A Banach space without a basis which has the bounded approximation property

by

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1. Introduction and the main results

Let X be a separable Banach space (real or complex). A sequence (x_n) of elements of X is called a (Schauder) basis iff every element of X can be uniquely written as a sum of a series $\Sigma_n t_n x_n$, where the t_n's are scalars. X is said to have the bounded approximation property (BAP) iff there exists a sequence (A_n) of finite rank operators on X such that $\lim_n \|A_n x - x\| = 0$ for $x \in X$, i.e. I_X—the identity operator on X—is a limit of a sequence of finite rank operators in the strong operator topology. Also recall that X has the approximation property (AP) iff I_X can be approximated by finite rank operators uniformly on compact sets. It is clear that

$$X \text{ has a basis} \quad \Rightarrow \quad X \text{ has BAP} \quad \Rightarrow \quad X \text{ has AP}$$

The fact that the converse implication to the second one does not hold in general was discovered by Figiel and Johnson [8] soon after Enflo's example [7] of a space without AP. The main purpose of this paper is to show that also the implication "BAP⇒basis" does not hold in general; this answers problems asked by a number of authors (e.g. [14], [18], [27]). Before stating the result, we recall more notation. For a given basis (x_n) of X one denotes by $\mathrm{bc}(x_n)$ (the basis constant of (x_n)) the smallest K such that $\|\Sigma_{n=1}^N t_n x_n\| \leq K \|\Sigma_{n=1}^M t_n x_n\|$ for all $N \leq M$ and all (t_n); one further denotes

$$\mathrm{bc}(X) = \inf \{\mathrm{bc}(x_n) \colon (x_n) \text{ a basis of } X\}.$$

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X is said to have the local basis structure (LBS) iff $X=\overline{\bigcup_n E_n}$, where $E_n \subset E_{n+1}$ are finite dimensional subspaces of X with $\mathrm{bc}(E_n) \leq C$, C a universal constant (this concept was studied under the name "finite dimensional supspace basis property" by L. Pujara [24], [25], it appears also in [27]; the author learned about it from A. Pelczynski). It is clear that if X has a basis, then it has LBS. Now we can state

THEOREM 1.1. *There exists a separable superreflexive Banach space with the bounded approximation property (in fact with unconditional finite dimensional decomposition) which does not have local basis structure and consequently does not have a basis.*

Recall that X is said to have *finite dimensional decomposition* (FDD) if there exists a sequence of finite dimensional subspaces (X_n) such that every $x \in X$ can be uniquely written as a sum of a series $\Sigma_n x_n$ with $x_n \in X_n$ (or, equivalently, if I_X is a limit of a sequence of commuting finite rank projections in the strong operator topology). An FDD is called *unconditional* if all series $x=\Sigma_n x_n$ converge unconditionally.

Theorem 1.1 shows that the property of having a basis is not inherited by complemented subspaces (cf. [14], [20]). Moreover, it follows from Theorem 1.1 and [13] that

COROLLARY 1.2. *There exist reflexive Banach spaces X, Y such that both Y and $X \oplus Y$ have bases but X does not.*

Since, by a result of Grothendieck [12], for reflexive (or separable dual) spaces AP and BAP coincide, our argument yields the first example of a reflexive space with AP but no basis. The space we construct to prove the theorem is a variant of a space not admitting complex structure from [32]. Similarly as in [32], the construction uses the ideas from [3], [31] and indirectly [10], [30]. The method we present is apparently the third (after [7] and [23]) essentially different approach to proving that an infinite dimensional space does not have a basis; for finite dimensional analogues see [11] or [30].

Let us note here that Theorem 1.1 settles also another open problem (asked e.g. by Pelczynski in the early seventies): does every Banach space have local basis structure? As an illustration of this concept and a motivation for further results we state the following observation (to be proved in Section 5), which was known to Pelczynski in the late seventies.

PROPOSITION 1.3. *If X contains l_∞^n's uniformly, then X has the local basis structure.*

We recall that X is said to contain l_∞^n's uniformly iff, for each $n \in \mathbf{N}$, there is a subspace $F_n \subset X$ with $d(F_n, l_\infty^n) \leq C_0$, where C_0 is a universal constant and $d(E,F) = \inf\{\|T\| \cdot \|T^{-1}\| : T: E \to F \text{ isomorphism}\}$ is the Banach–Mazur distance.

Let G be a compact metrizable Abelian group, Γ the dual group and $\Lambda \subset \Gamma$; denote $C_\Lambda = \{f \in C(G): \operatorname{supp} \hat{f} \subset \Lambda\}$, where $\hat{f}$ is the Fourier transform of f. It is well known that every C_Λ has BAP. A recent result of Bourgain and Milman [4] states that either Λ is a Sidon set, in which case C_Λ is isomorphic to $l_1(\Lambda)$, or C_Λ contains l_∞^n's uniformly. We thus get

COROLLARY 1.5. *Every space C_Λ has both* BAP *and the local basis structure.*

Since it is a well known open problem whether every C_Λ has a basis (and settling it in the case $\Lambda = \mathbf{Z}^+$ by Bockarev [1] was a major event), it is tempting to conjecture that a separable Banach space with BAP and local basis structure has a basis. This is, however, false, as follows from our next results (see [18, Vol. II] or [22] for the definition of cotype).

PROPOSITION 1.5. *There exists a superreflexive Banach space X with unconditional* FDD *such that whenever B is any Banach space of coptype 2, then $X \oplus B$ fails to have the local basis structure.*

We postpone the proof until Section 5. Now Propositions 1.3 and 1.5 imply

COROLLARY 1.6. *If X is as in Proposition 1.5, and $Z = X \oplus (\oplus_{n=1}^\infty l_1^n)_{l_2}$, then*

(a) *Z fails to have the local basis structure, hence basis*

(b) *Z^* has the local basis structure (and, obviously,* BAP*), but fails to have a basis.*

This shows that in general it may happen that Z has LBS while Z^* does not, even if Z is reflexive and has BAP (cf. Remark 4.2). There is, however, a related concept, which is inherited—at least in the reflexive case—by dual spaces and which we will now identify. We will say that a Banach space X has the local Π-basis structure (LΠBS) iff there exists a sequence $E_1 \subset E_2 \subset \dots$ of finite dimensional supspaces of X and a sequence of projections (P_n), $P_n: X \xrightarrow{\text{onto}} E_n$, such that $X = \overline{\bigcup_n E_n}$, $\sup \operatorname{bc}(E_n) < \infty$,

$\sup_n \|P_n\| < \infty$ (this was called "a B_ν-space" in [25]). It is clear that if a space has a basis, then it has LΠBS and that if it has LΠBS, then it has BAP and LBS. We thus have

COROLLARY 1.7. *The reflexive space Z^* from Corollary* 1.6 *has* FDD *and the local basis structure but fails to have the local* Π*-basis structure.*

All things considered, the right and natural question to ask in this direction is:

Problem 1.8. If a separable Banach space has LΠBS, does it have a basis?

