Forced vibrations of superquadratic Hamiltonian systems

by

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1. Introduction and main result

This paper is concerned with the existence of T-periodic solutions ($T\in\mathbf{R}$, $T>0$) of the following Hamiltonian system

$$\dot{z} = \mathcal{J}H'(z)+f(t). \tag{1.1}$$

Here,

$$\mathcal{J} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$

is the standard skewsymmetric matrix, $z=z(t)=(p,q)\colon \mathbf{R}\to\mathbf{R}^{2N}$, $\dot{z}=dz/dt$, $H\colon \mathbf{R}^{2N}\to\mathbf{R}$ is a given Hamiltonian and $f\colon\mathbf{R}\to\mathbf{R}^{2N}$ is a given function which is assumed to be T-periodic. The function $f(t)$ represents a forcing term and thus periodic solutions of (1.1) are called *forced vibrations* of the system. Here, H will be required to satisfy the following hypotheses.

(H 1) $H\in C^2(\mathbf{R}^{2N},\mathbf{R})$

(H 2) $0<H(z)\leqslant\theta H'(z)\cdot z,\quad \forall z\in\mathbf{R}^{2N},\quad |z|\geqslant R,\quad 0<\theta<\frac{1}{2}$

(H 3) $a|z|^{p+1}-b\leqslant H(z)\leqslant a'|z|^{q+1}+b'\quad$ with $\quad 1<p\leqslant q<2p+1$,

where $a,a'>0$, $b,b'\geqslant 0$ and $R>0$ are constants. $H'(z)\cdot z$ denotes the scalar product in $\mathbf{R}^{2N}$. Condition (H 2) is a usual way to express that the Hamiltonian is superquadratic

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as $|z|\to+\infty$. (Indeed, (H2) implies an inequality $H(z)\geqslant a|z|^{1/\theta}-b$, a and $b>0$ being constants.)

Our main result is the following.

THEOREM 1. *Let H satisfy conditions* (H1)–(H3), $T>0$ *be given and* $f\in C^1(\mathbf{R},\mathbf{R}^{2N})$ *be a given T-periodic function. Then,* (1.1) *has infinitely many distinct T-periodic solutions* $\{z_k\}_{k\in\mathbf{N}}$. *Moreover,* $\|z_k\|_{L^\infty}\to+\infty$ *as* $k\to+\infty$.

The proof of this result (which was announced in [9]) will take up most of this paper. In the last section we derive the existence of periodic solutions for different kinds of perturbations from autonomous systems.

Let us describe in a few words the principle of the method to prove Theorem 1. A first step is to construct critical values for the Lagrangian functional associated with the autonomous system (when $f\equiv 0$):

$$\dot{z}=\mathfrak{g}H'(z). \tag{1.2}$$

This construction is based on a "minimax" principle which relies on the S^1-invariance (through time shifts of (1.2)) and "semi-Galerkin" approximation of the space. Then we show that the critical values so constructed are *stable* in a topological sense. More precisely, we prove that some homotopy groups of level sets associated with those values are not trivial and remain so under "small" perturbations. Sharp estimates on the growth of the critical values are also required. Combining the preceding results and using Morse theory allow us to derive the existence of infinitely many critical values for some perturbations of the autonomous functional, thereby proving Theorem 1. In this argument we rely on recent results of A. Bahri [4, 5] in Morse theory that we recall together with their proofs in section 6.

The method employed here is to be compared with the perturbative approach in critical point theory we have used in [7, 8]. There, our purpose was to study some perturbations of *even* functionals. A "stability" result for the critical values defined in this context was obtained with a view to solving problems of the type $-\Delta u=g(u)+h(x)$ in Ω, $u=0$ on $\partial\Omega$, where $h\in L^2(\Omega)$ is given, $\Omega\subset\mathbf{R}^N$ is a bounded domain and $g:\mathbf{R}\to\mathbf{R}$ is odd and superlinear. (See also A. Bahri [6] for related results.)

