Hausdorff dimension and capacities of intersections of sets in n-space

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1. Introduction

Suppose A and B are Borel sets in the Euclidean n space $\mathbf{R}^n$. If they are sufficiently nice, for example C^1 submanifolds or rectifiable of dimensions k and m with $k+m\geq n$, then according to a well-known formula of integral-geometry

$$\int \mathcal{H}^{k+m-n}(A \cap fB)\, d\lambda_n f = c(k, m, n)\, \mathcal{H}^k(A)\, \mathcal{H}^m(B),$$

where $\mathcal{H}^s$ stands for the s dimensional Hausdorff measure and λ_n is an invariant measure on the group of isometries of $\mathbf{R}^n$. Thus in this case there is a precise relation between the measures of A and B and of those of the intersections $A \cap fB$. The object of this paper is to study to what extent there are such relations, necessarily less precise, if either A or both A and B are completely general except for measurability assumptions. Thus various Cantor type sets, graphs of nowhere differentiable functions etc. should be included in our theory. Particular examples are the self-similar fractals, which Mandelbrot [MB] has considered in connection of several physical phenomena and for which Hutchinson [H] has presented a unified theory.

First to consider this problem was Marstrand [MJ] who explored the geometric properties of fractional dimensional subsets of the plane $\mathbf{R}^2$. He proved that if $A\subset\mathbf{R}^2$ is $\mathcal{H}^s$ measurable with $0<\mathcal{H}^s(A)<\infty$, $1<s<2$, then for $\mathcal{H}^s$ almost all $x\in A$ $\dim A\cap l=s-1$ and $\mathcal{H}^{s-1}(A\cap l)<\infty$ for almost all lines l through x. He also showed by an example that $\mathcal{H}^{s-1}(A\cap l)$ may be zero for almost all lines l through any point of A. Marstrand's theorem was generalized to subsets of $\mathbf{R}^n$ with lines replaced by m planes in [MP1]. In [MP2] a potential-theoretic approach to this problem was presented. It was shown that by replacing the Hausdorff measure $\mathscr{H}^s$ by the capacity C_s corresponding to the kernel $|x-y|^{-s}$, one obtains more precise results and also a new proof for the above theorem. For example, if $n-m<s<n$, then for C_s almost all $x\in A$, $C_{s+m-n}(A\cap V)>0$ for almost all m planes V through x, and

$$bC_s(A)\leq \int C_{s+m-n}(A\cap V)\,d\lambda_{n,m}V\leq cC_s(A)$$

where b and c are positive constants independent of A and $\lambda_{n,m}$ is a rigidly invariant measure on the space of m planes in $\mathbf{R}^n$; the right hand inequality was proved in [MP3].

In Section 5 we prove analogous results when the intersections $A\cap V$ are replaced by $A\cap fB$, where f runs through the isometry group of $\mathbf{R}^n$ and B is an m rectifiable subset of $\mathbf{R}^n$, see Corollary 5.6, Theorem 5.8, Corollary 5.11 and Theorem 5.16. The examples of Section 7 indicate that these results are false if the assumption on the rectifiability of B is dropped. However in Section 6 we show that if we replace the isometry group by the similarity group, that is, maps composed of translations, rotations and homotheties, we get similar results without any rectifiability assumptions at all. To be more specific, let $\tau_z\colon\mathbf{R}^n\to\mathbf{R}^n$, $z\in\mathbf{R}^n$, be the translation, $\tau_z(x)=x+z$, $\delta_r\colon\mathbf{R}^n\to\mathbf{R}^n$, $r\in\mathbf{R}_+=\{t\colon 0<t<\infty\}$, the dilation, $\delta_r(x)=rx$, let $\mathscr{L}^n$ be the Lebesgue measure on $\mathbf{R}^n$ and θ_n the Haar measure on the orthogonal group $O(n)$ of $\mathbf{R}^n$. Suppose $0<s$, $t<n$ and $s+t>n$. We prove for example that (Theorem 6.8)

$$C_s(A)\,C_t(B)\leq c(n)\int_0^1 r^{t-1}\iint C_{s+t-n}(A\cap(\tau_z\circ g\circ\delta_r)B)\,d\mathscr{L}^n z\,d\theta_n g\,d\mathscr{L}^1 r,$$

and that if $0<\mathscr{H}^s(A)<\infty$ and $0<\mathscr{H}^t(B)<\infty$, then

$$\dim A\cap(\tau_x\circ g\circ\delta_r\circ\tau_{-y})B\geq s+t-n$$

for $\mathscr{H}^s\times\mathscr{H}^t\times\theta_n\times\mathscr{L}^1$ almost all $(x,y,g,r)\in A\times B\times O(n)\times\mathbf{R}_+$ (Corollary 6.12). The opposite inequality is in general false, see Example 7.2, but it holds if B has positive t dimensional lower density at all of its points (Theorem 6.13). Note that the effect of the map

$$\tau_x\circ g\circ\delta_r\circ\tau_{-y}\colon z\mapsto g(r(z-y))+x,$$

can be viewed as first applying rotation and homothety around y and then translating so that y goes to x. Here we allow the possibility $n=1$, in which case $O(1)$ can be neglected as it contains only $\mathrm{Id}_{\mathbf{R}^1}$ and $-\mathrm{Id}_{\mathbf{R}^1}$.

The basic method will be developed in Sections 3 and 4. There we first define natural intersection measures $\mu \cap \tau_{z\#} \nu$ of μ and $\tau_{z\#} \nu$ for $\mathcal{L}^n$ almost all $z \in \mathbf{R}^n$, where μ and ν are Radon measures on $\mathbf{R}^n$ and $\tau_{z\#} \nu : E \mapsto \nu(\tau_z^{-1} E)$. To do this we slice the product measure $\mu \times \nu$ by the n planes of $\mathbf{R}^n \times \mathbf{R}^n$ parallel to the diagonal; this idea comes from Federer's definition for intersections of currents, [F, 4.3.20]. We use the results of [MP2] on such slicing. Replacing ν by $g_{\#} \nu$ or $(g \circ \delta_r)_{\#} \nu$, $g \in O(n)$, $r \in \mathbf{R}_+$, we obtain the intersection measures $\mu \cap f_{\#} \nu$, where f is an isometry or similarity, respectively, and we study integral relations betwen these measures and μ and ν. For example, we give conditions on μ and ν which guarantee that the intersection measures $\mu \cap f_{\#} \nu$ form a kind of disintegration of μ and ν, that is, μ and ν can be recovered by integrating first with respect to $\mu \cap f_{\#} \nu$ and then over f's in the isometry or similarity group, see (4.4) combined with Theorem 4.8 and (6.4) combined with Theorem 6.6. In Sections 5 and 6 the measures $\mu \cap f_{\#} \nu$ will be used as test measures for the capacities of the sets $A \cap fB$.

The results concerning capacities in [MP2] were proved for general lower semi-continuous kernels $K(x, y)$ and $H(x, y) = |x-y|^{n-m} K(x, y)$ in place of $|x-y|^{-s}$ and $|x-y|^{n-s-m}$. Most of the results of Section 5 admit such a generalization in a straightforward manner (the possible exceptions being Theorem 5.8 and Corollary 5.9), whereas difficulties seem to arise in the case of Section 6. Another possible direction of generalization would be to replace the isometry and similarity groups by other sub-groups of the affine group of $\mathbf{R}^n$. Our method relies on the fact that we are using groups which are composed of a subgroup of the linear group with the translation group.

In [H 1] J. Hawkes has studied intersections from a different point of view. He has given conditions in terms of certain entropy concepts for two fixed sets A and B which lead to relations between the entropy and Hausdorff dimensions of A and B and their intersection $A \cap B$.

2. Measures and capacities on $\mathbf{R}^n$

2.1. *Measures*. The Lebesgue measure on $\mathbf{R}^n$ is denoted by $\mathcal{L}^n$, and the s dimensional Hausdorff measure by $\mathcal{H}^s$. In particular, $\mathcal{H}^n = \mathcal{L}^n$. We let $\alpha(n) = \mathcal{L}^n B(0, 1)$, where $B(x, r)$ stands for the closed ball in $\mathbf{R}^n$ with centre x and radius r. The Hausdorff dimension of $E \subset \mathbf{R}^n$ is

$$\dim E = \inf \{s : \mathcal{H}^s(E) = 0\} = \sup \{s : \mathcal{H}^s(E) = \infty\}.$$

Let μ be an (outer) measure on $\mathbf{R}^n$. If $f : \mathbf{R}^n \to \mathbf{R}^m$, the image measure $f_{\#} \mu$ is defined by

$$f_{\#} \mu(E) = \mu(f^{-1} E) \quad \text{for } E \subset \mathbf{R}^m.$$

If $g\colon \mathbf{R}^m \to \bar{\mathbf{R}} = \mathbf{R}^1 \cup \{\infty, -\infty\}$, then

$$\int g \circ f \, d\mu = \int g \, df_{\#}\, \mu,$$

provided the left hand side exists, [F, 2.4.18]. If φ is a μ measurable non-negative function on $\mathbf{R}^n$, the measure $\mu \llcorner \varphi$ is defined by

$$(\mu \llcorner \varphi)(E) = \int_E \varphi \, d\mu \quad \text{for } E \subset \mathbf{R}^n.$$

In case φ is the characteristic function of a μ measurable set A, we set $\mu \llcorner A = \mu \llcorner \varphi$. Then $(\mu \llcorner A)(E) = \mu(A \cap E)$.

Radon measure always means a non-negative (outer) Radon measure. $\mathcal{L}^n$ and $\mathcal{H}^s \llcorner A$, where A is $\mathcal{H}^s$ measurable with $\mathcal{H}^s(A) < \infty$, are Radon measures. If $f\colon \mathbf{R}^n \to \mathbf{R}^m$ is continuous, μ is a Radon measure on $\mathbf{R}^n$ and $f | \operatorname{supp} \mu$, the support of μ, is proper, then $f_{\#}\mu$ is a Radon measure on $\mathbf{R}^m$ [F, 2.2.17].

If μ is absolutely continuous with respect to a measure ν, that is $\nu E = 0$ implies $\mu E = 0$, we denote $\mu \ll \nu$. The following lemma, which follows from [F, 2.9.15], will be useful:

LEMMA 2.2. *Suppose that μ is a Borel regular measure on $\mathbf{R}^n$. If*

$$\liminf_{\delta \downarrow 0} \delta^{-n} \mu B(x, \delta) < \infty \text{ for } \mu \text{ almost all } x \in \mathbf{R}^n,$$

then $\mu \ll \mathcal{L}^n$.

2.3. *Capacities*. Let $0 < s < n$. The s-energy of a Radon measure μ on $\mathbf{R}^n$ is

$$I_s(\mu) = \int \int |x-y|^{-s} \, d\mu x \, d\mu y.$$

The inner s-capacity of a compact set $F \subset \mathbf{R}^n$ is

$$C_s(F) = \sup I_s(\mu)^{-1},$$

where the supremum is taken over all Radon measures μ with $\operatorname{supp} \mu \subset F$ and $\mu \mathbf{R}^n = 1$. For arbitrary $E \subset \mathbf{R}^n$ we set

$$C_s(E) = \sup \{ C_s(F) \colon F \text{ compact} \subset E \}.$$

The corresponding outer capacity C_s^* is defined by

$$C_s^*(E) = \inf\{C_s(G): E \subset G, G \text{ open}\}.$$

Then C_s is an outer measure on $\mathbf{R}^n$ and $C_s^*(E)=C_s(E)$ for Suslin, and hence Borel, sets $E\subset\mathbf{R}^n$ [L, Theorem 2.8, p. 156].