In other words, can we redo—for spaces with a basis instead of l_p^n's—the part of the theory of $\mathcal{L}_p$-spaces corresponding to [14, Theorem 5.1], even though we can not redo the part corresponding to [17, Theorem III(c)]? Notice that the methods of [14, Theorem 4.1, Lemma 4.2] show that, at least in the reflexive case, if X has LΠBS, then it has an FDD, say (X_n), and, moreover, we can assume that $E_n = X_1 + X_2 + \ldots + X_n$ works in the definitions of LΠBS. A positive answer to Problem 1.8, however unlikely, would add significance to the following.

Problem 1.9. Does every space C_Λ have LΠBS?

The papers [2] and [5] are relevant here. Probably a more interesting—and better known—question than Problem 1.8 is the following: Does a separable space with BAP have an FDD? Again, at least in the reflexive case, this reduces to: can we in the definition of BAP additionally require that the T_n's are projections? The reader may consult [27] for a list of related problems.

The organization of the paper is as follows:

Section 2 lists known facts and preliminary results.

Section 3 contains the proof of our main technical result, Proposition 3.1.

In Section 4 we deduce the real version of Theorem 1.1 from Proposition 3.1.

Section 5 contains the proof of Proposition 1.4, additional results, remarks and some open problems.

Finally, in Section 6 we show how to obtain the complex version of Theorem 1.1 and present some of its strengthenings.

We use the standard Banach space notation as can be found e.g. in [18]. The less widely used concepts are explained where they appear for the first time; the reader may also consult [32, Section 2]. To avoid unnecessary repetitions let us agree that

C, c, c', c_1 etc. will denote universal constants while e.g. $c(\delta)$ will denote a constant depending only on a parameter δ.

2. Known and preliminary results

We start by introducing some notation. We say that a normed space satisfies the Grothendieck theorem with constant C iff $\pi_1(T) \le C\|T\|$ for all $T \in L(X, Y)$ where $\pi_p(\cdot)$ denotes the usual p-absolutely summing norm (see [18, vol. II, p. 63–64]). If H is an inner product space and $T \in L(H)$ is compact, we denote by $(s_j(T))_{j=1}^{\dim H}$ the sequence of s-numbers of T (i.e. the eigenvalues of $|T| \overset{\text{def}}{=} (T^*T)^{1/2}$, counted with multiplicity and arranged in the nonincreasing order). We also define the quasi-norm $\|T\|_{C_0} = \Sigma_j \min\{s_j(T), 1\}$; $\|\cdot\|_{C_0}$ satisfies the triangle inequality (see [6, Proposition 16]), but is not positively homogeneous. In [31] and [32] we used a somewhat different $\|\cdot\|_{C_0}$, which, however, differs from the present one by a factor of at most 2.

Let us recall the following result from [31, Theorem 1.5], which was stated in the present form in [32].

THEOREM 2.1. *Given $\delta \in (0, 1)$ and $n \in \mathbf{N}$ there exists a norm $\|\cdot\|$ on $\mathbf{R}^n$ such that if $X = (\mathbf{R}^n, \|\cdot\|)$, then*

(i) *X is isometric to a quotient of l_1^N with $N \le 2n$,*

(ii) *$\|\cdot\|_2 \le \|\cdot\| \le \|\cdot\|_1 \le n^{1/2}\|\cdot\|_2$ (equivalently, $n^{-1/2}B_2^n \subset B_1^n \subset B(X) \subset B_2^n$, where $B(X)$ is the unit ball of X and B_p^n the unit ball of l_p^n),*

(iii) *$[\mathrm{vol}(B(X))/\mathrm{vol}(n^{-1/2}B_2^n)]^{1/n} \le 8$, where $\mathrm{vol}(\cdot)$ denotes the usual Lebesgue measure in $\mathbf{R}^n$,*

(iv) *X satisfies the Grothendieck theorem with constant C,*

(v) *if $\|T\|_{L(X)} \le c(\delta)\, n^{1/2}$, then $\|T - \lambda I\|_{C_0} \le \delta n$ for some $\lambda \in \mathbf{R}$.*

Recall that if $T \in L(Y, Z)$, then one denotes $\gamma_2(T) = \inf\{\|A\| \cdot \|B\| : A \in L(Y, l_2), B \in L(l_2, Z), BA = T\}$. We have the following

PROPOSITION 2.2. *Let $X = (\mathbf{R}^n, \|\cdot\|)$ be the space from Theorem 2.1 (for some $\delta \in (0, 1)$, $n \in \mathbf{N}$) and let $\alpha \in (0, 1)$. Then there exists a subspace $X_0 \subset \mathbf{R}^n$, $\operatorname{codim} X_0 \le \alpha n$, such that*

$$\|Tx\|_2 \le c_0(\alpha)\, \gamma_2(T)\, n^{-1/2} \|x\|_2 \quad \textit{for} \quad x \in X_0. \tag{1}$$

In particular $\|T\|_{C_0} \leqslant (\alpha + c_0(\alpha)\gamma_2(T)\, n^{-1/2})\, n.$

Proof. Clearly it is enough to prove (1), as it implies that at most $[\alpha n]$ s-numbers of T are $\geqslant c(\alpha)\gamma_2(T)\, n^{-1/2}$. To this end, choose $X_1 \subset \mathbf{R}^n$, $\operatorname{codim} X_1 \leqslant \frac{1}{3}\alpha n$, such that

$$c_1(\alpha)\, n^{1/2}\|x\|_2 \leqslant \|x\| \leqslant n^{1/2}\|x\|_2 \quad \text{for} \quad x \in X_1. \tag{2}$$

