On the invariant subspace problem for Banach spaces

by

PER ENFLO

Institute Mittag-Leffler
Djursholm, Sweden

Royal Institute of Technology
Stockholm, Sweden

0. Introduction

In this paper we will present a new approach to the invariant subspace problem for Banach spaces. Our main result will be that there exists a Banach space B and an operator T on B such that T has only trivial invariant subspaces. We feel though that the ideas of the approach can be used also to prove results about existence of invariant subspaces. As an example of this, see [1]. In Section 1 we give the general ideas of the approach. In this section we also reduce the problem of proving our main result to the problem of proving Theorem 1.3. In Section 2 we prove an inequality which will be the basic tool in the construction. In Section 3 we first reduce the problem of proving Theorem 1.3 to the problem of proving 6 statements. These statements contain a parameter k. We first give lemmas and propositions which give these statements for $k=1$ and $k=2$. We then give the induction hypothesis and the lemmas and propositions which give the statements for all positive integers k. In Section 4, finally, we give proofs of Theorem 1.2 of Section 1 and of the lemmas and propositions of Section 3. An outline of this construction was presented in Enflo [2]. This version is the same—except for some changes in the presentation—as was given in Enflo [3]. The author wishes to thank professor Enrico Bombieri for suggesting these changes.

1. Outline of the proof

We will below construct an operator with only trivial invariant subspaces on a Banach space. The Banach space in this example will be constructed at the same time as the operator and will be non-reflexive. There are very serious difficulties in carrying out a similar construction in a reflexive Banach space. So we feel that the construction gives some weak support to the conjecture that every operator on a Hilbert space has a non-trivial invariant subspace. We now turn to the basic considerations behind this approach. It is clear that every operator with a cyclic vector on a Banach space can be

14−878289 *Acta Mathematica* 158. Imprimé le 28 juillet 1987

represented as multiplication by x on the set of polynomials under some norm. So what we will do is to construct a norm on the space of polynomials and prove that multiplication by x under this norm has only trivial invariant subspaces. Our next basic consideration is based on the fact that one can have an operator with a dense set of cyclic vectors without having all vectors cyclic. In order to be able to make some limit procedure work we will construct the operator so that it has the following property:

(1.1) Let 1 be a cyclic vector of norm 1 in B. Let (p_j) be a sequence which is dense on the unit sphere of B. For every j and every m there is a positive number $C_{j,m}$ such that for every p_n with $\|p_j-p_n\|<1/2^{m+4}$ there is a polynomial $l(T)$ in T with $\|l(T)\|_{\mathrm{op}}\leq C_{j,m}$ such that $\|l(T)p_n-1\|\leq 1/2^m$. It is easily verified that such a T has only trivial invariant subspaces.

It follows easily from the fact that the spectrum of an operator is non-empy that there is no operator for which $C_{j,m}$ depends only on j or only on m.

If we have the operator T represented as multiplication by x, then $l(T)$ will just be multiplication by the polynomial l. From now on we shall identify B with the closure of the vector space of all polynomials with real (or complex) coefficients under a suitable norm $\|\ \|$. This leads us to the next basic consideration. Assume that we have a norm $\|\ \|$ on the space of polynomials. Assume that p is a polynomial of norm 1 and assume that $\|lp-1\|\leq\varepsilon$ and $\|l\|_{\mathrm{op}}\leq K$. This gives that for every polynomial h we have the inequality

$$\|h\|-K\|hp\|\leq\|hlp-h\cdot 1\|\leq\varepsilon\|h\|_{\mathrm{op}}.$$

And this implies that

$$\text{if } \|h\|_{\mathrm{op}}\leq\frac{1}{2\varepsilon}\|h\|, \quad \text{then } \|hp\|\geq\frac{\|h\|}{2K}. \tag{1.2}$$

In order that the operator also satisfies (1.1) it is of course necessary that the inequality $\|hp\|\geq\|h\|/2K$ holds uniformly in p in every ball of size $\varepsilon/16$ on the unit sphere. (At least if we put $\varepsilon=1/2^m$.)

There is a sense in which the inequality (1.2) is sufficient for p to be moved close to 1 by a polynomial with small operator norm. This is given by our Theorem 1.2 below. In order to describe this theorem we have to tell something about the way that we construct the final norm. It should be pointed out that this sufficiency of (1.2) depends on the fact that the norm constructed is non-reflexive. We do not know whether anything similar can be done in a reflexive space.

Consider all pairs (q,ε) where q is an arbitrary polynomial whose coefficients have real and imaginary parts rational, and ε is of form 2^{-k}. We enumerate all such pairs and call the sequence (q_n, ε_n). We also insist that for a fixed q, if $n_1, n_2, \ldots$ are all the integers such that $q_n = q$, then $\varepsilon_{n_1} > \varepsilon_{n_2} > \varepsilon_{n_3} \ldots$. Also we assume $\deg q_n \leq n$.

Our construction will be completely determined by a sequence of polynomials l_n and constants $C_n > 2$. $l_1, \ldots, l_k$ and $C_1, \ldots, C_k$ will determine a number α_{k+1} inductively as explained below and we define a sequence of norms as in the following definitions.

Definition 1. For any polynomial p, consider all representations $p = \Sigma\, a_{i,\beta}\, x^i l_1^{\beta_1} \ldots l_n^{\beta_n}$ and put

$$|p|_{\mathrm{op}\,n} = \inf \sum |a_{i,\beta}|\, 2^i (C_1 |l_1|_1)^{\beta_1} \ldots (C_n |l_n|_1)^{\beta_n}$$

where $|\ |_1$ denotes the usual l_1 norm equal to the sum of the absolute values of the coefficients.

Remark. In the final norm the operator x will have norm ≤ 2, and multiplication by l_k norm $\leq C_k |l_k|_1$.

Definition 2. For any p, consider all representations

$$p = r + \sum_1^n S_k (l_k \alpha_k q_k - 1).$$

Put $|p|^n = \inf |r|_1 + \Sigma\, |S_k|_{\mathrm{op}\,n}\, \varepsilon_k$. Put $|p|^0 = |p|_1$, and let α_k be determined inductively by the condition $|\alpha_k q_k|^{k-1} = 1$.

Remark. $|l_k \alpha_k q_k - 1|^n < \varepsilon_k$ and clearly the operator norm of multiplication by g is $|g|_{\mathrm{op}\,n}$. We see that $|\ |^n$ is the maximal norm satisfying the following four properties:

(1) $|\ |^n \leq |\ |_1$

(2) $|l_k \alpha_k q_k - 1|^n \leq \varepsilon_k,\ k = 1, 2, \ldots n,$

(3) $|g|_{\mathrm{op}} \leq |g|_{\mathrm{op}\,n},$

(4) $|x|_{\mathrm{op}} < 2.$

Observe that $|\ |^n$ and $|\ |_{\mathrm{op}\,n}$ are decreasing sequences of norms and hence converge to some pseudo-norms. We write $\|\ \| = \lim |\ |^n$.

THEOREM 1.1. *Assume that (C_n) and (l_n) are sequences which define norms $|\ |^m$ as above and assume that there are sequences of positive numbers $D_n \nearrow \infty$ and $L_n \nearrow \infty$ so that the following holds:*

(I) $|p|^m$ *is constant for* $m \geq (\deg p)-1$. *In particular* $|q_n|^m$ *is constant for* $m \geq n-1$.

(II) *For any* n, *consider all* $k \leq n$ *such that* $\varepsilon_n = \varepsilon_k$, *and* $|\alpha_k q_k - \alpha_n q_n|^{(n-1)} < \varepsilon_n/16$. *Let* K *be the least such* k. *Then* $|l_n|_1 = L_K$, $C_n = D_K$.

Then the resulting limit norm defines a space B, *for which multiplication by* x *has no invariant subspace.*

Proof. Let q be an element of B, which we recall is the closure of all polynomials and $\|q\|=1$. Let ε be a fixed negative power of 2. Choose increasing n_k such that $\varepsilon_{n_k}=\varepsilon$ and $\alpha_{n_k} q_{n_k} \to q$ in B. We can even insist that $\|\alpha_{n_k} q_{n_k} - q\| < \varepsilon/64$.

Hence for $k>1$,

$$|\alpha_{n_k} q_{n_k} - \alpha_{n_1} q_{n_1}|^{(n_k-1)} = \|\alpha_{n_k} q_{n_k} - \alpha_{n_1} q_{n_1}\| < \frac{\varepsilon}{32},$$

so by (II), $C_{n_k} \leq \max_{m \leq n_1} D_m$ and $|l_{n_k}|_1 \leq \max_{n \leq n_1} L_m$ so that $|l_{n_k}|_{\mathrm{op}\, n_k}$ is bounded $\leq A$. Therefore

$$\begin{aligned} \|l_{n_k} q - 1\| &\leq \|l_{n_k} \alpha_{n_k} q_{n_k} - 1\| + \|l_{n_k}(q - \alpha_{n_k} q_{n_k})\| \\ &\leq \varepsilon + |l_{n_k}|_{\mathrm{op}\, n_k} \|q - \alpha_{n_k} q_{n_k}\| \\ &\leq \varepsilon + A\|q - \alpha_{n_k} q_{n_k}\|. \end{aligned}$$

Letting k tend to infinity, we see that 1 is within distance ε of the space generated by q and hence, letting $\varepsilon \to 0$, we see that 1 is in that space and hence it equals B.

We will now drop α_k in our notation so when it is clear from the context we will denote $\alpha_k q_k$ by q_k and assume $|q_k|^{k-1}=1$.

Definition 3. $\operatorname{ord} p$=degree of lowest order term of the polynomial p.

Definition 4. Let f be a positive real valued function defined on $(0, \infty)$. We say that $l = \sum_{j \geq 0} a_j x^{n_j}$ is more lacunary than f if

$$\operatorname{ord} l = n_0 \geq f(0)$$

and

$$n_j \geq f(n_{j-1}) \quad \text{for every } j.$$

Our next theorem which we prove in Section 4, shows that, under the assumption of an inequality similar to (1.2) we can satisfy condition (I) as soon as the polynomials l_n are lacunary enough.

THEOREM 1.2. *Let $l_1, \ldots, l_N$, $C_1, \ldots, C_N$ be given with $C_k > 2$. Assume for all h and some B that*

$$\frac{|h|_{\mathrm{op}\,N}}{|h|^0} < \frac{1}{\varepsilon_{N+1}} \quad \Rightarrow \quad |hq_{N+1}|^N \geq \frac{|h|^0}{B}.$$

Then, given $K > 4B/\varepsilon_{N+1}$, there exists a lacunarity function f such that if

(1) $|l_{N+1}|_1 = K$,

(2) *the lacunarity of* $l_{N+1} \geq f$,

(3) $C_{N+1} > 2$,

then with this choice of l_{N+1} and C_{N+1} we have

$$|g|^{N+1} = |g|^N \quad \text{for all } g \text{ with } \deg g \leq N.$$

We now assume that we have two sequences $D_n \nearrow \infty$ and $L_n \nearrow \infty$. Assume that $|\ |^{n-1}$ is defined. We will then define $|\ |^n$ according to the following rule: consider all $k \leq n$ such that $\varepsilon_k = \varepsilon_n$ and $|q_k - q_n|^{n-1} < \varepsilon_n/16$. Let K be the least such k. Then $|l_n|_1 = L_K$, $C_n = D_K$. If this rule is fulfilled for all $n \leq N$, we say that $|\ |^N$ is defined in a compatible way from the sequences D_n and L_n. If for every N, $|\ |^N$ is defined in a compatible way from the sequences D_n and L_n, then obviously condition (II) of Theorem 1.1 is fulfilled. Our next theorem combined with Theorem 1.2 will now enable us to get also the condition (I) of Theorem 1.1 fulfilled. We first make

Definition 5. A growth function F is a function that for every n and every $3n$-tuple $D_1, \ldots, D_n$, $L_1, \ldots, L_n$, $l_1, \ldots, l_n$ gives a positive number $F(D_1, \ldots, D_n, L_1, \ldots, L_n, l_1, \ldots, l_n)$, and for every n and every $(3n+2)$-tuple $D_1, \ldots, D_{n+1}$, $L_1, \ldots, L_{n+1}$, $l_1, \ldots, l_n$ gives a lacunarity function f and a positive number δ. We say that the sequence $\{D_n, L_n, l_n, C_n\}$ grows faster than F if

(1) l_k and C_k are defined in a compatible way from the sequences D_n and L_n for every k.

(2) For every n, D_{n+1} and L_{n+1} are $> F(D_1, \ldots, D_n, L_1, \ldots, L_n, l_1, \ldots, l_n)$.

(3) For every n the lacunarity of $l_{n+1} \geq f$ and the moduli of the coefficients of l_{n+1} are $\leq \delta$ where f and δ are given by the growth function applied to $D_1, \ldots, D_{n+1}$, $L_1, \ldots, L_{n+1}$, $l_1, \ldots, l_n$.

We will, by slight abuse of language, say that a number depends only on $|\ |^m$ thus meaning that it is determined by $D_1, \ldots, D_m$, $L_1, \ldots, L_m$, $l_1, \ldots, l_m$, $C_1, \ldots, C_m$. (Obviously different such sequences could give the same $|\ |^m$.)

We now have

THEOREM 1.3. *There is a growth function F such that if $\{D_n, L_n, l_n, C_n\}$ grows faster than F, then for every n there exists B_n depending only on $|\ |^{n-1}$, such that for all $N \geq n$*

$$|q-q_n|^N < \frac{\varepsilon_n}{16} \quad \textit{and} \quad \frac{|h|_{\mathrm{op}\,N}}{|h|^0} < \frac{1}{\varepsilon_n} \quad \textit{imply} \quad |hq|^N \geq \frac{|h|^0}{B_n}.$$

We now combine this theorem with Theorem 1.2 to give also (I) of Theorem 1.1. The main difficulty in the construction is to prove Theorem 1.3. This will be done in the following sections.

Completion of the construction assuming Theorem 1.3.

$$\text{For every } N \geq 0, \text{ choose } L_{N+1} \text{ and} \\ D_{n+1} > \max\{F(D_1, \ldots, D_n, L_1, \ldots, L_n, l_1, \ldots, l_n), 4B_{N+1}/\varepsilon_{N+1}\}. \tag{A}$$

Now we assume that

$$\text{for every } r \leq N \text{ we have defined } L_1, \ldots, L_r, D_1, \ldots, D_r, l_1, \ldots, l_r, C_1, \ldots, C_r \\ \text{according to (A) and the growth function } F. \tag{A'}$$

Assuming this we will choose l_{N+1} by the following considerations (B)–(E):

Take the smallest $n \leq N+1$ such that $|q_{N+1}-q_n|^N < \varepsilon_n/16$ and $\varepsilon_n = \varepsilon_{N+1}$. Then by Theorem 1.3

$$\frac{|h|_{\mathrm{op}\,N}}{|h|^0} < \frac{1}{\varepsilon_{N+1}} \quad \Rightarrow \quad |hq_{N+1}|^N \geq \frac{|h|^0}{B_n}.$$

By the compatibility assumption and (A) we now choose

$$|l_{N+1}|_1 = L_n > 4B_n/\varepsilon_n. \tag{B}$$

$$C_{N+1} = D_n > 2. \tag{C}$$

By Theorem 1.2 by choosing l_{N+1} lacunary enough we then get

$$|p|^N = |p|^{N+1} \quad \text{for degree } p \leq N. \tag{D}$$

We choose l_{N+1} more lacunary and with smaller moduli of the coefficients than what is given by

$$F(D_1, \ldots, D_{N+1}, L_1, \ldots, L_{N+1}, l_1, \ldots, l_N). \tag{E}$$

By choosing the sequence $\{D_n, L_n, l_n, C_n\}$ according to (A)–(E), we thus get the following: $\{D_n, L_n, l_n, C_n\}$ is compatible by (B) and (C) and it grows faster than F by (A), (A′) and (E) so it satisfies the assumptions of Theorem 1.3.

By (B) and (C) it also satisfies (II) of Theorem 1.1 and by (D) it satisfies (I) of Theorem 1.1. Thus for the limit norm multiplication by x has only trivial invariant subspaces. So in fact Theorems 1.1, 1.2 and 1.3 give

THEOREM 1.4. *There is a growth function F such that if $\{D_n, L_n, l_n, C_n\}$ grows faster than F, then* $\lim |\ |^n$ *is a norm for which multiplication by x has only trivial invariant subspaces.*

2. An inequality for products of polynomials

Before continuing with the construction, we will use this section to prove the following theorem which we will need many times in Section 3, where the actual construction continues.

THEOREM 2.1. *Let A, B be homogeneous polynomials in many variables of degree d_1, d_2. Then*

$$|AB|_1 \geq K(d_1, d_2)\, |A|_1\, |B|_1.$$

Remark. The essential point is that K is independent of the number of variables. A bound depending on the number of variables is trivial since $AB \neq 0$ is not zero for $A, B \neq 0$, and the spaces of all A and B would then be finite dimensional. All norms in this section are l_1-norms.

In order to prove Theorem 2.1, we will analyze a more general situation. Let $A_1, \ldots, A_n$ denote homogeneous polynomials of fixed degree, but having no restriction on the number of variables. Let P be a polynomial of n variables of fixed degree which is "isobaric" so that $P(A_1, \ldots, A_n)$ is homogeneous of fixed degree. We shall study the case when $P(A_1, \ldots, A_n)$ is small.

From now on we consider a sequence of such $P, A_1, \ldots, A_n$ and drop any index to denote the term of the sequence. Quantities or polynomials whose norms tend to zero are denoted $o(1)$. Bounded quantities are denoted by $\ll 1$, and bounded from below by $\gg 1$. If we write $|P|$ we mean the l_1-norm of P as a function of n independent variables, and not $|P(A_1, \ldots, A_n)|$.

THEOREM 2.2. *Assume*

(1) $|A_i| \ll 1$, $|P| \ll 1$

(2) $|P(A_1, \ldots, A_n)| = o(1)$.

Then for some subsequences of the $P, A_1, \ldots, A_n$, *there exist an integer* m, *polynomials* $Q_1, \ldots, Q_n$, *in* m *variables and polynomials* $B_1, \ldots, B_m$ *of bounded degree such that*

(3) $A_i = Q_i(B_1, \ldots, B_m) + o(1)$, $|Q_i| \ll 1$, $|B_i| \ll 1$.

(4) *If* $P(Q_1(t_1, \ldots, t_m), \ldots, Q_n(t_1, \ldots, t_m)) = R(t_1, \ldots, t_m)$, *then* $|R| = o(1)$.

Remark. The number m and $\deg B_i$, $\deg Q_i$ admit bounds depending only on $\deg A_i$, $\deg P$. The polynomials B consist of various derivatives of the A_i. Note also that if (3) and (4) hold, clearly $|P(A)| = o(1)$.

Theorem 2.1 can now be deduced from Theorem 2.2. Assume $|A_1 A_2| = o(1)$, and $A_1 = Q_1(B_j)$, $A_2 = Q_2(B_j)$ satisfying (3) and (4), and $|A_i| = 1$. Then clearly $|Q_i| \gg 1$. But then $Q_1(t)\, Q_2(t)$ cannot be $o(1)$, since this violates Theorem 2.1 in the case in which the number of variables is bounded.

We use the following notation: If $R(t_1, \ldots, t_n)$ is a polynomial, $|R|(|A_1|, \ldots, |A_n|)$ denotes the value obtained by replacing each coefficient of R by its absolute value and substituting $|A_i|$ for t_i. If $A_i = \lambda_i B_i$, and $S(t_1, \ldots, t_m) = R(\lambda_1 t_1, \ldots, \lambda_m t_m)$, then clearly $|S|(|B_1|, \ldots, |B_m|) = |R|(|A_1|, \ldots, |A_m|)$. This will be used shortly.

For any polynomial $A(z_1, \ldots)$ let $A^{(i)}$ denote $\partial A / \partial z_i$. Now the following lemma is obvious:

LEMMA 2.1. *If* A *is homogeneous of degree* d, $\Sigma |A^{(i)}| = d|A|$.

We made the convention that all polynomials must have degree >0. For example, in the following lemma, if $\deg A_k = 1$, the variable $A_k^{(i)}$ does not occur, but is treated as a constant.

LEMMA 2.2. *Let*

$$[P(A_1, \ldots, A_n)]^{(i)} = \sum \frac{\partial P}{\partial A_k} A_k^{(i)} = R_i(A_1, \ldots, A_n, A_1^{(i)}, \ldots, A_n^{(i)}).$$

Then

$$\sum |R_i|(|A_1|, \ldots, |A_n|, |A_1^{(i)}|, \ldots, |A_n^{(i)}|) \leq K \cdot |P|(|A_1|, \ldots, |A_n|)$$

where K depends only on $\deg P$ *and* $\deg A_k$.

Proof. We can reduce to the case where P is a monomial where the proof is immediate.

Theorem 2.2 will be derived from Theorem 2.3 below.

THEOREM 2.3. *Assume again* $|A_i| \ll 1$, $|P| \ll 1$, *and let* $\deg A_1$ *be maximal among* $\deg A_i$. *Let*

$$P = A_1^r C_0(A_2, \ldots, A_n) + A_1^{r-1} C_1(A_2, \ldots, A_n) + \ldots + C_r(A_2, \ldots, A_n).$$

If $|C_0(A_2, \ldots, A_n)| \gg 1$, *then there are* Q *and* B_j *with* $\deg B_j < \deg A_1$, $|Q| \ll 1$, $|B_j| \ll 1$, *and* $A_1 = Q(A_2, \ldots, A_n, B_1, \ldots, B_m) + o(1)$. *Furthermore, the degree of each monomial in* $Q(A, B)$ *is equal to* $\deg A_1$.

Proof. Assume first that $\deg C_0 > 0$. Then

$$\begin{aligned}[P(A_1, \ldots, A_n)]^{(i)} &= A_1^r[C_0(A_2, \ldots, A_n)]^{(i)} + A_1^{r-1} \ldots \\ &= R_i(A_1, \ldots, A_n, A_1^{(i)}, \ldots, A_n^{(i)}).\end{aligned}$$

We have

(1) $\Sigma |[C_0(A_2, \ldots, A_n)]^{(i)}| \gg 1$,
(2) $\Sigma |[P(A_1, \ldots, A_n)]^{(i)}| = o(1)$, and
(3) $\Sigma |R_i|(|A_1|, \ldots, |A_n|, |A_1^{(i)}|, \ldots, |A_n^{(i)}|) \ll 1$.

From (1), (2), and (3) it follows that for some i, if $\alpha = |[C_0(A_2, \ldots, A_n)]^{(i)}|$, then

$$\frac{1}{\alpha}|[P(A_1, \ldots, A_n)]^{(i)}| = o(1)$$

and

$$\frac{1}{\alpha}|R_i|(|A_1|, \ldots, |A_n|, |A_1^{(i)}|, \ldots, |A_n^{(i)}|) \ll 1.$$

If we write $A_k^{(i)} = \lambda_k B_k$ so that $|B_k| = 1$, and put $S = R_i/\alpha$, it follows that

$$S = A_1^r D(A_2, \ldots, A_n, B) + A_1^{r-1} \ldots, \quad |S| \ll 1,$$

$$|D(A_1, \ldots, A_n, B)| \gg 1 \quad \text{and} \quad |S(A_1, \ldots, A_n, B)| = o(1).$$

S is a polynomial which, although involving more variables than P, clearly has "total degree" one less than that of P. By induction then the result will follow.

We now handle the case where C_0 is a constant. In this case $[P(A)]^{(i)} = A_1^{r-1}(rC_0 A_1{}^{(i)} + C_1^{(i)}) + A_1^{r-2} \ldots$. Now, if $rC_0 A_1 + C_1 = o(1)$, since $C_0 \gg 1$, we have $A_1 = (rC_0)^{-1} C_1 + o(1)$ and the result follows. If $|rC_0 A_1 + C_1| \gg 1$, then we have $\Sigma_i |rC_0 A_1^{(i)} + C_1^{(i)}| \gg 1$ and the proof proceeds as before. Thus Theorem 2.3 is proved.

Now we prove Theorem 2.2. We write, as in the proof of Theorem 2.3, $P = A_1^r C_0(A_2, \ldots, A_n) + \ldots$. If $|C_0(t_2, \ldots, t_n)| = o(1)$, then the first term can clearly be dropped and we have lowered the degree to which A_1 appears in P. If $|C_0(t_2, \ldots, t_n)| \gg 1$, and $|C_0(A_2, \ldots, A_n)| \gg 1$, then we apply Theorem 2.3. Replacing A_1 by $Q(A_2, \ldots, A_n, B) + o(1)$ leads to another polynomial R in which A_1 is absent and $|R(A_2, \ldots, A_n, B)| = o(1)$. If $|R(t_2, \ldots, t_n, u)| = o(1)$, we are done. If not, apply Theorem 2.3 again to R. Because R has one fewer variable whose degree is $\deg A_1$, we can deduce the theorem by induction. If $|C_0(t_2, \ldots, t_n)| \gg 1$, but $|C_0(A_2, \ldots, A_n)| = o(1)$, then C_0 involves fewer variables, so by induction we can assume $A_2 = Q_2(B) + o(1), \ldots, A_n = Q_n(B) + o(1)$ satisfying Theorem 2.2. Substituting in $P(A_1, \ldots, A_n)$, we get a polynomial $A_1^r C_0(Q_2(B), \ldots, Q_n(B)) + \ldots$. Now we have $|C_0(Q_2(t), \ldots)| = o(1)$ by the conclusion of Theorem 2.2, so we can neglect the first term and we have lowered the degree to which A_1 appears. The theorem follows again by induction.

Remark. Although the proof proceeds by contradiction, it is possible to reverse the implications and obtain effective bounds.

From Theorem 2.1 we get the following corollary for the case when the factors are not homogeneous. Here if A is a polynomial in many variables, let $[A]_n$ be the part of A that has degree $\leq n$.

COROLLARY. *Assume that A and B are polynomials in many variables such that*

$$|[A]_n|_1 = 1, \quad |[B]_m|_1 = 1, \quad |[A]_{n+m}|_1 \leq K, \quad |[B]_{n+m}|_1 \leq K$$

then

$$|[AB]_{n+m}|_1 \geq \alpha(n, m, K).$$

Proof. Assume $n \geq m$. Put $A_{(j)}$=the part of A which has degree j. Now there exists a smallest $j_1 \leq n$ such that $|A_{(j_1)}|_1 \geq 1/2n$ and a smallest $j_2 \leq m \leq n$ such that $|B_{(j_2)}|_1 \geq 1/2n$. According to Theorem 2.1, put

$$C = \inf_{\substack{j \leq n \\ l \leq n}} K(j, l)$$

We then have two cases: Either

$$|[AB]_{(j_1+j_2)}|_1 \geq \frac{1}{2} \cdot C \cdot \frac{1}{4n^2}$$

and the result is proved, or

$$|[AB]_{(j_1+j_2)}|_1 < \frac{1}{2} \cdot C \cdot \frac{1}{4n^2}.$$

In the latter case more than half of the contribution from $A_{(j_1)} B_{(j_2)}$ is cancelled. Obviously, this can only be done if either

$$|[A]_{j_1-1}|_1 \geq \frac{1}{2} \cdot \frac{1}{2} C \cdot \frac{1}{4n^2} \cdot \frac{1}{K+1}$$

or

$$|[B]_{j_2-1}|_1 \geq \frac{1}{2} \cdot \frac{1}{2} C \cdot \frac{1}{4n^2} \cdot \frac{1}{K+1}.$$

So for this case consider now the smallest number j_3 such that

$$|[A]_{(j_3)}|_1 \geq \frac{1}{2} \cdot \frac{1}{2} C \cdot \frac{1}{4n^2} \cdot \frac{1}{K+1} \cdot \frac{1}{2n}$$

and the smallest number j_4 such that

$$|[B]_{(j_4)}|_1 \geq \frac{1}{2} \cdot \frac{1}{2} C \cdot \frac{1}{4n^2} \cdot \frac{1}{K+1} \cdot \frac{1}{2n}.$$

Then we have $j_3 + j_4 < j_1 + j_2 \leq 2n$.

Now we have again two cases: Either

$$|[AB]_{(j_3+j_4)}|_1 \geq \frac{1}{2} C^3 \left(\frac{1}{4n^2} \cdot \frac{1}{K+1} \cdot \frac{1}{2n^2} \cdot \frac{1}{2} \right)^2$$

and the result is proved, or we have the opposite inequality. In the latter case we repeat the argument above and get $j_5+j_6<j_3+j_4$. This procedure must end after at most $2n-2$ steps, since $2\leqslant j_{2k-1}+j_{2k}$. Thus the theorem is proved.

3.

In this section we will give 6 statements which will prove Theorem 1.3. We will then give lemmas and propositions that prove these 6 statements. Most of these lemmas and propositions will then be proved in Section 4. Before giving the 6 statements we have to make some definitions.

Consider the sequence (n, ε_n). Let $1/2^m=\min\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$. We put $m=f(n)$. We consider a sequence $(\alpha_1, \alpha_2, \dots, \alpha_k)$ such that $\alpha_i=(n_i, \varepsilon_{n_i})$ and $\varepsilon_{n_{i+1}}<\varepsilon_{n_i}$. If $\varepsilon_{n_{i+1}}<\min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots, \varepsilon_{n_i}\}$ we say that the pair (α_i, α_{i+1}) is a jump.

Assume that $|\ |^N$ is defined in a compatible way. For every $k\leqslant N$ we consider the smallest number, say i, such that $|q_k-q_i|^{k-1}\leqslant\varepsilon_i/16$ and $\varepsilon_i=\varepsilon_k$. We then say that k belongs to the ith system. So with N given, a system is a subset of the integers $\{1,2,\dots,N\}$. To every system corresponds an ε in an obvious way. To every system corresponds in an obvious way q_i, D_i and L_i and if k belongs to the ith system, the compatibility assumption gives $C_k=D_i$, $|l_k|_1=L_i$.

Let $\alpha_1, \alpha_2, \dots, \alpha_k$ be a sequence of systems such that α_i is the n_ith system. We will consider such sequences where $\varepsilon_{n_{i+1}}<\varepsilon_{n_i}$ and we define jump as for the α's. If α is a system let $b(\alpha)$ be the number of the system (α is the $b(\alpha)$th system).

We will now give 6 statements from which Theorem 1.3 follows easily. We will then let the propositions of this section prove these 6 statements. To estimate $|hq|^N$ we form a representation V of hq,

$$hq=v_1(l_1q_1-1)+v_2(l_2q_2-1)+\dots+v_N(l_Nq_N-1)+v$$

and the estimate for $|hq|^N$ given by this representation V is

$$|V|^N_{\text{est}}=\sum_j |v_j|_{\text{op}\,N}\cdot\varepsilon_j+|v|_1.$$

Below we will not assume that V is the best representation of hq in the sense that it gives the $|\ |^N$-norm. It will turn out that when $|hq|^N$ is estimated many different cases can occur, the number of cases increasing with N. We will now write down a list of all cases that can occur, we will describe them later.

(1) $\alpha^*_{1,\mathrm{B}}, m^*_{2,\mathrm{B}}, \alpha^*_{2,\mathrm{B}}, m_{3,\mathrm{B}}, \dots, \alpha^*_{k-1,\mathrm{B}}, m_k$

(2) $\alpha^*_{1,\mathrm{B}}, m_{2,\mathrm{B}}, \alpha^*_{2,\mathrm{B}}, m_{3,\mathrm{B}}, \dots, \alpha^*_{k-1,\mathrm{B}}, m_{k,\mathrm{G}}$

(3) $\alpha^*_{1,B}, m_{2,B}, \alpha^*_{2,B}, m_{3,B}, \ldots, \alpha^*_{k-1,B}, m_{k,B}$

(4) $\alpha^*_{1,B}, m_{2,B}, \alpha^*_{2,B}, m_{3,B}, \ldots, m_{k,B}, \alpha_k$

(5) $\alpha^*_{1,B}, m_{2,B}, \alpha^*_{2,B}, m_{3,B}, \ldots, m_{k,B}, \alpha^{nc}_k$

(6) $\alpha^*_{1,B}, m_{2,B}, \alpha^*_{2,B}, m_{3,B}, \ldots, m_{k,B}, \alpha^{c}_{k,B}$

(7) $\alpha^*_{1,B}, m_{2,B}, \alpha^*_{3,B}, m_{2,B}, \ldots, m_{k,B}, \alpha^{nc}_{k,G}$

(8) $\alpha^*_{1,B}, m_{2,B}, \alpha^*_{2,B}, m_{3,B}, \ldots, m_{k,B}, \alpha^{nc}_{k,B}$

In this list the α's are systems with $b(\alpha)\leq N$. The ε of the system α_{j+1} is smaller than the ε of the system α_j. The m's correspond to ε's such that $1/2^{m_j}$ is the ε of α_j. The superscripts c and nc correspond to cancelled and non-cancelled and * takes the values c and nc. B and G should suggest bad and good. In this list (2) and (3) are subcases of (1), (4) is a subcase of (3), (5) and (6) are subcases of (4), (7) and (8) are subcases of (5). After listing the cases we now turn to the statements. We assume

$$|q-q_n|^N<\frac{\varepsilon_n}{16}, \quad |h|_1=1, \quad |h|_{\mathrm{op}\,N}<\frac{1}{\varepsilon_n}.$$

STATEMENT 1. *There is a growth function F such that if $\{D_n, L_n, l_n, C_n\}$ grows faster than F then for every n there is a B^1_n depending only on $|\ |^{n-1}$ such that if $|hq|^N$ is estimated by V then either*

case $\alpha^{nc}_{1,G}$: *The estimate given by V is $\geq B^1_n$,*

or

some case $\alpha^*_{1,B}, m_2$ *occurs with $1/2^{m_2}$ smaller than the ε corresponding to α_1.*

STATEMENT 2. *There is a growth function F such that if $\{D_n, L_n, l_n, C_n\}$ grows faster than F then for every n there is a B^2_n depending only on $|\ |^{n-1}$ such that if* case $\alpha^*_{1,B}, m_2$ *occurs when $|hq|^N$ is estimated by V then either*

case $\alpha^*_{1,B}, m_{2,G}$: *The estimate given by V is $\geq B^2_n$*

or

case $\alpha^*_{1,B}, m_{2,B}$: *Some* case $\alpha^*_{1,B}, m_{2,B}, \alpha_2$ *occurs.*

After these statements we can now pass to statements for cases which are represented by arbitrary long sequences.

STATEMENT 3. *There is a growth function F such that, if $\{D_n, L_n, l_n, C_n\}$ grows faster than F, then for every $k\geq 2$, n and N, if* case $\alpha^*_{1,B}, m_{2,B}, \alpha^*_{2,B}, m_{3,B}, \ldots, m_{k,B}, \alpha_k$ *occurs when $|hq|^N$ is estimated then either*

case $\alpha^*_{1,B}, m_{1,B}, \alpha^*_{2,B}, m_{2,B}, \ldots, m_{k,B}, \alpha^{nc}_k$

or

case $\alpha^*_{1,\mathrm{B}}, m_{1,\mathrm{B}}, \alpha^*_{2,\mathrm{B}}, m_{2,\mathrm{B}}, \dots, m_{k,\mathrm{B}}, \alpha^{\mathrm{c}}_{k,\mathrm{B}}$

occurs.

STATEMENT 4. *There is a growth function F such that, if $\{D_n, L_n, l_n, C_n\}$ grows faster than F, then for every $k \geq 2$, n and N, if the* case $\alpha^*_{1,\mathrm{B}}, m_{2,\mathrm{B}}, \alpha^*_{2,\mathrm{B}}, m_{3,\mathrm{B}}, \dots, m_{k,\mathrm{B}}, \alpha^{\mathrm{nc}}_k$ *occurs when $|hq|^N$ is estimated then either*

case $\alpha^*_{1,\mathrm{B}}, m_{1,\mathrm{B}}, \alpha^*_{2,\mathrm{B}}, m_{2,\mathrm{B}}, \dots, m_{k,\mathrm{B}}, \alpha^{\mathrm{nc}}_{k,\mathrm{G}}$: *The estimate given by V is ≥ 1*

or

some case $\alpha^*_{1,\mathrm{B}}, m_{1,\mathrm{B}}, \alpha^*_{2,\mathrm{B}}, m_{2,\mathrm{B}}, \dots, m_{k,\mathrm{B}}, \alpha^{\mathrm{nc}}_{k,\mathrm{B}}, m_{k+1}$

occurs.

STATEMENT 5. *There is a growth function F such that, if $\{D_n, L_n, l_n, C_n\}$ grows faster than F, then for every $k \geq 2$, n and N, if the* case $\alpha^*_{1,\mathrm{B}}, m_{1,\mathrm{B}}, \alpha^*_{2,\mathrm{B}}, m_{2,\mathrm{B}}, \dots, m_{k,\mathrm{B}}, \alpha^{\mathrm{c}}_{k,\mathrm{B}}$ *occurs when $|hq|^N$ is estimated then there is an m_{k+1} such that the*

case $\alpha^*_{1,\mathrm{B}}, m_{1,\mathrm{B}}, \alpha^*_{2,\mathrm{B}}, m_{2,\mathrm{B}}, \dots, m_{k,\mathrm{B}}, \alpha^{\mathrm{c}}_{k,\mathrm{B}}, m_{k+1}$

occurs.

STATEMENT 6. *There is a growth function F such that, if $\{D_n, L_n, l_n, C_n\}$ grows faster than F, then for every $k \geq 2$, n and N, if the* case $\alpha^*_{1,\mathrm{B}}, m_{1,\mathrm{B}}, \alpha^*_{2,\mathrm{B}}, m_{2,\mathrm{B}}, \dots, m_{k,\mathrm{B}}, \alpha^*_{k,\mathrm{B}}, m_{k+1}$ *occurs when $|hq|^N$ is estimated then either*

case $\alpha^*_{1,\mathrm{B}}, m_{1,\mathrm{B}}, \alpha^*_{2,\mathrm{B}}, m_{2,\mathrm{B}}, \dots, m_{k,\mathrm{B}}, \alpha^*_{k,\mathrm{B}}, m_{k+1,\mathrm{G}}$: *The estimate given by V is ≥ 1*

or

some case $\alpha^*_{1,\mathrm{B}}, m_{2,\mathrm{B}}, \alpha^*_{2,\mathrm{B}}, m_{3,\mathrm{B}}, \dots, m_{k,\mathrm{B}}, \alpha^*_{k,\mathrm{B}}, m_{k+1,\mathrm{B}}, \alpha_{k+1}$

occurs.

We now prove Theorem 1.3 from these statements. Assume that $\{D_n, L_n, l_n, C_n\}$ grows faster than F and let $B_n = \min(B^1_n, B^2_n)$. Let $|hq|^N$ be estimated by V. Then by Statements 1 and 2 either the estimate given by V is $\geq B_n$ or some case $\alpha_{1,\mathrm{B}}, m_{2,\mathrm{B}}, \alpha_2$ occurs. Then by Statements 3, 4 and 5 either the estimate given by V is ≥ 1 or some case $\alpha_{1,\mathrm{B}}, m_{2,\mathrm{B}}, \alpha_{2,\mathrm{B}}, m_3$ occurs. In the latter case by Statement 6 either the estimate given by V is ≥ 1 or some case $\alpha_{1,\mathrm{B}}, m_{2,\mathrm{B}}, \alpha_{2,\mathrm{B}}, m_{3,\mathrm{B}}, \alpha_3$ occurs. In the latter case we apply Statements 3, 4 and 5 again and the argument continues in an obvious manner.

