# Unitary derived functor modules with small spectrum

by

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## Contents

## §1. Introduction

For a real semisimple Lie group $G$, the description of the unitary dual remains an elusive question. One of the difficulties has been the lack of technique for constructing

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unitary representations. Unitary induction from parabolic subgroups of $G$ yields unitary representations by the very definition of these representations. However, all unitary irreducible representations of $G$ are not obtained by this type of induction. In addition, we need derived functor parabolic induction (cf. [25]) to describe all irreducible representations of $G$. For this second type of induction, the obvious analogues from parabolic subgroup induction regarding unitarity are false. In this article, we describe a setting where derived functor parabolic induction yields unitary representations of $G$. These results include proofs of unitarity for some of the representations conjectured to be unitary by Zuckerman [28] and also proofs of unitarity for some which lie outside the domain described in those conjectures.

The main results are described in section seven following the introduction of $\theta$-stable parabolic subalgebras and the associated generalized Verma modules (section two), results on invariant forms and complete reducibility for these modules (sections three through five) and a brief description of the derived functors introduced by Zuckerman (section six). The main results all take the same form: under certain hypotheses a derived functor applied to a generalized Verma module or quotient of one is shown to be either zero or unitarizable. These results include a proof of unitarity of Zuckerman's modules when the parabolic subalgebra has the property that inner products of compact and noncompact roots of the nilradical are nonnegative. Parabolic subalgebras satisfying this condition will be called quasi abelian.

The remaining sections describe various applications of the main results. In section eight, for each orthogonal group $SO(p,q)$ with $p+q$ even, we find a unitary representation which is multiplicity free as a $\mathfrak{k}$-module and has $\mathfrak{k}$-highest weights lying along a single line. We call these representations ladder representations of $SO(p,q)$. This result complements a result of Howe and Vogan [26] which asserts no such ladder representations exist if $p+q$ is odd and $p,q\geqslant 4$. A similar result is obtained in section nine. In this case, for each group $Sp(r,s)$, we construct a family of unitary representations each having the property that it is multiplicity free as a $\mathfrak{k}$-module and the highest weights of the $\mathfrak{k}$-submodules lie along a single line.

Section nine also includes an application to continuous cohomology. We construct a unitary representation of $Sp(n,\mathbf{R})$ which has nonzero cohomology at the rank. This implies that if $G$ is a connected, split over $\mathbf{R}$ classical group then there exists a nontrivial irreducible unitary representation $(\pi, H)$ of $G$ such that $H^l_{\text{cont}}(G,H)\neq 0$ for $l$ equal to the real rank of $G$ (cf. Theorem 9.8).

In section ten we consider the group $Sp(n,\mathbf{R})$ and construct families of unitary representations. Each of these representations is multiplicity free as a $\mathfrak{k}$-module and the set of its highest weights of $\mathfrak{k}$-submodules is given. This set is essentially the positive cone inside a lattice of rank $n$.

In section eleven we observe that $sl(2n, \mathbf{R})$ has a unique maximal parabolic subalgebra which is $\theta$-stable. Applying the results of section seven to this parabolic subalgebra, we obtain a family of unitary representations. These include the family of unitary representations $I(k)$, $k \in \mathbf{N}$, of Speh [24] which she constructed by analyzing certain poles of Eisenstein series associated to automorphic forms. Using reduction techniques to subgroups, these modules $I(k)$, $k \in \mathbf{N}$, prove the unitarity of all Zuckerman modules for $SL(m, \mathbf{R})$, [28]. In section eleven, we give the analogous series of unitary representations for $SU^*(2n)$. The reduction techniques of Vogan [28] then prove, as with $SL(m, \mathbf{R})$, the unitarity of the Zuckerman modules for $SU^*(2n)$.

In the classification of unitary highest weight modules [7] one case was handled by a difficult calculation. This case comprised a family of unitary representations for $E_6(-14)$. In section twelve, we obtain the unitarity of these modules as a corollary to our main results.

In [29], the third author studied the analytic continuation of the holomorphic discrete series representations having a one dimensional cyclic $\mathfrak{k}$-module. In the last section, we apply our main theorems and results of Jantzen [13] to prove analogous results for certain discrete series representations of $\mathfrak{g}$ with $(\mathfrak{g}, \mathfrak{k})$ not Hermitian symmetric. The results here prove unitarity for certain coherent continuations of discrete series representations out of the Borel-de Siebanthal Weyl chamber. These results are given in the form of a table in section thirteen. If we add a hypothesis of complete reducibility of a certain family of modules, then these results extend to any $\theta$-stable quasi abelian maximal parabolic whose complementary simple root has coefficient two in the maximal root (cf. Proposition 13.4).

There is a vast literature on various techniques for proving the unitarity of certain representations of $G$. Vogan and Zuckerman [28] have shown that all representations "having" nonzero continuous cohomology are Zuckerman representations. Thus these representations are a particularly important class of representations and their unitarity has been investigated in many articles (cf. [24], [11], [19], [1]). To date the main success has been in the cases where the representations are of holomorphic type ([14], [11], [7], [12]), where they can be related to Howe's theory of dual pairs ([1]) or where they can be related to automorphic forms ([24]).

Interesting classes of unitary representations have been constructed recently by Flensted-Jensen [10] and Schlichtkrull [20]. These representations are obtained by analytic methods by decomposing $L^2(G/H)$ for $H$ the fixed points of an involution of $G$.

The full determination of the discrete spectrum of $L^2(G/H)$ is given by Matsuki and Oshima [15].

Unitary representations have been constructed in many cases by geometric methods. In particular, the recent work of Rawnsley, Schmid and Wolf [19] develops a theory of $L^2$-cohomology based on harmonic forms for indefinite Kaehler metrics to unitarize representations on Dolbeault cohomology in a number of cases.

The direct algebraic approach to proving unitarizability for $\mathfrak{g}$-modules other than highest weight modules has been used only in a few cases: Parthasarathy's work on the discrete series [18], Vogan's work on representations associated to the minimal coadjoint orbit [26], and Enright's work comparing representations of Hermitian symmetric pairs and complex Lie groups [5]. The methods of this article are also algebraic. They are based on the duality theorem (cf. section six) and the study of Hermitian forms on Verma modules. The main results, although not all the applications, of this article were announced in [8].

Finally a few remarks on the proof of the main results. First, if the parabolic subalgebra is quasi abelian then the associated generalized Verma module is completely reducible as a $\mathfrak{k}$-module and the signature of the canonical invariant Hermitian form on the $\mathfrak{k}$-highest weight spaces is positive definite, whenever the generalized Verma modules highest weight is antidominant (cf. Lemma 3.1, Propositions 3.5, 4.1 and 5.4). Second, signature is preserved under the derived functor in the middle dimension. The precise formulation of this assertion is given as Proposition 6.6. Then in Proposition 6.9 these results are combined to prove the unitarity of derived functor modules.

We have recently received a preprint [27] from David Vogan containing general results on the unitarizability of derived functor modules. In particular his results include a proof of the conjecture given in his book [25]. Vogan's technique is more general than that developed in this article. However, in the cases studied here, the results of this article are the sharper.

## § 2. $\theta$-stable parabolic subalgebras

Let $G$ be a connected, simply connected semisimple Lie group and let $K$ be a maximal connected subgroup of $G$ whose image in $G$/center $G$ is compact. Let $\mathfrak{g}_0$ and $\mathfrak{k}_0$ be the corresponding Lie algebras of $G$ and $K$. Denote by $\theta$ the Cartan involution of $\mathfrak{g}_0$ giving the Cartan decomposition $\mathfrak{g}_0=\mathfrak{k}_0\oplus\mathfrak{p}_0$. Choose a Cartan subalgebra (CSA) $\mathfrak{t}_0$ of $\mathfrak{k}_0$ and let $\mathfrak{h}_0$ be the centralizer of $\mathfrak{t}_0$ in $\mathfrak{g}_0$. Then $\mathfrak{h}_0$ is a fundamental CSA of $\mathfrak{g}_0$. Let the complexification of a space be denoted by deleting the subscript 0. This gives: $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ and Cartan subalgebras $\mathfrak{t}$ and $\mathfrak{h}$.

Let $\mathfrak{q}$ denote a $\theta$-stable parabolic subalgebra of $\mathfrak{g}$ with decomposition $\mathfrak{q}=\mathfrak{m}\oplus\mathfrak{u}$ where $\mathfrak{u}$ is the nilradical of $\mathfrak{q}$. Assume that $\mathfrak{h}$ is contained in $\mathfrak{m}$. Since $\mathfrak{q}$ is $\theta$-stable, so is $\mathfrak{u}$. Also, since $\mathfrak{t}$ is fixed by $\theta$, so is $\mathfrak{h}$ and then so is $\mathfrak{m}$. For any $\theta$-stable vector space, let subscripts $c$ and $n$ denote the $+1$ and $-1$ eigenspaces for $\theta$. For example, $\mathfrak{u}_c=\mathfrak{u}\cap\mathfrak{k}$, $\mathfrak{u}_n=\mathfrak{u}\cap\mathfrak{p}$ giving the decomposition $\mathfrak{u}=\mathfrak{u}_c\oplus\mathfrak{u}_n$; and similarly, $\mathfrak{m}=\mathfrak{m}_c\oplus\mathfrak{m}_n$.

For any $\mathrm{ad}(\mathfrak{h})$ stable subspace of $\mathfrak{g}$, let $\Delta(E)$ denote the roots which occur in the root space decomposition of $E$. If $E$ is $\mathrm{ad}(\mathfrak{t})$-stable but not $\mathrm{ad}(\mathfrak{h})$-stable, let $\Delta(E)$ denote the nonzero $\mathfrak{t}$-weights which occur in $E$. Let $\Delta=\Delta(\mathfrak{g})$ be the set of roots of $\mathfrak{g}$ and fix a $\theta$-stable positive system of roots $\Delta^+$. For any $\mathrm{ad}(\mathfrak{h})$-stable $E$, let $\Delta^+(E)=\Delta^+\cap\Delta(E)$. Let $\mathfrak{b}$ be the Borel subalgebra corresponding to $\Delta^+$, i.e., $\mathfrak{b}=\mathfrak{h}\oplus\Sigma_{\alpha\in\Delta^+}\mathfrak{g}_\alpha$. Then $\mathfrak{b}$ is $\theta$-stable and $\mathfrak{b}_c=\mathfrak{b}\cap\mathfrak{k}$ is a Borel subalgebra of $\mathfrak{k}$. Let $\Delta^+(\mathfrak{k})$ be the positive system of $\Delta(\mathfrak{k})$ corresponding to $\mathfrak{b}_c$. If $E$ is $\mathrm{ad}(\mathfrak{t})$-stable, put $\Delta^+(E)=\Delta^+(\mathfrak{k})\cap\Delta(E)$.

Most of the results of this article involve a special class of $\theta$-stable parabolic subalgebras $\mathfrak{q}$ defined by the property: for all $\alpha\in\Delta(\mathfrak{u}_c)$ and $\beta\in\Delta(\mathfrak{u}_n)$, $\langle\alpha,\beta\rangle\geq 0$. These $\theta$-stable parabolic subalgebras will be called *quasi abelian*.

For any Lie algebra $\mathfrak{a}$, let $U(\mathfrak{a})$ denote the universal enveloping algebra of $\mathfrak{a}$. Let $Z(\mathfrak{a})$ denote the center of $U(\mathfrak{a})$. If $\lambda\in\mathfrak{h}^*$ is $\Delta^+(\mathfrak{m})$-dominant integral and $\mu\in\mathfrak{t}^*$ is $\Delta^+(\mathfrak{m}_c)$-dominant integral, let $F(\lambda)$ and $F_c(\mu)$ denote the irreducible finite dimensional $\mathfrak{m}$ and $\mathfrak{m}_c$ modules with highest weights $\lambda$ and $\mu$. Define generalized Verma modules by

$$N(\lambda)=U(\mathfrak{g})\underset{U(\mathfrak{q})}{\otimes}F(\lambda),\quad N_c(\mu)=U(\mathfrak{k})\underset{U(\mathfrak{q}_c)}{\otimes}F_c(\mu).$$

Let $L(\lambda)$ and $L_c(\mu)$ denote the unique irreducible quotients of $N(\lambda)$ and $N_c(\mu)$. There are many especially interesting cases where $\mathfrak{t}=\mathfrak{h}$ and, for this reason, we include the subscript $c$ to distinguish $\mathfrak{g}$ and $\mathfrak{k}$-modules. For convenience we single out two Weyl chambers associated to these generalized Verma modules. Let $\mathscr{C}$ (resp. $\mathscr{C}_c$) be the closed Weyl chamber in $\mathfrak{h}^*$ (resp. $\mathfrak{t}^*$) corresponding to the positive system $\Delta^+(\mathfrak{m})\cup-\Delta(\mathfrak{u})$ (resp. $\Delta^+(\mathfrak{m}_c)\cup-\Delta(\mathfrak{u}_c)$). Let $\varrho$ (resp. $\varrho_c$) be half the sum of elements in $\Delta^+$ (resp. $\Delta^+(\mathfrak{k})$). These Weyl chambers are distinguished by the property:

$$\text{If } \lambda+\varrho\in\mathscr{C} \text{ (resp. } \mu+\varrho_c\in\mathscr{C}_c\text{) then } N(\lambda) \text{ (resp. } N_c(\mu)\text{) is irreducible.} \tag{2.1}$$

Let $\mathfrak{u}^-$ (resp. $\mathfrak{u}_c^-$) be the sum of the root spaces $\mathfrak{g}_\alpha$ (resp. $\mathfrak{k}_\beta$) with $-\alpha\in\Delta(\mathfrak{u})$ (resp. $-\beta\in\Delta(\mathfrak{u}_c)$).

For any vector space $E$ we let $S(E)$ denote the symmetric algebra of $E$. Let $\mathcal{W}$ denote the Weyl group of $(\mathfrak{g}, \mathfrak{h})$ and $\mathcal{W}(\mathfrak{k})$ the Weyl group of $(\mathfrak{k}, \mathfrak{t})$. If $E$ is a $\mathfrak{t}$-module and $\nu \in \mathfrak{t}^*$ let $E_\nu$ denote the weight subspace of $E$ for weight $\nu$.

