Rigidity of time changes for horocycle flows

by

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Let T_t be a measure preserving (m.p.) flow on a probability space (X, μ) and let τ be a positive integrable function on X, $\int_X \tau\, d\mu = \bar{\tau}$. We say that a flow T_t^τ is obtained from T_t by the time change τ if

$$T_t^\tau(x) = T_{w(x,t)}(x)$$

for μ-almost every (a.e.) $x \in X$ and all $t \in \mathbf{R}$, where $w(x, t)$ is defined by

$$\int_0^{w(x,t)} \tau(T_u x)\, du = t.$$

The flow T_t^τ preserves the probability measure μ_τ on X defined by

$$d\mu_\tau(x) = (\tau/\bar{\tau})\, d\mu(x), \quad x \in X.$$

We say that two integrable functions $\tau_1, \tau_2\colon (X, \mu) \to \mathbf{R}$ are homologous along T_t if there is a measurable $v\colon X \to \mathbf{R}$ such that

$$\int_0^t (\tau_1 - \tau_2)(T_u x)\, du = v(T_t x) - v(x)$$

for μ-a.e. $x \in X$ and all $t \in \mathbf{R}$. One can check that two time changes τ_1 and τ_2 are homologous via v if and only if (iff) the map $\psi_v\colon X \to X$ defined by

$$\psi_v(x) = T_{\sigma(x)}\, x,$$

[1] Partially supported by NSF grant MCS 81-02262.

where $\int_0^{\alpha(x)} \tau_2(T_u x)\,du = v(x)$, is an invertible conjugacy between $T_t^{\tau_1}$ and $T_t^{\tau_2}$, i.e.

$$\psi_v T_t^{\tau_1}(x) = T_t^{\tau_2} \psi_v(x)$$

for a.e. $x \in X$ and all $t \in \mathbf{R}$. If T_t is ergodic and τ_1, τ_2 are homologous along T_t via some measurable functions v_1 and v_2 then $v_2 - v_1$ is equal to a constant a.e.

Let G denote the group $SL(2, \mathbf{R})$ equipped with a left invariant Riemannian metric and let $\mathbf{T}$ be the set of all discrete subgroups Γ of G such that the quotient space $M = \Gamma|G = \{\Gamma g : g \in G\}$ has finite volume. The horocycle flow h_t and the geodesic flow g_t on M are defined by

$$h_t(\Gamma g) = \Gamma g \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

$$g_t(\Gamma g) = \Gamma g \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

$g \in G$, $t \in \mathbf{R}$. The flows h_t and g_t preserve the normalized volume measure μ on M, are ergodic and mixing on (M, μ) and

$$g_t \circ h_s = h_{s e^{2t}} \circ g_t \qquad (*)$$

for all $s, t \in \mathbf{R}$.

In order to state our main theorem we shall need the following notations. Let

$$K = \left\{ K_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in (-\pi, \pi] \right\}$$

be the rotation subgroup of G. We say that a real valued function φ on $M = \Gamma|G$, $\Gamma \in \mathbf{T}$ is Hölder continuous in the direction of K with the Hölder exponent $\delta > 0$ if

$$|\varphi(x) - \varphi(y)| \leq C_\varphi |\theta|^\delta$$

for some $C_\varphi > 0$ and all $x, y \in M$ with $y = R_\theta(x)$, where $R_\theta(\Gamma g) = \Gamma g K_\theta$, $g \in G$. It was shown in [2] that if $\varphi \in L_2(M, \mu)$ is Hölder continuous in the direction of K with $\delta > \frac{1}{2}$ and $\bar{\varphi} = 0$ then

$$\left| \int_M \varphi(x)\, \varphi(h_t x)\, d\mu(x) \right| \leq D_\varphi |t|^{-\alpha_\varphi} \qquad (**)$$

for some $D_\varphi, \alpha_\varphi > 0$ and all $t \neq 0$. We shall denote by $\mathbf{K}(M)$ the set of all positive integrable functions τ on M such that τ and τ^{-1} are bounded and $\tau - \bar{\tau}$ satisfies (**) for some $D_\tau, \alpha_\tau > 0$.

THEOREM 1. *Let* $h_t^{(i)}$ *be the horocycle flow on* $(M_i=\Gamma_i|G,\mu_i)$, $\Gamma_i\in\mathbf{T}$, $i=1,2$ *and let* $h_t^{\tau_i}$ *be obtained from* $h_t^{(i)}$ *by a time change* $\tau_i\in K(M_i)$, $i=1,2$, *with* $\bar\tau_1=\bar\tau_2$. *Suppose that there is a measure preserving* $\psi\colon(M_1,\mu_{\tau_1})\to(M_2,\mu_{\tau_2})$ *such that*

$$\psi h_t^{\tau_1}(x)=h_t^{\tau_2}\psi(x)$$

for μ_{τ_1}-a.e. $x\in M_1$ *and all* $t\in\mathbf{R}$. *Then there are* $C\in G$ *and a measurable* $\sigma\colon M_2\to\mathbf{R}$ *such that*

$$C\Gamma_1C^{-1}\subset\Gamma_2\quad\text{and}\quad\psi(x)=h^{(2)}_{\sigma(\psi_C(x))}(\psi_C(x))$$

for μ_1-a.e. $x\in M_1$, *where* $\psi_C(\Gamma_1 g)=\Gamma_2 Cg$, $g\in G$.

The second conclusion of Theorem 1 says that τ_1 and τ_C defined by $\tau_C(x)=\tau_2(\psi_C(x))$, $x\in M_1$ are homologous along $h_t^{(1)}$ via v_C defined by

$$v_C(x)=\int_0^{\sigma(\psi_C(x))}\tau_C(h_u^{(1)}x)\,du,\quad x\in M_1.$$

Let us note that it follows from [1] that if $\psi\colon(M_1,\mu_{\tau_1})\to(M_2,\mu_{\tau_2})$ is an invertible measurable conjugacy between $h_t^{\tau_1}$ and $h_t^{\tau_2}$ then ψ is in fact measure preserving. The same is true when ψ is not invertible and M_2 is compact.

We assumed in Theorem 1 that $\bar\tau_1=\bar\tau_2$. Suppose now that $a=\bar\tau_1\neq\bar\tau_2=b$ and let

$$\tilde\tau_1(x)=\frac{b}{a}\tau_1(g_{-s}x),\ s=\tfrac{1}{2}\log\frac{a}{b},\quad x\in M_1,\ \bar{\tilde\tau}_1=b.$$

The commutation relation (*) shows that $h_t^{\tau_1}$ *and* $h_t^{\tilde\tau_1}$ are isomorphic via g_s, i.e. $g_s\circ h_t^{\tau_1}=h_t^{\tilde\tau_1}\circ g_s$, $t\in\mathbf{R}$. We get the following:

COROLLARY 1. *Let* $\tau_i\in\mathbf{K}(M_i)$, $i=1,2$ *and* $\bar\tau_1=a$, $\bar\tau_2=b$. *Suppose that* $h_t^{\tau_1}$ *is conjugate to* $h_t^{\tau_2}$ *via a measure preserving* $\psi\colon(M_1,\mu_{\tau_1})\to(M_2,\mu_{\tau_2})$. *Then there are* $C\in G$ *and a measurable* $\sigma\colon M_2\to R$ *such that* $C\Gamma_1C^{-1}\subset\Gamma_2$ *and* $\psi(x)=h^{(2)}_{\sigma(\psi_C(g_sx))}\psi_C(g_sx)$ *for* μ_1-a.e. $x\in M_1$, *where* ψ_C *is as in Theorem* 1 *and* $s=\frac{1}{2}\log(a/b)$.

COROLLARY 2. *Let* $\tau_i\in\mathbf{K}(M_i)$, $i=1,2$, $\bar\tau_1=a$, $\bar\tau_2=b$. *Then* $h_t^{\tau_1}$ *is isomorphic to* $h_t^{\tau_2}$ *if and only if there is* $C\in G$ *such that* $C\Gamma_1C^{-1}=\Gamma_2$ *and* $\tau_1(x)$ *and* $(a/b)\tau_2(\psi_C(g_sx))$, $x\in M_1$ *are homologous along* $h_t^{(1)}$, *where* $s=\frac{1}{2}\log(a/b)$. *Every isomorphism between* $h_t^{\tau_1}$ *and* $h_t^{\tau_2}$ *has the form as in Corollary* 1.

THEOREM 2. *Let $h_t^{(i)}$ be the horocycle flow on $(M_i=\Gamma_i|G,\mu_i)$, $\Gamma_i\in\mathbf{T}$ and let $h_t^{\tau_i}$ be obtained from $h_t^{(i)}$ by a time change $\tau_i\in\mathbf{K}(M_i)$, $i=1,2$.* Suppose that *$h_p^{\tau_i}$ is ergodic for some $p\neq 0$, $i=1,2$ and there is a measure preserving $\psi\colon(M_1,\mu_{\tau_1})\to(M_2,\mu_{\tau_2})$ such that*

$$\psi h_p^{\tau_1}(x)=h_p^{\tau_2}\psi(x)$$

*for μ_{τ_1}-*a.e. *$x\in M_1$. Then ψ is a conjugacy of the flows $h_t^{\tau_1}$ and $h_t^{\tau_2}$,* i.e.

$$\psi h_t^{\tau_1}(x)=h_t^{\tau_2}\psi(x)$$

*for μ_{τ_1}-*a.e. *$x\in M_1$ and all $t\in\mathbf{R}$.*

For $\Gamma\in\mathbf{T}$ let $\tilde\Gamma=\{C\in G\colon C\Gamma C^{-1}=\Gamma\}\in\mathbf{T}$ be the normalizer of Γ in G. Let $\mathbf{K}_1(M)$ be the set of all $\tau\in\mathbf{K}(M)$ with $\bar\tau=1$, $M=\Gamma|G$. We say that $\tau_1,\tau_2\in\mathbf{K}_1(M)$ are homologous modulo $\tilde\Gamma$ if there is $C\in\tilde\Gamma$ such that τ_1 and $\tau_C=\tau_2\circ\psi_C$ are homologous along h_t. Corollary 2 says that there is a one-to-one correspondence between the isomorphism classes of h_t^τ, $\tau\in\mathbf{K}_1(M)$ and the homology classes of $\tau\in\mathbf{K}_1(M)$ mod $\tilde\Gamma$.