This can be done since, in the terminology of [34], (ii) and (iii) of Theorem 2.1 mean that the "volume ratio" of X with respect to the ellipsoid $n^{-1/2}B_2^n$ does not exceed 8 and so we can apply [26, Theoreme 8] or [29, Remark 2]. Now set $X_2 = X_1 \cap T^{-1}X_1$, then $\operatorname{codim} X_2 \leqslant \frac{2}{3}\alpha n$. Choose $A\colon X \to l_2$, $B\colon l_2 \to X$ such that $BA = T$ and $\|A\| \cdot \|B\| = \gamma_2(T)$. By Theorem 2.1 (iv),

$$\pi_2(T) \leqslant \pi_1(T) \leqslant \pi_1(A) \cdot \|B\| \leqslant C\|A\| \cdot \|B\| = C\gamma_2(T)$$

and hence also $\pi_2(T|_{X_2}) \leqslant C\gamma_2(T)$. Now, by (2),

$$\pi_2(T\colon (X_2, \|\cdot\|_2) \to (X_2, \|\cdot\|_2)) = \operatorname{hs}(T|_{X_2}) \leqslant c_1(\alpha)^{-1} \cdot C\gamma_2(T)$$

(here $\operatorname{hs}(S) = (\Sigma_j [s_j(S)]^2)^{1/2}$ denotes the Hilbert–Schmidt norm; it is well known that for operators on a Hilbert space $\pi_2(\cdot) = \operatorname{hs}(\cdot)$). It then follows that at most $\alpha n/3$ of the s-numbers of $T|_{X_2}$ are $\geqslant \sqrt{3/\alpha n}\, c_1(\alpha)^{-1} \cdot C \cdot \gamma_2(T)$, which proves (1) for some $X_0 \subset X_2$ and $c_0(\alpha) = \sqrt{3/\alpha}\; c_1(\alpha)^{-1} \cdot C$.

COROLLARY 2.3. *Given* $n \in \mathbf{N}$, $\delta \in (0,1)$ *and* $q \in [2, \infty]$ *there exists* $Y_q = (\mathbf{R}^n, \|\cdot\|^{(q)})$ *such that*

(i) Y_q *is isometric to a subspace of* l_q^N with $N \leqslant 2n$

(ii) *if* $\|S\|_{L(Y_p)} \leqslant c'(\delta)\, n^{1/2-1/q}$, *then* $\|S - \lambda I\|_{C_0} \leqslant \delta n$ *for some* $\lambda \in \mathbf{R}$

(iii) *if* $\alpha \in (0,1)$ *and* $T \in L(Y_q)$, *then*

$$\|T\|_{C_0} \leqslant (\alpha + c_0'(\alpha)\gamma_2(T)\, n^{1/q-1/2})\, n.$$

Proof. If $q = \infty$, then Corollary 2.3 follows by duality from Theorem 2.1 and Proposition 2.2. Indeed, set $Y_\infty = X^*$; then (i) follows from Theorem 2.1 (i). Since $\|T\|_{C_0} = \|T^*\|_{C_0}$ (duality induced by the usual Euclidean structure),

$$\gamma_2(T^*\colon Y_\infty \to Y_\infty) = \gamma_2(T\colon X \to X) \quad \text{and} \quad \|S^*\|_{L(Y_\infty)} = \|S\|_{L(X)},$$

(ii) and (iii) follow from Theorem 2.1 (v) and Proposition 2.2 respectively.

If $q\in[2,\infty)$, we set Y_q to be the same linear submanifold of $\mathbf{R}^N$ as Y_∞, but endowed with the l_q^N-norm rather than l_∞^N-norm. (i) is then automatically satisfied. Since $\|x\|_\infty\leq\|x\|_q\leq N^{1/q}\|x\|_\infty\leq 2^{1/q}n^{1/q}\|x\|_\infty$, (ii) and (iii) follow from the corresponding statements for Y_∞. Let us also remark that for $q=2$ the assertion of Corollary 2.3 is trivial.

We will also need

LEMMA 2.4. *Let $\beta\in(0,1)$, $n\in\mathbf{N}$. If $T\in L(l_2^n)$ in such that $\|T-\frac{1}{2}I\|_{C_0}\leq\beta n$, then $\|T^2-\frac{1}{4}I\|_{C_0}\leq 3\beta^{1/2}n$.*

Proof. Since $\|T-\frac{1}{2}I\|_{C_0}\leq\beta n$, at most $\beta^{1/2}n$ s-numbers of $T-\frac{1}{2}I$ are $\geq\beta^{1/2}$ and so there exists a subspace $E\subset l_2^n$, $\operatorname{codim}E\leq\beta^{1/2}n$, such that

$$\|Tx-\tfrac{1}{2}x\|_2\leq\beta^{1/2}\|x\|_2 \quad \text{for} \quad x\in E.$$

Let $E_1=E\cap T^{-1}E$, then $\operatorname{codim}E_1\leq 2\beta^{1/2}n$. Now

$$T^2-\tfrac{1}{4}I=T(T-\tfrac{1}{2}I)+\tfrac{1}{2}(T-\tfrac{1}{2}I)$$

and so if $x\in E_1$ (hence $Tx-\frac{1}{2}x\in E$), then

$$\|T^2x-\tfrac{1}{4}x\|_2\leq\|T(Tx-\tfrac{1}{2}x)\|_2+\tfrac{1}{2}\|Tx-\tfrac{1}{2}x\|_2\leq[(\tfrac{1}{2}+\beta^{1/2})\beta^{1/2}+\tfrac{1}{2}\beta^{1/2}]\|x\|_2=(\beta+\beta^{1/2})\|x\|_2.$$

Therefore

$$\|T^2-\tfrac{1}{4}I\|_{C_0}\leq 2\beta^{1/2}n+(1-2\beta^{1/2})n(\beta+\beta^{1/2})<3\beta^{1/2}n,$$

as required. Of course this is not meant to be optimal, a more careful argument yields a bound $(6+\ln(1/\beta))\beta n$.