## § 3. Splitting criteria and signature

In this section we study the splitting of $N(\lambda)$ as a $\mathfrak{k}$-module. In cases where $N(\lambda)$ is completely reducible as a $\mathfrak{k}$-module we investigate the signature of the canonical invariant Hermitian form on subspaces of $N(\lambda)$. Throughout this section we use the notation of sections one and two and we assume that $\mathfrak{q}$ is a *quasi abelian* $\theta$-stable parabolic subalgebra. The main result in this section is Proposition 3.5.

LEMMA 3.1. *Let* $\lambda \in \mathfrak{h}^*$ *with* $\lambda|_\mathfrak{t} + \varrho_c \in \mathscr{C}_c$ *and* $F(\lambda)$ *irreducible as an* $\mathfrak{m}_c$*-module. Then* $N(\lambda)$ *and* $L(\lambda)$ *are completely reducible as* $\mathfrak{k}$*-modules. Each irreducible* $\mathfrak{k}$*-submodule is isomorphic to* $N_c(\mu_i)$ *with* $\mu_i + \varrho_c \in \mathscr{C}_c$.

*Proof.* Write $S(\mathfrak{u}_n^-) \otimes F(\lambda) = \Sigma_i F_c(\mu_i)$. Since $F(\lambda)$ is irreducible as an $\mathfrak{m}_c$-module, each highest weight $\mu_i$ has the form $\lambda$ plus a weight of $S(\mathfrak{u}_n^-)$. Now $\mu_i + \varrho_c$ is $\Delta^+(\mathfrak{m}_c)$-dominant; and so, to lie in $\mathscr{C}_c$, we need only evaluate its inner products with the $\beta$ in $\Delta(\mathfrak{u}_c^-)$. But $\mathfrak{q}$ is quasi abelian and $\lambda|_\mathfrak{t} + \varrho_c \in \mathscr{C}_c$ so $\langle \mu_i + \varrho_c, \beta \rangle \geqslant 0$. This proves $\mu_i + \varrho_c \in \mathscr{C}_c$.

$N(\lambda)$ is isomorphic as a $\mathfrak{k}$-module to $U(\mathfrak{k}) \otimes_{U(\mathfrak{q}_c)} (S(\mathfrak{u}_n^-) \otimes F(\lambda))$. For some indexing of the $\mu_i$, the decomposition above induces up to $U(\mathfrak{k})$ to give a filtration of $N(\lambda) = N^1(\lambda) \supset N^2(\lambda) \supset \ldots$ with $N^i(\lambda)/N^{i+1}(\lambda) \simeq N_c(\mu_i)$. Since $\mu_i + \varrho_c \in \mathscr{C}_c$, these $\mathfrak{k}$-modules are irreducible and $N(\lambda)$ splits as a direct sum. Finally, $L(\lambda)$ is a quotient of $N(\lambda)$ and so it also is completely reducible. This proves Lemma 3.1.

Next we introduce the invariant Hermitian forms on $N(\lambda)$. Let $p\colon U(\mathfrak{g}) \to U(\mathfrak{m})$ be the linear projection given by the decomposition $U(\mathfrak{g}) = U(\mathfrak{m}) \oplus (\mathfrak{u}^- U(\mathfrak{g}) + U(\mathfrak{g})\,\mathfrak{u})$. Let $X \mapsto \bar{X}$ be the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. For $X \in \mathfrak{g}$ define $X^* = -\bar{X}$ and extend this action to a conjugate linear antiautomorphism of $U(\mathfrak{g})$: i.e.,

$$(xy)^* = y^* x^*, \quad 1^* = 1 \quad \text{with} \quad x, y \in U(\mathfrak{g}).$$

A Hermitian form $(\cdot, \cdot)$ on a $\mathfrak{g}$-module $A$ is called *invariant* (with respect to $\mathfrak{g}_0$) if $(x \cdot a, b) = (a, x^* b), a, b \in A,\ x \in U(\mathfrak{g})$. The finite dimensional module $F(\lambda)$ admits an invariant Hermitian form (with respect to $\mathfrak{m} \cap \mathfrak{g}_0$) precisely when $\lambda(H) = -\bar{\lambda}(\bar{H})$ for all $H \in \mathfrak{h}$. In this case let $\xi_\lambda$ denote the Hermitian form which is unique up to real multiple.

Now define an invariant Hermitian form on $N(\lambda)$ by:

$$(x\otimes e, y\otimes f)_\lambda=\xi_\lambda(p(y^*x)\,e, f). \tag{3.2}$$

The radical of this form is the maximal submodule of $N(\lambda)$; and so, $(\cdot,\cdot)_\lambda$ induces a nondegenerate form on the quotient module $L(\lambda)$. We let $(\cdot,\cdot)_\lambda$ denote both these forms. We say that $F(\lambda)$ is *unitarizable* if $\xi_\lambda$ is definite. In this case we assume $\xi_\lambda$ is positive definite. For the remainder of this section assume that $F(\lambda)$ is unitarizable and note that this implies that $F(\lambda)$ is irreducible as an $\mathfrak{m}_c$-module.

For convenience, in this section we write:

$$\mathfrak{r}=\mathfrak{u}_c \quad\text{and}\quad \mathfrak{r}^-=\mathfrak{u}_c^-. \tag{3.3}$$

LEMMA 3.4. *Let assumptions be as in Lemma* 3.1. *Then* $L=L(\lambda)$ *has an orthogonal decomposition*

$$L=L^{\mathfrak{r}}\oplus\mathfrak{r}^-L.$$

*Proof.* First split $L$ into generalized eigenspaces for $Z(\mathfrak{k})$. These spaces are mutually orthogonal. Then by Lemma 3.1, each such subspace say $A$ is the sum of copies of one irreducible $N_c(\mu_i)$. Then $A^{\mathfrak{r}}$ is the full highest weight space in $A$. Weight spaces being orthogonal $A=A^{\mathfrak{r}}\oplus\mathfrak{r}^-A$ is an orthogonal direct sum. This proves Lemma 3.4.

For the remainder of this section we fix $\xi\in\mathfrak{h}^*$ with $F(\xi)$ one dimensional and unitarizable and with $\langle\xi,\alpha\rangle<0$ for all $\alpha\in\Delta(\mathfrak{u})$. We now consider signature questions by looking at the family of modules $N(\lambda+t\xi)$, $t\in\mathbf{R}$.

PROPOSITION 3.5. *Assume* $N(\lambda+t\xi)$ *is irreducible for all* $t>0$ *and* $\lambda|_{\mathfrak{t}}+\varrho_c\in\mathscr{C}_c$. *Then the restriction of* $(\cdot,\cdot)_\lambda$ *to* $L(\lambda)^{\mathfrak{r}}$ *is positive definite.*

*Proof.* Since $N(\lambda+t\xi)$ is induced from $\mathfrak{q}$,

$$N(\lambda+t\xi)\simeq U(\mathfrak{u}^-)\otimes F(\lambda+t\xi)\simeq S(\mathfrak{u}_c^-)\otimes S(\mathfrak{u}_n^-)\otimes F(\lambda)\otimes F(t\xi). \tag{3.6}$$

We may drop this last factor from the tensor product since $F(t\xi)$ is one-dimensional. So the underlying vector space for $N(\lambda+t\xi)$ is independent of $t$. Let $X_1,\dots,X_r$ be a basis for $\mathfrak{u}_c^-$ and $Y_1,\dots,Y_d$ a basis for $\mathfrak{u}_n^-$ with $X_i$ of weight $-\alpha_i$ and $Y_j$ a weight $-\beta_j$. Let $I$ (resp. $J$) be a multiindex of $r$ (resp. $d$) indices and let $X^I=X_1^{i_1}\dots X_r^{i_r}$, $Y^J=Y_1^{j_1}\dots Y_d^{j_d}$. We assume the bases are normalized so that the Killing form $B$ gives: $B(X_i,\bar{X}_j)=-\delta_{ij}$, $B(Y_i,\bar{Y}_j)=\delta_{ij}$. Now the usual computations and the identity $X^*=-\bar{X}$ give:

$$\begin{aligned}\nu([X_i^*, X_i])&=\langle\nu,\alpha_i\rangle\\ \nu([Y_j^*, Y_j])&=-\langle\nu,\beta_j\rangle,\quad \nu\in\mathfrak{h}^*.\end{aligned} \tag{3.7}$$

Let $e_1, \ldots, e_s$ be an orthonormal basis for the unitarizable module $F(\lambda)$ with $e_i$ of weight $\gamma_i$. From our formula for $(\cdot\,,\cdot)_\lambda$, it is clear that $(\cdot\,,\cdot)_{\lambda+t\xi}$ is a polynomial in $t$. Following Shapovalov [23], using (3.7) and an easy induction argument on the order $|I|+|J|$ we obtain an identity for the leading term of this polynomial.

$$(X^I \otimes Y^J \otimes e_q, X^{I'} \otimes Y^{J'} \otimes e_p)_{\lambda+t\xi} \equiv \delta_{II'}\,\delta_{JJ'}\,\delta_{qp} \prod_{1\leqslant l\leqslant r} \langle t\xi, \alpha_l\rangle^{i_l} \prod_{1\leqslant k\leqslant d} -\langle t\xi, \beta_k\rangle^{j_k} \quad (3.8)$$

modulo a polynomial in $t$ of degree less than $\min\{|I|+|J|, |I'|+|J'|\}$. Note that $\langle t\xi, \beta_k\rangle$ and $\langle t\xi, \alpha_l\rangle$ are both negative multiples of $t$.

Fix a weight $\nu \in \mathfrak{t}^*$ and consider the weight space $R = N(\lambda+t\xi)_{\lambda+t\xi+\nu}$. Let $\Psi = \{(I, J, l) \mid \Sigma_k - i_k\alpha_k + \Sigma_s - j_s\beta_s + \gamma_l = \lambda+\nu\}$. Then by (3.6) the set $\{X^I \otimes Y^J \otimes e_l \mid (I, J, l) \in \Psi\}$ is a basis for $R$. Let $R_1 = R \cap \mathfrak{r}^- N(\lambda+t\xi)$ and observe that by (3.6), $R$ as well as $R_1$ is independent of $t$ when expressed as a subspace of $S(\mathfrak{u}_c^-) \otimes S(\mathfrak{u}_n^-) \otimes F(\lambda)$.

Assume $N(\lambda)$ is irreducible. Then $(\cdot\,,\cdot)_{\lambda+t\xi}$ is nondegenerate for all $t \geqslant 0$ and the restrictions of $(\cdot\,,\cdot)_{\lambda+t\xi}$ to both $R$ and $R_1$ are nondegenerate by Lemma 3.4. Clearly, these forms depend polynomially on $t$; and thus, by nondegeneracy the signature is independent of $t$. Let $(p, q)$ be the signature on $R$ and $(p', q')$ the signature on $R_1$ with $p$ and $p'$ denoting the positive part. If we represent the restricted forms by matrices in the above bases then (3.8) implies that the diagonal terms dominate for large $t$. Therefore, for $t \gg 0$ we have:

$$\begin{aligned} p &= \#\{(I, J, l) \in \Psi \mid |I| \text{ is even}\} \\ q &= \#\{(I, J, l) \in \Psi \mid |I| \text{ is odd}\} \\ p' &= \#\{(I, J, l) \in \Psi \mid |I| \text{ is even and } \neq 0\} \\ q' &= q. \end{aligned} \quad (3.9)$$

The equality $q = q'$ implies that $(\cdot\,,\cdot)_{\lambda+t\xi}$ is positive definite on the orthogonal complement of $R_1$ in $R$. By Lemma 3.4 this is $N(\lambda+t\xi)^{\mathfrak{r}} \cap R$ and proves Proposition 3.5 if $N(\lambda)$ is irreducible.

To complete the proof we need:

LEMMA 3.10. *Assume* $\lambda|_{\mathfrak{t}} + \varrho_c \in \mathscr{C}_c$. *There are vectors* $v_1(t), \ldots, v_n(t)$ *in* $R$ *which satisfy the following:*

(i) $v_i(t) = \Sigma_\Psi\, a^i_{I,J,l}(t)\, X^I \otimes Y^J \otimes e_l$ *with coefficient functions* $a^i_{I,J,l}(t)$ *which are rational in* $t$ *and regular at* $t = 0$.

(ii) *For* $t$ *in an open set containing zero, these vectors are a basis for* $R \cap N(\lambda+t\xi)^{\mathfrak{r}}$.

*Proof.* Let $\alpha_1, \dots, \alpha_t$ be the simple roots of $\Delta^+(\mathfrak{k})$ and $X_i$ the corresponding root vectors, $1 \leq i \leq t$. Consider the map of $R$ into the $t$-fold product of $N(\lambda+t\xi)$ given by: $x \mapsto (X_1 \cdot x, \dots, X_t \cdot x)$. Let $A(t)$ represent this map as a matrix with respect to the bases of monomials for (3.6). This is a matrix of polynomials in $t$ and its kernel is precisely $R \cap N(\lambda+t\xi)^{\mathfrak{k}}$. By Lemma 3.1, $\operatorname{rank} A(t) \leq \operatorname{rank} A(0)$. Let $\operatorname{rank} A(0)=r$ and, by rearranging bases if necessary, assume the upper left $r\times r$ block of $A(t)$ has nonzero determinant $D(t)$ and $D(0)\neq 0$. Consider the $m\times m$ matrix

$$\left(\begin{array}{ccc} a_{11}(t) & \dots & a_{1m}(t) \\ \vdots & & \vdots \\ a_{r1}(t) & & a_{rm}(t) \\ \hline & 1 & 0 \\ 0 & & \ddots \\ & & 1 \end{array}\right).$$

This matrix has determinant $D(t)$; and so, is invertible in the ring $K=\mathbf{C}[T]_{(D(t))}$. Let $T(t)$ be the inverse matrix. Then $A(t)\,T(t)$ has the form

$$\left(\begin{array}{c|c} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & 0 \\ \hline \ast & 0 \end{array}\right).$$

The $(r+1)$-st through $m$th columns of $T(t)$ give the coefficient functions in (i). If $D(t)\neq 0$ then these vectors are a basis for $R\cap N^{\mathfrak{k}}(\lambda+t\xi)$. This proves (ii).