Let f_t be a m.p. flow on a probability space (X,μ) and let $\Psi(f_t)$ be the set of all isomorphisms $\psi\colon X\to X$ such that $\psi f_t(x)=f_t\psi(x)$ for μ-a.e. $x\in X$ and all $t\in\mathbf{R}$, i.e. ψ commutes with every f_t, $t\in\mathbf{R}$. We say that $\psi_1,\psi_2\in\Psi(f_t)$ are equivalent if $\psi_2=f_p\circ\psi_1$ a.e. for some $p\in\mathbf{R}$. Let $\varkappa(f_t)$ denote the set of equivalence classes in $\Psi(f_t)$. We define a group operation in $\varkappa(f_t)$ by $[\psi_1]\cdot[\psi_2]=[\psi_1\circ\psi_2]$, where $[\psi]$ denotes the equivalence class of ψ. The group $\varkappa(f_t)$ is called the commutant of f_t (see [6]).

It follows from Corollary 2 that if $\tau\in\mathbf{K}(M)$ and $\psi\in\Psi(h_t^\tau)$ then there are $C\in\tilde\Gamma$ and a measurable $\sigma_C\colon M\to M$ unique up to an additive constant such that τ and $\tau_C=\tau\circ\psi_C$ are homologous along h_t and $\psi=h^\tau_{\sigma_C}\circ\psi_C$ a.e. This implies that

$$\varkappa(h_t^\tau)=\{[h^\tau_{\sigma_C}\psi_C]\colon C\in\tilde\Gamma\}.$$

The map $\pi\colon\varkappa(h_t^\tau)\to\Gamma\setminus\tilde\Gamma$ defined by $\pi[h^\tau_{\sigma_C}\psi_C]=\Gamma C$, $C\in\tilde\Gamma$ is a group isomorphism from $\varkappa(h_t^\tau)$ onto a subgroup of $\Gamma\setminus\tilde\Gamma$. The group $\Gamma\setminus\tilde\Gamma$ is finite, since $\Gamma\in\mathbf{T}$. We get the following:

COROLLARY 3. *If $\tau\in\mathbf{K}(M)$ then the commutant $\varkappa(h_t^\tau)$ is finite and is isomorphic to a subgroup of $\Gamma\setminus\tilde\Gamma$. If $\Gamma=\tilde\Gamma$ or τ is not homologous to τ_C for any $C\in\tilde\Gamma$ different from the identity then the commutant $\varkappa(h_t^\tau)$ is trivial.*

In view of [2] we get:

COROLLARY 4. *All the above results hold for time changes Hölder continuous in the direction of K with the Hölder exponent greater than $\frac{1}{2}$ (in particular, C^1-functions in the direction of K) and bounded together with their reciprocals.*

Summarizing, we conclude that if $\tau \in \mathbf{K}(M)$ then h_t^τ inherits all the rigid properties of h_t found in [6].

Finally, let us note that for any $\Gamma_1, \Gamma_2 \in \mathbf{T}$ the horocycle flows $h_t^{(1)}$ and $h_t^{(2)}$ are Kakutani equivalent (see [4, 7]). This means that there is a time change $\tau_1: M_1 \to \mathbf{R}^+$ such that $h_t^{(2)}$ is isomorphic to $h_t^{\tau_1}$. It follows from [3] that τ_1 can be assumed differentiable and bounded on M_1, but some partial derivatives of τ_1 may be unbounded. Our Corollary 4 shows that there is no such a τ_1 with bounded τ_l^{-1} and bounded partial derivatives unless Γ_1 and Γ_2 are conjugate in G.

I am grateful to C. Moore for proving [2] at my request.

1. Preliminaries

Let $p: G \to M = \Gamma \backslash G$, $\Gamma \in \mathbf{T}$ be the covering projection $p(g) = \Gamma g$, $g \in G$. Let

$$G_t g = g \cdot \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \quad \text{and} \quad H_t g = g \cdot \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad g \in G,\ t \in \mathbf{R}$$

be the geodesic and the horocycle flows on G, covering g_t and h_t on M respectively. We shall also consider the flow $H_t^* g = g \cdot \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ on G, covering the flow $h_t^*(\Gamma g) = \Gamma H_t^* g$ on M. We have

$$\begin{aligned} G_t \circ H_s &= H_{se^{2t}} \circ G_t \\ G_t \circ H_s^* &= H_{se^{-2t}}^* \circ G_t \end{aligned} \tag{1.1}$$

$t, s \in \mathbf{R}$. We shall assume without loss of generality that the Riemannian metric in G is such that the length of the orbit intervals $[g, G_t g]$, $[g, H_t g]$ and $[g, H_t^* g]$ is t, $g \in G$. We shall denote by d the metric on G (or on M) induced by this Riemannian metric.

For $g \in G$ and $a, b, c > 0$ denote

$$U(g; a, b, c) = \{\bar{g} \in G: \bar{g} = H_r H_z^* G_p g \text{ for some } |p| \leq a,\ |z| \leq b,\ |r| \leq c\}$$

$$U(g; \varepsilon) = U(g; \varepsilon, \varepsilon, \varepsilon).$$

We have

$$U(g; a, b, c) = g \cdot U(\mathbf{e}; a, b, c)$$

where $\mathbf{e}$ denotes the identity element of G. It follows from (1.1) that

$$G_\tau U(g; a, b, c) = U(G_\tau g; a, b e^{-2\tau}, c e^{2\tau}), \quad \tau \in \mathbf{R}.$$

Denote $W(g) = \{H_s^* G_t g : t, s \in \mathbf{R}\}$. The set $W(g)$ is called the stable leaf of g for the geodesic flow G_t. Let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(\mathbf{e}; \varepsilon), \quad \varepsilon > 0.$$

Suppose that for $s>0$ there is $q(s)>0$ such that

$$H_{q(s)} g \in W(H_s \mathbf{e}).$$

The function $q(s)$ is uniquely defined by s and g and

$$H_s g = H_{r(s)} H_{z(s)}^* G_{p(s)}(H_s \mathbf{e})$$
$$q(s) = s + r(s)$$

where

$$\begin{aligned} e^{p(s)} &= (d-bs)^{-1} \\ z(s) &= b\, e^{p(s)} \\ r(s) &= -e^{p(s)}(bs^2+Ls-c) \\ L &= a-d. \end{aligned} \tag{1.2}$$

One can compute that if

$$g = G_p H_z^* \mathbf{e}$$

then

$$H_{q(s)} g = G_\alpha H_\beta^* (G_p H_z^* H_s \mathbf{e})$$

where

$$|\alpha| \leq L_1 |q(s)\, z|, \quad |\beta| \leq L_2 |z\alpha| \tag{1.3}$$

for some $0<L_1$, $L_2 \leq 2$, if z and p are sufficiently small.

For $0<\eta<1$ and

$$g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in U(\mathbf{e};\varepsilon)$$

denote

$$E=E(\mathbf{e},g,\eta)=\{s\in\mathbf{R}^+ : |bs^2+Ls|\leq 4s^{1-\eta}\}.$$

The set E consists of at most two connected components $E_0=E_0(\mathbf{e},g,\eta)=[0,l_0]$ and $E_1=E_1(\mathbf{e},g,\eta)=[l_1,l_2]$ for some $l_i=l_i(\mathbf{e},g,\eta)>0$ $i=0,1,2$ and $l_0\leq l_1\leq l_2$, where E_1 might be empty. One can compute that

$$|b|\leq\frac{\tilde{D}}{l_0^{1+\eta}},\quad |L|\leq\frac{\tilde{D}}{l_0^{\eta}}$$

for some $\tilde{D}>0$. This implies via (1.2) that

$$|z(s)|\leq\frac{D}{l_0^{1+\eta}},\quad |p(s)|\leq\frac{D}{l_0^{\eta}} \tag{1.4}$$

for some $0<D<100$ and all $0\leq s\leq l_0$, if $\varepsilon>0$ is sufficiently small.

For $x,y\in G$, $y\in U(x;\varepsilon)$ denote $l_0(x,y,\eta)=l_0(e,x^{-1}y,\eta)$ and for $0<r\leq l_0(x,y,\eta)$ denote

$$B(x,y,\eta)=\{(H_s x, H_{q(s)}y)\colon 0\leq s\leq r\}. \tag{1.5}$$

The set $B(x,y,\eta)$ will be called the (ε,η)-block of x,y of length r. Expression (1.4) shows that

$$H_{q(s)}y\in U\left(H_s x;\frac{D}{l_0^{\eta}},\frac{D}{l_0^{1+\eta}},0\right)$$

for all $0\leq s\leq r$.

2. Dynamical properties of h_t

In this section we shall prove the following

LEMMA 2.1 (Basic). *Let h_t be the horocycle flow on $(M=\Gamma|G,\mu)$, $\Gamma\in\mathbf{T}$. Given $0<\eta<1$, $0<\omega<1$ and $m>1$, there are $\gamma=\gamma(\eta)>0$, $0<\theta=\theta(\gamma)<1$, a compact $Y=Y(\gamma,\omega)\subset M$ with $\mu(Y)>1-\omega$ and $0<\varepsilon=\varepsilon(Y,m)<1$ possessing the following property. Let $u\in Y$, $v\in U(u;\varepsilon)$, and a subset $A\subset\mathbf{R}^+$ satisfy the following conditions* (i) $0\in A$, (ii) *if $s\in A$ then $h_s u\in Y$ and there is $t(s)>0$ increasing in s such that $h_{t(s)}v\in U(h_s u;\varepsilon)$*, (iii)

$|(t(s')-t(s))-(s'-s)|\leqslant(s'-s)^{1-\eta}$ *for all* $s,s'\in A$ *with* $\max\{(s'-s),\ (t(s')-t(s))\}\geqslant m$. *Then*

(1) *if* $\lambda\in A$ *and* $l(A\cap[0,\lambda])/\lambda>1-\theta/8$ *then there is* $s_\lambda\in A\cap[0,\lambda]$ *such that*

$$h_{t(s_\lambda)}v\in U\left(h_{s_\lambda}u;\frac{D}{\lambda^{2\gamma}},\frac{D}{\lambda^{1+2\gamma}},\varepsilon\right)$$

for some $D>0$, *where* $l(C)$ *denotes the length measure of* C,

(2) *if* $A\cap[0,\lambda]\neq\emptyset$ *for all* $\lambda\geqslant\lambda_0$ *and* $l(A\cap[0,\lambda])/\lambda>1-\theta/8$ *for all* $\lambda\in A$ *with* $\lambda\geqslant\lambda_0$ *then* $v=h_p u$ *for some* $p\in\mathbf{R}$.