In a separate paper [10], we study the case of some separable Hamiltonians: $H(p,q)=\frac{1}{2}|q|^2+V(p)$. This leads to a second order differential system of the type

$$\ddot{x}+V'(x)=h(t),\quad x(t)\in\mathbf{R}^N. \tag{1.3}$$

With no restriction on the growth of $V(x)$ (1) we show in [10] the existence of infinitely many periodic solutions by combining the methods developed here with different kinds of estimates.

Many works in the literature consider the particular case $f\equiv 0$, where (1.1) reduces to the autonomous Hamiltonian system (1.2). In the context of a superquadratic H, existence of *free vibrations* for (1.2) (i.e. non-constant T-periodic solutions for any given $T>0$) has been established in the recent years by Rabinowitz [29, 30, 33] and Benci–Rabinowitz [14]. The reader is also referred to the work of Ekeland [18] for an approach to (1.2) that uses convex analysis. (In the course of the proof of Theorem 1—see section 3—we will also have the occasion to derive the existence of free vibrations for (1.2).) The most general result in this direction is due to Rabinowitz [33] who shows the existence of a free vibration (2) in (1.2) under condition (H 2) and $H\in C^1(\mathbf{R}^{2N},\mathbf{R})$.

The methods employed in the above works do not seem to extend to the non-autonomous problem (1.1). In fact, Theorem 1 appears to be of a new type of result. Indeed, even the existence of *at least one* periodic solution of (1.1), for any given f was an open problem. (Compare for instance Fučik [23], Ekeland [19, 20] and the works previously mentioned.)

For different classes of non-autonomous superquadratic Hamiltonian systems a few partial results are known. In [29] Rabinowitz shows the existence of *one* periodic solution for a system of the type $\dot z=\mathfrak{g}\hat H'_z(t,z)$. There, $\hat H(t,z)$ is T-periodic in t and is such that $|\hat H(t,z)-H(z)|$ is bounded in the norm of $C^1(\mathbf{R}\times\mathbf{R}^{2N},\mathbf{R})$ (3). A much more general version of this result will be derived in the last section of the present paper. We show for instance that if H satisfies (H 1), (H 2) and has polynomial growth when $|z|\to+\infty$, then the system $\dot z=\mathfrak{g}\hat H'_z(t,z)$ possesses *infinitely many* T-periodic solutions.

When the Hamiltonian is *subquadratic* rather than superquadratic, the problem is of a different nature. The reader is referred to the works of Clarke–Ekeland [16] and Rabinowitz [31] for the existence of "subharmonic forced vibrations". Lastly, the limiting case, when H is exactly quadratic as $|z|\to+\infty$ has been studied by Amann and Zehnder [2, 3] who prove the existence of non-trivial solutions under "non-resonance"

(1) $V\in C^2(\mathbf{R}^N,\mathbf{R})$ just verifies an assumption like (H 2) on $\mathbf{R}^N$.

(2) The existence of free vibrations for some nonlinear wave equations in one spatial dimension has also been obtained by Rabinowitz [28] and Brézis, Coron and Nirenberg [15].

(3) Another result for systems of this kind is mentioned in [31] under different hypotheses. When specialized to (1.1) for instance, these hypotheses mean that $f\equiv 0$.

type assumptions. As for (1.1) with a superquadratic H, the only previously known result is due to Ekeland [20] (see also [19]). Under somewhat different conditions on H, he proved that for $\|f\|_{L^1(0,T)}$ *sufficiently small*, (1.1) possesses at least *two* solutions.