To complete the proof of Proposition 3.5 consider the restriction of $(\cdot,\cdot)_{\lambda+t\xi}$ to $R\cap N^{\mathfrak{k}}(\lambda+t\xi)$. From the first part of the proof, this restriction is positive definite for $t>0$. By Lemma 3.10 the restriction is continuous in $t$ near $t=0$; and so, the restriction is positive semi definite at $t=0$. The induced form on $L(\lambda)$ is nondegenerate; and so, the form is positive definite on $R\cap L(\lambda)^{\mathfrak{k}}$. This completes the proof of Proposition 3.5.

The hypothesis $\lambda|_{\mathfrak{t}}+\varrho_c \in \mathscr{C}_c$ has been used in our proofs only to imply that $N(\lambda+t\xi)$ is completely reducible over $U(\mathfrak{k})$ for all $t\leq 0$. For one special application we reformulate Proposition 3.5 as:

PROPOSITION 3.11. *Assume $N(\lambda+t\xi)$ is irreducible for all $t>0$ and completely reducible over $U(\mathfrak{k})$ for all $t\geq 0$. Then the restriction of $(\cdot,\cdot)_\lambda$ to $L(\lambda)^{\mathfrak{k}}$ is positive definite.*

*Moreover, if $\lambda$ is $\Delta(\mathfrak{k})$-integral and $L(\lambda)$ is free over $U(\mathfrak{u}_c^-)$ then every irreducible summand of $L(\lambda)$ is isomorphic to an irreducible $N_c(\mu)$ with $\mu+\varrho_c$ singular or an element of $\mathscr{C}_c$.*

*Proof.* For the first part, the proof is the same as that of Proposition 3.5. For the second part, since $N(\lambda)$ is completely reducible over $U(\mathfrak{k})$ so is $L(\lambda)$. Then any irreducible $U(\mathfrak{k})$-summand of $L(\lambda)$ is a free $U(\mathfrak{u}_c^-)$ module which is a quotient of some $N_c(\mu)$. But then this module must equal $N_c(\mu)$ and $N_c(\mu)$ is irreducible. By Jantzen's work [13] this implies $\mu+\varrho_c$ is either singular or an element of $\mathscr{C}_c$. This proves Proposition 3.11.

## § 4. Signature results when $\mathfrak{u}_c$ is abelian

The assumption $\lambda|_{\mathfrak{t}}+\varrho_c \in \mathscr{C}_c$ which appears in Proposition 3.5 is often too restrictive for our applications. In this section we prove a sharper result than Proposition 3.5 under the additional hypothesis that $\mathfrak{u}_c$ is abelian. We keep in force the standard assumptions of section three; i.e., $\mathfrak{q}$ is quasi abelian, $F(\xi)$ is one-dimensional, $\langle \xi, \alpha\rangle<0$ for all $\alpha \in \Delta(\mathfrak{u})$ and $F(\xi)$ and $F(\lambda)$ are unitarizable. In addition we assume $\mathfrak{u}_c$ is abelian. The main result in this section is:

PROPOSITION 4.1. *Assume $N(\lambda+t\xi)$ is irreducible for $t>0$. Then, the restriction of $(\cdot,\cdot)_\lambda$ to $L(\lambda)^{\mathfrak{r}}$ is positive definite.*

This result is proved by introducing a canonical Hermitian form on $N(\lambda+t\xi)$ which has certain $\mathfrak{k}$-invariance properties and is positive definite when $t>0$. We introduce this form and its properties in a series of lemmas.

Let $\tau$ be the automorphism of $U(\mathfrak{k})$ induced by the identity on $k\cap\mathfrak{m}$ and $(-1)\cdot$identity on $\mathfrak{u}_c^-\oplus\mathfrak{u}_c$. Since $\mathfrak{u}_c$ is abelian, $\tau$ is well defined. Let $\check{x}=\tau(x)^*$, $x\in U(\mathfrak{k})$. Then $x\mapsto\check{x}$ is a conjugate linear antiautomorphism of $U(\mathfrak{k})$.

LEMMA 4.2. *Let $M$ be a finite dimensional $\mathfrak{q}_c$-module with $\mathfrak{m}_c$-invariant form $(\cdot,\cdot)_M$. Let $N=U(\mathfrak{k})\otimes_{U(\mathfrak{q}_c)}M$. Then there exist unique forms $(\cdot,\cdot)$ and $\{\cdot,\cdot\}$ on $N$ such that*

(i) *$\{\cdot,\cdot\}$ and $(\cdot,\cdot)$ equal $(\cdot,\cdot)_M$ on $1\otimes M$.*
(ii) *$(x\cdot e, f)=(e, x^*f)$, $\{x\cdot e, f\}=\{e, \check{x}f\}$, $x\in U(\mathfrak{k}), e, f\in N$.*

*Moreover, if $(\cdot,\cdot)_M$ is Hermitian then so are $(\cdot,\cdot)$ and $\{\cdot,\cdot\}$.*

Forms which satisfy the first identity will be called *-invariant while those satisfying the second will be called $^{\vee}$-invariant. In either case we call the form on $N$ the extension of $(\cdot, \cdot)_M$ from $M$ to $N$.

The proof of Lemma 4.2 is identical to that of Proposition 6.12 in [4]; and so, we will not repeat the argument here. The remark on Hermitian follows by Proposition 6.13(b) in [4].

Recalling (3.6), let $M=S(\mathfrak{u}_n^-)\otimes F(\lambda)$ and let $\varphi_t$ be the restriction of $(\cdot, \cdot)_{\lambda+t\xi}$ to $M$. Then $\varphi_t$ is an $\mathfrak{m}_c$-invariant form. $N(\lambda+t\xi)$ is isomorphic to $U(\mathfrak{k})\otimes_{U(\mathfrak{q}_c)}M$ by (3.6), and so, the uniqueness assertion in Lemma 4.2 implies that $(\cdot, \cdot)_{\lambda+t\xi}$ is the unique extension of $\varphi_t$ on $M$ which is *-invariant. Let $\{\cdot, \cdot\}_{\lambda+t\xi}$ be the unique $^{\vee}$-invariant extension of $\varphi_t$ to $N(\lambda+t\xi)$. Let $\tau_0$ be the involutive linear isomorphism of $N(\lambda+t\xi)$ given by $\tau\otimes 1$ acting on $U(\mathfrak{u}_c^-)\otimes M$. Now for $x, y\in U(\mathfrak{u}_c^-)$, $e, f\in M$ we have $\{x\otimes e, y\otimes f\}_{\lambda+t\xi}=\varphi_t(p(y^{\vee}x)\cdot e, f)=(x\otimes e, \tau(y)\otimes f_{\lambda+t\xi})$. This proves

$$\{a, b\}_{\lambda+t\xi}=(a, \tau_0(b))_{\lambda+t\xi}, \quad a, b\in N(\lambda+t\xi). \tag{4.3}$$

By our assumptions, $(\cdot, \cdot)_{\lambda+t\xi}$ is nondegenerate for $t>0$; and so, since $\tau_0$ is bijective, $\{\cdot, \cdot\}_{\lambda+t\xi}$ is also nondegenerate. If we replace $\lambda$ by $\lambda+t\xi$ for $t>>0$ then (3.1) and (3.5) apply and we conclude: for $t>>0$, $N(\lambda+t\xi)$ is completely reducible as a $\mathfrak{k}$-module into an orthogonal direct sum and $(\cdot, \cdot)_{\lambda+t\xi}$ is positive definite on highest weight spaces of the irreducible summands. Also, each summand has the form $N_c(\mu_i)$ with $\mu_i+\varrho_i\in\mathscr{C}_c$.

Since both $(\cdot, \cdot)_{\lambda+t\xi}$ and $\{\cdot, \cdot\}_{\lambda+t\xi}$ are Hermitian, (4.3) implies

$$(a, \tau_0(b))_{\lambda+t\xi}=(\tau_0 a, b)_{\lambda+t\xi}. \tag{4.4}$$

$\tau_0$ is involutive and the $-1$ eigenspace is contained in $\mathfrak{u}_c^- N(\lambda+t\xi)$. By (4.4), the $+1$ and $-1$ eigenspaces are orthogonal; and so, (4.3) implies that $\{\cdot, \cdot\}_{\lambda+t\xi}$ and $(\cdot, \cdot)_{\lambda+t\xi}$ are equal on the highest weight spaces of the irreducible $\mathfrak{k}$-summands. So, $\{\cdot, \cdot\}_{\lambda+t\xi}$ is positive definite on the highest weight spaces for $t>>0$.

The $N_c(\mu_i)$ with $\mu_i+\varrho_c\in\mathscr{C}_c$ correspond to holomorphic discrete series representations for the real form of the Lie algebra $\mathfrak{k}$ having $\mathfrak{m}_c$ as the Lie algebra of a maximal compact subgroup. Moreover, the $^{\vee}$-invariant Hermitian forms defined above are the forms invariant with respect to this real form. Discrete series are unitary; and so, any $^{\vee}$-invariant form on $N_c(\mu_i)$ must have definite signature. By the preceding paragraph $\{\cdot, \cdot\}_{\lambda+t\xi}$ is positive definite on the highest weight spaces of the summands. Therefore, for $t>>0$, $\{\cdot, \cdot\}_{\lambda+t\xi}$ is positive definite. The forms are nondegenerate for $t>0$ and vary continuously in $t$. This proves

LEMMA 4.5. *$\{\cdot,\cdot\}_{\lambda+t\xi}$ is positive definite for $t>0$, positive semi definite for $t=0$ and positive definite for $t=0$ if $N(\lambda)$ is irreducible.*

By (4.3), $\{\cdot,\cdot\}_\lambda$ and $(\cdot,\cdot)_\lambda$ have the same radical; and so, both induce forms on $L(\lambda)$ which we denote by the same symbols. Also (4.3) implies that the radical is invariant under $\tau_0$ and thus $\tau_0$ induces a map also denoted $\tau_0$ on $L(\lambda)$.

LEMMA 4.6. *$L(\lambda)$ is completely reducible as a $\mathfrak{k}$-module and we have an orthogonal decomposition*

$$L(\lambda)=L(\lambda)^{\mathfrak{r}}\oplus\mathfrak{r}^{-}L(\lambda).$$

*Also, the restriction of $(\cdot,\cdot)_\lambda$ to $L(\lambda)^{\mathfrak{u}_c}$ is positive definite.*

*Proof.* From Lemma 4.5, $\{\cdot,\cdot\}_\lambda$ is positive definite on $L(\lambda)$. This form is $\mathfrak{k}$-invariant and so $L(\lambda)$ is completely reducible as a $\mathfrak{k}$-module into an orthogonal sum of irreducible $\mathfrak{k}$-modules. For any irreducible highest weight $\mathfrak{k}$-module $L_c(\mu)$, $L_c(\mu)$ is the orthogonal sum of $L_c(\mu)^{\mathfrak{r}}$ and $\mathfrak{r}^{-}L_c(\mu)$. Using (4.3) we conclude as above that $\{\cdot,\cdot\}_\lambda$ and $(\cdot,\cdot)_\lambda$ are equal on $L_c^{\mathfrak{r}}$, since the $-1$ eigenspace of $\tau_0$ is contained in $\mathfrak{r}^{-}L(\lambda)$. Since $\{\cdot,\cdot\}_\lambda$ is positive definite on $L(\lambda)$, $(\cdot,\cdot)_\lambda$ is positive definite on $L(\lambda)^{\mathfrak{r}}$. This proves Lemma 4.6 which in turn proves our main result Proposition 4.1.

## § 5. Signature results when $\mathfrak{u}$ is abelian

The sharpest signature results are available in the case where $\mathfrak{u}$ is abelian. The theory here is related to the unitarizability of highest weight modules for the Hermitian symmetric case, a problem which has been solved recently [7] and [12]. The main result in this section is Proposition 5.4.

Let notation be as in earlier sections. We assume that $\mathfrak{q}$ is $\theta$-stable, $\mathfrak{u}$ is abelian and $F(\lambda)$ is unitarizable. For convenience, also assume that $\mathfrak{g}$ is simple. Our assumptions on $\mathfrak{q}$ imply there exists $H_0\in\sqrt{-1}\,\mathfrak{t}_0$ with $H_0$ having eigenvalue $\pm1$ (resp. 0) on $\mathfrak{u}^{\pm}$ (resp. $\mathfrak{m}$). Let $\theta_1=\exp(\pi\sqrt{-1}\operatorname{ad}H_0)$. Then $\theta_1^2=1$ and $\theta$ and $\theta_1$ commute. Let $\sigma(X)=\theta_1\theta(\bar{X})$, $x\in\mathfrak{g}$ and let $\mathfrak{g}_1=\{x\in\mathfrak{g}\,|\,\sigma(x)=x\}$. Then $\mathfrak{g}_1$ is a real form of $\mathfrak{g}$ equipped with $\theta_1$ as a Cartan involution. Let $\mathfrak{g}_1=\mathfrak{k}_1\oplus\mathfrak{p}_1$ be the corresponding Cartan decomposition. Let subscripts C denote the complexification. Then by the definition of $\theta_1$

$$\mathfrak{m}=(\mathfrak{k}_1)_{\mathrm{C}},\quad \mathfrak{u}^{-}\oplus\mathfrak{u}^{+}=(\mathfrak{p}_1)_{\mathrm{C}}. \tag{5.1}$$

Now this implies that $(\mathfrak{g}_1,\mathfrak{k}_1)$ is a Hermitian symmetric pair.

Let $x^{\#}=-\sigma(x)$ and extend $x\mapsto x^{\#}$ to a conjugate linear antiautomorphism of $U(\mathfrak{g})$. A Hermitian form $\{\cdot,\cdot\}$ on a $\mathfrak{g}$-module $M$ is invariant with respect to $\mathfrak{g}_0$ (resp. $\mathfrak{g}_1$) if $\{x\cdot a, b\}=\{a, x^*b\}$ (resp. $\{x\cdot a, b\}=\{a, x^{\#}b\}$), $x\in U(\mathfrak{g})$, $a, b\in M$. For convenience we call the first invariance *-invariance and the second #-invariance.