Let us introduce some notations. Let I be an interval in $\mathbf{R}$ and let J_i, J_j be disjoint subintervals of I, $J_i=[x_i,y_i]$, $y_i<x_j$ if $i<j$. Denote $d(J_i,J_j)=l[y_i,x_j]=x_j-y_i$.

We shall use the following lemma whose proof in [5] is due to R. Solovay.

LEMMA 2.2. *Given* $\gamma>0$, *there is* $0<\theta=\theta(\gamma)<1$ *such that if* I *is an interval of length* t *(*t *is big) and* $\alpha=\{J_1,\dots,J_n\}$ *is a partition of* I *into black and white intervals such that*

(1) $d(J_i,J_j)\geqslant[\min\{l(J_i),l(J_j)\}]^{1+\gamma}$ *for any two black* $J_i,J_j\in\alpha$

(2) $l(J)\leqslant 3t/4$ *for any black* $J\in\alpha$

(3) $l(J)\geqslant 1$ *for any white* $J\in\alpha$

then $m_w(t,\alpha)\geqslant\theta$, *where* $m_w(t,\alpha)$ *denotes the total relative measure of white intervals of* α *on* I.

For given $0<\eta<1$, $0<\omega<1$ and $m>1$ we shall now specify the choice of γ, θ, Y and ε in Lemma 2.1.

First we choose $0<\gamma<\eta/2$ satisfying

$$\frac{2}{1+\gamma}-1+\eta>1+2\gamma. \tag{2.1}$$

The reason for this choice will be clear later.

Let $\theta=\theta(\gamma)$ be as in Lemma 2.2.

Since Γ is discrete, there are a compact $K\subset M$, $\mu(K)>1-0.1\min\{\gamma,\omega\}$ and $0<\Delta<1$ such that

$$\begin{gathered}\text{if } x\in p^{-1}(K)=\tilde{K},\ d(x,y)<\Delta \text{ and}\\ d(H_t x,\mathbf{D}H_s y)<\Delta \text{ for some } \mathbf{e}\neq\mathbf{D}\in\Gamma\\ \text{then } \max\{|t|,|s|\}\geqslant m.\end{gathered} \tag{2.2}$$

This implies that

$$\text{if } x\in \tilde{K},\ d(x,y)<\Delta \text{ and } d(x,\mathbf{D}\cdot y)<\Delta \\ \text{for some } \mathbf{D}\in\Gamma, \text{ then } \mathbf{D}=\mathbf{e}. \tag{2.3}$$

Since the geodesic flow g_t is ergodic on (M,μ), given $\omega>0$ there are a compact $\bar{Y}=\bar{Y}(\omega)\subset M$, $\mu(\bar{Y})>1-0.1\omega$ and $t_0=t_0(\bar{Y})>1$ such that

$$\text{if } w\in\bar{Y},\ t\geq t_0 \text{ then the relative length measure} \\ \text{of } K \text{ on } [w, g_{-t}w] \text{ is greater than } 1-0.2\gamma. \tag{2.4}$$

Set $Y=K\cap\bar{Y}$, $\mu(Y)>1-0.2\omega$.

Let $\varrho>1$ be such that

$$\tfrac{1}{2}\log\varrho>t_0 \quad\text{and}\quad 100\varrho^{-0.1\gamma}<\Delta/6. \tag{2.5}$$

Now we choose $0<\varepsilon<\Delta$ so small that if $g\in W_\varepsilon(\mathbf{e})$, $g\in G$, then

$$l_0(e,g,\eta)>\max\{\varrho,m\}. \tag{2.6}$$

(See (1.4).)

Thus $0<\gamma,\theta,\varepsilon<1$ and $Y\subset M$ have been chosen. The reason for these choices will become clear later.

Now let us describe a construction used in the proof of Lemma 2.3 below.

Let $u\in Y$, $v\in W_\varepsilon(u)$. We say that $(x,y)\in G\times G$ cover (u,v) if $y\in W_\varepsilon(x)$ and $p(x)=u$, $p(y)=v$. Let $B(x,y,\eta)$ be the (ε,η)-block of x,y of length r defined in (1.5). The set

$$B(u,v,\eta)=pB(x,y,\eta)=\{(h_s u, h_{q(s)}v)\colon 0\leq s\leq r\}$$

will be called the (ε,η)-block of u,v of length $r\leq l_0(x,y,\eta)=l_0(u,v,\eta)$. We shall write

$$B(u,v,\eta)=\{(u,v),(h_r u, h_{q(r)}v)\}=\{(u,v),(\bar{u},\bar{v})\}$$

emphasizing that (u,v) is the first and $(\bar{u},\bar{v})$ is the last pair of the block $B(u,v,\eta)$. It follows from (1.4) that $h_{q(s)}v=h^*_{z(s)}g_{p(s)}(h_s u)$ where

$$|p(s)|\leq\frac{D}{l_0^\eta},\quad |z(s)|\leq\frac{D}{l_0^{1+\eta}} \tag{2.7}$$

for all $s\in[0,r]$, where $l_0=l_0(u,v,\eta)$.

Henceforth the symbol D will always mean a positive constant which can be chosen less than 100 if $\varepsilon>0$ is sufficiently small.

Let $\beta=\{B_1,\dots,B_n\}$, $B_i=\{(u_i,v_i),(\bar{u}_i,\bar{v}_i)\}$ $i=1,\dots,n$ be a collection of pairwise disjoint (ε,η)-blocks on the orbit intervals $[u_1,h_\lambda u_1]$, $[v_1,h_{t(\lambda)}v_1]$ for some large $\lambda, t(\lambda)>0$, such that

$$\bar{u}_n=h_\lambda u_1,\quad \bar{v}_n=h_{t(\lambda)}v_1$$

$$u_i,\bar{u}_i\in Y,\quad v_i\in W_\varepsilon(u_i),\quad \bar{v}_i\in W_\varepsilon(\bar{u}_i)$$

$$u_i=h_{s_i}u_1,\quad v_i=h_{t_i}v_1,\quad \bar{u}_i=h_{\bar{s}_i}u_1,\quad \bar{v}_i=h_{\bar{t}_i}v_1$$

for some s_i, t_i, $\bar{s}_i$, $\bar{t}_i>0$, $\bar{s}_i<s_j\leq\lambda$, $\bar{t}_i<t_j\leq t(\lambda)$ if $i<j$, $i,j=1,\dots,n$.

Let $(x_i,y_i)\in G\times G$, $y_i\in W_\varepsilon(x_i)$ cover (u_i,v_i). Although $v_j\in W_\varepsilon(u_j)$ it is not necessarily true that $H_{t_j-t_i}y_i\in W_\varepsilon(H_{s_j-s_i}x_i)$, but there is a unique $\mathbf{D}\in\Gamma$ such that

$$\mathbf{D}\cdot y_j\in W_\varepsilon(x_j) \tag{2.8}$$

where $y_j=H_{t_j-t_i}y_i$, $x_j=H_{s_j-s_i}x_i$. We shall write

$$(u_i,v_i)\overset{\Gamma}{\sim}(u_j,v_j)\quad\text{if}\quad \mathbf{D}\neq\mathbf{e}\text{ in (2.8)}$$

$$(u_i,v_i)\overset{\mathbf{e}}{\sim}(u_j,v_j)\quad\text{if}\quad \mathbf{D}=\mathbf{e}\text{ in (2.8)}.$$

This definition does not depend on the choice of $(x_i,y_i)\in G\times G$ covering (u_i,v_i). For B_i, $B_j\in\beta$, $i<j$ we write

$$d(B_i,B_j)=s\quad\text{if}\quad u_j=h_s\bar{u}_i$$

$$B_i\overset{\Gamma}{\sim}B_j\quad\text{if}\quad (u_i,v_i)\overset{\Gamma}{\sim}(u_j,v_j)$$

$$B_i\overset{\mathbf{e}}{\sim}B_j\quad\text{if}\quad (u_i,v_i)\overset{\mathbf{e}}{\sim}(u_j,v_j).$$

We shall impose on β the following conditions

$$\begin{gathered} s_j-s_i>l_0(u_i,v_i,\eta)\\ |(t_j-t_i)-(s_j-s_i)|\leq 2(s_j-s_i)^{1-\eta}\\ |(\bar{t}_j-t_i)-(\bar{s}_j-s_i)|\leq 2(\bar{s}_j-s_i)^{1-\eta}\\ \text{if } i<j \text{ and } B_i\overset{\mathbf{e}}{\sim}B_j,\\ |(t_j-\bar{t}_i)-(s_j-\bar{s}_i)|\leq 2(s_j-\bar{s}_i)^{1-\eta}\\ \text{if } i<j \text{ and } B_i\overset{\Gamma}{\sim}B_j. \end{gathered} \tag{2.9}$$

Now let us construct a new collection $\beta_\gamma=\{\tilde{B}_1,\dots,\tilde{B}_k\}$ by the following procedure.

Take $B_1\in\beta$ and consider the following two cases. Case (i). There is no $j\in\{2,\dots,n\}$ such that $(u_1,v_1)\overset{\varepsilon}{\sim}(u_j,v_j)$. In this case we set $\tilde{B}_1=B_1$. Case (ii). There is $j\in\{2,\dots,n\}$ such that $(u_1,v_1)\overset{\varepsilon}{\sim}(u_j,v_j)$. Let $(x_1,y_1)\in G\times G$ cover (u_1,v_1) and let $x_j=H_sx_1$, $y_j=H_{q(s)}y_1$, where $s=s_j-s_1$. We have $t_j-t_1=q(s)$ and (x_j,y_j) cover (u_j,v_j). Let

$$E=E(x_1,y_1,\eta)=[0,l_0]\cup[l_1,l_2],\quad l_i=l_i(x_1,y_1,\eta),\quad i=0,1,2$$

be as in section 1. Expression (2.9) shows that $s\in[l_1,l_2]$. Denote

$$F(x_1,y_1,\eta)=\{s\in\mathbf{R}^+:|bs^2+Ls|\leq 4l_2^{1-\eta}\}$$

where

$$g=x_1^{-1}y_1=\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}$$

and $L=a-d$ (see (1.2)). The set $F(x_1,y_1,\eta)$ consists of at most two connected components $F_0=[0,l]$ and $F_1=[\bar{l},l_2]$ where $l>l_0(x_1,y_1,\eta)$, $\bar{l}<l_1$ and $l_2-\bar{l}=l$ if $F_1\neq\emptyset$.