3. The "local" result

Recall that a normed space Z is called D-Euclidean iff the Banach–Mazur distance $d(Z,l_2^{\dim Z})\leq D$. If Z,Z' are normed spaces, we denote by $Z\oplus_2 Z'$ their direct sum in the l_2-sense, i.e. endowed with the norm $\|(z,z')\|=(\|z\|^2+\|z'\|^2)^{1/2}$. We are now ready to state our main technical result, which appears to be of independent interest.

PROPOSITION 3.1. *Given $n\in\mathbf{N}$ and $q\in[2,\infty]$ there exists an n-dimensional sub-*

space $Y=Y_q^n$ *of* L_q *such that whenever* F *is a normed space such that all n-dimensional subspaces are D-Euclidean, then*

$$\mathrm{bc}\,(Y\oplus_2 F)\geqslant cn^{(1/2)(1/2-1/q)}D^{-1/2}.$$

In particular $\mathrm{bc}\,(Y\oplus_2 l_2)\geqslant cn^{(1/2)(1/2-1/q)}$.

Proof. Let $Y=Y_q$ be given by Corollary 2.3 (δ will be specified later). Let (x_j) be any basis of $Y\oplus_2 F$. Denote $b=\mathrm{bc}\,(x_j)$ and, to argue by contradiction, suppose that

$$b<cn^{(1/2)(1/2-1/q)}D^{-1/2} \tag{3}$$

(c to be specified later). Given $m\leqslant M=\dim(Y\oplus F)$ (possibly $M=\infty$), denote by P_m the mth partial sum projection: $P_m(\Sigma_{j=1}^M t_jx_j)=\Sigma_{j=1}^m t_jx_j$; then of course $b=\sup_m\|P_m\|$. Consider the matrix representation

$$P_m=\begin{bmatrix} S_m & B_m \\ A_m & C_m\end{bmatrix}\begin{matrix} Y \\ F\end{matrix}$$

i.e. S_m is a (linear) operator on Y, $A_m\in L(Y,F)$ etc. Clearly

$$\|S_m\|_{L(Y)}\leqslant b<cn^{(1/2)(1/2-1/q)}D^{-1/2}\leqslant c'(\delta)\,n^{1/2-1/q}$$

by (3), if c is chosen so that

$$c\leqslant c'(\delta) \tag{4}$$

(actually for any c if $n\geqslant n(c,\delta,q)$). Hence, by Corollary 2.3 (ii), $\|S_m-\lambda_m I\|_{C_0}\leqslant\delta n$ for some $\lambda_m\in\mathbf{R}$; we can certainly assume that $\|S_m-\lambda_m I\|_{C_0}=\inf_{\lambda\in\mathbf{R}}\|S_m-\lambda I\|_{C_0}$. Clearly $\lambda_0=0$ and $\lambda_M=1$ (or $\lim_{m\to\infty}\lambda_m=1$ if $M=\infty$). Notice also that since $P_{m+1}-P_m$ is a rank one operator, we have also rank $(S_{m+1}-S_m)\leqslant 1$ and so $\|S_{m+1}-S_m\|_{C_0}\leqslant 1$. Consequently $(|\lambda_{m+1}-\lambda_m|\wedge 1)\,n=\|\lambda_{m+1}I-\lambda_m I\|_{C_0}\leqslant\|S_{m+1}-\lambda_{m+1}I\|_{C_0}+\|S_m-\lambda_m I\|_{C_0}+\|S_{m+1}-S_m\|_{C_0}\leqslant 2\delta n+1$ and so $|\lambda_{m+1}-\lambda_m|<3\delta$ provided that

$$\delta\leqslant 1/3 \tag{5}$$

and $n\geqslant\delta^{-1}$. Since it is clearly enough to prove the proposition for sufficiently large n, we only need to worry about the condition in (5). It now follows that there exists $k\leqslant M$ such that $|\lambda_k-\frac{1}{2}|<2\delta$ (choose the smallest m such that $\lambda_m\geqslant\frac{1}{2}$, then $k=m$ or $k=m-1$ works). Hence $\|\lambda_k I-\frac{1}{2}I\|_{C_0}<2\delta n$ and so $\|S_k-\frac{1}{2}I\|_{C_0}\leqslant 3\delta n$. By Lemma 2.4,

$$\|S_k^2-\tfrac{1}{4}I\|_{C_0}\leq 3\sqrt{3\delta}\,n.$$

Since P_k is a projection, $P_k=P_k^2$ and so $S_k=S_k^2+B_kA_k$. Consequently

$$\tfrac{1}{4}I=(S_k^2-\tfrac{1}{4}I)+B_kA_k-(S_k-\tfrac{1}{2}I)$$

and

$$\tfrac{1}{4}n=\|\tfrac{1}{4}I\|_{C_0}\leq\|S_k^2-\tfrac{1}{4}I\|_{C_0}+\|S_k-\tfrac{1}{2}I\|_{C_0}+\|B_kA_k\|_{C_0}\leq(3\sqrt{3\delta}+\delta)\,n+\|B_kA_k\|_{C_0}. \tag{6}$$

Now $T=B_kA_k$ obviously factors through an n-dimensional subspace of F and so $\gamma_2(T)\leq\|B_k\|\cdot\|A_k\|\cdot D\leq b^2D$. Hence, by Corollary 2.3 (iii),

$$\|B_kA_k\|_{C_0}\leq(\alpha+c_0'(\alpha)\cdot b^2Dn^{1/q-1/2})\,n<(\alpha+c^2c_0'(\alpha))\,n$$

(the second inequality follows from (3)). This combined with (6) gives

$$1/4<(3\sqrt{3\delta}+\delta)+(\alpha+c^2c_0'(\alpha))$$

and if α, δ and c are chosen so that, in addition to (4) and (5),

$$\alpha=1/12,\quad 3\sqrt{3\delta}+\delta\leq 1/12,\quad c^2\leq c_0'(1/12)^{-1}\cdot 1/12,$$

a contradiction is obtained. This concludes the proof of Proposition 3.1.

Remark 3.2. A similar statement (with different constant c) can be shown for the "decomposition constant" of any finite dimensional decomposition with dimensions of factors not exceeding $n/2$ (cf. Proposition 5.1).