For $\nu\in\mathfrak{h}^*, N(\nu)$ admits a *-invariant (resp. #-invariant) Hermitian form precisely when $\nu(H)=-\bar{\nu}(\bar{H})$ (resp. $\nu(H)=-\overline{\nu(\sigma(H))}$), $H\in\mathfrak{h}$. For the remainder of this section we assume $\lambda$ satisfies both of these conditions. Equivalently, these conditions say that $\lambda$ is supported on $\mathfrak{t}$ and is pure imaginary on $\mathfrak{t}\cap\mathfrak{g}_0$. Let $(\cdot,\cdot)_\lambda$ (resp. $\{\cdot,\cdot\}_\lambda$) denote the *-invariant (resp. #-invariant) Hermitian form on $N(\lambda)$, as well as quotients of $N(\lambda)$. We now compare these two forms using the automorphism of $U(\mathfrak{g}), \gamma=\theta_1\circ\theta=\theta\circ\theta_1$.

By our assumptions, $\gamma(\mathfrak{q})=\mathfrak{q}$ and $\gamma(\lambda)=\lambda$. Therefore, $\gamma\otimes 1$ induces an action on $N(\lambda)$. Let $\gamma$ also denote this linear isomorphism of $N(\lambda)$. Since the image of a submodule by $\gamma$ is a submodule of $N(\lambda)$, the maximal submodule of $N(\lambda)$ is stable under $\gamma$. So $\gamma$ induces an action, also denoted $\gamma$, on $L(\lambda)$.

LEMMA 5.2. (i) *For both $N(\lambda)$ and $L(\lambda)$ the forms $(\cdot,\cdot)_\lambda$ and $\{\cdot,\cdot\}_\lambda$ are invariant by $\gamma$; i.e., $(\gamma(m),\gamma(n))_\lambda=(m,n)_\lambda$, $\{\gamma(m),\gamma(n)\}_\lambda=\{m,n\}_\lambda$, $m,n\in N(\lambda)$ or $L(\lambda)$.*

(ii) *Also we have the identity:*

$$\{m,n\}_\lambda=(m,\gamma(n))_\lambda, \quad m,n\in N(\lambda) \text{ or } L(\lambda).$$

*Proof.* $\gamma$ preserves $U(\mathfrak{m})$; and so, as above, induces an action on $F(\lambda)$. Since $F(\lambda)$ is unitarizable, $F(\lambda)$ is irreducible under $\mathfrak{m}\cap\mathfrak{k}$ and the induced action on $F(\lambda)$ is the identity. Recall from section three the projection $p$ and (3.2). Then $\gamma$ commutes with $*, \#$ and $p$. So for $x, y\in U(\mathfrak{g})$, $e, f\in F(\lambda)$,

$$\begin{aligned}(\gamma(x)\otimes e, \gamma(y)\otimes \mathrm{f})_\lambda &= \xi_\lambda(p(\gamma(y)^*\gamma(x))\cdot e, f)\\ &= \xi_\lambda(\gamma p(y^*x)\, e, f)\\ &= \xi_\lambda(p(y^*x)\, e, f)=(x\otimes e, y\otimes f)_\lambda.\end{aligned}$$

The invariance for $\{\cdot,\cdot\}_\lambda$ is proved in the same way. This proves (i).

We define $\{x\otimes e, y\otimes f\}_\lambda=\xi_\lambda(p(y^{\#}x)\, e, f)$. Since $y^{\#}=\gamma(y)^*$, $\{m,n\}_\lambda=(m,\gamma(n))_\lambda$ which proves (ii).

LEMMA 5.3. *Let $L(\lambda)^{-1}=\{x\in L(\lambda)\,|\,\gamma(x)=-x\}$. Then $L(\lambda)^{-1}\subset\mathfrak{r}^-L(\lambda)$ and $\mathfrak{r}^-L(\lambda)$ is stable by $\gamma$.*

*Proof.* $N(\lambda) \simeq U(\mathfrak{u}^-) \otimes F(\lambda)$. Since $\gamma$ acts by the identity on $F(\lambda)$ and $\mathfrak{u}_n^-$ and $(-1)$·identity on $\mathfrak{u}_c^-$ and since $U(\mathfrak{u}^-)=\mathfrak{u}_c^- U(\mathfrak{u}^-) \oplus U(\mathfrak{u}_n^-)$, $L(\lambda)^{-1}$ is contained in $\mathfrak{u}_c^- L(\lambda)$. Since $\gamma(\mathfrak{r}^-)=\mathfrak{r}^-$, $\mathfrak{r}^- L(\lambda)$ is stable by $\gamma$.

PROPOSITION 5.4. *Suppose that $L(\lambda)$ is a unitarizable representation of $\mathfrak{g}_1$; i.e., $\{\cdot,\cdot\}_\lambda$ is positive definite. Then*

(i) *relative to $(\cdot,\cdot)_\lambda$ we have the orthogonal direct sum $L(\lambda)=L(\lambda)^{\mathfrak{r}} \oplus \mathfrak{r}^- L(\lambda)$,*

(ii) *the restriction of $(\cdot,\cdot)_\lambda$ to $L(\lambda)^{\mathfrak{r}}$ is positive definite.*

*Proof.* Let $A=\mathfrak{r}^- L(\lambda)$. The restriction of $\{\cdot,\cdot\}_\lambda$ to $A$ is nondegenerate; and so, Lemma 5.2 (ii) and Lemma 5.3 imply that the restriction of $(\cdot,\cdot)_\lambda$ to $A$ is nondegenerate. Then relative to $(\cdot,\cdot)_\lambda$, $L(\lambda)=A \oplus A^\perp$. But $\mathfrak{r}^{-*}=\mathfrak{r}$ and so $A^\perp=L(\lambda)^{\mathfrak{r}}$. This gives (i). By Lemma 5.3, $A$ contains the $-1$ eigenspace of $\gamma$ and since distinct eigenspaces of $\gamma$ are orthogonal, $\gamma$ is the identity on $A^\perp$. Then by Lemma 5.2 (ii), $(\cdot,\cdot)_\lambda$ is positive definite on $A^\perp$. This proves (ii).

Under the hypotheses of this section the $\mathfrak{k}$-structure of $N(\lambda)$ is especially interesting. The remainder of the section concerns this question and begins with a basic lemma.

LEMMA 5.5. *Let $S(\mathfrak{u}_n^-) \otimes F(\lambda)=\Sigma_i F_c(\mu_i)$. Assume $N(\lambda)$ is completely reducible as a $\mathfrak{k}$-module. Then $N(\lambda) \simeq \Sigma_i N_c(\mu_i)$ and each $N_c(\mu_i)$ is irreducible.*

*Proof.* The proof is similar to Lemma 3.1. As in Lemma 3.1, $N(\lambda)$ has a filtration with $N^i/N^{i+1} \simeq N_c(\mu_i)$. Now $N(\lambda)$ is completely reducible and $N_c(\mu_i)$ is indecomposable. So, $N_c(\mu_i)$ is irreducible and we obtain Lemma 5.5.

Since $\mathfrak{u}$ is abelian, $\mathscr{L}=\mathfrak{u}_n^- \oplus \mathfrak{m}_c \oplus \mathfrak{u}_n$ is a Lie algebra. Also we may choose real forms $\mathscr{L}_1$ of $\mathscr{L}$ and $\mathfrak{m}_1$ of $\mathfrak{m}_c$ so that $(\mathscr{L}_1, \mathfrak{m}_1)$ is a Hermitian symmetric pair. Let $\Delta_1, \ldots, \Delta_q$ be the components of the root system $\Delta(\mathscr{L})$ not contained in $\Delta(\mathfrak{m}_c)$. Following the definition of Harish-Chandra (see [29] or [7]), let $\gamma_{i,1} \leqslant \ldots \leqslant \gamma_{i,r_i}$ be the strongly orthogonal roots associated to the pair $(\Delta_i, \Delta_i \cap \Delta(\mathfrak{m}_c))$, $1 \leqslant i \leqslant q$. Recall that $\gamma_{i,1}$ is the smallest element of $\Delta_i^+ \setminus \Delta(\mathfrak{m}_c)$. Then for $j \geqslant 2$, $\gamma_{i,j}$ is the minimal element of $\Delta_i^+ \setminus \Delta(\mathfrak{m}_c)$ subject to $\gamma_{i,j} \mp \gamma_{i,l}$, and $\gamma_{i,j} \pm \gamma_{i,l}$ is not a root for all $l$, $1 \leqslant l < j$. The sum $\Sigma_{1 \leqslant i \leqslant q} r_i$ is the split rank of $\mathscr{L}_1$.

PROPOSITION 5.6 [21]. $S(\mathfrak{u}_n^-)$ *is multiplicity free as an $\mathfrak{m}_c$-module and the highest weights are precisely those of the form*

$$-\sum_{1\leq i\leq q}\sum_{1\leq j\leq r_i} a_{ij}\gamma_{ij}$$

*where* $a_{ij}\in\mathrm{N}$ *and* $a_{i1}\geq\ldots\geq a_{ir_i}$, $1\leq i\leq q$.

Now combining Lemma 5.5 and Proposition 5.6 we have:

COROLLARY 5.7. *Assume* $F(\lambda)$ *is one-dimensional and* $N(\lambda)$ *is completely reducible as a* $\mathfrak{k}$*-module. Then* $N(\lambda)\simeq\oplus_\mu N_c(\mu)$ *where the sum is taken over the set* $\mu$ *of the form*

$$\lambda-\sum_{1\leq i\leq q}\sum_{1\leq j\leq r_i} a_{ij}\gamma_{ij}$$

*where* $a_{ij}\in\mathrm{N}$ *and* $a_{i1}\geq\ldots\geq a_{ir_i}$, $1\leq i\leq q$.

## § 6. Zuckerman functors

Here we describe briefly the derived functors introduced by Zuckerman [6], [25]. These results will be used in the next section to translate the results of sections three through five into our main results on Harish-Chandra modules.

Let $\mathscr{C}(\mathfrak{a},\mathfrak{b})$ be the category of $\mathfrak{a}$-modules which as $\mathfrak{b}$-modules are $U(\mathfrak{b})$-locally finite and completely reducible. For an object $A$ in $\mathscr{C}(\mathfrak{g},\mathfrak{m}_c)$, define $\Gamma A$ to be the subspace of $U(\mathfrak{k})$-locally finite vectors. Then $\Gamma$ is called the $\mathfrak{k}$-finite submodule functor and $\Gamma$: $\mathscr{C}(\mathfrak{g},\mathfrak{m}_c)\to\mathscr{C}(\mathfrak{g},\mathfrak{k})$. The category has enough injectives to construct injective resolutions; $0\to A\to I^*$. Now define $\Gamma^iA$ to be the $i$th cohomology group of the complex: $0\to\Gamma I^0\to\Gamma I^1\to\ldots\to\Gamma I^j\to\ldots$. The $\Gamma^i$ are the right derived functors of $\Gamma$. For all $i$, $\Gamma^iA$ is a $U(\mathfrak{k})$-locally finite $\mathfrak{g}$-module. If it has finite $\mathfrak{k}$-multiplicities, it is infinitesimally equivalent to an admissible representation of $G$.

Let $\mathscr{C}(\mathfrak{k},\mathfrak{m}_c)$ be the category defined as above with $\mathfrak{g}$ replaced by $\mathfrak{k}$. Let $F$ be the forgetful functor $F$: $\mathscr{C}(\mathfrak{g},\mathfrak{m}_c)\to\mathscr{C}(\mathfrak{k},\mathfrak{m}_c)$ which considers a $\mathfrak{g}$-module as a $\mathfrak{k}$-module. As is shown in [6], $F$ maps injectives in $\mathscr{C}(\mathfrak{g},\mathfrak{m}_c)$ into injectives in $\mathscr{C}(\mathfrak{k},\mathfrak{m}_c)$; and so, we have:

LEMMA 6.1. *For all* $i$, $\Gamma^i$ *and* $F$ *commute. More precisely:* $F\circ\Gamma^i$ *and* $\Gamma^i\circ F$ *are naturally equivalent.*

This lemma implies that the $\Gamma^i$ are frequently quite computable. The basis is the computation in $\mathscr{C}(\mathfrak{k},\mathfrak{m}_c)$ given as:

PROPOSITION 6.2. *Let* $\mu\in\mathfrak{t}^*$ *be* $\Delta^+(\mathfrak{k})$*-dominant and let* $\omega\in\mathcal{W}(\mathfrak{k})$ *satisfy:* $\omega\Delta^+(\mathfrak{k})\supset\Delta^+(\mathfrak{m}_c)$. *Then, for* $i\in\mathbf{N}$, $s=\dim\mathfrak{u}_c$,

$$\Gamma^{2s-i}N_c(\omega\mu-\varrho_c)=\begin{cases}L_c(\mu-\varrho) & \textit{if } i=\textit{length of } \omega \textit{ and } \mu \textit{ is integral and regular}\\ 0 & \textit{otherwise}\end{cases}$$

For a proof of Proposition 6.2 see [6].

For $A\in\mathscr{C}(\mathfrak{a},\mathfrak{b})$, let $\bar{A}^*$ be the space of all conjugate linear functionals on $A$. $\bar{A}^*$ is an $\mathfrak{a}$-module under the action $(x\cdot\varphi)(a)=\varphi(x^*\cdot a)$, $x\in\mathfrak{a}$, $\varphi\in\bar{A}^*$, $a\in A$. Let $\hat{A}$ denote the subspace of $\bar{A}^*$ of $U(\mathfrak{b})$-locally finite vectors. Then $\hat{A}\in\mathscr{C}(\mathfrak{a},\mathfrak{b})$. If $A$ is admissible then so is $\hat{A}$. In [6] a duality theorem for the admissible dual was proved. The proof works equally well in our case and gives the following variant for the conjugate dual. Let $\mathscr{A}(\mathfrak{a},\mathfrak{b})$ be the subcategory of $\mathscr{C}(\mathfrak{a},\mathfrak{b})$ of admissible objects.

THEOREM 6.3. *For all* $i$, $A\mapsto(\Gamma^iA)^{\wedge}$ *and* $A\mapsto\Gamma^{2s-i}(\hat{A})$ *are naturally equivalent functors from* $\mathscr{A}(\mathfrak{g},\mathfrak{m}_c)$ *to* $\mathscr{C}(\mathfrak{g},\mathfrak{k})$.

