Irregularities of distribution. I

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1. Introduction

The concept of *uniformly distributed sequences* plays a fundamental role in many branches of mathematics (ergodic theory, diophantine approximation, numerical integration, mathematical statistics, etc.). The object of the theory of Irregularities of Distribution is to measure the uniformity (or nonuniformity) of sequences and point distributions. For instance: how uniformly can an arbitrary distribution of n points in the unit cube be distributed relative to a given family of "nice" sets (e.g., boxes with sides parallel to the coordinate axes, balls, convex sets, etc.)?

This theory was initiated by the following conjecture of van der Corput. Let $\zeta=\{\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_3,\ldots\}$ be an infinite sequence of real numbers in the unit interval $U=[0,1]$. Given an x in U and a positive integer n, write $Z_n[\zeta;x]$ for the number of integers j with $1\leqslant j<n$ and $0\leqslant \mathbf{z}_j<x$ and put

$$D_n[\zeta;x]=Z_n[\zeta;x]-n\cdot x.$$

Let $\Delta_n[\zeta]$ be the supremum of $|D_n[\zeta;x]|$ over all numbers x in U. In 1935 van der Corput [6] conjectured that $\Delta_n[\zeta]$ cannot remain bounded as n tends to infinity. It was proved by Mrs T. van Aardenne-Ehrenfest [1] in 1945. Later her beautiful theorem was improved and extended in various directions by the work of K. F. Roth and Wolfgang M. Schmidt. There is now a vast literature on this subject. We refer the reader to Schmidt's book [13].

In this paper we continue the research started in Schmidt [11], [12]. We recall one of his basic results (Corollary of Theorem A3 in [12]): Let there be given n points

$\mathbf{z}_1, \ldots, \mathbf{z}_n$ in the K-dimensional unit cube $U^K=[0,1)^K$. Then there exists a ball A contained in U^K with "error"

$$|D[\mathbf{z}_1, \ldots, \mathbf{z}_n; A]| = \left| \sum_{\mathbf{z}_j \in A} 1 - n \cdot \mu(A) \right| > c_1(K, \varepsilon) \cdot n^{(K-1)/2K(K+2)-\varepsilon} \tag{1.1}$$

where μ denotes the K-dimensional Lebesgue measure and $c_1(K,\varepsilon)$ is a positive absolute constant depending only on the dimension K and $\varepsilon>0$.

In short, this theorem expresses the fact that no point distribution can, relative to balls in U^K, be too evenly distributed.

Note that Schmidt's theorem above guarantees the existence of a ball in U^K with "error" very large as compared to that of boxes in U^K with sides parallel to the axes. We recall: in 1954 K. F. Roth [9] proved the existence of a box B contained in U^K with sides parallel to the axes and with "error"

$$|D[\mathbf{z}_1, \ldots, \mathbf{z}_n; B]| = \left| \sum_{\mathbf{z}_j \in B} 1 - n \cdot \mu(B) \right| > c_2(K) \cdot (\log n)^{(K-1)/2}.$$

In the opposite direction, there is a distribution $\mathbf{w}_1, \ldots, \mathbf{w}_n$ of n points in U^K such that

$$|D[\mathbf{w}_1, \ldots, \mathbf{w}_n; B]| \leq c_3(K) \cdot (\log n)^{K-1}$$

for *any* box B in U^K with sides parallel to the axes (van der Corput–Hammersley–Halton sequence, see e.g. Schmidt [13] Theorem 1 E in Chapter I).

In the last section of his book Schmidt [13] raised the question of understanding the fascinating phenomenon that balls have much greater "error" than boxes with sides parallel to the axes. Our aim is to give a partial answer to this question.

We start with an essential improvement of Schmidt's bound (1.1) (observe that in (1.1) the exponent of n tends to zero as K tends to infinity).

THEOREM 1A. *Let ε be a positive real number and $\mathcal{P}$ be an arbitrary distribution of n points in $U^K=[0,1)^K$. Then there exists a ball A contained in U^K with error*

$$|D[\mathcal{P}; A]| = \left| \sum_{\mathbf{x} \in \mathcal{P} \cap A} 1 - n \cdot \mu(A) \right| > c_4(K, \varepsilon) \cdot n^{1/2-1/2K-\varepsilon}. \tag{1.2}$$

Here the exponent $(1/2-1/2K-\varepsilon)$ of n is essentially the best possible. Indeed, using probabilistic ideas it is not hard to show that (1.2) is certainly false if we replace the exponent by $(1/2-1/2K+\varepsilon)$ with $\varepsilon>0$ (for a quite analogous situation, see the proof of

Theorem 2 in Beck [3]). Observe that in (1.2) the exponent of n tends to 1/2 as K tends to infinity.

To avoid the technical difficulties caused by the requirement "contained in U^K", in what follows we shall study a new model.

Let $S=\{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \dots\}$ be a completely arbitrary infinite discrete set of points in Euclidean K-space $\mathbf{R}^K$. Given a compact set $A\subset\mathbf{R}^K$, write

$$\mathscr{D}[S;A] = \sum_{\mathbf{z}_j\in A} 1-\mu(A) \tag{1.3}$$

where μ denotes the K-dimensional Lebesgue measure. Observe that here the normalization is different from that in the previous results (compare the definitions of the error in (1.1) and (1.3)).

For arbitrary proper orthogonal transformation τ of K-space $\mathbf{R}^K$, real $\alpha\in(0,1]$ and vector $\mathbf{v}\in\mathbf{R}^K$ set

$$A(\tau,\alpha,\mathbf{v}) = \{\alpha(\tau\mathbf{x})+\mathbf{v} : \mathbf{x}\in A\}.$$

Clearly $A(\tau,\alpha,\mathbf{v})$ and A are similar to each other. Let

$$\Omega[S;A] = \sup_{\tau,\alpha,\mathbf{v}} |\mathscr{D}[S;A(\tau,\alpha,\mathbf{v})]|$$

and

$$\Omega[A] = \inf_{S} \Omega[S;A]$$

where the supremum is taken over all rotations τ, contractions α and translations $\mathbf{v}$, and the infimum is extended over all infinite discrete sets $S\subset\mathbf{R}^K$.

We say that $\Omega[A]$ is the *discrepancy* of the family $A(\tau,\alpha,\mathbf{v})$. We also say, in short, that $\Omega[A]$ is the *rotation discrepancy* of A.

Now assume that A is *convex*. Let ∂A be the boundary surface of A and $r(A)$ be the length of the radius of the largest inscribed ball in A. Let σ denote the $(K-1)$-dimensional surface area.

The next result (and the remark below) shows that for convex bodies the rotation discrepancy is always large and behaves like the square root of the surface area.

THEOREM 2A. *Let $S\subset\mathbf{R}^K$ be an arbitrary infinite discrete set and $A\subset\mathbf{R}^K$ be a K-dimensional compact convex body with $r(A)\geq 1$. Then*

$$\Omega[S;A] > c_5(K)\cdot(\sigma(\partial A))^{1/2},$$

i.e., there exist τ_0, $\alpha_0 \in (0,1]$ *and* $\mathbf{v}_0$ such that

$$|\mathscr{D}[S;A(\tau_0,\alpha_0,\mathbf{v}_0)]| > c_5(K)\cdot(\sigma(\partial A))^{1/2}.$$

Note that in the particular case $K=2$ and $A=$ "rectangle of size $n\times 2$" Theorem 2 A yields the existence of a tilted rectangle with error of "random size", that is, there is a tilted rectangle A with area$(A)\leq 2n$ and with error $|\mathscr{D}[S;A]|>$constant$\cdot n^{1/2}$.

(Throughout this paper constant stands for positive absolute constants depending only on the dimension K.)

Note that in the proof of Theorem 2 A we shall actually estimate from below the quadratic average of the "error" $\mathscr{D}[S;A(\tau,\alpha,\mathbf{v})]$. More precisely, we shall prove that

$$\liminf_{M\to\infty}(2M)^{-K}\int_{[-M,M)^K}\int_0^1\int_T(\mathscr{D}[S;A(\tau,\alpha,\mathbf{v})])^2\,d\tau\,d\alpha\,d\mathbf{v} > c_5(K)\cdot\sigma(\partial A),$$

where T is the group of proper orthogonal transformations in $\mathbf{R}^K$ and $d\tau$ is the volume element of the invariant measure on T, normalized such that $\int_T d\tau=1$.

We now explain that this stronger L^2-norm version of Theorem 2 A is already sharp apart from the constant factor. Let $\xi(\mathbf{l})$ denote an arbitrary point in the cube

$$Q(\mathbf{l})=\prod_{i=1}^{K}[l_i,l_i+1) \quad\text{where}\quad \mathbf{l}=(l_1,l_2,\dots,l_K)\in\mathbf{Z}^K\cap[-M,M)^K$$

(the parameter M will tend to infinity). Let

$$S_\xi=\{\xi(\mathbf{l})\colon \mathbf{l}\in\mathbf{Z}^K\cap[-M,M)^K\},$$

and for any $A(\tau,\alpha,\mathbf{v})\subset[-M,M)^K$, let

$$\tilde{A}(\tau,\alpha,\mathbf{v})=\bigcup_{\mathbf{l}:\,Q(\mathbf{l})\subset A(\tau,\alpha,\mathbf{v})}Q(\mathbf{l}).$$

By definition,

$$\mathscr{D}[S_\xi;\tilde{A}(\tau,\alpha,\mathbf{v})]=0.$$

Consequently, we obtain the trivial upper bound

$$\begin{aligned}\mathscr{D}[S_\xi;A(\tau,\alpha,\mathbf{v})] &\leq \operatorname{card}\{\mathbf{l}\in\mathbf{Z}^K\cap[-M,M)^K\colon Q(\mathbf{l})\cap(A(\tau,\alpha,\mathbf{v})\setminus\tilde{A}(\tau,\alpha,\mathbf{v}))\neq\emptyset\}\\ &< c_6(K)\cdot\sigma(\partial A)\end{aligned}$$

provided $A(\tau, \alpha, \mathbf{v}) \subset [-M, M)^K$. Now assume $\xi(\mathbf{l})$, $\mathbf{l} \in \mathbf{Z}^K \cap [-M, M)^K$ are *independent random variables* uniformly distributed in their cubes $Q(\mathbf{l})$. Since the L^2-norm of the "random error" is roughly the square root of the trivial error (Bessel inequality for orthogonal systems), there exists a $(2M)^K$-element set $S'(M) \subset [-M, M)^K$ such that the L^2-norm of the "error" of all sets $A(\tau, \alpha, \mathbf{v}) \subset [-M, M)^K$ is less than constant $\cdot (\sigma(\partial A))^{1/2}$. Using a simple compactness argument we conclude that there exists an infinite discrete set $S' \subset \mathbf{R}^K$ such that

$$\limsup_{M\to\infty} (2M)^{-K} \int_{[-M,M)^K} \int_0^1 \int_T (\mathscr{D}[S'; A(\tau, \alpha, \mathbf{v})])^2 \, d\tau \, d\alpha \, d\mathbf{v} < c_7(K) \cdot \sigma(\partial A),$$

as required.

Essentially the same random construction shows that for a suitable infinite discrete set $S'' \subset \mathbf{R}^K$ the L^∞-norm $\Omega[S'', A]$ of the errors is less than a sufficiently large constant multiple of

$$(\sigma(\partial A))^{1/2} \cdot (\log \sigma(\partial A))^{1/2}.$$

To prove it, choose $M = [\mathrm{diam}(A)] + 1$ (integral part) and apply the standard *large deviation theorem* of probability theory. The concrete calculation gives

$$\sup_{\tau, 0<\alpha\leq 1, \mathbf{v}} |\mathscr{D}[S_\xi; A(\tau, \alpha, \mathbf{v}) \cap [-M, M)^K]| < c_8(K) \cdot (\sigma(\partial A))^{1/2} \cdot (\log \sigma(\partial A))^{1/2}$$

with probability $\geq 1/2$ (for the details of this argument, see the proof of Theorem 2 in Beck [3]). Therefore, there must exist a $(2M)^K$-element set $S''(M) \subset [-M, M)^K$ such that

$$\sup_{\tau, 0<\alpha\leq 1, \mathbf{v}} |\mathscr{D}[S''(M); A(\tau, \alpha, \mathbf{v}) \cap [-M, M)^K]| < c_8(K) \cdot (\sigma(\partial A))^{1/2} \cdot (\log \sigma(\partial A))^{1/2}.$$

Finally, extend $S''(M)$ periodically modulo $[-M, M)^K$ over the whole K-space $\mathbf{R}^K$. We obtain an infinite discrete set $S'' \subset \mathbf{R}^K$ such that

$$\Omega[S'', A] = \sup_{\tau, 0<\alpha\leq 1, \mathbf{v}} |\mathscr{D}[S''; A(\tau, \alpha, \mathbf{v})]| < 2^K \cdot c_8(K) \cdot (\sigma(\partial A))^{1/2} \cdot (\log \sigma(\partial A))^{1/2}.$$

Here we used the simple fact that every set $A(\tau, \alpha, \mathbf{v})$ ($\tau \in T$, $\alpha \in (0, 1]$, $\mathbf{v} \in \mathbf{R}^K$) is the disjoint union of not more than 2^K sets of type

$$A(\tau, \alpha, \mathbf{v}) \cap ([-M, M)^K + 2M \cdot \mathbf{l}), \quad \mathbf{l} \in \mathbf{Z}^K.$$

This result indicates that Theorem 2 A is nearly best possible.

From Theorem 2 A one can immediately obtain results concerning the *unit torus*. Let $\mathcal{P}$ be a distribution of n points in the unit cube U^K. Extend $\mathcal{P}$ periodically over the whole K-space $\mathbf{R}^K$ modulo U^K. That is, let

$$\mathcal{P}^* = \{\mathbf{x}+\mathbf{l}\colon \mathbf{x} \in \mathcal{P},\ \mathbf{l} \in \mathbf{Z}^K\}.$$

Given a compact set $A \subset \mathbf{R}^K$, write $Z[\mathcal{P}^*; A]$ for the number of points of $\mathcal{P}^*$ in A, and put

$$D^*[\mathcal{P}; A] = Z[\mathcal{P}^*; A] - n \cdot \mu(A).$$

Finally, let

$$\Omega^*[\mathcal{P}; A] = \sup_{\tau, 0<\alpha\leq 1, \mathbf{v}} |D^*[\mathcal{P}; A(\tau, \alpha, \mathbf{v})]|$$

and

$$\Omega_n^*[A] = \inf \Omega^*[\mathcal{P}; A]$$

where the infimum is taken over all n-element sets $\mathcal{P} \subset U^K$.

If we rescale the periodic set $\mathcal{P}^*$ in ratio $n^{1/K}: 1$ and apply Theorem 2 A then we conclude that

COROLLARY 2B. *Let $\mathcal{P}$ be an arbitrary distribution of n points in the torus U^K, and let A be a compact convex body in $\mathbf{R}^K$. Suppose that $r(A) \geq n^{-1/K}$. Then*

$$\Omega^*[\mathcal{P}; A] > c_9(K) \cdot n^{1/2-1/2K} \cdot (\sigma(\partial A))^{1/2}.$$

In the particular cases A = "cube" and A = "ball", we get respectively Corollay 2 C and Corollary 2 D.

COROLLARY 2C. *Let δ be a positive real number and $\mathcal{P}$ be an arbitrary distribution of n points in the torus U^K. Then one can find a cube A in arbitrary position with diameter $\leq \delta$ and with "error"*

$$|D^*[\mathcal{P}; A]| > c_{10}(K) \cdot (n \cdot \delta^K)^{1/2-1/2K}.$$

We should mention here the pioneering result of Schmidt. For *boxes* in arbitrary position and $K=2, 3$ he proved the slightly weaker lower bound $(n \cdot \delta^K)^{1/2-1/2K-\varepsilon}$ (see Schmidt [12]); for arbitrary K it was hopeless to handle the very difficult integral equations that arise.

For balls Schmidt [12] was able to prove the lower bound $(n\cdot\delta^K)^{1/2-1/2K-\varepsilon}$ for *arbitrary* K. Here we obtain the following slight improvement.

COROLLARY 2D. *Let δ be a positive real number and $\mathcal{P}$ be an arbitrary distribution of n points in the torus U^K. Then one can find a ball A with diameter $\leqslant\delta$ and with "error"*

$$|D^*[\mathcal{P};A]|>c_11(K)\cdot(n\cdot\delta^K)^{1/2-1/2K}.$$

We note without proof that using the "truncation" technique in the proof of Theorem 1A it is not hard to show the following "contained in U^K" version of Corollary 2B: Let $\mathcal{P}$ be an arbitrary distribution of n points in U^K. Let $A\subset\mathbf{R}^K$ be a compact convex body of diameter less than one. Further suppose that $r(A)\geqslant n^{-1/K}$. Then there exist an orthogonal transformation τ_0, a real $\alpha_0\in(0,1]$ and a vector $\mathbf{v}_0\in\mathbf{R}^K$ such that $A(\tau_0,\alpha_0,\mathbf{v}_0)=A_0$ is *contained in UK* and has "error"

$$\left|\sum_{\mathbf{x}\in\mathcal{P}\cap A_0}1-n\cdot\mu(A_0)\right|>c_{12}(K,\varepsilon)\cdot n^{1/2-1/2K-\varepsilon}\cdot(\sigma(\partial A))^{1/2}.$$

In the forthcoming paper II we shall study the (traditional) case when rotation is forbidden (i.e., we may only contract and translate). We mention in advance that in this case the magnitude of the "error" depends mainly on the *smoothness* of the boundary surface ∂A of the given compact convex body $A\subset\mathbf{R}^K$.

The proofs are based on the so-called "Fourier transform method". As far as I know, the first appearance of this method is in Roth [10]. The same basic idea was later utilized in Baker [2] and Beck [4], [5].

We have learned that in the case $K=2$ results similar to our Corollary 2B, 2C and 2D have been proved, independently and about the same time by Montgomery [14].

We explain the machinery of the "Fourier transform method" with the following simple example.

THEOREM 3. *Let $\mathcal{P}$ be a distribution of n points in U^K. Then one can find a cube A in arbitrary position with diameter less than one such that*

$$\left|\sum_{\mathbf{x}\in\mathcal{P}\cap A}1-n\cdot\mu(A\cap U^K)\right|>c_{13}(K)\cdot n^{1/2-1/2K}.$$

Clearly Theorem 3 is an easy consequency of Corollary 2C, but in the next section we give a simple direct proof.

2. Illustration of the method

Proof of Theorem 3. The proof is based on an argument to "blow up" the "trivial error" (see (2.10) below). Let

$$\mathcal{P}=\{\mathbf{z}_1, \ldots, \mathbf{z}_n\}.$$

We introduce two measures.

For any $H \subset \mathbf{R}^K$ let

$$Z_0(H)=\sum_{\mathbf{z}_j \in H} 1,$$

i.e., Z_0 denotes the counting measure generated by the given point distribution $\mathcal{P}=\{\mathbf{z}_1, \ldots, \mathbf{z}_n\}$.

For any Lebesgue measurable $H \subset \mathbf{R}^K$ let

$$\mu_0(H)=\mu(H \cap U^K),$$

i.e., μ_0 denotes the restriction of the usual K-dimensional Lebesgue measure (volume) to the unit cube.