4. Construction of the space

If (W_n) is a sequence of normed spaces, one defines the space

$$(\oplus_n W_n)_{l_2}=\{(w_n):\sum_n\|w_n\|^2<\infty\},$$

endowed with the norm $\|(w_n)\|=(\Sigma_n\|w_n\|^2)^{1/2}$. We have the following result, of which (the real version of) Theorem 1.1 is an immediate corollary.

PROPOSITION 4.1. *For proper choice of sequences of reals $q_k\downarrow 2$ and integers $n_k\uparrow\infty$, the space $Z=(\oplus Y_{q_k}^{n_k})_{l_2}$ does not have a basis (with Y_q^n's given by Proposition 3.1). Moreover, it does not have the local basis structure.*

Proof. Define n_k, q_k inductively by

(1) $n_1=q_1=4$

(2) if (n_j), (q_j) were defined for $j<k$, choose $q_k \in (2, q_{k-1})$ such that

(a) $n_{k-1}^{1/2-1/q_k} \leq 2$

and then $n_k > n_{k-1}$ such that

(b) $n_k^{(1/2)(1/2-1/q_k)} \geq n_{k-1}$.

Now fix $k \geq 2$ and let $Z_0 \subset Z$ be a subspace such that, identifying $Y_{q_k}^{n_k}$ with its natural embedding into Z, we have $Y_{q_k}^{n_k} \subset Z_0$. Then $Z_0 = Y_{q_k}^{n_k} \oplus_2 F$, where $F \subset (\oplus_{j \neq k} Y_{q_j}^{n_j})_{l_2}$. We claim that if E is an n_k-dimensional subspace of F, then E is $n_{k-1}^{1/2}$-Euclidean. Indeed, denoting by Q_j the natural projection of Z onto the jth factor, we have $E \subset (\oplus_{j \neq k} Q_j E)_{l_2}$ and

$$d(Q_j E, l_2^{\dim Q_j E}) \leq (\dim Q_j E)^{1/2} \leq n_j^{1/2} \leq n_{k-1}^{1/2} \quad \text{for } j<k$$

(by John's theorem), while

$$d(Q_j E, l_2^{\dim Q_j E}) \leq (\dim Q_j E)^{1/2-1/q_j} \leq n_k^{1/2-1/q_{k+1}} \leq 2 \leq n_{k-1}^{1/2} \quad \text{for } j>k$$

by the result of Lewis [15], (1) and (2) (a). Hence, by Proposition 3.1 and (2) (b),

$$\operatorname{bc}(Z_0) \geq c n_k^{(1/2)(1/2-1/q_k)} n_{k-1}^{1/2} \geq c n_{k-1}^{1/2}.$$

As $n_k \uparrow \infty$, it follows that Z does not have neither basis nor local basis structure.

Remark 4.2. Clearly the assertion of Proposition 4.1 remains true if we replace Z by $Z \oplus_2 l_2$, Z^* or $Z^* \oplus_2 l_2$ (cf. Proposition 1.5 and Corollary 1.6).

Remark 4.3. The construction we present can be modified to yield a space which has all properties of the example from [32]: does not admit complex structure, is not isomorphic to the Cartesian square of any Banach space. Also, the space Z from Proposition 4.1 is naturally isomorphic to a subspace of a Banach lattice, which is 2 convex and q-concave for any $q>2$, in particular is of type 2 and cotype q for any $q>2$. This should be compared with the result of Szankowski [28] that if every subspace of X has the AP, then X is of type p for any $p<2$ and of cotype q for every $q>2$ and with Proposition 1.3 from this paper. Of course, in view of e.g. Proposition 1.5 this is not always the case for spaces without LBS, but one can still ask if every such space has the "coptype 2^+" property. Equivalently, if X contains l_q^n's uniformly for some $q>2$, does X have LBS?

5. Postponed proofs, additional results, remarks

Proof of Proposition 1.4. Suppose that X contains l_∞^n's uniformly, i.e. for each $n\in\mathbf{N}$ there exists $F_n\subset X$, $\dim F_n=n$, such that $d(F_n, l_\infty^n)\leq C_0$ (C_0 is a universal constant). Let E be any finite dimensional (say $\dim E=k$) subspace of X. It is enough to show that there exists E_0, $E\subset E_0\subset X$ such that $\mathrm{bc}(E_0)\leq C_1$ (C_1 is a universal constant). To this end, choose $n\geq 5^{k^2}(5^k+1)$ and $\mathcal{F}\subset B(X^*)$, $\#\mathcal{F}\leq 5^k$, which is 1/2-norming for E (i.e. $\max_{f\in\mathcal{F}}|f(x)|\geq 1/2\|x\|$ for $x\in E$). Let $E_1=F_n\cap\bigcap_{f\in\mathcal{F}}\ker f$, then the codimension of E_1 in F_n is $\leq 5^k$ and it follows that there exists a subspace $E_2\subset E_1$, $\dim E_2\geq 5^{k^2}$, with $d(E_2, l_\infty^{\dim E_2})\leq C_0$ (use the fact that if G is an m-codimensional subspace of $\mathbf{R}^n$, then G intersects nontrivially every subspace spanned by $m+1$ elements of the standard unit vector basis and hence G contains $[n/(m+1)]$ "disjoint" nonzero vectors; or one can apply [9, Corollary 6.2] with a slightly different choice of n). Since $\mathcal{F}$ is 1/2-norming for E, the projection of $E+E_2$ onto E with kernel E_2 is of norm ≤ 2. Finally recall that, by [20], there exists a k^2-dimensional space E_3 with $\mathrm{bc}(E_3)\leq 2$ such that E is isometric to a subspace of E_3, complemented by a projection of norm ≤ 2; denote by E_4 the kernel of that projection. Now E_4 is 2-isomorphic to a subspace of $l_\infty^{5^{k^2}}$ and hence $2C_0$-isomorphic to a subspace E_5 of E_2. Set $E_0=E+E_5$, then $d(E_0, E_3)\leq 50C_0$ by a standard argument; in particular $\mathrm{bc}(E_0)\leq 100C_0$.