2. Functional framework

From now on we assume that $T=2\pi$ which obviously causes no loss in the generality. (Indeed, a scale change in time reduces the problem to this case.) In the following, 2π-periodic functions on $\mathbf{R}$ will be thought of as functions defined on $S^1 \simeq \mathbf{R}/2\pi\mathbf{Z}$. We will work in the space $E=(H^{1/2}(S^1))^{2N}$. Recall that $H^{1/2}(S^1)$ is a fractional order Sobolev space obtained by interpolation between $L^2(S^1)$ and $H^1(S^1)$. (See Lions–Magenes [25] or Adams [1].) Equivalently, $H^{1/2}(S^1)$ can be identified as the completion of $C^\infty(S^1)$ under the norm

$$\|z\|^2_{H^{1/2}S^1} = \sum_{-\infty}^{+\infty} (1+|k|)\,|a_k|^2 \tag{2.1}$$

where $z(t)=\Sigma_{-\infty}^{+\infty}\ a_k e^{ikt}$ is the Fourier series expansion of z. It is well known that $H^{1/2}(S^1)$ is embedded with a *compact injection* into $L^r(S^1)$ for any finite $r \geq 1$.

The scalar product in $(L^2(S^1))^{2N}$ will be denoted $\langle\ ,\ \rangle$:

$$\langle v, w\rangle = \int_0^{2\pi} v\cdot w\, dt$$

where $v\cdot w$ is the product in $\mathbf{R}^{2N}$. The scalar product $\langle\ ,\ \rangle$ extends naturally as the duality pairing between E and $E'=(H^{-1/2}(S^1))^{2N}$. (One identifies $(L^2(S^1))^{2N}$ with its dual.) Thus, for any $z\in E$, the action integral $\frac{1}{2}A(z)$ is well defined, where

$$A(z)=\langle \dot{z}, \mathfrak{g}z\rangle.$$

It will often prove to be convenient in the sequel to identify $\mathbf{R}^{2N}$ with $\mathbf{C}^N$ through the isomorphism

$$(p_1,\ldots,p_N,q_1,\ldots,q_N)\in\mathbf{R}^{2N} \quad\leftrightarrow\quad (p_1+iq_1,\ldots,p_N+iq_N)\in\mathbf{C}^N.$$

Thus, multiplication by $\mathfrak{g}$ in $\mathbf{R}^{2N}$ is replaced by multiplication by the scalar i in $\mathbf{C}^N$. Similarly, functions into $\mathbf{R}^{2N}$ will often be thought of as functions valued in $\mathbf{C}^N$. Both the real and the complex notations will be used in an interchangeable way in the following.

Since we will need some "semi-finite" dimensional approximation, we introduce the classical orthogonal basis associated with A. The eigenspaces of the operator $z\to-\mathfrak{g}\dot{z}$ in E consist of the subspaces $\mathbf{C}^N\{e^{ikt}\}$ where $k\in\mathbf{Z}$. Let E_j^m be the span of these eigenspaces for $j\leq k\leq m$ and set $E^m=E^m_{-\infty}$, $E^+=E_1^\infty$, $E^-=E^{-1}_{-\infty}$ and $E^\circ=E_0^\circ\equiv\mathbf{C}^N$. Then $E=E^+\oplus E^-\oplus E^\circ$ and A is positive (resp. negative) definite on E^+ (resp. on E^-). If $z=z^++z^-+z^\circ$ denotes the decomposition of z along $E=E^+\oplus E^-\oplus E^\circ$, we define the norm in E as

$$\|z\|_E^2=A(z^+)-A(z^-)+|z^\circ|^2_{\mathbf{R}^{2N}}. \tag{2.2}$$

This norm is a Hilbert norm and is easily checked to be equivalent to the norm induced on E by $\|\cdot\|_{H^{1/2}(S^1)}$ defined in (2.1).