One can compute as in section 1 that if $H_{q(s)}y_1=H^*_{z(s)}G_{p(s)}(H_sx_1)$ then

$$|z(s)|\leq\frac{Dl_2^{1-\eta}}{l^2},\quad |p(s)|\leq\frac{Dl_2^{1-\eta}}{l}\tag{2.10}$$

for all $s\in[0,l]$. To define $\tilde{B}_1$ for the case (ii) we consider the following two possibilities: (a) $\bar{l}-l>l^{1+\gamma}$. In this case we set $\tilde{B}_1=B_1$. (b) $\bar{l}-l\leq l^{1+\gamma}$. Then

$$l\leq l_2\leq 3l^{1+\gamma}.$$

This implies via (2.1), (2.10) and (1.2) that

$$|z(s)|\leq\frac{D}{l_2^{1+2\gamma}},\quad |p(s)|\leq\frac{D}{l_2^{2\gamma}}\tag{2.11}$$

for all $s\in[0,l_2]$. We set in this case $\tilde{B}_1=\{(u_1,v_1),(\bar{u}_{j_1},\bar{v}_{j_1})\}$, where

$$j_1=\max\{j\in\{2,\dots,n\}:B_1\overset{\varepsilon}{\sim}B_j\}.$$

Thus $\tilde{B}_1\in\beta_\gamma$ has been constructed. Suppose that $\tilde{B}_m=\{(u_{j_{m-1}+1},v_{j_{m-1}+1}),(\bar{u}_{j_m},\bar{v}_{j_m})\}$, $j_0=0$ has been constructed. To define $\tilde{B}_{m+1}$ we apply the above construction to $B_{j_m+1}\in\beta$. Thus β_γ is completely defined. It follows from the construction that if $i<j$ and $\tilde{B}_i\overset{\varepsilon}{\sim}\tilde{B}_j$, $\tilde{B}_i$, $\tilde{B}_j\in\beta_\gamma$ then

$$d(\tilde{B}_i, \tilde{B}_j) > \varrho > 1 \quad \text{and} \quad d(\tilde{B}_i, \tilde{B}_j) > [l(\tilde{B}_i)]^{1+\gamma}. \tag{2.12}$$

It follows from (2.7), (2.11) and (2.6) that if $\tilde{B}_i=\{(u_i', v_i'), (\bar{u}_i', \bar{v}_i')\}$ then

$$\begin{aligned} v_i' &\in U\left(u_i'; \frac{D}{r_i^{2\gamma}}, \frac{D}{r_i^{1+2\gamma}}, 0\right) \\ \bar{v}_i' &\in U\left(\bar{u}_i'; \frac{D}{r_i^{2\gamma}}, \frac{D}{r_i^{1+2\gamma}}, 0\right) \end{aligned} \tag{2.13}$$

for some $r_i \geq \max\{\varrho, l(\tilde{B}_i)\}$, $i=1, \ldots, k$.

Let $u_i'=h_{\tau_i}u_1$, $\bar{u}_i'=h_{\bar{\tau}_i}u_1$. Denote $J_i=[\tau_i, \bar{\tau}_i]\subset[0,\lambda]$, $i=1,\ldots,k$. We shall call J_i the black interval induced by $\tilde{B}_i$. The collection β_γ induces a partition α of $I=[0,\lambda]$ into black and white intervals. We shall denote

$$m_w(\beta_\gamma) = m_w(\alpha, \lambda).$$

LEMMA 2.3. *Let* $0<\eta<1$, $0<\omega<1$ *and* $m>1$ *be given. Let* $\gamma=\gamma(\eta)>0$, $0<\theta=\theta(\gamma)<1$, $Y=Y(\gamma,\omega)\subset M$ *with* $\mu(Y)>1-\omega$ *and* $0<\varepsilon=\varepsilon(Y,m)<1$ *be chosen as above. Let* $\beta=\{B_1,\ldots,B_n\}$, $B_i=\{(u_i,v_i), (\bar{u}_i,\bar{v}_i)\}$, $v_i\in W_\varepsilon(u_i)$, $\bar{v}_i\in W_\varepsilon(\bar{u}_i)$, $i=1,\ldots,n$ *be a collection of pairwise disjoint* (ε,η)*-blocks on the orbit intervals* $[u_1, h_\lambda u_1]$, $[v_1, h_{t(\lambda)}v_1]$ *such that* u_i, $\bar{u}_i\in Y$, $i=1,\ldots,n$ *and (2.9) holds for* β. *Suppose that* $m_w(\beta)<\theta$. *Then there is* $B\in\beta_\gamma$ *such that* $l(B)>3\lambda/4$.

Proof. First let us show that

$$d(B', B'') > [\min\{l(B'), l(B'')\}]^{1+\gamma} \tag{2.14}$$

for any $B'\neq B''\in\beta_\gamma$. Indeed, suppose on the contrary that there are $B'\neq B''\in\beta_\gamma$ with $l(B')\leq l(B'')$ such that

$$d(B', B'') \leq [l(B')]^{1+\gamma}. \tag{2.15}$$

It follows then from (2.12) that $B' \overset{\Gamma}{\sim} B''$. Let

$$\begin{aligned} B' &= \{(u', v'), (\bar{u}', \bar{v}')\} \\ B'' &= \{(u'', v''), (\bar{u}'', \bar{v}'')\} \\ u'' &= h_s\bar{u}', \quad v'' = h_t\bar{v}'. \end{aligned}$$

We shall assume for simplicity that s, $t>0$. We have

$$s \leqslant [l(B')]^{1+\gamma}, \quad t \leqslant 3s \tag{2.16}$$

by (2.15) and (2.9).

Let $(x, y) \in G\times G$ cover (u'', v''), $(\bar{x}, \bar{y}) \in G\times G$ cover $(\bar{u}', \bar{v}')$ and $x = H_s\bar{x}$. We have

$$y = \mathbf{D}\cdot H_t\bar{y} \quad \text{for some } \mathbf{e} \neq \mathbf{D} \in \Gamma \tag{2.17}$$

since $B' \overset{\Gamma}{\sim} B''$. It follows from (2.13) that

$$\begin{aligned} &x \in U\left(\bar{y}; \frac{D}{r^{2\gamma}}, \frac{D}{r^{1+2\gamma}}, s\right) \quad \text{and} \\ &\mathbf{D}\cdot\bar{y} \in U\left(x; \frac{D}{r^{2\gamma}}, \frac{D}{r^{1+2\gamma}}, -t\right) \end{aligned} \tag{2.18}$$

for some $r \geqslant \max\{\varrho, l(B')\}$. Also

$$0 < s, t \leqslant 3r^{1+\gamma} \quad \text{by (2.16)} \tag{2.19}$$

Let $\tau_0 = \frac{1}{2}\log r^{1+1.5\gamma}$, $\tau_0 > t_0$ by (2.5). Since $u'' \in Y \subset \bar{Y}$ it follows from the definition of $\bar{Y}$ and t_0 in (2.4) that the relative length measure of K on $[u'', g_{-\tau_0}u'']$ is greater than $1-0.2\gamma$. This implies that there is τ satisfying

$$(1-0.2\gamma)\,\tau_0 < \tau \leqslant \tau_0$$

such that $g_{-\tau}u'' \in K$ and therefore

$$z = G_{-\tau}x \in p^{-1}(K) = \tilde{K}. \tag{2.20}$$

We have using (2.18) and (1.1)

$$\begin{aligned} &z \in U\left(G_{-\tau}\bar{y}; \frac{D}{r^{2\gamma}}, \frac{De^{2\tau}}{r^{1+2\gamma}}, \frac{s}{e^{2\tau}}\right) \\ &\mathbf{D}\cdot G_{-\tau}\bar{y} \in U\left(z; \frac{D}{r^{2\gamma}}, \frac{De^{2\tau}}{r^{1+2\gamma}}, \frac{-t}{e^{2\tau}}\right) \end{aligned} \tag{2.21}$$

where

$$r^{1+1.1\gamma} < r^{(1+1.5\gamma)(1-0.2\gamma)} < e^{2\tau} \leqslant e^{2\tau_0} = r^{1+1.5\gamma}.$$

This implies via (2.21), (2.19) and (2.5) that

$$d(G_{-\tau}\bar{y}, z) < \Delta \quad \text{and} \quad d(\mathbf{D}\cdot G_{-\tau}\bar{y}, z) < \Delta$$

and that

$$\mathbf{D}=\mathbf{e} \quad \text{by (2.3) and (2.20)}$$

which contradicts (2.17). Thus we proved (2.14). It also follows from the proof that if $B' \overset{\Gamma}{\sim} B''$, $B', B'' \in \beta_\gamma$, then

$$d(B', B'') > \varrho > 1.$$

This and (2.12) imply that

$$d(B', B'') > \varrho > 1$$

for all $B' \neq B'' \in \beta_\gamma$.

Now let α be the partition of $I=[0,\lambda]$ into black and white intervals induced by β_γ. We have using (2.12) and (2.14)

$$l(J) > 1 \text{ for every white } J \in \alpha$$
$$d(J_i, J_j) > [\min\{l(J_i), l(J_j)\}]^{1+\gamma}$$
$$\text{for any two black } J_i, J_j \in \alpha.$$

Also

$$m_w(\alpha, \lambda) \leq m_w(\beta) < \theta$$

by the condition of the lemma. It follows then from Lemma 2.2 that there is a black $J \in \alpha$ with $l(J) > 3\lambda/4$. This says that there is $B \in \beta_\gamma$ such that $l(B) > 3\lambda/4$. This completes the proof. Q.E.D.

Proof of basic Lemma 2.1. For given $0<\eta<1$, $0<\omega<1$ and $m>1$ we choose $\gamma=\gamma(\eta)>0$, $0<\theta=\theta(\gamma)<1$, a compact $Y=Y(\gamma,\omega)\subset M$, $\mu(Y)>1-\omega$ and $0<\varepsilon=\varepsilon(Y,m)<1$ as above.

Let $u \in Y$, $v \in U(u;\varepsilon)$ and let $A \subset \mathbf{R}^+$ satisfy (i)–(iii). For $\lambda \in A$ denote

$$A_\lambda = A \cap [0,\lambda]$$

and assume that

$$l(A_\lambda)/\lambda > 1 - \frac{\theta}{8}. \tag{2.22}$$

Let us construct a collection $\beta(\lambda)$ of pairwise disjoint (ε,η)-blocks as in Lemma 2.3. To do this take u, v and set $u_1=u$, $v_1=v$. Let $(x_1, y_1) \in G\times G$ cover (u_1, v_1) and let

$$\bar{s}_1 = \sup\{s \in A_\lambda \cap [0, l_0(x_1, y_1, \eta)]: H_{t(s)} y_1 \in U(H_s x_1, \varepsilon)\}.$$

Let B_1 be the (ε, η)-block of u_1, v_1 of length $\bar{s}_1$, $B_1 = \{(u_1, v_1), (\bar{u}_1, \bar{v}_1)\}$, where $\bar{u}_1 = h_{\bar{s}_1} u_1 \in Y$, since Y is compact.