The following relations between capacities and Hausdorff measures are well-known, see [L, pp. 196 and 200]:

$$C_s^*(E) = 0 \text{ implies } \mathcal{H}^t(E) = 0 \quad \text{for } t > s,$$

$$\mathcal{H}^s(E) < \infty \text{ implies } C_s(E) = 0.$$

Thus for Suslin sets E,

$$\dim E = \inf\{s: C_s(E) = 0\} = \sup\{s: C_s(E) > 0\}.$$

For the proof of the following lemma see e.g. [MP1, 6.3].

LEMMA 2.4. *If* $E\subset\mathbf{R}^n$, $\mathcal{H}^s(E)<\infty$ *and* $0<t<s$, *then* $\int_E |x-y|^{-t}\,d\mathcal{H}^s y<\infty$ *for* $\mathcal{H}^s$ *almost all* $x\in E$.

2.5. *Rectifiable sets.* A subset E of $\mathbf{R}^n$ is called m rectifiable if there are a bounded set $A\subset\mathbf{R}^m$ and a Lipschitzian map $f: A\to\mathbf{R}^n$ with $E=fA$. E is called $(\mathcal{H}^m, m)$ rectifiable if $\mathcal{H}^m(E)<\infty$ and there are m rectifiable subsets $E_1, E_2, \dots$ of E such that $\mathcal{H}^m(E\sim\bigcup_{i=1}^\infty E_i)=0$. By [F, 3.2.29] a set $E\subset\mathbf{R}^n$ with $\mathcal{H}^m(E)<\infty$ is $(\mathcal{H}^m, m)$ rectifiable if and only if $\mathcal{H}^m$ almost all of E can be covered with countably many m dimensional C^1 submanifolds of $\mathbf{R}^n$.

3. Intersections of measures

Let μ and ν be Radon measures on $\mathbf{R}^n$. We shall construct for $\mathcal{L}^n$ almost all $z\in\mathbf{R}^n$ Radon measures $\mu\cap\tau_{z\#}\nu$, which can be regarded as natural intersections of the measures μ and $\tau_{z\#}\nu$. Here τ_z is the translation

$$\tau_z: \mathbf{R}^n\to\mathbf{R}^n, \quad \tau_z(x) = x+z.$$

We denote for $z\in\mathbf{R}^n$, $\delta\in\mathbf{R}_+$

$$W = \{(x,y)\in\mathbf{R}^n\times\mathbf{R}^n: x=y\},$$

$$W_z = \{(x,y)\in\mathbf{R}^n\times\mathbf{R}^n: x=y+z\},$$

$$W_z(\delta) = \{(x,y)\in\mathbf{R}^n\times\mathbf{R}^n\colon |x-y-z|\le\delta\}$$
$$= \{(x,y)\in\mathbf{R}^n\times\mathbf{R}^n\colon \operatorname{dist}((x,y),\ W_z)\le\ \delta/\sqrt{2}\}.$$

Let S be the subtraction map,

$$S\colon \mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^n,\quad S(x,y)=x-y.$$

Then the orthogonal projection of $\mathbf{R}^n\times\mathbf{R}^n$ onto the orthogonal complement $W^\perp=\{(x,y)\colon x=-y\}$ of W is $h\circ S$ where $h\colon\mathbf{R}^n\to W^\perp$, $h(z)=(z,\ -z)/2$.

We use [MP2, 3.1–4] to slice the product measure $\mu\times\nu$ by the planes W_z, $z\in\mathbf{R}^n$. In [MP2] only Radon measures with compact support were considered, but with minor modifications this restriction can be removed. Note that $C^+(\mathbf{R}^n)$, which was erroneously claimed to be separable, should be replaced by $C_0^+(\mathbf{R}^n)$, the space of non-negative continuous functions on $\mathbf{R}^n$ with compact support. We obtain for $\mathcal{L}^n$ almost all $z\in\mathbf{R}^n$ Radon measures σ_z on $\mathbf{R}^n\times\mathbf{R}^n$ such that for any non-negative Borel function φ on $\mathbf{R}^n\times\mathbf{R}^n$ with $\int\varphi\, d\mu\times\nu<\infty$ the following four statements hold:

$$\operatorname{supp}\sigma_z\subset W_z\cap\operatorname{supp}\mu\times\nu. \tag{3.1}$$

$$\int\varphi\, d\sigma_z=\lim_{\delta\downarrow 0}\alpha(n)^{-1}\delta^{-n}\int_{W_z(\delta)}\varphi\, d\mu\times\nu \tag{3.2}$$

for $\mathcal{L}^n$ almost all $z\in\mathbf{R}^n$. Moreover, if $\varphi\in C_0^+(\mathbf{R}^n\times\mathbf{R}^n)$ the $\mathcal{L}^n$ exceptional set is independent of φ.

$$\text{The function } z\mapsto\int\varphi\, d\sigma_z \text{ is } \mathcal{L}^n \text{ measurable.} \tag{3.3}$$

$$\int_E\int\varphi\, d\sigma_z\, d\mathcal{L}^n z\le\int_{S^{-1}E}\varphi\, d\mu\times\nu \tag{3.4}$$

for any $\mathcal{L}^n$ measurable set $E\subset\mathbf{R}^n$. Here equality holds if $S_\#(\mu\times\nu)\ll\mathcal{L}^n$.

In [MP2] (3.4) was stated in the case $E=\mathbf{R}^n$, but the general case follows by the same proof.

Let $p\colon\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^n$, $p(x,y)=x$, be the projection. Whenever σ_z is defined, we define

$$\mu\cap\tau_{z\#}\nu=p_\#\sigma_z.$$

Then for any non-negative Borel functions φ and ψ on $\mathbf{R}^n$ with $\int\varphi\, d\mu<\infty$ and $\int\psi\, d\nu<\infty$ the following four statements hold:

$$\operatorname{supp}\mu \cap \tau_{z\#}\nu \subset \operatorname{supp}\mu \cap \tau_z(\operatorname{supp}\nu). \tag{3.5}$$

$$\int \varphi(x)\,\psi(x-z)\,d\mu \cap \tau_{z\#}\,\nu x = \int \varphi(x)\,\psi(y)\,d\sigma_z(x,y)$$

$$= \lim_{\delta\downarrow 0} \alpha(n)^{-1}\,\delta^{-n} \int_{W_z(\delta)} \varphi(x)\,\psi(y)\,d\mu\times\nu(x,y) \tag{3.6}$$

for $\mathscr{L}^n$ almost all $z\in\mathbf{R}^n$. Moreover, if $\varphi,\psi\in C_0^+(\mathbf{R}^n)$, the $\mathscr{L}^n$ exceptional set is independent of φ and ψ.

$$\text{The function } z\mapsto \int \varphi\cdot(\psi\circ\tau_{-z})\,d\mu \cap \tau_{z\#}\,\nu \text{ is } \mathscr{L}^n \text{ measurable.} \tag{3.7}$$

$$\int_E\int \varphi(\psi\circ\tau_{-z})\,d\mu \cap \tau_{z\#}\,\nu\, d\mathscr{L}^n z \leq \int_{S^{-1}E} \varphi(x)\,\psi(y)\,d\mu\times\nu(x,y) \tag{3.8}$$

for any $\mathscr{L}^n$ measurable set $E\subset\mathbf{R}^n$. Here equality holds if $S_\#(\mu\times\nu)\ll\mathscr{L}^n$.

Proofs. We get (3.5) from (3.1) since

$$\begin{aligned}\operatorname{supp}\mu \cap \tau_{z\#}\nu &\subset p(\operatorname{supp}\sigma_z) \subset p(W_z \cap \operatorname{supp}\mu\times\nu)\\ &\subset p(\{(x,y): x=y+z\} \cap (\operatorname{supp}\mu\times \operatorname{supp}\nu)) \subset \operatorname{supp}\mu \cap \tau_z(\operatorname{supp}\nu).\end{aligned}$$

To prove (3.6) we use (3.1) and (3.2) to get

$$\int \varphi(x)\,\psi(x-z)\,d\mu \cap \tau_{z\#}\,\nu x = \int \varphi(x)\,\psi(x-z)\,dp_\#\,\sigma_z x = \int_{W_z} \varphi(x)\,\psi(y)\,d\sigma_z(x,y)$$

$$= \lim_{\delta\downarrow 0} \alpha(n)^{-1}\,\delta^{-n} \int_{W_z(\delta)} \varphi(x)\,\psi(y)\,d\mu\times\nu(x,y).$$

The first formula of (3.6) combined with (3.3) yields (3.7) and combined with (3.4) it yields (3.8).

4. Intersections of measures and isometries

Let $O(n)$ be the orthogonal group of $\mathbf{R}^n$, and let θ_n be the unique invariant Radon measure on $O(n)$ with $\theta_n O(n)=1$. We denote by $I(n)$ the set of isometries of $\mathbf{R}^n$ equipped with the natural topology. Then each $f\in I(n)$ has a unique representation in the form

$$f=\tau_z\circ g \quad \text{with } z\in\mathbf{R}^n,\ g\in O(n).$$

There is an invariant Radon measure λ_n on $I(n)$ such that for any non-negative Borel function φ on $I(n)$

$$\int \varphi \, d\lambda_n = \iint \varphi(\tau_z \circ g) \, d\mathcal{L}^n z \, d\theta_n g.$$

For $z \in \mathbf{R}^n$, $g \in O(n)$, $f = \tau_z \circ g$ and $\delta \in \mathbf{R}_+$ we set

$$\begin{aligned} W_f = W_{g,z} &= \{(x,y): x = f(y)\} = \{(x,y): x = g(y) + z\}, \\ W_f(\delta) = W_{g,z}(\delta) &= \{(x,y): |x - f(y)| \leq \delta\}, \\ S_g: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n, &\quad S_g(x,y) = x - g(y). \end{aligned}$$

Let μ and ν be Radon measures on $\mathbf{R}^n$. For any $g \in O(n)$ we apply Section 3 to the measures μ and $g_\# \nu$ to find that

$$\mu \cap f_\# \nu = \mu \cap \tau_{z\#} g_\# \nu \quad \text{with } f = \tau_z \circ g$$

exists for $\mathcal{L}^n$ almost all $z \in \mathbf{R}^n$. Furthermore, one verifies as in [MP2, 3.3] that the set of all $f \in I(n)$ for which $\mu \cap f_\# \nu$ is defined is a Borel set. Hence by Fubini's theorem $\mu \cap f_\# \nu$ is defined for λ_n almost all $f \in I(n)$. It follows readily from (3.5)–(3.8) that for any non-negative Borel functions φ and ψ on $\mathbf{R}^n$ with $\int \varphi \, d\mu < \infty$ and $\int \psi \, d\nu < \infty$ the following four statements hold:

$$\operatorname{supp} \mu \cap f_\# \nu \subset \operatorname{supp} \mu \cap f(\operatorname{supp} \nu). \tag{4.1}$$

$$\int \varphi(x) \psi(f^{-1}(x)) \, d\mu \cap f_\# \nu x = \lim_{\delta \downarrow 0} \alpha(n)^{-1} \delta^{-n} \int_{W_f(\delta)} \varphi(x) \psi(y) \, d\mu \times \nu(x,y) \tag{4.2}$$

with $f = \tau_z \circ g$ for $\mathcal{L}^n$ almost all $z \in \mathbf{R}^n$. Moreover, if $\varphi, \psi \in C_0^+(\mathbf{R}^n)$, the $\mathcal{L}^n$ exceptional set is independent of φ and ψ.

$$\text{The function } f \mapsto \int \varphi \cdot (\psi \circ f^{-1}) \, d\mu \cap f_\# \nu \text{ is } \lambda_n \text{ measurable.} \tag{4.3}$$

$$\iint \varphi(\psi \circ f^{-1}) \, d\mu \cap f_\# \nu \, d\lambda_n f \leq \int \varphi \, d\mu \int \psi \, d\nu; \tag{4.4}$$

here equality holds if $S_{g\#}(\mu \times \nu) \ll \mathcal{L}^n$ for θ_n almost all $g \in O(n)$. Observe that $S_{g\#}(\mu \times \nu) = S_\#(\mu \times g_\# \nu)$.