Proof of Proposition 1.5. We only need to show a version of Proposition 3.1, i.e. if n, q, Y, D and F are as in Proposition 3.1 and, say, $q\leq 4$ (this is enough for the proof of Proposition 4.1), then, denoting $Z=Y\oplus_2 F\oplus_2 B$ and identifyng Y in a canonical way with a subspace of Z we have that whenever $Y\subset Z_0\subset Z$, then $\mathrm{bc}(Z_0)\geq c(B)\, n^{1/2(1/2-1/q)}D^{-1/2}$. Again let (P_m) be the sequence of partial sum projections for some basis of Z_0 with $\sup_m\|P_m\|=b$. Since clearly $Z_0=Y\oplus_2 Z_1$ for some $Z_1\subset F\oplus_2 B$, we can consider, for each m, the matrix decomposition

$$P_m=\begin{array}{c} \\ \end{array}\begin{bmatrix} S_m & B_m \\ A_m & C_m \end{bmatrix}\begin{array}{l} Y \\ Z_1 \end{array}.$$

To complete the argument as in the case of Proposition 3.1 (see (6) and the paragraph following it), we need to show that $\gamma_2(B_mA_m)\leq C'(B)\cdot b^2D$. To this end, observe first that Y, being a subspace of L_q with $q\leq 4$ has type 2 constant bounded independently of q, n. Next, notice that A_m takes values in an at most n-dimensional subspace of Z_1, say

Z_2, and then it is easily seen that the cotype 2 constant of Z_2 does not exceed the maximum of D and the cotype 2 constant of B. Now for an operator u between a type 2 and a cotype 2 space we have the estimate $\gamma_2(u) \leqslant C \cdot \|u\|$ (with C being the product of respective type 2 and cotype 2 constants; see [22] for the most general fact in this direction); since $\gamma_2(B_m A_m) \leqslant \|B_m\| \cdot \gamma_2(A_m) \leqslant b \cdot \gamma_2(A_m)$ and $\|A_m\| \leqslant b$, the required estimate follows.

Remark 5.1. An argument similar to, but simpler than, the proof of Proposition 2.2 shows that if in Proposition 1.5 we replace B by c_0 (or any $\mathcal{L}_\infty$-space) we still get that the resulting direct sum does not have a basis (nor LΠBS; use Grothendieck's result [15, Theorem 4.3]). This was observed during a conversation between Bill Johnson and the author and led to Proposition 1.5. We preferred to emphasize the version given in Proposition 1.5 as it yields reflexive examples.

PROPOSITION 5.2. *Given n there exists an n-dimensional normed space X such that whenever Y is another normed space and $A: X \to Y$, $B: Y \to X$ operators such that $BA = I_X$, then*

$$\dim Y \cdot \|A\| \cdot \|B\| \cdot \mathrm{bc}(Y) \geqslant c\left(\frac{n}{1+\ln n}\right)^{3/2}$$

Proof. Let $X = (\mathbf{R}^n, \|\cdot\|)$ be the space given by [31, Corollary 1.6]. Then, for any $T \in L(X)$,

$$\inf \|T - \lambda I\|_{C_1} \leqslant C[n(1+\ln n)^3]^{1/2} \|T\|_{L(X)}, \tag{7}$$

where $\|\cdot\|_{C_1} = \Sigma_j s_j(\cdot)$ is the usual trace class (nuclear) norm. Now let Y, A and B be as in the proposition and denote $b = \mathrm{bc}(Y)$. Assume for simplicity that $n = 2k$ and $\dim Y = mk$ for some $k, m \in \mathbf{N}$. Then $I_Y = \Sigma_{j=1}^m P_j$, where each P_j is a projection, $\operatorname{rank} P_j = k$ and $\|P_j\| \leqslant 2b$. Consequently, $I_X = \Sigma_{j=1}^m BP_jA$; denote $T_j = BP_jA$. Then $\operatorname{rank} T_j \leqslant k$, $\|T_j\| \leqslant 2b\|A\| \cdot \|B\|$ and so, by (7),

$$\inf_{\lambda \in \mathbf{R}} \|T_j - \lambda I\|_{C_1} \leqslant 2bC\|A\| \cdot \|B\| \, [n(1+\ln n)^3]^{1/2}. \tag{8}$$

Since $\operatorname{rank} T_j \leqslant k$, T_j vanishes on a k-dimensional subspace of $\mathbf{R}^n = \mathbf{R}^{2k}$ and hence, for any $\lambda \in \mathbf{R}$,

$$\|T_j - \lambda I\|_{C_1} \geqslant k|\lambda| = \tfrac{1}{2}\|\lambda I\|_{C_1}$$

Combining this with (8) we get

$$\|T_j\|_{C_1} \leqslant \inf_{\lambda \in \mathbf{R}} (\|T_j - \lambda I\|_{C_1} + \|\lambda I\|_{C_1}) \leqslant 6bC\|A\| \cdot \|B\| \, [n(1+\ln n)^3]^{1/2}.$$

Now

$$n = \|I\|_{C_1} = \left\| \sum_{j=1}^m T_j \right\|_{C_1} \leqslant \sum_{j=1}^m \|T_j\|_{C_1} \leqslant m \cdot 6bC\|A\| \cdot \|B\| \, [n(1+\ln n)^3]^{1/2},$$

whence the proposition follows (remember $b = \mathrm{bc}(Y)$ and $m = 2n^{-1} \dim Y$).

Remark 5.3. The assertion of Proposition 5.2 says roughly that if X is "well-isomorphic" to a "well-complemented" subspace of a space Y with a "good" basis, then $\dim Y \geqslant C(n/(1+\ln n))^{3/2} >> n = \dim X$. On the other hand, a modification of the argument from [20, Remark 2] shows that one can always have $\|A\| = 1$, $\|B\| \leqslant 2$, $\mathrm{bc}(Y) \leqslant 2$ and $\dim Y < (1+n^{1/2})^3$.

Remark 5.4. An interesting related open problem (which the author heard from A. Pelczynski) is whether every n-dimensional normed space X is isometric to a subspace of a $2n$-dimensional space Y with $\mathrm{bc}(Y) \leqslant C'$, C' a universal constant. We get this for free if $X = Y_q$ from Corollary 2.3. It would be interesting to verify this for X from Theorem 2.1.