Given any proper orthogonal transformation τ and real $r>0$, let $\chi_{\tau,r}$ denote the characteristic function of the rotated cube

$$\tau[-r, r]^K=\{\tau \mathbf{x}: \mathbf{x} \in [-r, r]^K\}.$$

Consider now the function

$$F_{\tau,r}=\chi_{\tau,r} * (dZ_0-n \cdot d\mu_0) \tag{2.1}$$

where $*$ denotes the *convolution* operation.

More explicitly,

$$\begin{aligned} F_{\tau,r}(\mathbf{x}) &= \int_{\mathbf{R}^K} \chi_{\tau,r}(\mathbf{x}-\mathbf{y})\,(dZ_0(\mathbf{y})-n \cdot d\mu_0(\mathbf{y}))) \\ &= \operatorname{card}(\mathcal{P} \cap (\tau[-r, r]^K+\mathbf{x}))-n \cdot \mu((\tau[-r, r]^K+\mathbf{x}) \cap U^K). \end{aligned} \tag{2.2}$$

In other words, $F_{\tau,r}(\mathbf{x})$ equals the "error" of the intersection $(\tau[-r, r]^K+\mathbf{x}) \cap U^K$. Since the "error function" $F_{\tau,r}$ has the form of a convolution (see (2.1), it is natural to utilize the theory of *Fourier transformation*. We recall some well known facts (see any textbook on harmonic analysis).

If $f \in L^2(\mathbf{R}^K)$ then

$$\hat{f}(\mathbf{t}) = (2\pi)^{-K/2} \int_{\mathbf{R}^K} e^{-i\mathbf{x}\cdot\mathbf{t}} \cdot f(\mathbf{x})\, d\mathbf{x}$$

denotes the Fourier transform of f (here i is the square root of minus one and $\mathbf{x}\cdot\mathbf{t}$ is the standard Euclidean inner product). It is well known that

$$(f * g)\hat{} = \hat{f} \cdot \hat{g} \tag{2.3}$$

and

$$\int_{\mathbf{R}^K} |f(\mathbf{x})|^2 d\mathbf{x} = \int_{\mathbf{R}^K} |\hat{f}(\mathbf{t})|^2 d\mathbf{t} \quad \text{(Parseval-Plancherel identity).} \tag{2.4}$$

Let T be the group of proper orthogonal transformations in $\mathbf{R}^K$ and $d\tau$ be the volume element of the invariant measure on T, normalized such that $\int_T d\tau = 1$.

Let q be a positive real parameter. Let

$$\Omega_0(q) = \frac{1}{q} \int_q^{2q} \int_T \int_{\mathbf{R}^K} (F_{\tau,r}(\mathbf{x}))^2\, d\mathbf{x}\, d\tau\, dr. \tag{2.5}$$

By (2.1), (2.3) and (2.4) we have

$$\Omega_0(q) = \int_{\mathbf{R}^K} \left(\frac{1}{q} \int_q^{2q} \int_T |\hat{\chi}_{\tau,r}(\mathbf{t})|^2\, d\tau\, dr \right) \cdot |(dZ_0 - n \cdot d\mu_0)\hat{}\ (\mathbf{t})|^2\, d\mathbf{t}. \tag{2.6}$$

For the sake of brevity, let

$$\omega_q(\mathbf{t}) = \frac{1}{q} \int_q^{2q} \int_T |\hat{\chi}_{\tau,r}(\mathbf{t})|^2\, d\tau\, dr \tag{2.7}$$

and

$$\varphi(\mathbf{t}) = (dZ_0 - n \cdot d\mu_0)\hat{}\ (\mathbf{t}) = (2\pi)^{-K/2} \int_{\mathbf{R}^K} e^{-i\mathbf{x}\cdot\mathbf{t}} (dZ_0 - n \cdot d\mu_0)(\mathbf{x}).$$

Thus we can rewrite (2.6) as follows:

$$\Omega_0(q) = \int_{\mathbf{R}^K} \omega_q(\mathbf{t}) \cdot |\varphi(\mathbf{t})|^2\, d\mathbf{t}. \tag{2.8}$$

We claim

$$\text{if } 0 < q < p \quad \text{then } \frac{\omega_p(\mathbf{t})}{\omega_q(\mathbf{t})} \gg \left(\frac{p}{q}\right)^{K-1} \textit{ uniformly} \text{ for all } \mathbf{t} \in \mathbf{R}^K. \tag{2.9}$$

(Throughout this paper the implicit constant in Vinogradov's notation $\gg$ is positive and depends only on the dimension K.)

Before verifying (2.9) we explain how it will be used to prove the theorem. As in Schmidt [12] we shall apply the following trivial observation:

$$\text{if } B \subset U^K \text{ satisfies } 0 < \frac{\delta}{n} < \mu(B) < \frac{1-\delta}{n}, \quad \text{then } B \text{ has "error" } \left| \sum_{\mathbf{z}_j \in B} 1 - n \cdot \mu(B) \right| > \delta. \tag{2.10}$$

Let $q = \frac{1}{6} n^{-1/K}$. Combining (2.10), (2.2) and (2.5) we see

$$\Omega_0(q) \gg 1. \tag{2.11}$$

Next let $p = \frac{1}{4} K^{-1/2}$. From (2.9), (2.8) and (2.11) it follows that

$$\Omega_0(p) = \int_{\mathbf{R}^K} \omega_p(\mathbf{t}) \cdot |\varphi(\mathbf{t})|^2 \, d\mathbf{t} \gg (n^{1/K})^{K-1} \int_{\mathbf{R}^K} \omega_q(\mathbf{t}) \cdot |\varphi(\mathbf{t})|^2 \, d\mathbf{t} = n^{(K-1)/K} \cdot \Omega_0(q) \gg n^{1-1/K}. \tag{2.12}$$

Therefore, by (2.12), (2.5) and (2.2) we obtain the existence of a cube A in arbitrary position such that the diameter of A is less than one and

$$\left| \sum_{\mathbf{z}_j \in A} 1 - n \cdot \mu(A \cap U^K) \right| \gg n^{1/2 - 1/2K},$$

which completes the proof of Theorem 3, provided that (2.9) is true.

It remains to check the assumption (2.9). It needs only elementary estimations. By definition,

$$\hat{\chi}_{\tau, r}(\mathbf{t}) = \hat{\chi}_r(\tau^{-1} \mathbf{t}) \tag{2.13}$$

where χ_r denotes the characteristic function of the cube $[-r, r]^K$. So it is sufficient to study the function $\hat{\chi}_r(\mathbf{u})$, $\mathbf{u} \in \mathbf{R}^K$. From the definition it follows via elementary calculation that

$$\hat{\chi}_r(\mathbf{u}) = \prod_{j=1}^{K} \frac{2 \sin(r \cdot u_j)}{(2\pi)^{1/2} \cdot u_j} \quad \text{where} \quad \mathbf{u} = (u_1, \ldots, u_K).$$

Since

$$\left|\frac{\sin(r\cdot u)}{u}\right| > \frac{r}{2} \quad \text{if} \quad |u| < \frac{1}{r},$$

we get

$$\frac{1}{q}\int_q^{2q} |\hat{\chi}_r(\mathbf{u})|^2\,dr \gg \left(q^{K-1}\cdot \min\left\{q, \frac{1}{|\mathbf{u}|}\right\}\right)^2 \quad \text{if} \quad |u_1| < \frac{1}{q}, |u_2| < \frac{1}{q}, \dots, |u_{K-1}| < \frac{1}{q}. \tag{2.14}$$

Here $|\mathbf{u}|$ denotes the usual Euclidean length, i.e., $|\mathbf{u}|=(\sum_{j=1}^K u_j^2)^{1/2}$. Let

$$V\left(\frac{1}{q}, K-1\right) = \left\{\mathbf{u}=(u_1,\dots,u_K)\in\mathbf{R}^K\colon\ |u_j| < \frac{1}{q},\ 1\le j\le K-1\right\}, \tag{2.15}$$

and for any $\mathbf{t}\in\mathbf{R}^K$,

$$W\left(\mathbf{t}, \frac{1}{q}, K-1\right) = \left\{\tau\in T\colon \tau^{-1}\mathbf{t}\in V\left(\frac{1}{q}, K-1\right)\right\}. \tag{2.16}$$

Simple calculation shows

$$\int_{W\left(\mathbf{t}, \frac{1}{q}, K-1\right)} d\tau \gg \min\left\{1, \left(\frac{1}{q\cdot|\mathbf{t}|}\right)^{K-1}\right\}. \tag{2.17}$$

Combining, (2.13), (2.14), (2.15), (2.16) and (2.17) we conclude that

$$\begin{aligned}\omega_q(\mathbf{t}) &= \frac{1}{q}\int_q^{2q}\int_T |\hat{\chi}_{\tau,r}(\mathbf{t})|^2\,d\tau\,dr \\ &\gg \left(q^{K-1}\cdot\min\left\{q,\frac{1}{|\mathbf{t}|}\right\}\right)^2\cdot\min\left\{1,\left(\frac{1}{q\cdot|\mathbf{t}|}\right)^{K-1}\right\} \\ &= \min\left\{q^{2K}, \frac{q^{K-1}}{|\mathbf{t}|^{K+1}}\right\}.\end{aligned} \tag{2.18}$$

A similar calculation shows the validity of the opposite inequality

$$\omega_q(\mathbf{t}) \ll \min\left\{q^{2K}, \frac{q^{K-1}}{|\mathbf{t}|^{K+1}}\right\}. \tag{2.19}$$

Indeed, let $l_1\ge 1, \dots, l_{K-1}\ge 1$ be arbitrary two-powers, that is, let $l_j=2^{s_j}$, $s_j\ge 0$, $1\le j\le K-1$. It is easily seen that

$$\frac{1}{q}\int_q^{2q} |\hat{\chi}_r(\mathbf{u})|^2\,dr \gg\ll \left\{\left(\prod_{j=1}^{K-1}\frac{q}{l_j}\right)\cdot\min\left\{q,\frac{1}{|\mathbf{u}|}\right\}\right\}^2$$

if

$$\frac{[l_j/2]}{q} \leqslant |u_j| < \frac{l_j}{q}, \quad 1 \leqslant j \leqslant K-1.$$

Note that here $\mathbf{u}=(u_1, \dots, u_K)$, $[x]$ denotes the lower integral part, i.e., the largest integer $\leqslant x$, and the notation $f \gg\ll g$ means that both $f \ll g$ and $f \gg g$.

Let $\mathbf{l}=(l_1, \dots, l_{K-1})$,

$$V\left(\frac{1}{q};\mathbf{l}\right) = \left\{\mathbf{u}=(u_1,\dots,u_K)\in\mathbf{R}^K:\ |u_j| < \frac{l_j}{q},\ 1\leqslant j\leqslant K-1\right\}$$

and for any $\mathbf{t}\in\mathbf{R}^K$.

$$W\left(\mathbf{t},\frac{1}{q};\mathbf{l}\right) = \left\{\tau\in T;\ \tau^{-1}\mathbf{t}\in V\left(\frac{1}{q};\mathbf{l}\right)\right\}.$$

It is not hard to see that

$$\int_{W(\mathbf{t},\frac{1}{q};\mathbf{l})} d\tau \gg\ll \prod_{j=1}^{K-1} \min\left\{1, \frac{l_j}{q\cdot|\mathbf{t}|}\right\}.$$

Summarizing, we obtain (2.19) as follows

$$\begin{aligned}
\omega_q(\mathbf{t}) &= \frac{1}{q}\int_q^{2q}\int_T |\hat{\chi}_{\tau,r}(\mathbf{t})|^2\, d\tau\, dr \\
&\ll \sum_{S_1\geqslant 0}\dots\sum_{S_{K-1}\geqslant 0} 2^{(S_1+\dots+S_{K-1})}\cdot\min\left\{q^{2K}, \frac{q^{K-1}}{|\mathbf{t}|^{K+1}}\right\} \\
&= \left(\sum_{S\geqslant 0} 2^{-S}\right)^{K-1}\cdot\min\left\{q^{2K}, \frac{q^{K-1}}{|\mathbf{t}|^{K+1}}\right\} \ll \min\left\{q^{2K}, \frac{q^{K-1}}{|\mathbf{t}|^{K+1}}\right\}.
\end{aligned}$$

From (2.18) and (2.19) the desired (2.9) follows immediately. Hence the proof is complete.

3. Proof of Theorem 1 A

First we renormalize Theorem 1 A as follows.

THEOREM 1B. *Let ε be a positive real number. Let there be given n points $\mathbf{z}_1, \dots, \mathbf{z}_n$ in the cube $[-M, M]^K$ where $M=\frac{1}{2}n^{1/K}$. If n is sufficiently large depending only on K and ε, then there exists a ball B contained in $[-M, M]^K$ such that*

$$\left|\sum_{\mathbf{z}_j\in B} 1-\mu(B)\right| > n^{1/2-1/2K-\varepsilon}$$

Proof of Theorem 1B. We recommend the reader to read first the proof of Theorem 3 in Section 2.

Throughout we assume that n is sufficiently large depending on K and $\varepsilon>0$ only. For notational convenience let $S=\{\mathbf{z}_1,\dots,\mathbf{z}_n\}$ and

$$Q(a)=[-a,a]^K, \quad a<0 \text{ real}.$$

Let m, $0<m\leq M=\frac{1}{2}n^{1/K}$ be a real parameter. Let

$$W_m(\mathbf{x})=\text{card}\,\{(U^K+\mathbf{x})\cap Q(m)\cap S\}.$$

We need

LEMMA 3.1. *For arbitrary sufficiently large n there exists a real m_0 with $M\cdot\exp\{-(\log n)^{2/3}\}\leq m_0\leq M/2$ such that either*

(i) $\text{card}\,\{Q(m_0)\cap S\}<\frac{1}{10}\cdot(2m_0)^K$,

or

(ii) $\text{card}\,\{Q(m_0)\cap S\}\geq\frac{1}{10}\cdot(2m_0)^K$ *and with $m_1=m_0/\log n$ we have*

$$\int_{\mathbf{R}^K}(w_{m_1}(\mathbf{x}))^2\,d\mathbf{x}>\exp\{-(\log n)^{2/3}\}\cdot\int_{\mathbf{R}^K}(w_{m_0}(\mathbf{x}))^2\,d\mathbf{x}.$$

Proof. Let $p_1=M/2$ and $p_{j+1}=p_j/\log n$, $j\geq 1$. Let

$$W_j=\int_{\mathbf{R}^K}(w_{p_j}(\mathbf{x}))^2\,d\mathbf{x}, \quad j\geq 1.$$

We may assume that for every $p_j\geq M\cdot\exp\{-(\log n)^{2/3}\}$

$$\text{card}\,\{Q(p_j)\cap S\}\geq\frac{1}{10}(2p_j)^K. \tag{3.1}$$

Indeed, if the opposite case (i) holds, we are done. From (3.1) we obtain via elementary estimations that for every $p_j\geq M\cdot\exp\{-(\log n)^{2/3}\}$

$$\frac{n}{(2\exp\{(\log n)^{2/3}\})^K} \leq (p_j)^K \ll W_j \leq n^2 \tag{3.2}$$

(we recall: $\ll$ is the Vinogradov's notation with positive implicit constants).

Now suppose, contrary to (ii), that

$$W_{j+1} \leq \exp\{-(\log n)^{2/3}\}\cdot W_j \quad \text{for all } j \text{ with } 1\leq j\leq l=(\log n)^{1/2}.$$

Then clearly

$$W_{l+1} \leq (\exp\{-(\log n)^{2/3}\})^l \cdot W_1,$$

and so by (3.2)

$$W_{l+1} \leq \exp\{-(\log n)^{7/6}\}\cdot W_1 \ll \frac{1}{n^2}\cdot W_1 \leq 1. \tag{3.3}$$

Let n be sufficiently large. Then (3.3) contradicts the second inequality in (3.2), since

$$p_{l+1} = p_1\cdot(\log n)^{-l} = \frac{1}{2}M\cdot(\log n)^{-1} \geq M\cdot\exp\{-(\log n)^{2/3}\}.$$

Lemma 3.1 follows.

If alternative (i) of Lemma 3.1 is true, then we are immediately done. Indeed, the cube $Q(m_0)$ contains less than 10% of the expected of the points $\mathbf{z}_j$, and by a standard averaging argument we get the existence of a ball B contained in $Q(m_0)$ with radius m_0/K such that $\Sigma_{\mathbf{z}_j\in B}1<\frac{1}{2}\mu(B)$. Thus B is certainly contained in $Q(M)=[-M,M]^K$ and has a huge "error"

$$\left|\sum_{\mathbf{z}_j\in B} 1-\mu(B)\right| \geq \frac{1}{2}\mu(B) \gg (m_0)^K \geq (M\cdot\exp\{-(\log n)^{2/3}\})^K$$
$$= n\cdot\left(\frac{1}{2}\exp\{-(\log n)^{2/3}\}\right)^K \geq n^{1-\varepsilon}.$$

Therefore, from now on we may assume the validity of alternative (ii) of Lemma 3.1.

We introduce two measures. For any $H\subset\mathbf{R}^K$ let

$$Z_0(H) = \mathrm{card}\,(S\cap H\cap Q(m_0))$$

where m_0 is defined to satisfy property (ii) in Lemma 3.1 (We recall: $S=\{\mathbf{z}_1,\dots,\mathbf{z}_n\}$.)

For any Lebesgue measurable $H\subset\mathbf{R}^K$ let

$$\mu_0(H) = \mu(H\cap Q(m_0)).$$

Let χ_r denote the characteristic function of the ball

$$B(\mathbf{0}, r) = \left\{ \mathbf{x} = (x_1, \ldots, x_K) \in \mathbf{R}^K : \sum_{j=1}^{K} x_j^2 \leq r^2 \right\}$$

centered at the origin and having radius r. The parameter r will be specified later.

Consider the function

$$F_r = \chi_r * (dZ_0 - d\mu_0) \tag{3.4}$$

where $*$ denotes the *convolution* operation. More explicitly,

$$\begin{aligned} F_r(\mathbf{x}) &= \int_{\mathbf{R}^K} \chi_r(\mathbf{x}-\mathbf{y})\,(dZ_0 - d\mu_0)(\mathbf{y}) \\ &= \operatorname{card}(S \cap B(\mathbf{x}, r) \cap Q(m_0)) - \mu(B(\mathbf{x}, r) \cap Q(m_0)), \end{aligned} \tag{3.5}$$

where $B(\mathbf{x}, r)$ denotes the translate $B(\mathbf{0}, r) + \mathbf{x}$ of $B(\mathbf{0}, r)$.

Let

$$E(\mathbf{x}) = \exp\{-|\mathbf{x}|^2 \cdot (m_1)^{-2}\} \tag{3.6}$$

where $|\mathbf{x}| = (\Sigma_{j=1}^{K} x_j^2)^{1/2}$ denotes the usual Euclidean distance and $= m_0/\log n$.