LEMMA 4.5. *Given $g \in O(n)$ there is a set $E_g \subset \mathbf{R}^n$ such that $\mathcal{L}^n(\mathbf{R}^n \sim E_g) = 0$ and for all $z \in E_g$ and for every non-negative lower semicontinuous function $\varphi: \mathbf{R}^n \to \bar{\mathbf{R}}$*

$$\int \varphi \, d\mu \cap (\tau_z \circ g)_{\#} \nu \leq \liminf_{\delta \downarrow 0} \alpha(n)^{-1} \delta^{-n} \int_{W_{g,z}(\delta)} \varphi(x) \, d\mu \times \nu(x, y).$$

Proof. Approximating φ from below by an increasing sequence of continuous functions with compact support, one derives this immediately from (4.2) and the monotone convergence theorem.

LEMMA 4.6. *If $S_{g\#}(\mu \times \nu) \ll \mathscr{L}^n$ for θ_n almost all $g \in O(n)$, then $\mu \cap (\tau_x \circ g \circ \tau_{-y})_{\#} \nu$ is defined and $\mu \cap (\tau_x \circ g \circ \tau_{-y})_{\#} \nu(\mathbf{R}^n) > 0$ for $\mu \times \nu \times \theta_n$ almost all $(x, y, g) \in \mathbf{R}^n \times \mathbf{R}^n \times O(n)$.*

Proof. Note that

$$\tau_x \circ g \circ \tau_{-y} = \tau_{x-g(y)} \circ g = \tau_{S_g(x,y)} \circ g.$$

For any $g \in O(n)$, $\mu \cap (\tau_z \circ g)_{\#} \nu$ exists for $\mathscr{L}^n$ almost all $z \in \mathbf{R}^n$. If $S_{g\#}(\mu \times \nu) \ll \mathscr{L}^n$, this implies $\mu \cap (\tau_{S_g(x,y)} \circ g)_{\#} \nu$ exists for $\mu \times \nu$ almost all (x, y). As in [MP2, 3.3] one sees that the set of those $(x, y, g) \in \mathbf{R}^n \times \mathbf{R}^n \times O(n)$ for which $\mu \cap \tau_{S_g(x,y)\#} \nu$ exists is a Borel set. Hence the first statement follows from Fubini's theorem, and the second as in [MP2, 3.3].

LEMMA 4.7. *There is a constant c depending only on n such that for any $x, y \in \mathbf{R}^n$, $x \neq 0$, and $\delta \in \mathbf{R}_+$*

$$\theta_n\{g : |x - g(y)| \leq \delta\} \leq c \delta^{n-1} |x|^{1-n}.$$

Moreover,

$$\theta_n\{g : |x - g(y)| \leq \delta\} = 0 \quad \text{if } ||x| - |y|| > \delta.$$

Proof. If $||x|-|y||>\delta$, then $\{g: |x-g(y)| \leq \delta\} = \varnothing$. Suppose $||x|-|y|| \leq \delta$, $x \neq 0 \neq y$. Then $|x-g(y)| \leq \delta$ implies $|x/|x| - g(y/|y|)| \leq 2\delta/|x|$, because

$$|x - g(|x| \, y/|y|)| \leq |x - g(y)| + |(1 - |x|/|y|) \, g(y)| = |x - g(y)| + ||x| - |y|| \leq 2\delta.$$

Thus we may assume $x, y \in S^{n-1} = \{z \in \mathbf{R}^n : |z| = 1\}$. Defining $\Phi : O(n) \to S^{n-1}$ by $\Phi(g) = g(y)$, we have by [F, 3.2.47]

$$\theta_n\{g : |x - g(y)| \leq \delta\} = \theta_n(\Phi^{-1}\{z : |x - z| \leq \delta\}) = c_1 \, \mathscr{H}^{n-1}\{z \in S^{n-1} : |x - z| \leq \delta\} \leq c_2 \delta^{n-1},$$

where c_1 and c_2 depend only on n.

Next we derive conditions under which the absolute continuity assumption of (4.4) is fulfilled.

THEOREM 4.8. *Suppose that* $0<s\leq n-1$,

(1) $$\int |x-u|^{-s}\,d\mu u<\infty \quad \text{for } \mu \text{ almost all } x\in\mathbf{R}^n,$$

and that there is b, $0<b<\infty$, *such that*

(2) $$\nu\{v: ||y-v|-r|\leq\delta\}\leq b\delta r^{n-s-1}$$

for ν *almost all* $y\in\mathbf{R}^n$ *and for* $\delta, r\in\mathbf{R}_+$. *Then* $S_{g\#}(\mu\times\nu)\ll\mathcal{L}^n$ *for* θ_n *almost all* $g\in O(n)$.

Proof. By standard methods (cf. [MP2, 2.5]) one verifies that the function

$$\begin{aligned}(x,y,g)\mapsto &\liminf_{\delta\downarrow 0}\delta^{-n}S_{g\#}(\mu\times\nu)(B(S_g(x,y),\delta))\\ &=\liminf_{\delta\downarrow 0}\delta^{-n}\mu\times\nu\{(u,v): |x-u-g(y-v)|\leq\delta\}\end{aligned}$$

is a Borel function on $\mathbf{R}^n\times\mathbf{R}^n\times O(n)$. If x satisfies (1) and y (2) we have by Fatou's lemma, Fubini's theorem and Lemma 4.7

$$\begin{aligned}&\int \liminf_{\delta\downarrow 0}\delta^{-n}S_{g\#}(\mu\times\nu)(B(S_g(x,y),\delta))\,d\theta_n g\\ &\leq\liminf_{\delta\downarrow 0}\delta^{-n}\int \mu\times\nu\{(u,v): |x-u-g(y-v)|\leq\delta\}\,d\theta_n g\\ &=\liminf_{\delta\downarrow 0}\delta^{-n}\int \theta_n\{g: |x-u-g(y-v)|\leq\delta\}\,d\mu\times\nu(u,v)\\ &\leq c\liminf_{\delta\downarrow 0}\delta^{-1}\int_{\{(u,v): ||x-u|-|y-v||\leq\delta\}} |x-u|^{1-n}\,d\mu\times\nu(u,v)\\ &=c\liminf_{\delta\downarrow 0}\delta^{-1}\int \nu\{v: ||x-u|-|y-v||\leq\delta\}\,|x-u|^{1-n}\,d\mu u\\ &\leq bc\int |x-u|^{-s}\,d\mu u<\infty.\end{aligned}$$

It follows from Fubini's theorem that for θ_n almost all $g\in O(n)$

$$\liminf_{\delta \downarrow 0} \delta^{-n} S_{g\#}(\mu \times \nu)(B(S_g(x,y),\delta)) < \infty$$

for $\mu \times \nu$ almost all (x,y), and by the definition of $S_{g\#}(\mu\times\nu)$

$$\liminf_{\delta \downarrow 0} \delta^{-n} S_{g\#}(\mu\times\nu)(B(z,\delta)) < \infty$$

for $S_{g\#}(\mu\times\nu)$ almost all $z \in \mathbf{R}^n$. This combined with Lemma 2.2 completes the proof.

COROLLARY 4.9. *Suppose that m is an integer,* $0<m<n$, $\int |x-u|^{m-n}\, d\mu u<\infty$ *for* μ *almost all* $x\in\mathbf{R}^n$, *and that B is an* $(\mathcal{H}^m, m)$ *rectifiable* $\mathcal{H}^m$ *measurable subset of* $\mathbf{R}^n$. *Then* $S_{g\#}(\mu\times(\mathcal{H}^m \llcorner B)) \ll \mathcal{L}^n$ *for* θ_n *almost all* $g\in O(n)$.

Proof. We can express B as $B=\bigcup_{i=0}^{\infty} A_i$ where $\mathcal{H}^m(A_0)=0$ and A_i is an $\mathcal{H}^m$ measurable set contained in an m dimensional C^1 submanifold M_i of $\mathbf{R}^n$. Each point of M_i has a neighborhood U in M_i such that there is a bilipschitzian map $h\colon U\to V\subset\mathbf{R}^m$. Then $\nu=\mathcal{H}^m\llcorner U$ satisfies the condition (2) of Theorem 4.8 with $s=n-m$. It follows that B can be written as $B=\bigcup_{i=1}^{\infty} B_i$ where each B_i is an $\mathcal{H}^m$ measurable subset of R^n and $S_{g\#}(\mu\times(\mathcal{H}^m\llcorner B_i)) \ll \mathcal{L}^n$ for θ_n almost all $g\in O(n)$. Then also $S_{g\#}(\mu\times(\mathcal{H}^m\llcorner B)) \ll \mathcal{L}^n$ for θ_n almost all $g\in O(n)$.

COROLLARY 4.10. *Let m and B as in Corollary* 4.9. *Suppose that* $n-m\leq s<n$ *and that A is an* $\mathcal{H}^s$ *measurable subset of* $\mathbf{R}^n$ *with* $\mathcal{H}^s(A)<\infty$. *If either* $n-m<s$ *or* $n-m=s$ *and A is* $n-m$ *rectifiable, then* $S_{g\#}((\mathcal{H}^s\llcorner A)\times(\mathcal{H}^m\llcorner B)) \ll \mathcal{L}^n$ *for* θ_n *almost all* $g\in O(n)$.

Proof. If $n-m<s$, then by Lemma 2.4 $\mathcal{H}^s(A)<\infty$ implies $\int_A |x-u|^{m-n}\, d\mathcal{H}^s u<\infty$ for $\mathcal{H}^s$ almost all $x\in A$, and the result follows from Corollary 4.9. Suppose $s=n-m$. Then $(\mathcal{H}^{n-m}\llcorner A)\times(\mathcal{H}^m\llcorner B)=\mathcal{H}^n\llcorner(A\times B)$ and $A\times B$ is $(\mathcal{H}^n, n)$ rectifiable by [F, 3.2.23]. A slight modification of the argument given in the last paragraph of [MP2, 5.1] shows that $S_{g\#}(\mathcal{H}^n\llcorner(A\times B)) \ll \mathcal{L}^n$ for θ_n almost all $g\in O(n)$.