PROPOSITION 5.5. *If* $\dim Y = n$ *and* $d(Y, l_2^n) \leqslant D$, *then*

$$\mathrm{bc}(Y \oplus_2 l_2) \leqslant \tfrac{1}{2}(1 + D^{1/2}).$$

Moreover, if $\varepsilon > 0$ *and* $N \geqslant C(\varepsilon) nD$, *then*

$$\mathrm{bc}(Y \oplus_2 l_2^N) \leqslant \tfrac{1}{2}(1+\varepsilon)(1 + D^{1/2}). \tag{9}$$

Consequently, the estimate $\mathrm{bc}(Y \oplus_2 l_2) \geqslant c n^{(1/2)(1/2 - 1/q)}$ *in Proposition* 3.1 *is sharp.*

Sketch of the proof (cf. [20], [21]). Let (x_j) be a basis of Y such that

$$D^{-1/2} \left(\sum |t_j|^2 \right)^{1/2} \leqslant \left\| \sum t_j x_j \right\| \leqslant D^{1/2} \left(\sum |t_j|^2 \right)^{1/2} \tag{10}$$

for all scalars (t_j). Let $m = 2^s$ for some $s \in \mathbf{N}$ and let $w = (w_{ij})_{i,j=1}^m$ be the $m \times m$ Walsh orthogonal matrix, in particular $w_{i1} = w_{1i} = m^{-1/2}$ for all i. Set $N = n(m-1)$ and if

$1 \leq k \leq N+n=mn$, say $k=(r-1)n+s$ with $1 \leq r \leq m$, $1 \leq s \leq n$, let

$$y_k = w_{1r} x_s + \sum_{t=2}^{m} w_{tr} e_{(s-1)(m-1)+t-1},$$

where (e_j) is the standard basis of l_2^N and $y_k \in Y \oplus_2 l_2^N$. Then (y_k) is a basis of $Y \oplus_2 l_2^N$. We claim that $\mathrm{bc}(y_k) \leq \frac{1}{2}(1+\varepsilon)(1+D^{1/2})$ if $m \geq c(\varepsilon) D$, whence (9) readily follows. To this end, observe that if $k=dn$, $d<m$, then the projection $P: \Sigma_1^{nm} t_i y_i \to \Sigma_1^k t_i y_i$ has the matrix representation

$$\begin{array}{cc} \begin{bmatrix} \lambda I & [\lambda(1-\lambda)]^{1/2} I & 0 \\ [\lambda(1-\lambda)]^{1/2} I & (1-\lambda) I & 0 \\ 0 & 0 & Q \end{bmatrix} & \begin{array}{c} Y \\ E \\ F \end{array} \end{array}$$

where $\lambda = d/m$, $E \oplus F$ is some orthogonal splitting of l_2^N with $\dim E = n$ and Q is some orthogonal projection in F. A standard argument (remember (10)) shows that

$$\|P\| \leq \left\| \begin{bmatrix} \lambda & [\lambda(1-\lambda)]^{1/2} D^{1/2} \\ [\lambda(1-\lambda)]^{1/2} D^{1/2} & 1-\lambda \end{bmatrix} \right\|_{L(l_2^2)} = \tfrac{1}{2}(1+D^{1/2}).$$

In the case of general $k=(r-1)n+s$, we get a very similar matrix representation, e.g. in the left upper corner the $n \times n$ submatrix λI is replaced by

$$\begin{array}{cc} \begin{bmatrix} \frac{r}{m} I & 0 \\ 0 & \frac{r-1}{m} M \end{bmatrix} & \begin{array}{l} \operatorname{span}\{x_j : 1 \leq j \leq s\} \\ \operatorname{span}\{x_j : s < j \leq n\} \end{array} \end{array}$$

and consequently its norm as an operator on Y does not exceed $r/m + D/m$ (use (10)). Similar statements hold for the remaining submatrices and so, if D/m is small when compared to ε, we get (9). The first statement from the proposition follows then right away.

COROLLARY 5.6. *If X is a Banach space, $\dim X = \infty$, and Y a finite dimensional subspace, then there exists Y_1, $Y \subset Y_1 \subset X$, such that* $\mathrm{bc}(Y_1) \leq d(Y, l_2^{\dim Y})^{1/2}$.

Proof. Use [19, Theorem 5.8] and Proposition 5.5.

6. The complex case

The complex case requires special treatment because [30, Theorem 15], stated in this paper as Theorem 2.1, was originally proved only in the real form. We have, however, PROPOSITION 6.1. *If X is the space from Theorem* 2.1, *Let $\tilde{X}$ be its complexification. More specifically, if $X=l_1^N(\mathbf{R})/E$ as provided by Theorem* 2.1 (i), *we set $\tilde{X}=l_1^N(\mathbf{C})/(E+iE)$. Then the conditions* (i)–(v) *of Theorem* 2.1 *hold with X, $\mathbf{R}$, $\mathbf{R}^n$, real l_p^k replaced by $\tilde{X}$, $\mathbf{C}$, $\mathbf{C}^n$, complex l_p^k respectively; the exponent* $1/n$ *in* (iii) *replaced by* $1/2n$ *and* (v) *restricted to* $\mathbf{C}$*-linear operators on $\tilde{X}$.*

Proof. (i) and (ii) follow directly from the definition of $\tilde{X}$. (iii) is geometrically obvious if we notice that $B(l_2^n(\mathbf{C}))=B(l_2^n(\mathbf{R})\oplus_2 l_2^n(\mathbf{R}))$, $B(l_1^N(\mathbf{C}))\subset B(l_1^N(\mathbf{R})\oplus_2 l_1^N(\mathbf{R}))$ and consequently, with canonical identifications, $B(\tilde{X})\subset B(X\oplus_2 iX)$. (iv) follows from the fact that in the identification $\tilde{X}=X\oplus iX$ the coordinate projections are of norm 1 (we may need the constant C to be twice bigger; note, however, that $\tilde{X}$ satisfies the Grothendieck theorem also for $\mathbf{R}$-linear operators). Finally, to show (v), we need to observe that any $\mathbf{C}$-linear operator T on $\tilde{X}$ can be written as

$$T=\begin{array}{c}\begin{bmatrix} A & -B \\ B & A \end{bmatrix}\end{array}\begin{array}{l} X \\ iX \end{array},$$

where $A,B\in L(X)$ and $\|A\|,\|B\|\leq\|T\|$.