Let $A\in\mathscr{C}(\mathfrak{g},\mathfrak{m}_c)$ and let $\psi$ be an invariant Hermitian form on $A$. We now apply Theorem 6.3 as follows to define $\Gamma^s\psi$. Let $\theta$ be the natural equivalence given by Theorem 6.3. Now $\psi$ defines a map $\psi^\vee\colon A\to\hat{A}$ by $(\psi^\vee(a))(b)=\psi(a,b)$. So $\Gamma^s\psi^\vee\colon\Gamma^sA\to\Gamma^s\hat{A}$. Finally, composing with $\theta$ we obtain

$$\theta\circ\Gamma^s\psi^\vee\colon\Gamma^sA\to(\Gamma^sA)^{\wedge}. \tag{6.4}$$

This map corresponds to an invariant sesquilinear form on $\Gamma^sA$ which we denote by $\Gamma^s\psi$.

The duality in Theorem 6.3 depends on the choice of one constant, a basis vector $\xi$ for $\Lambda^{2s}(\mathfrak{k}/\mathfrak{m}_c)$. From formulas (4.5) and (4.6) in [6] which describe the duality explicitly, we may and do choose $\xi$ so that if $\psi$ is Hermitian then so is $\Gamma^s\psi$.

In the remainder of this section we consider the action of $\Gamma^s$ on the canonical invariant forms on $N_c(\mu)$, $\mu\in\mathfrak{t}^*$.

Each Verma module $M(\mu)$ has a canonical cyclic vector $1\otimes1$. If $\mu$ is pure imaginary on $\mathfrak{t}_0$ then $M(\mu)$ admits invariant Hermitian forms (w.r.t.*) unique up to scalar multiple. Let $\varphi_\mu$ be the unique form with

$$\varphi_\mu(1\otimes1,\ 1\otimes1)=1. \tag{6.5}$$

Let $\varphi_\mu$ also denote the induced form on all quotients of $M(\mu)$. Let $r_0$ be the element of $\mathcal{W}(\mathfrak{k})$ such that $r_0\mathscr{C}_c$ is the positive chamber associated with $\Delta^+(\mathfrak{k})$ and let $s_0$ be the longest element of $\mathcal{W}(\mathfrak{k})$.

PROPOSITION 6.6. *Let* $\mu \in \mathfrak{t}^*$ *with* $\mu+\varrho_c$ *a* $\Delta(\mathfrak{k})$*-integral, regular element in* $\mathscr{C}_c$. *Define constants* $a_\mu$ *by*

$$\Gamma^s \varphi_\mu = a_\mu \varphi_{r_0(\mu+\varrho_c)-\varrho_c}.$$

*Then these constants are real, nonzero and all of the same sign.*

*Proof.* The forms defining $a_\mu$ are Hermitian; and so, the constants are real. To see that $a_\mu \neq 0$, note that $N_c(\mu)$ is irreducible. Then $\varphi_\mu^\vee$ is a nonzero multiple of the identity (cf. (6.4)) and the functor $\Gamma^s$ maps this to a nonzero multiple of the identity. So $a_\mu \neq 0$.

Let $\lambda$ be an integral element of $\mathscr{C}_c$. We now prove:

$$a_\mu \text{ and } a_{\mu+\lambda} \text{ have the same sign.} \tag{6.7}$$

Write $F=L(-s_0 r_0 \lambda)$, $\varphi=\varphi_{-s_0 r_0 \lambda}$, $N=N(\mu+\lambda)$ and $L=L(r_0(\mu+\lambda+\varrho_c)-\varrho_c)$. Let $\varphi_1$ (resp. $\varphi_2$) be the canonical form on $N$ (resp. $L$) and let $\Psi$ denote the Zuckerman translation functor which carries $N(\mu+\lambda)$ to $N(\mu)$.

The functors $A \mapsto \Gamma^i(F \otimes A)$ and $A \mapsto F \otimes \Gamma^i A$ are naturally equivalent [6]. Using this equivalence we now show that

$$\Gamma^s(\varphi \otimes \varphi_1) = \varphi \otimes \Gamma^s \varphi_1. \tag{6.8}$$

Let $\xi$ be the basis vector for $\Lambda^{2s}(\mathfrak{k}/\mathfrak{m}_c)$ chosen above. Then $\xi$ determines an $\mathfrak{m}_c$-invariant Hermitian form $\xi'$ on $\Lambda^s(\mathfrak{k}/\mathfrak{m}_c)$. For any $\mathfrak{k}$-module $X$, let $h(X)=\mathrm{Hom}_{U(\mathfrak{m}_c)}(\Lambda^s(\mathfrak{k}/\mathfrak{m}_c), X)$. Following the notation in [6], let $V_\gamma$ denote an irreducible finite dimensional $\mathfrak{k}$-module and $\varphi_\gamma$ its invariant Hermitian form (normalized as above). Then formulas (4.4), (4.5) and (4.6) in [6] show that $\Gamma^s\varphi_1$ is induced by the forms $\xi' \otimes \varphi_\gamma \otimes \varphi_1$ on $h(V_\gamma \otimes N)$, as we vary $\gamma$. Moreover, the natural equivalence described above (6.8) is induced by the equivalence of $F \otimes h(V_\gamma \otimes N)$ and $h(V_\gamma \otimes F \otimes N)$. But then this equivalence takes $\varphi \otimes \xi' \otimes \varphi_\gamma \otimes \varphi_1$ to $\xi' \otimes \varphi_\gamma \otimes \varphi \otimes \varphi_1$. The first form induces $\varphi \otimes \Gamma^s \varphi_1$ while the second induces $\Gamma^s(\varphi \otimes \varphi_1)$. This proves (6.8).

If $a$ (resp. $b$) is a highest weight vector of $F$ (resp. $N$), then $a \otimes b$ is a highest weight vector for $N(\mu)$. Since $\varphi \otimes \varphi_1\,(a \otimes b, a \otimes b) = \varphi(a,a)\,\varphi_1(b,b) > 0$, the restriction of $\varphi \otimes \varphi_1$ to $N(\mu)$ equals $a \cdot \varphi_\mu$ with $a>0$. The form $\varphi$ corresponds to the Hermitian form for a compact real form of $\mathfrak{k}$; and so, since finite dimensional representations of compact groups are unitarizable, $\varphi$ is positive definite. Therefore, $\varphi \otimes \Gamma^s \varphi_1$ is either positive or negative definite depending on whether $a_{\mu+\lambda}$ is positive or negative. Now $\Gamma^s(a \cdot \varphi_\mu) = a \cdot a_\mu \varphi_{r_0(\mu+\varrho_c)-\varrho_c}$ is positive (resp. negative) definite if $a_{\mu+\lambda}$ is positive (resp. negative). This proves (6.7).

To prove Proposition 6.6 let $\mu' \in \mathfrak{t}^*$ with $\mu'+\varrho_c$ a $\Delta(\mathfrak{k})$-integral, regular element of $\mathscr{C}_c$. Then there exist $\lambda$ and $\lambda'$ integral elements of $\mathscr{C}_c$ with $\mu+\lambda=\mu'+\lambda'$. Now applying (6.7) twice, $a_\mu$ and $a_{\mu+\lambda}$ and $a_{\mu'}$ have the same sign.

We should also note that Proposition 6.6 can be proved "directly" by a careful study of the resolution which gives the duality theorem.

PROPOSITION 6.9. *Assume* $L(\lambda)$ *is completely reducible as a* $\mathfrak{k}$*-module and all irreducible summands are isomorphic to* $N_c(\mu)$ *for* $\mu \in \mathfrak{t}^*$. *Assume the restriction of* $(\cdot, \cdot)_\lambda$ *to* $L(\lambda)^{\mathfrak{u}_c}$ *is positive definite. Then*

(i) $\Gamma^i L(\lambda)=0$ *for* $i \neq s$.
(ii) $\Gamma^s L(\lambda)$ *is either zero or a unitarizable representation of* $G$.

*Proof.* We can write $L(\lambda)=\Sigma_i N_c(\mu_i)$ as an orthogonal sum where each $N_c(\mu_i)$ is irreducible. If $\lambda$ is not $\Delta(\mathfrak{k})$-integral then $\Gamma^i L(\lambda)=0$ for all $i$ by Propositions 6.2 and 6.9 is true. So assume $\lambda$ is $\Delta(\mathfrak{k})$-integral. Then by [13] Corollary 4, the irreducibility of $N(\mu_i)$ implies that $\mu_i+\varrho_c \in \mathscr{C}_c$ or $\mu_i+\varrho_c$ is singular. Then Proposition 6.2 proves (i). The restriction of $(\cdot, \cdot)_\lambda$ to $N_c(\mu_i)$ is a positive multiple of $\varphi_{\mu_i}$; and so, applying Proposition 6.6, $\Gamma^s(\cdot, \cdot)_\lambda$ is a definite Hermitian form on $\Gamma^s L(\lambda)$. This proves (ii).

## §7. The main results

Here we apply the derived functors to the modules studied in sections three through five. Fix $\lambda, \xi \in \mathfrak{h}^*$ and assume $F(\lambda)$ and $F(\xi)$ are unitarizable (w.r.t. $\mathfrak{m}_0$) and $F(\xi)$ is one-dimensional. Also assume $\langle \xi, \alpha \rangle < 0$ for all $\alpha \in \Delta(\mathfrak{u})$, and $\mathfrak{q}$ is quasi abelian.

THEOREM 7.1. *Assume* $N(\lambda+t\xi)$ *is irreducible for all* $t>0$ *and either* (a) $\lambda|_{\mathfrak{t}}+\varrho_c \in \mathscr{C}_c$ *or* (b) $L(\lambda)$ *is free over* $U(\mathfrak{u}_c^-)$ *and* $N(\lambda+t\xi)$ *is completely reducible over* $U(\mathfrak{k})$ *for* $t \geq 0$. *Then*

(i) $\Gamma^i L(\lambda)=0$ *for* $i \neq s$.
(ii) $\Gamma^s L(\lambda)$ *is either zero or a unitarizable representation of* $G$.

*Note that* (a) *implies* (b).

*Proof.* Combine Lemma 3.1, Propositions 3.5 and 6.9 to prove the result under hypothesis (a). For (b), replace Proposition 3.5 by Proposition 3.11.

THEOREM 7.2. *Assume* $\mathfrak{u}_c$ *is abelian,* $N(\lambda+t\xi)$ *is irreducible for all* $t>0$ *and* $L(\lambda)$ *is a free module over* $U(\mathfrak{u}_c^-)$. *Then*

(i) $\Gamma^i L(\lambda)=0$ *if* $i \neq s$.

(ii) $\Gamma^s L(\lambda)$ *is either zero or a unitarizable representation of* $G$.

*Proof.* By Lemma 4.6, $L(\lambda)$ is completely reducible as a $\mathfrak{k}$-module. Since it is free over $U(\mathfrak{u}_c^-)$, each irreducible summand must be isomorphic to an irreducible $N_c(\mu)$. This fact, Propositions 4.1 and 6.9 complete the proof.

The sharpest results are available when $\mathfrak{u}$ is abelian. Let notation be as in section five. In particular, recall that $\mathfrak{g}_1$ is a Hermitian symmetric real form of $\mathfrak{g}$ and $G_1$ is the corresponding simply connected real Lie group.

THEOREM 7.3. *Assume* $\mathfrak{u}$ *is abelian. Assume* $L(\lambda)$ *is a unitarizable representation of* $G_1$ *and is free over* $U(\mathfrak{u}_c^-)$. *Then*

(i) $\Gamma^i L(\lambda)=0$ *if* $i \neq s$.

(ii) $\Gamma^s L(\lambda)$ *is either zero or a unitarizable representation of* $G$.

*Proof.* Since $L(\lambda)$ is unitarizable for $G_1$, it is completely reducible as a $\mathfrak{k}$-module. Since it is free over $U(\mathfrak{u}_c^-)$, each irreducible summand is isomorphic to a irreducible $N_c(\mu)$. This fact, Propositions 5.4 and 6.9 combine to prove Theorem 7.3.

Under the hypotheses of Theorem 7.3, $L(\lambda)$ is complete reducible: $L(\lambda)=\Sigma_i N_c(\mu_i)$ with each $N_c(\mu_i)$ irreducible.

PROPOSITION 7.4. *Under the hypotheses of Theorem* 7.3, *the* $\mathfrak{k}$*-multiplicities of* $\Gamma^s L(\lambda)$ *are given by: for* $\mu \in \mathfrak{t}^* \Delta^+(\mathfrak{k})$*-dominant integral,*

$$\dim \operatorname{Hom}(L_c(\mu), \Gamma^s L(\lambda)) = \dim \operatorname{Hom}(N_c(r_0^{-1}(\mu+\varrho_c)-\varrho_c), L(\lambda))$$

*with* $r_0$ *as in Proposition* 6.6.

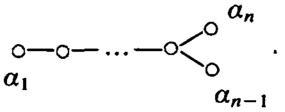
*Proof.* We may assume $\lambda$ is $\Delta(\mathfrak{k})$-integral. Since any $N_c(\mu_i)$ occurring in $L(\lambda)$ is irreducible, $\mu_i+\varrho_c$ is either in $\mathscr{C}_c$ or is singular. Then Proposition 6.2 proves Proposition 7.4.

*Remark.* As indicated by Corollary 5.7, the $\mathfrak{k}$-decomposition of $L(\lambda)$ is frequently quite explicit. So, in many applications, Proposition 7.4 will give explicit formulas for $\mathfrak{k}$-multiplicities in $\Gamma^s L(\lambda)$.

## § 8. Ladder representations for orthogonal groups $SO(p, q)$ with $p+q$ even

In this section we describe the first application of our main results. We define a distinguished unitary representation for each orthogonal group $SO(p, q)$ with $p+q$ even. We call these representations ladder representations since their $\mathfrak{k}$-types have highest weights which lie on a single line in $\mathfrak{t}^*$. Our results here follow from Theorem 7.3 when $\mathfrak{g}_0=so(p, q)$ and $\mathfrak{g}_1=so(2, 2n-2)$ with $2n=p+q$.