Remark 4.11. Suppose that k and m are positive integers, $n\leq k+m$, $A\subset\mathbf{R}^n$ is $\mathcal{H}^k$ measurable and k rectifiable, $B\subset\mathbf{R}^n$ is $\mathcal{H}^m$ measurable and $(\mathcal{H}^m, m)$ rectifiable. Using [F, 3.2.23] as in Corollary 4.10, the co-area formula [F, 3.2.22] and Lebesgue's theorem on differentiation of integrals [F, 2.9.8] as in [A, I.3(4)], one can compute the intersection measures $(\mathcal{H}^k\llcorner A)\cap f_\#(\mathcal{H}^m\llcorner B)$ for λ_n almost all $f\in I(n)$. One then finds

$$(\mathcal{H}^k \llcorner A)\cap f_\#(\mathcal{H}^m\llcorner B) = (\mathcal{H}^{k+m-n}\llcorner(A\cap fB))\llcorner\psi_f$$

where for $f=\tau_z \circ g$, $\psi_f : \mathbf{R}^n \to \bar{\mathbf{R}}$ is given by

$$\psi_f(x) = 2^{(k+m-n)/2}/\mathrm{ap}\, J_n(S_g|A\times B)(x, f^{-1}(x)).$$

For the definition of the approximate Jacobian $\mathrm{ap} J_n(S_g|A\times B)$ see [F, 3.2.22 and 3.2.16].

5. Capacities and Hausdorff measures of intersections of sets

In this section we study capacities and Hausdorff measures of the intersections $A\cap fB$, when $f\in I(n)$ and B is $(\mathscr{H}^m, m)$ rectifiable. Throughout this section we assume that μ and ν are Radon measures on $\mathbf{R}^n$ and $1\leq m<n$. Throughout 5.1–5.5 we assume that for some b, $0<b<\infty$,

$$\nu\{v: ||v-y|-r| \leq \delta\} \leq b\delta r^{m-1} \tag{5.1}$$

for ν almost all $y\in\mathbf{R}^n$ and for δ, $r\in\mathbf{R}_+$.

The following lemma can be verified as Lemma 4.2 in [MP2].

LEMMA 5.2. *Let* $0<s<\infty$. *The functions*

$$f\mapsto I_s(\mu \cap f_\# \nu), \quad f\in I(n),$$
$$(x,y,g)\mapsto I_s(\mu\cap(\tau_x\circ g\circ\tau_{-y})_\#\nu), \quad (x,y,g)\in\mathbf{R}^n\times\mathbf{R}^n\times O(n),$$
$$(x,f)\mapsto \int |x-u|^{-s}\, d\mu\cap f_\#\nu u, \quad (x,f)\in\mathbf{R}^n\times I(n),$$

are Borel functions.

Below we shall consider upper and lower integrals of the function $f\mapsto C_s(A\cap fB)$, $f\in I(n)$. In case A and B are compact, it is easily seen to be upper semicontinuous (cf. [MP2, 4.6]), whence it is a Borel function whenever A and B are σ-compact. We give the following general result on its measurability, which however will not be needed in this paper:

LEMMA 5.3. *If* $0<s<n$ *and A and B are Suslin subsets of* $\mathbf{R}^n$, *then the function* $f\mapsto C_s(A\cap fB)$ *is* λ_n *measurable.*

Proof. The set

$$E = A\times B\times I(n)\cap\{(x,y,f): x=f(y)\}$$

is a Suslin set in $\mathbf{R}^n \times \mathbf{R}^n \times I(n)$. Letting $E(f)=\{(x,y): (x,y,f)\in E\}$ we deduce from [D, III,12] that the function $f \mapsto C_s(E(f))$ is λ_n measurable. The map $\varphi_f: A\cap fB \to E(f)$, $\varphi_f(x)=(x,f^{-1}(x))$, is onto and $|\varphi_f(x)-\varphi_f(x')|=\sqrt{2}\,|x-x'|$ for all $x,x'\in A\cap fB$, whence $C_s(A\cap fB)=2^{-s/2}C_s(E(f))$, and the result follows.

THEOREM 5.4. *There is a constant c depending only on n such that for any s, $n-m<s<n$,*

$$\int I_{s+m-n}(\mu \cap f_\# \nu)\, d\lambda_n f \leq bcI_s(\mu)\, \nu(\mathbf{R}^n).$$

Proof. Set $q=s+m-n$. The various applications of Fubini's theorem can be justified by Lemma 5.2. By Lemma 4.5, Fatou's lemma and Fubini's theorem we have

$$\begin{aligned}
&\int I_q(\mu\cap f_\# \nu)\, d\lambda_n f\\
&= \iiint |x-u|^{-q}\, d\mu\cap f_\# \nu x\, d\mu\cap f_\# \nu u\, d\lambda_n f\\
&\leq \iiint \liminf_{\delta\downarrow 0} \alpha(n)^{-1}\delta^{-n}\int_{W_{g,z}(\delta)} |x-u|^{-q}\, d\mu\times\nu(x,y)\, d\mu\cap(\tau_z\circ g)_\# \nu u\, d\mathscr{L}^n z\, d\theta_n g\\
&\leq \liminf_{\delta\downarrow 0} \alpha(n)^{-1}\delta^{-n}\iiint\int_{W_{g,z}(\delta)} |x-u|^{-q}\, d\mu\times\nu(x,y)\, d\mu\cap(\tau_z\circ g)_\# \nu u\, d\mathscr{L}^n z\, d\theta_n g\\
&= \liminf_{\delta\downarrow 0} \alpha(n)^{-1}\delta^{-n}\iiint_{W_{g,z}(\delta)}\int |x-u|^{-q}\, d\mu\cap(\tau_z\circ g)_\# \nu u\, d\mu\times\nu(x,y)\, d\mathscr{L}^n z\, d\theta_n g.
\end{aligned}$$

Recalling that $W_{g,z}(\delta)=\{(x,y): |x-g(y)-z|\leq\delta\}$ we use Fubini's theorem and (3.8), with ν replaced by $g_\# \nu$, to get

$$\begin{aligned}
J(g) &= \iint_{W_{g,z}(\delta)}\int |x-u|^{-q}\, d\mu\cap(\tau_z\circ g)_\# \nu u\, d\mu\times\nu(x,y)\, d\mathscr{L}^n z\\
&= \iint_{\{z: |x-g(y)-z|\leq\delta\}}\int |x-u|^{-q}\, d\mu\cap(\tau_z\circ g)_\# \nu u\, d\mathscr{L}^n z\, d\mu\times\nu(x,y)\\
&\leq \iint_{\{(u,v): |x-g(y)-(u-g(v))|\leq\delta\}} |x-u|^{-q}\, d\mu\times\nu(u,v)\, d\mu\times\nu(x,y).
\end{aligned}$$

Denote

$$A_\delta = \{(u,v,x,y)\in(\mathbf{R}^n)^4: ||x-u|-|y-v||\leq\delta\}.$$

Integrating with respect to θ_n, using Fubini's theorem, Lemma 4.7 and (5.1) we obtain

$$\int J(g)\,d\theta_n g \leqslant \int\int \theta_n\{g: |x-u-g(y-v)|\leqslant\delta\}\,|x-u|^{-q}\,d\mu\times\nu(u,v)\,d\mu\times\nu(x,y)$$

$$\leqslant c_1\delta^{n-1}\int_{A_\delta}|x-u|^{1-s-m}\,d\mu\times\mu\times\nu\times\nu(u,x,v,y)$$

$$= c_1\delta^{n-1}\int\int\int \nu\{v: ||x-u|-|y-v||\leqslant\delta\}\,|x-u|^{1-s-m}\,d\mu x\,d\mu u\,d\nu y$$

$$\leqslant bc_1\delta^n\int\int|x-u|^{-s}\,d\mu x\,d\mu u\nu(\mathbf{R}^n),$$

where c_1 depends only on n. Combining this with the first inequality, we complete the proof.

As in [MP2, 4.6] this energy-inequality leads to an integral inequality for capacities:

THEOREM 5.5. *Let $n-m<s<n$ and let c be the constant of Theorem 5.4. If $A\subset\mathbf{R}^n$ and B is a ν measurable subset of $\mathbf{R}^n$, then*

$$C_s(A)\,\nu(B)\leqslant bc\int_* C_{s+m-n}(A\cap fB)\,d\lambda_n f.$$

Proof. We may assume A and B are compact. Then the integrand is λ_n measurable. Let $q=s+m-n$. We may assume $C_s(A)>0$. Let $\varepsilon>0$ and choose a Radon measure μ with $\operatorname{supp}\mu\subset A$, $\mu(\mathbf{R}^n)=1$ and $I_s(\mu)\leqslant C_s(A)^{-1}+\varepsilon$. Let J be the set of those $f\in I(n)$ for which $\mu\cap f_\#(\nu\llcorner B)(\mathbf{R}^n)>0$. For $f\in J$ put

$$\mu_f=(\mu\cap f_\#(\nu\llcorner B)(\mathbf{R}^n))^{-1}\mu\cap f_\#(\nu\llcorner B).$$

Then $\operatorname{supp}\mu_f\subset A\cap fB$ and $\mu_f(\mathbf{R}^n)=1$, whence $I_q(\mu_f)^{-1}\leqslant C_q(A\cap fB)$. Since $n-m<s$ and $I_s(\mu)<\infty$ imply $I_{n-m}(\mu)<\infty$, we deduce from Theorem 4.8 and (4.4)

$$\int\mu\cap f_\#(\nu\llcorner B)(\mathbf{R}^n)\,d\lambda_n f=\nu(B).$$

Thus by Hölder's inequality and Theorem 5.4

$$\nu(B)^2=\left(\int\mu\cap f_\#(\nu\llcorner B)(\mathbf{R}^n)\,d\lambda_n f\right)^2$$

$$\leq \int_J \mu \cap f_\#(\nu \llcorner B)(\mathbf{R}^n)^2 I_q(\mu_f)\, d\lambda_n f \int_J I_q(\mu_f)^{-1}\, d\lambda_n f$$

$$= \int_J I_q(\mu \cap f_\#(\nu \llcorner B))\, d\lambda_n f \int_J I_q(\mu_f)^{-1}\, d\lambda_n f$$

$$\leq bcI_s(\mu)\, \nu(B) \int C_q(A \cap fB)\, d\lambda_n f$$

$$\leq bc(C_s(A)^{-1}+\varepsilon)\, \nu(B) \int C_q(A \cap fB)\, d\lambda_n f.$$

Letting $\varepsilon \downarrow 0$ we obtain the result.

COROLLARY 5.6. *Suppose that m is an integer,* $1 \leq m < n$, $n-m < s < n$, *and that B is an* $\mathcal{H}^m$ *measurable* $(\mathcal{H}^m, m)$ *rectifiable subset of* $\mathbf{R}^n$ *with* $\mathcal{H}^m(B)>0$. *Then there is a positive number* $\beta(B)$ *such that for any* $A \subset \mathbf{R}^n$

$$\beta(B)\, C_s(A) \leq \int_* C_{s+m-n}(A \cap fB)\, d\lambda_n f.$$

Proof. As in the proof of Corollary 4.9 we find a compact subset D of B such that $\mathcal{H}^m(D)>0$ and $\nu=\mathcal{H}^m \llcorner D$ satisfies (5.1) with some b. We can then take $\beta(B)=\mathcal{H}^m(D)/(bc)$, where c is as in Theorem 5.4.