Consider now the following "truncated" version of F_r:

$$G_r = E \cdot F_r. \tag{3.7}$$

Clearly $G_r(\mathbf{x})$ is a good approximation of the "error function"

$$\begin{cases} \sum_{\mathbf{z}_j \in B(\mathbf{x}, r)} 1 - \mu(B(\mathbf{x}, r)) & \text{if } B(\mathbf{x}, r) \subset Q(M) \\ 0 & \text{otherwise,} \end{cases}$$

since the "weight" $E(\mathbf{x})$ is extremely small whenever $(B(\mathbf{x}, r)$ is *not* contained in $Q(m_0)$ (we mention in advance that $r < m_0/2$). In order to estimate the quadratic average of $G_r(\mathbf{x})$ we shall employ the theory of Fourier transformation. Besides identities (2.3) and (2.4) we need (see any textbook on harmonic analysis)

$$(f \cdot g)^\wedge = \hat{f} * \hat{g} \quad (f, g \in L^2(\mathbf{R}^K)). \tag{3.8}$$

By Parseval-Plancherel identity (2.4)

$$\int_{\mathbf{R}^K} (G_r(\mathbf{x}))^2 d\mathbf{x} = \int_{\mathbf{R}^K} |\hat{G}_r(\mathbf{t})|^2 d\mathbf{t}.$$

On combining (2.3), (3.4), (3.7) and (3.8), we conclude that

$$\hat{G}_r = \hat{E} * \hat{F}_r = \hat{E} * (\hat{\chi}_r \cdot (dZ_0 - d\mu_0)^\wedge). \tag{3.9}$$

Unfortunately, $\hat{G}_r$ has a rather difficult form, so we introduce the following auxiliary function

$$H_r = \chi_r * (E \cdot (dZ_0 - d\mu_0)), \tag{3.10}$$

that is,

$$\begin{aligned} H_r(\mathbf{x}) &= \int_{\mathbf{R}^K} \chi_r(\mathbf{x}-\mathbf{y}) (E(\mathbf{y}) \cdot dZ_0(\mathbf{y}) - E(\mathbf{y}) \cdot d\mu_0(\mathbf{y})) \\ &= \sum_{\mathbf{z}_j \in B(\mathbf{x}, r) \cap Q(m_0)} E(\mathbf{z}_j) - \int_{B(\mathbf{x}, r) \cap Q(m_0)} E(\mathbf{y})\, d\mathbf{y}. \end{aligned} \tag{3.11}$$

From (3.10), (2.3) and (3.8) we obtain

$$\hat{H}_r = \hat{\chi}_r \cdot (\hat{E} * (dZ_0 - d\mu_0)^\wedge). \tag{3.12}$$

For the sake of brevity, let

$$\varphi = (dZ_0 - d\mu_0)^\wedge \quad \text{and} \quad \psi = \hat{E} * (dZ_0 - d\mu_0)^\wedge = \hat{E} * \varphi. \tag{3.13}$$

Then, by (3.9), (3.10), (3.12) and (3.13)

$$\hat{G}_r = \hat{E} * (\hat{\chi}_r \cdot \varphi) \quad \text{and} \quad \hat{H}_r = \hat{\chi}_r \cdot \psi, \tag{3.14}$$

and we see that

$$\hat{H}_r(\mathbf{t}) - \hat{G}_r(\mathbf{t}) = \int_{\mathbf{R}^K} (\hat{\chi}_r(\mathbf{t}) - \hat{\chi}_r(\mathbf{t}-\mathbf{u})) \cdot \varphi(\mathbf{t}-\mathbf{u}) \cdot \hat{E}(\mathbf{u})\, d\mathbf{u}. \tag{3.15}$$

An outline of the proof of Theorem 1 B is as follows. Since $\hat{H}_r$ has the form of a simple product (i.e. $\hat{H}_r = \hat{\chi}_r \cdot \psi$), it is not hard to prove that the L^2-norm of $\hat{H}_r$ is "large". Moreover, we shall show that the difference $\hat{H}_r - \hat{G}_r$ is "predominantly small". Combining these arguments, we shall obtain a good lower bound to the L^2-norm of $\hat{G}_r$, or by Parseval-Plancherel identity, to the L^2-norm of the "truncated error function" $G_r(\mathbf{x})$.

We start with the investigation of the difference $\hat{H}_r(\mathbf{t})$-$\hat{G}_r(\mathbf{t})$. Using the following well-known result:

$$\text{if } f(x) = e^{-a^2x^2/2} \quad \text{then } \hat{f}(t) = \frac{1}{a} e^{-t^2/(2a^2)}, \tag{3.16}$$

by (3.6) we have

$$\hat{E}(\mathbf{u}) = \frac{(m_1)^K}{2^{K/2}} \cdot \exp\{-|\mathbf{u}|^2 \cdot (m_1)^2/4\}. \tag{3.17}$$

Clearly

$$|\varphi(\mathbf{t})| = |(dZ_0 - d\mu_0)^\wedge(\mathbf{t})| \leq \frac{1}{(2\pi)^{K/2}} \left\{ \left| \sum_{j=1}^{n} e^{-i\mathbf{z}_j \cdot \mathbf{t}} \right| + \int_{Q(M)} d\mathbf{x} \right\} \leq n,$$

and since the parameter r will be less than M,

$$|\hat{\chi}_r(\mathbf{t})| = \frac{1}{(2\pi)^{K/2}} \left| \int_{B(\mathbf{0},r)} e^{-i\mathbf{x}\cdot\mathbf{t}} d\mathbf{x} \right| \leq \frac{\mu(B(\mathbf{0},r))}{(2\pi)^{K/2}} \leq n.$$

Let $\delta_0 = (\log n)/m_1$. Then by (3.17)

$$\int_{\mathbf{R}^K \setminus Q(\delta_0)} \hat{E}(\mathbf{u})\, d\mathbf{u} \ll n^{-2}.$$

Using these upper estimates to (3.15) we see that

$$\begin{aligned} |\hat{H}_r(\mathbf{t}) - \hat{G}_r(\mathbf{t})| &\leq \left| \int_{Q(\delta_0)} (\hat{\chi}_r(\mathbf{t}) - \hat{\chi}_r(\mathbf{t}-\mathbf{u})) \cdot \varphi(\mathbf{t}-\mathbf{u}) \cdot \hat{E}(\mathbf{u})\, d\mathbf{u} \right| + c_{14}(K) \\ &\leq \max_{\mathbf{u} \in Q(\delta_0)} |\hat{\chi}_r(\mathbf{t}) - \hat{\chi}_r(\mathbf{t}-\mathbf{u})| \cdot \left| \int_{Q(\delta_0)} \varphi(\mathbf{t}-\mathbf{u}) \cdot \hat{E}(\mathbf{u})\, d\mathbf{u} \right| + c_{14}(K). \end{aligned} \tag{3.18}$$

We are going to study the Fourier transform of the characteristic function of the ball $\hat{\chi}_r(\mathbf{s})$, $\mathbf{s} \in \mathbf{R}^K$. For the sake of brevity, let $s = |\mathbf{s}|$. By definition

$$\begin{aligned} \hat{\chi}_r(\mathbf{s}) &= (2\pi)^{-K/2} \int_{\mathbf{R}^K} e^{-i\mathbf{x}\cdot\mathbf{s}} \cdot \chi_r(\mathbf{x})\, d\mathbf{x} = (2\pi)^{-K/2} \int_{B(\mathbf{0},r)} e^{-i\mathbf{x}\cdot\mathbf{s}} d\mathbf{x} \\ &= c_{15}(K) \int_{-r}^{r} e^{-iys} \cdot (r^2 - y^2)^{(K-1)/2} dy \\ &= c_{15}(K) \cdot r^K \cdot \int_{-1}^{1} \cos(s \cdot r \cdot h) \cdot (1-h^2)^{(K-1)/2} dh. \end{aligned} \tag{3.19}$$

The classical *Bessel-function* $J_\nu(x)$ (see e.g. in [8] p. 241) has the following integral representation (Poisson's integral)

$$J_\nu(x) = \frac{1}{\pi^{1/2} \cdot \Gamma(\nu + \frac{1}{2})} \cdot \left(\frac{x}{2}\right)^\nu \cdot \int_{-1}^{1} \cos(x \cdot h) \cdot (1-h^2)^{\nu - 1/2} dh \quad \left(\nu > -\frac{1}{2}\right) \tag{3.20}$$

Hence, by (3.19) and (3.20)

$$\hat{\chi}_r(\mathbf{s}) = c_{16}(K)\cdot\left(\frac{r}{s}\right)^{K/2} J_{K/2}(r\cdot s). \tag{3.21}$$

By Hankel's asymptotic expansion (see [8] p. 133)

$$J_\nu(x) = \left(\frac{2}{\pi x}\right)^{1/2}\cdot\cos\left(x-\frac{2\nu+1}{4}\pi\right)+O(x^{-3/2}) \tag{3.22}$$

where the implicit constant in the O-notation depends only on ν.

Therefore, by (3.21) and (3.22)

$$\hat{\chi}_r(\mathbf{s}) = c_{17}(K)\cdot\frac{r^{(K-1)/2}}{s^{(K+1)/2}}\cdot\cos(r\cdot s-(K+1)\pi/4)+O\left(\frac{r^{(K-3)/2}}{2^{(K+3)/2}}\right). \tag{3.23}$$

Here and in what follows the implicit constants in the O-notation depend only on the dimension K.

Combining (3.18) and (3.23) we obtain via elementary estimates

$$|\hat{H}_r(\mathbf{t})-\hat{G}_r(\mathbf{t})| \ll \left(\delta_0\cdot\left(\frac{r}{t}\right)^{(K+1)/2}+\frac{r^{(K-3)/2}}{t^{(K+3)/2}}\right)\cdot\left|\int_{Q(\delta_0)}\varphi(\mathbf{t}-\mathbf{u})\cdot\hat{E}(\mathbf{u})\,d\mathbf{u}\right|+O(1) \tag{3.24}$$

whenever $r\cdot t\geqslant 1$ (here $t=|\mathbf{t}|$). Since $\delta_0=(\log n)/m_1$, by (3.17) we have

$$\hat{E}(\mathbf{u}) \ll (\delta_0)^{-K}\cdot(\log n)^K \quad \text{for } \mathbf{u}\in Q(\delta_0).$$

Consequently,

$$\left|\int_{Q(\delta_0)}\varphi(\mathbf{t}-\mathbf{u})\cdot\hat{E}(\mathbf{u})\,d\mathbf{u}\right| \ll (\log n)^K\cdot(2\delta_0)^{-K}\cdot\left|\int_{Q(\delta_0)}\varphi(\mathbf{t}-\mathbf{u})\,d\mathbf{u}\right|. \tag{3.25}$$

By Cauchy-Schwarz inequality

$$(2\delta_0)^{-K}\left|\int_{Q(\delta_0)}\varphi(\mathbf{t}-\mathbf{u})\,d\mathbf{u}\right| \leqslant \left\{(2\delta_0)^{-K}\int_{Q(\delta_0)}|\varphi(\mathbf{t}-\mathbf{u})|^2 d\mathbf{u}\right\}^{1/2}. \tag{3.26}$$

Using the elementary inequality $(a+b)^2\leqslant 2a^2+2b^2$, a and b reals, (3.24), (3.25) and (3.26) we see

$$|\hat{H}_r(\mathbf{t})-\hat{G}_r(\mathbf{t})|^2 \ll \left((\delta_0)^2\cdot\left(\frac{r}{t}\right)^{K+1}+\frac{r^{K-3}}{t^{K+3}}\right)\cdot(\log n)^{2K}\cdot(2\delta_0)^{-K}\cdot\int_{Q(\delta_0)}|\varphi(\mathbf{t}-\mathbf{u})|^2 d\mathbf{u}$$
$$+O(1) \quad \text{whenever } r\cdot t\geqslant 1. \tag{3.27}$$

Let q, $0<q<\frac{1}{2}m_0\cdot(\log n)^{-3}$ be a real parameter to be fixed later. Let $T\subset\mathbf{R}^K$ be a Lebesgue measurable set such that the usual Euclidean distance of the origin $\mathbf{0}\in\mathbf{R}^K$ and T is greater than $1/q$ (note that in this section we do not use the group of proper orthogonal transformations, so this notation cannot cause any confusion).

Using the general inequality $a^2\geqslant\frac{1}{2}b^2-(a-b)^2$, a and b reals, (3.14) and (3.27) we have

$$\begin{aligned}\int_q^{2q}\left(\int_T|\hat{G}_r(\mathbf{t})|^2 d\mathbf{t}\right)dr &\geqslant \frac{1}{2}\int_q^{2q}\left(\int_T|\hat{H}_r(\mathbf{t})|^2 d\mathbf{t}\right)dr-\int_q^{2q}\left(\int_T|\hat{H}_r(\mathbf{t})-\hat{G}_r(\mathbf{t})|^2 d\mathbf{t}\right)dr\\ &\geqslant \frac{1}{2}\int_T\left(\int_q^{2q}|\hat{\chi}_r(\mathbf{t})|^2 dr\right)\cdot|\psi(\mathbf{t})|^2 d\mathbf{t}\\ &\quad -\text{const}\cdot\int_q^{2q}\left(\int_T\left\{\left((\delta_0)^2\left(\frac{r}{t}\right)^{K+1}+\frac{r^{K-3}}{t^{K+3}}\right)\cdot(\log n)^{2K}(2\delta_0)^{-K}\right.\right.\\ &\quad\times\left.\left.\int_{Q(\delta_0)}|\varphi(\mathbf{t}-\mathbf{u})|^2 d\mathbf{u}\right\}d\mathbf{t}\,dr-\text{const}\cdot\int_q^{2q}\left(\int_T d\mathbf{t}\right)dr.\end{aligned}\tag{3.28}$$

Note that by definition $r\cdot t\geqslant q\cdot\inf_{\mathbf{t}\in T}|t|>1$, and const stands for positive absolute constants depending only on the dimension K.

Next we need two lemmas concerning $\varphi(\mathbf{t})$ and $\psi(\mathbf{t})$. We recall:

$$W_m(\mathbf{x})=\text{card}\,\{(U^K+\mathbf{x})\cap Q(m)\cap S\},\quad S=\{\mathbf{z}_1,\ldots,\mathbf{z}_n\}$$

and $m_1=m_0/\log n$.

LEMMA 3.2A. $\int_{Q(100)}|\psi(\mathbf{t})|^2 d\mathbf{t}\gg\int_{\mathbf{R}^K}\{w_{m_1}(\mathbf{x})\}^2 d\mathbf{x}$.

For any real b, $0<b<200$, let

$$D_b(\mathbf{x})=\text{card}\left\{\left(Q\left(\frac{1}{b}\right)+\mathbf{x}\right)\cap Q(m_0)\cap S\right\}-\mu\left(\left(Q\left(\frac{1}{b}\right)+\mathbf{x}\right)\cap Q(m_0)\right).\tag{3.29}$$

Moreover, let

$$\Delta_b(\mathbf{x})=\sum_{\mathbf{z}_j\in(Q(1/b)+\mathbf{x})\cap Q(m_0)}E(\mathbf{x}-\mathbf{z}_j)-\int_{Q(1/b)+\mathbf{x}}E(\mathbf{x}-\mathbf{y})\,d\mu_0(\mathbf{y}).\tag{3.30}$$

LEMMA 3.3. *For any* $0<b\leqslant 200$

(i) $$\int_{Q(b)}|\varphi(\mathbf{t})|^2 d\mathbf{t}\ll\int_{\mathbf{R}^K}\{D_b(\mathbf{x})\}^2\cdot\left(\frac{b}{2}\right)^{2K}d\mathbf{x}$$

and

(ii) $$\int_{Q(b)} |\psi(\mathbf{t})|^2 d\mathbf{t} \ll \int_{\mathbf{R}^K} \{\Delta_b(\mathbf{x})\}^2 \cdot \left(\frac{b}{2}\right)^{2K} d\mathbf{x}.$$

We postpone the proof of these lemmas to the end of this section.

Combining Lemma 3.1 (ii) and Lemma 3.2 A we get

$$\int_{Q(100)} |\psi(\mathbf{t})|^2 d\mathbf{t} \gg \int_{\mathbf{R}^K} \{w_{m_1}(\mathbf{x})\}^2 d\mathbf{x} > \exp\{-(\log n)^{2/3}\} \cdot \int_{\mathbf{R}^K} \{w_{m_0}(\mathbf{x})\}^2 d\mathbf{x}. \quad (3.31)$$

By Lemma 3.1 (ii) we see

$$\int_{\mathbf{R}^K} \{w_{m_0}(\mathbf{x})\}^2 d\mathbf{x} \geq \int_{\mathbf{R}^K} w_{m_0}(\mathbf{x})\, d\mathbf{x} = \operatorname{card}\{Q(m_0) \cap S\} > \frac{1}{10} \cdot (2m_0)^K. \quad (3.32)$$

Clearly

$$w_{m_0}(\tilde{\mathbf{x}}) + \mu_0\left(Q\left(\frac{1}{200}\right) + \mathbf{x}\right) \geq |D_{200}(\mathbf{x})|$$

where $\tilde{\mathbf{x}} = (x_1 - \frac{1}{2}, x_2 - \frac{1}{2}, \ldots, x_K - \frac{1}{2})$, $\mathbf{x} = (x_1, \ldots, x_K)$, therefore, using the general inequality $(a+b)^2 \leq 2a^2 + 2b^2$, we see

$$2\int_{\mathbf{R}^K} (w_{m_0}(\mathbf{x}))^2 d\mathbf{x} + 2\int_{\mathbf{R}^K} \left(\mu_0\left(Q\left(\frac{1}{200}\right) + \mathbf{x}\right)\right)^2 d\mathbf{x} \geq \int_{\mathbf{R}^K} \{D_{200}(\mathbf{x})\}^2 d\mathbf{x}. \quad (3.33)$$

Since

$$\int_{\mathbf{R}^K} \left(\mu_0\left(Q\left(\frac{1}{200}\right) + \mathbf{x}\right)\right)^2 d\mathbf{x} = \left(\frac{1}{100}\right)^K \cdot (2m_0)^K, \quad (3.34)$$

from (3.32), (3.33) and (3.34) it follows that

$$\int_{\mathbf{R}^K} (w_{m_0}(\mathbf{x}))^2 d\mathbf{x} \gg \int_{\mathbf{R}^K} (D_{200}(\mathbf{x}))^2 d\mathbf{x}. \quad (3.35)$$

Combining (3.31), (3.35) and Lemma 3.3 (i) we obtain

$$\int_{Q(100)} |\psi(\mathbf{t})|^2 d\mathbf{t} \gg \exp\{-(\log n)^{2/3}\} \cdot \int_{Q(200)} |\varphi(\mathbf{t})|^2 d\mathbf{t}. \quad (3.36)$$

Moreover, from (3.31) and (3.32) we have

$$\int_{Q(100)} |\psi(\mathbf{t})|^2 d\mathbf{t} \gg \exp\{-(\log n)^{2/3}\} \cdot (m_0)^K. \quad (3.37)$$

Now let $\eta>0$ be arbitrarily small but fixed. We distinguish two cases (I) and (II).