Since the α's are all different and are among the N first systems (we recall moreover from above that the ε of α_{j+1} is smaller than the ε of α_j) this process must eventually stop. We have now in fact proved that if some case $\alpha_{1,B}, m_{2,B}, \alpha_2$ occurs then the estimate given by V is $\geqslant 1$. Since it is otherwise $\geqslant B_n$, Theorem 1.3 is proved.

Our next task is to give the propositions that prove the Statements 1 and 2. In order to give these propositions we have to do some preparations. Let $[\ ]^r$, $[\ ]_r$ operate on polynomials $\Sigma\, a_i x^i$ by means of

$$\left[\sum a_i x^i\right]^r = \sum_{i>r} a_i x^i$$

$$\left[\sum a_i x^i\right]_r = \sum_{i\leqslant r} a_i x^i.$$

We start with

LEMMA 3.1. *Suppose that* $|\ |^N$ *is defined in a compatible way. Then for every* n *there exist* R_n *and* S_n *depending only on* $|\ |^{n-1}$ *such that*

$$|q-q_n|^{n-1} < \frac{\varepsilon_n}{16} \quad \Rightarrow \quad |[q(x)]_{R_n}|_1 > \frac{3}{4} \quad \textit{and} \quad |q|_1 < S_n.$$

Proof. The existence of S_n is obvious since all norms $|\ |^m$ are equivalent to $|\ |_1$ and

$$|q-q_n|^{n-1} < \frac{\varepsilon_n}{16} \quad \Rightarrow \quad |q|^{n-1} < 1+\frac{\varepsilon_n}{16}.$$

To get R_n we write

$$q = q_n + \sum_{i,k} a_{1,i,k} x^i \bar{l}^{(k)}(l_1 q_1 - 1) + \ldots + \sum a_{n-1,i,k} x^i \bar{l}^{(k)}(l_{n-1} q_{n-1} - 1) + t.$$

Here

$$\bar{l}^{(k)} = \bar{l}_1^{k_1} \ldots \bar{l}_{n-1}^{k_{n-1}} \quad \text{and} \quad \bar{l}_j = \frac{l_j}{|l_j|_1}, \quad |k| = k_1 + k_2 + \ldots + k_{n-1}.$$

By Sublemma 4.1 of Section 4 we can assume that there is a uniform bound on $i+|k|$. Thus there exists R^n depending only on $|\ |^{n-1}$ such that $\deg(q-t)(x) \leqslant R_n$. We have

$$|q-t|_1 \geqslant |q-t|^{n-1} > 1 - \frac{\varepsilon_n}{16}.$$

Since $|t|_1<\varepsilon_n/16$ we get

$$|[q(x)]_{R_n}|_1>1-\frac{\varepsilon_n}{16}-\frac{\varepsilon_n}{16}>\frac{3}{4}.$$

We will now always assume ord $l_n>R_n$.

We will now define a pre-(n, m)-expansion of a polynomial q in the $|\ |^N$-norm and begin by defining the kth stage of a pre-(n, m)-expansion. The kth stage of a pre-(n, m)-expansion of q is a polynomial in 4 types of variables, q's, l's, s's, t's. The 0th stage of the pre-(n, m)-expansion is just q. The first stage of the expansion contains the following types of terms q_n, s_j, $s_j l_j q_j$, t. It is in fact a polynomial

$$q_n+s_1(l_1 q_1-1)+s_2(l_2 q_2-1)+\ldots+s_N(l_N q_N-1)+t.$$

The kth stage of the pre-(n, m)-expansion contains the following types of terms:

(1) $s_{j_1} l_{j_1} s_{j_2} l_{j_2} \ldots s_{j_r} l_{j_r} q_{j_r}$, $\quad r\leq k$

(2) $s_{j_1} l_{j_1} s_{j_2} l_{j_2} \ldots s_{j_r} l_{j_r} q_p$, $\quad r\leq k-1,\ p<j_r$

(3) $s_{j_1} l_{j_1} s_{j_2} l_{j_2} \ldots s_{j_r} l_{j_r} s_{j_{r+1}}$, $\quad r\leq k-1$

(4) $s_{j_1} l_{j_1} s_{j_2} l_{j_2} \ldots s_{j_r} l_{j_r} t$, $\quad r\leq k-1$.

The $(k+1)$st stage of the pre-(n, m)-expansion is obtained from the kth stage by replacing q's in the two first types of terms according to the following rules:

For terms of the first type.

Rule A. If q_{j_r} belongs to the vth system, $v<j_r$, $m<j_r$ it is replaced by a polynomial

$$q_v+s_1(l_1 q_1-1)+s_2(l_2 q_2-1)+\ldots+s_{j_r-1}(l_{j_r-1} q_{j_r-1}-1)+t.$$

Rule B. If q_{j_r} belong to the j_rth system, $m<j_r$, it is replaced by a polynomial

$$s_1(l_1 q_1-1)+s_2(l_2 q_2-1)+\ldots+s_{j_r-1}(l_{j_r-1} q_{j_r-1}-1)+t.$$

For terms of the second type (and the q_n that appears in the first stage).

Rule B$'$. If $m<p$ $(m<n)$, then q_p (q_n) is replaced by a polynomial

$$s_1(l_1 q_1-1)+s_2(l_2 q_2-1)+\ldots+s_{p-1}(l_{p-1} q_{p-1}-1)+t.$$

Moreover, we have the following rule: if a term in the ith stage of the expansion and a term in the jth stage of the expansion are both of the first type (second type), and end with the same $l_{j_r}q_{j_r}(l_{j_r}q_p)$ then q_{j_r} (q_p) is replaced in the same way for the two terms when passing to the $(i+1)$st, $(j+1)$st stage respectively. If a term of first or second type ends with q_n and is replaced according to rule B or B′, it is replaced in the same way as q_n that appears in the first stage.

Since the indices of the q's are going down at every new stage we see that there is a final stage of the pre-(n,m)-expansion. We will call this final stage the pre-(n,m)-expansion of q. We see that the parameter n only refers to q_n in the first stage of the expansion. The parameter m refers to the fact that $q_1, q_2, \ldots, q_m$ are not replaced. We see that to every term s_j in the first stage of the expansion corresponds an ε namely ε_j. In the same way there corresponds an ε to every term of the third type, namely $\varepsilon_{j_{r+1}}$.

We say that a pre-(n,m)-expansion is shortened with respect to $l_k q_k$ if all terms of the first type which end with $l_k q_k$ are not replaced further. Consider the final stage of a pre-(n,m)-expansion of q which is shortened with respect to $l_k q_k$. Let $S(l,s)\, l_k q_k$ be the sum of all terms which end with $l_k q_k$. We then say that $S(l,s)$ is the coefficient of $l_k q_k$ in the pre-(n,m)-expansion of q. Below we will define other types of shortened pre-expansions. (For every l_k we normalize it by putting $\bar{l}_k = l_k/|l_k|_1$.)

From the kth stage of a pre-(n,m)-expansion of q we get the kth stage of an (n,m)-expansion of q in the following way. For every variable s in the kth stage put

$$s = \sum a_{i,k} x^i \bar{l}^{(k)} \quad \text{with} \quad \bar{l}^{(k)} = \bar{l}_1^{k_1} \bar{l}_2^{k_2} \ldots \bar{l}_N^{k_N}$$

and every variable t in the kth stage put

$$t = \sum a_i x^i.$$

With these expressions substituted for s and t we have for replacements according to rule A

$$q_{j_r} = q_v + \sum_{i,k} a_{1,i,k} x^i \bar{l}^{(k)} (l_1 q_1 - 1) + \ldots + \sum_{i,k} a_{N,i,k} x^i \bar{l}^{(k)} (l_N q_N - 1) + \sum_i a_i x^i$$

where this representation shows $|q_{j_r} - q_v|^{j_r - 1} < \varepsilon_v/16$. For replacements according to the rules B and B′ the representations should show $|q_{j_r}|^{j_r - 1} = 1$ and $|q_p|^{p-1} = 1$.

Obviously every variable s or t that appears in the kth stage of the expansion has appeared for the first time at some stage $j \leq k$ as the result of some replacement of some q. An (n,m)-expansion of q is derived from a pre-(n,m)-expansion of q by making the substitution above in the final stage and replacing $q_1, q_2, \ldots, q_m$ by polynomials in x. We observe that an (n, m)-expansion of q is a polynomial in x, and l's. By making the substitution above we get the coefficient of $l_k q_k$ in an (n, m)-expansion of q. It will be a polynomial $S(x, \bar{l})$. We say that an (n, m)-expansion of q is r-substituted if every $l_j, j \leq r$ is substituted by a polynomial in x. More generally we say that we make an r-substitution in a polynomial $P(x, l, q, s, t)$ if the q's and t's are replaced by polynomials in x and s's are substituted by polynomials in x and l_i's with $i > r$ and every l_i with $i \leq r$ is substituted by a polynomial in x. Let us say that this gives $P(x, l, q, s, t) = P'(x, l)$. We then say that every monomial out of $P'(x, l)$ is derived from the polynomial $P(x, l, q, s, t)$. We will sometimes below use the notation $P(x, l, q, s, t) = P(x, l) = P(x)$, thus meaning that we get $P(x, l)$ by substituting q's, s's, and t's by polynomials in x and l's and $P(x)$ by substituting l's, q's, s's, and t's by polynomials in x.

As for a polynomial q we can define pre-m-expansion of V. We only need 1 parameter m since we do not use that q is close to q_n. The first stage of such a pre-m-expansion is the polynomial $v_1(l_1 q_1 - 1) + v_2(l_2 q_2 - 1) + \ldots + v_N(l_N q_N - 1) + v$. The replacements of q's are then done in exactly the same way as for pre-(n, m)-expansion of q.

Let $S(l, s)$ be the coefficient of $l_k q_k$ in a pre-(n, m)-expansion of q and $V(l, s)$ the coefficient of $l_k q_k$ in a pre-m-expansion of V. We then say that $hS(l, s) - V(l, s)$ is the coefficient of $l_k q_k$ in $hq - V$, with a pre-(n, m)-expansion of q and a pre-m-expansion of V.

Before going into the propositions of this section we will discuss growth functions in somewhat more detail. We say that a growth function is trivial up to the jth stage if it gives the value 1 for all $3m$-tuples, $m < j$, and gives the function f of lacunarity $\equiv 1$ and $\delta = 1$ for all $(3m+2)$-tuples, $m < j$. We say that two growth functions F_1 and F_2 coincide up to the jth stage if they give the same values for all $3m$-tuples and $(3m+2)$-tuples, $m < j$. We say F_1 dominates F_2 if for all $3m$-tuples $F_1(D_1, \ldots, D_m, L_1, \ldots, L_m, l_1, \ldots, l_m) \geq F_2(D_1, \ldots, D_m, L_1, \ldots, L_m, l_1, \ldots, l_m)$ and for all $(3m+2)$-tuples we have for the lacunarity functions f_1 and f_2 and the δ's, δ_1 and δ_2, that $f_1(x) \geq f_2(x)$ for all $x \in \mathbf{R}^+$ and $\delta_1 \leq \delta_2$. In the propositions and lemmas of this section we will meet different growth functions and so in order to have the assumptions of all propositions and lemmas fulfilled with the sequence $\{D_n, L_n, l_n, C_n\}$ we have to know that there is one growth function which dominates all of them. We now go into these considerations.

Let as above $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ be a sequence of systems such that $1/2^{m_i} = \varepsilon_{b(\alpha_i)} < \varepsilon_{b(\alpha_{i-1})}$. We will also consider sequences $(\alpha_1, \ldots, \alpha_k, m_{k+1})$ where $\varepsilon_{b(\alpha_i)} < \varepsilon_{b(\alpha_{i-1})}$ and $1/2^{m_{k+1}} <$

$1/2^{m_k}=\varepsilon_{b(\mathfrak{a}_k)}$. For such a sequence we consider

$$M=\{m_1, m_2, \dots, m_k\} \quad (\text{resp. } \{m_1, \dots, m_k, m_{k+1}\}).$$

Let $\mathfrak{A}$ be the set of $\mathfrak{a}_j$'s such that $(\mathfrak{a}_j, \mathfrak{a}_{j+1})$ is a jump—that is

$$\varepsilon_{b(\mathfrak{a}_{j+1})}<\min(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{b(\mathfrak{a}_j)}),$$

$\mathfrak{a}_k \in \mathfrak{A}$ and $(\mathfrak{a}_k, m_{k+1})$ is a jump if $1/2^{m_{k+1}}<\min(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{b(\mathfrak{a}_k)})$. Let $H \subset M$ be defined in the following way: $m_k \in H\,(m_{k+1} \in H)$ and if $m_j \in H$, let $i<j$ be the largest number such that $\min(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{b(\mathfrak{a}_i)}) \leq 1/2^{m_j}$, then also $m_i \in H$.

We now observe:

(3A) Given a set M there are only finitely many different sets $\mathfrak{A}$ and H that are derived from sequences $(\mathfrak{a}_1, \dots, \mathfrak{a}_k)$ or $(\mathfrak{a}_1, \dots, \mathfrak{a}_k, m_{k+1})$ with $\varepsilon_{b(\mathfrak{a}_i)}=1/2^{m_i}$.

For a sequence $(\mathfrak{a}_1, \dots, \mathfrak{a}_k, m_{k+1})$ we define $N(\mathfrak{a}_1, \dots, \mathfrak{a}_k, m_{k+1})$ to be the sequence of $M_j, \mathfrak{A}_j, H_j, j \leq k+1$ for the subsequences $(\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_j)$, $j \leq k$, and $M_{k+1}, \mathfrak{A}_{k+1}, H_{k+1}$ be the $M, \mathfrak{A}, H$ of $(\mathfrak{a}_1, \dots, \mathfrak{a}_k, m_{k+1})$.

We define $N(\mathfrak{a}_1, \dots, \mathfrak{a}_k)$ to be the sequence of $M_j, \mathfrak{A}_j, H_j, j \leq k$, and with the system $\mathfrak{a}_k$ as the $(k+1)$st element of the sequences. Below we will have a family of growth functions

$$F_{N(\mathfrak{a}_1, \dots, \mathfrak{a}_k)} \quad \text{and} \quad F_{N(\mathfrak{a}_1, \dots, \mathfrak{a}_k, m_{k+1})}.$$

We assume that if

$$N(\mathfrak{a}_1, \dots, \mathfrak{a}_k) = N(\mathfrak{b}_1, \dots, \mathfrak{b}_k) \quad (N(\mathfrak{a}_1, \dots, \mathfrak{a}_k, m_{k+1})=N(\mathfrak{b}_1, \dots, \mathfrak{b}_k, m_{k+1}))$$

then

$$F_{N(\mathfrak{a}_1, \dots, \mathfrak{a}_k)} = F_{N(\mathfrak{b}_1, \dots, \mathfrak{b}_k)} \quad (F_{N(\mathfrak{a}_1, \dots, \mathfrak{a}_k, m_{k+1})} = F_{N(\mathfrak{b}_1, \dots, \mathfrak{b}_k, m_{k+1})}).$$

We now have a family of growth functions $F_{N(\mathfrak{a}_1, \dots, \mathfrak{a}_k)}$ and $F_{N(\mathfrak{a}_1, \dots, \mathfrak{a}_k, m_{k+1})}$ which satisfy the following:

(3A1) If $(\mathfrak{a}_{k-1}, \mathfrak{a}_k)((\mathfrak{a}_k, m_{k+1}))$ is a jump $F_{N(\mathfrak{a}_1, \dots, \mathfrak{a}_k)}(F_{N(\mathfrak{a}_1, \dots, \mathfrak{a}_k; m_{k+1})}$ coincides with $F_{N(\mathfrak{a}_1, \dots, \mathfrak{a}_{k-1})}(F_{N(\mathfrak{a}_1, \dots, \mathfrak{a}_k)})$ up to the $(b(\mathfrak{a}_k)-1)$st stage $((b(m_{k+1})-1)$st stage) and depends only on $N(\mathfrak{a}_1, \dots, \mathfrak{a}_k)(N(\mathfrak{a}_1, \dots, \mathfrak{a}_k, m_{k+1}))$ and $|\ |^{b(\mathfrak{a}_k)-1}$ $(|\ |^{b(m_{k+1})-1})$.

(3A2) If $(\mathfrak{a}_{k-1}, \mathfrak{a}_k)$ $((\mathfrak{a}_k, m_{k+1}))$ is not a jump $F_{N(\mathfrak{a}_1,\dots,\mathfrak{a}_k)}$ $(F_{N(\mathfrak{a}_1,\dots,\mathfrak{a}_k, m_{k+1})})$ coincides with $F_{N(\mathfrak{a}_1,\dots,\mathfrak{a}_{k-1}; m_k)}$ up to the $(b(\mathfrak{a}_k)-1)$st stage (the $(b(m_{k+1})-1)$st stage) and depends only on $N(\mathfrak{a}_1, \dots, \mathfrak{a}_k)$ $(N(\mathfrak{a}_1, \dots, \mathfrak{a}_k, m_{k+1}))$ and $| \,|^{b(\mathfrak{a}_k)-1}$ $(| \,|^{b(m_k)-1})$.

We now show that for a family of growth functions which satisfies (3A1) and (3A2) there exists one growth function F which dominates all growth functions of the family.

(3B) In fact the conditions (3A1) and (3A2) give that every F_N agrees with some $F_{N(\mathfrak{a}_1,\dots,\mathfrak{a}_k)}$ (or $F_{N(\mathfrak{a}_1,\dots,\mathfrak{a}_{k-1}, m_k)}$) up to $(b(m_{k+1})-1)$st stage, where $b(\mathfrak{a}_k)<b(m_{k+1})$. Since $b(m_k) \leq b(\mathfrak{a}_k)<b(m_{k+1})$ there are by (3A) and the definition of N only finitely many such $N(\mathfrak{a}_1, \dots, \mathfrak{a}_k)$ and $N(\mathfrak{a}_1, \dots, \mathfrak{a}_{k-1}, m_k)$. Thus there is obviously one growth function F which dominates all F_N's.

For the next proposition consider a pre-(n, m)-expansion of q. In the final stage remove all terms of the third type for which the corresponding ε is $<\min(\varepsilon_1, \dots, \varepsilon_n)$. This will be called q'. Similarly we consider a pre-n-expansion of V and define V' by removing the terms of 3rd type for which the corresponding ε is $<\min(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$.

Now two cases can occur, either

$$|[q'(x)]_{R_n}|_1 > \frac{1}{2} \quad \text{or} \quad |[q'(x)]_{R_n}|_1 \leq \frac{1}{2}.$$

We denote them $\mathfrak{a}_1^{\mathrm{nc}}$ and $\mathfrak{a}_1^{\mathrm{c}}$ (low degrees of q are not cancelled resp. cancelled). We will need a couple of more definitions to state our first proposition. We say that a polynomial $\Sigma_{\alpha \in M_1} a_\alpha x_1^{\alpha_1} \dots x_N^{\alpha_N}$ contains a polynomial $\Sigma_{\alpha \in M_2} a_\alpha x_1^{\alpha_1} \dots x_N^{\alpha_N}$ if $M_1 \supset M_2$.

Below we will let $p_\varepsilon(\Sigma_{\alpha \in M} a_\alpha x_1^{\alpha_1} \dots x_N^{\alpha_N})$ denote a polynomial $\Sigma_{\alpha \in M} b_\alpha x_1^{\alpha_1} \dots x_N^{\alpha_N}$ where $\Sigma_{\alpha \in M} |b_\alpha - a_\alpha| < \varepsilon \Sigma_{\alpha \in M} |a_\alpha|$.

In the next proposition consider an $(n-1)$-substitution in q' and an $(n-1)$-substitution of V'. In the proposition we let $\mathfrak{a}_1$ denote the nth system so $b(\mathfrak{a}_1)=n$.

Proposition $\mathfrak{a}_1^{\mathrm{nc}}, N(\mathfrak{a}_1)$. *For all n, there is a growth function $F_{N(\mathfrak{a}_1)}$ which is trivial up to the $(n-1)$-st stage and depends only on $N(\mathfrak{a}_1)$ and $| \,|^{n-1}$ and there are numbers B'_n and m depending only on $| \,|^{n-1}$ such that for all $N \geq n$, if $\mathfrak{a}_1^{\mathrm{nc}}$ occurs then for all V either*

$\mathfrak{a}_{1,G}^{\mathrm{nc}}$: $|V|_{\mathrm{est}}^N \geq B'_n$

or

$\mathfrak{a}^{\mathrm{nc}}_{1,\mathrm{B}}$: *$(hq'-V')(x)$ contains $p_{1/25}(E(x))$*

where

$$E=\sum_{\substack{i+|\alpha\leq m\\ j_k\geq n \text{ for all } k}} e_{i,J,\alpha}x^i l_{j_1}^{\alpha_1} l_{j_2}^{\alpha_2}\dots l_{j_r}^{\alpha_r}$$

with $|E(x)|_1>B_n'$ and where j_k belongs to one of the $n-1$ first systems for all k.

Proof. We will give a detailed proof in Section 4. The strategy is the following: We first prove that hq' regarded as a polynomial in x contains $p_{1/50}(E(x))$ like in the proposition. We then show that if $|V|$ is too small, V' will not interfere very much with E. To prove that hq' is big we use that $|h|_{\mathrm{op}\,N}/|h|_1<1/\varepsilon_n$ implies that h is essentially a low degree polynomial, regarded as a polynomial in x and l's and that $|[q'(x)]_{R_n}|_1>1/2$ just says that q' has some concentration on low degrees regarded as a polynomial in x.

To get the Statements 1 and 2 we now study the case $\mathfrak{a}_1^{\mathrm{c}}$, that is $|[q'(x)]_{R_n}|_1\leq 1/2$.

(1) We first show that with $q=q_n+\Sigma_{j=1}^N s_j(l_jq_j-1)+t$,

$$[q'(x)]_{R_n}|_1\leq\frac{1}{2}\quad\text{implies}\quad\left|\left[\sum_{j\geq n}s_j(x)\right]_{R_n}\right|_1>\frac{1}{4}$$

where the summation is done only over those $s_j(x)$ for which $\varepsilon_j\geq\min\{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n\}$.

(2) To prove (1) we consider the first stage of a pre-(n,n)-expansion of q, $q=q_n+s_1(l_1q_1-1)+\dots+s_N(l_Nq_N-1)+t$. We observe that the ord of every term $s_jl_jq_j, j\geq n$—or of any term derived from such a term in a later stage of the pre-(n,n)-expansion—is $\geq R_n$ when expanded as a polynomial in x.

We have

$$\left|\left[\left(q_n+\sum_{j=1}^{n-1}s_j(l_jq_j-1)+t\right)\right]_{R_n}\right|_1>\frac{3}{4}. \tag{3}$$

To see (3) we write for $i<N-1$, $s_i(x,\bar{l})=s_i^{(1)}(x,\bar{l})+s_i^{(2)}(x,\bar{l})$ where $s_i^{(1)}(x,\bar{l})$ contains only l_r's with $1\leq r\leq n-1$ and every term in $s_i^{(2)}(x,\bar{l})$ contains an l_r with $r\geq n$. Then

$$\operatorname{ord}\left(\sum_{i=1}^{n-1}s_i^{(2)}(x)(l_iq_i-1)\right)>R_n. \tag{4}$$

We have

$$\left|\left[\left(q_n+\sum_{i=1}^{n-1}s_i^{(1)}(x)(l_iq_i-1)+t\right)\right]_{R_n}\right|_1>\frac{3}{4}. \tag{5}$$

by Lemma 3.1 since the distance between q_n and $q_n+\Sigma_{i=1}^{n-1} s_i^{(1)}(x)(l_i q_i-1)+t$ in the $|\ |^{n-1}$-norm is $<\varepsilon_n/16$.

Now (3) follows from (4) and (5) and (1) follows from (2), (3) and

$$\left|[q'(x)]_{R_n}\right| < \frac{1}{2}.$$

(6) We have that the condition $|q-q_n|^N<\varepsilon_n/16$ implies $\Sigma_{j\geq n}|s_j|_{\mathrm{op}\,N}\cdot\varepsilon_j<\varepsilon_n/16$ and this inequality obviously also holds if the summation is extended only over those j for which $\varepsilon_j\geq\min\{\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n\}$.

Now (1) and (6) give that if $|[q'(x)]_{R_N}]_1\leq\frac{1}{2}$, there is on average in j a bound on $|s_j|_{\mathrm{op}\,N}/|s_j|_1$ for $j\geq n$.

We have

$$\varepsilon_n\left|\sum_{\substack{j\geq n\\ \varepsilon_j\geq\varepsilon_n}} s_j(x)\right|_1 \leq \sum_{\substack{j\geq n\\ \varepsilon_j\geq\varepsilon_n}} \varepsilon_n|s_j|_{\mathrm{op}\,N} \leq \sum_{\substack{j\geq n\\ \varepsilon_j\geq\varepsilon_n}} \varepsilon_j|s_j|_{\mathrm{op}\,N} \leq \frac{\varepsilon_n}{16}.$$

Thus

$$\left|\sum_{\substack{j\geq n\\ \varepsilon_j\geq\varepsilon_n}} s_j(x)\right|_1 \leq \frac{1}{16}$$

and so by (1)

$$\left|\sum_{\varepsilon_j<\varepsilon_n} s_j^{(1)}(x)\right|_1 > \frac{1}{4}-\frac{1}{16}>\frac{1}{8}.$$

So with $\varepsilon_n=1/2^{m_1}$ there is a smallest $m_2>m_1$ such that for m_2 we have

$$\left|\sum_{\substack{j\geq n\\ \varepsilon_j=1/2^{m_2}}} s_j^{(1)}(x)\right|_1 > \frac{1}{8}\cdot\frac{1}{2^{m_2-m_1}}. \tag{3.1}$$

We then say that the case $\alpha_{1,\mathrm{B}}^{\mathrm{c}}, m_2$ occurs.

Remark. The argument above gives the reason for the requirement $|q-q_n|^N<\varepsilon_n/16$. The factor 1/16 in $\varepsilon_n/16$ gives the existence of an $m_2>m_1$ with

$$\left|\sum_{\substack{j\geq n\\ \varepsilon_j=1/2^{m_2}}} s_j^{(1)}(x)\right|_1 > \frac{1}{8}\cdot\frac{1}{2^{m_2-m_1}},$$

and it is important for the argument that we do not have $m_2\leq m_1$.

(3.2) We recall that $b(m)$ is the smallest j such that $\varepsilon_j = 1/2^m$. In order to give the next lemma and proposition we have to take into account not only s_j but the coefficient $S_j(x, \bar{l})$ of $l_j q_j$ in a $(b(m_2)-1)$-substituted $(n, b(m_2))$-expansion of q, $\varepsilon_j = 1/2^{m_2}$. To do this we first consider the coefficient of $l_j q_j$ in a pre-$(n, b(m_2))$-expansion of q. The terms of this coefficient have the forms

$$s_{j_1} l_{j_1} \dots l_{j_{r-1}} s_{j_r}, \quad r \geq 1, j_1 > j_2 > \dots > j_{r-1} > j \geq b(m_2).$$

We put

$$s_{j_1}(x, \bar{l})\, l_{j_1} \dots s_{j_{r-1}}(x, \bar{l})\, l_{j_{r-1}} s_{j_r}(x, \bar{l}) = l_{j_1} \dots l_{j_{r-1}} P(x, \bar{l}).$$

We say that every term (monomial) out of $l_{j_1} \dots l_{j_{r-1}} P(x, \bar{l})$ is derived from $s_{j_1} l_{j_1} \dots l_{j_{r-1}} s_{j_r}$. Now consider in $S_j(x, \bar{l})$ those terms which contain at most r' different l's (or $\bar{l}$'s) with index $>j$. Every such terms is obviously derived from a term $s_{j_1} l_{j_1} \dots s_{j_r}$ with $r' \leq r-1$.

Let $K_{j,r}$ be the sum of those terms out of $S_j(x, \bar{l})$ which

(1) do not contain any l_i (or $\bar{l}_i$) with both $i>j$ and i belonging to a system with number $\geq b(m_2)$,

(2) contain at most r different l's (or $\bar{l}$'s) with index $>j$.

Put

$$\left| \sum a_{i,\alpha} x^i l_1^{\alpha_1} \dots l_N^{\alpha_N} \right|_{\text{est op } N} = \sum |a_{i,\alpha}| 2^i (|l_1|_1 C_1)^{\alpha_1} \dots (|l_N|_1 C_N)^{\alpha_N}.$$

We now have

LEMMA 3.2. $\Sigma_j |K_{j,r}|_{\text{est op } N} \leq 2 \cdot (2^{m_2})^r \cdot L^r_{b(m_2)-1} \cdot D^r_{b(m_2)-1}$, *where the summation* Σ_j *is extended over all j for which the corresponding* ε *is* $1/2^{m_2}$.

This can be proved by induction on r by considering the observation (3.2). We give a complete proof in Section 4. With this preparation we can now define a semigood coefficient $S_j(x, \bar{l})$ of $l_j q_j$, $\varepsilon_j = 1/2^{m_2}$, assuming that the case $\mathfrak{a}^c_{1,\mathrm{B}}, m_2$ occurs.

We now let $S_{j,0}(x, \bar{l})$ denote those terms out of $s_j^{(1)}(x, \bar{l})$ which contain only $\bar{l}$'s with indices from the $b(m_2)-1$ first systems. We say that $S_j(x, \bar{l})$ is *semigood* if

$$\frac{|s_j^{(1)}|_{\text{op } N}}{|s_j(x)|_1} \leq 100 \cdot 4^{m_2 - m_1} \tag{1}$$

$$\frac{|s_j^{(2)}|_1}{|s_j^{(1)}|_1},\quad \frac{|s_j^{(2)}|_{\text{est op }N}}{|s_j^{(1)}|_{\text{est op }N}}\quad\text{and}\quad \frac{|s_j^{(2)}|_{\text{op }N}}{|s_j^{(1)}|_{\text{op }N}}\quad\text{are all} \leqslant 100\cdot 4^{m_2-m_1}. \tag{2}$$

$$\frac{|S_{j,0}|_1}{|s_j^{(1)}|_1} \geqslant \frac{1}{2},\quad \frac{|S_{j,0}|_{\text{op }N}}{|s_j^{(1)}|_{\text{op }N}} \geqslant \frac{1}{2} \tag{3}$$

$$|s_j(x)|_1 \geqslant \frac{1}{4}\cdot\frac{1}{2^{m_2-m_1}}\cdot\frac{1}{2^j} \tag{4}$$

$$|K_{j,r}|_{\text{est op }N} \leqslant |s_j(x)|_1\cdot 8\cdot 2^{m_2-m_1}\cdot 2(2^{m_2})^r\cdot L^r_{b(m_2)-1}\cdot D^r_{b(m_2)-1}\cdot 10^r. \tag{5}$$

Let the weight of S_j, be $w(S_j)=|s_j(x)|_1$.

In the next lemma we consider also a $(b(m_2)-1)$-substituted representation of h, say $h(x,\bar{l})$. That is, we consider an h with $|h(x)|_1=1$ and $|h|_{\text{op }N}<1/\varepsilon_n$ and we consider a representation $h=\Sigma_{i,\alpha}a_{i,\alpha}x^i\bar{l}_1^{\alpha_1}\dots\bar{l}_N^{\alpha_N}$ for which $|\Sigma_{i,\alpha}a_{i,\alpha}x^i\bar{l}_1^{\alpha_1}\dots\bar{l}_N^{\alpha_N}|_{\text{est op }N}<1/\varepsilon_n$. In this representation of h we substitute every $\bar{l}_i$ with $i\leqslant b(m_2)-1$ by a polynomial in x. So in $h(x,\bar{l})$ every $\bar{l}$ has index $\geqslant b(m_2)$.

In the next lemma we let

$$\sum_{|\alpha|=r_1}\bar{l}^{(\alpha)}\left(\sum_{i,(\beta)}c_{(\alpha),i,(\beta)}x^i\bar{l}^{(\beta)}\right)$$

denote the following terms out of the product $h(x,\bar{l})\,S_j(x,\bar{l})$:

In every $\bar{l}^{(\alpha)}=\bar{l}_{i_1}^{(\alpha_1)}\bar{l}_{i_2}^{(\alpha_2)}\dots\bar{l}_{i_p}^{(\alpha_p)}$ we have

(A) $\alpha_1+\alpha_2+\dots+\alpha_p=r_1$.

(B) $i_1,i_2,\dots,i_p$ are all $>j$.

(C) $i_1,i_2,\dots,i_p$ all belong to the $b(m_2)-1$ first systems.

(D) For every k_m, $1\leqslant m\leqslant s$, in $\bar{l}^{(\beta)}=\bar{l}_{k_1}^{(\beta_1)}\bar{l}_{k_2}^{(\beta_2)}\dots\bar{l}_{k_s}^{(\beta_s)}$ we have $k_m\leqslant j$.

LEMMA $\mathfrak{a}^{c}_{1,\mathrm{B}}, m_2, N(\mathfrak{a}_1, m_2)$. *For all $(\mathfrak{a}_1, m_2)$ there is a growth function $F'_{N(\mathfrak{a}_1,m_2)}$ which is trivial up to the $(b(m_2)-1)$-st stage and which depends only on $N(\mathfrak{a}_1,m_2)$ and $|\ |^{b(m_2)-1}$, an integer n_1, and a positive number E_1 depending only on $|\ |^{b(m_2)-1}$ such that if $\{D_n, L_n, l_n, C_n\}$ grows faster than $F_{N(\mathfrak{a}_1,m_2)}$, and the* case $\mathfrak{a}^{c}_{1,\mathrm{B}}, m_2$ *occurs when $|hq|^N$ is estimated by V then there is an $r_1\leqslant n_1$, such that for*

$$\sum_{|\alpha|=r_1}\bar{l}^{(\alpha)}\left(\sum_{i,(\beta)}c_{(\alpha),i,(\beta)}x^i\bar{l}^{(\beta)}\right)$$

we have

$$E_1|s_j(x)|_1 \geqslant \sum_{|\alpha|=r_1}\left(\left|\sum_{i,(\beta)} c_{(\alpha),i,(\beta)}x^i \bar{l}^{(\beta)}\right|_{\mathrm{est\,op}\,N}\right)$$
$$\geqslant \sum_{|\alpha|=r_1}\left|\left(\sum_{i,(\beta)} c_{(\alpha),i,(\beta)}x^i \bar{l}^{(\beta)}\right)(x)\right|_1$$
$$\geqslant \frac{1}{E_1}|s_j(x)|_1.$$

Proof. We give a detailed proof later. The ideas are the following. The condition $|h|_{\mathrm{op}\,N}/|h|_1 < 1/\varepsilon_n$ on h gives that h has a representation which has its main weight on low degrees (and early systems) regarded as a polynomial in x and $\bar{l}$'s. The condition (1) on s_j in the definition of semigood coefficient shows that the same holds for s_j. So by the theorem on multiplication of polynomials in many variables hs_j is big as a polynomial in x and $\bar{l}$'s and so also $|hs_j(x)|_1$ is big. On the other hand the submultiplicativity of est op N-norms give a bound on $|hs_j|_{\mathrm{est\,op}\,N}$. The extra complications in the statement and proof of Lemma $\mathfrak{a}^c_{1,\mathrm{B}}, m_2, N(\mathfrak{a}_1, m_2)$ are due to the fact that we consider hS_j instead of hs_j.

Now consider V and let $V_j(x, \bar{l})$ be the $(b(m_2)-1)$-substituted coefficient of $l_j q_j$ in a $b(m_2)$-expansion of V. Let

$$\sum_{|\alpha|=r_1} \bar{l}^{(\alpha)}\left(\sum_{i,(\beta)} d_{(\alpha),i,(\beta)}x^i \bar{l}^{(\beta)}\right) \tag{3.3}$$

denote the terms out of $h(x,\bar{l})\,S_j(x,\bar{l}) - V_j(x,\bar{l})$, which fulfill (A)–(D) above (as for $h(x,\bar{l})\,S_j(x,\bar{l})$).

We now say that $h(x,\bar{l})\,S_j(x,\bar{l}) - V_j(x,\bar{l})$ is a good coefficient of $l_j q_j$ if

$$\sum_{|\alpha|=r_1}\left(\left|\sum_{i,(\beta)} (d_{(\alpha),i,(\beta)} - c_{(\alpha),i,(\beta)})\, x^i \bar{l}^{(\beta)}\right|_{\mathrm{est\,op}\,N}\right)$$
$$\leqslant \frac{1}{10}\sum_{|\alpha|=r_1}\left(\left|\sum_{i,(\beta)} c_{(\alpha),i,(\beta)}\, x^i \bar{l}^{(\beta)}\right|_{\mathrm{est\,op}\,N}\right)$$

and

$$\sum_{|\alpha|=r_1}\left(\left|\sum_{i,(\beta)}(d_{(\alpha),i,(\beta)}-c_{(\alpha),i,(\beta)})\,x^i f^{(\beta)}(x)\right|_1\right)$$
$$\leq\frac{1}{10}\sum_{|\alpha|=r_1}\left(\left|\left(\sum c_{(\alpha),i,(\beta)}x^i f^{(\beta)}\right)(x)\right|_1\right)$$

We say that $|s_j(x)|_1$ is the weight of this good coefficient.
We now have

PROPOSITION $\alpha^c_{1,B}, m_2, N(\alpha_1, m_2)$. *For all* (α_1, m_2) *there is a growth function* $F_{N(\alpha_1,m_2)}$ *which is trivial up to the* $(b(m_2)-1)$*-st stage and which depends only on* $N(\alpha_1, m_2)$ *and* $|\ |^{b(m_2)-1}$, *and a number* $B(m_2)$ *depending only on* $|\ |^{b(m_2)-1}$ *such that if* $\{D_n, L_n, l_n, C_n\}$ *grows faster than* $F_{N(\alpha_1,m_2)}$ *and the* case $\alpha^c_{1,B}, m_2$ *occurs when* $|hq|^N$ *is estimated by* V *then either*

$\alpha^c_{1,B}, m_{2,G}$: $|V|^N_{est}\geq B(m_2)$

$\alpha^c_{1,B}, m_{2,B}$: *the sum of the weights of good coefficients of* $l_j q_j$'s, $\varepsilon_j=1/2^{m_2}$, *is*

$$>\frac{1}{2^{m_2-m_1}}\cdot\frac{1}{8}\cdot\frac{1}{2}.$$

Proof. The strategy is the following. If $|V|^N_{est}$ is very small, then the $d_{(\alpha),i,(\beta)}$'s are almost the same as the $c_{(\alpha),i,(\beta)}$'s. This gives many good coefficients.

Remark. We observe that the estimates and growth functions in Lemma $\alpha^c_{1,B}, m_2, N(\alpha_1, m_2)$ and Proposition $\alpha^c_{1,B}, m_2, N(\alpha_1, m_2)$ are uniform in α_1 as long as $\varepsilon_{b(\alpha_1)}=1/2^{m_1}$.

(3.4) If $\alpha^c_{1,B}, m_{2,B}$ occurs, let D be the sum of the weights of good coefficients of $l_j q_j$'s, $\varepsilon_j=1/2^{m_2}$. Among the systems $\alpha_{2,1}\,\alpha_{2,2}, \alpha_{2,3}\ldots b(\alpha_{2,1})<b(\alpha_{2,2})<\ldots$ for which the corresponding ε is $1/2^{m_2}$, there is a first system, say $\alpha_{2,p}=\alpha_2$, for which the sum of the weights of good coefficients of $l_j q_j$'s, $j\in\alpha_2$, is $>D/2^p$. We then say that the case $\alpha^c_{1,B}, m_{2,B}, \alpha_2$ occurs.