Remark 5.7. (1) In general it is not possible to choose $\beta(B)$ to be of the form $\beta\mathcal{H}^m(B)$ where β would depend only on m, n and s. In fact, the right hand side of Corollary 5.6 is bounded when A and B vary in a fixed ball. However, one can take $\beta(hB)=(\operatorname{Lip} h)^m\beta(B)$ whenever h is a similarity of $\mathbf{R}^n$. Thus for example for $(n-1)$-spheres B, $\beta(B)=\beta_n\mathcal{H}^{n-1}(B)$.

(2) The opposite inequality in Corollary 5.6 is false. For example, if A is a ball of radius r, then, for a fixed B, as $r \to \infty$ the left hand side behaves like r^s and the right hand side like r^n. Nevertheless, using the inequality of [MP3], we can prove the following:

THEOREM 5.8. *Suppose that m is an integer,* $1 \leq m < n$, $n-m < s < n$, $0 < R < \infty$, *and that B is a compact m dimensional* C^1 *submanifold of* $\mathbf{R}^n$. *Then there is a positive number* $\gamma(B, R)$ *such that for all sets* $A \subset \mathbf{R}^n$ *with* diam $A \leq R$,

$$\int^* C^*_{s+m-n}(A \cap fB)\, d\lambda_n f \leq \gamma(B,R)\, C^*_s(A).$$

Proof. By the definition of the outer capacity we may assume that A is open. We may also assume that A is contained in $B(0, R)$. Suppose that E is a compact subset of B such that diam $E \leq R$ and that there is an orthogonal projection $P: \mathbf{R}^n \to V$ of $\mathbf{R}^n$ onto a linear m dimensional subspace V with $P|E$ one-to-one and $\operatorname{Lip}(P|E)^{-1} \leq 2$. For $g \in O(n)$ define

$$E_{g,a} = (\tau_a \circ g) E \quad \text{for } a \in gV^\perp,$$
$$E_g = \bigcup \{E_{g,a} : a \in gV^\perp\},$$
$$\varphi_g : E_g \to gV^\perp, \quad \varphi_g(x) = a \quad \text{if } x \in E_{g,a}.$$

Note that $E_{g,a} \cap E_{g,b} = \emptyset$ for $a \neq b$ as $P|E$ is one-to-one. Then $\operatorname{Lip} \varphi_g \leq 3$ and $\varphi_g^{-1}\{a\} = E_{g,a}$. Let $q = s+m-n$. It follows from [MP3, Theorem 3.1] that

$$\int_{gV^\perp} C_q(A \cap E_{g,a})\, d\mathcal{H}^{n-m} a \leq c C_s(A \cap E_g) \leq c C_s(A)$$

where c depends only on m, n and s. Select x_g such that $-x_g \in gE$. Then $A \cap (\tau_z \circ g) E = \emptyset$ whenever $z \in \mathbf{R}^n \sim B(x_g, 2R)$. Hence

$$\begin{aligned}\int C_q(A \cap (\tau_z \circ g) E)\, d\mathcal{L}^n z &= \int_{gV} \int_{gV^\perp} C_q(A \cap (\tau_b \circ \tau_a \circ g) E)\, d\mathcal{H}^{n-m} a\, d\mathcal{H}^m b \\ &= \int_{g(V \cap B(Px_g, 2R))} \int_{gV^\perp} C_q((\tau_{-b} A) \cap (\tau_a \circ g) E)\, d\mathcal{H}^{n-m} a\, d\mathcal{H}^m b \\ &\leq c\alpha(m)\, 2^m R^m C_s(A).\end{aligned}$$

Integrating over $O(n)$ we get

$$\int C_q(A \cap fE)\, d\lambda_n f \leq c\alpha(m)\, 2^m R^m C_s(A).$$

We can cover B with finitely many sets $E_1, \ldots, E_k$ which satisfy the assumptions that were made on E. Therefore

$$\int C_q(A \cap fB)\, d\lambda_n f \leq \sum_{i=1}^{k} \int C_q(A \cap fE_i)\, d\lambda_n f \leq k c\alpha(m)\, 2^m R^m C_s(A).$$

COROLLARY 5.9. *Let m, s and B be as in Theorem 5.8. If $A \subset \mathbf{R}^n$ and $C_s^*(A) = 0$, then $C_{s+m-n}^*(A \cap fB) = 0$ for λ_n almost all $f \in I(n)$.*

THEOREM 5.10. *Suppose that* $n-m<s<n$, $\int|x-u|^{-s}d\mu u<\infty$ *for* μ *almost all* $x\in\mathbf{R}^n$ *and that* $\nu=\sum_{i=1}^{\infty}\nu_i$ *where each* ν_i *satisfies* (5.1) *for some* b_i. *If* A *is a* μ *measurable and* B *a* ν *measurable subset of* $\mathbf{R}^n$, *then*

$$C_{s+m-n}(A\cap(\tau_x\circ g\circ\tau_{-y})B)>0$$

for $\mu\times\nu\times\theta_n$ *almost all* $(x,y,g)\in A\times B\times O(n)$.

Proof. Obviously we may assume ν satisfies (5.1). Since A and B can be approximated in measure from within by compact sets, we may assume that they are compact. Considering the restrictions we may also assume that $\operatorname{supp}\mu\subset A$ and $\operatorname{supp}\nu\subset B$. By Theorem 4.8 $S_{g\#}(\mu\times\nu)\ll\mathscr{L}^n$ for λ_n almost all $g\in O(n)$, by Lemma 4.6 $\mu\cap(\tau_x\circ g\circ\tau_{-y})_\#\nu$ is defined and $\mu\cap(\tau_x\circ g\circ\tau_{-y})(\mathbf{R}^n)>0$ for $\mu\times\nu\times\theta_n$ almost all (x,y,g), and by (4.1) $\operatorname{supp}\mu\cap(\tau_x\circ g\circ\tau_{-y})_\#\nu\subset A\cap(\tau_x\circ g\circ\tau_{-y})B$. From Theorem 5.4 we infer that for θ_n almost all $g\in O(n)$

$$I_{s+m-n}(\mu\cap(\tau_z\circ g)_\#\nu)<\infty\quad\text{for }\mathscr{L}^n\text{ almost all }z\in\mathbf{R}^n,$$

which in view of the facts $S_{g\#}(\mu\times\nu)\ll\mathscr{L}^n$ and $\tau_x\circ g\circ\tau_{-y}=\tau_{S_g(x,y)}\circ g$ implies

$$I_{s+m-n}(\mu\cap(\tau_x\circ g\circ\tau_{-y})_\#\nu)<\infty\quad\text{for }\mu\times\nu\text{ almost all }(x,y)\in\mathbf{R}^n\times\mathbf{R}^n.$$

Recalling Lemma 5.2 we get from Fubini's theorem $I_{s+m-n}(\mu\cap(\tau_x\circ g\circ\tau_{-y})_\#\nu)<\infty$ for $\mu\times\nu\times\theta_n$ almost all $(x, y, g)\in\mathbf{R}^n\times\mathbf{R}^n\times O(n)$, and the assertion follows.

COROLLARY 5.11. *Suppose that* m *is an integer,* $n-m<s<n$, $A\subset\mathbf{R}^n$, *and* B *is an* $\mathscr{H}^m$ *measurable* $(\mathscr{H}^m,m)$ *rectifiable subset of* $\mathbf{R}^n$. *Then there is* $E\subset A$ *such that* $C_s(A\sim E)=0$ *and for* $x\in E$

$$C_{s+m-n}(A\cap(\tau_x\circ g\circ\tau_{-y})B)>0\quad\textit{for }\mathscr{H}^m\times\theta_n\textit{ almost all }(y,g)\in B\times O(n).$$

Proof. Otherwise the set of those $x\in A$ for which

$$\mathscr{H}^m\times\theta_n\{(y,g)\in B\times O(n): C_{s+m-n}(A\cap(\tau_x\circ g\circ\tau_{-y})B)=0\}>0$$

contains a compact set D with $C_s(D)>0$. Then there is a Radon measure μ with $\operatorname{supp}\mu\subset D$, $\mu(D)>0$ and $\int|x-u|^{-s}d\mu u<\infty$ for all $x\in\mathbf{R}^n$. As in Corollary 4.9 we see that $\nu=\mathscr{H}^m\llcorner B$ satisfies the condition of Theorem 5.10, and a contradiction follows.

COROLLARY 5.12. *Suppose in addition to the assumptions of Corollary* 5.11 *that*

A is $\mathcal{H}^s$ measurable with $\mathcal{H}^s(A)<\infty$. Then for $\mathcal{H}^s\times\mathcal{H}^m\times\theta_n$ almost all $(x,y,g)\in A\times B\times O(n)$

$$\dim A\cap(\tau_x\circ g\circ\tau_{-y})B\geqq s+m-n.$$

Proof. Apply Theorem 5.10 to the measures $\mu=\mathcal{H}^s\llcorner A$ and $\nu=\mathcal{H}^m\llcorner B$ with s replaced by an arbitrary t, $n-m<t<s$, and recall Section 2.3 and Lemma 2.4.

The following lemma and its proof are essentially from [LV, 6.3]:

LEMMA 5.13. *Suppose that m is an integer, $1\leqq m<n$, $n-m<s<n$, $E\subset\mathbf{R}^m$ and $\alpha\colon E\to\mathbf{R}^n$ is Lipschitzian. Then there is a positive number c such that for all $A\subset\mathbf{R}^n$*

$$\int^* \mathcal{H}^{s+m-n}(A\cap\tau_z(\alpha E))\,d\mathcal{L}^n z\leqq c\mathcal{L}^m(E)\,\mathcal{H}^s(A).$$

Proof. Define $\psi\colon\mathbf{R}^n\times E\to\mathbf{R}^n\times E$ by $\psi(z,y)=(z+\alpha(y),y)$. Then ψ is bilipschitzian, and putting $F=\psi^{-1}(A\times E)$ we have by [F, 2.10.45] for some $c_1\in\mathbf{R}_+$

$$\mathcal{H}^{m+s}(F)\leqq(\operatorname{Lip}\psi^{-1})^{m+s}\mathcal{H}^{m+s}(A\times E)\leqq c_1\mathcal{L}^m(E)\,\mathcal{H}^s(A).$$

Set $D(z)=\{y\in\mathbf{R}^n\colon(z,y)\in F\}$ for $z\in\mathbf{R}^n$. Then [F, 2.10.27] implies

$$\int^* \mathcal{H}^{m+s-n}(D(z))\,d\mathcal{L}^n z\leqq c_2\,\mathcal{H}^{m+s}(F)\leqq c_1 c_2\,\mathcal{L}^m(E)\,\mathcal{H}^s(A).$$

Letting $p\colon\mathbf{R}^n\times E\to\mathbf{R}^n$ be the projection, we have $A\cap\tau_z(\alpha E)\subset p\psi(\{z\}\times D(z))$, and the required inequality follows.