(I) $$\int_{Q(a_0)} |\psi(\mathbf{t})|^2 d\mathbf{t} < \frac{1}{2}\int_{Q(100)} |\psi(\mathbf{t})|^2 d\mathbf{t} \quad \text{where} \quad a_0 = n^{\eta}\cdot(m_0)^{-1}.$$

Then by (3.36)

$$\int_{Q(100)\setminus Q(a_0)} |\psi(\mathbf{t})|^2 d\mathbf{t} \gg \exp\{-(\log n)^{2/3}\}\cdot \int_{Q(200)} |\varphi(\mathbf{t})|^2 d\mathbf{t}. \tag{3.38}$$

From (3.38) it follows that there exists a real b_0 with $a_0 \leq b_0 \leq 50$ such that

$$\int_{T_0} |\psi(\mathbf{t})|^2 d\mathbf{t} \gg \frac{1}{\log n}\int_{Q(100)\setminus Q(a_0)} |\psi(\mathbf{t})|^2 d\mathbf{t} \quad \text{where} \quad T_0 = Q(2b_0)\setminus Q(b_0). \tag{3.39}$$

Let $q = m_0\cdot n^{-\eta/2} = n^{\eta/2}\cdot\frac{1}{a_0}$. Then clearly

$$\inf_{\mathbf{t}\in T_0} |\mathbf{t}| = b_0 \geq a_0 > \frac{1}{q},$$

i.e., the Euclidean distance of the origin $\mathbf{0}\in\mathbf{R}^K$ and T_0 is greater than $1/q$. By (3.28)

$$\begin{aligned}\int_q^{2q}\left(\int_{T_0} |\hat{G}_r(\mathbf{t})|^2 d\mathbf{t}\right) dr \geq \frac{1}{2}\int_{T_0}\left(\int_q^{2q} |\hat{\chi}_r(\mathbf{t})|^2 dr\right)\cdot|\psi(\mathbf{t})|^2 d\mathbf{t} \\ -\text{const}\cdot q\cdot\left\{(\delta_0)^2\cdot\left(\frac{q}{b_0}\right)^{K+1} + \frac{q^{K-3}}{b_0^{K+3}}\right\}\cdot(\log n)^{2K} \\ \times\int_{T_0}\left\{(2\delta_0)^{-K}\int_{Q(\delta_0)} |\varphi(\mathbf{t}-\mathbf{u})|^2 d\mathbf{u}\right\} d\mathbf{t} - \text{const}\cdot q\cdot\mu(T_0).\end{aligned} \tag{3.40}$$

Since

$$\delta_0 = (\log n)/m_1 = (\log n)^2/m_0 \leq (\log n)^2\cdot(\tfrac{1}{2}n^{1/K}\exp\{-(\log n)^{2/3}\})^{-1} \leq 100,$$

we have

$$\int_{T_0}\left\{(2\delta_0)^{-K}\int_{Q(\delta_0)} |\varphi(\mathbf{t}-\mathbf{u})|^2 d\mathbf{u}\right\} d\mathbf{t} \leq \int_{Q(200)} |\varphi(\mathbf{t})|^2 d\mathbf{t}. \tag{3.41}$$

Furthermore, by (3.23)

$$\inf_{\mathbf{t}\in T_0}\int_q^{2q} |\hat{\chi}_r(\mathbf{t})|^2 dr \gg q\cdot\frac{q^{K-1}}{b_0^{K+1}}. \tag{3.42}$$

We recall: $q=m_0\cdot n^{-\eta/2}$, $50\geqslant b_0\geqslant a_0=n^{\eta}\cdot(m_0)^{-1}$ and $\delta_0=(\log n)_2\cdot(m_0)^{-1}$. Combining (3.40), (3.41) and (3.42), via elementary calculations we have (n is sufficiently large)

$$\int_q^{2q}\left(\int_{T_0}|\hat{G}_r(\mathbf{t})|^2dt\right)dr\geqslant \text{const}\cdot\frac{q^K}{b_0^{K+1}}\cdot\int_{T_0}|\psi(\mathbf{t})|^2dt-n^{-\eta/2}\cdot\frac{q^K}{b_0^{K+1}}\cdot\int_{Q(200)}|\varphi(\mathbf{t})|^2dt-O(q). \tag{3.43}$$

From (3.43), (3.39) and (3.38) it follows that

$$\int_q^{2q}\left(\int_{T_0}|\hat{G}_r(\mathbf{t})|^2dt\right)dr\gg(1-O(n^{-\eta/4}))\cdot\frac{q^K}{b_0^{K+1}}\cdot\int_{T_0}|\psi(\mathbf{t})|^2dt-O(q). \tag{3.44}$$

By hypothesis (I), (3.37) and (3.39) we get

$$\int_{T_0}|\psi(\mathbf{t})|^2dt\gg\frac{1}{\log n}\cdot\exp\{-(\log n)^{2/3}\}\cdot(m_0)^K. \tag{3.45}$$

By (3.44) and (3.45) we see

$$\int_q^{2q}\left(\int_{T_0}|\hat{G}_r(\mathbf{t})|^2dt\right)dr\gg\frac{q^K}{b_0^{K+1}}\cdot(m_0)^{K-\varepsilon/2}\gg q^K\cdot(m_0)^{K-\varepsilon/2}\gg q\cdot(m_0)^{2K-1-\varepsilon}. \tag{3.46}$$

Now we are in the position to complete case (I). By Parseval-Plancherel identity (2.4) and (3.46)

$$\begin{aligned}\int_q^{2q}\left(\int_{\mathbf{R}^K}(G_r(\mathbf{x}))^2d\mathbf{x}\right)dr&=\int_q^{2q}\left(\int_{\mathbf{R}^K}|\hat{G}_r(\mathbf{t})|^2dt\right)dr\\&\geqslant\int_q^{2q}\left(\int_{T_0}|\hat{G}_r(\mathbf{t})|^2dt\right)dr\gg q\cdot(m_0)^{2K-1-\varepsilon}.\end{aligned} \tag{3.47}$$

We recall: $G_r=E\cdot F_r$ where

$$E(\mathbf{x})=\exp\{-|\mathbf{x}|^2\cdot(m_1)^{-2}\},\quad m_1=m_0/\log n$$

and

$$F_r(\mathbf{x})=\text{card}\,\{S\cap B(\mathbf{x},r)\cap Q(m_0)\}-\mu(B(\mathbf{x},r)\cap Q(m_0)).$$

Clearly

$$\begin{aligned}&E(\mathbf{x})\leqslant n^{-\text{const}\cdot\log n}\quad\text{whenever}\quad q\leqslant r\leqslant 2q\\&\text{and } B(\mathbf{x},r)=\{\mathbf{y}\in\mathbf{R}^K:|\mathbf{x}-\mathbf{y}|\leqslant r\}\not\subset Q(m_0)\end{aligned} \tag{3.48}$$

Now from (3.47) and (3.48) we obtain the existence of a ball $B(\mathbf{x}_0, r_0)$ such that $B(\mathbf{x}_0, r_0)$ is contained in $Q(m_0) \subset Q(M)$, $q \leq r_0 \leq 2q$ and

$$\left| \sum_{\mathbf{z}_j \in B(\mathbf{x}_0, r_0)} 1 - \mu(B(\mathbf{x}_0, r_0)) \right| \gg (m_0)^{K-1-\varepsilon}.$$

Since $m_0 \geq \frac{1}{2} n^{1/K} \cdot \exp\{-(\log n)^{2/3}\}$, we conclude that

$$\left| \sum_{\mathbf{z}_j \in B(\mathbf{x}_0, r_0)} 1 - \mu(B(\mathbf{x}_0, r_0)) \right| > n^{1/2 - 1/2K - \varepsilon}$$

if n is sufficiently large depending only on K and $\varepsilon > 0$. This completes case (I).

(II) $$\int_{Q(a_0)} |\psi(\mathbf{t})|^2 d\mathbf{t} \geq \frac{1}{2} \int_{Q(100)} |\psi(\mathbf{t})|^2 d\mathbf{t} \quad \text{where} \quad a_0 = n^{\eta} \cdot (m_0)^{-1}.$$

This is the simpler case. From (3.37) and Lemma 3.3 (ii) we obtain

$$\begin{aligned} \left(\frac{2}{a_0}\right)^{-2K} \int_{\mathbf{R}^K} \{\Delta_{a_0}(\mathbf{x})\}^2 d\mathbf{x} \gg \int_{Q(a_0)} |\psi(\mathbf{t})|^2 d\mathbf{t} \geq \frac{1}{2} \int_{Q(100)} |\psi(\mathbf{t})|^2 d\mathbf{t} \\ \gg \exp\{-(\log n)^{2/3}\} \cdot (2m_0)^K. \end{aligned} \tag{3.49}$$

From (3.49) standard averaging arguments yield that either

(II_1) there is a vector $\mathbf{x}_1 \in \mathbf{R}^K$ such that the translate $Q(1/a_0) + \mathbf{x}_1$ of the cube $Q(1/a_0)$ is contained in $Q(m_0)$ and $|\Delta_{a_0}(\mathbf{x})| \gg \exp\{-\frac{1}{2}(\log n)^{2/3}\} \cdot (2/a_0)^K$,

or

(II_2) there is another vector $\mathbf{x}_2 \in \mathbf{R}^K$ such that $Q(1/a_0) + \mathbf{x}_2$ is *not* contained in $Q(m_0)$ but $|\Delta_{a0}(\mathbf{x}_2)| > 4 \cdot (2/a_0)^K$.

Since by (3.6)

$$1 \geq E(\mathbf{y}) \geq 1 - \text{const} \frac{(\log n)^2}{n^{2\eta}} \quad \text{whenever} \quad \mathbf{y} \in Q\left(\frac{1}{a_0}\right),$$

elementary calculations give (see (3.29) and (3.30))

$$|D_{a_0}(\mathbf{x}_j)| \geq \frac{1}{2} |\Delta_{a_0}(\mathbf{x}_j)| \begin{cases} \gg \exp\left\{-\frac{1}{2}(\log n)^{2/3}\right\} \cdot \left(\frac{2}{a_0}\right)^K & \text{for } j = 1, \text{ case } (\text{II}_1) \\ \geq \left(\frac{2}{a_0}\right)^K & \text{for } j = 2, \text{ case } (\text{II}_2). \end{cases} \tag{3.50}$$

If alternative (II_1) holds then let $A_0=Q(1/a_0)+\mathbf{x}_1$; if alternative ($II_2$) holds then let A_0 be a translate of $Q(1/a_0)$ such that $(Q(1/a_0)+\mathbf{x}_2)\cap Q(m_0)\subset A_0\subset Q(m_0)$. In the latter case by (3.50) and (3.29)

$$\text{card}\left(\left(Q\left(\frac{1}{a_0}\right)+\mathbf{x}_2\right)\cap Q(m_0)\cap S\right)\geq |D_{a_0}(\mathbf{x}_2)|\geq 2\cdot\left(\frac{2}{a_0}\right)^K=2\mu(A_0),$$

and so

$$\sum_{\mathbf{z}_j\in A_0} 1\geq 2\cdot\mu(A_0).$$

Consequently, we have

$$\left|\sum_{\mathbf{z}_j\in A_0} 1-\mu(A_0)\right|\begin{cases}\gg \exp\left\{-\frac{1}{2}(\log n)^{2/3}\right\}\cdot\mu(A_0) & \text{in case (II}_1)\\ \geq\mu(A_0) & \text{in case (II}_2).\end{cases} \tag{3.51}$$

Now using (3.51) one can complete case (II) as follows. Let $r=m\cdot n^{-2\eta}=(1/a_0)\cdot n^{-\eta}$. Again we distinguish two cases (α) and (β).

(α) $\Sigma_{\mathbf{z}_j\in A_0}1-\mu(A_0)>0$ in (3.51). Then by a standard averaging argument we conclude that either

(α_1) there is a ball $B(\mathbf{x}_3,r)$ contained in $A_0\subset A(m_0)$ with

$$\sum_{\mathbf{z}_j\in B(\mathbf{x}_3,r)} 1-\mu(B(\mathbf{x}_3,r))>\text{const}\cdot n^{-K\cdot\eta}\cdot\left(\sum_{\mathbf{z}_j\in A_0}1-\mu(A_0)\right),$$

or

(α_2) there is another ball $B(\mathbf{x}_4,r)$ such that

$$B(\mathbf{x}_4,r)\cap Q(m_0)\neq\emptyset \quad\text{and}\quad \sum_{\mathbf{z}_j\in B(\mathbf{x}_4,r)\cap Q(m_0)} 1>2\cdot\mu(B(\mathbf{x}_4,r)).$$

In the case (α_2), since $2r<m_0\leq\frac{1}{2}M$ we have that $B(\mathbf{x}_4,r)\subset Q(M)$ and

$$\sum_{\mathbf{z}_j\in B(\mathbf{x}_4,r)} 1-\mu(B(\mathbf{x}_4,r))>\mu(B(\mathbf{x}_4,r)).$$

Summarizing, there exists a ball $B\subset Q(M)=[-M,M]^K$ of radius $r=m_0\cdot n^{-2\eta}\geq \frac{1}{2}n^{1/K}\cdot\exp\{-(\log n)^{2/3}\}\cdot n^{-2\eta}$ such that the "error" $\Sigma_{\mathbf{z}_j\in B}1-\mu(B)$ is greater than $n^{1-\varepsilon}$ if $\eta>0$ is sufficiently small depending only on $\varepsilon>0$ and K.

(β) $\mu(A_0)-\Sigma_{\mathbf{z}_j\in A_0}1>0$ in (3.51). Then there is only one alternative: one can find a ball $B(\mathbf{x}_5,r)$ contained in A_0 with

$$\mu(B(\mathbf{x}_5,r))-\sum_{\mathbf{z}_j\in B(\mathbf{x}_5,r)}1>\text{const}\cdot n^{-K\cdot\eta}\cdot\left(\mu(A_0)-\sum_{\mathbf{z}_j\in A_0}1\right).$$

This completes case (II).

It remains to prove Lemma 3.2A and Lemma 3.3. For later application we prove the following slight generalization of Lemma 3.2A.

LEMMA 3.2B. *Let $\psi_0=\varphi$ and $\psi_1=\psi$. Then*

$$\int_{Q(100)}|\psi_i(\mathbf{t})|^2\,d\mathbf{t}>>\int_{\mathbf{R}^K}\{w_{m_i}(\mathbf{x})\}^2\,d\mathbf{x},\quad i=0,1.$$

Proof of Lemma 3.2B. Let

$$f(\mathbf{x})=\prod_{j=1}^{K}\left(\frac{2\sin(bx_j)}{(2\pi)^{1/2}x_j}\right)^2$$

where the real parameter $b>0$ will be fixed later. From the general identity (3.8) we see that the Fourier transform, $\hat{f}$ of f equals the convolution of the characteristic function of the cube $Q(b)=[-b,b]^K$ with itself, i.e.,

$$\hat{f}(\mathbf{t})=(\chi_{Q(b)}*\chi_{Q(b)})(\mathbf{t})=\prod_{j=1}^{K}(2b-|t_j|)^+$$

where $(y)^+=y$ if $y>0$ and 0 otherwise.

Let $E_0(\mathbf{x})\equiv1$ and $E_1(\mathbf{x})=E(\mathbf{x})$. Then from (2.3) we obtain that the Fourier transform of the convolution

$$g_i(\mathbf{x})=\int_{\mathbf{R}^K}f(\mathbf{x}-\mathbf{y})\,(E_i(\mathbf{y})\cdot dZ_0(\mathbf{y})-E_i(\mathbf{y})\cdot d\mu_0(\mathbf{y}))$$

equals $\hat{f}\cdot\psi_i$, $i=0,1$ (see also (3.8) and (3.13)). By Parseval-Plancherel identity (2.4), for $i=0,1$

$$\begin{aligned}\int_{\mathbf{R}^K}g_i^2(\mathbf{x})\,d\mathbf{x}&=\int_{\mathbf{R}^K}|\hat{f}(\mathbf{t})|^2\cdot|\psi_i(\mathbf{t})|^2\,d\mathbf{t}\\&=\int_{\mathbf{R}^K}\left(\prod_{j=1}^{K}(2b-|t_j|)^+\right)^2\cdot|\psi_i(\mathbf{t})|^2\,d\mathbf{t}\le(2b)^{2K}\cdot\int_{Q(2b)}|\psi_i(\mathbf{t})|^2\,d\mathbf{t}.\end{aligned}\tag{3.52}$$

On the other hand, an elementary calculation shows that if b is sufficiently large, $b=50$ say, then

$$f(\mathbf{x}-\mathbf{z}_j)\cdot E_i(\mathbf{z}_j)-\int_{\mathbf{R}^K} f(\mathbf{x}-\mathbf{y})\,d\mathbf{y} > c_{18}(K)>0$$
$$\text{whenever}\quad \mathbf{z}_j\in\left(Q\left(\frac{1}{b}\right)+\mathbf{x}\right)\cap Q(m_i),\ i=0,1. \tag{3.53}$$

Therefore, using the fact that $f(\mathbf{x})$ is a *positive* function (i.e., a *Fejér kernel*), with (3.53) we get $(i=0,1)$

$$|g_i(\mathbf{x})| = \left|\int_{\mathbf{R}^K} f(\mathbf{x}-\mathbf{y})\,(E_i(\mathbf{y})\cdot dZ_0(\mathbf{y})-E_i(\mathbf{y})\cdot d\mu_0(\mathbf{y}))\right|$$
$$= \left|\sum_{j=1}^{n} f(\mathbf{x}-\mathbf{z}_j)\cdot E_i(\mathbf{z}_j)-\int_{\mathbf{R}^K} f(\mathbf{x}-\mathbf{y})\,E_i(\mathbf{y})\,d\mu_0(\mathbf{y})\right| > c_{18}(K)\cdot S_i(\mathbf{x}) \tag{3.54}$$

where $S_i(\mathbf{x})$ denotes the number of points $\mathbf{z}_j$ which lie in $(Q(1/50)+\mathbf{x})\cap Q(m_i)$ (note that $b=50$).

It is easy to see that $(i=0,1)$

$$\int_{\mathbf{R}^K} (S_i(\mathbf{x}))^2\,d\mathbf{x} \gg \int_{\mathbf{R}^K} (w_{m_i}(\mathbf{x}))^2\,d\mathbf{x}. \tag{3.55}$$

Combining (3.52), (3.54) and (3.55) Lemma 3.2B follows.

Proof of Lemma 3.3. We prove only (i). The proof of (ii) goes along exactly the same lines as that of (i).

Let $h(\mathbf{x})$ denote the characteristic function of the cube $Q(1/b)$. Then

$$\hat{h}(\mathbf{t}) = \prod_{j=1}^{K} \frac{2\sin(x_j/b)}{(2\pi)^{1/2}\cdot x_j}.$$

It is easily seen that

$$\int_{Q(b)} |\varphi(\mathbf{t})|^2\,d\mathbf{t} \ll b^{2K}\cdot\int_{\mathbf{R}^K} |\hat{h}(\mathbf{t})|^2\cdot|\varphi(\mathbf{t})|^2\,d\mathbf{t}. \tag{3.56}$$

By (2.3) and (2.4)

$$\int_{\mathbf{R}^K} |\hat{h}(\mathbf{t})|^2\cdot|\varphi(\mathbf{t})|^2\,d\mathbf{t} = \int_{\mathbf{R}^K}\left(\int_{\mathbf{R}^K} h(\mathbf{x}-\mathbf{y})\,(dZ_0-d\mu_0)\,(\mathbf{y})\right)^2 d\mathbf{x}. \tag{3.57}$$

Observe that (see (3.29))

$$D_b(\mathbf{x}) = \int_{\mathbf{R}^K} h(\mathbf{x}-\mathbf{y})\,(dZ_0 - d\mu_0)\,(\mathbf{y}). \tag{3.58}$$

Combining (3.56), (3.57) and (3.58) we conclude that

$$\int_{Q(b)} |\varphi(\mathbf{t})|^2\,d\mathbf{t} \ll b^{2K} \cdot \int_{\mathbf{R}^K} (D_b(\mathbf{x}))^2\,d\mathbf{x},$$

as required. Thus the proof of Theorem 1B is complete. Theorem 1A, being equivalent to Theorem 1B, follows immediately.