We will now study the case $\alpha^{nc}_{1,B}$ of Proposition $\alpha^{nc}_{1,N(\alpha_1)}$ and then consider how much we have proved of the Statements 1–6. We will first define the *cancellation effect*.

(3.5) If $P_1(y_1,\dots,y_n)=\Sigma a_{1,\alpha}y^\alpha$ and $P_2(y_1,\dots,y_n)=\Sigma a_{2,\alpha}y^\alpha, \alpha=(\alpha_1,\dots,\alpha_n), y=(y_1,\dots,y_n)$ and $a_{i,\alpha}\neq 0$ for all terms in the summation are two polynomials we say that *the cancellation effect* of P_1 on P_2 is $\Sigma_\alpha|a_{1,\alpha}|$ where the summation is extended only over those α for which $a_{2,\alpha}$ is $\neq 0$. Let $S(1/2^j)$ and $V(1/2^j)$ denote the sum of those terms of the 3rd type in a pre-(n,m)-expansion of q and a pre-n-expansion of V for which the corresponding ε is $1/2^j$. Let $S(1/2^j)(x)$ and $V(1/2^j)(x)$ denote the same sums after having been substituted as polynomials in x. If $\mathfrak{a}^{nc}_{1,B}$ occurs we know by Proposition $\mathfrak{a}^{nc}_{1,N(\mathfrak{a}_1)}$ that $(hq'-V')(x)$ contains $P_{1/25}(E)$. Since on the other hand $(hq-V)(x)\equiv 0$, we have that there is a smallest j say m_2, $1/2^{m_2}<\min(\varepsilon_1,\dots,\varepsilon_n)\leq 1/2^{m_1}$ such that the cancellation effect of $hS(1/2^{m_2})(x)-V(1/2^{m_2})(x)$ on $E(x)$ is $>|E(x)|_1/2\cdot 2^{m_2-m_1}$. We then say that the case $\mathfrak{a}^{nc}_{1,B}, m_2$ occurs.

In the next proposition we assume that we have an $(n, b(m_2))$-expansion of q and a $b(m_2)$-expansion of V and a representation $h(x,\bar{l})$ of h with $|h|_1=1$, $|h|_{\text{est op }N}<1/\varepsilon_n$ and that all of these are $(b(m_2)-1)$-substituted. We consider the $(b(m_2)$-1)-substituted coefficients $S_j(x,\bar{l})$ and $V_j(x,\bar{l})$ of l_jq_j's, $\varepsilon_j=1/2^{m_2}$.

Below we put $hS_j(x,\bar{l})-V_j(x,\bar{l})=S_j'(x,\bar{l})+S_j''(x,\bar{l})$ where $S_j'(x,\bar{l})$ consists of those terms which either are of degree $>m$, defined in Proposition $\mathfrak{a}^{nc}_{1,N(\mathfrak{a}_1)}$ in l_i's or contain an l_i with i from a system with number $\geq b(m_2)$. Below we consider $S_j''(x)$ (which we get from $S_j''(x,\bar{l})$ by substituting the l's by polynomials in x).

Proposition $\mathfrak{a}^{nc}_{1,B}, m_2, N(\mathfrak{a}_1, m_2)$. *For all n and m_2 there is a growth function $F_{N(\mathfrak{a}_1,m_2)}$ which coincides with $F_{N(\mathfrak{a}_1)}$ up to the $(b(m_2)-1)$-st stage and which depends only on $N(\mathfrak{a}_1,m_2)$ and $|\ |^{b(m_2)-1}$ such that if $\{L_n, D_n, l_n, C_n\}$ grows faster than $F_{N(\mathfrak{a}_1,m_2)}$ and the* case $\mathfrak{a}^{nc}_{1,B}, m_2$ *occurs when $|hq|^N$ is estimated by V then either*

$\mathfrak{a}^{nc}_{1,B}, m_{2,G}$: $|V|^N_{\text{est}}\geq 1$

or

$\mathfrak{a}^{nc}_{1,B}, m_{2,B}$: *The sum of the cancellation effects of all monomials $x^il^\alpha=P_\alpha(x)$ of all $S_j''(x,\bar{l})$'s on $E(x)$ is $<|E(x)|_1/(10\cdot 2\cdot 2^{m_2-m_1})$.*

Proof. The proposition is a simple consequence of the fact that the l's behave like independent variables. We give a detailed proof in Section 4.

(3.6) Let $T_j'(x,\bar{l})$ consist of the following monomials out of $hS_j(x,\bar{l})-V_j(x,\bar{l})$: Those which are of degree $\leq m$ in l_i's with $i>j$, and for which every l_i with $i>j$ is from one of the $b(m_2)-1$ first system. If $|V|_{\text{est}}^N \leq 1$, then by the previous proposition the cancellation effect of $[\Sigma_j(hS_j(x,\bar{l})-V_j(x,\bar{l})-T_j'(x,\bar{l}))](x)$ on $E(x)$ is

$$<\frac{|E(x)|_1}{10\cdot 2\cdot 2^{m_2-m_1}}$$

and so the cancellation effect of $[\Sigma_j T_j'(x,\bar{l})](x)$ on $E(x)$ is

$$>\frac{9}{10}\cdot\frac{|E(x)|_1}{2\cdot 2^{m_2-m_1}}>\frac{9}{10}\cdot\frac{B_n'}{2\cdot 2^{m_2-m_1}}.$$

We now have

LEMMA 3.3. *If* $|V|_{\text{est}}^N \leq 1$ *in the previous proposition, then*

$$\sum_j |T_j'(x,\bar{l})|_{\text{est op } N} \leq 2\cdot\sum_{r\leq m} L_{b(m_2)-1}^{r-1}\cdot D_{b(m_2)-1}^{r-1}\cdot(2^{m_2})$$
$$= K_{b(m_2)-1}.$$

Proof. This can be proved by considering the different stages of the pre-expansions of q and V and the terms of $T_j'(x,\bar{l})$ derived from them. We give details in Section 4.

With (3.6) and Lemma 3.3. we now give

Definition. We say that $hS_j(x,\bar{l})-V_j(x,\bar{l})$ is a good coefficient of l_jq_j in $hq-V$ if

$$(1)\ |T_j'(x,\bar{l})|_{\text{est op } N} \leq \frac{K_{b(m_2)-1}\cdot 100}{\frac{9}{10}\cdot\frac{B_n'}{2\cdot 2^{m_2-m_1}}}|[T_j'(x,\bar{l})](x)|_1$$

$$(2)\ |[T_j'(x,\bar{l})](x)|_1 \geq \frac{B_n'}{2\cdot 2^{m_2-m_1}}\cdot\frac{1}{10^j}. \tag{3.7}$$

(3.8) If $T_j'(x,\bar{l})$ fulfills (1) and (2) of (3.7), let $T_{j,r}'(x,\bar{l})$ be the terms out of $T_j(x,l)$ which are of degree r in $\bar{l}$'s with index $>j$. Let r_1 be the smallest number such that $|T_{j,r_1}'(x)|_1 \geq (1/m)|T_j'(x)|_1$—obviously $\Sigma_{r\leq m} T_{j,r}'(x,\bar{l})=T_j'(x,\bar{l})$. We say that $|T_{j,r_1}(x)|_1$ is the weight of this coefficient.

(3.9) If $\alpha_{1,B}^{nc}, m_{2,B}$ occurs let D be the sum of the weights of good coefficients of l_jq_j's, $\varepsilon_j=1/2^{m_2}$. There is among the systems $\alpha_{2,1},\alpha_{2,2},\alpha_{2,3}\ldots b(\alpha_{2,1})<b(\alpha_{2,2})<\ldots$ for which the corresponding ε is $1/2^{m_2}$ a first, say $\alpha_{2,p}=\alpha_2$, for which the sum of the weights of good coefficients of $l_j q_j$'s, $j \in \alpha_2$, is $>D/2^p$. We then say that the case $\alpha^{nc}_{1,B}, m_2, \alpha_2$ occurs.

We have now in fact proved the Statements 1 and 2. To see this we first observe that there is one growth function F which dominates all growth functions in the previous propositions.

The case $\alpha^{nc}_{1,G}$ of Statement 1 now follows from Proposition $\alpha^{nc}_{1,N(\alpha_1)}$ and the case $\alpha^*_{1,B}, m_2$ follows from the discussions above (3.1), (3.5) for $*=$c or nc. In Statement 2 the case $\alpha^*_{1,B}, m_{2,G}$ follows from Propositions $\alpha^c_{1,B}, m_2, N(\alpha_1, m_2)$ and $\alpha^{nc}_{1,B}, m_2, N(\alpha_1, m_2)$. The B^2_n here depends in fact only on $|\ |^{b(m_2)-1}$, $b(m_2)-1<n$. The case $\alpha^*_{1,B}, m_{2,B}, \alpha_2$ follows from these propositions and (3.4) and (3.9).

If the case $\alpha^*_{1,B}, m_{2,B}, \alpha_2$ occurs, $*=$c or nc, consider for each $l_j q_j, j \in \alpha_2$, which has a good coefficient a pre-$(b(\alpha_2), b(\alpha_2))$-expansion of q_j. (Since j belongs to the system with number $b(\alpha_2)$ we get $q_j = q_{b(\alpha_2)} + \dots$ in the first stage of this preexpansion.) We form $q'_j(s, l, q, t)$ by removing from the final stage of this preexpansion all terms of the 3rd type for which the corresponding ε is $<\min\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{b(\alpha_2)}\}$. We form $q'_j(x)$ by expanding the s's, l's, q's and t's as polynomials in x. For each j we have either

$$\left|[q'_j(x)]_{R_{b(\alpha_2)}}\right|_1 > \frac{1}{2} \quad \text{or} \quad \left|[q'_j(x)]_{R_{b(\alpha_2)}}\right|_1 \leq \frac{1}{2}.$$

(3.10) Let D' be the sum of the weights of good coefficients of $l_j q_j$'s, $j \in \alpha_2$. Let D_1 be the sum of the weights of the good coefficients of $l_j q_j$'s, $j \in \alpha_2$, for which

$$\left|[q'_j(x)]_{R_{b(\alpha_2)}}\right|_1 > \frac{1}{2}$$

and let D_2 be the sum of the weights for which

$$\left|[q'_j(x)]_{R_{b(\alpha_2)}}\right|_1 \leq \frac{1}{2}.$$

Then $D' = D_1 + D_2$. Now if $D_1 \geq D'/2$ we say that the case $\alpha^*_{1,B}, m_{2,B}, \alpha^{nc}_2$ occurs and if $D_2 > D'/2$ we say that the case $\alpha^*_{1,B}, m_{2,B}, \alpha^c_{2,B}$ occurs. These definitions now give the Statement 3 for $k=2$.

(3.11) Now we assume that the case $\alpha^*_{1,B}, m_{2,B}, \alpha^c_{2,B}$ occurs. For every j with

$$\left|[q'_j(x)]_{R_{b(\alpha_2)}}\right|_1 \leq \frac{1}{2}$$

consider the smallest m_j' (obviously $1/2^{m_j'} \geq \min(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{b(\alpha_2)})$, $m_j' > m_2$ will follow from (3.1) such that

$$\sum_{\varepsilon_j = 1/2^{m_j'}} |s_j|_1 > \frac{1}{8} \cdot \frac{1}{2^{m_j' - m_2}} \quad \text{(see (3.1))}.$$

Let w_j be the weight of the good coefficient of $l_j q_j$. Let m_3 be the smallest number such that $\Sigma_{m_j'=m_3} w_j > D_2/2^{m_3-m_2}$. We then say that the case $\alpha_{1,B}^*, m_2, \alpha_{2,B}^c, m_3$ occurs. This definition gives Statement 5 for $k=2$.

We will next give the Statements 4 and 6 for $k=2$ before going into the induction on k of the Statements 3–6.

Before the next proposition we will make the

Definition. We will say that a pre-(n_1, n_2)-expansion of q is shortened with respect to systems of number $\geq m$ if every term of the first type which ends with $l_j q_j$ where j belongs to a system with number $\geq m$ is not expanded further.

(3.12) In the next propositions we will assume that the case $\alpha_{1,B}^c, m_{2,B}, \alpha_2^{nc}$ or $\alpha_{1,B}^{nc}, m_{2,B}, \alpha_2^{nc}$ occurs when $|hq|^N$ is estimated by V. We consider a pre-$(n, b(\alpha_2))$-expansion of q which is shortened with respect to systems with number $\geq b(\alpha_2)$. Let $\Sigma S_j l_j q_j + R(s, l, q, t)$ be the final stage of this shortened expansion where j belongs to a system with number $\geq b(\alpha_2)$.

(3.13) We observe that S_j need not be the coefficient of $l_j q_j$ in a pre-$(n, b(\alpha_2))$-expansion of q, since the terms out of the coefficient which contain an l_j with j from a system with number $\geq b(\alpha_2)$ are missing.

(3.14) We also consider a pre-$(n, b(\alpha_2))$-expansion of V which is shortened with respect to systems with number $\geq b(\alpha_2)$, let $\Sigma V_j l_j q_j + R_V(s, l, q, t)$ be the final stage.

From (3.12) and (3.14) we get

$$hq(s, l, q, t) - V(s, l, q, t) = \sum (hS_j - V_j) l_j q_j + hR(s, l, q, t) - R_V(s, l, q, t). \tag{3.15}$$

For every q_j such that $l_j q_j$ has a good coefficient we now form a pre-$(b(\alpha_2), (b(\alpha_2))$-expansion of q_j and form q_j' by removing from this expansion all terms of the third type where the corresponding is $< \min(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{b(\alpha_2)})$. Remove also from R and R_V all terms of 3rd type for which $\varepsilon < \min(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{b(\alpha_2)})$, thus forming R' and R_V'.

(3.16) From (3.15) we now define $(hq-V)'(s,l,q,t)$ by

$$(hq-V)'(s,l,q,t)=\sum(hS_j-V_j)\,l_j q_j'+\sum(hS_j-V_j)\,l_j q_j+hR'(s,l,q,t)-R_V'(s,l,q,t)$$

where the sum $(hS_j-V_j)\,l_j q_j'$ is extended over those q_j's which have a good coefficient and for which

$$\left|[q_j'(x)]_{R_{b(\alpha_2)}}\right|_1>\frac{1}{2}.$$

(3.17) In (3.16) we consider the terms $(hS_j-V_j)\,l_j q_j'$. For every such j consider a $(b(m_2)-1)$-substitution of hS_j-V_j and consider the part

$$\sum_{|\alpha|=r_1}\bar{l}^{(\alpha)}\sum_{m,(\beta)}d'_{(\alpha),m,(\beta)}x^m\bar{l}^{(\beta)}\quad\text{(see 3.3)}$$

or

$$T'_{j,r_1}(x,\bar{l})=\sum_{|\alpha|=r_1}\bar{l}^{(\alpha)}\sum d'_{(\alpha),m,(\beta)}x^m\bar{l}^{(\beta)}.$$

We observe that (3.17) coincides with (3.3), (3.6). The reason is the following:

It follows from (3.13) that every term out of the total coefficient of $l_j q_j$ in $hq-V$ (in a pre-$(n,b(\alpha_2))$-expansion of q and a pre-$(b(\alpha_2))$-expansion of V) which is not in hS_j-V_j, contains an l_i with $i\geqslant j$ and i from a system with number $\geqslant b(\alpha_2)$. This obviously remains true for $(b(m_2)-1)$-substitutions of hS_j-V_j, and of the coefficient of $l_j q_j$ in $hq-V$.

We now make a $(b(\alpha_2)-1)$-substitution in (3.17). Since in $\bar{l}^{(\alpha)}$ the index of every l appearing is $>j\geqslant b(\alpha_2)$, $\bar{l}^{(\alpha)}$ remains unchanged when this replacement is done. So we get

$$\sum_{(\alpha)}\bar{l}^{(\alpha)}\sum_{m,(\beta)}d'_{(\alpha),m,(\beta)}x^m\bar{l}^{(\beta)}=\sum_{(\alpha)}\bar{l}^{(\alpha)}\sum_{m,(\beta)}d_{(\alpha),m,(\beta)}x^m\bar{l}^{(\beta)}.\tag{3.18}$$

(3.19) Now, using (3.16) we will define $(hq-V)'(x,\bar{l})$ in the following way: In every term $(hS_j-V_j)\,l_j q_j'$ make a $(b(\alpha_2)-1)$-substitution of hS_j-V_j and a $(b(\alpha_2)-1)$-substitution of q_j'. In every term $(hS_j-V_j)\,l_j q_j$ make a $(b(\alpha_2)-1)$-substitution of hS_j-V_j and expand q_j as a polynomial in x. In $hR'(s,l,q,t)$ we make a $(b(\alpha_2)-1)$-substitution of h. In $R'(s,l,q,t)$ and $R_V'(s,l,q,t)$ we expand every s as a $(b(\alpha_2)-1)$-substituted polynomial in x and l. Every t and every q is expanded as a polynomial in x. Every l with index $\geq b(\alpha_2)$ remains unchanged and every l with index $\leq(b(\alpha_2)-1)$ is substituted by a polynomial in x.

$(hq-V)'(x)$ is defined by expressing every l in $(hq-V)'(x, l)$ as a polynomial in x.

(3.20) Here we observe that every q that appears in $R'(s, l, q, t)$ or $R_V(s, l, q, t)$ has index $\leq b(\alpha_2)-1$.

The reason is the following: Every term of the first type which ends with $l_j q_j$ where q_j belongs to a system with number $\geq b(\alpha_2)$ is not expanded further and the term does not go into $R(s, l, q, t)$ or $R_V(s, l, q, t)$. Thus if a term of the first type is expanded further q_j is replaced by a q with index $\leq b(\alpha_2)-1$. Moreover if $n>b(\alpha_2)$ (obviously $n \neq b(\alpha_2)$) then q_n will be expanded and so in particular q_n will not appear in $R(s, l, q, t)$ and from the shortened expansion of q_n we only get q's with index $\leq b(\alpha_2)-1$.

In the next proposition we will let

$$E=\sum_j l_j\left(\sum_{(\alpha)} \bar{l}^{(\alpha)} \sum e_{(\alpha), k, (\beta)} x^k \bar{l}^{(\beta)}\right)$$

stand for a polynomial where

(1) l_j runs over those l_j's such that $l_j q_j$ has a good coefficient and for which $\|[q_j'(x)]_{R_{b(\alpha_2)}}|_1>\frac{1}{2}$.

(2) For every j, (α) runs over a subset of the set of (α)'s appearing in (3.17).

(3) Only l's from the $b(\alpha_2)-1$ first systems appear in $\bar{l}(\beta)$ and $b(\alpha_2)\leq i\leq j$ for every l_i there.

We now have

PROPOSITION $\alpha_{1,B}^*, m_{2,B}, \alpha_2^{nc}, N(\alpha_1, \alpha_2)$. *For all* (α_1, α_2) *there is a growth function* $F_{N(\alpha_1, \alpha_2)}$ *which coincides with* $F_{N(\alpha_1, m_2)}$ *up to the* $(b(\alpha_2)-1)$*-st stage and which depends only on* $N(\alpha_1, \alpha_2)$ *and* $|\ |^{b(\alpha_2)-1}$ *and numbers* $B'_{b(\alpha_2)}$ *and* n_2 *depending only on* $|\ |^{b(\alpha_2)-1}$ *and* j, *such that if* $\{L_n, D_n, l_n, C_n\}$ *grows faster than* $F_{N(\alpha_1, \alpha_2)}$ *and the* case $\alpha_{1,B}^*, m_{2,B}, \alpha_2^{nc}$ *occurs then either*

$\alpha_{1,B}^*, m_{2,B}, \alpha_{2,G}^{nc}$: $|V|_{est}^N \geq 1$

or

$\alpha^*_{1,B}, m_{2,B}, \alpha^{nc}_{2,B}$: *$(hq-V)'(x)$ contains $p_{1/20}(E)$ where $E=\Sigma_j l_j(\Sigma\, \bar{l}^{(\alpha)} \Sigma\, e_{(\alpha),k,(\beta)} x^k \bar{l}^{(\beta)})$ has the properties* 1–3 *above and in addition*

(4) *$k+|\beta|\leq n_2$ for all k and β*

(5) *$|E|_1 \geq DB'_{b(\alpha_2)} L_{b(\alpha_2)}$ where D is the sum of the weights of the good coefficients of $l_j q_j$'s with j in the system α_2.*

We give the proof of this in Section 4.

As before it follows from (3B) that there is one growth function which dominates all growth functions $F_{N(\alpha_1,\alpha_2)}$. Since $(hq-V)(x)=0$ we have the following:

There is a smallest m_3 with $1/2^{m_3}<\min(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{b(\alpha_2)})$ such that the following holds: Let in (3.16) $S_{1,j}$ be the sum of the terms of 3rd type in $q_j'-q_j$ which the corresponding ε is $1/2^{m_3}$. Let S and S_V be the sum of the terms of 3rd type in R and R_V for which the corresponding ε is $1/2^{m_3}$. Then the cancellation effect of

$$\sum_j (hS_j - V_j)\, S_{1,j}(x) + hS(x) - S_V(x) \quad \text{on } E \text{ is } \; > \frac{|E|_1}{2\cdot 2^{m_3-m_2}}.$$

We then say that the case $\alpha^*_{1,B}, m_{2,B}, \alpha^{nc}_2, m_3$ occurs. So the Proposition $\alpha^*_{1,B}, m_{2,B}, \alpha^{nc}_2, N(\alpha_1,\alpha_2)$ now gives Statement 4 for $k=2$.

To get Statement 6 for $k=2$ we now assume that $\alpha^*_{1,B}, m_{2,B}, \alpha^{nc}_{2,B}, m_3$ occurs when $|hq|^N$ is estimated by V. We consider a $j\in\alpha_2$ such that $l_j q_j$ has a good coefficient and such that $|[q_j'(x)]_{R_{b(\alpha_2)}}|_1 \leq \frac{1}{2}$ and $m_j'=m_3$.

(3.18) We define a semigood coefficient of $l_k q_k$, $\varepsilon_k=1/2^{m_3}$, in a $(b(m_3)-1)$-substituted $(b(\alpha_2), b(m_3))$-expansion of q_j like above (see before Lemma 3.2). We now give a lemma which corresponds to Lemma $\alpha^c_{1,B}, m_2, N(\alpha_1, m_2)$: h in Lemma $\alpha^c_{1,B}, m_2, N(\alpha_1, m_2)$ corresponds to the "good" part of the coefficient of $l_j q_j$ below and S_j of Lemma $\alpha^c_{1,B}, m_2, N(\alpha_1, m_2)$ corresponds to S_k below.

Let $S_k(x,\bar{l})$ be the semi-good coefficient of $l_k q_k$ in a $(b(m_3)-1)$-substituted $(n, b(m_3))$-expansion of q_j. We observe that the indices of all l's appearing in $S_k(x,\bar{l})$ are $<j$.

(3.19) Put $S_{1,j}(x,\bar{l})=h(x,\bar{l})\,S_j(x,\bar{l})-V_j(x,\bar{l})$ where $S_j(x,\bar{l})$ and $V_j(x,\bar{l})$ are the $(b(m_3)-1)$-substituted coefficients of $l_j q_j$ in the $(n, b(m_3))$- resp. $b(m_3)$-expansions of q and V and $h(x,\bar{l})$ is a $(b(m_3)-1)$-substituted representation of h. We observe that $S_j(x,\bar{l})$ and $V_j(x,\bar{l})$ would be the same if we instead considered $(n, b(m_2))$- resp. $b(m_2)$-

expansions of q and V. This is since $j>k\geqslant b(m_3)$ which gives that the expansions of q's with index between $b(m_2)$ and $b(m_3)$ will never contain any $l_j q_j$.

In the next lemma let

$$\sum_{|\alpha|=r_1} \bar{l}^{(\alpha)} \sum_{|\beta|=r_2} \bar{l}^{(\beta)} \sum_{i,(\gamma)} c_{(\alpha),(\beta),i,(\gamma)} x^i \bar{l}^{(\gamma)}$$

denote the following terms out of the product $S_{1,j}(x,\bar{l})\, S_k(x,\bar{l})$: For every $\bar{l}^{(\alpha 0)}$ we have the conditions A, B and C of semigood coefficients (given after Lemma 3.2). In every $\bar{l}^{(\beta)}=\bar{l}_{i_1}^{(\beta_1)},\dots \bar{l}_{i_p}^{(\beta_p)}$, we have (A_1) $\beta_1+\beta_2+\dots+\beta_p=r_2$, $(B_1)\, i_1, i_2, \dots, i_p$ are all $>k$ but $<j$, $(C_1)\, i_1, i_2, \dots, i_p$ all belong to the $b(m_3)-1$ first systems—(since j belongs to a system with number $\geqslant b(m_3)$ we get $<j$ instead of $\leqslant j$ in (B_1). (D_1) For every m_r, $1\leqslant |r|\leqslant s$, in $\bar{l}^{(\gamma)}=\bar{l}^{\gamma_1}\dots \bar{l}^{\gamma_s}$ we have $m_r\leqslant k$.

In the next lemma let w_j denote the weight of the good coefficient of $l_j q_j$ and let $q_j=q_{b(\alpha_2)}+\Sigma_{r\leqslant j-1}\, s_r(l_r q_r-1)+t$ be the first stage of the expansion of q_j.

In the lemma below, if $*=\text{nc}$, r_1 is the number given by (3.8) and if $*=\text{c}$, r_1 is the number given by Lemma $\alpha_{1,\text{B}}^{\text{c}}, m_2, N(\alpha_1, \alpha_2)$.

Lemma $\alpha_{1,\text{B}}^{*}, m_{2,\text{B}}, \alpha_{2,\text{B}}^{\text{c}}, m_3, N(\alpha_1, \alpha_2, m_3)$. *For all $(\alpha_1, \alpha_2, m_3)$ there is a growth function $F'_{N(\alpha_1,\alpha_2,m_3)}$ which coincides with $F_{N(\alpha_1,m_2)}$ up to the $(b(m_3)-1)$-st stage and which depends only on $N(\alpha_1,\alpha_2,m_3)$ and $|\ |^{b(m_3)-1}$ (and numbers E_2 and n_2 depending only on $|\ |^{b(m_3)-1}$) such that if $\{D_n, L_n, l_n, C_n\}$ grows faster than $F'_{N(\alpha_1,\alpha_2,m_3)}$ and $\alpha_{1,\text{B}}^{*}, m_{2,\text{B}}, \alpha_{2,\text{B}}^{\text{c}}, m_3$ occurs then for some $r_2\leqslant n_2$ we have for*

$$\sum_{|\alpha|=r_1} \bar{l}^{(\alpha)} \sum_{|\beta|=r_2} \bar{l}^{(\beta)} \sum_{i,(\gamma)} c_{(\alpha),(\beta),i,(\gamma)} x^i \bar{l}^{(\gamma)}$$

$$\begin{aligned} E_2 w_j |s_k(x)|_1 &\geqslant \sum_{|\alpha|=r_1} \sum_{|\beta|=r_2} \left(\left| \sum_{i,(\gamma)} c_{(\alpha),(\beta),i,(\gamma)} x^i \bar{l}^{(\gamma)} \right|_{\text{est op } N} \right) \\ &\geqslant \sum_{|\alpha|=r_1} \sum_{|\beta|=r_2} \left| \left(\sum_{i,(\gamma)} c_{(\alpha),(\beta),i,(\gamma)} x^i \bar{l}^{(\gamma)} \right)(x) \right|_1 \\ &\geqslant \frac{1}{E_2} w_j |s_k(x)|_1 \end{aligned}$$

Proof. The proof of this is essentially the same as the proof of Lemma $\alpha_{1,\text{B}}^{\text{c}}, m_2, N(\alpha_1, m_2)$. We give the details in Section 4.

Now we can use this lemma to define *a good partial coefficient of* $l_k q_k$ *that contains* l_j and define its weight.

Consider again $(b(m_3)-1)$-substituted $(n, b(m_3))$- resp. $b(m_3)$-expansions of q and V and let $S_{1,k}(x,\bar{l})$ resp. $V_{1,k}(x,\bar{l})$ be the coefficients of $l_k q_k$ in these expansions. Consider also a $(b(m_3)-1)$-substituted representation $h(x,\bar{l})$ of h.

Consider out of $h(x,\bar{l})\, S_{1,k}(x,\bar{l}) - V_{1,k}(x,\bar{l})$ the terms

$$l_j \sum_{|\alpha|=r_1} \bar{l}^{(\alpha)} \sum_{|\beta|=r_2} \bar{l}^{(\beta)} \sum_{i,(\gamma)} d_{(\alpha),(\beta),i,(\gamma)}\, x^i \bar{l}^{(\gamma)}$$

where (α) and (β) run through the same sets as in the lemma above. Every l in $\bar{l}^{(\gamma)}$ has index $\leqslant k$ but i and (γ) need not run through the same sets as in the lemma above. We say that

$$l_j \sum_{|\alpha|=r_1} \bar{l}^{(\alpha)} \sum_{|\beta|=r_2} \bar{l}^{(\beta)} \sum_{i,(\gamma)} d_{(\alpha),(\beta),i,(\gamma)}\, x^i \bar{l}^{(\gamma)}$$

is a good partial coefficient of $l_k q_k$ that contains l_j with weight $w_j |s_k(x)|_1$ if

(1) $S_{1,j}(x,\bar{l})$ is a good coefficient of $l_j q_j$ with weight w_j.

(2) $S_k(x,\bar{l})$ is a semigood coefficient of $l_k q_k$.

(3) $$\sum_{|\alpha|=r_1} \sum_{|\beta|=r_2} \left| \sum_{i,(\gamma)} (d_{(\alpha),(\beta),i,(\gamma)})\, x^i \bar{l}^{(\gamma)} \right|_{\text{est op } N} \leqslant \frac{1}{10} \sum_{|\alpha|=r_1} \sum_{|\beta|=r_2} \left| \sum_{i,(\gamma)} c_{(\alpha),(\beta),i,(\gamma)}\, x^i \bar{l}^{(\gamma)} \right|_{\text{est op } N}$$

and

(4) $$\sum_{|\alpha|=r_1} \sum_{|\beta|=r_2} \left| \left(\sum_{i,(\gamma)} (d_{(\alpha),(\beta),i,(\gamma)} - c_{(\alpha),(\beta),i,(\gamma)})\, x^i \bar{l}^{(\gamma)} \right)(x) \right|_1 \leqslant \frac{1}{10} \sum_{|\alpha|=r_1} \sum_{|\beta|=r_2} \left| \left(\sum_{i,(\gamma)} c_{(\alpha),(\beta),i,(\gamma)}\, x^i \bar{l}^{(\gamma)} \right)(x) \right|_1 .$$

In the next proposition let D' be the sum of the w_j's. The proposition is analogous to Proposition $\mathfrak{a}^c_{1,\mathrm{B}}, m_2, N(\mathfrak{a}_1, m_2)$.

Proposition $\mathfrak{a}^*_{1,\mathrm{B}}, m_{2,\mathrm{B}}, \mathfrak{a}^c_{2,\mathrm{B}}, m_3, N(\mathfrak{a}_1, \mathfrak{a}_2, m_3)$. *For all* $(\mathfrak{a}_1, \mathfrak{a}_2, m_3)$ *there is a growth function* $F_{N(\mathfrak{a}_1,\mathfrak{a}_2,m_3)}$ *which coincides with* $F_{N(\mathfrak{a}_1,m_2)}$ *up to the* $(b(m_3)-1)$*-st stage and which depends only on* $N(\mathfrak{a}_1,\mathfrak{a}_2,m_3)$ *and* $|\ |^{b(m_3)-1}$ *such that if* $\{D_n, L_n, l_n, C_n\}$ *grows faster than* $F_{N(\mathfrak{a}_1,\mathfrak{a}_2,m_3)}$ *and* $\mathfrak{a}^*_{1,\mathrm{B}}, m_{2,\mathrm{B}}, \mathfrak{a}^c_{2,\mathrm{B}}, m_3$ *occurs then either*

$\alpha^*_{1,B}, m_2, \alpha^c_{2,B}, m_{3,G}$: $|V|^N_{est} \geq 1$

or

$\alpha^*_{1,B}, m_2, \alpha^c_{2,B}, m_{3,B}$: *The sum of the weights of good partial coefficients of* $l_k q_k$*'s,* $\varepsilon_k = 1/2^{m_3}$, *that contain* l_j*'s,* $j \in a_2$, *is*

$$> D' \cdot \frac{1}{2^{m_3 - m_2}} \cdot \frac{1}{8} \cdot \frac{1}{2}.$$

Proof. The strategy of the proof of this proposition is similar to that of Proposition $\alpha^c_{1,B}, m_2, N(a_1, m_2)$. We give details in Section 4.

(3.20) Let D be the sum of the weights of good partial coefficients of $l_k q_k$'s that contain l_j's—assuming that $\alpha^*_{1,B}, m_{2,B}, \alpha^c_{2,B}, m_{3,B}$ occurs. There is among the systems $\alpha_{3,1}, \alpha_{3,2}, \ldots, b(\alpha_{3,1}) < b(\alpha_{3,2}) < \ldots$ for which the corresponding ε is $1/2^{m_3}$, a first, say $\alpha_{3,p} = \alpha_3$ for which the sum of the weights of good partial coefficients of $l_k q_k$'s, $k \in \alpha_3$, that contain l_j's, $j \in \alpha_2$, is $> D/2^p$. We then say that the case $\alpha^*_{1,B}, m_{2,B}, \alpha^c_{2,B}, m_{3,B}, \alpha_3$ occurs.

(3.21) It now follows from (3B) that there is one growth function F which dominates all $F_{N(\alpha_1, \alpha_2, m_3)}$. This gives Statement 6 for $k=2$.

We now turn to the induction on k to get Statements 3–6. We draw a "flow chart" (next page) to illustrate how the argument goes.

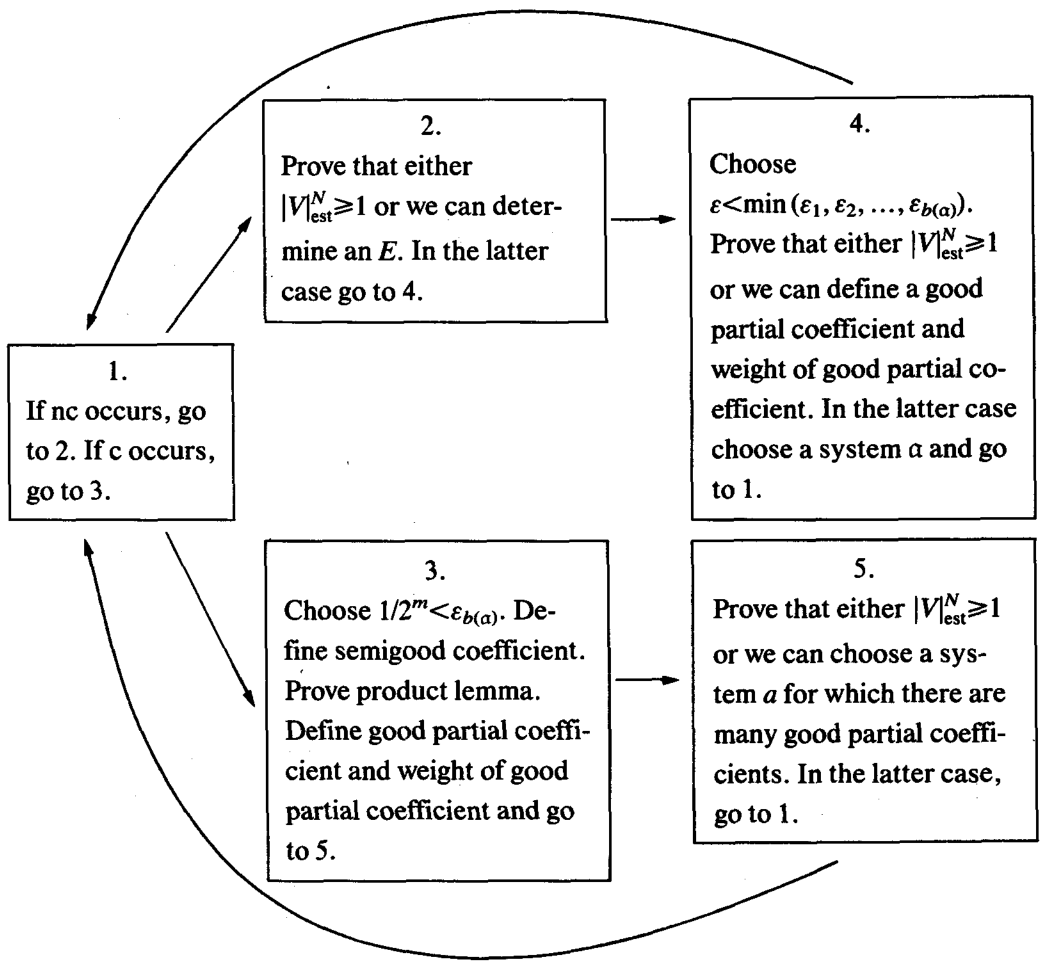
In the induction hypothesis below we let $\alpha^*_{1,B}, m_{2,B}, \alpha^*_{2,B}, m_{3,B}, \ldots, \alpha_k$ define $M\mathfrak{A}H$ and $N(\alpha_1, \ldots, \alpha_k)$. With $M = \{m_1, m_2, \ldots, m_k\}$ we let $\{m_{j_1}, m_{j_2}, \ldots, m_{j_r}, m_k\} = \{m_2, m_3, \ldots, m_k\} \cap H$. Let l_{sj_i} denote an l with sj_i in the system α_{j_i} (and l_{sk} an l with sk in the system a_k). We assume that $|h|_1 = 1, h = \Sigma\, b_{i,(j)} x^i l^{(j)}$ is a representation of h in x, and l's which shows $|h|_{\mathrm{op}\,N} < 1/\varepsilon_n$. We have already proved the induction hypothesis below for $k=1,2,3$. We use it below to prove Lemma

$$\alpha^*_{1,B}, \ldots, \alpha^c_{k,B}, m_{k+1}, N(\alpha_1, \ldots, \alpha_k, m_{k+1})$$

and Propositions

$$\alpha^*_{1,B}, \ldots, \alpha^c_{k,B}, m_{k+1}, N(\alpha_1, \ldots, \alpha_k, m_{k+1})$$

$$\alpha^*_{1,B}, \ldots, \alpha^{nc}_k, N(\alpha_1, \ldots, \alpha_k),$$



$$\alpha^*_{1,\mathrm{B}}, \ldots, \alpha^{\mathrm{nc}}_{k,\mathrm{B}}, m_{k+1}, N(\alpha_1, \ldots, \alpha_k, m_{k+1})$$

and to get the induction hypothesis verified for $k+1$.