Consider the Dynkin diagram for $D_n$



with $\alpha_i=\varepsilon_i-\varepsilon_{i+1}$, $1\leqslant i<n$, and $\alpha_n=\varepsilon_{n-1}+\varepsilon_n$. Let $\mathfrak{q}$ be the parabolic subalgebra $\mathfrak{q}=\mathfrak{m}\oplus\mathfrak{u}$ with $\Delta^+(\mathfrak{m})$ having simple roots $\alpha_2, \ldots, \alpha_n$. Assume $n\geqslant 4$ and $l$ is an integer $2\leqslant l\leqslant n-2$. Let all $\alpha_i$ be compact roots except $\alpha_l$ which is noncompact. Then $\mathfrak{h}$ is the complexification of a compact Cartan subalgebra of $so(2l, 2n-2l)$. The Cartan involution $\theta$ equals the identity on $\mathfrak{h}$; and so, $\mathfrak{q}$ is $\theta$-stable. In the $\varepsilon_i$ coordinates, let $\lambda=(z, 0, \ldots, 0)$, $z\in\mathbf{R}$. From [7], $N(\lambda)$ is irreducible and unitarizable for $so(2, 2n-2)$ if and only if $z<-n+2$. At $z=-n+2$, $N(\lambda)$ is reducible. Let $N=N(\lambda)$ and $L=L(\lambda)$ for $z=-n+2$. From [29], p. 29, we have:

$$L\simeq N/N(\lambda-2\varepsilon_1). \tag{8.1}$$

Now in the notation of section five, $\mathscr{L}=\mathfrak{u}_n^-\oplus\mathfrak{m}_c\oplus\mathfrak{u}_n$ is the complexification of $so(2l-2)\times so(2, 2r)$ with $r=n-l$. The strongly orthogonal roots here are $\gamma_1=\varepsilon_1-\varepsilon_{l+1}$ and $\gamma_2=\varepsilon_1+\varepsilon_{l+1}$. So, by Corollary 5.7, we have

$$N(\lambda)\simeq\sum_{a,b\in\mathbf{N}} N_c(\lambda-(2a+b)\varepsilon_1+b\varepsilon_{l+1}). \tag{8.2}$$

From (8.1)

$$L\simeq\sum_{b\in\mathbf{N}} N_c(-n+2-b, 0, \ldots, 0, b, 0, \ldots, 0) \tag{8.3}$$

where the $b$ occurs as the $l+1$ coordinate.

If we change $\alpha_{n-1}$ and $\alpha_n$ to complex roots and let $\theta$ act on $\mathfrak{h}$ by the identity on $\varepsilon_i$, $1\leqslant i<n$, and $(-1)\cdot$identity on $\varepsilon_n$, then instead of $\mathfrak{g}_0\simeq so(2l, 2n-2l)$ we have $\mathfrak{g}_0\simeq so(2l+1, 2n-2l-1)$. Now $\mathfrak{t}^*$ is spanned by $\varepsilon_i$, $1\leqslant i<n$. In this case, $\mathscr{L}$ is the complexification of $so(2l-1)\times so(2, 2n-2l-1)$ and formulas (8.2) and (8.3) still hold. Note that since $l+1<n$ the weights in these formulas are supported on $\mathfrak{t}$.

The module $L$ is unitarizable for $SO(2,2n-2)$, and so, by Theorem 7.3 and Proposition 7.4 we obtain:

PROPOSITION 8.4. *Let* $2n=p+q$, $l=[p/2]$ *and assume* $p$ *and* $q$ *are positive integers,* $n\geq 4$ *and* $2\leq l\leq n-2$. *Let* $G=SO(p,q)$. *Then* $X=\Gamma^s L$ *is a unitarizable representation of* $G$. *Moreover,* $X$ *is multiplicity free as a* $\mathfrak{k}$*-module with highest weights*

$$(n-p+j,0,\ldots,0,j,0,\ldots,0)$$

*with* $n-p+j\geq 0, j\in\mathbf{N}$ *and* $j$ *occurring as the* $l+1$ *coordinate.*

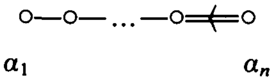
Note that in the case $l=1$, $\mathfrak{g}_0=\mathfrak{g}_1$, $s=0$ and $\Gamma^s L=L$.

In the case $p+q$ odd, $p,q\geq 4$, results of Howe and Vogan [26] assert that no such ladder representations exist.

## §9. Ladder representations for $Sp(r,s)$ and special representations for $Sp(n,\mathbf{R})$

First we construct some special unitary representations of $Sp(r,s)$. As in the previous section, we shall call these representations ladder representations since the highest weights of their $\mathfrak{k}$-types lie on a single line in $\mathfrak{t}^*$. Here our results follow from Theorem 7.1 for a quasi abelian parabolic subalgebra which, however, does not satisfy the hypotheses of Theorems 7.2 and 7.3.

Let $\mathfrak{g}_0=sp(r,s)$, $n=r+s$ and let $\mathfrak{h}_0$ be a compact CSA of $\mathfrak{g}_0$. As usual let $\mathfrak{g}$ and $\mathfrak{h}$ be the complexifications of these Lie algebras and let $\Delta^+$ be any positive system of roots for $(\mathfrak{g},\mathfrak{h})$. Consider the Dynkin diagram for $C_n$:



where $\alpha_i$ are the simple roots of $\Delta^+$. Since all long roots of $\mathfrak{g}$ are compact, $\alpha_n$ is compact. In coordinates we write $\alpha_i=\varepsilon_i-\varepsilon_{i+1}$, $1\leq i<n$, $\alpha_n=2\varepsilon_n$. Let $\mathfrak{g}$ be the maximal $\theta$-stable parabolic subalgebra with $\Delta^+(\mathfrak{m})$ having simple roots $\alpha_i$, $2\leq i\leq n$. Then $\mathfrak{u}$ is a Heisenberg algebra and $\mathfrak{q}$ is quasi abelian. In this case $\mathfrak{k}\simeq sp(r)\times sp(s)$, $\mathfrak{m}\simeq\mathfrak{u}(1)\times \mathrm{sp}(n-1)$.

LEMMA 9.1. *If* $\mathfrak{m}$ *is not contained in* $\mathfrak{k}$ *then* $\mathfrak{m}_c$ *is isomorphic to either* $\mathfrak{u}(1)\times sp(r-1)\times sp(s)$ or $\mathfrak{u}(1)\times sp(r)\times sp(s-1)$.

*Proof.* We decompose $\mathfrak{u}$ with $\mathfrak{u}=\mathfrak{u}_1\oplus\mathfrak{u}_2$, $\Delta(\mathfrak{u}_2)=2\varepsilon_1$ and $\Delta(\mathfrak{u}_1)=\{\varepsilon_1\pm\varepsilon_j|2\leq j\leq n\}$. Now split $\mathfrak{u}_1$ into its compact and noncompact parts $\mathfrak{u}_1=\mathfrak{u}_{1,c}\oplus\mathfrak{u}_{1,n}$. Let $2d=$

$\dim \mathfrak{u}_{1,c}$, $2e=\dim \mathfrak{u}_{1,n}$. So $d+e=n-1$. If $\varepsilon_1+\varepsilon_j$ is compact then so is $\varepsilon_1-\varepsilon_j$ and conversely. So if $\varepsilon_1\pm\varepsilon_j$ and $\varepsilon_1\pm\varepsilon_k$ are all in $\Delta(\mathfrak{u}_{1,c})$ or all in $\Delta(\mathfrak{u}_{1,n})$, then $\varepsilon_j\pm\varepsilon_k \in \Delta(\mathfrak{m}_c)$. Since all long roots are compact this implies $\mathfrak{m}\cap\mathfrak{k}$ contains $sp(d)\times sp(e)$. But $\mathfrak{m}\cap\mathfrak{k}$ has rank $n$ with a nontrivial center; and so, $\mathfrak{m}\cap\mathfrak{k}\simeq \mathfrak{u}(1)\times sp(d)\times sp(e)$ or $\mathfrak{u}(1)\times sp(n-1)$. This second case is excluded if $\mathfrak{m}$ is not contained in $\mathfrak{k}$. Since $\mathfrak{m}\cap\mathfrak{k}$ is a rank $n$ subalgebra of $\mathfrak{k}$, either $d\leqslant r$, $e\leqslant s$ or $d\leqslant s$, $e\leqslant r$. Now $d+e=r+s-1$ implies $\{d, e\}$ equals the set $\{r-1, s\}$ or $\{r, s-1\}$. This proves Lemma 9.1.

The action of $sp(e)$ on $\mathfrak{u}_{1,n}$ is the defining action of $sp(e)$ on $\mathbf{C}^{2e}$. The other factor $sp(d)$ acts trivially on $\mathfrak{u}_{1,n}$. Let $\gamma$ be the lowest weight in $\Delta(\mathfrak{u}_{1,n})$.

LEMMA 9.2. *As an* $\mathfrak{m}_c$*-module,* $S(\mathfrak{u}_n^-)$ *is multiplicity free and* $S(\mathfrak{u}_n^-)\simeq \Sigma_{n\in\mathbf{N}} F_c(-n\gamma)$.

*Proof.* This action is equivalent to the action of $sp(e)$ on $S(\mathbf{C}^{2e})$. Here the result is classical [30].

LEMMA 9.3. *Let* $\lambda=(z, 0, \ldots, 0)$, $z\in\mathbf{R}$. *Then, for* $z<0$, $N(\lambda)$ *is irreducible and as a* $\mathfrak{k}$*-module*

$$N(\lambda)\simeq\sum_{n\in\mathbf{N}} N_c(\lambda-n\gamma).$$

*Note.* This is an especially curious result since reducibility does not occur until $z=0$ where $L(\lambda)$ is the trivial representation.

*Proof.* The irreducibility can be verified by using Jantzen's criteria (cf. Table 13.2). Let $\mathfrak{k}=\mathfrak{k}_1\oplus\mathfrak{k}_2$ with $\mathfrak{k}_2$ contained in $\mathfrak{m}$. Then since $\gamma$ is not orthogonal to $\Delta(\mathfrak{k}_2)$, $N_c(\lambda-n\gamma)$ and $N_c(\lambda-m\gamma)$ have different infinitesimal characters for $n\neq m$. Therefore the filtration on $N(\lambda)$ induced by the decomposition of $S(\mathfrak{u}_n^-)\otimes\mathbf{C}_\lambda$ Lemma 9.2, actually splits. This proves Lemma 9.3.

Recall the element $r_0\in\mathscr{W}(\mathfrak{k})$ with $r_0\mathscr{C}_c$ the positive chamber for $\Delta^+(\mathfrak{k})$.

PROPOSITION 9.4. *Let* $\lambda=(z, 0, \ldots, 0)$. *For each integer* $z<0$, $\Gamma^s N(\lambda)$ *is a unitarizable representation of* $Sp(r, s)$. *Moreover, the* $\mathfrak{k}$*-types are multiplicity free and highest weights are those elements* $r_0(\lambda-n\gamma+\varrho_c)-\varrho_c$, $n\in\mathbf{N}$, *which are* $\Delta^+(\mathfrak{k})$*-dominant.*

*Proof.* By Theorem 7.1, $\Gamma^s N(\lambda)$ is unitarizable or zero. However by Lemma 9.3, Propositions 3.11 and 6.2 the $\mathfrak{k}$-structure is as described and a short calculation shows this set is not empty. So $\Gamma^s N(\lambda)$ is not zero.

We now turn to the case of $Sp(n, \mathbf{R})$. Let $\mathfrak{g}_0=sp(n, \mathbf{R})$ and $\mathfrak{h}_0$ a compact CSA of $\mathfrak{g}_0$.

Let $\mathfrak{q}$ be the maximal parabolic as above; but now all the long roots are noncompact. The restriction of $\theta$ to $\mathfrak{m}$ is a Cartan involution. Since $\mathfrak{m}$ is not contained in $\mathfrak{k}$ (long roots being noncompact), $(\mathfrak{k}, \mathfrak{m}_c)$ corresponds to the symmetric pair $(u(n), u(1)\times u(n-1))$. Therefore, $\mathfrak{u}_c$ has dimension $n-1$. Then $\mathfrak{u}_n$ has dimension $n$. Let $\gamma_1=2\varepsilon_1$ and let $\gamma_2$ be the minimal short root in $\Delta^+(\mathfrak{u}_n)$. The analogue to Lemma 9.2 for $sp(n,\mathbf{R})$ is:

LEMMA 9.5. *As an* $\mathfrak{m}_c$*-module,* $S(\mathfrak{u}_n^-)$ *is multiplicity free and*

$$S(\mathfrak{u}_n^-)\simeq \sum_{a,b\in\mathbf{N}} F_c(-a\gamma_1-b\gamma_2).$$

*Proof.* $\mathfrak{u}_n^-$ contains the one-dimensional $F_c(-\gamma_1)$ spanned by the $-\gamma_1$ root space. Also $-\gamma_2$ is a highest weight of $\mathfrak{u}_n^-$; and so, $F(-\gamma_2)$ also occurs in $\mathfrak{u}_n^-$. But $F(-\gamma_2)$ has dimension $=n-1$. So $\mathfrak{u}_n^-=F_c(-\gamma_1)\oplus F_c(-\gamma_2)$. Now $F_c(-\gamma_2)$ is the defining representation of $u(n-1)$ on $\mathbf{C}^{n-1}$. Then $S(\mathfrak{u}_n^-)\simeq S(F_c(-\gamma_1))\otimes S(F(-\gamma_2))$ and since $F_c(-\gamma_1)$ is one-dimensional, the classical decomposition [30] of $S(F_c(-\gamma_2))$ gives Lemma 9.5.

LEMMA 9.6. *Let* $\lambda=(z,0,\dots,0)$, $z\in\mathbf{R}$. *Then for* $z<0$, $N(\lambda)$ *is irreducible and as a* $\mathfrak{k}$*-module*

$$N(\lambda)\simeq \bigoplus_{a,b\in\mathbf{N}} N_c(\lambda-a\gamma_1-b\gamma_2).$$

*Proof.* The irreducibility is given in Lemma 9.3. By Lemma 4.6, $N(\lambda)$ is completely reducible as a $\mathfrak{k}$-module. Then as in Lemma 9.3, the decomposition in Lemma 9.5 induces to give Lemma 9.6.