Consequently, by Theorem 2.1 (v), there exist $\alpha,\beta\in\mathbf{R}$ such that $\|A-\alpha I\|_{C_0}\leq\delta n$, $\|B-\beta I\|_{C_0}\leq\delta n$. Set $\lambda=\alpha+\beta i$, then

$$\lambda I_{\tilde{X}}=\begin{bmatrix} \alpha I & -\beta I \\ \beta I & \alpha I \end{bmatrix}\begin{array}{l} X \\ iX \end{array}$$

and so $\|T-\lambda I_{\tilde{X}}\|_{C_0}\leq 2\delta n$. This shows (i) with $c(\cdot)$ replaced by $c(\frac{1}{2}\cdot)$. Note that we are using here the complex ideal quasinorm $\|\cdot\|_{C_0}$; for the real one an additional factor 1/2 would be needed.

The complex version of Theorem 1.1 follows now exactly as the real one; with proper care we can also make the complex example to be the complexification of the real example. As a bonus, we get a complex version of Corollary 1.2 from [32] (cf. Remark 4.3).

COROLLARY 6.2. *The complex space from Theorem* 1.1 *can be constructed to be nonisomorphic to the Cartesian square of any complex Banach space.*

In fact, a statement stranger than Proposition 6.1 holds.

Proposition 6.3. *Given $\delta\in(0,1)$ and $n\in\mathbf{N}$ there exists a complex normed space $X=(\mathbf{C}^n,\|\cdot\|)$ such that the conditions* (i)–(v) *of Theorem* 2.1 *are satisfied in the same sense as in Proposition* 6.1 *with* (v) *(and* (iv)*) holding for* **R**-*linear operators.*

Sketch of the proof. Of course X can not be a complexification of a real space. Instead, we have to use [31, Theorem 1.4] with $\mathbf{C}^n$ identified with $\mathbf{R}^{2n}$ and $\Gamma=\{I, iI, -I, -iI\}$. The properties (i)–(iv) follow then, as in Theorem 2.1, from the construction in [31] (see [31, Section 5 (a)]). To get (v), it is certainly enough to prove the following.

Lemma 6.4. *If $\delta\in(0,1)$ and T is* **R**-*linear on $\mathbf{C}^n$ such that $\|T-\lambda I\|_{C_0}\geq\delta n$ for every $\lambda\in\mathbf{C}$, then there exists a* **C**-*linear subspace $H\subset\mathbf{C}^n$, $k=\dim H\geq 10^{-5}\delta n$ such that if $\alpha=10^{-3}\delta$, then*

$$\|P_{H^\perp}Tx\|_2\geq\alpha\|x\|_2 \quad \text{for} \quad x\in H. \tag{11}$$

where $P_{H^\perp}$ denotes the orthogonal projection onto $H^\perp$. In the language of [31] *this means that T satisfies the "mixing" condition $(M_{k,\alpha})$.*

Proof. Let

$$T=\begin{array}{cc}\begin{bmatrix}T_{11} & T_{12}\\ T_{21} & T_{22}\end{bmatrix} & \begin{array}{l}\mathbf{R}^n\\ i\mathbf{R}^n\end{array}\end{array}$$

be the matrix representation of T. We consider the following dychotomy:

(a) there exist reals $a_{\mu\nu}$, $\mu,\nu=1,2$, such that $\|T_{\mu\nu}-a_{\mu\nu}I\|_{C_0}\leq\delta n/50$ for $\mu,\nu=1,2$

(b) for some μ,ν, $\|T_{\mu\nu}-aI\|_{C_0}>\delta n/50$ for all $a\in\mathbf{R}$.

The case (a) is easy to settle: if $A=(a_{\mu\nu}I)_{\mu,\nu}\in L(\mathbf{R}^{2n})$ then

$$\|T-A\|_{C_0}\leq 2\delta n/25 \tag{12}$$

and hence $\|A-\lambda I\|_{C_0}\geq 0.92\delta n$. An elementary argument shows then that a version of (11) with T replaced by A, $k=\dim H=[\frac{1}{2}n]$ and $\alpha=0.23\delta$ holds. Since, by (12), T is very close to A on a subspace of small codimension, (11) readily follows. The case (b) is somewhat harder. First, replacing Tx by $T(ix)$, iTx or $iT(ix)$ if necessary, we may assume that $\|T_{11}-aI\|_{C_0}>\delta n/50$ for all $a\in\mathbf{R}$. By [31, Proposition 6.1], there is $E\subset\mathbf{R}^n$, $\dim E\geq\delta n/1800$, such that

$$\|P_{E^\perp} T_{11} x\|_2 \geqslant \frac{\delta}{300} \|x\|_2 \quad \text{for} \quad x \in E.$$

Now choose by induction (cf. [31, Lemma 4.1]) an orthonormal sequence (x_j), $j=1, \ldots, m$ $(m \geqslant \frac{1}{4} \dim E)$, in E such that if $E_j = P_{E^\perp}[T_{11} x_j, T_{12}(ix_j)] \subset \mathbf{R}^n$, then E_j's are mutually orthogonal. By enumerating x_j's appropriately, we can assume that, for $j=1, 2, \ldots, 2k$ $(k \geqslant \frac{1}{8} m - 1)$, all $(P_{E^\perp} T_{11} x_j, P_{E^\perp} T_{12} ix_j))$ are of the same sign and the same is true about $\|P_{E^\perp} T_{11} x_j\|_2 - \|P_{E^\perp} T_{12}(ix_j)\|_2$. These conditions finally guarantee that if we set $h_s = 2^{-1/2}(x_{2s-1} + ix_{2s})$, then $H = [h_s : s = 1, 2, \ldots, k]$ (complex span) works in (11).

Arguing exactly as before we deduce the following strengthening of the complex version of Theorem 1.1 and Corollary 6.2.

COROLLARY 6.5. *These exists a complex superreflexive Banach space with (complex) unconditional finite dimensional decomposition which does not have a basis even in the real sense and is not real-isomorphic to a Cartesian square of any real Banach space.*

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