4. Proof of Theorem 2A

For notational convenience let $Q(a)$ denote the cube $[-a, a]^K$, $a>0$ real.

Let $M>0$ be a parameter to be fixed later.

We recall: $S=\{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \ldots\}$ is the given infinite discrete subset of $\mathbf{R}^K$.

We introduce two measures. For any $E\subset\mathbf{R}^K$ let

$$Z_0(E) = \operatorname{card}(S \cap E \cap Q(M)),$$

i.e., Z_0 denotes the counting measure generated by the discrete set $S\cap Q(M)$.

For any Lebesgue measurable set $E\subset\mathbf{R}^K$ let

$$\mu_0(E) = \mu(E \cap Q(M)),$$

i.e., μ_0 denotes the restriction of the usual K-dimensional volume to the cube $Q(M)$.

Let $\chi_{\tau,\alpha}$ denote the characteristic function of the set

$$A(\tau, \alpha, \mathbf{0}) = \{\alpha(\tau\mathbf{x}) : \mathbf{x} \in A\},$$

where A is the given compact convex body, τ is a proper orthogonal transformation and $\alpha \in (0, 1]$ is a real number.

Consider now the function

$$F_{\tau,\alpha} = \chi_{\tau,\alpha} * (dZ_0 - d\mu_0) \tag{4.1}$$

where $*$ denotes the *convolution* operation. More explicitly,

$$\begin{aligned} F_{\tau,\alpha}(\mathbf{x}) &= \int_{\mathbf{R}^K} \chi_{\tau,\alpha}(\mathbf{x}-\mathbf{y})(dZ_0 - d\mu_0)\,(\mathbf{y}) \\ &= \operatorname{card}(S \cap A(\tau, \alpha, \mathbf{x}) \cap Q(M)) - \mu(A(\tau, \alpha, \mathbf{x}) \cap Q(M)), \end{aligned} \tag{4.2}$$

where $A(\tau, \alpha, \mathbf{x}) = A(\tau, \alpha, \mathbf{0}) + \mathbf{x}$ is the translate of $A(\tau, \alpha, \mathbf{0})$. Therefore,

$$\text{if} \quad A(\tau, \alpha, \mathbf{x}) \subset Q(M) \quad \text{then} \quad F_{\tau,\alpha}(\mathbf{x}) = \sum_{\mathbf{z}_j \in A(\tau, \alpha, \mathbf{x})} 1 - \mu(A(\tau, \alpha, \mathbf{x})). \tag{4.3}$$

By the Parseval-Plancherel identity (2.4)

$$\int_{\mathbf{R}^K} (F_{\tau,\alpha}(\mathbf{x}))^2 \, d\mathbf{x} = \int_{\mathbf{R}^K} |\hat{F}_{\tau,\alpha}(\mathbf{t})|^2 \, d\mathbf{t} \tag{4.4}$$

where $\hat{F}_{\tau,\alpha}$ denotes the Fourier transform of $F_{\tau,\alpha}$. By (2.3) and (4.1)

$$\hat{F}_{\tau,\alpha} = (\chi_{\tau,\alpha} * (dZ_0 - d\mu_0))\hat{} = \hat{\chi}_{\tau,\alpha} \cdot (dZ_0 - d\mu_0)\hat{}\,, \tag{4.5}$$

and so by (4.4)

$$\int_T \int_0^1 \int_{\mathbf{R}^K} (F_{\tau,\alpha}(\mathbf{x}))^2 \, d\mathbf{x} \, d\alpha \, d\tau = \int_{\mathbf{R}^K} \left(\int_T \int_0^1 |\hat{\chi}_{\tau,\alpha}(\mathbf{t})|^2 \, d\alpha \, d\tau \right) \cdot |(dZ_0 - d\mu_0)\hat{}\,(\mathbf{t})|^2 \, d\mathbf{t} \tag{4.6}$$

where T is the group of proper orthogonal transformations in $\mathbf{R}^K$ and $d\tau$ is the volume element of the invariant measure on T, normalized such that $\int_T d\tau = 1$.

We mention in advance that $M \geqq 100 \cdot \operatorname{diam}(A)$ where diam stands for diameter. Thus we may assume that

$$\frac{1}{2} \cdot (2M)^K < \operatorname{card}(S \cap Q(M) < 2 \cdot (2M)^K. \tag{4.7}$$

Indeed, in the opposite case we are immediately done via standard averaging arguments.

In what follows we shall employ both the $\gg$ notation and the O-notation, with constants which may depend on the dimension K only.

Let $\varphi = (dZ_0 - d\mu_0)\hat{}$. We need

LEMMA 4.1. $\int_{Q(100)} |\varphi(\mathbf{t})|^2 \, d\mathbf{t} \gg M^K$.

Proof. Comparing the definitions of the measures Z_0 and μ_0 in Sections 3 and 4, we see that parameter M in Section 4 plays the same role as that of parameter m_0 in Section 3. Thus by Lemma 3.2B with $i=0$ we have

$$\int_{Q(100)} |\varphi(\mathbf{t})|^2 \, d\mathbf{t} \gg \int_{\mathbf{R}^K} (w_M(\mathbf{x}))^2 \, d\mathbf{x}$$

where $w_M(\mathbf{x})=\text{card}((U^K+\mathbf{x})\cap Q(M)\cap S)$. Moreover, by (4.7)

$$\int_{\mathbf{R}^K}(w_M(\mathbf{x}))^2\,d\mathbf{x}\geq\int_{\mathbf{R}^K}(w_M(\mathbf{x}))\,d\mathbf{x}=\text{card}\,(Q(M)\cap S)>\frac{1}{2}\cdot(2M)^K.$$

Summarizing,

$$\int_{Q(100)}|\varphi(\mathbf{t})|^2\,d\mathbf{t}>>\frac{1}{2}\cdot(2M)^K,$$

which proves Lemma 4.1.

Clearly (see also (4.7))

$$\begin{aligned}|\varphi(\mathbf{t})|&=(2\pi)^{-K/2}\cdot\left|\int_{\mathbf{R}^K}e^{-i\mathbf{x}\cdot\mathbf{t}}\cdot(dZ_0-d\mu_0)(\mathbf{x})\right|\\&\leq(2\pi)^{-K/2}\cdot\{\text{card}\,(Q(M)\cap S)+\mu(Q(M))\}<<M^K.\end{aligned}\tag{4.8}$$

Therefore, if $c_{19}(K)$ is a sufficiently small positive constant, with Lemma 4.1 and (4.8) we obtain

$$\int_{Q(c_{19}(K)/M)}|\varphi(\mathbf{t})|^2\,d\mathbf{t}<\frac{1}{2}\int_{Q(100)}|\varphi(\mathbf{t})|^2\,d\mathbf{t}.\tag{4.9}$$

From Lemma 4.1 and (4.9) it follows that there is an integer m satisfying $1\leq m\leq O(\log M)$ such that

$$\int_{Q_m}|\varphi(\mathbf{t})|^2\,d\mathbf{t}>>\frac{M^K}{m^2}\quad\text{where}\quad Q_m=Q(200\cdot 2^{-m})\setminus Q(100\cdot 2^{-m}).\tag{4.10}$$

Here is the outline of the proof of Theorem 2A. By (4.6) and (4.10) it suffices to give a suitable lower bound to

$$\inf_{\mathbf{t}\in Q_m}\int_0^1\int_T|\hat{\chi}_{\tau,a}(\mathbf{t})|^2\,d\tau\,da.\tag{4.11}$$

Obviously

$$\hat{\chi}_{\tau,a}(\mathbf{t})=a^K\cdot\hat{\chi}_A(\tau^{-1}(a\mathbf{t}))$$

where χ_A denotes the characteristic function of $A\subset\mathbf{R}^K$ and τ^{-1} is the inverse rotation. Let $G(\mathbf{x})$, $\mathbf{x}\in\mathbf{R}^K$ be a function satisfying

$$|\hat{G}(\mathbf{t})|\begin{cases}=0, & \text{for } \mathbf{t}\in Q(\delta\cdot 2^{-m})\\ \leqslant 2, & \text{for } \mathbf{t}\in \tilde{Q}=Q(100\cdot 2^{-m}\cdot K^{-1/2})\setminus Q(\delta\cdot 2^{-m})\\ =0, & \text{for } \mathbf{t}\notin Q(100\cdot 2^{-m}\cdot K^{-1/2})\end{cases}$$

where $\delta>0$ is a sufficiently small constant depending only on K. Then for every $\mathbf{t}^*\in Q_m$ (we recall: $\tilde{Q}=Q(100\cdot 2^{-m}\cdot K^{-1/2})\setminus Q(\delta\cdot 2^{-m})$)

$$\begin{aligned}\int_0^1\int_T |\hat{\chi}_{\tau,\alpha}(\mathbf{t}^*)|^2\,d\tau\,d\alpha &= \int_0^1 \alpha^{2K}\left(\int_T |\hat{\chi}_A(\tau^{-1}(\alpha\mathbf{t}^*))|^2\,d\tau\right)d\alpha\\ &= \int_0^1 \alpha^{2K}\cdot(\sigma(B(\mathbf{0},\alpha|\mathbf{t}^*|)))^{-1}\cdot\int_{|\mathbf{t}|=\alpha|\mathbf{t}^*|}|\hat{\chi}_A(\mathbf{t})|^2\,d\sigma(\mathbf{t})\,d\alpha\\ &= \frac{1}{|\mathbf{t}^*|}\int_0^{|\mathbf{t}^*|}\left(\frac{y}{|\mathbf{t}^*|}\right)^{2K}\cdot(\sigma(B(\mathbf{0},y)))^{-1}\int_{|\mathbf{t}|=y}|\hat{\chi}_A(\mathbf{t})|^2\,d\sigma(\mathbf{t})\,dy\\ &= \frac{1}{|\mathbf{t}^*|}\int_{B(\mathbf{0},|\mathbf{t}^*|)}\left(\frac{|\mathbf{t}|}{|\mathbf{t}^*|}\right)^{2K}\cdot(\sigma(B(\mathbf{0},|\mathbf{t}|)))^{-1}\cdot|\hat{\chi}_A(\mathbf{t})|^2\,d\mathbf{t}\\ &\geqslant \frac{1}{|\mathbf{t}^*|}\int_{\tilde{Q}}\left(\frac{|\mathbf{t}|}{|\mathbf{t}^*|}\right)^{2K}\cdot(\sigma(B(\mathbf{0},|\mathbf{t}|)))^{-1}\cdot|\hat{\chi}_A(\mathbf{t})|^2\,d\mathbf{t}\\ &{}_{\delta}\gg (\mu(\tilde{Q}))^{-1}\cdot\int_{\tilde{Q}}|\hat{\chi}_A(\mathbf{t})|^2\,d\mathbf{t}\\ &\gg (\mu(Q_m))^{-1}\cdot\int_{\mathbf{R}^K}|\hat{\chi}_A(\mathbf{t})|^2\cdot|\hat{G}(\mathbf{t})|^2\,d\mathbf{t},\end{aligned}$$

where σ denotes the $(K-1)$-dimensional surface area and $B(\mathbf{0},r)$ is the ball $\{\mathbf{x}\in\mathbf{R}^K: |\mathbf{x}|\leqslant r\}$.

Choosing $f=\chi_A * G$, by (2.3) and (2.4) we have

$$\int_{\mathbf{R}^K}|\hat{\chi}_A(\mathbf{t})|^2\cdot|\hat{G}(\mathbf{t})|^2\,d\mathbf{t}=\int_{\mathbf{R}^K}\left(\int_{\mathbf{R}^K}\chi_A(\mathbf{x}-\mathbf{y})\cdot G(\mathbf{y})\,d\mathbf{y}\right)^2 d\mathbf{x}.$$

Therefore, in order to give a lower bound to (4.11), it suffices to investigate the right-hand term of the last equality.

We shall construct the desired function G in the form of a difference $G=h-H$. The functions h and H will satisfy the following properties:

(i) $\int_{\mathbf{R}^K} h(\mathbf{y})\,d\mathbf{y}=\int_{\mathbf{R}^K} H(\mathbf{y})\,d\mathbf{y}=1,$

(ii) both functions h and H are "predominantly" positive.

Let $r(h)$ be the smallest radius such that

$$\int_{B(\mathbf{0},\, r(h))} h(\mathbf{y})\, d\mathbf{y} \geqslant \frac{99}{100},$$

and similarly, let $r(H)$ be the smallest radius such that

$$\int_{B(\mathbf{0},\, r(H))} H(\mathbf{y})\, d\mathbf{y} \geqslant \frac{99}{100}.$$

We also need the following property:

(iii) $r(H)$ is "much smaller" than $r(h)$. In other words, the integral of $H(\mathbf{y})$, $\mathbf{y} \in \mathbf{R}^K$ is "essentially concentrated" on a much smaller ball centered at the origin than that of $h(\mathbf{y})$, $\mathbf{y} \in \mathbf{R}^K$.

The geometric heuristics of the proof is as follows. Assume $\mathbf{x}_0 \in A$, and further that the Euclidean distance of $\mathbf{x}_0$ and the boundary ∂A of A is in the interval $[r(H), 2r(H)]$. Since the ball $B(\mathbf{x}_0, r(H)) = \{\mathbf{x}_0 - \mathbf{y}\colon |\mathbf{y}| \leqslant r(H)\}$ is *contained in* A, it is expected that

$$\int_{\mathbf{R}^K} \chi_A(\mathbf{x}_0 - \mathbf{y})\, H(\mathbf{y})\, d\mathbf{y} \geqslant \frac{9}{10}$$

(note that $H(\mathbf{z})$ is not necessarily positive everywhere on the set $\{\mathbf{x}_0 - \mathbf{z}\colon \mathbf{z} \in A\}$, but "predominantly" positive). On the other hand, the intersection $B(\mathbf{x}_0, r(h)) \cap A$ forms, roughly speaking, a half-ball, so it is rightly expected that the integral

$$\int_{\mathbf{R}^K} \chi_A(\mathbf{x}_0 - \mathbf{y})\, h(\mathbf{y})\, d\mathbf{y}$$

is also about the half of the integral

$$\int_{B(\mathbf{0},\, r(h))} h(\mathbf{y})\, d\mathbf{y} = \frac{99}{100},$$

i.e. about $\frac{1}{2}$. Summarizing, for these values of $\mathbf{x}_0$ we expect that the integral

$$\left| \int_{\mathbf{R}^K} \chi_A(\mathbf{x}_0 - \mathbf{y}) \cdot G(\mathbf{y})\, d\mathbf{y} \right| = \int_{\mathbf{R}^K} \chi_A(\mathbf{x}_0 - \mathbf{y})\, (H - h)(\mathbf{y})\, d\mathbf{y}$$

is greater than a positive absolute constant, which implies the desired lower bound to (4.11).

After the heuristics we give the explicit form of h and H. Let

$$h(\mathbf{x})=\frac{1}{(2\varepsilon_1)^K}\cdot\left\{\prod_{j=1}^{K}\frac{2\sin(\varepsilon_1 x_j)}{(2\pi)^{1/2}\cdot x_j}\right\}\cdot\left\{\prod_{j=1}^{K}\frac{2\sin(\varepsilon_2 x_j)}{(2\pi)^{1/2}\cdot x_j}\right\},\quad \mathbf{x}\in\mathbf{R}^K \tag{4.12}$$

and

$$H(\mathbf{x})=\frac{1}{(2\varepsilon_3)^K}\cdot\left\{\prod_{j=1}^{K}\frac{2\sin(\varepsilon_3 x_j)}{(2\pi)^{1/2}\cdot x_j}\right\}\cdot\left\{\prod_{j=1}^{K}\frac{2\sin(\varepsilon_4 x_j)}{(2\pi)^{1/2}\cdot x_j}\right\},\quad \mathbf{x}\in\mathbf{R}^K \tag{4.13}$$

where $0<\varepsilon 1<\varepsilon 2<\varepsilon<\varepsilon 4$ and $\varepsilon_2-\varepsilon_1=\varepsilon_4-\varepsilon_3$. These four parameters ε_i, $1\leqslant i\leqslant 4$ will be specified later.

For later application we list the basic properties of h and H (see (4.18), (4.19), (4.22)–(4.29), (4.31) and (4.32) below).

Since

$$\left(\frac{2\sin(bx)}{(2\pi)^{1/2}\cdot x}\right)^{\wedge}=\chi_{[-b,b]}(t) \tag{4.14}$$

where $\chi_{[-b,b]}$ denotes the characteristic function of the interval $[-b,b]$, by (3.8) we have

$$\hat{h}(\mathbf{t})=\frac{1}{(2\varepsilon_1)^K}\prod_{j=1}^{K}(\chi_{[-\varepsilon_1,\varepsilon_1]}*\chi_{[-\varepsilon_2,\varepsilon_2]})(t_j) \tag{4.15}$$

and

$$\hat{H}(\mathbf{t})=\frac{1}{(2\varepsilon_3)^K}\prod_{j=1}^{K}(\chi_{[-\varepsilon_3,\varepsilon_3]}*\chi_{[-\varepsilon_4,\varepsilon_4]})(t_j) \tag{4.16}$$

where $*$ denotes the convolution operation (note that h and H are de la Vallee Poussin type kernels).

If $0<a<b$ then obviously

$$(\chi_{[-a,a]}*\chi_{[-b,b]})(t)=\begin{cases}2a & \text{for } |t|\leqslant b-a\\ \leqslant 2a & \text{for } b-a\leqslant|t|\leqslant b+a.\\ 0 & \text{for } |t|\geqslant b+a.\end{cases} \tag{4.17}$$

Combining (4.15), (4.16) and (4.17) we see

$$|(\hat{h}-\hat{H})(\mathbf{t})|=\begin{cases}0 & \text{for } \mathbf{t}\in Q(\varepsilon_4-\varepsilon_3)=Q(\varepsilon_2-\varepsilon_1)\\ \leqslant 2 & \text{for } \mathbf{t}\in Q(\varepsilon_3+\varepsilon_4)\setminus Q(\varepsilon_4-\varepsilon_3).\\ 0 & \text{for } \mathbf{t}\notin Q(\varepsilon_3+\varepsilon_4).\end{cases} \tag{4.18}$$

Let ε_3 and ε_4 be defined by the equations

$$\left.\begin{array}{l}\varepsilon_3+\varepsilon_4=100\cdot 2^{-m}\cdot K^{-1/2} \text{ (see (4.10))}, \ \varepsilon_3=\left(1-\frac{1}{c^4}\right)\varepsilon_4. \\ \text{Moreover, let} \quad \varepsilon_2=\frac{1}{c^2}\cdot\varepsilon_4, \quad \varepsilon_1=\varepsilon_2-\frac{1}{c^4}\cdot\varepsilon_4=\left(1-\frac{1}{c^2}\right)\varepsilon_2\end{array}\right\}. \tag{4.19}$$

Here the parameter $c\geq 4$ will be specified later as a sufficiently large absolute constant depending only on K. Observe that $\varepsilon_2-\varepsilon_1=\varepsilon_4-\varepsilon_3$.