Assuming $sj_1 > sj_2 > sj_r > sk$, we let

$$\sum l_{sj_1} l_{sj_2} \ldots l_{sj_r} \sum_{\substack{|\alpha_1|, \ldots, |\alpha_r|, |\beta| \\ = r_1, \ldots, r_r, r_k}} \bar{l}^{(\alpha_1)} \ldots \bar{l}^{(\alpha_r)} \bar{l}^{(\beta)} \sum_{i_1, (\gamma)} d_{(\alpha), (\beta), i, (\gamma)} x^i \bar{l}^{(\gamma)}$$

stand for a polynomial where each $\bar{l}^{(\alpha_i)}$ is a monomial of degree r_i in l's from the $b(m_{j_i})-1$ first systems and the index of every $\bar{l}^{(\alpha_i)}$ appearing is $> sj_i$ but $> sj_{i-1}$ (for $\bar{l}^{(\beta)}$

we have l's from the $b(m_k)-1$ first systems and with indices $>sk$ but $<sj_r$). For every sj_i or sk we let $R_i(x, l)$ bet the coefficient of $l_{sj_i} q_{sj_i}$ in $hq-V$, in an $(n, b(m_{j_i}))$-expansion of q and a $b(m_{j_i})$-expansion of V which is $(b(m_{j_i})-1)$-substituted. Since $sj_i > sk \geq b(m_k) \geq b(m_{j_i})$, $R_i(x, l)$ would not change if we consider an $(n, b(m_k))$-expansion of q and a $b(m_k)$-expansion of V.

We define the coefficient $R_i(h, s, l, q)$ of $l_{sj_i} q_{sj_i}$ in $hq-V$ as follows:

Let S_{sj_i} and V_{sj_i} be the coefficients of $l_{sj_i} q_{sj_i}$ in q and V. Then $R_i = hS_{sj_i} - V_{sj_i}$.

Let

$$U_h(x, \bar{l}) = l_{sj_1} l_{sj_2} \cdots l_{sj_{h-1}} \sum_{\substack{|\alpha_1|, |\alpha_2|, \ldots, |\alpha_h| \\ = r_1, \ldots, r_h}} \bar{l}^{(\alpha_1)} \bar{l}^{(\alpha_2)} \cdots \bar{l}^{(\alpha_{h-1})} \sum_{i, (\gamma)} d_{(\alpha), i, (\gamma)} x^i \bar{l}^{(\gamma)}$$

be the sum of those terms out of $R_h(x, \bar{l})$ which contain the product $l_{sj_1} l_{sj_2} \cdots l_{sj_{h-1}}$ and where $\bar{l}^{(\alpha_m)}$ is a monomial of degree r_m in $\bar{l}$'s from the $b(m_{j_m})-1$ first systems and with indices $>sj_m$ but $<sj_{m-1}$. The indices appearing in

$$\sum_{i, (\gamma)} d_{(\alpha), i, (\gamma)} x^i \bar{l}^{(\gamma)}$$

are $\leq s_{j_i}$. We observe that this implies that no l_{sj_m} appears in any monomial $\bar{l}^{(\alpha_m)}$.

Now consider a pair $(m_{j_{h-1}}, m_{j_h})$ of adjacent integers of H such that $(m_{j_{h-1}}, m_{j_h})$ is also a pair of adjacent integers of M (that is $j_h = j_{h-1} + 1$).

Let $S_h(x, \bar{l})$ be the coefficient of l_{sj_h} in a $(b(\alpha_{k-1}), b(m_k))$-expansion of $q_{sj_{h-1}}$.

Consider the part

$$W_h = l_{sj_1} l_{sj_2} \cdots l_{sj_h} \sum_{\substack{|\alpha_1|, \ldots, |\alpha_h| \\ = r_1, \ldots, r_h}} \bar{l}^{(\alpha_1)} \bar{l}^{(\alpha_2)} \cdots \bar{l}^{(\alpha_h)} \sum_{i, (\gamma)} d_{(\alpha), i, (\gamma)} x^i \bar{l}^{(\gamma)}$$

of the product $l_{sj_h} R_{h-1}(x, \bar{l}) \cdot S_h(x, \bar{l})$, with notations as above. With these notations we can now state the Induction Hypothesis. We work with a fixed N.

INDUCTION HYPOTHESIS. (1) *There is a growth function* $F_{N(\alpha_1, \ldots, \alpha_{k-1}, m_k)}$ *such that if* $\{D_n, L_n, l_n, C_n\}$ *grows faster than* $F_{N(\alpha_1, \ldots, \alpha_{k-1}, m_k)}$ *then it is well defined that the case* $\alpha^*_{1,B}, m_{2,B}, \alpha^*_{2,B}, m_3, \ldots, \alpha_k$ *occurs when* $|\ |^N$ *is estimated by* V.

(2) *There are numbers* $n_1, n_2, \ldots, n_r, n_k, E_1, E_2, \ldots, E_r, E_k, G_1$ *and* G_2 *where* n_i *and* E_i *depend only upon* $|\ |^{b(m_{j_i})-1}$ *and* n_k, E_k, G_1 and G_2 *depend only upon* $|\ |^{b(m_k)-1}$ *so that*

with these numbers we have (1), (2), (3), 3, 4, 5 *below and the lemmas below.*

(3) *Now assume that* $r_m \leq n_m$, $m \leq k$. *We assume that it is defined that* $l_{sk} q_{sk}$ *has a good partial coefficient* $U_k(x, \bar{l})$ *containing* $l_{sj_1} l_{sj_2} \dots l_{sj_r}$ *and that this implies that there exist* $r_1, \dots, r_r, r_k$ *such that* (1)–(3) *below are satisfied.*

(a) *For every* $m \leq k$ *we have for* $U_m(x, \bar{l})$

$$\sum_{\substack{|\alpha_1|, |\alpha_2|, \dots, |\alpha_m| \\ = r_1, r_2, \dots, r_m}} \left| \sum_{i, (\gamma)} d_{(\alpha), i, (\gamma)} x^i \bar{l}^{(\gamma)} \right|_{\text{est op } N} \leq E_m^2 \sum_{\substack{|\alpha_1|, |\alpha_2|, \dots, |\alpha_m| \\ = r_1, r_2, \dots, r_m}} \left| \sum_{i, (\gamma)} d_{(\alpha) i, (\gamma)} x^i \bar{l}^{(\gamma)} \right|_1 .$$

(b) *For every* h *the coefficient* $S_h(x, \bar{l})$ *is semigood.*

(c) *For every* $W_h(x, \bar{l})$ *and* $U_h(x, \bar{l})$ *we have*

$$\sum_{(\alpha_1), \dots, (\alpha_h)} \left| \sum_{i, (\gamma)} (d_{(\alpha), i, (\gamma)} - c_{(\alpha), i, (\gamma)}) x^i \bar{l}^{(\gamma)} \right|_{\text{est op } N} \leq \frac{1}{10} \sum_{(\alpha_1), \dots, (\alpha_h)} \left| \sum_{i, (\gamma)} c_{(\alpha), i, (\gamma)} x^i \bar{l}^{(\gamma)} \right|_{\text{est op } N} .$$

(4) *If* $U_k(x, \bar{l})$ *is a good partial coefficient of* $l_{sk} q_{sk}$ *then* $U_m(x, \bar{l})$ *is a good partial coefficient of* $l_{sj_m} q_{sj_m}$ *for every* $m \leq r$, *and the definition of good partial coefficient of* $l_{sj_m} q_{sj_m}$ *is given by the numbers* $n_1, n_2, \dots, n_m$ *and* $E_1, E_2, \dots, E_m$, *and* $U_m(x, \bar{l})$ *is given by* $r_1, r_2, \dots, r_m$.

(5) *The weight* $w(U_m)$ *of* $U_m(x, \bar{l})$, $m = 1, 2, \dots, r, k$ *is defined inductively after* m *as follows.*

If $(\alpha_{j_m - 1}, \alpha_{j_m})$ *is a jump, then*

$$w(U_m) = \sum_{(1), \dots, (m)} \sum_{i, (j)} |d_{i, (j)}|$$

If $(\alpha_{j_m - 1}, \alpha_{j_m})$ *is not a jump (that is if* $j_m - 1 = j_{m-1}$*) then* $w(U_m) = w(U_{m-1}) w(S_m)$.

LEMMA 11.
$$\frac{1}{G_1} w(U_k) \leq \sum |d_{i, (j)}| \leq G_1 \cdot w(U_k).$$

For the next lemmas let α_k be the p_kth system for which the ε is $1/2^{m_k}$. Assume first that (α_{k-1}, α_k) is not a jump (that is $k - 1 = j_r$) and form the sum $\Sigma_c w(U_r)$ where the summation is extended over all different combinations $l_{sj_1} l_{sj_2} \dots l_{sj_r}$ for which

$$\left| [q'_{sj_r}(x)]_{R_{b(\alpha_{j_r})}} \right|_1 \leq \frac{1}{2}.$$

For $m=1,2,\ldots,r,k,$ let $\Sigma\, w(U_m)$ be the sum where the summation is extended over all combinations $l_{sj_1} l_{sj_2} \ldots l_{sj_m}$.

LEMMA 2I. *If* (α_{k-1},α_k) *is not a jump, then*

$$\sum w(U_k) \geq \left(\sum_{c} w(U_r)\right)\cdot\frac{1}{2^{p_k}}\cdot\frac{1}{2^{m_k-m_{k-1}}}\cdot\frac{1}{16}.$$

LEMMA 3I. *If* (α_{k-1},α_k) *is not a jump, then*

$$\sum w(U_k) \geq V_2\cdot\left(\sum w(U_r)\right)\cdot\frac{1}{2^{p_k}}.$$

Assuming the induction hypothesis for $N(\alpha_1,\ldots,\alpha_{k-1},m_k)$ we will by the definition below verify Statement 3 for those sequences $\alpha^*_{1,B}, m_{2,B},\ldots,\alpha_k$ which give $N(\alpha_1,\ldots,\alpha_k)$.

Definition (3.21). We assume that the case $\alpha^*_{1,B}, m_{2,B},\ldots,\alpha_k$ occurs. Then if

$$\sum_{|[q'_{sk}(x)]_{R_{b(\alpha_k)}}|_1>\frac{1}{2}} w(U_k) \geq \frac{1}{2}\sum w(U_k)$$

we say that the case $\alpha^*_{1,B}, m_{2,B},..,\alpha_k^{nc}$ occurs. Otherwise we say that the case $\alpha^*_{1,B}, m_{2,B},..,\alpha^c_{k,B}$ occurs. Thus, if we can prove that there is one growth function that dominates all growth functions that appear in the lemmas and propositions below, and verify the Induction hypothesis for $k+1$ then this gives Statement 3.

(3.22) Now we assume that the case $\alpha^*_{1,B}, m_{2,B},..,\alpha^c_{k,B}$ occurs. For every sk with

$$\left|[q'_{sk}(x)]_{R_{b(\alpha_k)'}}\right|_1 \leq \frac{1}{2}$$

consider the smallest m'_{sk} such that

$$\sum_{\varepsilon_j=1/2^{m'_{sk}}} |s_j(x)|_1 > \frac{1}{8}\cdot\frac{1}{2^{m'_{sk}-m_k}}.$$

Let D'_{sk} be the sum of the weights of good partial coefficients of $l_{sk}q_{sk}$. Let m_{k+1} be the smallest number such that

$$\sum_{m'_{sk}=m_{k+1}} D'_{sk} > \frac{\sum w(U_k)}{2^{m_{k+1}-m_k}}$$

where $\Sigma w(U_k)$ as above is the sum of the weights of good partial coefficients of $l_{sk} q_{sk}$'s, $sk \in \alpha_k$. We then say that the case $\alpha^*_{1,\mathrm{B}}, m_{2,\mathrm{B}}, \ldots, \alpha^c_{k,\mathrm{B}}, m_{k+1}$ occurs. This definition gives Statement 5 as a consequence of the induction hypothesis.

In the next lemma we assume that the case $\alpha^*_{1,\mathrm{B}}, m_{2,\mathrm{B}}, \ldots, \alpha^c_{k,\mathrm{B}}, m_{k+1}$ occurs. We let $S_{s(k+1)}$ be a semigood coefficient of $l_{s(k+1)} q_{s(k+1)}$ in a $(b(m_{k+1})-1)$-substituted $(b(\alpha_k), b(m_{k+1}))$-expansion of q_{sk}. Let $U_k(x, \bar{l})$ be a good partial coefficient of $l_{sk} q_{sk}$ in $hq-V$ with an $(n, b(m_k))$-expansion of q and $b(m_k)$-expansion of V, which contains $l_{sj_1} \ldots l_{sj_r}$ and which is $(b(m_{k+1})-1)$-substituted. We observe that since $sk \geq b(m_{k+1}) > b(m_k)$ we could as well consider an $(n, b(m_{k+1}))$-expansion of q and $b(m_{k+1})$-expansion of V.

We now have

LEMMA $\alpha^*_{1,\mathrm{B}}, m_{2,\mathrm{B}}, \ldots, \alpha^c_{k,\mathrm{B}}, m_{k+1}, N(\alpha_1, \ldots, \alpha_k, m_{k+1})$. *There is a growth function* $F'_{N(\alpha_1, \ldots, \alpha_k, m_{k+1})}$ *which coincides with* $F_{N(\alpha_1, \ldots, \alpha_{k-1}, m_k)}$ *up to the* $(b(m_{k+1})-1)$*-st stage and numbers* n_{k+1} *and* E_{k+1} *depending only on* $| \,|^{b(m_{k+1})-1}$ *such that if* $\{D_n, L_n, l_n, C_n\}$ *grows faster than* $F'_{N(\alpha_1, \ldots, \alpha_k, m_{k+1})}$ *and the* case $\alpha^*_{1,\mathrm{B}}, m_{2,\mathrm{B}}, \ldots, \alpha^c_{k,\mathrm{B}}, m_{k+1}$ *occurs then we have the following:*

There is an $r_{k+1} \leq n_{k+1}$ *such that if we consider the part*

$$W_{k+1}(x, \bar{l}) = l_{sj_1} \ldots l_{sj_r} l_{sk} \sum_{\substack{|\alpha_1|, |\alpha_2|, \ldots, |\alpha_k|, |\alpha_{k+1}| \\ = r_1, r_2, \ldots, r_k, r_{k+1}}} \bar{l}^{(\alpha_1)} \bar{l}^{(\alpha_2)} \ldots \bar{l}^{(\alpha_{k-1})} \sum_{i, (\gamma)} c_{(\alpha), i, (\gamma)} x^i \bar{l}^{(\gamma)}$$

out of the product $l_{sk} U_k(x, \bar{l}) S_{s(k+1)}(x, \bar{l})$ *then we have*

$$\begin{aligned} E_{k+1} \cdot w(U_k) \cdot |S_{(k+1)}(x)|_1 &\geq \sum_{(\alpha_1), \ldots, (\alpha_{k+1})} \left| \sum_{i, (\gamma)} c_{(\alpha), i, (\gamma)} x^i \bar{l}^{(\gamma)} \right|_{\mathrm{est\,op}\,N} \\ &\geq \sum_{(\alpha_1), \ldots, (\alpha_{k+1})} \left| \left(\sum_{i, (\gamma)} c_{(\alpha), i, (\gamma)} x^i \bar{l}^{(\gamma)} \right)(x) \right|_1 \\ &\geq \frac{1}{E_{k+1}} w(U_k) \cdot |S_{s(k+1)}(x)|_1. \end{aligned}$$

Here $S_{s(k+1)}$ is given by

$$q_{sk} = q_{b(a_k)} + \ldots + s_{s(k+1)}(l_{s(k+1)}q_{s(k+1)}) - 1) + \ldots + t.$$

Proof. The proof of this is essentially the same as the proof of Lemma $\mathfrak{a}^{c}_{1,B}, m_2, N(\mathfrak{a}_1, m_2)$. We give details in Section 4.

(3.23) Now consider the part

$$l_{sj_1} \cdots l_{sj_r} l_{sk} \sum_{\substack{|\alpha_1|,\ldots,|\alpha_{k+1}| \\ = r_1,\ldots,r_{k+1}}} (\bar{l}^{(\alpha_1)} \ldots \bar{l}^{(\alpha_r)} \bar{l}^{(\alpha_k)} \bar{l}^{(\alpha_{k+1})}) \left(\sum_{i,(\gamma)} d_{(\alpha),i,(\gamma)} x^i \bar{l}^{(\gamma)} \right) = U_{k+1}(x, \bar{l})$$

out of the coefficient of $l_{s(k+1)} q_{s(k+1)}$ in $hq - V$ where we consider $(b(m_{k+1}) - 1)$-substituted $(n, b(m_{k+1}))$- resp. $b(m_{k+1})$-expansions of q and V and a $(b(m_{k+1} - 1)$-substituted representation of h. Let $\bar{l}^{(\alpha_1)} \ldots \bar{l}^{(\alpha_r)} \bar{l}^{(\alpha_k)}$ run through the same set as in the lemma above. Every index appearing in $\bar{l}^{(j)}$ is $\leq s(k+1)$ but the monomials $x^i \bar{l}^{(j)}$ need not run through the same set as in the lemma above. We say that $U_{k+1}(x, \bar{l})$ is *a good partial coefficient of* $l_{s(k+1)} q_{s(k+1)}$ *that contains* $l_{sj_1} l_{sj_2} \cdots l_{sj_r} l_{sk}$ with weight $w(U_k)|s_{s(k+1)}(x)|_1$ if

(1) $U_k(x, \bar{l})$ is a good partial coefficient of $l_{sk} q_{sk}$ that contains $l_{sj_1} \ldots l_{sj_r}$ with weight $w(U_k)$.

(2) $S_{s(k+1)}$ is a semigood coefficient of $l_{s(k+1)} q_{s(k+1)}$.

(3) $\sum_{\alpha} \left| \left(\sum_{i,(\gamma)} (d_{(\alpha),i,(\gamma)} - c_{(\alpha),i,(\gamma)}) x^i \bar{l}^{(\gamma)} \right) \right|_{\text{est op } N} \leq \frac{1}{10} \sum_{\alpha} \left| \sum_{i,(\gamma)} c_{(\alpha),i,(\gamma)} x^i \bar{l}^{(\gamma)} \right|_{\text{est op } N}$

and

$$\sum_{\alpha} \left| \left(\sum_{i,(\gamma)} (d_{(\alpha),i,(\gamma)} - c_{(\alpha),i,(\gamma)}) x^i \bar{l}^{(\gamma)} \right)(x) \right|_1 \leq \frac{1}{10} \sum_{\alpha} \left| \left(\sum_{i,(\gamma)} c_{(\alpha),i,(\gamma)} x^i \bar{l}^{(\gamma)} \right)(x) \right|_1.$$

Let $\Sigma w(U_k)$ be the sum of the weights of good partial coefficients of $l_{sk} q_{sk}$'s, $sk \in \mathfrak{a}_k$. We now have

PROPOSITION $\mathfrak{a}^{*}_{1,B}, m_{2,B}, \ldots, \mathfrak{a}^{c}_{k,B}, m_{k+1}, N(\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_k, m_{k+1})$. *There is a growth function* $F_{N(\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_k, m_{k+1})}$ *which coincides with* $F_{N(\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_{k-1}, m_k)}$ *up to the* $(b(m_{k+1}) - 1)$*-st stage such that if* $\{D_n, L_n, l_n, C_n\}$ *grows faster than* $F_{N(\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_k, m_{k+1})}$ *and the* case $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_k, m_{k+1}$ *occurs then either*

case $\mathfrak{a}^{*}_{1,B}, m_{2,B}, \ldots, \mathfrak{a}^{*}_{k,B}, m_{k+1,G}$: $|V|^{N}_{\text{est}} \geq 1$

or

case $\mathfrak{a}^*_{1,B}, m_{2,B}, \ldots, \mathfrak{a}^*_{k,B}, m_{k+1,B}$: *The sum D_{k+1} of the weights of all good partial coefficients of all $l_{s(k+1)}q_{s(k+1)}$'s containing $l_{sj_1}l_{sj_2}\ldots l_{sk}$'s is*

$$\geqslant \sum w(U_k)\cdot\frac{1}{2^{m_{k+1}-m_k}}\cdot\frac{1}{8}\cdot\frac{1}{2}.$$

(3.24) The proof of this will be given later in Section 4. If case $\mathfrak{a}^*_{1,B}, \ldots, m_{k+1,B}$ occurs, then there is among the systems $\mathfrak{a}_{k+1,1}, \mathfrak{a}_{k+1,2}\ldots$ for which ε is $1/2^{m_{k+1}}$ a first, say $\mathfrak{a}_{k+1,p}$, for which the sum of the weights of good partial coefficients for $l_{s(k+1)}q_{s(k+1)}$'s, $s(k+1)\in\mathfrak{a}_{k+1}$ is $>D_{k+1}/2^p$. We then say that the case $\mathfrak{a}_{1,B}, \ldots, m_{k+1,B}, \mathfrak{a}_{k+1}$ occurs.

(3.25) The Proposition $\mathfrak{a}^*_{1,B}, m_{2,B}, \ldots, \mathfrak{a}^c_{k,B}, m_{k+1}, N(\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_k, m_{k+1})$ and (3.24) now give Statement 6.

(3.26) In the next proposition we assume that the case $\mathfrak{a}^*_{1,B}, m_{2,B}, \ldots, \mathfrak{a}^{nc}_k$ occurs when $|hq|^N$ is estimated by V. Let $b=\max(b(\mathfrak{a}_1), b(\mathfrak{a}_2), \ldots, b(\mathfrak{a}_k))$. We say that a pre-$(n, b(\mathfrak{a}_k))$-expansion of q (pre-$(b(\mathfrak{a}_k))$-expansion of V) is shortened with respect to the case $\mathfrak{a}^*_{1,B}, m_{2,B}, \ldots, \mathfrak{a}^{nc}_k$, if every term which ends with $l_{sk}q_{sk}$, $sk\in\mathfrak{a}_k$ of sk belongs to a system with number $>b$, is not further expanded. With such a shortened expansion we get

$$(hq-V)(h, s, l, q, t) = \sum_j S_j l_j q_j + R,$$

where every j belongs to a system with number $>b$ or to the system $\mathfrak{a}_k$.

(3.27) In (3.26) we make a pre-$(b(\mathfrak{a}_k), b(\mathfrak{a}_k))$-expansion of every q_j, $j\in\mathfrak{a}_k$ and form q'_j by removing all terms of the 3rd type for which the corresponding ε is $<\min\{\varepsilon_1, \varepsilon_2, .., \varepsilon_{b(\mathfrak{a}_k)}\}$. We do this also in R thus forming R'. Now we form the sum

$$\sum_j S_j l_j q'_j + \sum_j S_j l_j q_j + R' = (hq-V)'$$

by letting the first sum run over all those j in (3.26) for which q'_j has a good partial coefficient and for which $|q'_j(x)|_{R_{b(\mathfrak{a}_k)}}|_1 > \frac{1}{2}$ and let the second sum run over the other j's.

Since for every q_j' in (3.27) $j \in \alpha_k$ we change the notation and write $\Sigma S_j l_j q_j' = S_{sk} l_{sk} q_{sk}'$ when sk runs over the same set as j. By making a $(b(\alpha_k)-1)$-substitution we form $(hq-V)'(x, \bar{l})$.

(3.28) We get $(hq-V)'(x)$ by substituting all l's in $(hq-V)'(x, \bar{l})$ by polynomials in x. We have for every sk, $S_{sk}(x, \bar{l}) = \Sigma_i U_{i,sk}(x, \bar{l}) + R_{sk}(x, \bar{l})$ where $U_{i,sk}$ runs over the good partial coefficients of $l_{sk} q_{sk}$ containing $l_{sj_1} l_{sj_2} \dots l_{sj_r}$, for different combinations $l_{sj_1} l_{sj_2} \dots l_{sj_r}$

We observe that the part

$$l_{sj_1} l_{sj_2} \dots l_{sj_r} \sum_{\substack{|\alpha_1|, \dots, |\alpha_k| \\ = r_1 \dots r_k}} \left(\bar{l}^{(\alpha_1)} \bar{l}^{(\alpha_2)} \dots \bar{l}^{(\alpha_r)} \bar{l}^{(\alpha_k)} \left(\sum_{i, (\gamma)} d_{(\alpha), i, (\gamma)} x^i \bar{l}^{(\gamma)} \right) \right)$$

out of $S_{sk}(x, \bar{l})$ coincides with the good partial coefficient of $l_{sk} q_{sk}$ that contains $l_{sj_1} l_{sj_2} \dots l_{sj_r}$. This is since in a good partial coefficient none of the sj_m's or indices in $l_{(m)}$'s belong to the system α_k or to any system with number $> b$.

We now have

Proposition $\alpha_{1,\mathrm{B}}^*, m_{2,\mathrm{B}}, \dots, \alpha_k^{\mathrm{nc}}, N(\alpha_1, \alpha_2, \dots, \alpha_k)$. *There is a growth function* $F_{N(\alpha_1, \alpha_2, \dots, \alpha_k)}$ *which coincides with* $F_{N(\alpha_1, \alpha_2, \dots, m_k)}$ *up to the* $(b(\alpha_k)-1)$*-st stage and numbers* $B'_{b(\alpha_k)}$ *and* n_k *depending only on* $|\ |^{b(\alpha_k)-1}$ *and* $N(\alpha_1, \alpha_2, \dots, \alpha_k)$ *such that if* $\{D_n, L_n, l_n, C_n\}$ *grows faster than* $F_{N(\alpha_1, \alpha_2, \dots, \alpha_k)}$ *and the* case $\alpha_{1,\mathrm{B}}^*, m_{2,\mathrm{B}}, \dots, \alpha_k^{\mathrm{nc}}$ *occurs then either*

case $\alpha_{1,\mathrm{B}}^*, m_{2,\mathrm{B}}, \dots, \alpha_{k,\mathrm{G}}^{\mathrm{nc}}$: $|V|_{\mathrm{est}}^N > 1$

or

case $\alpha_{1,\mathrm{B}}^*, m_{2,\mathrm{B}}, \dots, \alpha_{k,\mathrm{B}}^{\mathrm{nc}}$: $(hq-V)'(x)$ *contains* $p_{1/20}(E)$ *with*

$$E = \sum_{sk} l_{sk} \left(\sum_{sj_1, \dots, sj_j} \left(l_{sj_1} l_{sj_2} \dots l_{sj_r} \sum_{\substack{|\alpha_1|, \dots, |\alpha_k| \\ = r_1, \dots, r_k}} \left(\bar{l}^{(\alpha_1)} \dots \bar{l}^{(\alpha_r)} \bar{l}^{(\alpha_k)} \sum_{i, (\gamma)} e_{(\alpha), i, (\gamma)} x^i \bar{l}^{(\gamma)} \right) \right) \right)$$

and with

(1) $|E|_1 \geq D \cdot B'_{b(\alpha_k)} \cdot L_{b(\alpha_{j_1})} \cdot L_{b(\alpha_{j_2})} \dots L_{b(\alpha_k)}$ *where* D *is the sum of the weights of good partial coefficients of* $l_{sk} q_{sk}$*'s containing* $l_{sj_1} l_{sj_2} \dots l_{sj_r}$.

(2) $i + |\gamma| \leq n_k$ *for all* i *and* γ.

Proof. The strategy of the proof of this proposition is similar to that of Proposition $\alpha^*_{1,B}, m_{2,B}, \alpha_2^{nc}, N(\alpha_1, \alpha_2)$. We give details in Section 4.

(3.29) For the final proposition we now assume that the case $\alpha^*_{1,B}, m_{2,B}, \ldots, \alpha^{nc}_{k,B}$ occurs. Since $(hq-V)(x)\equiv 0$ we get by the previous proposition that there is a smallest m_{k+1} with $1/2^{m_{k+1}} \min(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{b(\alpha_k)}) = 1/2^{j'}$ such that the cancellation effect of the terms of 3rd type for which $\varepsilon = 1/2^{m_{k+1}}$ on $E(x)$ is $>|E|_1/2\cdot 2^{m_{k+1}-j'}$. We then say that the case $\alpha^*_{1,B}, m_{2,B}, \ldots, \alpha^{nc}_{k,B}, m_{k+1}$ occurs.

(3.30) We observe that the set H' given by the sequence $\alpha^*_{1,B}, \ldots, \alpha^{nc}_{k,B}, m_{k+1}$ is not uniquely determined by the corresponding set H for $\alpha^*_{1,B}, m_{2,B}, \ldots, \alpha_k$. We have, however, obviously the following: If $m_{j_i} \in H \cap H'$, then $m_{j_h} \in H \cap H'$ for $h \leq i$. We put $H' = \{m_{j_1}, m_{j_2}, \ldots, m_{j_{r'}}, m_{k+1}\}$. Given $l_j q_j, \varepsilon_j \in 1/2^{m_{k+1}}$, we consider the $(b(m_{k+1})-1)$-substituted coefficient $S_j(x, \bar{l})$ of $l_j q_j$ in an $(n, b(m_{k+1}))$-expansion of $hq-V$. For every $l_{sj_{r'}}$ with $sj_{r'} \in \alpha_{j_{r'}}$ (and $m_{j_{r'}} \in H'$) and every $l_{sj_1} l_{sj_2} \ldots l_{sj_{r'-1}}$ such that $l_{sj_{r'}}$ has a good partial coefficient

$$l_{sj_1} l_{sj_2} \ldots l_{sj_{r'-1}} \sum_{\substack{|\alpha_1|, \ldots, |\alpha_{r'}| \\ = r_1, \ldots, r_{r'}}} \bar{l}^{(\alpha_1)} \bar{l}^{(\alpha_2)} \ldots \bar{l}^{(\alpha_{r'-1})} \bar{l}^{(\alpha_{r'})} \sum d_{(\alpha), i, (\gamma)} x^i \bar{l}^{(\gamma)}$$

containing $l_{sj_1} l_{sj_2} \ldots l_{sj_{r'-1}}$, we consider the terms

$$T'_{sj_1, sj_2, \ldots, sj_{r'}, n'_{k+1}} = l_{sj_1} l_{sj_2} \ldots l_{sj_{r'}} \sum_{\substack{|\alpha_1|, \ldots, |\alpha_{r'}| \\ = r_1, \ldots, r_{r'} \\ r'_{k+1} \leq n_{k+1}}} \bar{l}^{(\alpha_1)} \ldots \bar{l}^{(\alpha_{r'})} \bar{l}^{(\alpha_{k+1})} \sum d_{(\alpha), i, (\gamma)} x^i \bar{l}^{(\gamma)}$$

out of the coefficient of $l_j q_j$ where $n'_{k+1} = r_{r'+1} + \ldots + r_k + n_k$.

For every such $l_{sj_{r'}}$ and every $l_{sj_1} l_{sj_2} \ldots l_{sj_{r'-1}}$, consider also the terms

$$T''_{sj_1, sj_2, \ldots, sj_{r'}, n'_{k+1}} = l_{sj_1} l_{sj_2} \ldots l_{sj_{r'}} \sum_{\substack{|\alpha_1|, \ldots, |\alpha_{r'}| \\ = r_1, \ldots, r_{r'} \\ r'_{k+1} \leq n_{k+1}}} \bar{l}^{(\alpha_1)} \ldots \bar{l}^{(\alpha_{r'})} \bar{l}^{(\alpha_{k+1})} \sum_{i, (\gamma)} c_{(\alpha), i, (\gamma)} x^i \bar{l}^{(\gamma)}$$

out of the product $S'_{sj_{r'}} \cdot S'_{r'j} \cdot l_{sj_{r'}}$, where $S'_{sj_{r'}}$ is the $(b(m_{k+1})-1)$-substituted coefficient of $l_{sj_r} q_{sj_r}$ in $hq-V$ in an $(n, m_{k+1}))$-expansion of q and a $b(m_{k+1})$-expansion of V, and

$S'_{r'j}$ is the $(b(m_{k+1})-1)$-substituted coefficient of $l_j q_j$ in a $(b(\alpha_{j_r}), b(m_{k+1}))$-expansion of $q_{sj_{r'}}$. Now let $\mathcal{S}\mathcal{J}$ be the subset of all combinations $sj_1, sj_2, \ldots, sj_r$, such that

$$\sum_{\alpha}\sum_{i,(\gamma)} \left|c_{(\alpha),i,(\gamma)}-d_{(\alpha),i,(\gamma)}\right| \leq \frac{1}{10}\cdot\frac{1}{r'_{k+1}}\sum_{\alpha}\sum_{i,(\gamma)}\left|c_{(\alpha),i,(\gamma)}\right|$$

and such that

$$\sum_{\alpha}\sum_{i,(\gamma)} \left|(c_{(\alpha),i,(\gamma)}-d_{(\alpha),i,(\gamma)})\, x^i \bar{l}^{(\gamma)}\right|_{\text{est op } N} \leq \frac{1}{10}\cdot\frac{1}{r'_{k+1}}\sum_{\alpha}\sum_{i,(\gamma)}\left|c_{(\alpha),i,(\gamma)}\, x^i \bar{l}^{(\gamma)}\right|_{\text{est op } N}.$$

Now let

$$T'_j(x, \bar{l}) = \sum_{\mathcal{S}\mathcal{J}} T'_{sj_1, sj_2, \ldots, sj_{r'}, n_{k+1}}(x, \bar{l}).$$

With these notations we have the following

Proposition $\alpha^*_{1,\mathrm{B}}, m_{2,\mathrm{B}}, \ldots, \alpha^{\mathrm{nc}}_{k,\mathrm{B}}, N(\alpha_1, \alpha_2, \ldots, \alpha_k, m_{k+1})$. *There is a growth function* $F_{N(\alpha_1, \alpha_2, \ldots, \alpha_k, m_{k+1})}$ *which coincides with*
$F_{N(\alpha_1, \alpha_2, \ldots, \alpha_k)}$ *up to the* $(b(m_{k+1})-1)$*-st stage such that if* $\{L_n, D_n, l_n, C_n\}$ *grows faster than* $F_{N(\alpha_1, \alpha_2, \ldots, \alpha_k, m_{k+1})}$ *and the* case $\alpha^*_{1,\mathrm{B}}, m_{2,\mathrm{B}}, \ldots, \alpha^{\mathrm{nc}}_{k,\mathrm{B}}, m_{k+1}$ *occurs then either*

case $\alpha^*_{1,\mathrm{B}}, m_{2,\mathrm{B}}, \ldots, \alpha^{\mathrm{nc}}_{k,\mathrm{B}}, m_{k+1,\mathrm{G}}$: $|V|^N_{\text{est}} \geq 1$

or

case $\alpha^*_{1,\mathrm{B}}, m_{2,\mathrm{B}}, \ldots, \alpha^{\mathrm{nc}}_{k,\mathrm{B}}, m_{k+1,\mathrm{B}}$: *The sum of the cancellation effects of all monomials* $x^i \bar{l}^{\alpha} = P_a(x)$ *of all* $S_j(x, \bar{l}) - T'_j(x, \bar{l})$, $\varepsilon_j = 1/2^{m_{k+1}}$, *is*

$$\leq \frac{|E(x)|_1}{10\cdot 2\cdot 2^{m_{k+1}-j'}} \leq \frac{|E(x)|_1}{10\cdot 2\cdot 2^{m_{k+1}-m_k}}.$$

Proof. The proof will be given below in Section 4.

(3.31) We now let $\Sigma w(U_{r'})$ be the sum of the weights of good partial coefficients of all $l_{sj_{r'}} q_{sj_{r'}}$'s.

We now have

Lemma 3.4. *There is a number* $K'_{b(m_{k+1})}$ *depending only on* $|\ |^{b(m_{k+1})-1}$ *such that if* $|V|^N_{\text{est}} < 1$ *in the previous proposition then*

$$\sum_{\alpha}\sum_{i,(\gamma)}\left|d_{(\alpha),i,(\gamma)}x^i\bar{l}^{(\gamma)}\right|_{\text{est op }N}\leqslant\left(\sum w(U_{r'})\right)\cdot K'_{b(m_{k+1})} \tag{3.32}$$

where the summation is extended over all $T'_{sj_1,\dots,sj_{r'},n'_{k+1}}$ for all $\mathcal{S}\mathcal{J}$ and j.

(3.33) From Lemmas 1 I, 2 I and 3 I of the induction hypothesis it follows that the cancellation effects of $T'_j(x,\bar{l})$'s on $E(x)$ is

$$>\frac{9}{10}\frac{|E(x)|_1}{2\cdot 2^{m_{k+1}-m_k}}\geqslant\left(\sum w(U_{r'})\right)\cdot L_{b(\alpha_{j_1})}\cdot\ldots\cdot L_{b(\alpha_{j_{r'}})}\cdot C'_{b(m_{k+1})},$$

where $C'_{b(m_{k+1})-1}$ depends only on $|\ |^{b(m_{k+1})-1}$.

So with Lemma 3.4. and (3.33) we can define good partial coefficient of $l_j q_j$. For this let r_{k+1} be the smallest number $\leqslant n'_{k+1}$ such that for

$$T'_{sj_1,sj_2,\dots,sj_{r'}}=l_{sj_1}l_{sj_2}\cdots l_{sj_{r'}}\sum_{\substack{|\alpha_1|,\dots,|\alpha_k|\\ =r_1,\dots,r_k}}\bar{l}^{(\alpha_1)}\cdots\bar{l}^{(\alpha_{r'})}\bar{l}^{(\alpha_{k+1})}\sum_{i,(\gamma)}d_{(\alpha),i,(\gamma)}x^i\bar{l}^{(\gamma)}$$

we have

$$\left|T'_{sj_1,sj_2,\dots sj_{r'}}(x)\right|_1\geqslant\frac{1}{n'_{k+1}}\left|T'_{sj_1,\dots,sj_{r'},n'_{k+1}}(x)\right|_1.$$

We now say that $T'_{sj_1,(sj_2,\dots sj_{r'})}(x,\bar{l})$ is a good partial coefficient of $l_j q_j$ that contains $l_{sj_1}l_{sj_2}\cdots l_{sj_{r'}}$ if

$$\sum_{\alpha}\sum_{i,(\gamma)}\left|d_{(\alpha),i,(\gamma)}x^i\bar{l}^{(\gamma)}\right|_{\text{est op }N}\leqslant\frac{K'_{b(m_{k+1})}\cdot n'_{k+1}\cdot 100}{C'_{b(m_{k+1})}}\sum_{\alpha}\sum_{i,(\gamma)}\left|d_{(\alpha),i,(\gamma)}x^i\bar{l}^{(\gamma)}\right|_1. \tag{3.34}$$

(3.35) We say that $\Sigma_\alpha\Sigma_{i,(\gamma)}|d_{(\alpha),i,(\gamma)}|$ is the weight of this partial coefficient.

(3.36) Let D' be the sum of the weights of good partial coefficents of $l_j q_j$'s, $\varepsilon_j\in 1/2^{m_{k+1}}$. There is among the systems $\alpha_{k+1,1}\,\alpha_{k+1,2}\ldots b(\alpha_{k+1,1})<b(\alpha_{k+1,2})<\ldots$ for which the corresponding ε is $1/2^{m_{k+1}}$ a first say $\alpha_{k+1,p}=\alpha_{k+1}$ for which the sum of the weights of good coefficients of $l_j q_j$'s, $\varepsilon_j=1/2^{m_{k+1}}$, is $>D'/2^p$. We then say that the case $\alpha^*_{1,B},\dots,m_{k+1,B},\alpha_k$ occurs.

With this we will now see that we have verified the Induction Hypothesis for $k+1$.