For $z\in E$, the Lagrangian associated with (1.1) is

$$I(z)=\tfrac{1}{2}A(z)-\int_0^{2\pi}H(z)\,dt-\langle f,z\rangle.$$

The 2π-periodic solutions of (1.1) are the critical points of I on E. Indeed it will be seen later on that the critical points of I on E actually are in $(L^\infty(S^1))^{2N}$ and in fact are C^1 classical solutions of (1.1). Similarly, solutions of the autonomous Hamiltonian system (1.2) are the critical points in E of the Lagrangian

$$I^*(z)=\tfrac{1}{2}A(z)-\int_0^{2\pi}H(z)\,dt. \qquad \square$$

3. S^1-action and free vibrations

An element of S^1 will be denoted either by $e^{i\tau}$ or by τ ($\tau \in \mathbf{R}/2\pi\mathbf{Z}$). The group $S^1 \simeq \mathbf{R}/2\pi\mathbf{Z}$ acts naturally on E by $(T_\tau z)(t) = z(t+\tau)$. Notice that $T_\tau z = z$, $\forall \tau \in \mathbf{R}/2\pi\mathbf{Z}$, just means that $z \in E^\circ$. This action leaves invariant each subspace E_j^m of E. We recall that S^1 also acts on odd dimensional spheres S^{2k-1}, $k \in \mathbf{N}^* = \mathbf{N} \setminus \{0\}$ in the following standard way: Identify S^{2k-1} with $\{\zeta \in \mathbf{C}^k;\ \zeta = (\zeta_1, \ldots, \zeta_k),\ \Sigma_{i=1}^k |\zeta_i|^2 = 1\}$ through the same identification $\mathbf{R}^{2k} \simeq \mathbf{C}^k$ as in section 2 above. Then, for $e^{i\theta} \in S^1$, and $\zeta \in S^{2k-1}$ define $\hat{T}_\theta \zeta = e^{i\theta}\zeta = (e^{i\theta}\zeta_1, \ldots, e^{i\theta}\zeta_k)$; $\hat{T}_\theta \zeta \in S^{2k-1}$. A mapping $h: S^{2k-1} \to E$ is said to be *equivariant* if $h \circ \hat{T}_\tau = T_\tau \circ h$ for all $e^{i\tau} \in S^1$, and similarly for a mapping $h: S^{2k-1} \to S^{2k'-1}$ ($k, k' \in \mathbf{N}^*$).

A crucial property of the action $\hat{T}_\tau$ that will be used later is the following result which extends Borsuk's theorem to the S^1-action.

PROPOSITION 3.1. *Let j and k be integers with $1 \leq j \leq k$. There does not exist any continuous mapping $h: S^{2k-1} \to S^{2j-1}$ such that $h(e^{i\theta}\zeta) = e^{in\theta}h(\zeta)$, $\forall \zeta \in S^{2k-1}$, $\forall e^{i\theta} \in S^1$ where $n \in \mathbf{N} \setminus \{0\}$ is any fixed integer.*

For an elegant proof of this proposition, we refer the reader to Nirenberg [27]. (See also the proofs in Benci [12] and in Fadell–Rabinowitz [22].) Further extensions of the Borsuk theorem are developed in [21, 22, 27]. Note that as a particular case of Proposition 3.1 we see that there is no continuous mapping: $S^{2k-1} \to S^{2j-1}$ which is equivariant when $k > j$. Actually, we require the following slightly more general statement.

PROPOSITION 3.2. *Let $j, k \in \mathbf{N}^*$, $1 \leq j < k$. There does not exist any continuous mapping $h: S^{2k-1} \to S^{2j-1}$ such that*

$$h(e^{i\theta}\zeta) = \left(e^{in_1\theta}h_1(\zeta),\quad e^{in_2\theta}h_2(\zeta), \ldots, e^{in_j\theta}h_j(\zeta)\right)$$

for all $\zeta \in S^{2k-1}$ and all $e^{i\theta} \in S^1$, where

$$h_1(\zeta), \ldots, h_j(\zeta) \in \mathbf{C}, \quad h(\zeta) = (h_1(\zeta), \ldots, h_j(\zeta)) \in S^{2j-1}$$

and $n_1, \ldots, n_j \in \mathbf{N}^*$ are arbitrarily given.