Remark 5.14. If B is an m dimensional C^1 submanifold of $\mathbf{R}^n$, one can modify the above proof to show that $\int^*\mathcal{H}^{s+m-n}(A\cap\tau_z(B))\,d\mathcal{L}^n z\leqq c\mathcal{H}^m(B)\,\mathcal{H}^s(A)$, where c depends only on m, n and s. Probably this is true also if B is m rectifiable.

LEMMA 5.15. *If A and B are Suslin subsets of $\mathbf{R}^n$ and $0<t<\infty$, then for every real number α the set*

$$\{(x,y,g)\in\mathbf{R}^n\times\mathbf{R}^n\times O(n)\colon\mathcal{H}^t(A\cap(\tau_x\circ g\circ\tau_{-y})B)>\alpha\}$$

is a Suslin set.

Proof. Denote

$$E=A\times B\times\mathbf{R}^n\times\mathbf{R}^n\times O(n)\cap\{(u,v,x,y,g)\colon u=(\tau_x\circ g\circ\tau_{-y})(v)\},$$

$$E(x,y,g)=\{(u,v)\colon(u,v,x,y,g)\in E\}\text{ for }(x,y,g)\in\mathbf{R}^n\times\mathbf{R}^n\times O(n).$$

Then E is a Suslin set and by [D, VI,21] the set $\{(x,y,g)\colon \mathscr{H}^t(E(x,y,g))>\alpha\}$ is a Suslin set for every $\alpha\in\mathbf{R}^1$. But since $u\in A\cap(\tau_x\circ g\circ\tau_{-y})B$ if and only if $(u,(\tau_x\circ g\circ\tau_{-y})^{-1}(u))\in E(x,y,g)$, we have

$$\mathscr{H}^t(A\cap(\tau_x\circ g\circ\tau_{-y})B)=2^{-t/2}\mathscr{H}^t(E(x,y,g)),$$

and the lemma follows.

THEOREM 5.16. *Suppose that m is an integer, $1\leq m<n$, $n-m<s<n$, A and B are Suslin subsets of $\mathbf{R}^n$, $\mathscr{H}^s(A)<\infty$ and B is m rectifiable. Then for $\mathscr{H}^s\times\mathscr{H}^m\times\theta_n$ almost all $(x,y,g)\in A\times B\times O(n)$*

(1) $\mathscr{H}^{s+m-n}(A\cap(\tau_x\circ g\circ\tau_{-y})B)<\infty$

and

(2) $\dim A\cap(\tau_x\circ g\circ\tau_{-y})B=s+m-n.$

Proof. Due to Corollary 5.12 it suffices to prove (1). This follows since Lemma 5.13 implies for any $g\in O(n)$, $\mathscr{H}^{s+m-n}(A\cap(\tau_z\circ g)B)<\infty$ for $\mathscr{L}^n$ almost $z\in\mathbf{R}^n$, since Corollary 4.10 implies $S_{g\#}((\mathscr{H}^s\llcorner A)\times(\mathscr{H}^m\llcorner B))\ll\mathscr{L}^n$ for θ_n almost all $g\in O(n)$, and since Lemma 5.15 implies that the function $(x,y,g)\mapsto\mathscr{H}^{s+m-n}(A\cap(\tau_x\circ g\circ\tau_{-y})B)$ is $\mathscr{H}^s\times\mathscr{H}^m\times\theta_n$ measurable.

6. Similarities and intersections

Here we shall consider intersections $A\cap fB$, where f is a similarity map of $\mathbf{R}^n$, without any smoothness or rectifiability assumptions on either A or B. By a similarity we mean a map $f\colon\mathbf{R}^n\to\mathbf{R}^n$ such that there is $r\in\mathbf{R}_+$ with $|f(x)-f(y)|=r|x-y|$ for all $x,y\in\mathbf{R}^n$. Then $r=\operatorname{Lip}f$ and f has a unique decomposition as

$$f=\tau_z\circ g\circ\delta_r,\quad z\in\mathbf{R}^n,\ g\in O(n),\ r\in\mathbf{R}_+,$$

where

$$\delta_r\colon\mathbf{R}^n\to\mathbf{R}^n,\ \delta_r(x)=rx.$$

We denote by $S(n)$ the space of all similarities of $\mathbf{R}^n$ and for $0\leq a<b\leq\infty$

$$S_{a,b}(n)=\{f\in S(n)\colon a\leq\operatorname{Lip}f\leq b\}.$$

For any $t\in\mathbf{R}_+$ we define the Borel measure $\sigma^t_{a,b}$ on $S_{a,b}(n)$ such that

$$\int \varphi\, d\sigma^t_{a,b} = \int_a^b r^{t-1} \int\int \varphi(\tau_z \circ g \circ \delta_r)\, d\mathcal{L}^n z\, d\theta_n g\, d\mathcal{L}^1 r$$

for any non-negative Borel function φ on $S_{a,b}(n)$, and we put $\sigma^t = \sigma^t_{0,\infty}$ and $\sigma = \sigma^1$. For $f \in S(n)$ and $\delta \in \mathbf{R}_+$ we let

$$W_f(\delta) = \{(x,y) \in \mathbf{R}^n \times \mathbf{R}^n : |x - f(y)| \leq \delta\}.$$

If $g \in O(n)$ and $r \in \mathbf{R}_+$ we define

$$S_{g,r} : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n \quad \text{by} \quad S_{g,r}(x,y) = x - g(ry).$$

Let μ and ν be Radon measures on $\mathbf{R}^n$. For any $r \in \mathbf{R}_+$ we can apply Section 4 to the measures μ and $\delta_{r\#}\nu$ to conclude that for $g \in O(n)$ the intersection measures

$$\mu \cap f_\# \nu = \mu \cap (\tau_z \circ g)_\# (\delta_{r\#} \nu) \quad \text{with} \quad f = \tau_z \circ g \circ \delta_r$$

exist for $\mathcal{L}^n$ almost all $z \in \mathbf{R}^n$. As in Section 4 we see that $\mu \cap f_\# \nu$ is defined for σ almost all $f \in S(n)$, and from (4.1)–(4.4) we infer that the following four statements hold whenever φ and ψ are non-negative Borel functions on $\mathbf{R}^n$:

$$\operatorname{supp}\mu \cap f_\# \nu \subset \operatorname{supp}\mu \cap f(\operatorname{supp}\nu). \tag{6.1}$$

$$\int \varphi(x)\,\psi(f^{-1}(x))\, d\mu \cap f_\# \nu x = \lim_{\delta \downarrow 0} \alpha(n)^{-1} \delta^{-n} \int_{W_f(\delta)} \varphi(x)\,\psi(y)\, d\mu \times \nu(x,y) \tag{6.2}$$

with $f = \tau_z \circ g \circ \delta_r$ for any $g \in O(n)$ for $\mathcal{L}^n$ almost all $z \in \mathbf{R}^n$.

$$\text{The function } f \mapsto \int \varphi(\psi \circ f^{-1})\, d\mu \cap f_\# \nu \text{ is } \sigma^t \text{ measurable.} \tag{6.3}$$

$$\int\int \varphi(\psi \circ f^{-1})\, d\mu \cap f_\# \nu\, d\sigma^t_{a,b} f \leq t^{-1}(b^t - a^t) \int \varphi\, d\mu \int \psi\, d\nu; \tag{6.4}$$

here equality holds if $S_{g,r\#}(\mu \times \nu) \ll \mathcal{L}^n$ for $\theta_n \times \mathcal{L}^1$ almost all $(g,r) \in O(n) \times \mathbf{R}_+$. Note that $S_{g,r\#}(\mu \times \nu) = S_{g\#}(\mu \times \delta_{r\#}\nu)$.

For the rest of this section s, t and q will be real numbers such that

$$0 < s < n, \quad 0 < t < n, \quad 0 \leq q = s + t - n.$$

LEMMA 6.5. *Suppose that α and β are Borel measures on $\mathbf{R}^n$ such that*

$$\int |x|^{-s}\, d\alpha x < \infty \quad \textit{and} \quad \int |y|^{-t}\, d\beta y < \infty.$$

Then for $0<a<b<\infty$

$$\limsup_{\delta\downarrow 0}\delta^{-n}\int\int_a^b r^{t-1}\theta_n\{g\colon |x-g(ry)|\le\delta\}\,d\mathscr{L}^1r|x|^{-q}\,d\alpha\times\beta(x,y)$$
$$\le c(a,b)\int_{A(a,b)}|x|^{-s}|y|^{-t}\,d\alpha\times\beta(x,y)$$

where

$$A(a,b)=\{(x,y)\in\mathbf{R}^n\times\mathbf{R}^n\colon a|y|\le|x|\le b|y|\},$$
$$c(a,b)=c_0\max\{(b/a)^{t-1},(a/b)^{t-1}\}$$

and c_0 *depends only on* n.

Proof. Let $0<\delta<1$ and define

$$J_\delta(x,y)=\int_a^b r^{t-1}\theta_n\{g\colon |x-g(ry)|\le\delta\}\,d\mathscr{L}^1r$$

for $(x,y)\in\mathbf{R}^n\times\mathbf{R}^n$. Using Lemma 4.7 we find that

$$J_\delta(x,y)=\int_{[a,b]\cap\{r\colon ||x|-r|y||\le\delta\}}r^{t-1}\theta_n\{g\colon |x-g(ry)|\le\delta\}\,d\mathscr{L}^1r.$$

Letting

$$A_\delta=\{(x,y)\in\mathbf{R}^n\times\mathbf{R}^n\colon a|y|-\delta\le|x|\le b|y|+\delta\}$$

we have $A(a,b)=\bigcap_{\delta>0}A_\delta$ and $[a,b]\cap\{r\colon ||x|-r|y||\le\delta\}=\emptyset$ for $(x,y)\notin A_\delta$, whence

$$\int J_\delta\,d\alpha\times\beta=\int_{A_\delta}J_\delta\,d\alpha\times\beta.$$

Set

$$B_\delta=\{(x,y)\in A_\delta\colon |x|\le 2\delta\},$$
$$C_\delta=\{(x,y)\in A_\delta\colon a|y|\le 2\delta\},$$
$$D_\delta=\{(x,y)\in A_\delta\colon |x|>2\delta,\ a|y|>2\delta\}.$$

Then $A_\delta=B_\delta\cup C_\delta\cup D_\delta$. For $(x,y)\in B_\delta$ we have $|x|\le 2\delta$ and $|y|\le a^{-1}(|x|+\delta)\le 3a^{-1}\delta$. These inequalities imply

$$\delta^{-n}|x|^{-q}\le 2^{n-t}3^t a^{-t}|x|^{-s}|y|^{-t},$$

from which

$$\text{(1)}\qquad \limsup_{\delta\downarrow 0}\delta^{-n}\int_{B_\delta} J_\delta(x,y)|x|^{-q}\,d\alpha\times\beta(x,y)$$
$$\leq \limsup_{\delta\downarrow 0}\int_a^b r^{t-1}dr 2^{n-t}3^t a^{-t}\int_{B_\delta}|x|^{-s}|y|^{-t}\,d\alpha\times\beta(x,y)=0.$$