Choosing $f=\chi_A*(h-H)$, by Parseval-Plancherel identity (2.4) we have

$$\int_{\mathbf{R}^K}\left(\int_{\mathbf{R}^K}\chi_A(\mathbf{x}-\mathbf{y})\cdot(h-H)(\mathbf{y})\,d\mathbf{y}\right)^2 d\mathbf{x}=\int_{\mathbf{R}^K}|\hat{\chi}_A(\mathbf{t})|^2\cdot|(\hat{h}-\hat{H})(\mathbf{t})|^2\,d\mathbf{t} \tag{4.20}$$

where χ_A denotes the characteristic function of the given convex body A. By (4.18) we obtain

$$\int_{\mathbf{R}^K}|\hat{\chi}_A(\mathbf{t})|^2\cdot|(\hat{h}-\hat{H})(\mathbf{t})|^2\,d\mathbf{t}\leq 2\cdot\int_{Q(\varepsilon_3+\varepsilon_4)\setminus Q(\varepsilon_4-\varepsilon_3)}|\hat{\chi}_A(\mathbf{t})|^2\,d\mathbf{t}. \tag{4.21}$$

In order to give a lower bound to the right-hand term of (4.21) (and via this to (4.11)), it suffices to investigate the left-hand term of (4.20).

Using the well known general identity (see any textbook on harmonic analysis)

$$\int_{\mathbf{R}^K} f(\mathbf{x})\cdot\overline{g(\mathbf{x})}\,d\mathbf{x}=\int_{\mathbf{R}^K}\hat{f}(\mathbf{t})\cdot\overline{\hat{g}(\mathbf{t})}\,d\mathbf{t} \quad (f,g\in L^2(\mathbf{R}^K)),$$

from (4.12), (4.13) and (4.14) we get

$$\int_{\mathbf{R}^K} h(\mathbf{y})\,d\mathbf{y}=\int_{\mathbf{R}^K}\frac{1}{(2\varepsilon_1)^K}\cdot\chi_{Q(\varepsilon_1)}(\mathbf{t})\cdot\chi_{Q(\varepsilon_2)}(\mathbf{t})\,d\mathbf{t}=1, \tag{4.22}$$

and

$$\int_{\mathbf{R}^K} H(\mathbf{y})\,d\mathbf{y}=\int_{\mathbf{R}^K}\frac{1}{(2\varepsilon_3)^K}\cdot\chi_{Q(\varepsilon_3)}(\mathbf{t})\cdot\chi_{Q(\varepsilon_4)}(\mathbf{t})\,d\mathbf{t}=1. \tag{4.23}$$

Besides (4.18), (4.22) and (4.23) we shall also use the following properties of $h(\mathbf{y})$ and $H(\mathbf{y})$: if the parameter c is sufficiently large, then, roughly speaking, both functions h and H are positive "nearly everywhere", and the integral $\int_{\mathbf{R}^K} H(\mathbf{y})\,d\mathbf{y}$ of H is "essentially supported" by a much smaller ball centered at the origin than that of h. More explicitly, using (4.12), (4.13) and (4.19), via elementary estimates we obtain

$$\text{if} \quad \mathbf{y}\in Q\left(\frac{1}{\varepsilon_2}\right) \quad \text{then} \quad h(\mathbf{y}) \text{ is positive and } h(\mathbf{y}) \gg (\varepsilon_2)^K, \tag{4.24}$$

$$\text{if} \quad \mathbf{y}\in Q\left(\frac{1}{\varepsilon_4}\right) \quad \text{then} \quad H(\mathbf{y}) \text{ is positive and } H(\mathbf{y}) \gg (\varepsilon_4)^K, \tag{4.25}$$

$$\begin{gathered}\text{if} \quad \mathbf{y}\in \prod_{j=1}^{K}\left[\frac{l_j-\frac{1}{2}}{\varepsilon_2},\frac{l_j+\frac{1}{2}}{\varepsilon_2}\right] \quad \text{where} \quad \mathbf{l}=(l_1,\ldots,l_K)\in \mathbf{Z}^K \\ \text{then} \quad |h(\mathbf{y})| \ll (\varepsilon_2)^K\cdot \prod_{j=1}^{K}\frac{1}{(1+|l_j|)^2},\end{gathered} \tag{4.26}$$

$$\begin{gathered}\text{if} \quad \mathbf{y}\in \prod_{j=1}^{K}\left[\frac{l_j-\frac{1}{2}}{\varepsilon_4},\frac{l_j+\frac{1}{2}}{\varepsilon_4}\right] \quad \text{where} \quad \mathbf{l}=(l_1,\ldots,l_K)\in \mathbf{Z}^K \\ \text{then} \quad |H(\mathbf{y})| \ll (\varepsilon_4)^K\cdot \prod_{j=1}^{K}\frac{1}{(1+|l_j|)^2}.\end{gathered} \tag{4.27}$$

For notational convenience let

$$Q\left(\frac{1}{2\varepsilon_i};\mathbf{l}\right)=\prod_{j=1}^{K}\left[\frac{l_j-\frac{1}{2}}{\varepsilon_i},\frac{l_j+\frac{1}{2}}{\varepsilon_i}\right], \quad \mathbf{l}=(l_1,\ldots,l_K)\in \mathbf{Z}^K,\ i=1,\ldots,4.$$

Using (4.26) we have for any $\beta\geq 1$,

$$\begin{aligned}\int_{\mathbf{R}^K\setminus Q(\beta/\varepsilon_2)}|h(\mathbf{y})|\,dy &\leq \sum_{\substack{\mathbf{l}\in \mathbf{Z}^K:\\ \max_{1\leq j\leq K}|l_j|\geq\beta-\frac{1}{2}}}\int_{Q\left(\frac{1}{2\varepsilon_2};\mathbf{l}\right)}|h(\mathbf{y})|\,dy \ll \sum_{\substack{\mathbf{l}\in \mathbf{Z}^K:\\ \max_{1\leq j\leq K}|l_j|\geq\beta-\frac{1}{2}}}\prod_{j=1}^{K}\frac{1}{(1+|l_j|)^2}\\ &=\left(\sum_{l\in \mathbf{Z}}\frac{1}{(1+|l|)^2}\right)^K-\left(\sum_{\substack{l\in\mathbf{Z}:\\ |l|<\beta-\frac{1}{2}}}\frac{1}{(1+|l|)^2}\right)^K\\ &\ll \sum_{\substack{l\in\mathbf{Z}:\\ |l|\geq\beta-\frac{1}{2}}}\frac{1}{(1+|l|)^2}\ll\frac{1}{\beta}.\end{aligned} \tag{4.28}$$

Repeating the same argument we get

$$\int_{\mathbf{R}^K\setminus Q(\beta/\varepsilon_4)}|H(\mathbf{y})|\,dy \ll \frac{1}{\beta} \quad \text{for every real } \beta\geq 1. \tag{4.29}$$

We need the following elementary observation:

$$\text{if}\quad 0<a<b \text{ and } \sin(ax)\cdot\sin(bx)<0 \quad\text{then}\quad |\sin(ax)\cdot\sin(bx)|\le\frac{1}{4}(b-a)^2\cdot x^2. \tag{4.30}$$

Indeed, then for some d, $a<d<b$, $\sin(dx)=0$, and so we have that

$$|\sin(ax)\cdot\sin(bx)| = |\sin((d-a)x)\cdot\sin((b-d)x)|\le(d-a)(b-d)\cdot x^2\le\frac{1}{4}(b-a)^2\cdot x^2.$$

Let $\mathbf{y}\in Q(1/2\varepsilon_2;\mathbf{l})$ where $\mathbf{l}=(l_1,\dots,l_K)\in\mathbf{Z}^K$ with $|l_j|\le c+\frac{1}{2}$, $1\le j\le K$. If $h(\mathbf{y})<0$, then for some index ν, $1\le\nu\le K$, $\sin(\varepsilon_1\cdot y_\nu)\cdot\sin(\varepsilon_2\cdot y_\nu)<0$ (see (4.12)). Therefore, by (4.12), (4.19) and (4.30) we obtain that

$$\begin{aligned}|(h(\mathbf{y}))^-| &\ll \frac{|\sin(\varepsilon_1\cdot y_\nu)\cdot\sin(\varepsilon_2\cdot y_\nu)|}{\varepsilon_1\cdot y_\nu\cdot\varepsilon_2\cdot y_\nu}\cdot(\varepsilon_2)^K\cdot\prod_{\substack{j=1\\ j\ne\nu}}^{K}\frac{1}{(1+|l_j|)^2}\\ &\le\frac{1}{4}\frac{(\varepsilon_2-\varepsilon_1)^2}{\varepsilon_1\cdot\varepsilon_2}\cdot(\varepsilon_2)^K\cdot\prod_{\substack{j=1\\ j\ne\nu}}^{K}\frac{1}{(1+|l_j|)^2}\ll\left(\frac{1}{c^2}\right)^2\cdot(\varepsilon_2)^K\cdot\prod_{\substack{j=1\\ j\ne\nu}}^{K}\frac{1}{(1+|l_j|)^2}\\ &\ll\frac{1}{c^2}\cdot(\varepsilon_2)^K\cdot\prod_{j=1}^{K}\frac{1}{(1+|l_j|)^2}\end{aligned} \tag{4.31}$$

where $(h(\mathbf{y}))^-=h(\mathbf{y})$ for $h(\mathbf{y})<0$ and 0 otherwise. Similarly, if $\mathbf{y}\in Q(1/2\varepsilon_4;\mathbf{l})$ where $\mathbf{l}=(l_1,\dots,l_K)\in\mathbf{Z}^K$ with $|l_j|\le c+\frac{1}{2}$, $1\le j\le K$ and $H(\mathbf{y})<0$, then for some index ν, $1\le\nu\le K$, $\sin(\varepsilon_3\cdot y_\nu)\cdot\sin(\varepsilon_4\cdot y_\nu)<0$. Therefore, by (4.13), (4.19) and (4.30) we obtain that

$$\begin{aligned}|(H(\mathbf{y}))^- &\ll \frac{|\sin(\varepsilon_3\cdot y_\nu)\cdot\sin(\varepsilon_4\cdot y_\nu)|}{\varepsilon_3\cdot y_\nu\cdot\varepsilon_4\cdot y_\nu}\cdot(\varepsilon_4)^K\cdot\prod_{\substack{j=1\\ j\ne\nu}}^{K}\frac{1}{(1+|l_j|)^2}\\ &\le\frac{1}{4}\frac{(\varepsilon_4-\varepsilon_3)^2}{\varepsilon_3\cdot\varepsilon_4}(\varepsilon_4)^K\cdot\prod_{\substack{j=1\\ j\ne\nu}}^{K}\frac{1}{(1+|l_j|)^2}\ll\left(\frac{1}{c^4}\right)^2\cdot(\varepsilon_4)^K\cdot\prod_{\substack{j=1\\ j\ne\nu}}^{K}\frac{1}{(1+|l_j|)^2}\\ &\ll\frac{1}{c^6}\cdot(\varepsilon_4)^K\cdot\prod_{j=1}^{K}\frac{1}{(1+|l_j|)^2}.\end{aligned} \tag{4.32}$$

For any compact convex body $P\subset\mathbf{R}^K$ and real ϱ, $0<\varrho\le r(P)$ (we recall: $r(P)$ denotes the length of the radius of the largest inscribed ball in P), let $P^-[\varrho]$ be the set of all centres of balls of radius ϱ contained in P. It is obvious that $P^-[\varrho]$ is also compact and convex.

We shall apply the following result from discrete geometry (Hadwiger [7]): for any compact and convex body $A\subset\mathbf{R}^K$ there exist boxes in arbitrary position B and D with parallel edges such that

$$B\subset A\subset D \quad\text{and}\quad \frac{\mu(D)}{K!}\leq\mu(A)\leq K^K\cdot\mu(B). \tag{4.33}$$

Let b_j and d_j, $j=1,2,\dots,K$ denote the length of the parallel edges of B and D, respectively. Without loss of generality we may assume that

$$b_1\leq b_2\leq\dots\leq b_K. \tag{4.34}$$

From (4.33) it immediately follows that

$$b_j\leq d_j\leq K^K\cdot K!\cdot b_j,\quad 1\leq j\leq K. \tag{4.35}$$

Furthermore, we shall apply without proof the following well-known geometric fact:

$$\begin{gathered}\text{if } P_1 \text{ and } P_2 \text{ are compact convex bodies in } \mathbf{R}^K\\ \text{and } P_1\subset P_2 \text{ then } \sigma(\partial P_1)\leq\sigma(\partial P_2).\end{gathered} \tag{4.36}$$

Here ∂P denotes the boundary surface of P, and σ is the $(K-1)$-dimensional surface area.

Let $\varepsilon=c\cdot\varepsilon_2=(1/c)\cdot\varepsilon_4$. Now we are ready to estimate from below the left-hand term of (4.20). We distinguish two cases (I) and (II).

(I) $1/\varepsilon\leq b^1/8$. Let $A_1=A^-[1/\varepsilon]$ and $A_2=A_1\setminus A_1^-[1/\varepsilon]$. in order to estimate the volume of A_2, it suffices to use the following very crude lower bound.

Lemma 4.2. *If $P\subset\mathbf{R}^K$ is a compact convex body and $0<\varrho\leq r(P)$, then*

$$\mu(P\setminus P^-[\varrho])\geq c_{20}(K)\cdot\varrho\cdot\sigma(\partial P^-[\varrho]),$$

where the positive absolute constant $c_{20}(K)$ depends only on the dimension K.

Proof. Let

$$E=\partial P^-[\varrho],\quad F=B(\mathbf{0},\varrho)=\{\mathbf{y}\in\mathbf{R}^K\colon|\mathbf{y}|\leq\varrho\}\quad\text{and}\quad G=P\setminus P^-[\varrho].$$

Simple double integration argument gives

$$\int_E \mu((F+\mathbf{x})\cap G)\,d\sigma(\mathbf{x}) = \int_E\left(\int_G \chi_\varrho(\mathbf{y}-\mathbf{x})\,d\mathbf{y}\right)d\mathbf{x} = \int_G\left(\int_E \chi_\varrho(\mathbf{y}-\mathbf{x})\,d\mathbf{x}\right)d\mathbf{y}$$
$$= \int_G\left(\int_E \chi_\varrho(\mathbf{y}-\mathbf{x})\,d\mathbf{x}\right)d\mathbf{y} = \int_G \sigma((F+\mathbf{y})\cap E)\,d\mathbf{y} \tag{4.37}$$

where χ_ϱ denotes the characteristic function of the ball F. Since $P^-[\varrho]$ is convex, for any $\mathbf{x}\in E$ the intersection $(F+\mathbf{x})\cap G$ certainly contains a half-ball of radius ϱ. Hence

$$\mu((F+\mathbf{x})\cap G) \geqslant \frac{1}{2}\mu(F) \quad \text{for any } \mathbf{x}\in E. \tag{4.38}$$

On the other hand, by (4.36)

$$\sigma((F+\mathbf{y})\cap E) \leqslant \sigma(\partial F) \quad \text{for arbitrary } \mathbf{y}\in \mathbf{R}^K. \tag{4.39}$$

Combining (4.37), (4.38) and (4.39) we obtain

$$\sigma(E)\cdot\frac{1}{2}\mu(F) \leqslant \int_E \mu((F+\mathbf{x})\cap G)\,d\sigma(\mathbf{x}) = \int_G \sigma((F+\mathbf{y})\cap E)\,d\mathbf{y} \leqslant \mu(G)\cdot\sigma(\partial F),$$

and so

$$\mu(G) \geqslant \sigma(E)\cdot\frac{1}{2}\frac{\mu(F)}{\sigma(\partial F)} = \sigma(E)\cdot c_{20}(K)\cdot\varrho.$$

Lemma 4.2 follows.

By definition (see (4.33))

$$B = \tau_1\left(\prod_{j=1}^K\left[-\frac{b_j}{2},\frac{b_j}{2}\right]\right)+\mathbf{x}_1$$

with some appropriate orthogonal transformation τ_1 and translation $\mathbf{x}_1$. By hypothesis $2/\varepsilon\leqslant b_1/4\leqslant b_2/4\leqslant\ldots\leqslant b_K/4$, and so we have

$$A_1^-\left[\frac{1}{\varepsilon}\right] \supset \tau_1\left(\prod_{j=1}^K\left[-\frac{b_j}{4},\frac{b_j}{4}\right]\right)+\mathbf{x}_1. \tag{4.40}$$

By (4.36) and (4.40)

$$\sigma\left(\partial A_1^-\left[\frac{1}{\varepsilon}\right]\right) \gg \sigma(\partial B), \tag{4.41}$$

and by (4.35) and (4.36)

$$\sigma(\partial B) \gg \sigma(\partial D) \geq \sigma(\partial A). \tag{4.42}$$

That is, by (4.41) and (4.42)

$$\sigma\left(\partial A_1^-\left[\frac{1}{\varepsilon}\right]\right) \gg \sigma(\partial A). \tag{4.43}$$

By Lemma 4.2 and (4.43)

$$\mu(A_2) = \mu\left(A_1 \setminus A_1^-\left[\frac{1}{\varepsilon}\right]\right) \gg \frac{1}{\varepsilon} \cdot \sigma(\partial A). \tag{4.44}$$

After these preparations we are ready to realize the heuristics mentioned above. Let $\mathbf{x}_0 \in A_2$, and estimate the integral

$$\left| \int_{\mathbf{R}^K} \chi_A(\mathbf{x}_0 - \mathbf{y})\,(h-H)(\mathbf{y})\,d\mathbf{y} \right|$$

from below. Since $\mathbf{x}_0 \in A_1 = A^-[1/\varepsilon]$, we have

$$\left(Q\left(\frac{1}{K\cdot\varepsilon}\right) + \mathbf{x}_0\right) \subset A.$$

Therefore

$$\int_{\mathbf{R}^K} \chi_A(\mathbf{x}_0 - \mathbf{y})\,H(\mathbf{y})\,d\mathbf{y} \geq \int_{Q(\frac{1}{K\cdot\varepsilon})} H(\mathbf{y})\,d\mathbf{y} - \int_{\mathbf{R}^K \setminus Q(\frac{1}{K\cdot\varepsilon})} |H(\mathbf{y})|\,d\mathbf{y}. \tag{4.45}$$

By (4.23)

$$\int_{Q(\frac{1}{K\cdot\varepsilon})} H(\mathbf{y})\,d\mathbf{y} = 1 - \int_{\mathbf{R}^K \setminus Q(\frac{1}{K\cdot\varepsilon})} H(\mathbf{y})d\mathbf{y} \geq 1 - \int_{\mathbf{R}^K \setminus Q(\frac{1}{K\cdot\varepsilon})} |H(\mathbf{y})|\,d\mathbf{y}. \tag{4.46}$$