(1) follows from Proposition $\alpha^*_{1,B}, m_{2,B}, \ldots, \alpha^c_{k,B}, m_{k+1}, N(\alpha_1, \alpha_2, \ldots, \alpha_k, m_{k+1})$ and (3.24) and Proposition $\alpha^*_{1,B}, m_{2,B}, \ldots, \alpha^{nc}_{k,B}, m_{k+1}, N(\alpha_1, \alpha_2, \ldots, \alpha_k, m_{k+1})$ and (3.36). The existence of E_{k+1} which works for (3a) in the definition of good partial coefficient follows from Lemma $\alpha^*_{1,B}, m_{2,B}, \ldots, \alpha^c_{k,B}, m_{k+1}, N(\alpha_1, \ldots, \alpha_k, m_{k+1})$ and (3.23) and from (3.34). (3b) and (3c) of the definition of good partial coefficient follow from (3.23). (3) of the Induction Hypothesis follows from (3.23) and (3.30) and (3.34). (4) of the Induction Hypothesis follows from (3.35) and (3.23) and (5) after Lemma 3.2. The existence of V_1 and Lemma 1I, follow from Lemma $\alpha^*_{1,B}, m_{2,B}, \ldots, \alpha^c_{k,B}, m_{k+1}, N(\alpha_1, \alpha_2, \ldots, \alpha_k, m_{k+1})$ and (3.23) and (3.35). Lemma 2I follows from Proposition $\alpha^*_{1,B}, m_{2,B}, \ldots, \alpha^c_{k,B}, m_{k+1}, N(\alpha_1, \alpha_2, \ldots, \alpha_k, m_{k+1})$ and (3.24). The existence of V_2 and Lemma 3I follow from (3.31)–(3.36). So we have verified the Induction Hypothesis for $k+1$.

Now the family of growth functions in the propositions, lemmas and in the Induction Hypothesis satisfy (3A1) and (3A2) so there is, by (3B) one growth function F that dominates all of them. So to conclude the construction we now verify that Statements 3–6 follow from the propositions and the Induction Hypothesis. Statement 3 follows from (3.21), Statement 4 follows from Proposition $\alpha^*_{1,B}, m_{2,B}, \ldots, \alpha^{nc}_k, N(\alpha_1, \alpha_2, \ldots, \alpha_k)$ and (3.29). Statement 5 follows from (3.22). Statement 6 follows from Proposition $\alpha^*_{1,B}, m_{2,B}, \ldots, \alpha^c_{k,B}, m_{k+1}$ and (3.24). This concludes the construction.

4.

In this section we will give the complete proofs left from Sections 1 and 3. We start by proving Theorem 1.2 from Section 1 and then we turn to proofs of lemmas and propositions from the 3rd section.

Proof of Theorem 1.2.

SUBLEMMA 4.1. *Let $l_1, \ldots, l_n$, $C_1, \ldots, C_n$ be given, $C_k > 2$. There exists a constant M depending only on $|l_1|_1, \ldots, |l_n|_1$ such that for all y there is a representation*

$$y = r + \sum_{k=1}^{n} S_k(l_k q_k - 1); \quad S_k = \sum a_{k,i,\alpha} x^i l_1^{\alpha_1} \ldots l_n^{\alpha_n}$$

and

$$|y|^n = |r|_1 + \sum_{k=1}^{n} |a_{k,i,\alpha}| 2^i (C_1|l_1|_1)^{\alpha_1} \dots (C_n|l_n|_1)^{\alpha_n} \varepsilon_k$$

and for each term $a_{k,i,\alpha} \neq 0$, *we have* $|i+\alpha_1+\dots+\alpha_n| \leq M$.

Remark. This means that the norm $|\ |^n$ is attained and with S_k having bounded powers of x and l_i this shows that $|p|^n$ never vanishes for any polynomial $p \neq 0$.

Proof. We have

$$|y|^n = \inf\{|r|_1 + \sum_k |a_{k,i,\alpha}| 2^i (C_1|l_1|_1)^{\alpha_1} \dots (C_n|l_n|_1)^{\alpha_n} \varepsilon_k.$$

If we take a given term corresponding to $a_{k,i,\alpha}$ and remove it from the sum and place it in the term r, we decrease the sum but increase $|r|_1$. If any i or $\alpha_j > M$, we decrease the sum by at least $2^M \varepsilon_k |a_{k,i,\alpha} x^i l_1^{\alpha_1} \dots l_n^{\alpha_n}$. On the other hand we increase $|r|_1$ by at most $|a_{k,i,\alpha} x^i l_1^{\alpha_1} \dots l_n^{\alpha_n}| \cdot |l_k q_k - 1|_1$. Remembering that there is by compactness a bound on $|q_k|_1$ depending only on $|l_1|_1 \dots |l_n|_1$ it follows that we can assume $|i+\alpha_1+\dots+\alpha_n| < M$ for suitable M. Compactness yields that the inf is attained.

COROLLARY. *If* $l_1, \dots, l_n$, $C_1, \dots, C_n$ *are given there exist* M, *(depending on* $|l_k|_1$ *and degree of* l_k, $1 \leq k \leq n$*) such that if* $\operatorname{ord} g > M$, $|g|^n = |g|^0 = |g|_1$.

Continuation of proof of Theorem 1.2. Fix K and N. Let g be any polynomial with $\deg g \leq N$. Suppose g has a representation

$$g = \sum_{k=1}^{n+1} S_k (l_k q_k - 1) + r \tag{4.1}$$

$$S_k = \sum a_{k,i,\alpha} x^i l_1^{\alpha_1} \dots l_{n+1}^{\alpha_{n+1}}.$$

So $|g|^{n+1}$ is defined as

$$\inf\{|r|_1 + \sum_1^{n+1} |a_{k,i,\alpha}| 2^i (C_1|l_1|_1)^{\alpha_1} \dots (C_{n+1}|l_{n+1}|_1)^{\alpha_{n+1}} \varepsilon_k\}$$

and we can assume by Sublemma 4.1, that $|g|^{n+1}$ equals the above expression.

Our goal is to show that we can find another representation of g in which l_{n+1} does not appear and with a lower "norm". Recall that we denote by $[g]_m$ the polynomial g "cut-off at m", i.e. with all terms of degree $> m$ removed. We shall be comparing the representation (1) with the same representation with all terms "cut-off" at some suitable degree.

Let us expand S_k by powers of l_{n+1}, so $S_k = S_{0,k} + l_{n+1} S_{1,k} + l_{n+1}^2 S_{2,k} + \ldots$ where each $S_{i,k}$ is a polynomial in $x, l_1, \ldots, l_n$ and by Sublemma 4.1, since $|l_{n+1}|_1$ is given S_k and in particular $S_{i,k}$ have bounded degrees. Thereby, if ord $l_{n+1} = \omega + 1$ for sufficiently large ω, we have by "cutting down" to ω,

$$g = [r]_\omega + \sum_{k=1}^{n} S_{0,k}(l_k q_k - 1) - S_{0,n+1}. \tag{4.2}$$

We see that the "norm" of (4.2) where we put together the terms $r - S_{0,n+1}$ is smaller than the "norm" of (4.1) by at least the amount

$$\varepsilon_{n+1} |S_{0,n+1}|_{\mathrm{op}\,n} - |S_{0,n+1}|_1$$

Hence, this quantity cannot be positive and so we deduce that

$$\frac{|S_{0,n+1}|_{\mathrm{op}\,n}}{|S_{0,n+1}|_1} \leqslant \frac{1}{\varepsilon_{n+1}}. \tag{4.3}$$

Therefore, $S_{0,n+1}$ satisfies the condition for our theorem. So, denoting $\bar{S} = S_{0,n+1}$,

$$|\bar{S} q_{n+1}|^n \geqslant |\bar{S}|_1 \cdot B. \tag{4.4}$$

Now again comparing (4.2) with (4.1) above we see that the norm in (4.2) is less than that of (4.1) by at least

$$[r]^\omega + C_{n+1} |l_{n+1}|_1 \left(\sum_{k=1}^{n} |S_{1,k}(l_k q_k - 1)|^n + \varepsilon_{n+1} |S_{1,n+1}|_{\mathrm{op}\,n} \right) - |\bar{S}|_1$$

with $[r]^\omega = r - [r]_\omega$.

Again, since this quantity cannot be positive, and $|\ |_{\mathrm{op}\,n} \geqslant |\ |_1$,

$$[r]^\omega + C_{n+1} |l_{n+1}|_1 \left(\sum_{k=1}^{n} |S_{1,k}(l_k q_k - 1)|^n + \varepsilon_{n+1} |S_{1,n+1}|_1 \right) < |\bar{S}|_1. \tag{4.5}$$

We need a sublemma.

SUBLEMMA 4.2. *Let $A>0$. There exists a lacunarity function f such that if l is a polynomial of lacunarity $\geq f$, and* $\deg G_i \leq A$, $1 \leq i \leq A$, *then*

$$|lG_1+l^2G_2+\ldots+l^AG_A|_1=\sum_{k=1}^{A}|l^kG_k|_1.$$

Proof. If $n_1<n_2<\ldots$ are the exponents occurring in l, then $n_{i_1}+\ldots+n_{i_r}$ are the exponents in l^r. Hence, the lemma is true if we know

$$|n_{i_1}+\ldots+n_{i_r}-(n_{j_1}+\ldots+n_{j_s})|>A \quad \text{if } r\neq s. \tag{4.6}$$

We can clearly assume $i_r\neq j_s$, since otherwise we drop these terms. If $i_r>j_s$, we can ensure (4.6) if $n_{i_r}>n_{i_{r-1}}+\ldots+n_1+A$. This is a lacunarity condition.

Now since $\deg g<\omega$ we have by looking at (4.1) and considering terms with $\operatorname{ord}>\omega$,

$$0=[r]^{\omega}+l_{n+1}\left(\sum_{k=1}^{n}S_{1,k}(l_kq_k-1)+S_{0,n+1}q_{n+1}-S_{1,n+1}\right)+l_{n+1}^2G_2+l_{n+1}^3G_3\ldots l_{n+1}^AG_A, \tag{4.7}$$
$$\text{where } \deg G_i\leq A$$

for some constant A by our application of Sublemma 4. Thus from our lemma, if l_{n+1} is sufficiently lacunary,

$$\left|l_{n+1}\left(\sum_{k=1}^{n}S_{1,k}(l_kq_k-1)+S_{0,n+1}q_{n+1}-S_{1,n+1}\right)\right|_1\leq|[r]^{\omega}|_1. \tag{4.8}$$

The left side equals the product norm of the norm of the two factors if l_{n+1} is sufficiently lacunary, since the second factor has bounded degree. So, the left side of (4.8) is $\geq|l_{n+1}|_1$ times the $|\ |^n$ norm of the second factor so, since the $|\ |^n$ norm is $\leq|\ |_1$,

$$\geq|l_{n+1}|_1\left\{|S_{0,n+1}q_{n+1}|^n-\left|\sum_{k=1}^{n}S_{1,k}(l_kq_k-1)\right|^n-|S_{1,n+1}|_1\right\}.$$

Using (4.8),

$$\geq|l_{n+1}|_1\frac{|S_{0,n+1}|_1}{B}-\frac{|l_{n+1}|_1}{\varepsilon_{n+1}}\left[\left|\sum_{k=1}^{n}S_{1,k}(l_kq_k-1)\right|^n+\varepsilon_{n+1}|S_{1,n+1}|_1\right].$$

Using this and

$$|l_{n+1}|_1 > \frac{4B}{\varepsilon_{n+1}} \geqslant \frac{4|S_{0,n+1}|_1}{\varepsilon_{n+1}} - \frac{1}{\varepsilon_{n+1}}[|S_{0,n+1}|_1 - |[r]^{\omega}|_1] > |[r]^{\omega}|_1$$

contradicting (4.8). Q.E.D.

For the proof of Proposition $\mathfrak{a}_1^{nc}, N(\mathfrak{a}_1)$ we need a sequence of lemmas which will also be used later on. The general idea behind these lemmas is that if a sequence of l_i's is sufficiently lacunary and has sufficiently small coefficients then the cancellation effect of a monomial which contains l_i is small on a polynomial in which the terms do not contain l_i. We make here the trivial but important observation that the sequences (l_i) in the Lemmas 4.1–4.12 below do not need to start with l_1 but can start with any l_j with $j \geqslant 1$.

We also make the remark that for the proofs of these lemmas it is more convenient to write indices of l's in increasing order as opposed to what we have done in Section 3. So let $J=(j_1, j_2, \dots, j_r)$ below denote a finite sequence of integers such that $j_k < j_m$ if $k<m$. Put $|J|=r$. We recall $\bar{l}_i = l_i/|l_i|_1$. We will below assume that for each i the moduli of coefficients of l_i are constant.

LEMMA 4.1. *Given $\varepsilon>0$, a sequence (F_j) of real numbers and a $w>0$ and $r_0>0$ and $m_0>0$, there is a lacunarity function f and a sequence δ_i such that if the sequence (l_i) is more lacunary than f and has moduli of coefficients $\leqslant \delta_i$ then the following holds:*

Assume that in

$$\sum_{i,J,\alpha} a_{i,J,\alpha} x^i \bar{l}_{j_1}^{\alpha_{j_1}} \bar{l}_{j_2}^{\alpha_{j_2}} \dots \bar{l}_{j_r}^{\alpha_{j_r}}$$

we have

$$\sum_{\substack{i,J,\alpha \\ |J|=r}} |a_{i,j,\alpha}| \leqslant F_r.$$

Then for every polynomial

$$S(x) = \sum_{\substack{i,J,\alpha \\ i<m_0 \\ |J|\leqslant r_0 \\ |\alpha|\leqslant w}} b_{i,j,\alpha} x^i \bar{l}_{j_1}^{\alpha_{j_1}} \bar{l}_{j_2}^{\alpha_{j_2}} \dots \bar{l}_{j_p}^{\alpha_{j_p}}$$

the cancellation effect of

$$P_{r_0}(x) = \sum_{\substack{i,J,\alpha \\ |J|>r_0}} a_{i,j,\alpha} x^i \bar{l}_{j_1}^{\alpha_{j_1}} \bar{l}_{j_2}^{\alpha_{j_2}} \dots \bar{l}_{j_r}^{\alpha_{j_r}}$$

on $S(x)$ *is* $<\varepsilon$.

Proof. Let $s_{1,1}, s_{1,2}, \dots, s_{1,N_1}, s_{2,1}, s_{2,2}, \dots s_{2,N_2}, \dots$ be the exponents of x appearing in $l_1, l_2, \dots$ and written in increasing order. Then the moduli of the coefficients of $\bar{l}_1, \bar{l}_2, \dots$ will be $1/N_1, 1/N_2 \dots$. Now we write

$$P_{r_0}(x) = \sum_{\substack{I,J,\alpha \\ |I|>r_0}} (a_{i,J,\alpha} x^i \bar{l}_{j_1}^{\alpha_{j_1}-1} \bar{l}_{j_2}^{\alpha_{j_2}-1} \dots \bar{l}_{j_r}^{\alpha_{j_r}-1}) \bar{l}_{j_1} \bar{l}_{j_2} \dots \bar{l}_{j_r}$$

$$= \sum_{|J|>r_0} c_{i,J} x^i \bar{l}_{j_1} \bar{l}_{j_2} \dots \bar{l}_{j_r}.$$

(4.9) We observe that if the sequence l_i is sufficiently lacunary then the moduli of the coefficients in $\bar{l}_{j_1} \bar{l}_{j_2} \dots \bar{l}_{j_r}$ are all $1/N_{j_1} 1/N_{j_2} \dots 1/N_{j_r}$.

This is all right only if the l_i's are so lacunary that the monomials in $l_{i_1} \dots$ can arise in only one way (for example, we may ask that the monomials in the l_i's be powers of 2, all distinct).

Now let

$$s_{i,j} > 4w\left(\sum_{p<i} s_{p,q} + \sum_{q<j} s_{i,q}\right). \tag{4.10}$$

It is clear that (4.10) holds if we put the lacunarity function $f(k)=8wk$. Now we fix i_0 and i_1 and we study the cancellation effect of the monomial

$$C = c_{i_1,J} x^{i_1} \bar{l}_{j_1} \bar{l}_{j_2} \dots \bar{l}_{j_r}$$

on

$$B = \sum_{a,J} b_{i_0,J_a} x^{i_0} \bar{l}_{j_1}^{\alpha_{j_1}} \dots \bar{l}_{j_2}^{\alpha_{j_2}} \dots \bar{l}_{j_m}^{\alpha_{j_m}}.$$

The exponents appearing in C have the form $i_1 + \Sigma_{k\le r} s_{j_k,q_k}$ and the exponents appearing in B have the form $i_0 + \Sigma_{h\le r_0} d_{j_h,q_h} s_{j_h,q_h}$ where every d_{j_h,q_h} is an integer with $1 \le d_{j_h,q_h} \le w$. Now assume that there are solutions $\Sigma_{k\le r} s_{1,j_k,q_k}$ and $\Sigma_{k\le r} s_{2,j_k,q_k}$ of equations

$$i_1+\sum_{k\le r} s_{1,j_k,q_k}=i_0+\sum_{h\le r_0} d_{1,j_h,q_h}s_{j_h,q_h} \tag{4.11}$$

and

$$i_1+\sum_{k\le r} s_{2,j_k,q_k}=i_0+\sum_{h\le r_0} d_{2,j_h,q_h}s_{j_h,q_h}. \tag{4.12}$$

Then (4.11) and (4.12) will give

$$\sum_{k\le r} s_{1,j_k,q_k}-\sum_{h\le r_0} d_{1,j_h,q_h}s_{1,j_h,q_h}=\sum_{k\le r} s_{2,j_k,q_k}-\sum_{h\le r_0} d_{2,j_h,q_h}s_{2,j_h,q_h}. \tag{4.13}$$

Since $r>r_0$, $\Sigma_{k\le 0}s_{1,j_k,q_k}$ contains at least $r-r_0$ different terms s_{1,j_i,q_i} which do not appear in $\Sigma_{k\le r_0}d_{1,j_h,q_h}s_{1,j_h,q_h}$ and correspondingly for $\Sigma_{k\le r}s_{2,j_k,q_k}$. Now (4.10) and (4.13) implies that these s_{1,j_i,q_i}'s must be the same as the s_{2,j_i,q_i}'s. And this implies, since $N_1<N_2<$ that for i_0 and i_1 fixed there can be at most $N_{j_{r-r_0+1}}N_{j_{r-r_0+2}}\dots N_{j_r}$ different sums $\Sigma_{k\le r}s_{1,j_k,q_k}$ which solve equations of the form (4.11) and (4.12).

This together with (4.9) now implies that the cancellation effect of

$$c_{i_1,J}x^i l_{j_1}l_{j_2}\dots l_{j_r} \quad \text{on} \quad \sum_{\alpha,J} b_{i_0,J}x^i l_{j_1}^{\alpha_{j_1}}l_{j_2}^{\alpha_{j_2}}\dots l_{j_m}^{\alpha_{j_m}}$$

is

$$\le |c_{i_1,J}|\frac{1}{N_{j_1}}\cdots\frac{1}{N_{j_{r-r_0}}}\le |c_{i_1,J}|\frac{1}{N_1}\cdot\frac{1}{N_2}\cdot\ \cdots\frac{1}{N_{r-r_0}}.$$

Now summing this over all i_1 and the m possible i_0's gives that the cancellation effect of

$$\sum_{\substack{i,J\\|J|=r}} c_{i,J}x^i l_{j_1}l_{j_2}\dots l_{j_r}$$

on $S(x)$ is

$$\le m\left(\sum_{|J|=r}|c_{i,J}|\right)\frac{1}{N_1}\cdot\frac{1}{N_2}\cdot\ \cdots\frac{1}{N_{r-r_0}}\le\frac{mF_r}{N_1N_2\dots N_{r-r_0}}.$$

And so the cancellation effect of $P_{r_0}(x)$ on $S(x)$ is

$$\leqslant \sum_r \frac{mF_r}{N_1 N_2 \ldots N_{r-r_0}}.$$

By choosing

$$\delta_{r-r_0} = \frac{1}{N_{r-r_0}} < \frac{\varepsilon}{m \cdot F_r \cdot 2^r}$$

we get the lemma.

LEMMA 4.2. *Given w there is a lacunarity function f such that if the sequence (l_i) is more lacunary than f, then for all polynomials*

$$h(x) = \sum_{\substack{i,J,\alpha \\ i+|\alpha| \leqslant w}} a_{i,J,\alpha} x^i l_{j_1}^{\alpha_{j_1}} l_{j_2}^{\alpha_{j_2}} \ldots l_{j_r}^{\alpha_{j_r}}$$

we have $|h|_1 = \Sigma |a_{i,J,\alpha}|$.

The proof of this is obvious.

LEMMA 4.3. *Given integers m and w and positive real numbers K and ε there is a lacunarity function f and a sequence $\delta_i \searrow 0$ such that if (l_i) is more lacunary than f and the moduli of the coefficients of l_i are $<\delta_i$ then the following holds: Put*

$$S(x) = \sum_{\substack{i,J,\alpha \\ i \leqslant m \\ |\alpha| = w}} a_{i,J,\alpha} x^i l_{j_1}^{\alpha_{j_1}} l_{j_2}^{\alpha_{j_2}} \ldots l_{j_r}^{\alpha_{j_r}}$$

and let

$$P(x) = \sum_{\substack{i,J,\alpha \\ i > m \\ J, |\alpha| = w}} a_{i,J,\alpha} x^i l_{j_1}^{\alpha_{j_1}} l_{j_2}^{\alpha_{j_2}} \ldots l_{j_r}^{\alpha_{j_r}}$$

satisfy

$$\sum_{\substack{i > m \\ J, |\alpha| = w}} |a_{i,J,\alpha}| \leqslant K.$$

Then the cancellation effect of $P(x)$ on $S(x)$ is $<\varepsilon$.

Proof. The proof of this lemma is quite similar to that of Lemma 4.1. We form $s_{1,1}, s_{1,2}, \ldots, s_{1,N_1}, s_{2,1}, s_{2,2}, \ldots, s_{2,N_2}, \ldots$ as there. Let the $s_{i,j}$'s satisfy the condition (4.10) of Lemma 4.1. Then it follows from the binomial theorem that the moduli of the coefficients of

$$x^i t_{j_1}^{\alpha_{j_1}} t_{j_2}^{\alpha_{j_2}} \ldots t_{j_r}^{\alpha_{j_r}}$$

are

$$\leqslant \frac{\alpha_{j_1}!}{N_{j_1}^{\alpha_{j_1}}} \cdot \frac{\alpha_{j_2}!}{N_{j_2}^{\alpha_{j_2}}} \ldots \frac{\alpha_{j_r}!}{N_{j_r}^{\alpha_{j_r}}} \leqslant \frac{w!}{N_{j_1}^{\alpha_{j_1}} N_{j_2}^{\alpha_{j_2}} \ldots N_{j_r}^{\alpha_{j_r}}}. \tag{4.14}$$

Now fix $i_0 \leqslant m$ and $i_1 > m$ and consider the cancellation effect of the monomial

$$A = x^{i_1} t_{j_1}^{\alpha_{j_1}} t_{j_2}^{\alpha_{j_2}} \ldots t_{j_r}^{\alpha_{j_r}}$$

on

$$B = \sum_{J, |\alpha| = w} a_{i_0, J, \alpha} x^{i_0} t_{j_1}^{\alpha_{j_1}} t_{j_2}^{\alpha_{j_2}} \ldots t_{j_r}^{\alpha_{j_r}}.$$

The exponents appearing in A have the form $i_1 + \Sigma\, a_{j_k, q_k} s_{j_k, q_k}$ with $\Sigma\, a_{j_k, q_k} = w$ and the exponents appearing in B have the form $i_0 + \Sigma\, d_{j_h, q_h} s_{j_h, q_h}$ with $\Sigma\, d_{j_h, q_h} = w$. Now assume that there are solutions $\Sigma\, a_{1,j_k,q_k} s_{1,j_k,q_k}$ and $\Sigma\, a_{2,j_k,q_k} s_{2,j_k,q_k}$ of equations

$$i_0 + \Sigma\, a_{1,j_k,q_k} s_{1,j_k,q_k} = i_0 + \Sigma\, d_{1,j_h,q_h} s_{1,j_h,q_h} \tag{4.15}$$

and

$$i_1 + \Sigma\, a_{2,j_k,q_k} s_{2,j_k,q_k} = i_0 + \Sigma\, d_{2,j_h,q_h} s_{2,j_h,q_h}. \tag{4.16}$$

Since $i_1 > i_0$ there is in (4.15) at least one s_{1,j_i,q_i} which either does not appear on the right hand side of the equation or appears but is multiplied by a smaller number than on the left hand side. Analogously we find at least one s_{2,j_i,q_i} from equation (4.16). As in Lemma 4.1 above, (4.10) implies that the s_{1,j_i,q_i}'s must be the same as the s_{2,j_i,q_i}'s. Since $N_{j_1} < N_{j_2} \ldots$ this gives that for fixed i_0 and i_1 there are at most

$$w \cdot N_{j_1}^{\alpha_{j_1}} N_{j_2}^{\alpha_{j_2}} \ldots N_{j_r}^{\alpha_{j_r}}$$

different sums $\Sigma\, a_{j_k, q_k} s_{j_k, q_k}$ which solve equations of the form (4.15) or (4.16). By (4.14) this gives that the cancellation effect of A on B is at most $w \cdot w!/N_{j_1} \leq w \cdot w!/N_1$.

Since there are only m different i_0's and since

$$\sum_{\substack{i>m\\|\alpha|=w}} |a_{i,J,\alpha}| \leq K$$

we get that the cancellation effect of $P(x)$ on $S(x)$ is at most $K \cdot m \cdot w \cdot w!/N_1$. This gives the lemma if we choose $1/N_1 < \varepsilon/K \cdot m \cdot w \cdot w!$.

LEMMA 4.4. *Given integers m and w a positive number ε and a function γ such that* $\gamma(k) \to 0$ *as* $k \to \infty$, *there is a lacunarity function f and a sequence* $\delta_j \searrow 0$ *such that if* $\{l_i\}$ *is more lacunary than f and the moduli of the coefficients of* l_i *are* $<\delta_i$ *then the following holds: Put*

$$S(x) = \sum_{\substack{i,J,\alpha\\ i\leq m\\ |\alpha|=w}} a_{i,J,\alpha} x^i l_{j_1}^{\alpha_{j_1}} l_{j_2}^{\alpha_{j_2}} \dots l_{j_r}^{\alpha_{j_r}}$$

and let

$$P(x) = \sum_{\substack{i,J,\alpha\\ |\alpha|>w}} a_{i,J,\alpha} x^i l_{j_1}^{\alpha_{j_1}} l_{j_2}^{\alpha_{j_2}} \dots l_{j_r}^{\alpha_{j_r}}$$

satisfy

$$\sum_{\substack{i,J,\alpha\\ |\alpha|\geq k}} |a_{i,J,\alpha}| \leq \gamma(k).$$

Then the cancellation effect of $P(x)$ *on* $S(x)$ *is* $<\varepsilon$.

Proof. We first choose W so that $\gamma(W) < \varepsilon/2$. Then it is obviously enough to prove that the cancellation effect of

$$\sum_{w<|\alpha|\leq W} a_{i,J,\alpha} x^i l_{j_1}^{\alpha_{j_1}} l_{j_2}^{\alpha_{j_2}} \dots l_{j_r}^{\alpha_{j_r}}$$

on $S(x)$ is $\leq \varepsilon/2$.

If in (4.10) (of the Lemmas 4.1 and 4.3 above) we replace w by W we can now just repeat the argument of Lemma 4.1. The only difference is the following: In this lemma we use the condition $|\alpha|>w$ in the terms of $P(x)$ to conclude that the left hand side of (4.15) has an s_{j_i,q_i} which either does not appear on the right hand side or appears multiplied by a smaller number than on the left hand side.

Let D be a subset of the integers. Let

$$P=\sum_{i,J,\alpha} a_{i,J,\alpha}\, x^i l_{j_1}^{\alpha_{j_1}} l_{j_2}^{\alpha_{j_2}} \dots l_{j_r}^{\alpha_{j_r}}$$

be a polynomial such that in every term

$$x^i l_{j_1}^{\alpha_{j_1}} l_{j_2}^{\alpha_{j_2}} \dots l_{j_r}^{\alpha_{j_r}}$$

out of P there is at least one k with $j_k \in D$. Then we can write

$$P=\sum_{j\geq 1}\left(\sum_{m\geq 0} P_{(m,j)}\right)$$

where $P_{(m,j)}$ consists of the following terms out of P:

The term

$$x^i l_{j_1}^{\alpha_{j_1}} l_{j_2}^{\alpha_{j_2}} \dots l_{j_{r-m}}^{\alpha_{j_{r-m}}} l_{j_{r-m+1}}^{\alpha_{j_{r-m+1}}} \dots l_{j_r}^{\alpha_{j_r}}$$

belongs to $P_{(m,j)}$ if $j=j_{r-m}$ and j is the highest index of an l appearing in the term with $j\in D$. So for such a term none of the numbers $j_{r-m+1}, j_{r-m+2}, \dots, j_r$ is in D. Now every term out of $P_{(m,j)}$ we will below rewrite in the following way:

$$x^i l_{j_1}^{\alpha_{j_1}} l_{j_2}^{\alpha_{j_2}} \dots l_{j_{r-m}}^{\alpha_{j_{r-m}}} l_{j_{r-m+1}}^{\alpha_{j_{r-m+1}}} \dots l_{j_r}^{\alpha_{j_r}} = \left(x^i l_{j_1}^{\alpha_{j_1}} l_{j_2}^{\alpha_{j_2}} \dots l_{j_{r-m-1}}^{\alpha_{j_{r-m-1}}} l_{j_{r-m}}^{\alpha_{j_{r-m}}-1}\right) l_{j_{r-m}}\, l_{j_{r-m+1}}^{\alpha_{j_{r-m+1}}} \dots l_{j_r}^{\alpha_{j_r}}.$$

We now write

$$x^i l_{j_1}^{\alpha_{j_1}} l_{j_2}^{\alpha_{j_2}} \dots l_{j_{r-m-1}}^{\alpha_{j_{r-m-1}}} l_{j_{r-m}}^{\alpha_{j_{r-m}}-1}$$

as a polynomial in x (by expanding the l's). We also put

$$\alpha_{(m)}=(\alpha_{j_{r-m+1}},\dots,\alpha_{j_r}),\quad J_{(m)}=(j_{r-m}, j_{r-m+1},\dots,j_r).$$

Then we get

$$P_{(m,j)} = \sum_{\substack{i \\ J_{(m)}, \alpha_{(m)}}} c_{i,J_{(m)},\alpha_{(m)}} x^i l_{j_{r-m}} l_{j_{r-m+1}}^{\alpha_{j_{r-m+1}}} \dots l_{j_r}^{\alpha_{j_r}}.$$

We will put

$$|P_{(m,j)}|_{1,f} = \sum |c_{i,J_{(m)},\alpha_{(m)}}|.$$

We put

$$P_{(m,j,w)} = \sum_{\substack{i, J_{(m)} \\ |\alpha|_{(m)} \geq w}} c_{i,J_{(m)},\alpha_{(m)}} x^i l_{j_{r-m}} l_{j_{r-m+1}}^{\alpha_{j_{r-m+1}}} \dots l_{j_r}^{\alpha_{j_r}}$$

and

$$|P_{(m,j,w)}|_{1,f} = \sum_{|\alpha_{(m)}| \geq w} |c_{i,J_{(m)},\alpha_{(m)}}|.$$

In order to prove the more important Lemma 4.6 we use the following

LEMMA 4.5. *Given integers m and w there is a lacunarity function f such that if (l_i) is more lacunary than f then the following holds:*

Let D be any subset of the integers and $\{G_j\}$ any sequence of real numbers. Assume that in

$$S(x) = \sum_{\substack{i,J,\alpha \\ i \leq m \\ |\alpha| \leq w}} a_{i,J} x^i l_{j_1}^{\alpha_{j_1}} l_{j_2}^{\alpha_{j_2}} \dots l_{j_r}^{\alpha_{j_r}}$$

for all j_k in all terms we have $j_k \notin D$. Assume that in $P(x) = \Sigma_{i,j} c_{i,j} x^i l_j$ we have $\Sigma_i |c_{i,j}| \leq G_j$ for every j. Assume that the moduli of the coefficients of l_i are $\leq \varepsilon/(4^i m G_j)$. Then the cancellation effect of $P(x)$ on $S(x)$ is $< \varepsilon$.

Proof. Let the lacunary condition be (4.10) of Lemma 4.1. We fix i_0, i_1 and j and consider the cancellation effect of $x^{i_1} l_j$ on

$$\sum_{J,\alpha} a_{i_0,J,\alpha} x^{i_0} l_{j_1}^{\alpha_{j_1}} l_{j_2}^{\alpha_{j_2}} \dots l_{j_r}^{\alpha_{j_r}}.$$

The equations (4.11) and (4.12) take the form

$$i+s_{j,q_1}=i_0+\Sigma\, d_{1,j_h,q_h}s_{1,j_h,q_h}$$

and

$$i_1+s_{j,q_2}=i_0+\Sigma\, d_{2,j_h,q_h}s_{2,j_h,q_h}.$$

Since by assumption the index j does not appear in any of the terms out of $S(x)$, (4.10) now gives $s_{j,q_1}=s_{j,q_2}$. Since there are only m different i_0 we get that the cancellation effect of $x^{i_1}l_j$ on $S(x)$ is $\leq m\varepsilon/(4^jG_jm)$. And by the assumptions this gives the lemma.

LEMMA 4.6. *Given an integer m_0, a positive number ε, an increasing sequence of integers $\{w_m\}_{m\geq 0}$ and an increasing sequence $\{F_m\}$ of real numbers then there is a lacunarity function f such that for every subset D of the integers and every increasing sequence $\{G_j\}$ of real numbers we have the following:*

Assume

$$S(x)=\sum_{\substack{i+r\leq m_0\\ |\alpha|\leq w_0}} a_{i,J,\alpha}x^i l_{j_1}^{\alpha_{j_1}} l_{j_2}^{\alpha_{j_2}}\dots l_{j_r}^{\alpha_{j_r}}$$

where for all j_k of all terms $j_k\notin D$. Assume

$$P=\sum_j\left(\sum_m P_{(m,j)}\right)$$

where

$$|P_{(m,j)}|_{1,f}\leq G_jF_m \quad and \quad |P_{m,j,w_m}|_{1,f}\leq\frac{G_j}{2^m}.$$

Assume also that $\{l_i\}$ is more lacunary than f and that the moduli $1/N_i$ of the coefficients of l_i are

$$\frac{1}{N_i}\leq\left(\prod_{k<i}\left(\frac{1}{N_k}\right)^{w_{i+m_0}}\right)\frac{\varepsilon}{(w_{i+m_0})!w_{i+m_0}}\cdot\frac{1}{4^{i+m_0}}\cdot\frac{1}{m_0G_i(F_0+\dots+F_{i+m_0}+1)}.$$

Then the cancellation effect of $P(x)$ on *$S(x)$ is $<\varepsilon$.*

Proof. By assumption we can write

$$P_{(m,j)} = \sum_{\substack{i \\ J_{(m)}, \alpha_{(m)}}} c_{i,J_{(m)},\alpha_{(m)}} x^i l_{j_{r-m}} l_{j_{r-m+1}}^{\alpha_{j_{r-m+1}}} \dots l_{j_r}^{\alpha_{j_r}} = \sum_i c_{i,j} x^i l_j$$

with

$$\sum_i |c_{i,j}| \leq \sum_{\substack{i \\ J_{(m)}, \alpha_{(m)}}} |c_{i,J_{(m)},\alpha_{(m)}}| \leq G_j F_m.$$

Also we can write

$$P_{(m,j,w_m)} = \sum_{\substack{i, J_{(m)}, \alpha_{(m)} \\ \alpha_{(m)} \geq w_m}} c_{i,J_{(m)},\alpha_{(m)}} x^i l_j l_{j_{r-m+1}}^{\alpha_{j_{r-m+1}}} \dots l_{j_r}^{\alpha_{j_r}} = \sum_i c'_{i,j} x^i l_j$$

with

$$\sum_i |c'_{i,j}| \leq \sum |c_{i,J_{(m)},\alpha_{(m)}}| \leq G_j.$$

We now consider the sum

$$\sum_{m \leq m_0} P_{(m,j)} + \sum_{m > m_0} P_{(m,j,w_m)} = \sum_i d_{i,j} x^i l_j.$$

And so we get

$$\sum_i |d_{i,j}| \leq G_j(F_0 + F_1 + \dots + F_{m_0} + 1).$$

And this gives by the previous Lemma 4.5 that the cancellation effect of

$$\sum_j \left(\sum_{m \leq m_0} P_{(m,j)} + \sum_{m > m_0} P_{(m,j,w_m)} \right)$$

on $S(x)$ is $<\varepsilon/2$.

So we now consider the cancellation effect of

$$\sum_j \left(\sum_m (P_{(m,j)} - P_{(m,j,w_m)}) \right)$$

on $S(x)$.

We assume

$$s_{i,j} > 4w_{i+m_0}\left(\sum_{p<i} s_{p,q} + \sum_{j<q} s_{i,q}\right). \tag{4.17}$$

We observe that (4.17) is stronger than (4.10) since the w_m's increase with m. We also observe that (4.17) is given by the lacunarity function $f(k)=8w_{k+m_0}k$.

We now fix j, $m>m_0$, i_0 and i_1 and we investigate the cancellation effect of $x^{i_1}l_j l_{j_{r-m+1}}^{a_{j_{r-m+1}}} \dots l_{j_r}^{a_{j_r}}$ on

$$\sum_{\substack{J,a \\ i_0+r\leq m_0 \\ |a|\leq w_0}} a_{i_0,J,a} x^{i_0} l_{j_1}^{a_{j_1}} l_{j_2}^{a_{j_2}} \dots l_{j_r}^{a_{j_r}}.$$

We observe that the largest absolute value of a coefficient out of $l_j l_{j_{r-m+1}}^{a_{j_{r-m+1}}} \dots l_{j_r}^{a_{j_r}}$ is smaller than or equal to the largest absolute value of a coefficient out of $l_{j_{r-m+m-m_0}}^{a_{j_{r-m+m-m_0}}} \dots l_{j_r}^{a_{j_r}}$. And the largest absolute value of a coefficient out of $l_{j_{r-m_0}}^{a_{j_{r-m_0}}} \dots l_{j_r}^{a_{j_r}}$ is, by (4.17) and the binomial theorem, and since $r-m>0$,

$$\leq \frac{(a_{j_{r-m_0}})!}{N_{j_{r-m_0}}^{a_{j_{r-m_0}}}} \dots \frac{(a_{j_r})!}{N_{j_r}^{a_{j_r}}} \leq \frac{w_m!}{N_{j_{r-m_0}}^{a_{j_{r-m_0}}} \dots N_{j_r}^{a_{j_r}}}.$$

As above we now form the equations

$$i_1 + s_{j,q_1} \Sigma\, a_{1,j_k,q_k} s_{1,j_k,q_k} = i_0 + \Sigma\, d_{1,j_h,q_h} s_{1,j_h,q_h} \tag{4.18}$$

$$i_1 + s_{j,q_2} \Sigma\, a_{2,j_k,q_k} s_{2,j_k,q_k} = i_0 + \Sigma\, d_{2,j_h,q_h} s_{2,j_h,q_h}. \tag{4.19}$$

There are at least $m-m_0$ different indices j_k which appear on the left hand side of (4.18) and which do not appear on the right hand side. And the correspondingly is true for the equation (4.19). So for at least one of these we get an s_{1,j_i,q_i} so that the index of the corresponding l is $j_{r-m+(m-m_0)} > m-m_0$. And so by (4.17) there must be some k so that $s_{1,j_i,q_i} = s_{2,j_k,q_k}$. Thus for fixed $j, m,$ i_0 and i_1 there are at most

$$w_m N_j \cdot N_{j_{r-m+1}}^{a_{j_{r-m+1}}} \dots N_{j_{r-m_0}}^{a_{j_{r-m_0}}-1} \dots N_{j_r}^{a_{j_r}}$$

sums that can satisfy equations of the form (4.18) and (4.19). Thus the cancellation effect of $x^{i_1}l_j l_{j_{r-m+1}}^{\alpha_{j_{r-m+1}}} \dots l_{j_r}^{\alpha_{j_r}}$ on $S(x)$ is

$$\leq w_m! m_0 w_m \cdot N_j \cdot N_{j_{r-m+1}}^{\alpha_{j_{r-m+1}}} \dots N_{j_{r-m_0-1}}^{\alpha_{j_{r-m_0-1}}} \frac{1}{N_{j_{r-m_0}}}.$$

Since $j_{r-m_0} \geq j+m-m_0$ this is

$$\leq \frac{w_m! m_0 w_m}{(w_{m+j})! m_0 w_{m+j}} \cdot \frac{\varepsilon}{4^{j+m}} \cdot \frac{1}{G_{j+m-m_0}(F_0+\dots+F_{j+m}+1)} \leq \frac{\varepsilon}{4^{j+m}} \cdot \frac{1}{G_j F_m}.$$

This gives the lemma.