For $(x,y)\in C_\delta$, $|y|\leq 2a^{-1}\delta$ and $|x|\leq(2a^{-1}b+1)\delta$, and we get in the same way

$$\text{(2)}\qquad \limsup_{\delta\downarrow 0}\delta^{-n}\int_{C_\delta} J_\delta(x,y)|x|^{-q}\,d\alpha\times\beta(x,y)=0.$$

For $(x,y)\in D_\delta$ we have $a|y|/2\leq|x|\leq 2b|y|$. Therefore by Lemma 4.7, with c_1 depending only on n,

$$\begin{aligned} J_\delta(x,y)\,|x|^{-q} &\leq c_1\,\delta^{n-1}|x|^{1-n}\int_{[a,b]\cap\{r:\,||x|-r|y||\leq\delta\}} r^{t-1}\,d\mathscr{L}^1 r|x|^{-q}\\ &\leq c_1\,\delta^{n-1}\max\{a^{t-1},b^{t-1}\}\,2\delta|y|^{-1}\,|x|^{1-s-t}\\ &\leq 2c_1\,\delta^n\max\{a^{t-1},b^{t-1}\}\max\{(a/2)^{1-t},(2b)^{1-t}\}\,|x|^{-s}\,|y|^{-t}\\ &\leq c(a,b)\,\delta^n|x|^{-s}\,|y|^{-t}\end{aligned}$$

where $c(a,\,b)=c_1 2^{n+1}\max\{(a/b)^{t-1},\,(b/a)^{t-1}\}$. Hence

$$\limsup_{\delta\downarrow 0}\delta^{-n}\int_{D_\delta} J_\delta(x,y)|x|^{-q}\,d\alpha\times\beta(x,y)\leq c(a,b)\limsup_{\delta\downarrow 0}\int_{A_\delta}|x|^{-s}\,|y|^{-t}\,d\alpha\times\beta(x,y)$$
$$=c(a,b)\int_{A(a,b)}|x|^{-s}\,|y|^{-t}\,d\alpha\times\beta(x,y).$$

Combining this with (1) and (2) we get the lemma.

THEOREM 6.6. *If $I_s(\mu)<\infty$ and $I_t(\nu)<\infty$, then $S_{g,r\#}(\mu\times\nu)\ll\mathscr{L}^n$ for $\theta_n\times\mathscr{L}^1$ almost all $(g,r)\in O(n)\times\mathbf{R}_+$.*

Proof. We may suppose that μ and ν have compact supports, and consequently that $q=s+t-n=0$. For $0<a<b<\infty$ we have by Fatou's lemma, Fubini's theorem and Lemma 6.5 applied to the measures $\alpha=S_\#(\mu\times\mu)$ and $\beta=S_\#(\nu\times\nu)$ (recall that $S(x,y)=x-y$)

$$\iint_a^b r^{t-1}\int \liminf_{\delta\downarrow 0}\delta^{-n}S_{g,r\#}(\mu\times\nu)(B(S_{g,r}(x,y),\delta))\,d\theta_n g\,d\mathscr{L}^1 r\,d\mu\times\nu(x,y)$$

$$\leq \liminf_{\delta\downarrow 0}\delta^{-n}\iint_a^b r^{t-1}\int \mu\times\nu\{(u,v): |x-u-g(r(y-v))|\leq\delta\}\,d\theta_n g\,d\mathscr{L}^1 r\,d\mu\times\nu(x,y)$$

$$= \liminf_{\delta\downarrow 0}\delta^{-n}\iiint_a^b r^{t-1}\theta_n\{g: |x-u-g(r(y-v))|\leq\delta\}\,d\mathscr{L}^1 r\,d\mu\times\mu(x,u)\,d\nu\times\nu(y,v)$$

$$= \liminf_{\delta\downarrow 0}\delta^{-n}\iint_a^b r^{t-1}\theta_n\{g: |x-g(ry)|\leq\delta\}\,d\mathscr{L}^1 r\,d\alpha\times\beta(x,y)$$

$$\leq c(a,b)\int |x|^{-s}|y|^{-t}\,d\alpha\times\beta(x,y)$$

$$= c(a,b)\,I_s(\mu)\,I_t(\nu)<\infty.$$

It follows that for $\theta_n\times\mathscr{L}^1$ almost all $(g,r)\in O(n)\times\mathbf{R}_+$

$$\liminf_{\delta\downarrow 0}\delta^{-n}S_{g,r\#}(\mu\times\nu)(B(S_{g,r}(x,y),\delta))<\infty$$

for $\mu\times\nu$ almost all $(x,y)\in\mathbf{R}^n\times\mathbf{R}^n$, which implies

$$\liminf_{\delta\downarrow 0}\delta^{-n}S_{g,r\#}(\mu\times\nu)(B(S_{g,r}(z,\delta))<\infty$$

for $S_{g,r\#}(\mu\times\nu)$ almost all $z\in\mathbf{R}^n$, and further by Lemma 2.2, $S_{g,r\#}(\mu\times\nu)\ll\mathscr{L}^n$.

THEOREM 6.7. *There is a constant c depending only on n such that*

$$\int I_{s+t-n}(\mu\cap f_\#\,\nu)\,d\sigma^t f\leq c I_s(\mu)\,I_t(\nu).$$

Proof. From the proof of Theorem 5.4, with ν replaced by $\delta_{r\#}\,\nu$, we see immediately that for any $r\in\mathbf{R}_+$

$$\iint I_q(\mu\cap(\tau_z\circ g\circ\delta_r)_\#\,\nu)\,d\mathscr{L}^n z\,d\theta_n g$$

$$\leq \liminf_{\delta\downarrow 0}\alpha(n)^{-1}\delta^{-n}\iint \theta_n\{g: |x-u-g(r(y-v))|\leq\delta\}\,|x-u|^{-q}\,d\mu\times\nu(u,v)\,d\mu\times\nu(x,y).$$

Integrating over an interval $[a, 2a]$, $0<a<\infty$, using Fatou's lemma, Fubini's theorem and Lemma 6.5, we obtain

$$\int_a^{2a} r^{t-1} \int\int I_q(\mu \cap (\tau_z \circ g \circ \delta_r)_\# \nu)\, d\mathcal{L}^n z\, d\theta_n g\, d\mathcal{L}^1 r$$

$$\leq c_2 \int_{A(a)} |x-u|^{-s} |y-v|^{-t}\, d\mu\times\mu\times\nu\times\nu(x, u, y, v),$$

where

$$A(a) = \{(x, u, y, v) \in (\mathbf{R}^n)^4 \colon a|y-v| \leq |x-u| \leq 2a|y-v|\}$$

and c_2 depends only on n. Applying this inequality to the intervals $[2^{-i}, 2^{-i+1}]$, summing over all integers i and observing that $A(2^{-i}) \cap A(2^{-j}) = \emptyset$ unless $|i-j| \leq 1$, we obtain the required inequality.

THEOREM 6.8. *Suppose that $n<s+t$ and let c be the constant of Theorem 6.7. Then for any A, $B\subset\mathbf{R}^n$ and $0<a<b<\infty$*

$$t^{-2}(b^t-a^t)^2\, C_s(A)\, C_t(B) \leq c \int_* C_{s+t-n}(A \cap fB)\, d\sigma^t_{a,b} f$$

Proof. We may assume that A and B are compact with $C_s(A)>0$ and $C_t(B)>0$. Let $\varepsilon>0$. Then there are Radon measures μ and ν such that $\operatorname{supp}\mu\subset A$, $\operatorname{supp}\nu\subset B$, $\mu(\mathbf{R}^n)=\nu(\mathbf{R}^n)=1$, $I_s(\mu)<C_s(A)^{-1}+\varepsilon$ and $I_t(\nu)<C_t(B)^{-1}+\varepsilon$. Replacing λ_n by $\sigma^t_{a,b}$ and using Theorem 6.6, (6.4), Theorem 6.7 and (6.1) we argue exactly as in the proof of Theorem 5.5 to get

$$t^{-2}(b^t-a^t)^2 = \left(\int \mu \cap f_\# \nu(\mathbf{R}^n)\, d\sigma^t_{a,b} f\right)^2$$

$$\leq c(C_s(A)^{-1}+\varepsilon)(C_t(B)^{-1}+\varepsilon) \int C_q(A \cap fB)\, d\sigma^t_{a,b} f,$$

from which the theorem follows.

Remark 6.9. If B is a compact m dimensional C^1 submanifold of $\mathbf{R}^n$ with $\mathcal{H}^m(B)>0$, it follows from Theorems 5.6 and 5.8 that the right hand side in Theorem 6.8 behaves like $b^{2t}-a^{2t}$ as a function of a and b. For $b-a$ large this is like $(b^t-a^t)^2$ but for $b-a$ small it gives a better lower bound than Theorem 6.8.

THEOREM 6.10. *Suppose that $n<s+t$, $\int |x-u|^{-s}\, d\mu u<\infty$ for μ almost all $x\in\mathbf{R}^n$, $\int |y-v|^{-t}\, d\nu v<\infty$ for ν almost all $y\in\mathbf{R}^n$, A is a μ measurable and B a ν measurable subset of $\mathbf{R}^n$. Then for $\mu\times\nu\times\theta_n\times\mathcal{L}^1$ almost all $(x, y, g, r)\in A\times B\times O(n)\times R_+$*

$$C_{s+t-n}(A \cap (\tau_x \circ g \circ \delta_r \circ \tau_{-y})B) > 0.$$

Proof. Approximating μ and ν we may assume $I_s(\mu)<\infty$ and $I_t(\nu)<\infty$. As in the proof of Theorem 5.10 we may also assume that A and B are compact and that supp $\mu\subset A$ and supp $\nu\subset B$. By Theorem 6.6 $S_{g,r\#}(\mu\times\nu)\ll\mathscr{L}^n$ for $\theta_n\times\mathscr{L}^1$ almost all $(g,r)\in O(n)\times\mathbf{R}_+$ and by Theorem 6.7 $I_q(\mu\cap(\tau_z\circ g\circ\delta_r)_\#\nu)<\infty$ for $\mathscr{L}^n\times\theta_n\times\mathscr{L}^1$ almost all $(z,g,r)\in\mathbf{R}^n\times O(n)\times\mathbf{R}_+$. Using these facts we complete the proof as in Theorem 5.10.

COROLLARY 6.11. *Suppose that $n<s+t$, $A\subset\mathbf{R}^n$ and $B\subset\mathbf{R}^n$ with $C_t(B)>0$. Then there is a subset E of A with $C_s(A\sim E)=0$ and with the following property: For every $x\in E$ there is a subset B_x of B such that $C_t(B_x)>0$ and for all $y\in B_x$*

$$C_{s+t-n}(A \cap (\tau_x \circ g \circ \delta_r \circ \tau_{-y})B) > 0$$

for $\theta_n\times\mathscr{L}^1$ almost all $(g,r)\in O(n)\times\mathbf{R}_+$.