In this case $\mathfrak{u}_c$ is abelian and $\mathfrak{q}$ is quasi abelian. So we may apply Theorem 7.2 and Proposition 6.2. This gives:

PROPOSITION 9.7. *Let* $\lambda=(z,0,\dots,0)$ *in the* $\varepsilon_i$ *coordinates. For each integer* $z<0$, $\Gamma^s N(\lambda)$ *is a unitarizable representation of* $Sp(n,\mathbf{R})$. *Moreover, it is multiplicity free as a* $\mathfrak{k}$*-module and the highest weights are those element* $r_0(\lambda-a\gamma_1-b\gamma_2+\varrho_c)-\varrho_c$, $a,b\in\mathbf{N}$, *which are* $\Delta^+(\mathfrak{k})$*-dominant.*

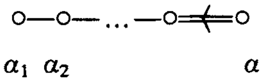
The representation $\Gamma^s N(\lambda)$ for $z=-2n$ is the Zuckerman module $A_{\mathfrak{q}}$ for $\mathfrak{q}$ as above. These modules have computable relative Lie algebra cohomology or equivalently continuous cohomology (cf. [28] Theorem 3.3).

THEOREM 9.8. *Let* $G$ *be a connected, split over* $\mathbf{R}$ *classical group. Then there exists an irreducible unitary non-trivial representation* $(\pi, H)$ *of* $G$ *such that the continuous cohomology group* $H^l_{\mathrm{cont}}(G,H)\neq 0$ *for* $l$ *equal to the real rank of* $G$.

*Proof.* From the formula for the cohomology of $A_{\mathfrak{q}}$ ([28] Theorem 3.3), the cohomology is nonzero for $l=\dim \mathfrak{u}_n$. For the example above $l=n$ which is the split rank of $Sp(n, \mathbf{R})$. For $G=SL(n, \mathbf{R})$ the result follows by the unitarity of the Speh representations (cf. section eleven and [28] table 8.2). The remaining cases are $G=SO(p, p+1)$ and $SO(p, p)$. Then by table 8.2 in [28] the $\theta$-stable parabolic one must choose is a maximal parabolic subalgebra with abelian nilradical. So in this case also the parabolic is quasi abelian and Theorem 7.3 gives the unitarity of the representation $A_{\mathfrak{q}}$.

## § 10. Special multiplicity free unitary representations of $Sp(n, \mathbf{R})$

Let $\mathfrak{g}_0=sp(n, \mathbf{R})$ and let $\mathfrak{h}_0$ be a compact CSA of $\mathfrak{g}_0$. Let $\Delta^+$ be any positive system. Consider the Dynkin diagram



$\alpha_i=\varepsilon_i-\varepsilon_{i+1}$, $1 \leqslant i<n$, and $\alpha_n=2\varepsilon_n$ with $\alpha_i$ the simple roots of $\Delta^+$. All long roots are noncompact so $\alpha_n$ is noncompact. The remaining $\alpha_i$ are compact or noncompact as specified by $\Delta^+$. Conversely, for any assignment of compact and noncompact for $\alpha_i$, $1 \leqslant i<n$, there is a corresponding positive system.

Let $\mathfrak{q}=\mathfrak{m} \oplus \mathfrak{u}$ be the $\theta$-stable parabolic subalgebra with $\Delta^+(\mathfrak{m})$ having simple roots $\alpha_i$, $1 \leqslant i<n$. Then $\mathfrak{m} \simeq \mathfrak{m}(n)$. Since the unitary highest weight representations for $\mathfrak{g}_0$ are known we avoid this case; and so, assume not all $\alpha_i$, $1 \leqslant i<n$, are compact. The restriction of $\theta$ to $\mathfrak{m}$ is a Cartan involution of $\mathfrak{m}$. Thus $\mathfrak{m}_c \simeq \mathfrak{m}(r) \times \mathfrak{m}(s)$ with $r+s=n$, $r, s>0$. Now $\mathscr{L}=\mathfrak{u}_n^- \oplus \mathfrak{m}_c \oplus \mathfrak{u}_n$ is a Hermitian symmetric pair. All the positive long roots lie in $\Delta(\mathfrak{u}_n)$ and $\mathfrak{m}_c$ has a two-dimensional center. Therefore the Hermitian symmetric pair is $sp(r, \mathbf{R}) \times sp(s, \mathbf{R})$. These algebras have split rank $r$ and $s$ and we let $\gamma_1<\ldots<\gamma_r$, $\gamma_1'<\ldots<\gamma_s'$ be the strongly orthogonal roots described in Proposition 5.6.

PROPOSITION 10.1. *Let $\lambda=z(1,1, \ldots, 1)$ in the $\varepsilon_i$ coordinates. Assume $2z \in \mathbf{N}$ and $z<-(n-1)/2$. Then $\Gamma^s N(\lambda)$ is a unitarizable representation of the universal covering of $Sp(n, \mathbf{R})$. Moreover, as a $\mathfrak{k}$-module it is multiplicity free with highest weights all elements in the $\Delta^+(\mathfrak{k})$-dominant chamber of the form*

$$r_0\left(\lambda+\varrho_c-\sum_{1 \leqslant i \leqslant r} a_i \gamma_i-\sum_{1 \leqslant j \leqslant s} b_j \gamma_j'\right)-\varrho_c$$

*with* $a_i, b_j \in \mathbf{N}$, $a_1 \geqslant \ldots \geqslant a_r$, $b_1 \geqslant \ldots \geqslant b_s$. *Also, none of these representations has a highest weight.*

*Proof.* Jantzen's criteria [13] (cf. [7]) imply that $N(\lambda)$ is irreducible. So Theorem 7.3 asserts that $\Gamma^s N(\lambda)$ is either zero or unitarizable. Then Proposition 7.4 and Corollary 5.7 give the $\mathfrak{k}$-multiplicity formula. In turn this formula implies $\Gamma^s L(\lambda)$ is not zero since all weights of $L(\lambda)$ are $\Delta(\mathfrak{k})$-integral.

Using the $\varepsilon_i$ coordinates, the center of $\mathfrak{m}$ is spanned by $\zeta=(1,1,\ldots,1)$ while the center of $\mathfrak{k}$ is spanned by $\zeta'=(a_1,\ldots,a_n)$ with $a_i=\pm 1$. Since $\mathfrak{m} \neq \mathfrak{k}$, $\zeta$ and $\zeta'$ are linearly independent. So the $a_i$ are not all of the same sign. A $(\mathfrak{g},\mathfrak{k})$-module is a highest weight module precisely when the set of inner products of all $\mathfrak{k}$-highest weights with $\zeta'$ is bounded either above or below. The strongly orthogonal roots in our case are all long so $\{\gamma_i\} \cup \{\gamma'_j\} = \{2\varepsilon_i \mid 1 \leqslant i \leqslant n\}$. The $\gamma_i$ are strongly orthogonal roots for $sp(r,\mathbf{R})$ with the compactly embedded subalgebra corresponds to a factor of $\mathfrak{m} \cap \mathfrak{k}$. All the $\gamma_i$ are linear combinations of $\gamma_1$ and elements of $\Delta(\mathfrak{m} \cap \mathfrak{k})$. Now $\zeta'$ is orthogonal to $\Delta(\mathfrak{k})$, so $\langle \zeta', \gamma_i \rangle$ all have the same sign, $1 \leqslant i \leqslant r$. Likewise $\langle \zeta', \gamma'_j \rangle$ all have the same sign, $1 \leqslant j \leqslant s$. If these signs are equal then $a_i$, $1 \leqslant i \leqslant n$, are all $+1$ or all $-1$. This contradiction implies that $\langle \zeta', \gamma_1 \rangle$ and $\langle \zeta', \gamma'_1 \rangle$ have different signs. So, from our formula for $\mathfrak{k}$-highest weights and the fact that $r_0 \zeta' = \zeta'$, the set of inner products $\zeta'$ with a $\mathfrak{k}$-highest weight is unbounded above and below. This proves $\Gamma^s L(\lambda)$ is not a highest weight representation.

## § 11. The Speh representations for $SL(2n,\mathbf{R})$ and their analogues for $SU^*(2n)$

First we describe an alternate proof of unitarity for the Speh representations [24]. Following this we describe the analogous set of unitary representations for $SU^*(2n)$. In both cases, we shall be applying the results of section five and Theorem 7.3 with $\mathfrak{g}_0 \simeq sl(2n,\mathbf{R})$ or $su^*(2n)$ and $\mathfrak{g}_1 \simeq su(n,n)$.

Consider the Dynkin diagram for $A_{2n-1}$:

o—o— ... —o— ... —o—o

$\alpha_1$ $\alpha_n$ $\alpha_{2n-1}$

with $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $1 \leqslant i < 2n$. Let $\mathfrak{g}_0 = sl(2n,\mathbf{R})$ and $\mathfrak{h}_0$ be the fundamental CSA with positive root system $\Delta^+$ having simple roots $\alpha_i$, $1 \leqslant i < 2n$. Assume $\Delta^+$ is $\theta$-stable. Then $\theta$ flips the diagram; i.e., $\theta\varepsilon_i = -\varepsilon_{2n-i}$. In this case, $\alpha_n$ is a noncompact root and all other $\alpha_i$ are complex roots. Let $\mathfrak{q}$ be the $\theta$-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{u}$ with

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$\Delta^+(\mathfrak{m})$ having simple roots $\alpha_i$, $1\leqslant i<2n$ and $i\neq n$. For $1\leqslant i\leqslant n$, define $2\delta_i=\alpha_i+\alpha_{i+1}+\ldots+\alpha_{2n-i}$ restricted to $\mathfrak{t}$. Then the $\delta_i$ span $\mathfrak{t}^*$ and

$$\begin{aligned}\Delta^+(\mathfrak{k}) &= \{\delta_i\pm\delta_j \mid 1\leqslant i<j\leqslant n\},\\ \Delta(\mathfrak{u}) &= \{\varepsilon_i-\varepsilon_j \mid 1\leqslant i\leqslant n<j\leqslant 2n\},\\ \Delta(\mathfrak{u}_c) &= \{\delta_i+\delta_j \mid 1\leqslant i<j\leqslant n\},\\ \Delta(\mathfrak{u}_n) &= \{\delta_i+\delta_j \mid 1\leqslant i\leqslant j\leqslant n\}.\end{aligned} \tag{11.1}$$

Let $\lambda=\frac{1}{2}z(1,\ldots,1,-1,\ldots,-1)$ with both 1 and $-1$ occurring $n$ times. Then $\lambda|_{\mathfrak{t}}=z(\delta_1+\ldots+\delta_n)$. Let $G$ and $G_1$ be the simply connected covering groups of $SL(2n,\mathbf{R})$ and $SU(n,n)$. We know [7] that $N(\lambda)$ is irreducible and unitary for $G'$ if and only if $z<-n+1$. Put $X(\lambda)=\Gamma^s N(\lambda)$.

PROPOSITION 11.2. *For any half integer $z<-n+1$, $X(\lambda)$ is a unitarizable representation of $G$. Moreover, $X(\lambda)$ is multiplicity free as a $\mathfrak{k}$-module and the highest weights are:*

$$(-z-n+1)(\delta_1+\ldots+\delta_{n-1}\pm\delta_n)+2a_1\delta_1+\ldots\pm 2a_n\delta_n,\quad a_i\in\mathbf{N},\ a_1\geqslant a_2\geqslant\ldots\geqslant a_n,$$

*and $+$ (resp. $-$) is taken if $n$ is even (resp. odd).*

*Proof.* This follows from Theorem 7.3, Corollary 5.7 and Proposition 7.4. The strongly orthogonal roots used in Corollary 5.7 are $\gamma_i=2\delta_i$, $1\leqslant i\leqslant n$.

In the case above, $\lambda+\varrho$ lies in $\mathscr{C}$ for $z\leqslant -2n+1$.

Speh has constructed a family of unitary representations [24] denoted $I(k)$, $k\in\mathbf{N}$, which are the Zuckerman derived functor modules for the maximal parabolic subalgebra above. Comparing $\mathfrak{k}$-types and indices in the two cases we obtain:

PROPOSITION 11.3. *For $k\in\mathbf{N}$ and $z=-k-n$, $X(\lambda)$ and $I(k)$ are isomorphic $\mathfrak{g}$-modules.*

We note there is an easy proof that $X(\lambda)$ is irreducible. To prove this we observe that if $\mu\in\mathfrak{t}^*$ is a $\Delta(\mathfrak{k})^+$ highest weight in $N(\lambda)$ then $\mu+\varrho_c\in\mathscr{C}_c$. Moreover, if $\alpha\in\Delta$ then $\mu+\varrho_c+\alpha|_{\mathfrak{t}}\in\mathscr{C}_c$. From [5] it follows that if $A$ is a $\mathfrak{k}$-submodule of $N(\lambda)$, $\Gamma^s A$ is a $\mathfrak{g}$-submodule of $X(\lambda)$ if and only if $A$ is a $\mathfrak{g}$-submodule of $N(\lambda)$. So $X(\lambda)$ is irreducible.

PROPOSITION 11.4. *Let $z=-n+1$ and $L=L(\lambda)$. Then $\Gamma^s L$ is a unitarizable representation of $SL(2n,\mathbf{R})$. Moreover, it is multiplicity free as a $\mathfrak{k}$-module with highest weights*

$$2a_1\delta_1+\ldots+2a_{n-1}\delta_{n-1},\quad a_i\in\mathbf{N},\ a_1\geqslant\ldots\geqslant a_{n-1}.$$

*Proof.* Since $\lambda|_{\mathfrak{t}}+\varrho_c\in\mathscr{C}_c$, $L(\lambda)$ is completely reducible. By [29], $L(\lambda)=N(\lambda)/N(\lambda')$ where $\lambda'$ corresponds to $\lambda$ with $z=-n-1$. $N(\lambda)$ and $N(\lambda')$ are completely reducible as $\mathfrak{k}$-modules with decompositions given by Corollary 5.7. Now applying Theorem 7.3 and Proposition 7.4, we obtain Proposition 11.4.