For the next lemma we introduce the following notation. With

$$H = \sum_{i,J,\alpha} a_{i,J,\alpha} x^i l_{j_1}^{\alpha_{j_1}} l_{j_2}^{\alpha_{j_2}} \dots l_{j_r}^{\alpha_{j_r}}$$

we put

$$H_r = \sum_{\substack{i,J,\alpha \\ |J|=r}} a_{i,J,\alpha} x^i l_{j_1}^{\alpha_{j_1}} l_{j_2}^{\alpha_{j_2}} \dots l_{j_r}^{\alpha_{j_r}}$$

and

$$H_{r,w} = \sum_{\substack{i \\ |\alpha|=w \\ |J|=r}} a_{i,J,\alpha} x^i l_{j_1}^{\alpha_{j_1}} l_{j_2}^{\alpha_{j_2}} \dots l_{j_r}^{\alpha_{j_r}}.$$

We will also put

$$|H|_{1,f} = \sum |a_{i,J,\alpha}|.$$

LEMMA 4.7. *Given an integer R, an increasing sequence $\{F_j\}$, $j \geq 0$ of real numbers and a function γ, $\gamma(k) \to 0$ as $k \to \infty$, then there is a lacunarity function f, a sequence $\delta_j \searrow 0$, an integer m and a positive number B so that if $\{l_i\}$ is more lacunary than f and has moduli of coefficients $\leq \delta_j$ then the following holds:*

Assume that H is a polynomial, $|H|_{1,f} = 1$ and $\Sigma_{i+|\alpha| \geq k} |a_{i,J,\alpha}| \leq \gamma(k)$.

Assume that $Q=\Sigma_{j\geq 0}Q_j$ *defined like* H_r *above, is a polynomial such that* $|Q_j|_{1,f}\leq F_j$, $j\geq 0$ *and* $|[Q_0]_R|_1=1$.

Then there is a polynomial

$$E=\sum_{i+|\alpha|\leq m} e_{i,J,\alpha}x^i l_{j_1}^{\alpha_{j_1}} l_{j_2}^{\alpha_{j_2}}\dots l_{j_r}^{\alpha_{j_r}}$$

such that $|E(x)|_1\geq B$ *and* $HQ(x)$ *contains* $p_{1/100}(E)$.

Proof. We choose k_1 so that $\gamma(k_1)\leq 1/2$. Then we can find r_1 and w_1 with $r_1\leq w_1\leq k_1$ so that $|H_{r_1,w_1}|_{1,f}\geq 1/2k_1^2$. We consider H_{r_1,w_1}. There is a K_1 so that in H_{r_1,w_1} we have $\Sigma_{i>K_1}|a_{i,J,\alpha}|<1/4k_1^2$ (we just choose K_1 so that $\gamma(K_1)<1/4k_1^2$). We now form the product

$$H_{r_1,w_1}(x,\bar{l})\cdot Q_0(x)=\sum_{\substack{|J|=r_1\\|\alpha|=w_1}} e_{1,i,J,\alpha}x^i l_{j_1}^{\alpha_{j_1}} l_{j_2}^{\alpha_{j_2}}\dots l_{j_r}^{\alpha_{j_r}}.$$

Now it is easy to see that there is a B_1 such that $\Sigma_{i\leq K_1+R}|e_{1,i,J,\alpha}|\geq B_1$ and obviously $\Sigma_{i>K_1+R}|e_{1,i,J,\alpha}|\leq 1\cdot F_0$. Thus if we put

$$E_1=\sum_{i\leq K_1+R} e_{1,i,J,\alpha}x^i l_{j_1}^{\alpha_{j_1}} l_{j_2}^{\alpha_{j_2}}\dots l_{j_r}^{\alpha_{j_r}},$$

then by Lemma 4.2. $|E_1(x)|_1=\Sigma\,|e_{1,i,J,\alpha}|$ and by Lemma 4.3. $(H_{r_1,w_1}Q_0)(x)$ contains $p_{1/200}(E_1(x))$.

Now by Lemma 4.1 the cancellation effect of $(\Sigma_{\max(m,p)>r_1}H_m Q_p)(x)$ on E_1 is $<(1/400)\cdot B_1$. So either $(HQ)(x)$ contains $p_{1/100}(E_1)$, in which case the lemma is proved, or the cancellation effect of

$$\left(\sum_{\substack{\max(m,p)\leq r_1\\(m,w)\neq(r_1;w_1)}} H_{m,w}Q_p\right)(x)$$

on E_1 is $>(1/400)B_1$.

By Lemma 4.4 the cancellation effect of $(\Sigma_{w>w_1}H_{r_1,w}\Sigma_{j\leq r_1}Q_j)(x)$ on E_1 is $<(1/800)B_1$. We see this by writing $\Sigma_{j\leq r_1}Q_j$ as a polynomial in x. So in this case the cancellation effect of $(\Sigma_{r<r_1, p\leq r_1} H_r Q_r)(x) + (\Sigma_{w<w_1} H_{r_1,w} \Sigma_{j\leq r_1} Q_j)(x)$ on E_1 is $>(1/800)\,B_1$. And this implies that either for some $r<r_1$ we have

$$|H_r|_{1,f} > \frac{1}{1600}\cdot\sum_{j\leq r_1}\frac{B_1}{F_j\cdot r_1}$$

or for some $w<w_1$ we have

$$|H_{r_1,w}|_{1,f} > \frac{1}{1600}\cdot\frac{B_1}{w_1\sum_{j\leq r_1}F_j}.$$

In the first of these cases we can choose k_2 to be so big that

$$\gamma(k_2) < \frac{1}{3200}\cdot\frac{B_1}{\sum_{j\leq r_1}F_j\cdot r_1}.$$

This gives that in either case we find a new H_{r_2,w_2} for which we can repeat the same argument as for H_{r_1,w_1}. Obviously this process has to stop after at most k_1^2 steps which only depends on γ. This proves the lemma.

For the next lemma consider an $(n-1)$-substituted (n,n)-expansion of q and form q'. Put $q'=q_1'+q_2'$ where q_1' consists of those terms which contain only x's and $\bar{l}$'s form the $n-1$ first systems. And so in q_2' every term contains at least one l from a system with number n. As in Lemma 4.7 above with $q_1'=\Sigma\, a_{i,J,\alpha}\, x^i \bar{l}_{j_1}^{\alpha_{j_1}} \bar{l}_{j_2}^{\alpha_{j_2}} \dots \bar{l}_{j_r}^{\alpha_{j_r}}$, we use the notation

$$q'_{1,m} = \Sigma\, a_{i,J,\alpha}\, x^i \bar{l}_{j_1}^{\alpha_{j_1}} \bar{l}_{j_2}^{\alpha_{j_2}} \dots \bar{l}_{j_m}^{\alpha_{j_m}}.$$

LEMMA 4.8. *There are constants C and D depending only on* $|\ |^{n-1}$ *such that for all* $N\geq n$ *we have the following:*

Consider q with $|q-q_n|^N<\varepsilon_n/16$. *Then* $|q'_{1,m}|_{1,f}\leq C\cdot D^m$ *for all* $m\geq 0$.

Proof. (4.20) We consider a pre-(n,n)-expansion of q. We say that a term in the final stage of the pre-expansion contributes to $q'_{1,m}$ if—when the s's are written as polynomials in x and $\bar{l}$'s and $l_1, l_2, \dots, l_{n-1}$ and $\bar{l}_1, \bar{l}_2, \dots, \bar{l}_{n-1}$ are written as polynomials in x—we get the terms that enter into $q'_{1,m}$. We observe that a term of the second type can contribute only if $p \leqslant n-1$. Since otherwise l_{j_r} belongs to a system with number $\geqslant n$.

(4.21) We observe that a term which is derived from a term of the second type cannot contribute to $q'_{1,m}$. Since every such term will contain an l_{j_r} from a system with number $\geqslant n$.

(4.22) We observe that no term of the types 1–4 with $r \geqslant m+2$ can contribute to $q'_{1,m}$. Since all such terms contain at least $m+1$ different l_{j_i}'s with the index $j_i \geqslant n$. The same is obviously true also for terms which are derived from such terms.

(4.23) We finally observe that a term of type 1 can contribute only if $j_r \leqslant n$. Since otherwise the term will be replaced either by the first or the second rule.

Put $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$ and fix an integer $r \leqslant m+1$. Consider those terms $s_{j_1} l_{j_1} s_{j_2} l_{j_2} \ldots s_{j_r} l_{j_r} q_{j_r}$ of the first type, such that either the term itself or some term derived from it contributes to $q'_{1,m}$. We have

$$\Sigma\, |s_{j_1} s_{j_2} \ldots s_{j_r}|_{\mathrm{op}\,N} \leqslant \left(\frac{1}{\varepsilon}\right)^r, \tag{4.24}$$

where the sum is extended over all such terms.

To prove (4.24) we first recall that in forming q' we only consider terms in which every s_{j_i} has its corresponding $\varepsilon_{j_i} \geqslant \varepsilon$.

We prove (4.24) by induction on r. For $r=1$ we only have to consider the first stage of the pre-(n, n)-expansion of q, $q = q_n + \Sigma_i s_i(l_i q_i - 1) + t$. It is clear that in this case all terms from later stages of the expansion contain both an l_{j_1} and an l_{j_2}. We get

$$\sum_{\varepsilon_i \geqslant \varepsilon} |s_i|_{\mathrm{op}\,N} \cdot \varepsilon_i \leqslant \frac{\varepsilon_n}{16} \leqslant 1,$$

and this gives

$$\sum_{\varepsilon_i \geqslant \varepsilon} |s_i|_{\mathrm{op}\,N} \leqslant \frac{1}{\varepsilon}.$$

We now assume that (4.24) is true for r. By the observations (4.20) and (4.21) above, for $r+1$ we only have to consider those terms which are obtained by replacements according to the first rule. So consider a term $s_{j_1} l_{j_1} s_{j_2} l_{j_2} \ldots s_{j_r} l_{j_r} q_{j_r}$. Assume that we have

$q_{j_r}=q_p+\Sigma_i s_{r+1,i}(l_i q_i-1)+t$, where $p\leq n-1$ (otherwise l_{j_r} would belong to a system with number $\geq n$). Then

$$\sum_{\varepsilon_{r+1,i}\leq\varepsilon} |s_{r+1,i}|_{\mathrm{op}\,N}\cdot\varepsilon_{r+1}\leq\frac{\varepsilon_p}{16}\leq 1.$$

And so

$$\sum_{\varepsilon_{r+1,i}\leq\varepsilon} |s_{r+1,i}|_{\mathrm{op}\,N}\leq\frac{1}{\varepsilon}.$$

And since the op N-norm is submultiplicative this gives (4.24) for $r+1$.

We now have

$$\sum |s_{j_1}s_{j_2}\dots s_{j_r}|_1\leq\sum |s_{j_1}s_{j_2}\dots s_{j_r}|_{\mathrm{op}\,N}.$$

We also have that all l_{j_i} which appear in terms that contribute to $q'_{1,m}$ are from the $n-1$ first systems and so $|l_{j_i}|_1\leq L_{n-1}$. Thus given r we get by (4.23) that for terms of the first type that contribute to $q'_{1,m}$

$$\sum |s_{j_1}l_{j_1}s_{j_2}\dots s_{j_r}l_{j_r}q_{j_r}|_1\leq\left(\max_{j\leq n}(|q_j|_1)\right)\left(\frac{1}{\varepsilon}\right)^r L^r_{n-1}.$$

Obviously for terms of the second type we get the same estimate

$$\sum |s_{j_1}l_{j_1}s_{j_2}l_{j_2}\dots s_{j_r}l_{j_r}q_p|_1\leq\left(\max_{j\leq n}(|q_j|_1)\right)\left(\frac{1}{\varepsilon}\right)^r L_{n-1}.$$

For terms of the third type we get by (4.24)

$$\sum |s_{j_1}l_{j_1}s_{j_2}l_{j_2}\dots s_{j_r}l_{j_r}s_{j_{r+1}}|_1\leq\left(\frac{1}{\varepsilon}\right)^{r+1}L^r_{n-1},$$

and finally since $|t|\leq 1$ we get by (4.24) for terms of the fourth type

$$\sum |s_{j_1}l_{j_1}s_{j_2}l_{j_2}\dots s_{j_r}l_{j_r}t|\leq\left(\frac{1}{\varepsilon}\right)^r L^r_{n-1}.$$

Now $r\leq m+1$ by (4.22). So summing these estimates over all $r\leq m+1$ we get

$$|q'_{1,m}|_1 \leqslant \sum_r \left(2 \max_{j\leqslant n} |q_j|_1 + \frac{1}{\varepsilon} + 1\right)\left(\frac{1}{\varepsilon}\right)^r L_{n-1}^r$$
$$\leqslant \left\{\left(2 \max_{j\leqslant n} |q_j|_1 + \frac{1}{\varepsilon} + 1\right)\frac{1}{\varepsilon}\cdot L_{n-1}\right\}\frac{2^m L_{n-1}^m}{\varepsilon^m}.$$

It is easy to check that this holds also for $m=0$.

This gives the lemma with

$$C = \left(2 \max_{j\leqslant n} |q_j|_1 + \frac{1}{\varepsilon} + 1\right)\frac{L_{n-1}}{\varepsilon}$$

and

$$D = \frac{2L_{n-1}}{\varepsilon}.$$

In the next lemma we will let D denote the subset of integers $\leqslant N$ so that $j \in D$ if and only if l_j belongs to a system with number $\geqslant n$. We can then as in the Lemmas 4.5 and 4.6 above write

$$q'_2 = \sum_j \left(\sum_{m\geqslant 0} P_{(m,j)}\right).$$

LEMMA 4.9. *There is a number D and a sequence of integers $\{W_m\}$ depending only on $|\ |^{n-1}$ and numbers K_j, $j\geqslant n$, K_j depending only on $|\ |^{j-1}$ such that for all $N\geqslant n$, with $|q-q_n|^N<\varepsilon_n/16$ we have for q'_2*

$$|P_{(m,j)}|_{1,f} \leqslant D^m K_j \quad \textit{and} \quad |P_{(m,j,w_m)}|_{1,f} \leqslant K_j/2^m.$$

Proof. We consider a pre-(n,n)-expansion of q. For every k we consider terms of the first type at the stage k which have the form $s_{j_1} l_{j_1} s_{j_2} l_{j_2} \dots s_{j_{r-1}} l_{j_{r-1}} s_j l_j q_j$ and which have the property that some term derived from them contributes to $P_{(m,j)}$. Let $P'_{(m,j)}$ be the sum of these contributions. Obviously all terms of any of the four types which in the final stage contains l_j is derived from such a term.

(4.25) Now consider a q_j in a term $s_{j_1} l_{j_1} s_{j_2} l_{j_2} \dots s_{j_{r-1}} l_{j_{r-1}} s_j l_j q_j$ and consider what expression q_j is replaced by in the final stage of the pre-(n,n)-expansion say $q_{j,n,n}(s,l,q)$. Then remove from $q_{j,n,n}(s,l,q)$ all terms of the third type so that the corresponding ε is $<\min\{\varepsilon_1,\dots,\varepsilon_n\}$. Let $q'_{j,n,n}(s,l,q)$ be what remains. Now we expand

$q'_{j,n,n}(s,l,q)$ as a polynomial in x, say $q'_{j,n,n}(x)$. Then we obviously get $|q'_{j,n,n}|_1 \leq K_{1,j}$ where $K_{1,j}$ is determined by the sequence $l_1, l_2, \ldots, l_{j-1}$, $C_1, C_2, \ldots, C_{j-1}$, $D_1, D_2, \ldots, D_{j-1}$, $L_1, L_2, \ldots, L_{j-1}$.

We make here the following remark: When q_j is replaced in the pre-(n,n)-expansion of q and then in the final stage the s's are replaced by polynomials in x and l's, then only q's and l's with index $<j$ will appear. So for computing $|P_{(m,j)}|_{1,f}$ all these will be replaced by polynomials in x.

(4.26) We observe that in order that a term of 1st, 2nd or 3rd type of the form $s_{j_1} l_{j_1} s_{j_2} l_{j_2} \ldots$ has a term which is derived from it and which contributes to $P_{(m,j)}$ it is necessary that $r-1 \leq m$, that the ε's corresponding to $s_{j_1}, s_{j_2}, \ldots, s_{j_{r-1}}$ are all $\geq \varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\}$ and that $l_{j_1}, l_{j_2}, \ldots, l_j$ all belong to the $n-1$ first systems.

We remark that ε_j which corresponds to s_j can possibly be $<\varepsilon$.
With $J = j_1, j_2, \ldots, j_{r-1}, j$ we consider all products

$$s_{j_1} s_{j_2} \ldots s_{j_{r-1}} s_j = \sum a_{J,i,\alpha} x^i l_{k_1}^{\alpha_{k_1}} l_{k_2}^{\alpha_{k_2}} \ldots l_{k_p}^{\alpha_{k_p}}$$

where for every k_l, $k_l \geq j$, $1 \leq l \leq p$. We get as in (4.24) of Lemma 4.8 that

$$\sum_J \left(\sum_{i,\alpha} |a_{J,i,\alpha}| \right) \leq \left(\frac{1}{\varepsilon} \right)^{r-1} \frac{1}{\varepsilon_j}.$$

By (4.25) and (4.26) we get by summing over all r

$$|P'_{(m,j)}|_{1,f} \leq \frac{K_{1,j}}{\varepsilon_j} \sum_r \left(\frac{L_{n-1}}{\varepsilon} \right)^{r-1} \leq \frac{K_{1,j}}{\varepsilon_j} \left(\frac{2L_{n-1}}{\varepsilon} \right)^m \cdot |l_j|_1. \tag{4.27}$$

Consider

$$s_j l_j \ldots s_{j_{r-1}} l_{j_{r-1}} = \sum c_{J,i,\alpha} x^i l_{k_1}^{\alpha_{k_1}} \ldots l_{k_p}^{\alpha_{k_p}}$$

and let $w_{1,m}$ be defined by

$$\sum_{J,r} |s_{j_1} l_{j_1} \ldots s_{j_{r-1}} l_{j_{r-1}}|_{\mathrm{op}\,N} \leq \left(\frac{2L_{n-1}}{\varepsilon} \right)^m \Rightarrow \sum_{J,r} \left(\sum_{|\alpha|>w} |c_{J,i,\alpha}| \right) < \frac{1}{2} \cdot \frac{1}{2^m}.$$

Then we get

$$|P'_{(m,j,w_{1,m})}|_{1,f} \leq \frac{1}{2} \cdot \frac{K_{1,j}}{\varepsilon_j} \cdot \frac{1}{2^m} \cdot |l_j|_1.$$

We fix an $r \leq m$ and we consider the terms derived from terms of the form $s_{j_1} l_{j_1} s_{j_2} l_{j_2} \dots s_{j_r} l_{j_r} q_{j_r}$ where $j_r > j$. Obviously we can assume that all the l_{j_i}'s are from the $n-1$ first systems. Then by passing to the next stage of the expansion we have

$$q_{j_r} = q_p + \sum_i s_i(l_i q_i - 1) + t, \quad p \leq n-1.$$

Now we consider the terms of the first type $s_{j_1} l_{j_1} s_{j_2} l_{j_2} \dots s_{j_r} l_{j_r} s_i l_i q_i$ and we first consider the terms where $i<j$. For these terms we form as above $q'_{i,n,n}(x)$ and we put $K_{2,j} = \max_{i<j} |q'_{i,n,n}(x)|_1$. Let $P''_{(m,j)}$ be the contribution to $P_{(m,j)}$ from terms derived from these terms. Then we get as above

$$|P''_{(m,j)}|_{1,f} \leq \sum_r \left(\frac{L_{n-1}}{\varepsilon}\right) \cdot \frac{K_{1,j}}{\min\{\varepsilon_1, \dots, \varepsilon_{j-1}\}} \leq \left(\frac{2L_{n-1}}{\varepsilon}\right)^m + \frac{K_{2,j}}{\min\{\varepsilon_1, \dots, \varepsilon_{j-1}\}}.$$

As above we find a $w_{2,m}$ so that

$$|P''_{(m,j,w_{2,m})}|_{1,f} \leq \frac{K_{2,j}}{\min\{\varepsilon_1, \dots, \varepsilon_{j-1}\}} \cdot \frac{1}{2^m} \cdot \frac{1}{4}.$$

Now we consider terms of the first type $s_{j_1} l_{j_1} s_{j_2} l_{j_2} \dots s_i l_i q_i$ with $i>j$. We see that these terms will appear when we consider $r+1$.

So now we fix $r \leq m$ and consider terms of the second type $s_{j_1} l_{j_1} s_{j_2} l_{j_2} \dots s_{j_r} l_{j_r} q_p$ where we can assume $p \leq n-1$, otherwise l_{j_r} would belong to a system with number $\geq n$. For the contributions $P'''_{(m,j)}$ from these terms we get as above after summing over r

$$|P'''_{(m,j)}|_{1,f} \leq \left(\frac{2L_{n-1}}{\varepsilon}\right)^m \max_{p \leq n-1} |q_p|_1 \leq \left(\frac{2L_{n-1}}{\varepsilon}\right)^m \cdot K_{3,j}$$

with $K_{3,j} = \max_{p \leq n-1} |q_p|_1$.

As above we also get a number $w_{3,m}$ such that

$$|P'''_{(m,j,w_{3,m})}|_{1,f} \leq K_{3,j} \cdot \frac{1}{2 \cdot 2^m}.$$

For terms of the third type $s_{j_1} l_{j_1} l_{j_2} \dots s_{j_r} l_{j_r} s_i$ we observe that only s_i's with $\varepsilon_i \geqslant \min\{\varepsilon_i, \dots, \varepsilon_n\} = \varepsilon$ will enter. So with $K_{4,j} = 1/\varepsilon$ we get

$$|P^{\mathrm{IV}}_{(m,j)}|_{1,f} \leqslant \frac{1}{\varepsilon}\left(\frac{2L_{n-1}}{\varepsilon}\right)^m$$

and we find a $w_{4,m}$ such that

$$|P^{\mathrm{IV}}_{(m,j,w_{4,m})}|_{1,f} \leqslant K_{4,j}\cdot\frac{1}{2^m}.$$

And for terms of the fourth type we have $|t| \leqslant 1$. So with $K_{5,j} = 1$ we get

$$|P^{\mathrm{V}}_{(m,j)}|_{1,f} \leqslant K_{5,j}\left(\frac{2L_{n-1}}{\varepsilon}\right)^m$$

and we get an $w_{5,m}$ such that

$$|P^{\mathrm{V}}_{(m,j,w_{5,m})}|_{1,f} \leqslant \frac{K_{5,j}}{2^m}.$$

Finally for terms s_{j_1}, t and terms derived from $s_{j_1} l_{j_1} q_{j_1}$ with $j_1 < j$, we get $K_{6,j}$ and $w_{6,j}$ so that the contributions to $P_{(m,j)}$ satisfy

$$|P^{\mathrm{VI}}_{(m,j)}|_{1,f} \leqslant K_{6,j}\left(\frac{2L_{n-1}}{\varepsilon}\right)^m$$

and

$$|P^{\mathrm{VI}}_{(m,j,w_{6,j})}|_{1,f} \leqslant \frac{K_{6,j}}{2^m}.$$

This gives the lemma with $w_m = \max\{w_{1,m}, w_{2,m}, \dots, w_{6,m}\}$, $D = 2L_{n-1}/\varepsilon$, and

$$K_j = \left(\frac{K_{1,j}}{\varepsilon_j} + \frac{K_{2,j}}{\min(\varepsilon_1, \dots, \varepsilon_{j-1})} + K_{3,j} + K_{4,j} + K_{5,j} + K_{6,j}\right).$$

In the next lemma we consider

$$H = \sum b_{i,\alpha} x^i l_1^{\alpha_1} l_2^{\alpha_2} \dots l_N^{\alpha_N} = \sum_{j_i \geqslant n} a_{i,J,\alpha} x^i l_{j_1}^{\alpha_{j_1}} l_{j_2}^{\alpha_{j_2}} \dots l_{j_r}^{\alpha_{j_r}},$$

$$H_{(k)} = \sum_{i+|\alpha| \geqslant k} a_{i,J,\alpha} x^i l_1^{j_1} \dots l_r^{j_r}.$$

The last representation is of course the $(n-1)$-expanded representation of H. We recall $|H|_{1,f}=\Sigma|a_{i,J,\alpha}|$. We also assume that the representation

$$\sum b_{i,\alpha}x^i l_1^{\alpha_1} l_2^{\alpha_2}\dots l_N^{\alpha_N}$$

shows $|H|_{\mathrm{op}\,N}<1/\varepsilon$ which in particular implies $\Sigma_{i+|\alpha|\geq k}|b_{i,\alpha}|\leq 1/(\varepsilon\cdot 2^k)$.

LEMMA 4.10. *Given $\varepsilon>0$ and $|\ |^{n-1}$ there is a function $\gamma_{\varepsilon,n-1}$ such that $\gamma_{\varepsilon,n-1}(k)\to 0$ as $k\to\infty$ so that the following holds:*

Let $|H|_{\mathrm{op}\,N}<1/\varepsilon$ hold. Then

$$\sum_{i+|\alpha|\geq k}|a_{i,J,\alpha}|\leq\gamma_{\varepsilon,n-1}(k)$$

Proof. We consider the representation

$$H=\sum_{i,\alpha} b_{i,\alpha}x^i l_1^{\alpha_1} l_2^{\alpha_2}\dots l_N^{\alpha_N}.$$

We have that the only terms that can contribute to $H_{(k)}$ have the property

$$i+\sum_{j\leq n-1}\alpha_j\deg l_j+\sum_{j=n}^{N}\alpha_j\geq k,$$

which implies

$$i+|\alpha|\deg l_{n-1}\geq k,$$

which implies

$$i+|\alpha|>\frac{k}{\deg l_{n-1}}.$$

This gives the lemma with

$$\gamma_{\varepsilon,n-1}(k)=\frac{1}{\varepsilon\cdot 2^{k/\deg l_{n-1}}}.$$

Remark. The assumption $|H|_{1,f}=1$ would be obviously superfluous in the lemma above.

In the next lemma we assume $|q-q_n|^N\leq\bar{\varepsilon}_n/16$. We consider an $(n-1)$-replaced

(n, n)-expansion of q. We consider $q'=q'_1+q'_2$ as in Lemmas 4.8 and 4.9. We assume that the case α_1^{nc} occurs that is $|[q']_{R_n}|_1>\frac{1}{2}$. We recall that a growth function is trivial up to the mth stage if it takes the value 1 for all $3k$-tuples, $k\leq m-1$, and has the lacunarity function $f\equiv 1$ and the $\delta=1$ for all $(3k+2)$-tuples, $k\leq m-1$.

We will assume that $|h|_1=1$ and that $|h|_{\mathrm{op}\,N}\leq 1/\varepsilon_n$. We let

$$h=\sum_{\substack{i,\alpha\\ j_k\geq n \text{ for all } k}} a_{i,J,\alpha}\, x^i l_{j_1}^{\alpha_{j_1}} l_{j_2}^{\alpha_{j_2}}\dots l_{j_r}^{\alpha_{j_r}}$$

be an $(n-1)$-substituted representation of h as for H in the previous lemma.

LEMMA $\alpha_1^{nc}, N(\alpha_1)$. *There is a growth function $F'_{N(\alpha_1)}$ which is trivial up to the $(n-1)$-st stage and numbers B'_n and m depending only on $|\ |^{n-1}$ such that if $\{D_n, L_n, l_n, C_n\}$ grows faster than $F'_{N(\alpha_1)}$ and the* case α_1^{nc} *occurs then there is an*

$$E=\sum_{\substack{i+|\alpha|\leq m\\ j_k\geq n \text{ for all } k}} e_{i,J,\alpha}\, x^i l_{j_1}^{\alpha_{j_1}} l_{j_2}^{\alpha_{j_2}}\dots l_{j_r}^{\alpha_{j_r}}$$

such that $|E(x)|_1\geq B'_n$ and $(hq')(x)$ contains $p_{1/50}(E)$.

Proof. We put $q'=q'_1+q'_2$ and put

$$h=\sum a_{i,J,\alpha}\, x^i l_{j_1}^{\alpha_{j_1}} l_{j_2}^{\alpha_{j_2}}\dots l_{j_r}^{\alpha_{j_r}}=h_1+h_2$$

where h_1 consists of those terms which contain only l's from the $n-1$ first systems and where each term in h_2 contains an l from a system with number $\geq n$. Now if $D_n>10/\varepsilon_n$ then $|h_2|_{1,f}\leq\frac{1}{10}$ and so $|h_1|_{1,f}\geq\frac{9}{10}$. Then by the Lemmas 4.8 and 4.10, h_1 and q'_1 satisfy the assumptions of Lemma 4.7 and so $(h_1 q'_1)(x)$ contains a $p_{1/100}(E)$ where E is as above. Now by writing $h_1 q'_2+h_2(q'_1+q'_2)$ in the form $\Sigma_j(\Sigma_m P'_{(m,j)})$ it follows easily from Lemma 4.9 that we have $|P'_{(m,j)}|_{1,f}\leq(D')^m K'_j$ and $|P'_{(m,j,w'_m)}|_{1,f}\leq K'_j/2^m$. Thus by Lemma 4.6 if $\{D_n, L_n, l_n, C_n\}$ grows sufficiently fast then the cancellation effect of $h_1 q'_2+h_2(q'_1+q'_2)$ on E is $<(1/100)|E|_1$. This concludes the proof of the lemma.

To estimate $|hq|^N$ we write $\Sigma_{j=1}^N v_j(l_j q_j-1)+v=V$ and the estimate $|V|_{\mathrm{est}}^N$ of $|hq|^N$ given by V is $\Sigma_{j=1}^N |v_j|_{\mathrm{op}\,N}\cdot\varepsilon_j+|v|_1$. We consider a pre-$n$-expansion of V. We form V' by removing all the terms of the third type such that the corresponding ε is $<\min\{\varepsilon_1, \ldots, \varepsilon_n\}$. We then form an $(n-1)$-replaced n-expansion of V'. We write $V'=V_1'+V_2'$ where V_1' consists of those terms out of the $(n-1)$-replaced n-expansion of V' which contain only l's from the $n-1$ first systems and where every term out of V_2' contains an l from a system with number $\geq n$. We write $V_1'=\Sigma\, V_{1,m}'$ and $V_2'=\Sigma_j(\Sigma_m P_{(m,j)})$ with the same notations as for q_1' and q_2'. We now have

LEMMA 4.11. *There are constants C' and D' depending only on $|\ |^{n-1}$ such that for all $N\geq n$ we have $|V_{1,m}'|_{1,f}\leq |V|C'\cdot(D')^m$.*

Proof. The proof is essentially the same as for Lemma 4.8. We observe that (4.24) will be replaced by

$$\sum |s_{j_2}s_{j_2}\ldots s_{j_r}|_{\mathrm{op}\,N}\leq |V|_{\mathrm{est}}^N\left(\frac{1}{\varepsilon}\right)^r,$$

and we have corresponding modifications later in the proof.

LEMMA 4.12. *There is a number D' and a sequence of integers w_m' depending only on $|\ |^{n-1}$ and numbers $K_j', j\geq n$, K_j' depending only on $|\ |^{j-1}$ such that if $|V|_{\mathrm{est}}^N\leq 1$ then for all $N\geq n$ we have for V_2',*

$$|P_{m,j}|_{1,f}\leq K_j'(D')^m\cdot|V|_{\mathrm{est}}^N$$

$$|P_{m,j,w_m'}|_{1,f}\leq \frac{K_j'}{2^m}\cdot|V|_{\mathrm{est}}^N$$

The proof is essentially the same as for Lemma 4.9.

With these lemmas we now easily complete the proof of Proposition $\mathfrak{a}_1^{\mathrm{nc}}, N(\mathfrak{a}_1)$.

Proof of Proposition $\mathfrak{a}_1^{\mathrm{nc}}, N(\mathfrak{a}_1)$. hq' contains $p_{1/50}(E)$ by Lemma $\mathfrak{a}_1^{\mathrm{nc}}, N(\mathfrak{a}_1)$. Now if $|V|_{\mathrm{est}}^N\leq 1$ then by the Lemmas 4.6 and 4.12 the cancellation effect of V_2' on E is $\leq(1/100)|E|$ and by Lemma 4.1 the same is true for $\Sigma_{j>m}V_{1,j}'$. So if $(hq'-V')(x)$ does not contain $p_{1/25}(E)$ then the cancellation effect of $\Sigma_{j\leq m}V_{1,j}'$ on E is $>(1/25)|E|_1$. By Lemma 4.11 this concludes the proof of the propostion with

$$B_n''=\frac{B_n'}{25\cdot C'\cdot\sum_{j\leq m}(D')^j}.$$

Proof of Lemma 3.2. We prove that the estimate holds for the terms derived from terms of the 3rd type in a shortened pre-$(n, b(m_2))$-expansion of q. These terms should have the form $s_{j_1} l_{j_1} s_{j_2} l_{j_2} \dots s_{j_{p-1}} l_{j_{p-1}} s_{j_p}$ with $p \leq r+1$, where $j_i > j$ for all $i \leq p-1$, $j_p \geq j$ and j_i, $i \leq p$, is from one of the $b(m_2)-1$ first systems. Since for every term $s_{j_1} l_{j_1} \dots l_{j_{p-1}} s_{j_p} l_j q_j$ in the shortened pre-$(n, b(m_2))$-expansion of q there is a term $s_{j_1} l_{j_1} \dots s_{j_{p-1}} s_{j_p}$ in the shortened pre-$(n, b(m_2))$-expansion of q this obviously gives the lemma.

We thus prove the estimate

$$\sum_{\substack{j_1, \dots, j_p \\ j_i > j \\ \text{system of } j_i \leq b(m_2)-1}} |s_{j_1} l_{j_1} \dots s_{j_p}(x, l)|_{\text{est op } N} \leq (2 \cdot 2^{m_2})^p \cdot L^p_{b(m_2)-1} \cdot D^p_{b(m_2)-1}, \tag{4.27}$$

which completes, the proof of the lemma.

We prove (4.27) by induction on p. To get it for $p=1$ we consider the first stage of the pre-$(n, b(m_2))$-expansion of q

$$q = q_n + \sum_{i=1}^{N} s_i(l_i q_i - 1) + t.$$

We observe that, since $j > n$ no l_i with $i > j$ will appear in any stage of the further expansion of q_n (or q_i, $i < n$), not even if the s's are substituted by polynomials in x and l's.

We observe further that any term of the 3rd type that "ends" with s_{j_p}, $j_p \geq j$, and which does not appear in the first stage of the pre-$(n, b(m_2))$-expansion, must contain at least one l_i with $i \geq j$. So to get (4.27) for $p=1$, it is enough to prove

$$\sum_{\substack{j_p \geq j \\ \varepsilon_{j_p} \geq \frac{1}{2^{m_2}}}} |s_{j_p}|_{\text{est op } N} \leq 2 \cdot 2^{m_2}$$

which is immediate from the assumption $|q-q_n|^N \leq \varepsilon_n/16 < 1$. Now we assume that (4.27) holds for an integer p and we prove it for $p+1$.

Every term of the 3rd type, say $s_{j_1} l_{j_1} \dots l_{j_p} s_{j_{p+1}}$ in the pre-$(n, b(m_2))$-expansion of q is obtained by expanding q_{j_p} in the term $s_{j_1} l_{j_1} \dots l_{j_p} q_{j_p}$. Since we assume $j_p > j$ and that j_p belongs to a system with number $\leqslant b(m_2)-1<n<j$ we have

$$q_{j_p} = q_r + \sum_{i=1}^{j_p-1} s_i'(l_i q_i - 1) + t', \quad \text{with } r \leqslant b(m_2)-1.$$

So, as above, we get that for fixed $s_{j_1} l_{j_1} \dots s_{j_p}$ we have

$$\sum |s_{j_{p+1}}|_{\text{est op } N} = \sum_{\substack{j_{p+1} \geqslant j \\ \varepsilon_{j_{p+1}} \geqslant \frac{1}{2^{m_2}}}} |s'_{j_{p+1}}|_{\text{est op } N} \leqslant 2 \cdot 2^{m_2}.$$

Since $|l_{j_p}|_{\text{est op } N} \leqslant D_{b(m_2)-1} L_{b(m_2)-1}$ we get (4.27) and the lemma is proved.

Proof of Lemma $\alpha^c_{1,\mathrm{B}}, m_2, N(\alpha_1, m_2)$. We write $h(x, \bar{l}) = h_1(x, \bar{l}) + h_2(x, \bar{l})$ with

$$h_1(x, \bar{l}) = \sum_{(j)} \bar{l}^{(j)}_{h_1} h_{1,(j)}(x, \bar{l}) \quad \text{and} \quad h_2(x, \bar{l}) = \sum_{(j)} \bar{l}^{(j)}_{h_2} h_{2,(j)}(x, \bar{l})$$

and

$$S_j(x, \bar{l}) = S_{1,j}(x, \bar{l}) + S_{2,j}(x, \bar{l}) = \sum_{(j)} \bar{l}^{(j)}_{S_1} S_{1,(j)}(x, \bar{l}) + \sum_{(j)} \bar{l}^{(j)}_{S_2} S_{2,(j)}(x, \bar{l})$$

where we have the following: Every l in $h_{1,(j)}, h_{2,(j)}, S_{1,(j)}$ or $S_{2,(j)}$ has index $\leqslant j$ and every l in $\bar{l}^{(j)}_{h_1}, \bar{l}^{(j)}_{h_2}, \bar{l}^{(j)}_{S_1}$, or $\bar{l}^{(j)}_{S_2}$, has index $>j$. For every $\bar{l}_i$ in $\bar{l}^{(j)}_{h_1}$, $\bar{l}^{(j)}_{S_1}$, $h_{1,(j)}$ or $S_{1,(j)}$, i belongs to some of the $b(m_2)-1$ first systems and in every $\bar{l}^{(j)}_{h_2}$, $\bar{l}^{(j)}_{S_2}$, $h_{2,(0)}$ or $S_{2,(0)}$ there is an $\bar{l}_i$ with i belonging to a system with number $\geqslant b(m_2)$. For $|j|=0$ say $(j)=0$, we put $S_{2,(0)}(x, \bar{l}) \equiv 0$ and we observe that $S_{1,(0)}(x, \bar{l})$ consists of terms out of $S_j(x, \bar{l})$ and that in fact $S_{1,(0)} = S_{j,0}$.

