be the grid consisting of closed dyadic squares on sidelength ϱ, and divide $\mathcal{G}_N$ into B^2 periodic classes (modulo $B\times 2^{-N}$ along the coordinate axes) which we call $\mathcal{G}_N^j$, $1\le j\le B^2$. Let E_j be the intersection of E with the collection of all $Q\in\mathcal{G}_N^j$, $\varrho(Q)=\varrho$. Then $E=\bigcup_{j=1}^{B^2}E_j$. Let $\Omega_j=\mathbf{R}^2\setminus E_j$ and let ω_j denote harmonic measure on Ω_j. Then $\omega^*\le\sum_{j=1}^{B^2}\omega_j$, where ω^* is the original harmonic measure on Ω. We may therefore assume from the start that Ω satisfies the (B) annulus condition for all cubes Q, $l(Q)=\varrho$.

We now start altering E (and hence Ω) by the procedure developed in section 3. First perform the (2) disk construction for all Q, $l(Q)=\varrho$. By the Main Lemma 1.3, the harmonic measure of each Q decreases by at most a bounded factor, i.e.,

$\omega^*(Q) \leqslant C\omega(Q)$ for the new harmonic measure ω. Our new set consists of disks of radii$\leqslant \varrho$ well separated from each other. Furthermore each of these disks is contained in its associated Q, $l(Q)=\varrho$.

We now proceed with the following algorithm.

Step I. Let ω and E be given. Choose the largest dyadic square Q with $l(Q) \geqslant \varrho$ for which

$$\omega(Q) \geqslant Ml(Q).$$

If no such Q exists, stop. If such a Q exists, proceed to Step II.

Step II. Perform the (B) annulus construction on Q. This gives a new E and a new ω. Proceed to Step III.

Step III. Perform the (A) disk construction on Q to obtain a new E and a new ω. Go back to Step I.

Special Rule 1. A square Q on which Steps II, III have been performed should not be subdivided at some future stage. In other words, in applying Step I we should not consider Q's inside squares which have already been modified.

Special rule 2. Let Q_1, Q_2, Q_3, ... be the squares acted upon in Steps II, III, enumerated by the order in which they are considered. Let $\mathcal{A}_j$ be the (B) annulus about Q_j and let Δ_j be the disk associated to Q_j constructed by Step III. If $Q_j \not\subset \mathcal{A}_k$, $k>j$, then do not delete from E at stage k any part of the disk Δ_j.

Upon analyzing this algorithm we first notice that it stops. The reason for this is simply that by *Special Rule* 1 the possible number of constructions is finite. When the algorithm stops it has produced a certain number (possibly zero) of disjoint squares Q_j. Each satisfies (Step II) the (B) annulus condition of size $l(Q_j) \geqslant \varrho$, and has inside of it a disk Δ_j of radius r_j. This follows from *Special Rule* 2 because if a disk is constructed at some stage, it is either wholly untouched or totally deleted at each subsequent stage of the construction. (If $A \sim 1$, a small piece of Δ_j may lie outside of Q_j.) There also remain a certain number of original squares Q_j' and disks Δ_j' of radii r_j'. For each Q_j',

$$\omega(Q_j') \geqslant \omega^*(Q_j'),$$

where ω^* is the original harmonic measure.

The original set, E^*, also lies near the new set E in a quantifiable sense. Let Q be some dyadic square, $l(Q)=\varrho$, and let $E_Q=E\cap Q$. Then if E_Q is not inside a Q_j or a Q_j', E_Q must be in the $2B$ annulus of some Q_j. The reason for doubling the annulus is that we at some stage consider a square $Q=Q^0$. This square may at a later stage of the algorithm be included in a (B) annulus A^1 corresponding to a larger square Q^1 and Q^1 may similarly relate to Q^2, etc. The squares Q^0, Q^1, Q^2, ... satisfy $l(Q^k)\geq 2l(Q^{k-1})$, so a doubling of the annulus suffices to include E^*.