Proof of Proposition 3.2. Suppose such an h exists. Let $\omega = \Pi_{i=1}^j n_i$ and $\omega_i = \omega/n_i$. Define $\hat{h}: S^{2k-1} \to S^{2j-1}$ by setting $\hat{h}(\zeta) = (\hat{h}_1(\zeta), \ldots, \hat{h}_j(\zeta))$ and $\hat{h}_s(\zeta) = |h_s(\zeta)|\, [h_s(\zeta)/$

$|h_s(\zeta)|]\,\omega_s(\hat{h}_s(\zeta)=0$ if $h_s(\zeta)=0)$ for $1\leq s\leq j$. Then $\hat{h}=S^{2k-1}\to S^{2j-1}$ is continuous and satisfies

$$\hat{h}(e^{i\theta}\zeta)=e^{i\omega\theta}\hat{h}(\zeta),\quad \forall\zeta\in S^{2k-1},\quad \forall e^{i\theta}\in S^1. \tag{3.1}$$

Since (3.1) is impossible by Proposition 3.1, the proof is complete. □

An essential feature of I^* in the following construction is its invariance under the S^1-action:

$$I^*(T_\tau z)=I^*(z),\quad \forall\tau\in\mathbf{R}/2\pi\mathbf{Z}.$$

We adapt to the present framework the method of Krasnosel'skii [24, Chapter VI] which concerned critical points of even functionals. Here, the S^1-action will play the role assumed by the $\mathbf{Z}/2\mathbf{Z}$-action in the even case. We use a "semi-Galerkin" method, that is, we first construct critical values for $I^*|_{E^m}$ and we then obtain critical values for I^* on E by letting $m\to+\infty$.

Define a set of mappings and a family of sets in E^m for $m\in\mathbf{N}^*$, $k\in\mathbf{N}$, $k\leq m-1$ by letting:

$$\mathcal{H}_m(k)=\{h\colon S^{2Nm-2k-1}\to E^m\setminus\{0\};\ \ h \text{ is continuous and equivariant with respect to the } S^1\text{-action}\}.$$

$$\mathcal{A}_m(k)=\{A\subset E^m\setminus\{0\};\ \ A=h(S^{2Nm-2k-1}),\quad h\in\mathcal{H}_m(k)\}.$$

We now let

$$c_m(k)=\sup_{A\in\mathcal{A}_m(k)}\min_{z\in A} I^*(z). \tag{3.2}$$

For the next result, weaker assumptions than (H 1) and (H 3) will suffice. In particular, (H 3) will be replaced here by

(H 4) $H(z)\leq a'|z|^{q+1}+b',\quad \forall z\in\mathbf{R}^{2N}$

where $q>1$, and $a', b'>0$ are some constants.

Let us recall that (H 2) implies that H is superquadratic at infinity, that is:

$$H(z)\geq a|z|^{p+1}-b,\quad \forall z\in\mathbf{R}^{2N} \tag{3.3}$$

where $p+1=1/\theta$, $(p>1)$, and $a, b>0$ are constants. Thus, the difference with (H 3) is that no restriction is imposed here on p and q. Without loss of generality we may assume in the following that $H(0)=0$ so that $I^*(0)=0$.

PROPOSITION 3.3. *Suppose* $H \in C^1(\mathbf{R}^{2N}, \mathbf{R})$ *satisfies* (H 2) *and* (H 4). *Then*

(i) $\forall k \leq m-1$, $0 \leq c_m(k-1) \leq c_m(k) < \infty$.

(ii) *For any* $k \in \mathbf{N}$, *there exist* $0 \leq \mu(k) \leq \nu(k) < +\infty$ (μ *and* ν *are independent of* m) *such that*

$$\forall m \geq k+1, \quad \mu(k) \leq c_m(k) \leq \nu(k) \quad \textit{and} \quad \lim_{k \to +\infty} \mu(k) = +\infty.$$

(iii) $\mu(k)$ *and* $\nu(k)$ *only depend on the constants* p, q, a', b', a *and* b *in* (3.3) *and* (H 4). ($\mu(k)$ *and* $\nu(k)$ *do not depend on the particular* H *satisfying* (H 2) *and* (H 4).)