To define B_2 we take

$$s_2 = \inf\{s \in A_\lambda: s > \bar{s}_1\}$$

$$t(s_2) = \inf\{t(s): s \in A_\lambda, s > \bar{s}_1\}$$

and apply the above procedure to

$$u_2 = h_{s_2} u_1, \quad v_2 = h_{t(s_2)} v_1.$$

It is clear that $u_2 \in Y$, since Y is compact. This process defines a collection $\beta(\lambda) = \{B_1, \dots, B_n\}$ of (ε, η)-blocks on the orbit intervals $[u_1, h_\lambda u_1]$, $[v_1, h_{t(\lambda)} v_1]$, $B_i = \{(u_i, v_i), (\bar{u}_i, \bar{v}_i)\}$, $u_i, \bar{u}_i \in Y$, $i = 1, \dots, n$. Let

$$u_i = h_{s_i} u_1, \quad \bar{u}_i = h_{\bar{s}_i} u_1$$

$$v_i = h_{t_i} v_1, \quad \bar{v}_i = h_{\bar{t}_i} v_1, \quad i = 1, 2, \dots, n.$$

Suppose that $B_i \overset{\Gamma}{\sim} B_j$, $i<j$. Then

$$\max\{s_j - \bar{s}_i, t_j - \bar{t}_i\} \geq m$$

by (2.2) and our choice of ε. This implies via (iii) that

$$|(t_j - \bar{t}_i) - (s_j - \bar{s}_i)| \leq (s_j - \bar{s}_i)^{1-\eta}.$$

Suppose that $B_i \overset{\varepsilon}{\sim} B_j$, $i<j$. It follows from the construction of B_i, B_j that

$$s_j - s_i \geq l_0(u_i, v_i, \eta) > m$$

and therefore

$$|(t_j - t_i) - (s_j - s_i)| \leq (s_j - s_i)^{1-\eta}$$

$$|(\bar{t}_j - t_i) - (\bar{s}_j - s_i)| \leq (\bar{s}_j - s_i)^{1-\eta}$$

by (iii). This implies that

$$s_j - s_i > l_0(u_i, v_i, \eta)$$

and that B_i and B_j are disjoint.

Thus $\beta(\lambda)$ satisfies all conditions of Lemma 2.4. We have

$$m_w(\beta(\lambda)) < \theta$$

by (2.22), since each $s \in A_\lambda$ belongs to a black interval induced by $\beta(\lambda)$. This implies by Lemma 2.3 that there is $B_\lambda \in \beta_\gamma(\lambda)$, $B_\lambda = \{(u_\lambda, v_\lambda), (\bar{u}_\lambda, \bar{v}_\lambda)\}$ such that

$$l(B_\lambda) > 3\lambda/4.$$

It follows then from (2.13) that

$$v_\lambda \in U\left(u_\lambda; \frac{D}{\lambda^{2\gamma}}, \frac{D}{\lambda^{1+2\gamma}}, 0\right). \tag{2.23}$$

This proves (1) with s_λ such that $h_{s_\lambda} u = u_\lambda$.

Now let $A_\lambda \neq \emptyset$ for all $\lambda \geq \lambda_0$ and let

$$l(A_\lambda)/\lambda > 1 - \frac{\theta}{8} \tag{2.24}$$

for all $\lambda \in A$ with $\lambda \geq \lambda_0$. It follows from (2.24) that there are $\lambda_n \in A$, $\lambda_n \geq \lambda_0$, $n = 1, 2, \ldots$, $\lambda_n \to \infty$, $n \to \infty$ such that

$$\lambda_n < \lambda_{n+1} < \tfrac{9}{8}\lambda_n, \quad n = 1, 2, \ldots. \tag{2.25}$$

Let $B_{\lambda_n} \in \beta_\gamma(\lambda_n)$ be as above. We have

$$l(B_{\lambda_n}) > 3\lambda_n/4.$$

This and (2.25) imply that

$$B_{\lambda_n} \cap B_{\lambda_{n+1}} \neq \emptyset, \quad n = 1, 2, \ldots$$

and therefore

$$B_{\lambda_n} \subset B_{\lambda_{n+1}}, \quad n = 1, 2, \ldots.$$

This implies via (2.23) that

$$v_{\lambda_1} \in U\left(u_{\lambda_1}; \frac{D}{\lambda_n^{2\gamma}}, \frac{D}{\lambda_n^{1+2\gamma}}, 0\right)$$

for all λ_n, $n = 1, 2, \ldots$. This says that $v_{\lambda_1} = u_{\lambda_1}$ and therefore $v = h_p u$ for some $p \in \mathbf{R}$. Q.E.D.

3. The class $K(M)$

Let us recall that a positive measurable function τ on $M=\Gamma\backslash G$, $\Gamma\in\mathbf{T}$ belongs to $\mathbf{K}(M)$, if τ and τ^{-1} are bounded and

$$\left|\int_M \varphi(x)\,\varphi(h_t x)\,d\mu\right| \leq D|t|^{-\alpha}. \tag{3.1}$$

for some $D=D(\tau)>0$, $0<\alpha=\alpha(\tau)<1$ and all $t\neq 0$, where $\varphi=\tau-\bar{\tau}$.

LEMMA 3.1. *Let $\varphi: M\to\mathbf{R}$ be measurable, bounded, $\bar{\varphi}=0$ and let* (3.1) *hold for φ with some $D(\varphi)$, $\alpha(\varphi)>0$. Then given $\omega>0$ there are $P=P(\omega)\subset M$ with $\mu(P)>1-\omega$ and $m=m(P)>0$ such that if $x\in P$ then*

$$\left|\int_0^t \varphi(h_u x)\,du\right| \leq t^{1-\alpha'}$$

for all $t\geq m$, where $\alpha'=\alpha'(\varphi)=\alpha(\varphi)/8$.

Proof. Denote

$$s_t(x)=\int_0^t \varphi(h_u x)\,du$$

$$C(t)=\int_M \varphi(x)\,\varphi(h_t x)\,d\mu.$$

We claim that

$$\int_M [s_t(x)]^2\,d\mu \leq \bar{D}t^{2-\alpha} \tag{3.2}$$

for some $\bar{D}>0$ and all $t>0$, where $\alpha=\alpha(\varphi)$ is as in (3.1). Indeed, we have using (3.1)

$$\begin{aligned}\int_M [s_t(x)]^2\,d\mu &= \int_M\left(\int_0^t\int_0^t \varphi(h_s x)\,\varphi(h_u x)\,ds\,du\right)d\mu\\ &= \int_0^t\int_0^t C(u-s)\,ds\,du \leq 2\int_0^t\left(\int_0^t |C(v)|\,dv\right)ds\\ &\leq \frac{2D}{1-\alpha}t^{2-\alpha}=\bar{D}t^{2-\alpha}.\end{aligned}$$

It follows from (3.2) that

$$\mu\{x\in M: |s_t(x)|\geq t^{1-\alpha/4}\}\leq \bar{D}t^{-\alpha/2}. \tag{3.3}$$

Denote

$$A_t = \{x \in M : |s_t(x)| < t^{1-\alpha/4}\}, \quad t > 0$$
$$p_n = n^{4/\alpha}, \quad n = 1, 2, \ldots .$$

We have using (3.3)

$$\mu(A_{p_n}) \geq 1 - \frac{\bar{D}}{n^2}, \quad n = 1, 2, \ldots .$$

Given $\omega > 0$, let $k_0 = k_0(\omega)$ be such that

$$\bar{D} \sum_{k \geq k_0} \frac{1}{k^2} < \omega$$

and let $P = P(\omega) = \bigcap_{k \geq k_0} A_{p_k}$. We have

$$\mu(P) > 1 - \omega$$

and if $x \in P$ then

$$|s_{p_k}(x)| < p_k^{1-\alpha/4}$$

for all $k \geq k_0$.

Now let $t \geq p_{k_0}$ and let $k \geq k_0$ be such that

$$p_k < t \leq p_{k+1}.$$

One can compute that

$$p_{k+1} - p_k = Q p_k^{1-\alpha/4}$$

for some $Q > 0$ and all $k = 1, 2, \ldots$. This implies that

$$t = p_k + q$$

where $0 < q \leq Q p_k^{1-\alpha/4}$. For $x \in P$ we have using (3.2)

$$|s_t(x)| \leq |s_{p_k}(x)| + \left| \int_{p_k}^{p_k+q} \varphi(h_u x)\, du \right| \leq \bar{Q} p_k^{1-\alpha/4} < \bar{Q} t^{1-\alpha/4}$$

for some $\bar{Q} > 0$, since φ is bounded. This completes the proof. Q.E.D.

4. Time changes and a conjugacy ψ

In this section we shall prove Theorem 1.

Let $M_i=\Gamma_i|G$, $\Gamma_i\in\mathbf{T}$ and let $\tau_i: M_i\to\mathbf{R}^+$ be a time change for the horocycle flow $h_t^{(i)}$ on (M_i,μ_i), $i=1,2$. Suppose that $\tau_i\in\mathbf{K}(M_i)$ and let

$$\int_{M_i}\tau_i\,d\mu_i=a>0$$

$$\varphi_i=\tau_i-a$$

$$\sup_{x\in M_i}\{\tau_i(x),\tau_i^{-1}(x)\}\leqslant K \tag{4.1}$$

for some $K>1, i=1,2$.

We shall assume without loss of generality that $a=1$. Let $h_t^{\tau_i}$ be obtained from $h_t^{(i)}$ by the time change τ_i and let $\psi:(M_1,\mu_{\tau_1})\to(M_2,\mu_{\tau_2})$ be measure preserving and

$$\psi h_t^{\tau_1}(x)=h_t^{\tau_2}\psi(x) \tag{4.2}$$

for μ_{τ_1}-a.e. $x\in M_1$ and all $t\in\mathbf{R}$, where $d\mu_{\tau_i}(x)=\tau_i(x)\,d\mu_i(x)$, $i=1,2$.