To prove Lemma $\alpha^c_{1,\mathrm{B}}, m_2, N(\alpha_1, m_2)$ it is obviously enough to prove the conclusion for the product $h_1(x, \bar{l}) S_{1,j}(x, \bar{l})$. To do this we first observe that $|h_2(x)| < \frac{1}{10}$ if $D_{b(m_2)} > 10 \cdot 2^{m_2} > 10/\varepsilon_n$ since $|h|_{\text{est op } N} / |h(x)| \leqslant 1/\varepsilon_n$ and so $|h_1(x)|_1 > \frac{9}{10}$.

Since $|h|_{\text{est op } N} \leqslant 1/\varepsilon_n$, we have that

$$\sum_{|j|>r} |h_{1,(j)}(x)|_1 < \frac{1}{\varepsilon_n \cdot 2^r}$$

and so for some $r_{1,1}<10m_2$ we have

$$\left|\sum_{|j|=r_{1,1}} (l^{(j)} h_{1,(j)})(x)\right|_1 > \left(\frac{9}{10}-\frac{1}{10}\right)\cdot\frac{1}{10m_2}.$$

Obviously for the same $r_{1,1}$ we have

$$\sum_{|j|=r_{1,1}} \left|l^{(j)} h_{1,(j)}(x,l)\right|_{\text{est op } N} \leq \frac{1}{\varepsilon_n}.$$

This gives by the multiplication theorem that there is a constant $E_{1,1}>2$ depending only on m_2 such that for the product

$$\left(\sum_{|j|=r_{1,1}} l_{h_1}^{(j)} h_{1,(j)}\right) s_{1,(0)} = \sum_{|j|=r_{1,1}} l_{h_1}^{(j)} c_{1,(j),i,(k)} x^i l^{(j)}$$

we have

$$E_{1,1}\left|s_j(x)\right|_1 \geq \sum_{|j|=r_{1,1}} \left(\left|\sum c_{1,(j),i,(k)} x^i l^{(k)}\right|_{\text{est op } N}\right)$$

$$\geq \sum_{|j|=r_{1,1}} \left|\left(\sum_{i,(k)} c_{1,(j),i,(k)} x^i l^{(k)}\right)(x)\right|_1 \geq \frac{1}{E_{1,1}}\left|s_j(x)\right|_1.$$

Now two possibilities can occur. Either the terms

$$\sum_{|j|=r_{1,1}} l^{(j)}\left(\sum_{i,(k)} c'_{1,(j),i,(k)} x^i l^{(k)}\right)$$

out of the product

$$\left(\sum_{|j|<r_{1,1}} l_{h_1}^{(j)} h_{1,(j)}\right)\left(\sum_{|j|>0} l_1^{(j)} s_{1,(j)}\right)$$

satisfy

$$\sum_{|j|=r_{1,1}} \left|\sum_{i,(k)} c'_{1,(j),i,(k)} x^i l^{(k)}\right|_{\text{est op } N} \leq \frac{1}{2E_{1,1}}\left|s_j(x)\right|_1$$

or satisfy

$$\sum_{|j|=r_{1,1}} \left|\sum_{i,(k)} c'_{1,(j),i,(k)} x^i l^{(k)}\right|_{\text{est op } N} > \frac{1}{2E_{1,1}}\left|s_j(x)\right|_1.$$

In the first case the lemma follows with $E_1=2E_{1,1}$ since always $|\ |_{\mathrm{est\,op}\,N}\geq |\ |_1$, and for the other terms out of $h_1 S_{1,j}$ we have $|j|>r_{1,1}$. In the second case we have

$$\left|\left(\sum_{|j|<r_{1,1}} l^{(j)} \sum h_{1,(j)}\right)(x)\right|_1 \cdot \sum_{r<r_{1,1}} |K_{j,r}|_{\mathrm{est\,op}\,N} > \frac{1}{2E_{1,1}} |s_j(x)|_1$$

which gives

$$\left|\left(\sum_{|j|,r_{1,1}} l^{(j)} \sum h_{1,(j)}(x)\right)\right|_1 > \frac{1}{2E_{1,1}} \cdot \frac{1}{8\cdot 2^{m_2-m_1} \sum\limits_{r<r_{1,1}} (2\cdot 2^{m_2})^r L^r_{b(m_2)-1} D^r_{b(m_2)-1} \cdot 10^r}$$

$$> \frac{1}{2E_{1,1}} \cdot C_{m_2} \cdot \frac{1}{L^{m_2}_{b(m_2)-1} D^{m_2}_{b(m_2)-1}}$$

with obvious notations.

Thus there is a number $r_{1,2}<r_{1,1}$ such that

$$\left|\left(\sum_{|j|=r_{1,2}} l^{(j)} \sum h_{1,(j)}\right)(x)\right|_1 > \frac{1}{2E_{1,1}} \cdot \frac{C_{m_2}}{m_2} \cdot \frac{1}{L^{m_2}_{b(m_2)-1} D^{m_2}_{b(m_2)-1}}.$$

We can now repeat the argument above with $r_{1,2}$ instead of $r_{1,1}$ and so the result follows.

Proof of Proposition $\mathfrak{a}^{c}_{1,B}, m_2, N(\mathfrak{a}_1, m_2)$. We observe that the only terms out of coefficients of $l_j q_j$'s in V, $\varepsilon_j=1/2^{m_2}$, that can contribute to (3.3) are those which are derived from terms of the form $v_{j_1} l_{j_1} s_{j_2} l_{j_2} \dots l_{j_r} s_j$ where $j_1, j_2, \dots, j_r$ are in the $b(m_2)-1$ first systems and $r\leq r_1\leq n_1$. The sum of the est op N-norms of all such terms is

$$\leq |V| \cdot L^{n_1}_{b(m_2)-1} D^{n_1}_{b(m_2)-1} (2\cdot 2^{m_2})^{n_1+1}$$

by similar arguments as above. This gives the propositon.

Proof of Proposition $\mathfrak{a}^{nc}_{1,B}, m_2, N(\mathfrak{a}_1, m_2)$. We assume $|V|^N_{\mathrm{est}}<1$. First consider the sum S_1 of the est op N-norms of all monomials derived by a $(b(m_2)-1)$-replacement from the following types of terms: $hs_{j_1} l_{j_1} \dots l_{j_r} s_{j_{r+1}}$ or $v_{j_1} l_{j_1} \dots l_{j_r} s_{j_{r+1}}$ obtained in the pre-$(n, b(m_2))$-expansion of q and $b(m_2)$-expansion of V, with $r\leq m$ and with every l_{j_i} from one of the $b(m_2)-1$ first systems. Like in the proof of Lemma 3.2. we obtain

$$S_1 \leqslant 2\cdot(2\cdot 2^{m_2})^{m+1}\cdot L^m_{b(m_2)-1} D^m_{b(m_2)-1}.$$

This gives that there is a function $\gamma(k)\to 0$ as $k\to\infty$ such that the sum of l_1-norms of monomials of degree k in x and l's is $<\gamma(k)$. Thus by Lemmas 4.1, 4.4 and 4.5 the sum of the cancellation effects on $E(x)$ of the monomials which are of degree $>m$ in $\bar{l}_i$'s or contain an $\bar{l}_i$ from a system with number $\geqslant b(m_2)$ is

$$<\frac{|E(x)|_1}{30\cdot 2\cdot 2^{m_2-m_1}}.$$

Now for each $r>m$ we consider the sum S_{2r} of the est op N-norms of all monomials derived by a $(b(m_2)-1)$-replacement from the following types of terms: $hs_{j_1}l_{j_1}\dots l_{j_r}s_{j_{r+1}}$ or $v_{j_1}l_{j_1}\dots l_{j_r}s_{j_r}+1$ obtained in the pre-$(n, b(m_2))$-expansion of q and $b(m_2)$-expansion of V with every j_i from one of the $b(m_2)-1$ first systems.

Like in the proof of Lemma 3.2 we obtain

$$S_{2r}\leqslant(2\cdot 2^{m_2})^{r+1} L^r_{b(m_2)-1} D^r_{b(m_2)-1}.$$

Thus we can use Lemma 4.1 with $F_r=(2\cdot 2^{m_2})^{r+1} L^r_{b(m_2)-1} D^r_{b(m_2)-1}$. and get that the sum of the cancellation effects on $E(x)$ of the monomials which are of degree $>m$ in $\bar{l}_i$'s or contain an $\bar{l}_i$ from a system with number $\geqslant b(m_2)$ is

$$<\frac{|E(x)|_1}{30\cdot 2\cdot 2^{m_2-m_1}}.$$

We finally consider the sum P of all monomials derived by a replacement described below from the following types of terms: $s_{j_1}l_{j_1}\dots l_{j_r}s_{j_{r+1}}$ or $v_{j_1}l_{j_1}\dots l_{j_r}s_{j_{r+1}}$ where some j_i belongs to a system with number $\geqslant b(m_2)$. In every j_i which belongs to a system with number $\geqslant b(m_2)$ we consider those terms $s_{j_1}l_{j_1}\dots l_{j_i}\dots l_{j_r}s_{j_{r+1}}$ or $v_{j_1}l_{j_1}s_{j_2}\dots l_{j_i}\dots l_{j_r}s_{j_{r+1}}$ where $j_1,j_2,\dots,j_{i-1}$ all belong to a system with number $\leqslant b(m_2)-1$. For every such term we rewrite $s_{j_1}s_{j_2}\dots s_{j_{i-1}}s_{j_i}$ and $v_{j_1}s_{j_1}s_{j_2}\dots s_{j_i}$ as a polynomial in x, we rewrite every $s_{j_{i+1}}l_{j_{i+1}}\dots l_{j_r}s_{j_{r+1}}$ as a polynomial in x ($l_{j_1}, l_{j_2},\dots, l_{j_r}, s_{j_{r+1}}$ we do not rewrite). With $j_i=j$ and $i-1=m$ we then rewrite $P=\Sigma_j(\Sigma_{m\geqslant 0}P_{m,j})$ where $P_{m,j}$ consists of all monomials $x^i l_{j_1}\dots l_{j_m}l_j$. We can now apply Lemma 4.6 in the following way:

D is the set of integers $\{j\}$ such that j belongs to a system with number $\geqslant b(m_2)$. G_j

is the l_1-sum as polynomials in x of all terms $s_{j_{i+1}} l_{j_{i+1}} \dots s_{j_{r+1}}$ which appear in the expansion of q_{j_i}. G_j is obviously determined by $| \, |^{j_i-1}$.

$$F_m = 2 \cdot L^m_{b(m_2)-1} D^m_{b(m_2)-1} (2 \cdot 2^{m_2})^{m+1}.$$

Then $|P_{m,j}|_{1,f} \leqslant G_j F_m$ by the same arguments as in Lemma 3.2 and obviously $|P_{m,j,w_m}|_{1,f} < G_j$ for $w_m \geqslant m+1$. Thus the cancellation effect of $P(x)$ on $E(x)$ is

$$\leqslant \frac{|E(x)|_1}{30 \cdot 2 \cdot 2^{m_2 - m_1}}$$

and the proposition is proved.

Proof of Lemma 3.3. We omit this proof since it is the same as of Lemma 3.2. We now turn to the

Proof of Proposition $\mathfrak{a}^*_{1,\mathrm{B}}, m_{2,\mathrm{B}}, \mathfrak{a}_2^{\mathrm{nc}}, N(\mathfrak{a}_1, \mathfrak{a}_2)$. We consider (3.16) and we assume $|V|^N_{\mathrm{est}} < 1$. The proof of the proposition will be completed from Lemma 4.17 below. To prove that lemma we first prove Lemma 4.13. To prove Lemma 4.13 we use the Lemmas 4.14 and 4.15.

For every j there is an $n_{1,j} \leqslant n_1$ such that for the part

$$G_i = \sum_{|\alpha| = n_{1,j}} l^{(\alpha)} \sum d_{(\alpha), m, (\beta)} x^m l^{(\beta)}$$

of $hS_j - V_j$ we have

$$|2E_1^2| \sum \left(\sum \left| d_{(\alpha), m, (\beta)} x^m l^{(\beta)} \right|_1 \right) \geqslant \sum \left| \sum d_{(\alpha), m, (\beta)} x^m l^{(\beta)} \right|_{\mathrm{est\,op}\,N}$$

where n_1 is given by Lemma $\mathfrak{a}^{\mathrm{c}}_{1,\mathrm{B}}, m_2, N(\mathfrak{a}_1, m_2)$ and $n_{1,j}$ is the r_1 of that lemma. We put $G_j = G_{j,1} + G_{j,2}$ where $G_{j,1} = \Sigma\, l^{(\alpha)} \Sigma\, d_{(\alpha), m, (\beta)} x^m l^{(\beta)}$ where the summation is extended over those (α) for which

$$10 E_1^2 \left| \sum d_{(\alpha), m, (\beta)} x^m l^{(\beta)} \right|_1 > \left| \sum d_{(\alpha), m, (\beta)} x^m l^{(\beta)} \right|_{\mathrm{est\,op}\,N}$$

and $G_{i,2}$ is the sum extended over the other (j)'s. We let B_j be defined by

$$HS_j - V_j = G_{j,1} + G_{j,2} + B_j.$$

We can now apply Lemma $\alpha_{\mathrm{I}}^{\mathrm{nc}}, N(\alpha_1)$ to the product $(\Sigma\, d_{(\alpha),m,(\beta)} x^m l^{(\beta)})\, q_j'$ for every fixed (α) appearing in $G_{j,1}$. Then $\Sigma\, d_{(\alpha),m,(\beta)} x^m l^{(\beta)}$ will play the role of h and q_j' the role of q'. Here, of course, $\Sigma\, d_{(\alpha),m,(\beta)} x^m l^{(\beta)}$ is not normalized as h in Lemma $\alpha_{\mathrm{I}}^{\mathrm{nc}}, N(\alpha^1)$, but the important thing is that $\Sigma\, d_{(\alpha),m,(\beta)} x^m l^{(\beta)}$ like h in Lemma $\alpha_{\mathrm{I}}^{\mathrm{nc}}, N(\alpha^1)$, has a bound $10\, E_1^2$ on the ratio between its $|\;|_{\mathrm{op}\,\mathcal{N}}$-norm and its $|\;|_1$-norm. The $(n-1)$st stage of Lemma $\alpha_{\mathrm{I}}^{\mathrm{nc}}, \mathrm{N}(\alpha^1)$, will here be the $(b(\alpha_2)-1)$st stage. So this gives us that there exist n_2 and $B''_{b(\alpha_2)}$ depending only on $|\;|^{b(\alpha_2)-1}$ such that $\Sigma\, d_{(\alpha),m,(\beta)} x^m l^{(\beta)}\, q_j'$ contains

$$P_{1/50}(E_{(\alpha),1,j}) = p_{1/50} \sum e_{(\alpha),k,(\beta)} x^k l^{(\beta)}, \quad k+|\beta| \leqslant n_2$$

where

$$|E_{(\alpha),1,j}|_1 \geqslant B''_{b(\alpha_2)} \left| \sum d_{(\alpha),m,(\beta)} x^m l^{(\beta)} \right|_1 .$$

Now by Lemma $\alpha_{\mathrm{I},\mathrm{B}}^{\mathrm{c}}, m_3, N(\alpha_1, m_2)$ and the definition of weight of good coefficient and the definition of $(hS_j - V_j)\, l_j q_j'$ in (3.16) and the definition of $G_{j,1}$ there is a constant C depending only on $|\;|^{b(m_2)-1}$ such that

$$\sum_j \left(\sum_{(\alpha)} \left(\left| \sum d_{(\alpha),m,(\beta)} x^m l^{(\beta)} \right|_1 \right) \right) \geqslant C \cdot D,$$

where D is the sum of the weights of good coefficients of $l_j q_j'$s with q_j in the system a_2. Now we define

$$E = \sum_j l_j \left(\sum_{(\alpha)} l^{(\alpha)} E_{(\alpha),1,j} \right).$$

With this definition we get $|E|_1 \geqslant D \cdot C \cdot B''_{b(\alpha_2)} \cdot L_{b(\alpha_2)}$ and so this gives that E has the properties 1–5 above with $B'_{b(\alpha_2)} = C \cdot B''_{b(\alpha_2)}$. To prove that $(hq-V)'(x)$ contains $p_{1/20}(E)$ we start by proving

LEMMA 4.13. *For every j, $(hS_j - V_j) l_j q_j'$ contains*

$$p_{1/24}\left(l_j \sum_{(\alpha)} l^{(j)} E_{(\alpha),1,j} \right).$$

To prove Lemma 4.13 we first prove

LEMMA 4.14. *For every j*

$$\sum_{(\alpha)} \bar{l}^{(\alpha)} l_j \left(\sum d_{(\alpha),m,(\beta)} x^k l^{(\beta)} q_i' \right)(x) \quad \textit{contains} \quad p_{1/30}\left(\sum_{(\alpha)} (l^{(\alpha)} l_j E_{(\alpha),1,j}) \right).$$

To prove this we put

$$\sum d_{(\alpha),m,(\beta)} x^m l^{(\beta)} q_j' = \sum_{i=0}^{m'} a_i x^i + \sum_{i>m'} a_i x^i$$

where m' is chosen so that $\deg(\Sigma e_{k,(\beta)} x^k l^{(\beta)}) \leq m'$. We can choose $m' = n_2 \deg l_{j-1}$. Then we obviously have that $\Sigma_0^{m'} a_i x^i$ contains $p_{1/50}(E_{(\alpha),1,j})$ and

$$\sum_{(\alpha)} \left(\bar{l}^{(\alpha)} l_j \sum_{i=0}^{m'} a_i x^i \right)$$

contains

$$p_{1/50}\left(\sum_{(\alpha)} (\bar{l}^{(\alpha)} l_j E_{(\alpha),1,j}) \right) \quad \text{if} \quad \operatorname{ord} l_j > n_2 \deg l_{j-1}$$

which certainly is true if the sequence is sufficiently lacunary. We have

$$\left| \sum_{i>m'} a_i x^i \right|_1 \leq \left| \sum_{i\geq 0} a_i x^i \right|_1 \leq \left| \sum d_{(\alpha),m,(\beta)} x^m l^{(\beta)} \right|_1 \cdot |q_j'|_1 \leq \frac{1}{B''_{b(m_2)}} \cdot |E_{(\alpha),1,j}|_1 \cdot |q_j'|_1.$$

Now we apply Lemma 4.3 in the following way:

$S(x)$ of the lemma is $l_j \Sigma_{(\alpha)} (\bar{l}^{(\alpha)} \Sigma_{i\leq m'} a_i x^i)$. $P(x)$ of the lemma is $l_j \Sigma_{(\alpha)} (\bar{l}^{(\alpha)} \Sigma_{i>m'} a_i x^i)$, $w = |i| + 1$. Since we have not normalized we get that the cancellation effect of $P(x)$ on $S(x)$ is $\leq \varepsilon \cdot |E_{(\alpha),1,j}|_1 \cdot L_{b(a_2)}$ if the sequence $\{l_i\}_{i\geq j}$ is sufficiently lacunary and the moduli of the coefficients decrease sufficiently rapidly. With $\varepsilon = 1/30 - 1/50$ this gives Lemma 4.14.

To continue the proof of Lemma 4.13 we will now study the cancellation effect of $(G_{j,2} + B_j) l_j q_j'$ on $l_j (\Sigma_{(\alpha)} l^{(\alpha)} E_{(\alpha),1,j})$. To do that we will use the Lemma 4.15 below. That lemma will be used both in studying $(hS_j - V_j) l_j q_j'$ and $(hS_j - V_j) l_j q_j$.

Fix a j and consider $hS_j - V_j$ as in (3.16). Expand $hS_j - V_j$ as a polynomial in x and $\bar{l}$'s and do not substitute any $\bar{l}$ by a polynomial in x. We can now write

$$(hS_j-V_j)(x,\bar{l}) = \sum_{r\geq 0}\left(\sum_{\substack{J,\gamma \\ |J|=r}} F_{J,\gamma}(x,\bar{l})\, l_{j_1}^{\gamma_{j_1}} l_{j_2}^{\gamma_{j_2}} \dots l_{j_r}^{\gamma_{j_r}}\right)$$
$$= \sum_{r\geq 0}\left(\sum_{\substack{J,\gamma \\ |J|=r}} F'_{J,\gamma}(x,\bar{l})\, l_{j_1}^{\gamma_{j_1}} l_{j_2}^{\gamma_{j_2}} \dots l_{j_r}^{\gamma_{j_r}}\right) + \sum_{r\geq 0}\left(\sum_{\substack{J,\gamma \\ |J|=r}} F''_{J,\gamma}(x,\bar{l})\, l_{j_1}^{\gamma_{j_1}} l_{j_2}^{\gamma_{j_2}} \dots l_{j_r}^{\gamma_{j_r}}\right)$$

with the following notations: $l_{j_1}, l_{j_2}, \dots, l_{j_r}$ are in systems with number $\leq b(\mathfrak{a}_2)-1$ and $j_1, j_2, \dots, j_r$ are all $>j$. $F_{J,\alpha}(x,l)$ is a polynomial in x and $\bar{l}$'s there for each $\bar{l}_i$, either i belongs to a system with number $\geq b(\mathfrak{a}_2)$ or $i\leq j$ (or both). $F'_{J,\gamma}(x,\bar{l})$ is a polynomial in x and $\bar{l}$'s where all l's have index $\leq j$. $F''_{J,\gamma}(x,\bar{l})$ is a polynomial in x and $\bar{l}$'s where in each term for at least one $\bar{l}_i$, $i>j$ and $\bar{l}_i$ belongs to a system with number $\geq b(\mathfrak{a}_2)$.

(4.28) We observe that for those j for which $G_{j,1}$ is defined we have

$$G_{j,1} = \sum_{J,\gamma} F'_{J,\gamma}(x,l)\, l_{j_1}^{\gamma_{j_1}} l_{j_2}^{\gamma_{j_2}} \dots l_{j_r}^{\gamma_{j_r}}$$

if the sum is extended over appropriate J and α and every l with index $\leq b(\mathfrak{a}_2)-1$ in F' is substituted by a polynomial in x.

(4.29) We also observe that for every r (with no l's substituted)

$$\sum \left| F_{J,\gamma}(x,l)\, l_{j_1}^{\gamma_{j_1}} l_{j_2}^{\gamma_{j_2}} \dots l_{j_r}^{\gamma_{j_r}} \right|_{\text{est op } N}$$
$$= \sum \left| F'_{J,\gamma}(x,l)\, l_{j_1}^{\gamma_{j_1}} l_{j_2}^{\gamma_{j_2}} \dots l_{j_r}^{\gamma_{j_r}} \right|_{\text{est op } N} + \sum \left| F''_{J,\gamma}(x,l)\, l_{j_1}^{\gamma_{j_1}} l_{j_2}^{\gamma_{j_2}} \dots l_{j_r}^{\gamma_{j_r}} \right|_{\text{est op } N}.$$

LEMMA 4.15. *For every i and every fixed* $r\geq 0$,

$$\sum_{\substack{J,\gamma \\ |J|=r}} \left| F_{J,\alpha}(x,l)\, l_{j_1}^{\gamma_{j_1}} l_{j_2}^{\gamma_{j_2}} \dots l_{j_r}^{\gamma_{j_r}} \right|_{\text{est op } N} \leq 4\cdot\left(\frac{1}{\varepsilon}\right)^{r+1} \left(L_{b(\mathfrak{a}_2)-1} D_{b(\mathfrak{a}_2)-1}\right)^r,$$

where $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_{b(\mathfrak{a}_2)}\}$.

The proof is the same as for Lemma 3.2.

Now we use this lemma to estimate the cancellation effect of $(G_{j,2}+B_j)\,l_j q_j'$ on $l_j \Sigma_{(\alpha)} \bar{l}^{(\alpha)} E_{(\alpha),1,j}$ which will give Lemma 4.13. By (4.28) and (4.29) we have if in $G_{j,2}$ and B_j no l is replaced by a polynomial in x,

$$G_{j,2}+B_j=\sum_{r\geq 0}\sum_{\substack{J,\gamma\\ |J|=r\\ J,\gamma\notin G_{j,1}}} F'_{J,\gamma}(x,\gamma)\,l_{j_1}^{\gamma_{j_1}} l_{j_2}^{\gamma_{j_2}}\dots l_{j_r}^{\gamma_{j_r}}+\sum_{r\geq 0}\left(\sum_{\substack{J,\gamma\\ |J|=r}} F''_{J,\gamma}(x,\bar{l})\,l_{j_1}^{\gamma_{j_1}} l_{j_2}^{\gamma_{j_2}}\dots l_{j_r}^{\gamma_{j_r}}\right),$$

where the first sum is extended only over those J,γ which do not enter in $G_{j,1}$. Now with

$$F'''=\sum_{r=0}^{n_1+n_2+1}\sum F'_{J,\gamma}(x,\bar{l})\,l_{j_1}^{\gamma_{j_1}} l_{j}^{\gamma_{j_2}}\dots l_{j_r}^{\gamma_{j_r}}+\sum\sum F''_{J,\gamma}(x,\bar{l})\,l_{j_1}^{\gamma_{j_1}} l_{j_2}^{\gamma_{j_2}}\dots l_{j_r}^{\gamma_{j_r}}$$

we have by Lemma 4.15

$$|F'''|_{\text{est op}\,N}\leq 8\cdot\left(\frac{1}{\varepsilon}\right)^{n_1+n_2+2}\left(L_{b(a_2)-1}D_{b(a_2)-1}\right)^{n_1+n_2+1}.$$

This gives that for every $\varepsilon>0$ there is an m'' depending only on ε and $|\ |^{j-1}$ so that with $F'''=\Sigma_{i,\gamma}\,a_{i,\gamma}x^i\,\bar{l}_1^{\gamma_1}\,\bar{l}_2^{\gamma_2}\dots\bar{l}_N^{\gamma_N}$ we have that

$$\sum_{j+|\gamma|>m''}|a_{j,\gamma}|<\varepsilon\left(\sum_{(\alpha)}|E_{(\alpha),1,j}|_1\right)\cdot\frac{1}{q_j'}. \tag{4.32}$$

To see that m'' only depends on ε and $|\ |^{j-1}$ we observe that given $|\ |^{j-1}$ we have an estimate from below on $\Sigma_{(\alpha)}|E_{(\alpha),1,j}|_1$ and precise information on $|q_j'|_1$. Now we have that the cancellation effect of

$$\left(\sum_{j+|\gamma|\leq m''}a_{i,\gamma}x^i\,\bar{l}_1^{\gamma_1}\,\bar{l}_2^{\gamma_2}\dots\bar{l}_N^{\gamma_N}\right)l_j q_j' \quad\text{on}\quad l_j\sum_{(\alpha)}\bar{l}^{(\alpha)}E_{(\alpha),1,j}$$

is 0. To see this we can argue as follows:

Make an $(j-1)$-substitution in $l_j\Sigma\bar{l}^{(\alpha)}E_{(\alpha),1,j}$ and in

$$\left(\sum_{j+|\gamma|\leq m''}a_{i,\gamma}x^i\,\bar{l}_1^{\gamma_1}\,\bar{l}_2^{\gamma_2}\dots\bar{l}_N^{\gamma_N}\right)l_j q_j'.$$

Then both these expressions have bounded degrees as polynomials in x and l's, and the bounds depend on $|\ |^{j-1}$. And by definition, a monomial that appears in one of them does not appear in the other. Combining this with (4.30) we get that the cancellation effect of $F''' l_j q_j'$ on $l_j \Sigma_{(\alpha)} \bar{l}^{(\alpha)} E_{(\alpha),1,j}$ is $\leq L_{b(a_2)}(\Sigma |E_{(\alpha),1,j}|_1)$.

To prove Lemma 4.13 we have finally to prove that the cancellation effect of

$$F^{\mathrm{IV}} l_j q_j' = \sum_{r>n_1+n_2+1} \sum \left(\sum_{\substack{J,\gamma \\ |J|=r}} F_{J,\gamma}(x,l)\, \bar{l}_{j_1}^{\gamma_{j_1}} \bar{l}_{j_2}^{\gamma_{j_2}} \dots \bar{l}_{j_r}^{\gamma_{j_r}} \right) l_j q_j'$$

on $l_j \Sigma_{(\alpha)} \bar{l}^{(\alpha)} E_{(\alpha),1,j}$ is sufficiently small. To see this we make an $(j-1)$-substitution of $F^{\mathrm{IV}} l_j q_j'$ say

$$F^{\mathrm{IV}} l_j q_j' = l_j \sum a_{i,J,\gamma} x^i \bar{l}_{j_1}^{\gamma_{j_1}} \bar{l}_{j_2}^{\gamma_{j_2}} \dots \bar{l}_{j_r}^{\gamma_{j_r}}$$

and an $(j-1)$-substitution of $l_j \Sigma_{(\alpha)} \bar{l}^{(\alpha)} E_{(\alpha),1,j}$, say

$$l_j \sum b_{i,J,\alpha} x^i \bar{l}_{j_1}^{\alpha_{j_1}} \bar{l}_{j_2}^{\alpha_{j_2}} \dots \bar{l}_{j_m}^{\alpha_{j_m}}.$$

Now by Lemma 4.15 the conditions of Lemma 4.1 are fulfilled (except for normalization of l_j) and so the cancellation effect of $F^{\mathrm{IV}} l_j q_j'$ on $l_j \sum l^{(\alpha)} E_{(\alpha),1,i}$ is

$$< \delta L_{b(a_2)} \left(\sum_{(\alpha)} |E_{(\alpha),1,j}|_1 \right).$$

With $\delta = 1/10^4$ this completes the proof of Lemma 4.13.

The next step is to prove that $\Sigma_j (hS_j - V_j)\, l_j q_j'$ contains $p_{1/23}(\Sigma l_j (\Sigma \bar{l}^{(\alpha)} E_{(\alpha),1,j}))$. To do this we first prove

LEMMA 4.16. *Let the support of a polynomical in x be the set of exponents. Assume that $\{A_i\}$ and $\{B_i\}$ are sequences, finite or infinite, of polynomials in x. Assume that the B_i's are mutually disjointly supported. Assume also that A_i contains $p_\delta(B_i)$ and that the cancellation effect of A_i on $\Sigma_{j\neq i} B_j$ is $<\varepsilon |B_i|_1$. Then ΣA_i contains $p_{\delta+\varepsilon}(\Sigma B_i)$.*

Proof. Put $A_i = p_\delta(B_i) + C_i + D_i$ where

$$\operatorname{supp} C_i \subset \operatorname{supp} \Sigma_{j\neq i} B_j \quad \text{and} \quad \operatorname{supp} D_i \cap \operatorname{supp} \Sigma_j B_j = \emptyset$$

Then by assumption the cancellation effect of C_i on ΣB_j is $\leqslant e|B_i|_1$. We have $\Sigma A_i = p_\delta(B_i) + \Sigma C_i + \Sigma D_i$. The lemma follows since the cancellation effect of ΣD_i on ΣB_i is 0 and the cancellation effect of ΣC_i on ΣB_i is $<\varepsilon|B_i|_1$. We now prove

LEMMA 4.17. *$\Sigma_j (hS_j - V_j)\, l_j q_j'$ contains $p_{1/23}(\Sigma l_j \Sigma \bar{l}^{(a)} E_{(a),1,j})$.*

To prove this we first consider the cancellation effect of $(hS_j - V_j)\, l_j q_j'$ on

$$\sum_{r \neq j} l_r \left(\sum l^{(a)} E_{(a),1,r} \right) = \sum_{r \neq j} l_r \left(\sum \bar{l}^{(a)} \sum e_{k,(m)} x^k \bar{l}^{(m)} \right).$$

We make a $(j-1)$-substitution in $(hS_j - V_j)\, l_j q_j'$ and in

$$\sum_{r \neq j} l_r \left(\sum_{(a)} \bar{l}^{(a)} \sum e_{k,(m)} x^k l^{(m)} \right)$$

and replace q_j' by a polynomial in x. Then obviously l_j will not appear in

$$\sum_{r \neq j} l_r \left(\sum_{(a)} \bar{l}^{(a)} \sum_{k,(m)} e_{k,(m)} x^k l^{(m)} \right)$$

since every l in every $l^{(\alpha)}$ belongs to a system with number $\leqslant b(m_2) - 1$ and every l in every $\bar{l}^{(m)}$ belongs to a system with number $\leqslant b(a_2) - 1$. This makes it possible to apply Lemma 4.6 where the set D consists of the one integer j. $S(x)$ of the lemma will be the $(j-1)$-substitution of

$$\sum_{r \neq j} l_r \left(\sum_{(a)} \bar{l}^{(a)} \sum_{k,(m)} e_{k,(m)} x_k \bar{l}^{(m)} \right)$$

and $P_{(m,j)}$ will be the sum of all monomials out of the $(j-1)$-substituted expansion of $(hS_j - V_j)\, l_j q_j'$ with q_j' replaced by a polynomial in x, which are of degree m in l's with index $>j$. Sequences W_m and F_m depending only on $|\ |^{j-1}$ can now be determined by Lemma 4.12.

(4.33) Thus, given δ the cancellation effect of $(hS_j - V_j)\, l_j q_j'$ on

$$\sum_{r \neq j} l_r \left(\sum l^{(a)} \sum_{k,(m)} e_{k,(m)} x_k \bar{l}^{(m)} \right)$$

will be

$$\leqslant \delta L_{b(a_2)}\left(\sum\nolimits_{(\alpha)}|E_{(\alpha),1,j}|_1\right).$$

By applying Lemma 4.16 we now get Lemma 4.17.

(4.34) We now investigate the cancellation effect of $\Sigma_j(hS_j-V_j)\,l_j q_j$ on E. Given any sequence $\delta_j\searrow 0$ we prove that the cancellation effect of $(hS_j-V_j)\,l_j q_j$ on E is $<\delta_j$ in exactly the same way as (4.33) is proved.

We now finally investigate the cancellation effect of $hR'-R'_V$ on E. We first observe that all terms in hR' and R'_V are derived from terms in which the l_i's that appear have i from one of the $b(a_2)-1$ first systems and the q_i's that appear have $i\leqslant b(a_2)-1$. Thus, assuming $|V|<1$, we get by the same argument as in the proof of Lemma 3.2 that the sum of l_1-norms of all terms derived from terms that contain at most n_1+n_2+1 $(m+n_2+1)$ different l_i's in hR' or R'_V is

$$\leqslant \max_{j\leqslant b(a_2)-1}(|q_j|_1)\cdot(2\cdot 2^{m_k})^{n_1+n_2+2}\cdot L_{b(a_2)-1}^{n_1+n_2+1}\cdot\left(\max_{j\leqslant b(a_2)-1}(|q_j|_1)\cdot(2\cdot 2^{m_k})^{m+n_2+2}\cdot L_{b(a_2)-1}^{m+n_2+1}\right)$$

and this is much smaller than $|E|_1/200$ since $|E|_1$ contains the factor $L_{b(a_2)}$.

Now Lemma 4.1 gives that the cancellation effect of the terms derived from terms that contain $>n_1+n_2+1$ $(>m+n_2+1)$ different l's is $|E|_1/200$ with F_r of that lemma

$$=\max_{j\leqslant b(a_2)-1}|q_j|_1\cdot(2\cdot 2^{m_k})^{r+1}L_{b(a_2)-1}^{r}.$$

This completes the proof of Proposition $\mathfrak{a}^*_{1,\mathrm{B}}, m_{2,\mathrm{B}}, \mathfrak{a}_2^{\mathrm{nc}}, N(\alpha_1,\mathfrak{a}_2)$.

Proof of Lemma $\mathfrak{a}^*_{1,\mathrm{B}}\, m_{2,\mathrm{B}}, \mathfrak{a}^{\mathrm{c}}_{2,B}, m_3, N(\alpha_1,\mathfrak{a}_2,m_3)$. Consider the terms (3.3) or (3.8)

$$G_j(x,\bar l)=\sum_{|\alpha|=r_1}\bar l^{(\alpha)}\sum_{i,(\beta)}d_{(\alpha),i,(\beta)}x^i\bar l^{(\beta)}$$

out of $S_{1,j}$ or T'_{j,r_1}. Since every l in $S_k(x,\bar l)$ has index $<j$, the terms

$$\sum_{|\alpha_1|=r_1}=r_1\bar l^{(\alpha_1)}\sum_{|\alpha_2|=r_2}\bar l^{(\alpha_2)}\sum c_{(\alpha_1),(\alpha_2),i,(\gamma)}x^i\bar l^{(\gamma)}$$

from $S_{1,j}(x,\bar{l})\,S_k(x,\bar{l})$ or $T_j(x,\bar{l})\,S_k(x,\bar{l})$ are obtained from the product $G_j(x,\bar{l})\,S_k(x,\bar{l})$. We now by (3.3) and Lemma $\mathfrak{a}^c_{1,B}, m_2, N(\mathfrak{a}_1, m_2)$ or by (3.8)) consider those (α) for which

$$\left|\sum_{i,(\gamma)} c_{(\alpha),i,(\gamma)} x^i \bar{l}^{(\gamma)}\right|_{\text{est op } N} < 20\cdot E_1^2 \left|\sum_{i,(\gamma)} d_{(\alpha),i,(\gamma)} x^i \bar{l}^{(\gamma)}(x)\right|_1$$

or

$$\left|\sum_{i,(\gamma)} c_{(\alpha),i,(\gamma)} x^i \bar{l}^{(\gamma)}\right|_{\text{est op } N} \leqslant \frac{10m\cdot K_{b(m_2)-1}\cdot 100}{\frac{9}{10}\cdot B'_n\cdot\frac{1}{2}\cdot\frac{1}{2^{m_2-m_1}}\cdot L^{r_1}_{b(m_2)-1}\cdot D^{r_1}_{b(m_2)-1}}\left|\sum d_{(\alpha),i,(\gamma)} x^i \bar{l}^{(\gamma)}(x)\right|.$$

Apparently this is so for "most" of the (α)'s. Now for every such (α) we can exactly repeat the proof of Lemma $\mathfrak{a}^c_{1,B}, m_2, N(\mathfrak{a}_1, m_2)$ with

$$\sum_{i,(\gamma)} d_{(\alpha),i,(\gamma)} x^i \bar{l}^{(\gamma)}$$

playing the role of h and $S_k(x,\bar{l})$ playing the role of $S_j(x,\bar{l})$. This will give us a different r_2 for each (α) but by losing at most a factor n_2 on E_2 we can fix one of them. This completes the proof.