Proof. Let ν be a Radon measure with compact support such that $K=\operatorname{supp}\nu\subset B$, $\nu(\mathbf{R}^n)=1$ and $I_t(\nu)<\infty$. Then $C_t(F)>0$ whenever $F\subset B$ is ν measurable with $\nu(F)>0$. Let D be the set of those $x\in A$ for which the set of all $y\in B$ such that

$$\theta_n\times\mathscr{L}^1\{(g,r)\in O(n)\times\mathbf{R}_+: C_q(A \cap (\tau_x\circ g\circ\delta_r\circ\tau_{-y})B)=0\}>0$$

has positive ν measure. It suffices to show $C_s(D)=0$. If $C_s(D)>0$, there is a Radon measure μ with compact support such that $H=\operatorname{supp}\mu\subset D$, $\mu(\mathbf{R}^n)=1$ and $I_s(\mu)<\infty$. By the definition of D and the fact that the function $(x,y,g,r)\mapsto C_q(H\cap(\tau_x\circ g\circ\delta_r\circ\tau_{-y})K)$ is a Borel function, we infer from Fubini's theorem that the set of those $(x,y,g,r)\in H\times K\times O(n)\times\mathbf{R}_+$ for which $C_q(H\cap(\tau_x\circ g\circ\delta_r\circ\tau_{-y})K)=0$ has positive $\mu\times\nu\times\theta_n\times\mathscr{L}^1$ measure. But this contradicts Theorem 6.10.

I do not know if $C_t(B_x)>0$ can be replaced by $C_t(B\sim B_x)=0$.

COROLLARY 6.12. *Suppose that A is an $\mathscr{H}^s$ measurable and B an $\mathscr{H}^t$ measurable subset of $\mathbf{R}^n$ with $\mathscr{H}^s(A)<\infty$ and $\mathscr{H}^t(B)<\infty$. Then*

$$\dim(A\cap(\tau_x\circ g\circ\delta_r\circ\tau_{-y})B \geqslant s+t-n$$

for $\mathscr{H}^s\times\mathscr{H}^t\times\theta_n\times\mathscr{L}^1$ almost all $(x,y,g,r)\in A\times B\times O(n)\times\mathbf{R}_+$.

Proof. Combine Theorem 6.10 with Lemma 2.4 and Section 2.3.

In general the opposite inequality is false, but it holds if B has positive t dimensional lower density at all of its points:

THEOREM 6.13. *Suppose that A and B are Suslin subsets of $\mathbf{R}^n$ with $\mathcal{H}^s(A)<\infty$ and $\mathcal{H}^t(B)<\infty$ and that*

$$\liminf_{\delta\downarrow 0} \delta^{-t}\mathcal{H}^t(B\cap B(x,\delta))>0 \textit{ for all } x\in B.$$

Then

$$\dim A\cap(\tau_x\circ g\circ\delta_r\circ\tau_{-y})B=s+t-n$$

for $\mathcal{H}^s\times\mathcal{H}^t\times\theta_n\times\mathcal{L}^1$ almost all $(x,y,g,r)\in\mathbf{R}^n\times\mathbf{R}^n\times O(n)\times\mathbf{R}_+$.

Proof. Let $g\in O(n)$ and $r\in\mathbf{R}_+$. A result of Besicovitch and Moran [BM] implies $\dim A\times((g\circ\delta_r)B)=\dim A+\dim B$, and so $\mathcal{H}^{s+u}(A\times((g\circ\delta_r)B)=0$ for $t<u$. Using this, [F, 2.10.27] and the similarity of the sets $A\cap\tau_z(g\circ\delta_r)B)$ and $(A\times(g\circ\delta_r)B)\cap W_z$, we obtain

$$\int^* \mathcal{H}^{s+u-n}(A\cap\tau_z((g\circ\delta_r)B))\,d\mathcal{L}^n z=2^{(n-s-u)/2}\int^* \mathcal{H}^{s+u-n}((A\times((g\circ\delta_r)B))\cap W_z)\,d\mathcal{L}^n z$$
$$\leq c_1\,\mathcal{H}^{s+u}(A\times(g\circ\delta_r)B)=0$$

for $t<u$. Thus $\dim A\cap(\tau_z\circ g\circ\delta_r)B\leq s+t-n$ for $\mathcal{L}^n$ almost all $z\in\mathbf{R}^n$. From Lemma 2.4 and Theorem 6.6 we see that $S_{g,r\#}((\mathcal{H}^s\llcorner A)\times(\mathcal{H}^t\llcorner B))\ll\mathcal{L}^n$ for $\theta_n\times\mathcal{L}^1$ almost all $(g,r)\in O(n)\times\mathbf{R}_+$. For any such (g,r) $\dim A\cap(\tau_x\circ g\circ\delta_r\circ\tau_{-y})B\leq s+t-n$ for $\mathcal{H}^s\times\mathcal{H}^t$ almost all $(x,y)\in A\times B$. The theorem follows now from Fubini's theorem, the obvious modification of Lemma 5.15 and Corollary 6.12.

Remark 6.14. (1) The lower density assumption on B holds if B is self-similar, see [H, 5.3(1)].

(2) Clearly the above proof requires only the equality $\dim A\times B=\dim A+\dim B$. A recent sufficient condition for this, more general than the positiveness of lower density, was given by Tricot [T].

7. Examples

We construct two examples to illustrate the sharpness of the preceding results. For simplicity we consider only subsets of the real line, but quite likely such examples could be given in any $\mathbf{R}^n$. Of course the situation is somewhat different and more complicated for $n\geq 2$ as rotations play then an essential role.

The first example shows that Corollaries 5.6, 5.11 and 5.12 do not hold for general sets B. We could also use Hawkes' work on the Cantor ternary set [H 2], but it does not display the worst possible situation.

EXAMPLE 7.1. *For any s, $0 \leq s \leq 1$, there are compact subsets A and B of $\mathbf{R}^1$ such that* $\dim A = \dim B = s$ *and for any $z \in \mathbf{R}^1$, $A \cap \tau_z B$ contains at most one point.*

Proof. We assume $0<s<1$. It is clear that only slight modifications are required to handle the extreme cases. Take a strictly increasing sequence (n_k) of positive integers with $n_1=1$ and define positive numbers d_k by

$$d_1 = 1, \quad n_{k+1} d_{k+1}^s = d_k^s.$$

Choosing the sequence (n_k) sufficiently rapidly increasing, the numbers c_k defined by

$$(n_{k+1}-1)\, c_k + 3(n_k - 1)\, d_{k+1} + d_{k+1} = d_k$$

satisfy

(1) $4 n_k d_{k+1} \leq c_k$.

We define inductively intervals $I_{i_1 \dots i_m}$ and $J_{i_1 \dots i_m}$ of length d_k for $1 \leq i_k \leq n_k$, $1 \leq k \leq m$, $m = 1, 2, \dots$, as follows: Take $I_1 = J_1 = [0, 1]$. Assuming that $I_{i_1 \dots i_k} = [a, a+d_k]$ and $J_{i_1 \dots i_k} = [b, b+d_k]$ have been selected, we define for $i = 1, \dots, n_{k+1}$

$$I_{i_1 \dots i_k i} = [a+(i-1)\, c_k,\ a+(i-1)\, c_k + d_{k+1}],$$

$$J_{i_1 \dots i_k i} = [b+(i-1)\, c_k + 3(i_k - 1)\, d_{k+1},\ b+(i-1)\, c_k + 3(i_k-1)\, d_{k+1} + d_{k+1}].$$

Letting

$$A = \bigcap_{k=1}^{\infty} \bigcup_{i_1 \dots i_k} I_{i_1 \dots i_k} \quad \text{and} \quad B = \bigcap_{k=1}^{\infty} \bigcup_{i_1 \dots i_k} J_{i_1 \dots i_k},$$

we have $\dim A = \dim B = s$, cf. [F, 2.10,18]. Using (1) one verifies that for any $z \in \mathbf{R}^1$ and $k = 1, 2, \dots$ there is at most one pair of sequences $i_1, \dots, i_k$ and $j_1, \dots, j_k$ such that

$$\left(\bigcup_{i=1}^{n_{k+1}} I_{i_1 \dots i_k i} \right) \cap \left(\bigcup_{j=1}^{n_{k+1}} \tau_z J_{j_1 \dots j_k j} \right) \neq \varnothing.$$

From this it follows that $A \cap \tau_z B$ is either empty or a singleton.

The second example shows that the lower density assumption cannot be removed in Theorem 6.13.

EXAMPLE 7.2. *For any s, $0\leq s\leq 1$, there are compact subsets A and B of $\mathbf{R}^1$ such that* $\dim A=\dim B=s$ *and* $\dim A\cap\tau_z(\delta_r B)=s$ *for all* $z\in[0,1/2]$ *and* $r\in[1/2,1]$.

Proof. Again we only treat the case $0<s<1$. Let $0<s_i<s<1$, $s_i\uparrow s$, $I=[0,1]$, and let L be the set of the lines $\{(x,y)\colon y=ax+b\}$ with $1\leq a\leq 2$, $-1/2\leq b\leq 0$. We say that a line l intersects a rectangle $R\subset\mathbf{R}^2$ maximally if $\mathcal{H}^1(l\cap\tau_z R)\leq\mathcal{H}^1(l\cap R)$ for all $z\in\mathbf{R}^2$. We choose uniformly distributed disjoint subintervals $I_1,\dots,I_{m_1}$ of I of length d_1 such that $m_1 d_1^s=1$. We can take d_1 so small that the following condition is also satisfied:

If $l\in L$ and p is the number of those rectangles $I_i\times I$, $i=1,\dots,n_1$, which l intersects maximally, then $p\geq d_1^{s-s_1}m_1$. Thus $pd_1^{s_1}\geq 1$.

Next we choose uniformly distributed disjoint intervals $J_1,\dots,J_{n_1}$ of I of length e_1 such that $n_1 e_1^s=1$. We can take e_1 so small that the following condition also holds:

If $l\in L$ intersects maximally a rectangle $I_i\times I$ and q is the number of those rectangles $I_i\times J_j$, $j=1,\dots,n_1$, which l intersects maximally, then $q\geq e_1^{s-s_1}n_1$. Thus $qe_1^{s_1}\geq 1$.

Next for each $i=1,\dots,m_1$ we select uniformly distributed disjoint subintervals $I_{i1},\dots,I_{im_2}$ of I_i with length d_2 such that $m_2 d_2^s=d_1^s$. We can again take d_2 so small that the following condition holds.

If $l\in L$ intersects maximally a rectangle $I_i\times J_j$ and p is the number of those rectangles $I_{ik}\times J_j$, $k=1,\dots,n_2$, which l intersects maximally, then $p\geq d_1^{-s}d_2^{s-s_2}m_2$. Thus $pd_2^{s_2}\geq 1$.

It is now obvious how we continue to find the intervals $I_{i_1}\supset I_{i_1 i_2}\supset\dots$ and $J_{i_1}\supset J_{i_1 i_2}\supset\dots$ Defining A and B as in Example 7.1, we have $\dim A=\dim B=s$. Using the facts about the maximal intersections in the above constructions and [F, 2.10.28], one finds $\dim(A\times B)\cap l\geq s$ for all $l\in L$, whence $\dim(A\times(\delta_r B))\cap l_z\geq s$ for all $r\in[1/2,1]$, $z\in[0,1/2]$, where $l_z=\{(x,y)\colon y=x-z\}$. This implies $\dim A\cap\tau_z(\delta_r B)=s$.

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Received September 9, 1982