By (4.45) and (4.46)

$$\int_{\mathbf{R}^K} \chi_A(\mathbf{x}_0 - \mathbf{y})\,H(\mathbf{y})\,d\mathbf{y} \geq 1 - 2\int_{\mathbf{R}^K \setminus Q(\frac{1}{K\cdot\varepsilon})} |H(\mathbf{y})|\,d\mathbf{y}. \tag{4.47}$$

Since $\varepsilon = (1/c)\,\varepsilon_4$, from (4.29) it follows that

$$\int_{\mathbf{R}^K \setminus Q(\frac{1}{K\cdot\varepsilon})} |H(\mathbf{y})|\,d\mathbf{y} \ll \frac{1}{c},$$

thus by (4.47)

$$\int_{\mathbf{R}^K} \chi_A(\mathbf{x}_0-\mathbf{y})\,H(\mathbf{y})\,d\mathbf{y} \geq 1-O\left(\frac{1}{c}\right) \quad \text{for any } \mathbf{x}_0 \in A_2. \tag{4.48}$$

On the other hand, since $\mathbf{x}_0 \in A_2 = A_1 \setminus A_1^-[1/\varepsilon]$ where $A_1 = A^-[1/\varepsilon]$, we see that the usual Euclidean distance of $\mathbf{x}_0$ and the complement $\mathbf{R}^K \setminus A$ of A is $\leq 2/\varepsilon$. Using this observation and the convexity of A it is easily seen that

$$\left(Q\left(\frac{1}{\varepsilon_2}\right)+\mathbf{x}_0\right)\setminus A \quad \text{contains a ball of radius } \frac{1}{2\varepsilon_2}. \tag{4.49}$$

Here we also used that $\varepsilon = c\cdot\varepsilon_2$ and $c \geq 4$. Clearly

$$\int_{\mathbf{R}^K} \chi_A(\mathbf{x}_0-\mathbf{y})\,h(\mathbf{y})\,d\mathbf{y} \leq \int_{Q\left(\frac{1}{\varepsilon_2}\right)} \chi_A(\mathbf{x}_0-\mathbf{y})\,h(\mathbf{y})\,d\mathbf{y} + \int_{\mathbf{R}^K\setminus Q\left(\frac{1}{\varepsilon_2}\right)} h(\mathbf{y})\,d\mathbf{y} + \left|\int_{\mathbf{R}^K\setminus Q\left(\frac{1}{\varepsilon_2}\right)} (h(\mathbf{y}))^-\,d\mathbf{y}\right|$$

$$\text{where } (h(\mathbf{y}))^- = \begin{cases} h(\mathbf{y}) & \text{for } h(\mathbf{y})<0 \\ 0 & \text{otherwise.} \end{cases} \tag{4.50}$$

By (4.22), (4.24) and (4.49)

$$\begin{aligned}\int_{Q\left(\frac{1}{\varepsilon_2}\right)} \chi_A(\mathbf{x}_0-\mathbf{y})\,h(\mathbf{y})\,d\mathbf{y} + \int_{\mathbf{R}^K\setminus Q\left(\frac{1}{\varepsilon_2}\right)} h(\mathbf{y})\,d\mathbf{y} &= \int_{\mathbf{R}^K} h(\mathbf{y})\,d\mathbf{y} + \int_{Q\left(\frac{1}{\varepsilon_2}\right)} (\chi_A(\mathbf{x}_0-\mathbf{y})-1)\,h(\mathbf{y})\,d\mathbf{y} \\ &\leq 1-c_{21}(K)\cdot(\varepsilon_2)^K\cdot\mu\left(\left\{\mathbf{y}\in\mathbf{R}^K\colon |\mathbf{y}|\leq\frac{1}{2\varepsilon_2}\right\}\right) \\ &= 1-c_{22}(K)\end{aligned} \tag{4.51}$$

where $c_{22}(K)>0$.

Moreover,

$$\left|\int_{\mathbf{R}^K\setminus Q\left(\frac{1}{\varepsilon_2}\right)} h(\mathbf{y}))^-\,d\mathbf{y}\right| \leq \int_{\mathbf{R}^K\setminus Q\left(\frac{c}{\varepsilon_2}\right)} |h(\mathbf{y})|\,d\mathbf{y} + \sum_{\substack{l_1\in\mathbf{Z}:\\ |l_1|\leq c+\frac{1}{2}}} \cdots \sum_{\substack{l_K\in\mathbf{Z}:\\ |l_K|\leq c+\frac{1}{2}}} \left|\int_{Q\left(\frac{1}{2\varepsilon_2};\mathbf{l}\right)} (h(\mathbf{y}))^-\,d\mathbf{y}\right| \tag{4.52}$$

where

$$Q\left(\frac{1}{2\varepsilon_2};\mathbf{l}\right) = \prod_{j=1}^{K}\left[\frac{l_j-\frac{1}{2}}{\varepsilon_2},\frac{l_j+\frac{1}{2}}{\varepsilon_2}\right], \quad \mathbf{l}\in\mathbf{Z}^K.$$

By (4.28)

$$\int_{\mathbf{R}^K \setminus Q\left(\frac{\varepsilon}{\varepsilon_2}\right)} |h(\mathbf{y})|\, d\mathbf{y} \ll \frac{1}{c}. \tag{4.53}$$

By (4.31)

$$\begin{aligned}
&\sum_{\substack{l_1 \in \mathbf{Z}:\\ |l_1| \le c+\frac{1}{2}}} \cdots \sum_{\substack{l_K \in \mathbf{Z}:\\ |l_K| \le c+\frac{1}{2}}} \left| \int_{Q\left(\frac{1}{2\varepsilon_2}; \mathbf{l}\right)} (h(\mathbf{y}))^- \, d\mathbf{y} \right| \\
&\ll \sum_{\substack{l_1 \in \mathbf{Z}:\\ |l_1| \le c+\frac{1}{2}}} \cdots \sum_{\substack{l_K \in \mathbf{Z}:\\ |l_K| \le c+\frac{1}{2}}} \frac{1}{c^2} \cdot \prod_{j=1}^{K} \frac{1}{(1+|l_j|)^2} \cdot (\varepsilon_2)^K \cdot \mu\left(Q\left(\frac{1}{2\varepsilon_2}; \mathbf{l}\right)\right) \qquad (4.54) \\
&\le \frac{1}{c^2} \sum_{\substack{l_1 \in \mathbf{Z}:\\ |l_1| \le c+\frac{1}{2}}} \cdots \sum_{\substack{l_K \in \mathbf{Z}:\\ |l_K| \le c+\frac{1}{2}}} \prod_{j=1}^{K} \frac{1}{(1+|l_j|)^2} \le \frac{1}{c^2} \left(\sum_{l \in \mathbf{Z}} \frac{1}{(1+|l|)^2} \right)^K \ll \frac{1}{c^2}.
\end{aligned}$$

Combining (4.50), (4.51), (4.52), (4.53) and (4.54) we obtain

$$\int_{\mathbf{R}^K} \chi_A(\mathbf{x}_0 - \mathbf{y})\, h(\mathbf{y})\, d\mathbf{y} \le 1 - c_{22}(K) + O\left(\frac{1}{c}\right) + O\left(\frac{1}{c^2}\right) \quad \text{for any } \mathbf{x}_0 \in A_2. \tag{4.55}$$

From (4.48) and (4.55) it follows that

$$\int_{\mathbf{R}^K} \chi_A(\mathbf{x}_0 - \mathbf{y})\, (H-h)(\mathbf{y})\, d\mathbf{y} \ge c_{22}(K) - O\left(\frac{1}{c}\right) > \frac{1}{2} c_{22}(K) > 0 \quad \text{if } c \ge c_{23}(K) \text{ and } \mathbf{x}_0 \in A_2. \tag{4.56}$$

Using (4.44) and (4.56) we see that if $c \ge c_{23}(K)$ then

$$\begin{aligned}
\int_{\mathbf{R}^K} \left(\int_{\mathbf{R}^K} \chi_A(\mathbf{x}-\mathbf{y})\, (h-H)(\mathbf{y})\, d\mathbf{y} \right)^2 d\mathbf{x} &\ge \int_{A_2} \left(\int_{\mathbf{R}^K} \chi_A(\mathbf{x}-\mathbf{y})\, (h-H)(\mathbf{y})\, d\mathbf{y} \right)^2 d\mathbf{x} \\
&\gg \mu(A_2) \gg \frac{1}{\varepsilon} \cdot \sigma(\partial A).
\end{aligned} \tag{4.57}$$

(II) $1/\varepsilon > b_1/8$. Since the boxes B and D have parallel edges, we have

$$B = \tau_1\left(\prod_{j=1}^{K} \left[-\frac{b_j}{2}, \frac{b_j}{2} \right] \right) + \mathbf{x}_1 \quad \text{and} \quad D = \tau_1\left(\prod_{j=1}^{K} \left[-\frac{d_j}{2}, \frac{d_j}{2} \right] \right) + \mathbf{x}_2$$

where τ_1 is an orthogonal transformation and $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{R}^K$.

Let

$$B_1=\tau_1\left(\prod_{j=1}^{K}\left[-\frac{b_j}{2}-\frac{1}{2K\cdot\varepsilon_4},\frac{b_j}{2}+\frac{1}{2K\cdot\varepsilon_4}\right]\right)+\mathbf{x}_1.$$

Let $\mathbf{x}_0\in B_1$. We are going to estimate

$$\left(\int_{\mathbf{R}^K}\chi_A(\mathbf{x}_0-\mathbf{y})\,(h-H)(\mathbf{y})\,d\mathbf{y}\right)^2$$

from below. Clearly

$$\int_{\mathbf{R}^K}\chi_A(\mathbf{x}_0-\mathbf{y})\,H(\mathbf{y})\,d\mathbf{y}\geqslant\int_{Q(1/\varepsilon_4)}\chi_A(\mathbf{x}_0-\mathbf{y})\,H(\mathbf{y})\,d\mathbf{y}-\left|\int_{\mathbf{R}^K\setminus Q(1/\varepsilon_4)}\chi_A(\mathbf{x}_0-\mathbf{y})\,(H(\mathbf{y}))^-\,d\mathbf{y}\right|. \tag{4.58}$$

Since $\mathbf{x}_0\in B_1$, we see that the usual Euclidean distance of $\mathbf{x}_0$ and B is $<1/2\varepsilon_4$. Hence

$$\mu\left(\left(Q\left(\frac{1}{\varepsilon_4}\right)+\mathbf{x}_0\right)\cap B\right)>>\prod_{j=1}^{K}\min\left\{\frac{1}{\varepsilon_4},b_j\right\}. \tag{4.59}$$

By (4.25) and (4.59)

$$\begin{aligned}\int_{Q(1/\varepsilon_4)}\chi_A(\mathbf{x}_0-\mathbf{y})\,H(\mathbf{y})\,d\mathbf{y}&\geqslant c_{24}(K)\cdot(\varepsilon_4)^K\cdot\mu\left(\left(Q\left(\frac{1}{\varepsilon_4}\right)+\mathbf{x}_0\right)\cap A\right)\\&\geqslant c_{24}(K)\cdot(\varepsilon_4)^K\cdot\mu\left(\left(Q\left(\frac{1}{\varepsilon_4}\right)+\mathbf{x}_0\right)\cap B\right)\\&\geqslant c_{25}(K)\cdot(\varepsilon_4)^K\cdot\prod_{j=1}^{K}\min\left\{\frac{1}{\varepsilon_4},b_j\right\}\\&=c_{25}(K)\cdot\prod_{j=1}^{K}\min\{1,\varepsilon_4\cdot b_j\}.\end{aligned} \tag{4.60}$$

Moreover

$$\begin{aligned}\left|\int_{\mathbf{R}^K\setminus Q(1/\varepsilon_4)}\chi_A(\mathbf{x}_0-\mathbf{y})\,(H(\mathbf{y}))^-\,d\mathbf{y}\right|&\leqslant\int_{\mathbf{R}^K\setminus Q(c/\varepsilon_4)}\chi_A(\mathbf{x}_0-\mathbf{y})\cdot|H(\mathbf{y})|\,d\mathbf{y}\\&\quad+\sum_{\substack{l_1\in\mathbf{Z}:\\|l_1|\leqslant c+\frac12}}\cdots\sum_{\substack{l_K\in\mathbf{Z}:\\|l_K|\leqslant c+\frac12}}\left|\int_{Q(1/2\varepsilon_4;\mathbf{l})}\chi_A(\mathbf{x}_0-\mathbf{y})\,(H(\mathbf{y}))^-\,d\mathbf{y}\right|\end{aligned} \tag{4.61}$$

where

$$Q\left(\frac{1}{2\varepsilon_4};\mathbf{l}\right)=\prod_{j=1}^{K}\left[\frac{l_j-\frac{1}{2}}{\varepsilon_4},\frac{l_j+\frac{1}{2}}{\varepsilon_4}\right],\quad \mathbf{l}\in\mathbf{Z}^K.$$

By (4.27)

$$\begin{aligned}\int_{\mathbf{R}^K\setminus Q(c/\varepsilon_4)}\chi_A(\mathbf{x}_0-\mathbf{y})\,|H(\mathbf{y})|\,d\mathbf{y}&\leq\sum_{\substack{\mathbf{l}\in\mathbf{Z}^K:\\ \max\limits_{1\leq j\leq K}|l_j|\geq c-\frac{1}{2}}}\int_{Q(1/2\varepsilon_4;\mathbf{l})}\chi_A(\mathbf{x}_0-\mathbf{y})\,|H(\mathbf{y})|\,d\mathbf{y}\\ &\ll\sum_{\substack{\mathbf{l}\in\mathbf{Z}^K:\\ \max\limits_{1\leq j\leq K}|l_j|\geq c-\frac{1}{2}}}(\varepsilon_4)^K\cdot\prod_{j=1}^{K}\frac{1}{(1+|l_j|)^2}\cdot\mu\left(\left(Q\left(\frac{1}{2\varepsilon_4};\mathbf{l}\right)+\mathbf{x}_0\right)\cap A\right).\end{aligned}\tag{4.62}$$

Since

$$\mu\left(\left(Q\left(\frac{1}{2\varepsilon_4};\mathbf{l}\right)+\mathbf{x}_0\right)\cap A\right)\leq\mu\left(\left(Q\left(\frac{1}{2\varepsilon_4};\mathbf{l}\right)+\mathbf{x}_0\right)\cap D\right)\ll\prod_{j=1}^{K}\min\left\{\frac{1}{\varepsilon_4},d_j\right\},\tag{4.63}$$

by (4.62) we have

$$\begin{aligned}\int_{\mathbf{R}^K\setminus Q(c/\varepsilon_4)}\chi_A(\mathbf{x}_0-\mathbf{y})\,|H(\mathbf{y})|\,d\mathbf{y}&\ll\left(\prod_{j=1}^{K}\min\left\{\frac{1}{\varepsilon_4},d_j\right\}\right)\cdot(\varepsilon_4)^K\cdot\sum_{\substack{\mathbf{l}\in\mathbf{Z}^K:\\ \max\limits_{1\leq j\leq K}|l_j|\geq c-\frac{1}{2}}}\frac{1}{(1+|l_j|)^2}\\ &\ll\left(\prod_{j=1}^{K}\min\{1,\varepsilon_4\cdot d_j\}\right)\\ &\quad\times\left\{\left(\sum_{l\in\mathbf{Z}}\frac{1}{(1+|l|)^2}\right)^K-\left(\sum_{\substack{l\in\mathbf{Z}:\\ |l|<c-\frac{1}{2}}}\frac{1}{(1+|l|)^2}\right)^K\right\}\\ &\ll\left(\prod_{j=1}^{K}\min\{1,\varepsilon_4\cdot d_j\}\right)\cdot\left(\sum_{\substack{l\in\mathbf{Z}:\\ |l|\geq c-\frac{1}{2}}}\frac{1}{(1+|l|)^2}\right)\\ &\ll\frac{1}{c}\cdot\prod_{j=1}^{K}\min\{1,\varepsilon_4\cdot d_j\}.\end{aligned}\tag{4.64}$$

By (4.32) and (4.63)

$$\begin{aligned}
&\sum_{\substack{l_1\in\mathbf{Z}:\\ |l_1|\le c+\frac12}}\cdots\sum_{\substack{l_K\in\mathbf{Z}:\\ |l_K|\le c+\frac12}}\left|\int_{Q(1/2\varepsilon_4;\mathbf{l})}\chi_A(\mathbf{x}_0-\mathbf{y})\cdot(H(\mathbf{y}))^-\,d\mathbf{y}\right|\\
&\ll\sum_{\substack{l_1\in\mathbf{Z}:\\ |l_1|\le c+\frac12}}\cdots\sum_{\substack{l_K\in\mathbf{Z}:\\ |l_K|\le c+\frac12}}\frac{1}{c^6}\left(\prod_{j=1}^{K}\frac{1}{(1+|l_j|)^2}\right)\cdot(\varepsilon_4)^K\cdot\mu\left(\left(Q\left(\frac{1}{2\varepsilon_4};\mathbf{l}\right)+\mathbf{x}_0\right)\cap A\right)\\
&\le\frac{1}{c^6}\cdot(\varepsilon_4)^K\cdot\left(\prod_{j=1}^{K}\min\left\{\frac{1}{\varepsilon_4},d_j\right\}\right)\cdot\sum_{l_1\in\mathbf{Z}}\cdots\sum_{l_K\in\mathbf{Z}}\frac{1}{(1+|l_j|)^2}\qquad(4.65)\\
&=\frac{1}{c^6}\cdot\left(\prod_{j=1}^{K}\min\{1,\varepsilon_4\cdot d_j\}\right)\cdot\left(\sum_{l\in\mathbf{Z}}\frac{1}{(1+|l|)^2}\right)^K\\
&\ll\frac{1}{c^6}\cdot\left(\prod_{j=1}^{K}\min\{1,\varepsilon_4\cdot d_j\}\right).
\end{aligned}$$

Combining (4.61), (4.64) and (4.65) we see

$$\left|\int_{\mathbf{R}^K\setminus Q(1/\varepsilon_4)}\chi_A(\mathbf{x}_0-\mathbf{y})(H(\mathbf{y}))^-\,d\mathbf{y}\right|\ll\frac{1}{c}\prod_{j=1}^{K}\min\{1,\varepsilon_4\cdot d_j\}.\qquad(4.66)$$

Therefore, by the inequality $d_j\le K^K\cdot K!\cdot b_j$ (see (4.35)), (4.58), (4.60) and (4.66) we obtain

$$\begin{aligned}
\int_{\mathbf{R}^K}\chi_A(\mathbf{x}_0-\mathbf{y})H(\mathbf{y})\,d\mathbf{y}&\ge c_{25}(K)\cdot\prod_{j=1}^{K}\min\{1,\varepsilon_4\cdot b_j\}-\frac{c_{26}(K)}{c}\prod_{j=1}^{K}\min\{1,\varepsilon_4\cdot d_j\}\\
&\ge\left\{c_{25}(K)-\frac{c_{26}(K)\cdot K^K\cdot K!}{c}\right\}\prod_{j=1}^{K}\min\{1,\varepsilon_4\cdot b_j\}\qquad(4.67)\\
&\ge\frac{1}{2}c_{25}(K)\cdot\prod_{j=1}^{K}\min\{1,\varepsilon_4\cdot b_j\}\quad\text{if } c\ge c_{27}(K).
\end{aligned}$$