Special Rule 1 together with the Main Lemma 1.3 implies that

$$\omega(Q_j) \geq \text{const.}\, Ml(Q_j),$$

and consequently,

$$\sum_j l(Q_j) \leq \frac{C}{M}\sum_j \omega(Q_j) \leq \frac{C}{M}. \tag{2.1}$$

Therefore $h_\varphi(\bigcup_j Q_j)\leq C/M$ and by the last paragraph

$$h_\varphi(E^*\cap \textstyle\bigcup_j 2BQ_j) \leq \frac{C'B}{M}$$

for any φ such that $\varphi(r)/r$ is an increasing function.

Finally, notice that by the algorithm

$$\omega(\{|z-z_0|<r\}) \leq CMr \tag{2.2}$$

holds for any $z_0\in Q_j$ or Q_j' and $r\geq l(Q_j)$ or $l(Q_j')$.

§ 3. The level set

To prove the theorem we will integrate $|\nabla G|\log|\nabla G|$ around certain level curves of G, Green's function for Ω with pole at ∞, to obtain an estimate on the set where $|\nabla G|$ is small. We first require a lemma on such integrations.

LEMMA 3.1. *Let Γ surround E and assume that E and all of Γ lie inside $\{|z|<1\}$. Γ is assumed to be composed of level lines Γ_i where $G=c_i$. Then*

$$I(\Gamma) = \frac{1}{2\pi}\int_\Gamma \frac{\partial G}{\partial n}\log|\nabla G|\, ds \geq -\log 2.$$

Proof. Let ϱ_j be the critical points of G which lie outside of Γ. Then by Green's formula

$$\begin{aligned} I(\Gamma) &= \sum_j G(\varrho_j)+\sum_i \frac{1}{2\pi}\int_{\Gamma_i} G\frac{\partial}{\partial n}\log|\nabla G|\,ds+\gamma \\ &= \sum_j G(\varrho_j)+\sum_i \frac{c_i}{2\pi}\int_{\Gamma_i} \frac{\partial}{\partial n}\log|\nabla G|\,ds+\gamma. \end{aligned}$$

Let α_i be the number of boundary components of E inside Γ_i, and let β_i be the number of critical points of G inside Γ_i. Then

$$\frac{1}{2\pi}\int_{\Gamma_i}\frac{\partial}{\partial n}\log|\nabla G|\,ds=-(\alpha_i-\beta_i)=-1,$$

and consequently

$$I(\Gamma)=\sum_j G(\varrho_j)-\sum_i c_i+\gamma.$$

Let $n(c)$ equal the number of components of $\{z\colon G(z)=c\}$ which lie outside Γ. Also, let $c_0=\sup_i c_i$. Then

$$\sum_j G(\varrho_j)=\int_0^\infty (n(c)-1)\,dc$$

and

$$\sum_i c_i=\int_0^\infty \sum_i \chi_{\{c\le c_i\}}(c)\,dc\le\int_0^{c_0} n(c)\,dc.$$

Therefore

$$I(\Gamma)>\gamma-c_0.$$

But for $|z|<1$, $G(z)=\int\log|z-\varrho|\,d\omega(\varrho)+\gamma$ satisfies $G(z)\le\log 2+\gamma$ so that $c_0\le\log 2+\gamma$. (All critical points of G lie inside $\{|z|<1\}$.)

Now let z belong to $Q\setminus\Delta$ where Q is either a Q_j or a Q'_j, and Δ is the corresponding disk. Then Green's function can be written as

$$\begin{aligned} G(z) &= \int_\Delta \log|z-\varrho|\,d\omega(\varrho)+\int_{E\setminus\Delta}\log|z-\varrho|\,d\omega(\varrho)+\gamma \\ &= u(z)+v(z)+\gamma. \end{aligned}$$

Observe that by (2.2),

$$|\nabla v(z)| \leq \int_{E\setminus\Delta} \frac{d\omega(\varrho)}{|z-\varrho|} \leq CM \int_{l(Q)}^{1} \frac{dr}{r}$$
$$= CM \log \frac{1}{l(Q)}.$$

We define

$$S = S(\Delta) = \operatorname{Max}\left(\frac{\omega(\Delta)}{M^2 \log 1/l(Q)}, r(\Delta)\right),$$

where $r(\Delta)$ is the radius of Δ.