Let $0<\alpha_i'=\alpha'(\varphi_i)<1$ be as in Lemma 3.1 for $\varphi_i=\tau_i-1$, $i=1,2$ and let

$$\eta=\tfrac{1}{2}\min\{\alpha_1',\alpha_2'\}.$$

Let $\gamma=\gamma(\eta)>0$ and $0<\theta=\theta(\gamma)<1$ be chosen as in Lemma 2.1.

Since ψ is measure preserving and μ_{τ_i} is equivalent to μ_i, $i=1,2$, there is $0<\omega<\theta/(200K^4)$ such that

$$\mu_1(\psi^{-1}(A))<\frac{\theta}{200K^4}$$

whenever $\mu_2(A)<\omega$.

Let $P_i=P_i(\omega)\subset M_i$, $\mu_i(P_i)>1-\omega$ and $m_i=m_i(P_i)>0$ be as in Lemma 3.1 for φ_i, $i=1,2$. If $x\in P_i$ then

$$\left|\int_0^t\varphi_i(h_u^{(i)}x)\,du\right|\leqslant t^{1-2\eta}$$

for all $t\geqslant\max\{m_1,m_2\}$, $i=1,2$. This implies that there is $m_0\geqslant\max\{m_1,m_2\}$ such that

$$\left| \int_0^t \varphi_i(h_u^{(i)} x)\, du \right| \leqslant \frac{1}{200K^4} t^{1-\eta} \tag{4.3}$$

for all $x \in P_i$ and all $t \geqslant m_0$, $i = 1, 2$.

Set $m = 2K^4 m_0$ and let $Y = Y(\gamma, \omega) \subset M_2$, $\mu_2(Y) > 1 - \omega$ and $0 < \varepsilon = \varepsilon(Y, m) < 1$ be as in Lemma 2.1 for $h_t^{(2)}$ on (M_2, μ_2).

Since $\psi: M_1 \to M_2$ is measurable, there is a compact $\Lambda \subset M_1, \mu_1(\Lambda) > 1 - \omega$ such that ψ is uniformly continuous on Λ. Let $0 < \delta < \varepsilon/2$ be such that if u, $v \in \Lambda$, $d(u, v) < \delta$ then $d(\psi(u), \psi(v)) < \varepsilon/2$. Let now $0 < \delta' < \delta$ be so small that if $x \in G$ and $y \in W_{\delta'}(x)$ then

$$H_{q(t)} y \in W_{\delta/2}(H_t x) \quad \text{and} \quad |q(t) - t| \leqslant \delta t \tag{4.4}$$

for all $0 \leqslant t \leqslant 2K^2$ (see section 1 for the definition of $q(t)$ and W_δ).

Let $X = P_1 \cap \Lambda \cap \psi^{-1}(P_2 \cap Y)$. We have

$$\mu_1(X) > 1 - \frac{\theta}{50K^4}$$

For $x \in M_i$ and $t \in \mathbf{R}$ denote

$$\xi_i(x, t) = \int_0^t \tau_i(h_u^{(i)} x)\, du, \quad i = 1, 2.$$

For $u \in M_1$ and $t \in \mathbf{R}$ let $z(u, t)$ be defined by

$$\xi_1(u, t) = \xi_2(\psi(u), z(u, t)). \tag{4.5}$$

It follows from (4.2) that

$$\psi h_t^{(1)}(u) = h_{z(u,t)}^{(2)} \psi(u)$$

for μ_1-a.e. $u \in M_1$ and all $t \in \mathbf{R}$. Expression (4.1) implies that

$$\frac{1}{K^2} t \leqslant z(u, t) \leqslant K^2 t \tag{4.6}$$

for all $u \in M_1$, $t \geqslant 0$.

Since $h_t^{(1)}$ is ergodic, there are $V_n \subset M$, $\mu_1(V_n) > 1 - 2^{-n}$ and $t_n > 1$, $t_n \nearrow \infty$, $n \to \infty$ such that if $u \in V_n$ and $|t| \geqslant t_n/2$ then

$$|z(u, t) - t| \leqslant |t| n^{-1} \tag{4.7}$$

and

$$\text{the relative length measure of } X \text{ on } [u, h_t^{(1)}u] \text{ is at least } 1-\frac{\theta}{40K^4}. \tag{4.8}$$

We shall use (4.7) in the proof of Lemma 4.2 below and (4.8) in the proof of Lemma 4.1.

Let $r_n=\frac{1}{2}\log t_n^{1+\gamma}$ and let $V=\bigcap_n g_{-r_n}^{(1)} V_n$, $\mu_1(V)>0$.

LEMMA 4.1. *Let $u, v\in V$ and $v=g_\alpha^{(1)}h_\beta^{*(1)}u$ for some $|\alpha|, |\beta|<\delta'$. Then*

$$d(\bar v_n, g_\alpha^{(2)}h_\beta^{*(2)}\bar u_n)\to 0,\quad n\to\infty,$$

where $\bar u_n=g_{-r_n}^{(2)}\psi g_{r_n}^{(1)}u$, $\bar v_n=g_{-r_n}^{(2)}\psi g_{r_n}^{(1)}v$.

Proof. Denote

$$u_n=g_{r_n}^{(1)}u,\quad v_n=g_{r_n}^{(1)}v$$
$$u_n'=\psi(u_n),\quad v_n'=\psi(v_n).$$

We have using (4.4) and (1.3)

$$\begin{gathered} v_n=g_\alpha^{(1)}h_{\beta_n}^{*(1)}u_n \\ h_t^{(1)}u_n\in U\left(h_{-\beta_n}^{*\,(1)}g_{-\alpha}^{(1)}h_{q(t)}^{(1)}v_n;\frac{4}{t_n^\gamma},\frac{4}{t_n^{1+2\gamma}},0\right)\end{gathered} \tag{4.9}$$

for all $0\le t\le 2t_n$, where $\beta_n=\beta t_n^{-(1+\gamma)}$, $n=1,2,\dots$.

For $p\in\mathbf{R}$ denote

$$u_n(p)=h_p^{(1)}u_n,\quad v_n(p)=h_p^{(1)}v_n$$
$$s(p)=z(u_n,p),\quad a(p)=z(v_n,p).$$

We have

$$u_n'(s(p))=h_{s(p)}^{(2)}u_n'=\psi u_n(p)$$
$$v_n'(a(p))=h_{a(p)}^{(2)}v_n'=\psi v_n(p).$$

Let

$$B_n=\{p\in[0,t_n]: u_n(p)\in X, v_n(q(p))\in X\},\quad n=1,2,\dots.$$

It follows from (4.8) and (4.9) that

$$l(B_n)/t_n>1-\frac{\theta}{18K^4},\quad n=1,2,\dots \tag{4.10}$$

if $\delta'>0$ is sufficiently small. It follows from the definition of X that if $p\in B_n$ then

$$u_n(p), v_n(q(p))\in P_1\cap\Lambda$$
$$u'_n(s(p)), v'_n(a(q(p)))\in Y\cap P_2$$

and

$$v'_n(a(q(p)))\in U(u'_n(s(p));\varepsilon/2),\quad n=1,2,\dots. \tag{4.11}$$

Suppose that

$$s(p')-s(p)\geq m$$

for some p, $p'\in B_n$, $p<p'$. It follows then from (4.6) and (4.9) that

$$p'-p\geq m/K^2=2K^2m_0$$
$$q(p')-q(p)\geq K^2m_0$$
$$a(q(p'))-a(q(p))\geq m_0$$

and therefore

$$\begin{aligned}|(s(p')-s(p))-(p'-p)|&\leq 0.01(s(p')-s(p))^{1-\eta}/K^2\\ |(a(q(p'))-a(q(p)))-(q(p')-q(p))|&\leq 0.01(q(p')-q(p))^{1-\eta}/K^2\end{aligned} \tag{4.12}$$

by (4.3) and (4.6), since $u_n(p)$, $v_n(q(p))\in P_1$ and $u'_n(s(p))$, $v'_n(a(q(p)))\in P_2$.

Denote

$$p_0=p_0(n)=\inf B_n,\quad \bar p=\bar p(n)=\sup B_n$$
$$s_0=s_0(n)=s(p_0),\quad \bar s=\bar s(n)=s(\bar p),\quad \bar s-s_0=\lambda_n$$
$$a_0=a_0(n)=a(q(p_0)),\quad \bar a=\bar a(n)=a(q(\bar p))$$
$$B'_n=s(B_n)\subset[s_0,\bar s],\quad n=1,2,\dots.$$

We can assume without loss of generality that p_0, $\bar p\in B_n$. We have using (4.10) and (4.6)

$$\begin{gathered}\left(1-\frac{\theta}{18K^4}\right)t_n\leq\bar p-p_0\leq t_n\\ \frac{\left(1-\frac{\theta}{18K^4}\right)}{K^2}t_n\leq\lambda_n\leq K^2t_n\\ l(B'_n)/\lambda_n\geq 1-\frac{\theta}{18}.\end{gathered} \tag{4.13}$$

Let

$$A'_n = \{s_0\} \cup (B'_n \cap [s_0+m, \bar{s}]).$$

We have

$$l(A'_n)/\lambda_n \geq 1 - \frac{\theta}{15} \tag{4.14}$$

if n is sufficiently large. It follows from (4.12) that

$$|(a(q(p)) - a_0) - (q(p) - q(p_0))| \leq 0.01(q(p) - q(p_0))^{1-\eta}/K^2$$

for all p with $s(p) \in A'_n$.

Denote

$$\begin{gathered} x_n = u_n(p_0), \quad y_n = v_n(q(p_0)) \\ x'_n = \psi(x_n) = u'_n(s_0), \quad y'_n = \psi(y_n) = v'_n(a_0) \\ y'_n \in U(x'_n; \varepsilon/2). \end{gathered} \tag{4.15}$$

We have

$$x_n = g^{(1)}_{c_n} h^{*\,(1)}_{b_n} y_n$$

for some $b_n, c_n \in \mathbf{R}$, $n=1,2,\ldots$. Denote

$$w_n = g^{(2)}_{c_n} h^{*\,(2)}_{b_n} y'_n \in W_{\delta/2}(y'_n).$$

We have

$$w_n \in U(x'_n; \varepsilon)$$

by (4.15). Let

$$A_n = \{s - s_0 : s \in A'_n\} \subset [0, \lambda_n].$$

We have

$$0, \lambda_n \in A_n, \quad l(A_n)/\lambda_n > 1 - \frac{\theta}{15} \quad \text{and if } s \in A_n \text{ then } h^{(2)}_s x'_n \in Y. \tag{4.16}$$

Let $\chi: [0, 2K^2 t_n] \to \mathbf{R}$ be defined by

$$H_{\chi(p)} \bar{w}_n \in W_{\delta/2}(H_p \bar{y}'_n)$$

where $(\bar{w}_n, \bar{y}'_n) \in G \times G$ cover (w_n, y'_n). The function χ for w_n, y'_n is analogous to the function q for u_n, v_n. One can see that

$$\chi(q(p')-q(p))=p'-p$$

for every $p, p'\geqslant p_0$. For $s=s(p)-s_0\in A_n$ let

$$t(s)=\chi(a(q(p))-a_0).$$

We have using (4.11)

$$h^{(2)}_{t(s)}w_n\in W_{\delta/2}(h^{(2)}_{a(q(p))}y'_n) \quad \text{and} \quad h^{(2)}_{t(s)}w_n\in U(h^{(2)}_s x'_n;\varepsilon) \tag{4.17}$$

for all $s\in A_n$ with $s=s(p)-s_0$.