Now we consider the similar series for $SU^*(2n)$. Let notation be as above except now $\alpha_n$ is compact. So all long roots are compact and we have

$$\begin{aligned}\Delta^+(\mathfrak{k})&=\{\delta_i\pm\delta_j\,|\,1\leqslant i<j\leqslant n\}\cup\{2\delta_i\,|\,1\leqslant i\leqslant n\},\\ \Delta(\mathfrak{u}_c)&=\{\delta_i+\delta_j\,|\,1\leqslant i\leqslant j\leqslant n\},\\ \Delta(\mathfrak{u}_n)&=\{\delta_i+\delta_j\,|\,1\leqslant i<j\leqslant n\}.\end{aligned}\tag{11.5}$$

Recall $\mathscr{L}=\mathfrak{u}_n^-\oplus\mathfrak{m}_c\oplus\mathfrak{u}_n$. Then $(\mathscr{L},\mathfrak{m}_c)$ corresponds to $so^*(2n)$ which has split rank $l=[n/2]$. The strongly orthogonal roots are $\gamma_i$, $1\leqslant i\leqslant l$, with $\gamma_i=\delta_{n-2i+1}+\delta_{n-2i+2}$. Exactly as in Proposition 11.2 we prove:

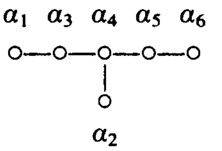
PROPOSITION 11.6. *Let $z$ be an integer $<-n+1$, $X(\lambda)=\Gamma^s N(\lambda)$. Then $X(\lambda)$ is a unitarizable representation for $SU^*(2n)$. Moreover, it is multiplicity free as a $\mathfrak{k}$-module with highest weights all $\Delta(\mathfrak{k})^+$-dominant elements of the form:*

$$(-z-n-1)(\delta_1+\ldots+\delta_n)+a_1(\delta_1+\delta_2)+a_2(\delta_3+\delta_4)+\ldots+a_l(\delta_{2l-1}+\delta_{2l}),\quad a_i\in\mathbf{N},\ a_1\geqslant\ldots\geqslant a_l.$$

## § 12. An application to unitary highest weight modules

In the classification of unitary highest weight modules [7] one case was handled by a long and difficult calculation. This occurred in the exceptional group $E_6(-14)$. Here we give an alternate proof of the unitarity of these representations.

Following Bourbaki notation consider the Dynkin diagram:



with $\alpha_2=\varepsilon_1+\varepsilon_2$, $\alpha_i=\varepsilon_{i-1}-\varepsilon_{i-2}$ $(3\leq i\leq 6)$ and $\alpha_1=\frac{1}{2}(\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4-\varepsilon_5-\varepsilon_6-\varepsilon_7+\varepsilon_8)$. Let $\alpha_1$ be noncompact and $\alpha_i$ be compact, $2\leq i\leq 6$. Let $\mathfrak{q}=\mathfrak{m}\oplus\mathfrak{u}$ be the $\theta$-stable parabolic subalgebra with $\Delta^+(\mathfrak{m})$ having simple roots $\alpha_i$, $1\leq i\leq 5$. Now the pair $(\mathfrak{m},\mathfrak{m}_c)$ corresponds to $\mathfrak{u}(1)\times so(2,8)$; and so $\dim\mathfrak{u}_c=8$. Since $\dim\mathfrak{u}=16$, $\dim\mathfrak{u}_n=8$. As in section five, let $\mathscr{L}=\mathfrak{u}_n^-\oplus\mathfrak{m}_c\oplus\mathfrak{u}_n$. Then $(\mathscr{L},\mathfrak{m}_c)$ is a Hermitian symmetric pair and since the center of $\mathfrak{m}_c$ has dimension two, $\mathscr{L}$ has at most two simple ideals not contained in $\mathfrak{k}$. Since $[\mathfrak{m}_c,\mathfrak{m}_c]\simeq SO(8)$, $(\mathscr{L},\mathfrak{m}_c)$ corresponds to $\mathfrak{u}(1)\times so(2,8)$. The split rank of $so(2,8)$ is two, and so we let $\gamma_1$ and $\gamma_2$ be the strongly orthogonal roots in $\Delta(\mathfrak{u}_n)$ as in Corollary 5.7. A quick check of the roots gives in $\varepsilon_i$ coordinates:

$$\begin{aligned}\gamma_1&=\tfrac{1}{2}(-1,-1,-1,-1,+1,-1,-1,+1)\\ \gamma_2&=\tfrac{1}{2}(+1,+1,+1,+1,+1,-1,-1,+1).\end{aligned}\tag{12.1}$$

Now put $\lambda=(0,0,0,0,z,-z/3,-z/3,z/3)$, $z\in\mathbf{R}$. With $r_0$ as in Proposition 6.6, $\varrho_c=(0,1,2,3,4,0,0,0)$ and $r_0(\varepsilon_i)=\mp\varepsilon_i$ depending as $i=1$ or 5 or not. So, in particular,

$$r_0(\lambda+\varrho_c)-\varrho_c=(0,0,0,0,-z-8,-z/3,-z/3,z/3).\tag{12.2}$$

From [7], $N(\lambda)$ is irreducible for $z<-3$; and so Theorem 7.3, Proposition 6.6 and Corollary 5.7 combine to give:

PROPOSITION 12.3. *For any integer* $z\leq-4$, $X(\lambda)=\Gamma^s N(\lambda)$ *is unitarizable. Moreover,* $X(\lambda)$ *is multiplicity free as a* $\mathfrak{k}$*-module with highest weights all* $\Delta^+(\mathfrak{k})$*-dominant elements of the form:*

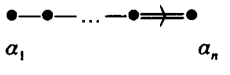
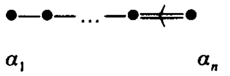
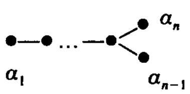
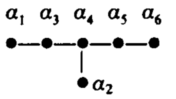
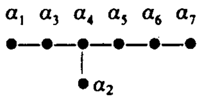
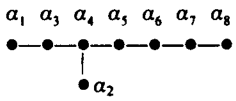
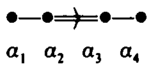
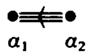
$$r_0(\lambda-n_1\gamma_1-n_2\gamma_2+\varrho_c)-\varrho_c\quad\textit{with } n_i\in\mathbf{N},\ n_1\geq n_2.$$

*Remark* 12.4. Let $\zeta=(0,0,0,0,0,-1,-1,1)$. Then $\zeta$ is orthogonal to elements in $\Delta(\mathfrak{k})$ and $r_0\zeta=\zeta$. Now $\langle\gamma_i,\zeta\rangle>0$ for both $i=1$ and $i=2$. So, $X(\lambda)$ is a highest weight module. By (12.2), for $z$ an integer $<-8$ the highest weight of $X(\lambda)$ takes the form $(0,0,0,0,a,-b,-b,b)$ with $a+3b=-8$. These representations $X(\lambda)$ are precisely those considered in Proposition 12.5 and Corollary 12.6 in [7] which was the especially difficult case in that article.

## § 13. Coherent continuation of Borel de Siebenthal discrete series representations

In [29], Wallach described the analytic continuation of the holomorphic discrete series representations having a one-dimensional cyclic $\mathfrak{k}$-module. In this section we apply

Table 13.2. First reduction point

| Root system | Diagram | Complementary simple root $\alpha_l$ | First reduction point $z=a$ |
|---|---|---|---|
| $B_n$ |  | $2\leqslant l\leqslant n-1$ | $n-l/2-1/2$ if $l$ is odd and $n-l\leqslant(l-1)/2$<br>$n-l/2$ otherwise |
| | | $l=n$ | $2\left[\frac{n}{2}\right]+1$ |
| $C_n$ |  | $l=1$ | $2n-1$ |
| | | $2\leqslant l\leqslant n-1$ | $n-l/2$ if $l$ is even and $n-l\geqslant l/2$<br>$n-l/2+1/2$ otherwise |
| $D_n$ |  | $2\leqslant l\leqslant n-2$ | $n-l/2-1$ if $l$ is even and $n-l\leqslant l/2$<br>$n-l/2-1/2$ otherwise |
| $E_6$ |  | 3 or 5 | 5 |
| | | 2 | 11/2 |
| $E_7$ |  | 1 | 17/2 |
| | | 2 | 7 |
| | | 6 | 6 |
| $E_8$ |  | 1 | 23/2 |
| | | 8 | 29/2 |
| $F_4$ |  | 1 | 4 |
| | | 4 | 5 |
| $G_2$ |  | 2 | 4/3 |

Theorem 7.2 to prove analogous results for certain discrete series representations of $\mathfrak{g}$ with $(\mathfrak{g},\mathfrak{k})$ not Hermitian symmetric.

Let notation be as in earlier sections. Let $\alpha_1,\ldots,\alpha_n$ be the simple roots of $\Delta^+$. Assume $\alpha_i$ are all compact except $\alpha=\alpha_l$ which is noncompact. Let $\Delta^+(\mathfrak{m})$ have simple roots $\alpha_i$, all $i\neq l$, and *assume* the coefficient of $\alpha$ in the expansion of the maximal root as a sum of simple roots is two. Then $\mathfrak{q}=\mathfrak{m}\oplus\mathfrak{u}$ is a maximal $\theta$-stable parabolic subalgebra with $\mathfrak{u}=\mathfrak{u}_c\oplus\mathfrak{u}_n$. $\Delta(\mathfrak{u}_n)$ (resp. $\Delta(\mathfrak{u}_c)$) is the set of roots $\beta$ whose coefficient of $\alpha$ in the

Table 13.3. $\mathfrak{k}$-Integrality

| Root system | Condition for $\Delta(\mathfrak{k})$-integrality | Number of unitary $\Gamma^s N(\lambda)$, $0 \leq z < a$ | Unitarity of $\Gamma^s L(\lambda)$ at $z = a$ |
|---|---|---|---|
| $B_n$ | $2z \in \mathbf{Z}$ | $2n-l-1$ | In both cases, yes if $a \leq 2n-2l$. |
| | | $2n-l$ | ? otherwise |
| | $z \in \mathbf{Z}$ | $2\left[\frac{n}{2}\right]+1$ | ? |
| $C_n$ | $z \in \mathbf{Z}$ | $2n-1$ | $\Gamma^s L(a)=0$ |
| | $z \in \mathbf{Z}$ | $n-l/2$ | If $a \notin \mathbf{Z}$ then $\Gamma^s L(a)=0$. |
| | | $n-[(l-1)/2]$ | $\Gamma^s L(a)$ is unitary if $a \in \mathbf{Z}$ and $a<2n-2l$. |
| | | | ? otherwise |
| $D_n$ | $2z \in \mathbf{Z}$ | $2n-2-l$ | In both cases, yes if $a \leq 2n-2l-1$. |
| | | $2n-1-l$ | ? otherwise |
| $E_6$ | $2z \in \mathbf{Z}$ | 10 | Yes |
| | | 11 | |
| $E_7$ | $2z \in \mathbf{Z}$ | 17 | Yes |
| | | 14 | |
| | | 12 | |
| $E_8$ | $2z \in \mathbf{Z}$ | 23 | Yes |
| | | 29 | |
| $F_4$ | $2z \in \mathbf{Z}$ | 8 | Yes |
| | $z \in \mathbf{Z}$ | 5 | ? |
| $G_2$ | $2z \in \mathbf{Z}$ | 3 | $\Gamma^s L(a)=0$ |

expansion of $\beta$ as a sum of simple roots is one (resp. two). In this case, weight vectors in $[\mathfrak{u}, \mathfrak{u}_c]$ would have a weight with coefficient of $\alpha$ greater than two. So $[\mathfrak{u}, \mathfrak{u}_c]=0$ and Theorem 7.2 applies in this setting.

Let $\zeta \in \mathfrak{h}^*$ be orthogonal to $\Delta(\mathfrak{m})$ and normalized by $2\langle \alpha, \zeta\rangle / \langle \alpha, \alpha\rangle = 1$. Consider the line $z\zeta$, $z \in \mathbf{R}$, and let $\lambda_0$ be the unique point on this line such that $\lambda_0+\varrho$ lies on a wall of $\mathscr{C}$. Let $\lambda=\lambda_0+z\zeta$. We consider the modules $N(\lambda)$ for various values of $z$. Let $a$ be the smallest value of $z$ with $N(\lambda)$ reducible. We call this the *first reduction point*.

PROPOSITION 13.1. *For $z<a$, $N(\lambda)$ is completely reducible as a $\mathfrak{k}$-module and if $\lambda$ is $\Delta(\mathfrak{k})$-integral, $X(\lambda)=\Gamma^s N(\lambda)$ is a unitary representation of $G$. Moreover, for $z<0$ and $\lambda$ $\Delta(\mathfrak{k})$-integral, $X(\lambda)$ is a discrete series representation of $G$.*

*Proof.* A short calculation shows that $X(\lambda)\neq 0$ if $\lambda$ is $\Delta(\mathfrak{k})$-integral. Then the first part follows from Theorem 7.2. Let $r_0$ be the Weyl group element in Proposition 6.6. For $z<<0$, $X(\lambda)$ contains the $\mathfrak{k}$-module $L_c(r_0(\lambda+\varrho_c)-\varrho_c)$ but not $L_c((r_0(\lambda+\alpha+\varrho_c)-\varrho_c)$ for any noncompact positive root $\alpha$. By Schmid's result [22], $X(\lambda)$ is infinitesimally equivalent to a discrete series representation. Now applying translation functors, we extend this to all $z<0$.

Jantzen [13] has computed the determinant of the canonical form on $N(\lambda)$. Using this product formula for the determinant one can compute the value $a$ above. See [9] for details of how this calculation is performed. Tables 13.2 and 13.3 describe the outcome.

The results above hold in a more general setting if we add an additional hypothesis. Let $\mathfrak{q}=\mathfrak{m}\oplus\mathfrak{u}$ be any maximal $\theta$-stable quasi abelian parabolic subalgebra whose complementary simple root has coefficient two in the maximal root (as above). Theorem 7.1 now implies:

PROPOSITION 13.4. *Assume $F(\zeta)$ is a unitarizable $\mathfrak{m}_0$-module and $N(\lambda)$ is completely reducible as a $\mathfrak{k}$-module for $z<a$. Then, for $z<a$, $\Gamma^s N(\lambda)$ is either zero or unitarizable.*

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