Let r denote the radius from the center, z_0, of Δ. Then

$$-\frac{\partial u}{\partial r}(z) \geq \frac{\omega(\Delta)}{|z-z_0|} - \frac{Cr(\Delta)\,\omega(\Delta)}{|z-z_0|^2}.$$

If

$$S = \frac{\omega(\Delta)}{M^2 \log 1/l(Q)} > r(\Delta)$$

we see that $|\partial u/\partial r|$ dominates $|\nabla v|$ when $|z-z_0| \sim S$, and consequently there is an essentially circular level line encircling Δ where $|z-z_0| \sim S$. On this level line

$$|\nabla G(z)| \sim \frac{\omega(\Delta)}{S}. \tag{3.1}$$

If $S=r(\Delta)$, we take $\partial\Delta$ (i.e., $G=0$) as the level line, and again (3.1) holds.

The above procedure gives a collection of level lines whose union, Γ, surrounds E. By (3.1),

$$\frac{1}{2\pi} \int_\Gamma \frac{\partial G}{\partial n} \log^+ |\nabla G|\, ds \leq C \sum_{\text{all } \Delta} \omega(\Delta) \log\left(+\frac{\omega(\Delta)}{S(\Delta)}\right)$$
$$\leq C \sum \omega(\Delta) \log\left(M^2 \log \frac{1}{l(Q)}\right)$$
$$\leq C' \log\log \frac{1}{\varrho},$$

if $M^2 \leq \log 1/\varrho$. By Lemma (3.1),

$$\frac{1}{2\pi} \int_\Gamma \frac{\partial G}{\partial n} \log^- |\nabla G|\, ds \geq -C \log\log \frac{1}{\varrho}. \tag{3.2}$$

We now use the Hausdorff function

$$\varphi(r) = r^{1+\delta}, \quad \delta = 2\varepsilon > 0.$$

We then set $A=1+2\varepsilon$ (the disk constant) and fix $B=B(2\varepsilon)$ (the annulus constant) as in the Main Lemma 1.3. By (2.1) we need only worry about the squares Q_j' (those of sidelength ϱ which have not been altered). Fix such a Q_j' and its associated disk Δ_j', and assume first that

$$\omega(\Delta_j') \geqslant r_j' \varrho^{\varepsilon/2}, \tag{3.3}$$

where r_j is the radius of Δ_j'. Call such a disk Type 1. Then if $E_j^*=E\cap Q_j'$, (1.5) shows

$$\begin{aligned}\sum_{\Delta_j' \in \text{Type 1}} h_\varphi(E_j^*) &\leqslant C \sum_{\text{Type 1}} (r_j')^{1+\varepsilon} \\ &\leqslant C\varrho^{\varepsilon/2} \sum_{\text{Type 1}} \omega(\Delta_j') \\ &\leqslant C\varrho^{\varepsilon/2}.\end{aligned}$$

In the remaining disks where (3.3) fails (Type 2 disks), we have $S_j'=r_j'$ and by (3.1), $|\nabla G| \leqslant \varrho^{\varepsilon/2}$ on $\partial\Delta_j'$. By (3.2),

$$\begin{aligned}\sum_{\Delta_j' \in \text{Type 2}} \omega(\Delta_j') &= \sum_{\text{Type 2}} \frac{1}{2\pi} \int_{\partial\Delta_j'} \frac{\partial G}{\partial n}\, ds \\ &\leqslant \frac{C}{\varepsilon \log 1/\varrho} \sum_{\text{Type 2}} \int_{\partial\Delta_j'} \frac{\partial G}{\partial n} \log \frac{1}{|\nabla G|}\, ds \\ &\leqslant \frac{C \log\log 1/\varrho}{\varepsilon \log 1/\varrho}.\end{aligned}$$

The proof of the theorem is complete.

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