On the other hand, by (4.26)

$$\begin{aligned}
\left|\int_{\mathbf{R}^K}\chi_A(\mathbf{x}_0-\mathbf{y})\,h(\mathbf{y})\,d\mathbf{y}\right|&\le\sum_{\mathbf{l}\in\mathbf{Z}^K}\int_{Q(1/2\varepsilon_2;\mathbf{l})}\chi_A(\mathbf{x}_0-\mathbf{y})\,|h(\mathbf{y})|\,d\mathbf{y}\\
&\ll\sum_{\mathbf{l}\in\mathbf{Z}^K}(\varepsilon_2)^K\cdot\left(\prod_{j=1}^{K}\frac{1}{(1+|l_j|)^2}\right)\cdot\mu\left(\left(Q\left(\frac{1}{2\varepsilon_2};\mathbf{l}\right)+\mathbf{x}_0\right)\cap A\right).\qquad(4.68)
\end{aligned}$$

Since

$$\mu\left(\left(Q\left(\frac{1}{2\varepsilon_2};\mathbf{1}\right)+\mathbf{x}_0\right)\cap A\right)\leqslant\mu\left(\left(Q\left(\frac{1}{2\varepsilon_2};\mathbf{1}\right)+\mathbf{x}_0\right)\cap D\right)\ll\prod_{j=1}^{K}\min\left\{\frac{1}{\varepsilon_2},d_j\right\},$$

by (4.68) we have

$$\begin{aligned}\left|\int_{\mathbf{R}^K}\chi_A(\mathbf{x}_0-\mathbf{y})\,h(\mathbf{y})\,d\mathbf{y}\right| &\ll \sum_{\mathbf{l}\in\mathbf{Z}^K}(\varepsilon_2)^K\cdot\left(\prod_{j=1}^{K}\frac{1}{(1+|l_j|)^2}\right)\cdot\left(\prod_{j=1}^{K}\min\left\{\frac{1}{\varepsilon_2},d_j\right\}\right)\\ &=\left(\prod_{j=1}^{K}\min\{1,\varepsilon_2\cdot d_j\}\right)\cdot\sum_{\mathbf{l}\in\mathbf{Z}^K}\prod_{j=1}^{K}\frac{1}{(1+|l_j|)^2} \qquad (4.69)\\ &=\left(\prod_{j=1}^{K}\min\{1,\varepsilon_2\cdot d_j\}\right)\cdot\left(\sum_{l\in\mathbf{Z}}\frac{1}{(1+|l|)^2}\right)^K\ll\prod_{j=1}^{K}\min\{1,\varepsilon_2\cdot d_j\}.\end{aligned}$$

By hypothesis $1/\varepsilon>b_1/8$, and so $1>\varepsilon\cdot b_1/8$. Furthermore, $d_j\leqslant K^K\cdot K!\cdot b_j$ and $\varepsilon=\varepsilon_4/c=c\cdot\varepsilon_2$, thus by (4.67) and (4.69)

$$\begin{aligned}\int_{\mathbf{R}^K}\chi_A(\mathbf{x}_0-\mathbf{y})\,(H-h)(\mathbf{y})\,d\mathbf{y} &\geqslant\frac{1}{2}c_{25}(K)\cdot\prod_{j=1}^{K}\min\{1,\varepsilon_4\cdot b_j\}-c_{28}(K)\cdot\prod_{j=1}^{K}\min\{1,\varepsilon_2\cdot d_j\}\\ &\geqslant\frac{1}{2}c_{25}(K)\cdot\min\{1,c\cdot\varepsilon\cdot b_1\}\cdot\prod_{j=2}^{K}\min\{1,\varepsilon_4\cdot b_j\}\\ &\quad-c_{28}(K)\cdot\min\left\{1,\frac{K^K\cdot K!\cdot\varepsilon\cdot b_1}{c}\right\}\cdot(K^K\cdot K!)^{K-1}\cdot\prod_{j=2}^{K}\min\{1,\varepsilon_4\cdot b_j\}\\ &=\left\{\frac{1}{2}c_{25}(K)\cdot\min\{1,c\cdot\varepsilon\cdot b_1\}-c_{29}(K)\cdot\min\left\{1,\frac{K^K\cdot K!}{c}\cdot\varepsilon\cdot b_1\right\}\right\}\\ &\qquad\times\prod_{j=2}^{K}\min\{1,\varepsilon_4\cdot b_j\}\\ &\geqslant\frac{1}{4}c_{25}(K)\cdot\min\{1,c\cdot\varepsilon\cdot b_1\}\cdot\prod_{j=2}^{K}\min\{1,\varepsilon_4\cdot b_j\}\\ &=\frac{1}{4}c_{25}(K)\cdot\prod_{j=1}^{K}\min\{1,\varepsilon_4\cdot b_j\}\end{aligned} \qquad (4.70)$$

whenever $c\geqslant c_{30}(K)\geqslant c_{27}(K)$ (we recall: $\mathbf{x}_0$ is an arbitrary point in B_1). Obviously

$$\mu(B_1) >> \prod_{j=1}^{K} \max\left\{b_j, \frac{1}{\varepsilon_4}\right\}. \tag{4.71}$$

Therefore, using (4.70) and (4.71) we have

$$\begin{aligned}\int_{\mathbf{R}^K}\left(\int_{\mathbf{R}^K}\chi_A(\mathbf{x}-\mathbf{y})(h-H)(\mathbf{y})\,d\mathbf{y}\right)^2 d\mathbf{x} &\geq \int_{B_1}\left(\int_{\mathbf{R}^K}\chi_A(\mathbf{x}-\mathbf{y})(h-H)(\mathbf{y})\,d\mathbf{y}\right)^2 d\mathbf{x}\\ &>> \left\{\prod_{j=1}^{K}\min\{1, \varepsilon_4\cdot b_j\}\right\}^2\cdot\left\{\prod_{j=1}^{K}\max\left\{b_j, \frac{1}{\varepsilon_4}\right\}\right\}\\ &\text{for } c \geq c_{30}(K).\end{aligned} \tag{4.72}$$

Let j_0 be the largest index j, $1 \leq j \leq K$ such that $\varepsilon_4 \cdot b_j \leq 1$. Then by (4.72)

$$\int_{\mathbf{R}^K}\left(\int_{\mathbf{R}^K}\chi_A(\mathbf{x}-\mathbf{y})(h-H)(\mathbf{y})\,d\mathbf{y}\right)^2 d\mathbf{x} >> \left(\prod_{1\leq i\leq j_0}\varepsilon_4\cdot b_i^2\right)\cdot\left(\prod_{j_0<j\leq K} b_j\right) \quad \text{if } c \geq c_{30}(K). \tag{4.73}$$

Let $c = \max\{4, c_{23}(K), c_{27}(K), c_{30}(K)\}$. Then from (4.20), (4.21), (4.57) and (4.73) it follows that

$$\begin{aligned}2\cdot\int_{Q(\varepsilon_3+\varepsilon_4)\setminus Q(\varepsilon_4-\varepsilon_3)}|\hat{\chi}_A(\mathbf{t})|^2\,dt &\geq \int_{\mathbf{R}^K}|\hat{\chi}_A(\mathbf{t})|^2\cdot|(\hat{h}-\hat{H})(\mathbf{t})|^2\,dt\\ &= \int_{\mathbf{R}^K}\left(\int_{\mathbf{R}^K}\chi_A(\mathbf{x}-\mathbf{y})(h-H)(\mathbf{y})\,d\mathbf{y}\right)^2 d\mathbf{x}\\ &>> \begin{cases}\dfrac{1}{\varepsilon}\cdot\sigma(\partial A) & \text{in case (I)}\\ \left(\prod\limits_{1\leq i\leq j_0}\varepsilon_4\cdot b_i^2\right)\cdot\left(\prod\limits_{j_0<j\leq K} b_j\right) & \text{in case (II)}\end{cases}\end{aligned} \tag{4.74}$$

where j_0 is the largest index j such that $\varepsilon_4 \cdot b_j \leq 1$.

Now we return to (4.11). By definition

$$\hat{\chi}_{\tau,\alpha}(\mathbf{t}) = \alpha^K\cdot\hat{\chi}_A(\tau^{-1}(\alpha\mathbf{t}))$$

where $\chi_{\tau,\alpha}$ denotes the characteristic function of $A(\tau, \alpha, \mathbf{0})$. Consequently, for *every* $\mathbf{t}^* \in Q_m = Q(2K^{1/2}(\varepsilon_3+\varepsilon_4)) \setminus Q(K^{1/2}(\varepsilon_3+\varepsilon_4))$ (see (4.10) and (4.19)),

$$\begin{aligned}\int_T\int_0^1 |\hat{\chi}_{\tau,\alpha}(\mathbf{t}^*)|^2\,d\alpha\,d\tau &= \int_0^1 \alpha^{2K}\left(\int_T |\hat{\chi}_A(\tau^{-1}(\alpha\mathbf{t}^*))|^2\,d\tau\right)d\alpha\\ &\gg (\mu(Q(\varepsilon_3+\varepsilon_4)\setminus Q(\varepsilon_4-\varepsilon_3)))^{-1}\cdot\int_{Q(\varepsilon_3+\varepsilon_4)\setminus Q(\varepsilon_4-\varepsilon_3)} |\hat{\chi}_A(\mathbf{t})|^2\,d\mathbf{t} \qquad (4.75)\\ &\gg (\varepsilon_4)^{-K}\int_{Q(\varepsilon_3+\varepsilon_4)\setminus Q(\varepsilon_4-\varepsilon_3)} |\hat{\chi}_A(\mathbf{t})|^2\,d\mathbf{t}.\end{aligned}$$

By (4.74) and (4.75): for *every* $\mathbf{t}^*\in Q_m$

$$\int_T\int_0^1 |\hat{\chi}_{\tau,\alpha}(\mathbf{t}^*)|^2\,d\alpha\,d\tau \gg \begin{cases}(\varepsilon_4)^{-K-1}\cdot\sigma(\partial A) & \text{in case (I)}\\ \prod_{1\le i\le j_0} b_i^2\cdot\prod_{j_0<j\le K}\left(\dfrac{b_j}{\varepsilon_4}\right) & \text{in case (II).}\end{cases} \qquad (4.76)$$

Combining (4.6), (4.10), (4.19) and (4.76) we obtain

$$\begin{aligned}\int_T\int_0^1\int_{\mathbf{R}^K} F^2_{\tau,\alpha}(\mathbf{x})\,d\mathbf{x}\,d\alpha\,d\tau &= \int_{\mathbf{R}^K}\left(\int_T\int_0^1 |\hat{\chi}_{\tau,\alpha}(\mathbf{t})|^2\,d\alpha\,d\tau\right)\cdot|\varphi(\mathbf{t})|^2\,d\mathbf{t}\\ &\ge \int_{Q_m}\left(\int_T\int_0^1 |\hat{\chi}_{\tau,\alpha}(\mathbf{t})|^2\,d\alpha\,d\tau\right)\cdot|\varphi(\mathbf{t})|^2\,d\mathbf{t}\\ &\ge \left(\min_{\mathbf{t}\in Q_m}\int_T\int_0^1 |\hat{\chi}_{\tau,\alpha}(\mathbf{t})|^2\,d\alpha\,d\tau\right)\cdot\int_{Q_m}|\varphi(\mathbf{t})|^2\,d\mathbf{t} \qquad (4.77)\\ &\gg \begin{cases}(\varepsilon_4)^{-K-1}\cdot\sigma(\partial A)\dfrac{1}{m^2}\cdot M^K \gg \dfrac{2^{(K+1)m}}{m^2}\cdot\sigma(\partial A)\cdot M^K & \text{in case (I)}\\ \prod_{1\le i\le j_0} b_i^2\cdot\prod_{j_0<j\le K}(2^m\cdot b_j)\cdot\dfrac{M^K}{m^2} & \text{in case (II)}\end{cases}\end{aligned}$$

where $\varphi=(dZ_0-d\mu_0)^\wedge$, j_0 is the largest index j with $\varepsilon_4\cdot b_j\le 1$ and m is an integer with $1\le m=O(\log M)$.

Clearly

$$\prod_{j=1}^K b_j=\mu(B)\gg\mu(A) \quad\text{and}\quad \prod_{j=2}^K b_j\gg\sigma(\partial B)\gg\sigma(\partial D)\ge\sigma(\partial A)$$

by (4.33), (4.35) and (4.36); moreover,

$$b_K\ge b_{K-1}\ge\ldots\ge b_1$$

by (4.34), and finally

$$b_1 \gg d_1 \geqslant r(A) \geqslant 1$$

by (4.35) and the hypothesis of Theorem 2A.

Therefore, in view of (4.77) we have

$$\int_T \int_0^1 \int_{\mathbf{R}^K} F^2_{\tau,\alpha}(\mathbf{x})\, d\mathbf{x}\, d\alpha\, d\tau \gg \begin{cases} \sigma(\partial A)\cdot M^K & \text{in case (I) and in case (II), } j_0 < K \\ \dfrac{(\mu(A))^2\cdot M^K}{(\log M)^2} & \text{in case (II), } j_0 = K. \end{cases} \tag{4.78}$$

We recall (4.2):

$$F_{\tau,\alpha}(\mathbf{x}) = \operatorname{card}(S \cap A(\tau, \alpha, \mathbf{x}) \cap Q(M)) - \mu(A(\tau, \alpha, \mathbf{x}) \cap Q(M)). \tag{4.79}$$

Hence

$$F_{\tau,\alpha}(\mathbf{x}) = 0 \quad \text{whenever} \quad \mathbf{x} \notin Q(M + \operatorname{diam}(A)).$$

(Here diam stands for diameter.) Thus from (4.78) we obtain

$$\int_T \int_0^1 \int_{Q(M+\operatorname{diam}(A))} F^2_{\tau,\alpha}(\mathbf{x})\, d\mathbf{x}\, d\alpha\, d\tau \gg \min\left\{\sigma(\partial A), \frac{(\mu(A))^2}{(\log M)^2}\right\}\cdot M^K. \tag{4.80}$$

Inequality (4.80) gives that either

$$\int_{Q^*} \int_T \int_0^1 F^2_{\tau,\alpha}(\mathbf{x})\, d\alpha\, d\tau\, d\mathbf{x} \gg \min\left\{\sigma(\partial A), \frac{(\mu(A))^2}{(\log M)^2}\right\}\cdot M^K \tag{4.81a}$$

$$\text{with} \quad Q^* = Q(M + \operatorname{diam}(A)) \setminus Q(M - \operatorname{diam}(A)),$$

or

$$\int_{Q^{**}} \int_T \int_0^1 F^2_{\tau,\alpha}(\mathbf{x})\, d\alpha\, d\tau\, d\mathbf{x} \gg \min\left\{\sigma(\partial A), \frac{(\mu(A))^2}{(\log M)^2}\right\}\cdot M^K \tag{4.81b}$$

$$\text{with} \quad Q^{**} = Q(M - \operatorname{diam}(A)).$$

Now we specify the value of the parameter M: let

$$M = (\operatorname{diam}(A))^{2K+2}.$$

If alternative (4.81a) holds, then it follows that there exist

$$\tau_0 \in T,\ \alpha_0 \in (0,1] \quad \text{and} \quad \mathbf{x}_0 \in \mathbf{R}^K \text{ such that}$$

$$(F_{\tau_0,\alpha_0}(\mathbf{x}_0))^2 \cdot M^{K-1} \operatorname{diam}(A) \gg \min\left\{\sigma(\partial A), \frac{(\mu(A))^2}{(\log M)^2}\right\} \cdot M^K;$$

that is

$$(F_{\tau_0,\alpha_0}(\mathbf{x}_0))^2 \gg \min\left\{\sigma(\partial A), \frac{(\mu(A))^2}{(\log M)^2}\right\} \cdot \frac{M}{\operatorname{diam}(A)}. \tag{4.82}$$

Since by hypothesis $r(A) \geq 1$, we see that

$$\mu(A) \gg \sigma(\partial A) \gg \operatorname{diam}(A) = M^{1/(2K+2)}. \tag{4.83}$$

Thus by (4.82) and (4.83)

$$(F_{\tau_0,\alpha_0}(\mathbf{x}_0))^2 \gg \frac{M}{\operatorname{diam}(A)} = (\operatorname{diam}(A))^{2K+1} \gg \operatorname{diam}(A) \cdot (\mu(A))^2,$$

and so

$$|F_{\tau_0,\alpha_0}(\mathbf{x}_0)| > 2\mu(A) \quad \text{if} \quad \operatorname{diam}(A) \geq c_{31}(K). \tag{4.84}$$

From (4.79) and (4.84) we obtain that the cardinality of

$$S \cap A(\tau_0, \alpha_0, \mathbf{x}_0) \cap Q(M)$$

is greater than $2\mu(A)$. Consequently (we recall: $S = \{\mathbf{z}_1, \mathbf{z}_2, \ldots\}$)

$$\sum_{\mathbf{z}_j \in A(\tau_0,\alpha_0,\mathbf{x}_0)} 1 - (A(\tau_0, \alpha_0, \mathbf{x}_0)) > 2\mu(A) - \mu(A) = \mu(A) \gg \sigma(\partial A) \geq (\sigma(\partial A))^{1/2},$$

which was to be proved.

If alternative (4.81b) holds, then there exist $\tau_1 \in T$, $\alpha_1 \in (0,1]$ and $\mathbf{x}_1 \in \mathbf{R}^K$ such that

$$A(\tau_1, \alpha_1, \mathbf{x}_1) \subset Q(M) \quad \text{and} \quad (F_{\tau_1}, \alpha_1(\mathbf{x}_1))^2 \gg \min\left\{\sigma(\partial A), \frac{(\mu(A))^2}{(\log M)^2}\right\}. \tag{4.85}$$

By (4.3), (4.83) and (4.85) we conclude that if $\operatorname{diam}(A) \geq c_{31}(K)$ (i.e., $\operatorname{diam}(A)$ is sufficiently large) then

$$\begin{aligned}\left|\sum_{\mathbf{z}_j \in A(\tau_1,\alpha_1,\mathbf{x}_1)} 1 - \mu(A(\tau_1, \alpha_1, \mathbf{x}_1))\right|^2 &= (F_{\tau_1,\alpha_1}(\mathbf{x}_1))^2 \\ &\gg \min\{\sigma(\partial A), \mu(A) \cdot M^{1/(2K+2)} \cdot (\log M)^{-2}\} \\ &> \min\{\sigma(\partial A), \mu(A)\} \gg \sigma(\partial A),\end{aligned}$$

which was to be proved.

Finally, if $\operatorname{diam}(A)<c_{31}(K)$ then we are done by the following trivial argument. Choosing $\beta=(2\mu(A))^{-1/K}\in(0,1]$ we have $\mu(\beta A)=\frac{1}{2}$, and so certainly

$$\Omega[S;A]\geq\frac{1}{2}.$$

The proof of Theorem 2A is complete.

Remark. The proof actually gives that there exists a set $A_0=A(\tau_0,\alpha_0,\mathbf{v}_0)$ such that

$$\left|\sum_{z_j\in A_0}1-\mu(A_0)\right|\gg(\sigma(\partial A))^{1/2}$$

and $1\geq\alpha_0>c_{32}(K)$ (i.e. the contraction factor is larger than a positive absolute constant).

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