Proof of Proposition $\mathfrak{a}^*_{1,B}, m_{2,B}, \mathfrak{a}^c_{2,B}, m_3, N(\mathfrak{a}_1, \mathfrak{a}_2, m_3)$. We consider for a fixed k

$$\sum_j l_j \sum_{|\alpha_1|=r_1} \bar{l}^{(\alpha_1)} \sum_{|\alpha_2|=r_2} \bar{l}^{(\alpha_2)} \sum_{i,(\gamma)} d_{(\alpha_1),(\alpha_2),i,(\gamma)} x^i \bar{l}^{(\gamma)}. \tag{4.35}$$

We consider a pre-$(n, b(m_3))$-expansion of q and a pre-$(b(m_3))$-expansion of V. And we consider terms which are derived from terms $hs_{j_1} l_{j_1} \dots l_{j_r} s_{j_{r+1}}$ or $v_{j_1} l_{j_1} \dots l_{j_r} s_{j_{r+1}}$ and which contribute to (4.35). Terms derived from the following types of terms $hs_{j_1} l_{j_1} \dots l_{j_r} s_{j_{r+1}}$ or $v_{j_1} l_{j_1} \dots l_{j_r} s_{j_{r+1}}$ can contribute to (4.35):

(1) Terms derived from $hs_{j_1} l_{j_1} \dots l_{j_r} s_{j_{r+1}}$ or $v_{j_1} l_{j_1} s_{j_2} \dots l_{j_r} s_{j_{r+1}}$ where for some i, $l_{j_1} = l_j$ for j out of a_2. The sum of the contributions of all these terms will obviously be precisely

$$\sum l_j \sum \bar{l}^{(\alpha_1)} \sum \bar{l}^{(\alpha_2)} \sum_{i,(\gamma)} c_{(\alpha_1),(\alpha_2),i,(\gamma)} x^i \bar{l}^{(\gamma)} \tag{4.36}$$

as defined in Lemma $\mathfrak{a}^*_{1,B}, m_{2,B}, \mathfrak{a}^c_{2,B}, m_3, N(\mathfrak{a}_1, \mathfrak{a}_2, m_3)$.

(2) Terms derived from $hs_{j_1}\dots l_{j_r}s_{j_{r+1}}$ or $v_{j_1}l_{j_1}s_{j_2}\dots l_{j_r}s_{j_{r+1}}$ where $l_{j_1}\neq l_j$ for all i and j. Since in (4.35) for all l_i with $i>k$, i belongs to one of the $b(m_3)-1$ first systems we must have that $j_1,\dots,j_r$ (which all are $>k$) belong to the $b(m_3)-1$ first systems. Moreover from (4.35) we get $r\leqslant r_1+r_2\leqslant n_1+n_2$. Assuming that $|V|^N_{\mathrm{est}}\leqslant 1$ we get that the sum of est op N-norms of all these terms after summation over k is

$$\leqslant 2\cdot L^{n_1+n_2}_{b(m_3)-1}D^{n_1+n_2}_{b(m_3)-1}(2\cdot 2^{m_3})^{n_1+n_2+1} \tag{4.37}$$

by the same argument as in Lemma 3.2.

The sum of the l_1-norms is

$$\leqslant 2\cdot L^{n_1+n_2}_{b(m_3)-1}(2\cdot 2^{m_3})^{n_1+n_2+1} \tag{4.38}$$

by the same argument.

(4.39) Let $D'_{1,j}$ be the weight of the good coefficient of l_jq_j, $j\in\mathfrak{a}_2$. Let D'_1 the sum of the $D'_{1,j}$'s. For every j and k we get that the sum of the est op N-norms of the terms

$$l_j\sum_{(\alpha_1)} l^{(\alpha_1)}\sum_{(\alpha_2)} l^{(\alpha_2)}\sum_{i,(\gamma)} c_{(\alpha_1),(\alpha_2),i,(\gamma)}x^i l^{(\gamma)}$$

is

$$\geqslant D'_{1,j}\cdot E'_2\cdot |S_k|_{\mathrm{est\,op}\,N}\cdot L_{b(\mathfrak{a}_2)}D_{b(\mathfrak{a}_2)}$$

where E'_2 depends only on $|\ |^{b(m_3)-1}$. Also the sum of the l_1-norms is $\geqslant D'_{1,j}\cdot E''_2\cdot |S_k|_1\cdot L_{b(\mathfrak{a}_2)}$ where E''_2 depends only on $|\ |^{b(m_3)-1}$. Now there is a C depending only on $|\ |^{b(\mathfrak{a}_2)-1}$ such that $D'_1>C$. This follows from Propositions $\mathfrak{a}^*_{1,\mathrm{B}}, m_2$ and the definition of $\mathfrak{a}^*_{1,\mathrm{B}}, m_{2,\mathrm{B}}, \mathfrak{a}_2$.

Since $\mathfrak{a}^*_{1,\mathrm{B}}, m_{2,\mathrm{B}}, \mathfrak{a}^c_{2,\mathrm{B}}, m_3$ occurs we have $\Sigma_j D'_{1,j}>\frac{1}{2}\cdot D'_1$ if we sum only over those j for which

$$\sum |S_k|_{\mathrm{est\,op}\,N}\geqslant \sum |S_k(x)|_1>\frac{1}{10^{m_3-m_2}}.$$

Thus we get that the sum of the est op N-norms of all terms

$$\sum_j l_j \sum_{(\alpha_1)} l^{(\alpha_1)} \sum_{(\alpha_2)} l^{(\alpha_2)} \sum c_{(\alpha_1),(\alpha_2),i,(\gamma)} x^i l^{(\gamma)}$$

summed also over all α_2's is

$$\geqslant \frac{D_1'}{2} \cdot \frac{1}{10^{m_2-m_2}} \cdot E_2' \cdot L_{b(\alpha_2)} D_{b(\alpha_2)}. \tag{4.40}$$

The sum of l_1-norms is

$$\geqslant \frac{D_1'}{2} \cdot \frac{1}{10^{m_3-m_2}} \cdot E_2'' \cdot L_{b(\alpha_2)}. \tag{4.41}$$

Since $b(\alpha_2) > b(m_3)-1$ these numbers are much bigger than (4.37) and (4.38) and this gives the proposition.

Proof of Lemma $\alpha_{1,\mathrm{B}}^*, m_{2,\mathrm{B}}, \ldots, \alpha_{k,\mathrm{B}}^c, m_{k+1}, N(\alpha_1, \ldots, \alpha_k, m_{k+1})$. By in the Induction Hypothesis we have a bound E_k^2 on the ratio between est op N-norm and $|\ |_1$-norm depending only on $|\ |^{b(m_k)-1}$ for

$$\sum_{\alpha} \sum_{i,(\gamma)} d_{(\alpha),i,(\gamma)} x^i l^{(\gamma)}$$

in $U_k(x, l)$. We observe that the assumption that $U_k(x, l)$ is $(b(m_{k+1})-1)$-substituted only effects the terms $d_{(\alpha),i,(\gamma)} x^i l^{(\gamma)}$ where all indices are $\leqslant sk$ since in

$$l_{sj_1} \cdots l_{sj_r} \sum_{(\alpha_1),\ldots,(\alpha_k)} l^{(\alpha_1)} \ldots l^{(\alpha_r)} l^{(\alpha_k)}$$

every index that appears is $> sk > b(m_{k+1})$. We observe that in $S_{s(k+1)}(x, l)$ all indices that appear are $< sk$. Now for most combinations

$$l_{sj_1} \cdots l_{sj_r} \sum_{(\alpha_1),\ldots,(\alpha_k)} l^{(\alpha_1)} \ldots l^{(\alpha_r)} l^{(\alpha_k)}$$

we have a bound $100\, E_k^2$ on the ratio between the est op N-norm and $|\ |_1$-norm of

$$\sum_{i,(\gamma)} d_{(\alpha),i,(\gamma)} x^i l^{(\gamma)}.$$

The proof is now the same as for Lemma $\alpha_{1,\mathrm{B}}^*, \ldots, \alpha_{2,\mathrm{B}}^c, m_3, N(\alpha_1, \alpha_2, m_3)$.

Proof of Proposition $\mathfrak{a}^*_{1,\mathrm{B}} m_{2,\mathrm{B}}, \ldots, m_{k+1}, N(\mathfrak{a}_1, \ldots \mathfrak{a}_k, m_{k+1})$. For the proof of this proposition we will consider the part

$$U^{s(k+1)}_{1,(\mathscr{J})} = l_{sj_1} l_{sj_2} \ldots l_{sk} \sum_{r_1, \ldots, r_{k+1}} \bar{l}^{(\alpha_1)} \bar{l}^{(\alpha_2)} \ldots \bar{l}^{(\alpha_k)} \bar{l}^{(\alpha_{k+1})} \sum d_{(\alpha), i, (\gamma)} x^i \bar{l}^{(\gamma)}$$

of the $(b(m_{k+1}))$-substituted coefficient of $l_{s(k+1)} q_{s(k+1)}$ in $hq-V$ as above. $(\mathscr{J})= sj_1, sj_2, \ldots, sk$. We will do this for different $(\mathscr{J})$ and $s(k+1)$ and we will show that either V gives a big estimate of $|hq|^N$ or the sum of the weights of those $U^{s(k+1)}_{1,(\mathscr{J})}$ which are good partial coefficients is big. We will only consider such combinations $l_{sj_1} l_{sj_2} \ldots l_{sk}$ where

$$U_k = l_{sj_1} l_{sj_2} \ldots l_{sj_r} \sum_{(\alpha_1), \ldots, (\alpha_k)} \bar{l}^{(\alpha_1)} \bar{l}^{(\alpha_2)} \ldots \bar{l}^{(\alpha_r)} \bar{l}^{(\alpha_k)} \sum_{i, (\gamma)} d_{(\alpha), i, (\gamma)} x^i \bar{l}^{(\gamma)}$$

is a good partial coefficient of $l_{sk} q_{sk}$ and where the coefficient $S_{s(k+1)}$ of $l_{s(k+1)} q_{(k+1)}$ in the $(sk, b(m_{k+1}))$-expansion of q_{sk} is semigood. We first observe that with the definition below

$$U^{s(k+1)}_{1,(\mathscr{J})} = W^{s(k+1)}_{1,0,(\mathscr{J})} + \sum_h W^{s(k+1)}_{1,h,(\mathscr{J})} \tag{4.42}$$

$W^{s(k+1)}_{1,h,(\mathscr{J})}$ is defined in the following way:

Consider, for fixed $\mathscr{J}$ and $s(k+1)$ out of the coefficient of $l_{s(k+1)} q_{s(k+1)}$ in a pre-$(n, b(m_{k+1}))$-expansion of q and a pre-$(b(m_{k+1}))$-expansion of V, those terms which contain l_{sj_h} but do not contain any l_{sj_p} with $p>h$, call them $T_h(s, l)$ and $V_h(s, l)$, respectively. By substituting the s's we get $T_h(s, l)=T_h(x, \bar{l})$ and $V_h(s, l)=V_h(s, \bar{l})$. Now let $W^{s(k+1)}_{1,h,(\mathscr{J})}$ be the part

$$l_{sj_1} l_{sj_2} \ldots l_{sk} \sum_{\substack{|\alpha_1|, \ldots, |\alpha_{k+1}| \\ = r_1, \ldots, r_{k+1}}} \bar{l}^{(\alpha_1)} \bar{l}^{(\alpha_2)} \ldots \bar{l}^{(\alpha_{k+1})} \sum_{i, (\gamma)} e_{(\alpha), i, (\gamma)} x^i \bar{l}^{(\gamma)}$$

of $h \cdot T_h(x, l) - V_h(x, l)$ after having in this expression made substitutions between l's and $\bar{l}$'s as above. $W^{s(k+1)}_{1,0,(\mathscr{J})}$ consists of those terms which do not contain any l_{sj_i} at all. With this definition (4.42) is obvious.

We see that we get $W_{k+1} = W^{s(k+1)}_{1,k,(\mathscr{J})}$ where W_{k+1} is defined in

Lemma $\mathfrak{a}^*_{1,\mathrm{B}}, m_2, \ldots, \mathfrak{a}^c_{k,\mathrm{B}}, m_{k+1}, N(\mathfrak{a}_1, \ldots, \mathfrak{a}_k, m_{k+1})$.

Let $\mathscr{J}_1=\{sj_1, sj_2, \ldots, sj_h\}$ and $\mathscr{J}_2=\{sj_{h+1}, \ldots, sk\}$. Let $S^{1,h,\mathscr{J}_2}_{s(k+1)}$ be the terms out of the coefficient of $l_{s(k+1)}\,q_{s(k+1)}$ in a pre-$(b(a_h), b(m_{k+1}))$-expansion of q_{j_h} which do not contain any l_i with i from a system with number $\geq b_{(m_{k+1})}$ and not any l_{sj_p}, $p>h$. Form $S^{1,h,\mathscr{J}_2}_{s(k+1)}(x, \bar{l})$ by $b_{(m_{k+1})}$ substituting the s's. Then it is easy to see that $W^{s(k+1)}_{1,h,(\mathscr{J})}$ is the part

$$l_{sj_1} l_{sj_2} \ldots l_{sk} \sum_{\substack{|\alpha_1|,\ldots,|\alpha_{k+1}| \\ =r_1,\ldots,r_{k+1}}} \bar{l}^{(\alpha_1)} \bar{l}^{(\alpha_2)} \ldots \bar{l}^{(\alpha_{k+1})} \sum_{i,(\gamma)} e'_{(\alpha),i,(\gamma)} x^i \bar{l}^{(\gamma)}$$

of the product

$$l_{sj_h} U_h(x, \bar{l})\, S^{1,h,\mathscr{J}_2}_{s(k+1)}(x, \bar{l}).$$

Now consider a pre-$(b(\alpha_h), b(m_{k+1}))$-expansion of q_{sj_h}. Consider those terms out of the coefficient of $l_{s(k+1)}\,q_{s(k+1)}$ which contain only l's from the $b(m_{k+1})-1$ first systems and which are of degree at most $\Sigma^{k+1}_{m=n} n_m+\text{card } H$ in the l's. They form the polynomial $S^{s(k+1)}_h$, say. We see that for fixed $\mathscr{J}_1$ no other term out of the coefficient of any $l_{s(k+1)}\,q_{s(k+1)}$ will, when multiplied by $U^{sj_h}_{1,(\mathscr{J}_1)}$ and with s's expanded in x and l's, give any contribution to $W^{s(k+1)}_{1,h,(\mathscr{J})}$ for any $s(k+1)$ or any $\mathscr{J}_2$. This is clear since they will all either have too high degree in l's with index strictly between sj_h and $s(k+1)$ or contain an l from a system with number $\geq b(m_{k+1})$. The only possibility for that would be an l_{sj_p}, $p>h$—according to the definition of $U^{s(k+1)}_{1,(\mathscr{J})}$—but this would contradict the definition of $W^{s(k+1)}_{1,h,(\mathscr{J})}$.

By the previous argument of the proof of Lemma 3.2 we have

$$\sum_{s(k+1)} \left|S^{s(k+1)}_h(x, \bar{l})\right|_{\text{est op } N} \leq C_{b(m_{k+1})-1}$$

and this estimate obviously also holds even if we sum the estimated op N norms of all monomials $m(\bar{l}, x)$ in x and l's out of all $S^{s(k+1)}_h$'s. We observe that every such monimial $m(\bar{l}, x)$ can contribute to $W^{s(k+1)}_{h,(\mathscr{J})}$ for several different $\mathscr{J}_2$'s when multiplied by $U^{sj_h}_{1,(\mathscr{J}_1)}$.

For fixed $\mathscr{J}_1$ and $s(k+1)$ but different $\mathscr{J}_2$'s we consider the parts

$$\sum_{\mathscr{J}_2} l_{sj_1} \ldots l_{sj_h} l_{sj_{h+1}} \ldots l_{sk} \sum_{\substack{|\alpha_1|,\ldots,|\alpha_{k+1}| \\ =r_1,\ldots,r_{k+1}}} \bar{l}^{(\alpha_1)} \bar{l}^{(\alpha_2)} \ldots \bar{l}^{(\alpha_{k+1})} \sum_{i,(\gamma)} f_{(\alpha),i,(\gamma)} x^i \bar{l}^{(\gamma)}$$

out of the product $U^{sj_h}_{1,(\mathscr{J}_1)} m(\bar{l}, x)$. Here we have

$$\sum_{\mathscr{I}_2}\left|\sum_{(\alpha),i,(\gamma)} f_{(\alpha),i,(\gamma)}x^i\bar{l}^{(\gamma)}\right|_{\text{est op }N}$$
$$\leq \frac{1}{L_{b(\alpha_{j_{h+1}})}}\cdot\frac{1}{L_{b(\alpha_{j_{h+2}})}}\cdot\ldots\cdot\frac{1}{L_{b(\alpha_k)}}\cdot\left(\sum_\alpha\left|\sum d^h_{(\alpha),i,(\gamma)}x^i\bar{l}^{(\gamma)}\right|_{\text{est op }N}\right)\cdot|m(\bar{l},x)|_{\text{est op }N}$$
$$\leq\frac{1}{L_r}\left(\sum\left|\sum d^h_{i,(j)}x^i\bar{l}^{(j)}\right|_{\text{op }N}\right)|m(\bar{l},x)|_{\text{est op }N}$$

where

$$r=\max\left(b(\alpha_{j_{h+1}}),b(\alpha_{j_{h+2}}),\ldots,b(\alpha_k)\right). \tag{4.43}$$

The factor

$$\frac{1}{L_{b(\alpha_{j_{n+1}})}}\cdot\frac{1}{L_{b(\alpha_{j_{n+2}})}}\cdot\ldots\cdot\frac{1}{L_{b(\alpha_k)}}$$

comes in since all the l's in $l_{sj_{h+1}}\ldots l_{sk}$ have to appear in the products $(\Sigma d^h_{i,(j)}x^i\bar{l}^{(j)}m(\bar{l},x)$.

By summing (4.43) over all different $m(l,x)$ and then over all $s(k+1)$ we get for fixed $\mathscr{I}_1$

$$\sum_{s(k+1)}\left(\sum_{\mathscr{I}_2}\left|\sum_\alpha\sum_{i,(\gamma)}e^h_{(\alpha),i,(\gamma)}x^i l^{(\gamma)}\right|_{\text{op }N}\right)\leq C_{b(m_{k+1})-1}\left(\sum\left|\sum d^h_{i,(j)}x^i\bar{l}^{(j)}\right|_{\text{op }N}\right)\frac{1}{L_r}. \tag{4.44}$$

By finally summing (4.44) over all different $\mathscr{I}_1$ (which end with an l in $b(a_{j_h})$) we obtain

$$\sum_{\mathscr{I}_1}\sum_{s(k+1)}\sum_{\mathscr{I}_2}\left(\sum_\alpha\left|\sum_{i,(\gamma)}e^h_{(\alpha),i,(\gamma)}x^i\bar{l}^{(\gamma)}\right|_{\text{op }N}\right)\leq\sum w(U_h)\cdot\frac{1}{L_r}\cdot C_{b(m_{k+1})-1}K_{b(m_{k+1})-1} \tag{4.45}$$

where $\Sigma w(U_h)$ is the sum of the weights of good partial coefficients of l_{sj_h}'s, $l_{sj_h}\in$ system $b(\alpha_{j_h})$.

To finish the proof we first observe that every $W^{s(k+1)}_{1,k,(\mathscr{I})}$ satisfies the conditions in the definition of good partial coefficient and so if we sum the "weights" over $\mathscr{I}$ we get $\Sigma w(U_k)\cdot|s_{s(k+1)}|_1$ and then over $s(k+1)$ we get a number

$$\sum w(U_k)\cdot\frac{1}{2^{m_{k+1}-m_k}}\cdot\frac{1}{16},$$

or by Lemma $\alpha^*_{1,B},\dots,m_{k+1},N(\alpha_1,\dots,m_{k+1})$

$$\sum_{\mathscr{I},\,s(k+1)}\sum_{\alpha}\left|c_{(\alpha),i,(\gamma)}x^i f^{(\gamma)}\right|_1\geq\frac{1}{16}\cdot\frac{1}{E_{k+1}}\cdot\frac{1}{2^{m_{k+1}-m_k}}\cdot\sum w(U_k).$$

Now by Lemma 2I and 3I of the induction hypotesis there exists a C' depending only on $|\;|^{b(m_{k+1})-1}$ such that we have

$$\sum_{\mathscr{I},\,s(k+1)}\sum_{\alpha}\left|c_{(\alpha),i,(\gamma)}x^i f^{(\gamma)}\right|_1\geq C'\cdot D_h\cdot\frac{1}{2^{m_{k+1}}}\cdot\left(\frac{1}{16}\right)^k\cdot\frac{1}{2^{p_{h+1}}}\cdot\frac{1}{2^{p_k}}\cdot\frac{1}{E_{k+1}}. \tag{4.46}$$

By (4.46) above we have

$$\sum_{\alpha}\sum_{i}\left|e^h_{(\alpha),i,(\gamma)}x^i f^{(\gamma)}\right|_{\text{est op }N}\leq D_h\cdot\frac{1}{L_r}\cdot C_{b(m_{k+1})-1}\cdot K_{b(m_{k+1})-1}$$

which is much smaller than

$$D_h\cdot\frac{1}{E_{k+1}}\cdot\frac{1}{2^{p_{h+1}}}\cdot\frac{1}{2^{p_k}}\cdot\frac{1}{2^{m_{k+1}}}\cdot\left(\frac{1}{V_1}\right)^{k+1}V_2^{k+1}$$

if the L's grow sufficiently fast since $r\geq\max\{p_{h+1},\dots,p_k\}$. This is obviously also true even if we sum (4.46) over all h if the L's grow sufficiently fast. To complete the proof we now assume that the estimate given by V is ≤ 1. Then it is clear that $|\;|_{\text{est op }N}$ of the perturbation caused by the $W^{s(k+1)}_{1,0,(\mathscr{I})}$'s is much smaller than

$$\left(\sum w(U_k)\right)\cdot\frac{1}{2^{m_{k+1}-m_k}}\cdot\frac{1}{E_{k+1}}\cdot\frac{1}{16}$$

by the same argument as in Lemma 3.2. This completes the proof of the proposition.

Proof of Proposition $\alpha^*_{1,B},m_{2,B},\dots,\alpha^{nc}_k,N(\alpha_1,\dots,\alpha_k)$. We consider the first sum $\Sigma_j S_j l_j q'_j=\Sigma_{sk}S_{sk}l_{sk}q'_{sk}$ in (3.27). With $S_{sk}=\Sigma_i U_{i,sk}+R_{sk}$ (see (3.28)) we get $\Sigma_{sk}S_{sk}l_{sk}q'_{sk}=\Sigma_{sk}(\Sigma_i U_{i,sk}l_{sk}q'_{sk})$.

For every set $l_{sj_1}l_{sj_2}\dots l_{sj_r}$ we consider the good partial coefficient

$$l_{sj_1}\dots l_{sj_r}\sum_{\substack{|\alpha_1|,\dots,|\alpha_k|\\=r_1,\dots,r_k}} \bar{l}^{(\alpha_1)}\dots \bar{l}^{(\alpha_r)}\bar{l}^{(\alpha_k)}\left(\sum_{i,(\gamma)} d_{(\boldsymbol{\alpha}),i,(\gamma)}x^i\bar{l}^{(\gamma)}\right)$$

out of $V_{i,sk}$. We consider those $\bar{l}^{(\alpha_1)}\dots\bar{l}^{(\alpha_r)}\bar{l}^{(\alpha_k)}$ for which

$$\left|\sum_{i,(\gamma)} c_{(\boldsymbol{\alpha}),i,(\gamma)}x^i\bar{l}^{(\gamma)}\right|_{\text{est op }N}\leq 100E_k^2\left|\sum_{i,(\gamma)} d_{(\boldsymbol{\alpha}),i,(\gamma)}\right|_1.$$

For these there exists $B''_{b(\alpha_k)}$ and n_k depending only on $|\;|^{b(\alpha_k)-1}$ such that $(\Sigma d_{(\boldsymbol{\alpha}),i,(\gamma)}x^i\bar{l}^{(\gamma)}q'_{sk})(x)$ contains

$$p_{1/50}\left(E_{(\alpha_1),\dots,(\alpha_k),sj_1,\dots,sj_r}\right)=p_{1/50}\sum_{i,(\gamma)} e_{(\boldsymbol{\alpha}),i,(\gamma)}x^i\bar{l}^{(\gamma)},\quad i+|m|\leq n_k$$

with

$$\left|E_{(\alpha_1),\dots,(\alpha_k),sj_1,\dots,sj_r}\right|(x)_1\geq B''_{b(\alpha_k)}\cdot\left|\sum_{i,(\gamma)} d_{(\boldsymbol{\alpha}),i,(\gamma)}x^i\bar{l}^{(\gamma)}\right|_1.$$

Now by Lemma 1I in the Induction Hypothesis we have

$$\sum_{\substack{(\alpha_1),\dots,(\alpha_k)\\=sj_1,\dots,sj_r}}\left|\sum_{i,(\gamma)} d_{(\boldsymbol{\alpha}),i,(\gamma)}x^i\bar{l}^{(\gamma)}\right|_1>\frac{1}{V_1}\cdot D,$$

where V_1 depends only upon $|\;|^{b(\alpha_k)-1}$ and D is the sum of the weights of good partial coefficients of $l_{sk}q'_{sk}$'s, $sk\in\alpha_k$.

Now we define E by

$$E=\sum_{sk} l_{sk}\sum_{\substack{(\alpha_1),\dots,(\alpha_k)\\=sj_1,\dots,sj_r}} l_{sj_1}\dots l_{sj_r}\bar{l}^{(\alpha_1)}\dots\bar{l}^{(\alpha_k)}E_{(\alpha_1),\dots,(\alpha_k),sj_1,\dots,sj_r}.$$

Then

$$|E|_1\geq D\cdot B''_{b(\alpha_k)}\cdot\frac{1}{V_1}L_{b(\alpha_{j_1})}\dots L_{b(\alpha_k)}$$

and so E satisfies the conclusion in the proposition. We now prove that for every sk, $\Sigma_i U_{i,sk}l_{sk}q'_{sk}(x)$ contains

$$p_{1/30}\left(l_{sk}\sum l_{sj_1}\dots l_{sj_r}\bar{l}^{(\alpha_1)}\dots\bar{l}^{(\alpha_k)}E_{(\alpha_1),\dots,(\alpha_k),sj_1,\dots,sj_r}\right)$$

and for this we apply Lemma 4.3 in the same way as in the proof of Lemma 4.14. The strategy of the proof will now be the following:

We first prove that the cancellation effect of

$$\sum_{sk} R_{sk} l_{sk} q'_{sk} + R' + \sum_j S_j l_j q_j \quad \text{on } E \text{ is } < \frac{|E|_1}{1\,000}. \tag{4.47}$$

Then we prove that the cancellation effect of $\Sigma_i U_{i,sk} l_{sk} q'_{sk}$ on

$$E' = E - l_{sk} \sum l_{sj_1} \dots l_{sj_r} l^{(\alpha_1)} \dots l^{(\alpha_k)} E_{(\alpha_1),\dots,(\alpha_k),sj_1,\dots,sj_r}$$

is

$$\leq \frac{1}{1\,000} |(E'-E)(x)|_1.$$

That will by Lemma 4.16 give that

$$\sum_{sk} \left(\sum_i U_{i,sk} l_{sk} q'_{sk} \right) \quad \text{contains} \quad p_{1/30+1/1\,000}(E). \tag{4.48}$$

For the proofs of (4.47) and (4.48) we introduce the following notation: Let $\alpha_{j_{i_1}}, \alpha_{j_{i_2}}, \dots, \alpha_{j_{i_k}}$ by the subsequence of $\alpha_{j_1}, \alpha_{j_2}, \dots, \alpha_{j_r}, \alpha_k$ with the following property

$$b(\alpha_{j_{i_1}}) = \max \left\{ b(\alpha_{j_1}), b(\alpha_{j_2}), \dots, b(\alpha_{j_r}), b(\alpha_k) \right\}$$

and if $b(\alpha_{j_{i_m}})$ is defined we put

$$b(\alpha_{j_{i_{m+1}}}) = \max \left\{ b(\alpha_{j_{i_m+1}}) \, b(\alpha_{j_{i_m+2}}), \dots, b(\alpha_{j_r}), b(\alpha_k) \right\}.$$

We say that a monomial M contains exactly a combination

$$l_{sj_1} \dots l_{sj_r} l^{(\alpha_1)} \dots l^{(\alpha_r)}$$

if l_{sj_r} is in the monomial and if the l's with index $> sj_r$ in M agree with

$$l_{sj_1} \dots l_{sj_{r-1}} l^{(\alpha_1)} \dots l^{(\alpha_r)}.$$

(If $M = a_i x^i l_1^{\alpha_1} \dots l_N^{\alpha_N}$ then of course $a_i x^i$ or the l's with index $< sj_r$ do not influence whether M exactly contains a certains combination or not.) We say that a monomial M contains exactly a combination

$$l_{sj_1} \dots l_{sj_{r-1}} \bar{l}^{(\alpha_1)} \dots \bar{l}^{(\alpha_r)}$$

if the l's with index $> l_{sj_r}$ in M agree with

$$l_{sj_1} \dots l_{sj_{r-1}} \bar{l}^{(\alpha_1)} \dots \bar{l}^{(\alpha_r)}.$$

We say that a monomial is obtained by multiplying terms out of A by terms out of B, thus meaning that the monomial is one of the terms in the product AB.

(4.49) We consider now the shortened pre-expansion used in defining $(hq-V)'$ above. Assume that at some stage of the preexpansion some term of the first type ends with $l_i q_i$. We now define $q_i''(s, l, q, t)$. In the final stage of the shortened pre-expansion the q_i will be replaced by a polynomial in s's, l's, t's, and q's. Let q_i''be the polynomial we get from $q_i = q_i(s, l, q, t)$ by removing all terms of the 3rd type for which $\varepsilon < \min\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{b(\alpha_k)}\}$. Now obviously there is a number K_i depending only on $| \, |^{i-1}$ such that the sum of l_1-norms of all terms out of the expansion of q_i is $< K_i$, in particular $|q_i''(x)|_1 < K_i$.

Moreover $|l_i|_1$ depends only on $| \, |^{i-1}$ and D_i and L_i. To estimate the cancellation effect in (4.47) we consider out of

$$\sum_{sk} R_{sk} l_{sk} q'_{sk} + \sum S_j l_j q_j + R'$$

the following types of monomials M_m: Those which contain exactly a combination

$$l_{sj_1} l_{sj_2} \dots l_{sj_{i_m}} \bar{l}_{(1)} \dots \bar{l}_{(i_m)}$$

which appears in some term of E but which do not contain exactly any combination

$$l_{sj_1} l_{sj_2} \dots l_{sj_{i_m}} l_{sj_{i_{m+1}}} \bar{l}^{(\alpha_1)} \bar{l}^{(\alpha_{i_{m+1}})}$$

that appears in any term out of E. First we study for fixed sj_{i_m} the sum of the cancellation effects on E of all monomials $M_{m, sj_{i_m}}$ out of M_m which are derived from terms that contain an $l_{sj_{i_m}}$. Monomials out of $M_{m, sj_{i_m}}$ are obtained only by multiplying a good partial coefficient of $l_{sj_{i_m}}$ by terms out of $q''_{sj_{i_m}}$. To study this we substitute $q''_{j_{i_m}}(s, l, q, t)$ as a polynomial in x and l's in the following way:

We get, from the shortened pre-$(n, b(\alpha_k))$-expansion of q a shortened pre-expansion of $q_{sj_{i_m}}$—just consider what $q_{sj_{i_m}}$ in terms that end with $l_{sj_{i_m}} q_{sj_{i_m}}$ is replaced by in the final stage in the shortened pre-$(n, b(\alpha_k))$-expansion of q. We shorten this pre-expansion of $q_{sj_{i_m}}$ further by not expanding terms which end with $l_i q_i$, $i \in$ system with number $\geqslant b(\alpha_{j_{i_{m+1}}})$, any further. Let then $S_i l_i q_i$ be the sum of all terms ending with $l_i q_i$. For all these i we substitute q''_i as a polynomial in x. For all other terms out of $q''_{sj_{i_m}}$ we make a $(b(\alpha_k)-1)$-substitution.

(4.50) Now we fix i and study the cancellation effect of all monomials $M_{m, sj_{i_m}, i}$ of $M_{m, sj_{i_m}}$ obtained by multiplying a good partial coefficient of $l_{sj_{i_m}}$ by $S_i l_i q''_i$, where S_i is $(b(\alpha_k)-1)$-substituted and $q''_i = q''_i(x)$. We get that the sum for fixed m and i over all sj_{i_m} of the l_1-norms of all $M_{msj_{i_m}}$'s which contain $< n_1+n_2+\ldots+n_{i_m}+i_m+r$ different l's with index $>i$ is

$$\leqslant V_1 \cdot D(m) L_{b(\alpha_{j_1})} \ldots L_{b(\alpha_{j_{i_m}})} \left(2^{m_{i_{m+1}}+1}\right)^r L^r_{b(\alpha_{j_{i_{m+1}}})-1} \cdot |l_i|_1 K_i \tag{4.51}$$

where $D(m) = \Sigma w(U_{i_m})$ is the sum of the weights of all good partial coefficients of all $l_{sj_{i_m}} q_{sj_{i_m}}$'s by the same argument as in Lemma 3.2.

(4.52) Now by Lemmas 1I, 2I and 3I we have

$$|E|_1 > D(m) \cdot L_{b(\alpha_{j_1})} \ldots L_{b(\alpha_{j_{i_m}})} \cdot L_{b(\alpha_{j_{i_{m+1}}})} \cdot L_{b(\alpha_{j_{i_{m+1}}})} \cdot C,$$

where C depends only on $|\,|^{b(\alpha_{j_{i_{m+1}}})-1}$. Now this gives by Lemma 4.1 that the sum over sj_{i_m} of the cancellation effects on E of the $M_{m, sj_{i_m}}$'s which contain $> n_1+n_2+\ldots+n_k$ different l's with index $>i$ is

$$< \frac{|E|_1}{m \cdot 2000 \cdot 2^i}.$$

For the other $M_{m,sj_{i_m}}$'s it is also

$$<\frac{|E|_1}{m\cdot 2000\cdot 2^i}.$$

We can see this in the same way as we prove Lemma 4.4 by observing that the cancellation effect is $=0$ except for those terms which contain some $l_j^{\alpha_j}$ with $\alpha_j > \omega_i$, ω_i depending only on $|\ |^{i-1}$ and the sum of l_1-norms of those is much smaller than

$$\frac{|E|_1}{m\cdot 2000\cdot 2^i}$$

if ω_i is large enough (depending on $|\ |^{i-1}$).

Now we consider the monomials out of M_{m,sj_m} obtained by multiplying a good partial coefficient of $l_{sj_{i_m}}$ by other terms of $q''_{sj_{i_m}}$ than those in (4.50). These terms out of $q''_{sj_{i_m}}$ are derived from terms that contain only l's from the $b(a_{j_{i_{m+1}}})-1$ first systems.

So instead of the estimate (4.50) we will get the estimate

$$V_1\cdot D(m)\, L_{b(a_{j_1})}\cdots L_{b(a_{j_{i_m}})}\big(2^{m_{i_{m+1}}+1}\big)^r L_{b(a_{j_{i_{m+1}}})-1}\cdot C',$$

where C' depends only on $|\ |^{b(\alpha_k)-1}$. So for small r's we can here just use that this is much smaller than $|E|_1/1000$ and for large r's we use Lemma 4.1 as above.

We now study the sum of the cancellation effects of monomials $M_{m,sj_{i_r}}$ out of M_m which are derived from terms that contain an $l_{sj_{i_r}}$, $r<m$, but do not contain any $l_{sj_{i_p}}$, $m\geq p>r$. Such terms are only obtained by multiplying a good partial coefficient of $l_{sj_{i_r}}$ by terms out of $q''_{sj_{i_r}}$, which are derived from terms which do not contain any $l_{sj_{i_p}}$, $m\geq p>r$. So also here we shorten the pre-expansion of $q_{sj_{i_r}}$, by not expanding further terms which end with $l_i q_i$, i belongs to a system with number $>b(\alpha_{j_{i_{m+1}}})$. Then we proceed as for the terms $M_{msj_{i_m}}$ but we only need to use

$$|E|_1 > D(r)\cdot L_{b(\alpha_{j_1})}\cdots L_{b(\alpha_{j_{i_r}})}\cdot L_{b(\alpha_{j_{i_m}})}\cdot L_{b(\alpha_{j_{i_{m+1}}})}\cdot C$$

with obvious notations, C depending only on $|\ |^{b(\alpha_{j_{i_{m+1}}})-1}$.

To prove (4.48) we use Lemma 4.6 letting the set D consist of the single integer sk. We can thus make the cancellation effect of $\Sigma_i U_{i,sk} l_{sk} q'_{sk}$ on

$$E' < D_{sk} \cdot \frac{1}{V_1} \cdot B''_{b(\alpha_k)} \cdot L_{b(\alpha_{j_1})} \dots L_{b(\alpha_{j_r})}$$

where D_{sk} is the sum of the weights of good partial coefficient of $l_{sk} q_{sk}$. Summing this over all sk obviously gives the result. This is since the estimate $|q_i(x)|_1 < K_i$ is obviously also valid for the sum of the l_1-norms of terms of 3rd type in the pre-expansion of q_i gotten from the shortened $(n, b(\alpha_k))$-expansion of q.

To complete the proof we have now only to prove that the sum of the cancellation effects on E of the following $(b(m_{k+1})-1)$-substituted terms of 3rd type in $hq-V$ with $\varepsilon = 1/2^{m_{k+1}}$ is smaller than $(|E|_1/1\,000) \cdot 1/2^{m_{k+1}}$.

(4.53) Every monomial that contains exactly a combination

$$l_{sj_1} \dots l_{sj_{r'}} l^{(\alpha_1)} \dots l^{(\alpha_{r'})} l^{(\alpha_{r'+1})}$$

out of E but is derived from a term, which does not contain $l_{sj_{r'}}$.

Assume that it is derived from a term that contains $l_{sj_{i_m}}$ but does not contain $l_{sj_{i_{m+k}}}$ for any $k \geq 1$. (Obviously $l_{sj_{r'}} = l_{sj_{i_p}}$, for some p. Then it is obtained by multiplying a good partial coefficient of $l_{sj_{i_m}}$, by terms derived from terms of 3rd type out of the expansion of $q_{sj_{i_m}}$ gotten from the $(b(m_{k+1})-1)$-substituted shortened $(n, b(\alpha_k))$-expansion of q. These terms of 3rd type cannot contain an l_i with i from a system with number $> b(\alpha_{j_{r'}})-1$. By the same argument as for the estimate (4.51) we get that the sum of l_1-norms of terms which contain few different l's is much smaller than $(1/2^{m_{k+1}}) \cdot (|E|_1/1\,000)$ and by Lemma 4.1 the sum of the cancellation effect of the others is also $\leq (|E|_1/1\,000) \cdot 1/2^{m_{k+1}}$. This completes the proof.

Proof of Lemma 3.4. This proof is exactly the same as for the last part of the previous proposition.

References

[1] Beauzamy, B. Sous-espaces invariant de type fonctionel dans les espaces de Banach. *Acta Math.*, 144 (1980), 65–82.
[2] Enflo, P. On the invariant subspace problem for Banach spaces. *Seminaire Maurey-Schwarz* (1975–76).
[3] — *On the invariant subspace problem for Banach spaces*. Institute Mittag-Leffler report no. 9 (1980).

Received February 10, 1981
Received in revised form March 21, 1985