Expressions (4.16) and (4.17) show that the subset $A_n\subset[0,\lambda_n]$ satisfies conditions (i)–(ii) of Lemma 2.1 with x'_n, w_n instead of u, v respectively. We claim that A_n satisfies (iii), too. Indeed, let us show that if $s, s'\in A_n$, $s<s'$ and

$$\max\{(t(s')-t(s)),\,(s'-s)\}\geqslant m$$

then

$$|(t(s')-t(s))-(s'-s)|\leqslant(s'-s)^{1-\eta}. \tag{4.18}$$

So let $s'=s(p')$, $s=s(p)$, $s<s'$, s, $s'\in A_n$ and suppose that

$$s'-s\geqslant m.$$

Denote

$$a'=a(q(p')),\quad a=a(q(p)).$$

We have using (4.12)

$$\begin{aligned}|(s'-s)-(p'-p)|&\leqslant 0.01(s'-s)^{1-\eta}/K^2\\ |(a'-a)-(q(p')-q(p))|&\leqslant 0.01(q(p')-q(p))^{1-\eta}/K^2.\end{aligned} \tag{4.19}$$

Also

$$t(s')-t(s)=\chi(a')-\chi(a)$$

$$|(a-a_0)-(q(p)-q(p_0))|\leqslant 0.01(q(p)-q(p_0))^{1-\eta}/K^2.$$

This implies that

$$\chi(a)=(p-p_0)+f \tag{4.20}$$

where $|f|\leqslant 0.02t_n^{1-\eta}/K^2$. Let

$$h^{(1)}_{p-p_0}x_n=g^{(1)}_{c(p)}h^{*(1)}_{b(p)}(h^{(1)}_{q(p)-q(p_0)}y_n).$$

It follows from (4.9), (1.3) and (4.20) that

$$h^{(2)}_{\chi(a)}w_n \in U\left(g^{(2)}_{c(p)}h^{*(2)}_{b(p)}(h^{(2)}_a y'_n); \frac{0.02t_n^{-\eta}}{K^2}; \frac{0.02t_n^{1-\eta}}{K^2}, 0\right).$$

This implies that

$$|(\chi(a')-\chi(a))-(p'-p)| \leqslant 0.08(s'-s)^{1-\eta}$$

by (4.19), (1.2) and (4.6). This and (4.19) show that

$$|(\chi(a')-\chi(a))-(s'-s)| \leqslant (s'-s)^{1-\eta}$$

or

$$|(t(s')-t(s))-(s'-s)| \leqslant (s'-s)^{1-\eta}.$$

Thus we have proved (4.18) assuming that $s'-s \geqslant m$. Similarly, we can prove (4.18) assuming that $t(s')-t(s) \geqslant m$.

Thus $A_n \subset [0, \lambda_n]$ satisfies all conditions of Lemma 2.1. Using this lemma and (4.14) we conclude that there is $s_n \in A_n$ with

$$h^{(2)}_{t(s_n)}w_n \in U\left(h^{(2)}_{s_n}x'_n; \frac{D}{\lambda_n^{2\gamma}}, \frac{D}{\lambda_n^{1+2\gamma}}, \varepsilon\right). \tag{4.21}$$

Let $s_n = s(p_n) - s_0$, $a_n = a(q(p_n)) - a_0$. We have via (4.9)

$$h^{(2)}_{t(s_n)}w_n \in U\left(h^{*(2)}_{-\beta_n}g^{(2)}_{-\alpha}h^{(2)}_{a_n}y'_n; \frac{2K^2}{t_n^{\gamma}}, \frac{2K^2}{t_n^{1+2\gamma}}, 0\right)$$

This implies via (4.21) that if we denote $s(p_n) = \bar{s}_n$, $a(q(p_n)) = \bar{a}_n$ then

$$h^{(2)}_{\bar{a}_n}v'_n \in U\left(g^{(2)}_{\alpha}h^{*(2)}_{\beta_n}h^{(2)}_{\bar{s}_n}u'_n; \frac{DK^2}{\lambda_n^{\gamma}}, \frac{DK^2}{\lambda_n^{1+2\gamma}}, \varepsilon\right). \tag{4.22}$$

We have

$$\bar{v}_n = g^{(2)}_{-r_n}v'_n, \quad \bar{u}_n = g^{(2)}_{-r_n}u'_n, \quad e^{2r_n} = t_n^{1+\gamma}.$$

This implies that

$$d(\bar{v}_n, g^{(2)}_{\alpha}h^{*(2)}_{\beta}\bar{u}_n) \to 0, \quad n \to \infty$$

by (4.22), (4.13) and (1.1), since $0 < \bar{a}_n \leqslant 2K^2 t_n$ and $0 < \bar{s}_n \leqslant t_n$. This completes the proof. Q.E.D.

LEMMA 4.2. *If* $u\in V$ *and* $v=h_p^{(1)}u$ *for some* $p\in\mathbf{R}$ *then* $d(\bar{u}_n, h_p^{(2)}\bar{v}_n)\to 0$ *when* $n\to\infty$, *where* $\bar{u}_n, \bar{v}_n$ *are as in Lemma* 4.1.

Proof. Let $p\neq 0$ and let u_n, v_n, u'_n, v'_n and $s\colon \mathbf{R}\to\mathbf{R}$ be as in the proof of Lemma 4.1. We have

$$u_n\in V_n$$
$$v_n=h^{(1)}_{pt_n^{1+\gamma}}u_n$$
$$v'_n=h^{(2)}_{s(pt_n^{1+\gamma})}u'_n.$$

It follows from (4.7) that

$$|s(pt_n^{1+\gamma})-pt_n^{1+\gamma}|\leq |p|\,t_n^{1+\gamma}n^{-1}$$

if n is so big that $|p|\,t_n^{1+\gamma}\geq t_n$. This implies that

$$d(\bar{v}_n, h_p^{(2)}\bar{u}_n)\leq |p|\,n^{-1}$$

if n is sufficiently large. This completes the proof. Q.E.D.

COROLLARY 4.1. *There are an* $h_t^{(1)}$*-invariant subset* $\Omega\subset M_1$ *with* $\mu_1(\Omega)=1$ *and a subsequence* $\{n_k; k=1,2,\ldots\}\subset\{n\colon n=1,2,\ldots\}$ *such that if* $u\in\Omega$ *then* $\lim_{k\to\infty}\bar{u}_{n_k}=\zeta(u)\in M_2$ *exists and* $\zeta(h_p^{(1)}u)=h_p^{(2)}\zeta(u)$ *for all* $p\in\mathbf{R}$, $u\in\Omega$.

Proof. Let $M_2=\bigcup_{n=1}^{\infty}K_n$, where K_n are compact and $\mu_2(M_2-K_n)<2^{-n}$, $n=1,2,\ldots$. Denote

$$\tilde{K}_n=M_2-K_n$$
$$F_n=g^{(1)}_{-r_n}\psi^{-1}g^{(2)}_{r_n}\tilde{K}_n,\quad n=1,2,\ldots.$$

We have

$$\sum_{n=1}^{\infty}\mu_1(F_n)<\infty.$$

Let

$$F=\{u\in M_1\colon u \text{ belongs to finitely many } F_n\}.$$

By the Borel-Cantelli lemma

$$\mu(F)=1. \tag{4.23}$$

If $u \in F$ then $\bar{u}_n$ belongs to finitely many $\tilde{K}_n$. This implies that there is a subsequence $n_k(u)$, $k=1,2,\ldots$ such that $\bar{u}_{n_k(u)}$ converges in M_2.

Let $V \subset M_1$, $\mu_1(V)>0$ be as in Lemmas 4.1 and 4.2. In view of (4.23) we can assume that $V \subset F$. Since $\mu_1(V)>0$, there is $u^0 \in V$ such that

$$\nu(V \cap W_{\delta'}(u^0)) > 0$$

where ν denotes the Riemannian volume on the stable leaf $W(u^0)$. Since $u^0 \in F$, there is a subsequence $n_k = n_k(u^0)$ such that $\bar{u}^0_{n_k}$ converges in M_2. Let

$$\Omega = \{h_p^{(1)} w : p \in \mathbf{R},\ w \in V \cap W_{\delta'}(u^0)\}.$$

The set Ω is $h_t^{(1)}$-invariant and $\mu_1(\Omega)>0$. Since $h_t^{(1)}$ is ergodic, $\mu_1(\Omega)=1$. It follows then from Lemmas 4.1 and 4.2 that $\lim_{k\to\infty} \bar{u}_{n_k} = \zeta(u)$ exists for every $u \in \Omega$ and $\zeta(h_p^{(1)} u) = h_p^{(2)} \zeta(u)$ for all $p \in \mathbf{R}$, $u \in \Omega$. This completes the proof. Q.E.D.

Proof of Theorem 1. Let $\Omega \subset M_1$, $\mu_1(\Omega)=1$ and a subsequence $\{n_k\} \subset \{n\}$ be as in Corollary 4.1. We can assume without loss of generality that $\Omega = M_1$ and $\{n_k\} = \{n\}$. Thus

$$\lim_{n\to\infty} \bar{u}_n = \zeta(u)$$

exists for all $u \in M_1$ and

$$\zeta(h_p^{(1)} u) = h_p^{(2)} \zeta(u)$$

for all $p \in \mathbf{R}$, $u \in M_1$. This says that the map

$$\zeta : (M_1, \mu_1) \to (M_2, \mu_2)$$

is a measurable conjugacy between $h_t^{(1)}$ and $h_t^{(2)}$. In fact, ζ is measure preserving (see [6]). It follows from the rigidity theorem [6] that there are $C \in G$, $a \in \mathbf{R}$ such that

$$C\Gamma_1 C^{-1} \subset \Gamma_2 \quad \text{and} \quad \zeta(u) = h_a^{(2)} \psi_C(u) \tag{4.24}$$

for μ_1-a.e. $u \in M_1$, where $\psi_C(\Gamma_1 g) = \Gamma_2 C g$, $g \in G$. It follows from Lemma 4.1 that if u, $v \in V$, $v = g_\alpha^{(1)} h_\beta^{*(1)} u$ for some $|\alpha|$, $|\beta| < \delta'$ then

$$\zeta(v) = g_\alpha^{(2)} h_\beta^{*(2)} \zeta(u).$$

This implies that $a=0$ in (4.24) and therefore

$$\zeta(u)=\psi_C(u)$$

for μ_1-a.e. $u\in M_1$.

Now we have to show that

$$\psi(u)=h^{(2)}_{\sigma(u)}\psi_C(u)$$

for some $\sigma(u)\in\mathbf{R}$ and μ_1-a.e. $u\in M_1$.

Let $0<\eta,\gamma,\theta,\omega,\varepsilon<1$, $m\geq 1$, Y, $P_2\subset M_2$ and P_1 be chosen as above.

Let $S\subset M_1$, $\mu_1(S)>1-\omega$ and $n_0\geq 1$ be such that if $u\in S$ and $n\geq n_0$ then $d(\bar{u}_n,\zeta(u))<\varepsilon$.

Let $n\geq n_0$ be fixed. Denote

$$\bar{X}=g^{(1)}_{-r_n}(P_1\cap\psi^{-1}P_2)\cap S\cap\zeta^{-1}(Y).$$

We have

$$\mu_1(\bar{X})>1-\frac{\theta}{50}.$$

Let $Q\subset M_2$, $\mu_1(Q)=1$ be the generic set of $\bar{X}$ for $h^{(1)}_t$. This means that if $u\in Q$ then the relative length measure of $\bar{X}$ on $[u,h^{(1)}_t u]$ tends to $\mu_1(\bar{X})$ when $t\to\infty$. Denote $\bar{Q}=Q\cap\bar{X}$, $\mu_1(\bar{Q})>0$.

Let $u\in\bar{Q}$ and let

$$A=A(u)=\{s\in\mathbf{R}^+ : h^{(1)}_s u\in\bar{X}\}.$$

We have

$$l(A\cap[0,\lambda])/\lambda\to 1-\frac{\theta}{50} \tag{4.25}$$

when $\lambda\to\infty$. Denote

$$v(u)=\bar{u}_n\in M_2.$$

For $s\in\mathbf{R}$ define $t(s)$ by

$$h^{(2)}_{t(s)}v(u)=v(h^{(2)}_s u).$$

We have

$$h_s^{(2)}\zeta(u) \in Y$$
$$h_{t(s)}^{(2)} v(u) \in U(h_s^{(2)}\zeta(u); \varepsilon) \tag{4.26}$$

for all $s \in A$. Also $0 \in A$. This and (4.26) show that A satisfies conditions (i)–(ii) of Lemma 2.1 with $\zeta(u)$ and $v(u)$ instead of u and v respectively.

Let us show that A satisfies (iii), too. Indeed, let s, $s' \in A$, $s < s'$ and let

$$\max\{s'-s, t(s')-t(s)\} \geqslant m.$$

Suppose for definiteness that

$$s'-s \geqslant m$$

and show that

$$|(t(s')-t(s))-(s'-s)| \leqslant (s'-s)^{1-\eta}. \tag{4.27}$$

Let

$$u_n(s) = g_{r_n}^{(1)}(h_s^{(1)}u)$$

and let $z: \mathbf{R} \to \mathbf{R}$ be defined by

$$z(p) = z(u_n(s), p)$$

where $z(u, p)$ is defined in (4.5). We have

$$u_n(s) \in P_1, \quad \psi(u_n(s)) \in P_2$$
$$u_n(s') = h_{t_n^{1+\gamma}(s'-s)}^{(1)} u_n(s)$$
$$\psi(u_n(s')) = h_{z(t_n^{1+\gamma}(s'-s))}^{(2)} \psi(u_n(s))$$
$$t(s')-t(s) = t_n^{-(1+\gamma)} z(t_n^{1+\gamma}(s'-s)).$$

It follows from (4.3) that

$$|z(t_n^{1+\gamma}(s'-s)) - t_n^{1+\gamma}(s'-s)| \leqslant [t_n^{1+\gamma}(s'-s)]^{1-\eta}$$

and therefore

$$|(t(s')-t(s))-(s'-s)| \leqslant (s'-s)^{1-\eta}.$$

This proves (4.27) when $s'-s \geqslant m$. Similarly, we prove (4.27) when $t(s')-t(s) \geqslant m$.

Thus $A=A(u)$, $u\in\bar{Q}$ satisfies all conditions of Lemma 2.1. Using this lemma and (4.25) we conclude that

$$v(u)=\bar{u}_n \text{ lies on the } h_t^{(1)}\text{-orbit of } \zeta(u) \text{ for every } u\in\bar{Q}.$$

We have

$$\zeta(g_{r_n}^{(1)}u)=g_{r_n}^{(2)}\zeta(u)$$

for μ_1-a.e. $u\in M_1$. This implies that if we denote

$$Q_n=g_{r_n}^{(1)}\bar{Q},\quad \mu_1(Q_n)>0$$

then

$$\psi(u)=h_{\sigma(u)}^{(2)}\zeta(u)$$

for some $\sigma(u)\in\mathbf{R}$ and all $u\in Q_n$. The set

$$\bar{\Omega}=\{u\in M_1:\psi(u)=h_{\sigma(u)}^{(2)}\zeta(u) \text{ for some } \sigma(u)\in\mathbf{R}\}$$

is $h_t^{(1)}$-invariant and contains Q_n. This implies that

$$\mu_1(\bar{\Omega})=1$$

since $h_t^{(1)}$ is ergodic and $\mu_1(Q_n)>0$. This completes the proof. Q.E.D.

Proof of Theorem 2. We can assume without loss of generality that $p=1$ in the theorem. So let $\tau_i\in\mathbf{K}(M_i)$ and $h_1^{\tau_i}$ be ergodic, $i=1,2$. Let $\psi:(M_1,\mu_{\tau_1})\to(M_2,\mu_{\tau_2})$ be m.p. and

$$\psi h_1^{\tau_1}(x)=h_1^{\tau_2}\psi(x)$$

for μ_{τ_1}-a.e. $x\in M_1$.

Let $0<\eta$, γ, θ, ω, $\varepsilon<1$, $m>1$, Y, $P_2\subset M_2$ and $P_1\subset M_1$ be as above.

Since ψ is measurable, there is $\Lambda\subset M_1$, $\mu_1(\Lambda)>1-\omega$ such that ψ is uniformly continuous on Λ. Let $\delta>0$ be such that if $u,v\in\Lambda$, $d(u,v)<\delta$ then $d(\psi(u),\psi(v))<\varepsilon$. Let

$$Z=\Lambda\cap P_1\cap\psi^{-1}(P_2\cap Y),\ \mu_1(Z)>1-\frac{\theta}{50K^2}$$

and let Q be the generic set of Z for $h_1^{\tau_1}$, $\mu_1(Q)=1$. Let $\bar{Q}=Q\cap Z$, $\mu_1(\bar{Q})>0$. We claim that

$$\text{if } u,\, v \in \bar{Q} \text{ and } v = h_p^{(1)} u \text{ for some } |p| < \delta$$
$$\text{then } \psi(v) = h_q^{(2)} \psi(u) \text{ for some } |q| < \varepsilon. \tag{4.28}$$

Indeed, let $\xi(p)$, $r(p)$, $p \in \mathbf{R}$ be defined by

$$\int_0^{\xi(p)} \tau_2(h_s^{(2)} \psi(u))\, ds = p = \int_0^{r(p)} \tau_2(h_s^{(2)} \psi(v))\, ds$$

and let

$$B = \{n \in \mathbf{Z}^+ : h_n^{\tau_1} u,\, h_n^{\tau_2} v \in Z\}$$
$$A = \{\xi(n+p) : n \in B,\ 0 \leqslant p \leqslant 1\}.$$

We have

$$l(A \cap [0, \lambda])/\lambda > 1 - \frac{\theta}{50} \tag{4.29}$$

for all $\lambda \geqslant \lambda_0$. Also

$$h_{\xi(n)}^{(2)} \psi(u) \in Y \tag{4.30}$$

for all $\xi(n) \in A$ with $n \in B$. For $\xi = \xi(n+p) \in A$ define

$$t(\xi) = r(n+p).$$

If $\xi = \xi(n)$ for some $n \in B$ then

$$h_{t(\xi)}^{(2)} \psi(v) \in U(h_\xi^{(2)} \psi(u); \varepsilon). \tag{4.31}$$

As in the proof of Theorem 1 we show that if $\xi = \xi(n) < \xi' = \xi(n')$, n, $n' \in B$ then

$$|(t(\xi') - t(\xi)) - (\xi' - \xi)| \leqslant (\xi' - \xi)^{1-\eta} \tag{4.32}$$

whenever

$$\max\{(t(\xi') - t(\xi)),\, (\xi' - \xi)\} \geqslant m.$$

Arguing as in the proof of Lemma 2.1 we show that (4.29), (4.30), (4.31) and (4.32) imply that

$$\psi(v) = h_q^{(2)} \psi(u)$$

for some $|q| < \varepsilon$.

Thus we proved (4.28). We omit the rest of the proof, since it is completely similar to the proof of Theorem 3 in [6]. Q.E.D.

References

[1] DANI, S. G., Dynamics of the horospherical flow. *Bull. Amer. Math. Soc.*, 3 (1980), 1037–1039.

[2] MOORE, C., Exponential decay of matrix coefficients. To appear.

[3] ORNSTEIN, D. & SMORODINSKY, M., Continuous speed changes for flows. *Israel J. Math.*, 31 (1978), 161–168.

[4] RATNER, M., Horocycle flows are loosely Bernoulli. *Israel J. Math.*, 31 (1978), 122–132.

[5] — The Cartesian square of the horocycle flow is not loosely Bernoulli. *Israel J. Math.*, 34 (1979), 72–96.

[6] — Rigidity of horocycle flows. *Ann. of Math.*, 115 (1982), 587–614.

[7] — Ergodic theory in hyperbolic space. *Contemp. Math.*, 26.

Received March 26, 1984