Proof of Proposition 3.3. Proof of (i). Given any $A \in \mathscr{A}_m(k-1)$, there exists $A' \in \mathscr{A}_m(k)$ with $A' \subset A$. Indeed, think of $S^{2Nm-2k-1}$ as standardly imbedded in $S^{2Nm-2k+1}$ so that $S^{2Nm-2k-1}$ is invariant under the S^1 action on $S^{2Nm-2k+1}$. Now let $A = h(S^{2Nm-2k+1})$ with $h \in \mathscr{H}_m(k-1)$. Clearly, the restriction of h to $S^{2Nm-2k-1}$ belongs to $\mathscr{H}_m(k)$ and thus $A' = h(S^{2Nm-2k-1}) \in \mathscr{A}_m(k)$. It follows that $c_m(k-1) \leq c_m(k)$.

Choose any $A \in \mathscr{A}_m(k)$ (in the proof of (ii) below we indeed check that $\mathscr{A}_m(k)$ is not empty). For any real $r > 0$, $rA \in \mathscr{A}_m(k)$. Therefore,

$$c_m(k) \geq \min_{z \in rA} I^*(z).$$

Letting $r \searrow 0$, yields $c_m(k) \geq 0$. Lastly, that $c_m(k)$ is finite will be shown in the next argument.

Proof of (ii) and (iii). By (3.3) and (H 4),

$$a|z|^{p+1} - b \leq H(z) \leq a'|z|^{q+1} + b'.$$

Let

$$\Phi(z) = \tfrac{1}{2}A(z) - a' \int_0^{2\pi} |z|^{q+1} \quad (^1)$$

$$\Psi(z) = \tfrac{1}{2}A(z) - a \int_0^{2\pi} |z|^{p+1}.$$

Hence

$$\Phi(z) - 2\pi b' \leq I^*(z) \leq \Psi(z) + 2\pi b$$

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(1) From now on the measure dt is understood in integrals over $[0, 2\pi]$.

and thus

$$\gamma_m(k) - 2\pi b' \leq c_m(k) \leq \delta_m(k) + 2\pi b \tag{3.4}$$

where

$$\delta_m(k) = \sup_{A \in \mathscr{A}_m(k)} \min_{z \in A} \Psi(z) \tag{3.5}$$

$$\gamma_m(k) = \sup_{A \in \mathscr{A}_m(k)} \min_{z \in A} \Phi(z). \tag{3.6}$$

To prove the existence of $\nu(k)$ (thus also showing $c_m(k)$ to be finite) we require the following consequence of Proposition 3.2.

LEMMA 3.4. *For any $A \in \mathscr{A}_m(Nk)$, one has*

$$A \cap E^{k+1} \neq \emptyset.$$

Proof of Lemma 3.4. Let $A \in \mathscr{A}_m(Nk)$, i.e. $A = h(S^{2N(m-k)-1})$ with $h \in \mathscr{H}_m(Nk)$. Suppose $A \cap E^{k+1} = \emptyset$.

For $z \in E$, $j \in \mathbf{N}$, we denote $Q_j z = (1/2\pi) \int_0^{2\pi} z(t) e^{-ijt}$, $Q_j z \in \mathbf{C}^N$ is the jth coefficient in the Fourier series expansion of z. It is easily seen that

$$Q_j(T_\tau z) = e^{ij\tau} Q_j(z). \tag{3.7}$$

Since $A \cap E^{k+1} = \emptyset$ and $0 \notin A$, at least one of the coefficients $Q_j z$ is not zero, with $k+2 \leq j \leq m$. Therefore, setting

$$\sigma(z) = (|Q_{k+2}(z)|^2 + \ldots + |Q_m(z)|^2)^{-1/2} (Q_{k+2}(z), \ldots, Q_m(z)),$$

σ is well defined, continuous on A and $\sigma: A \to S^{2N(m-k-1)-1}$. Now let $\hat{h} = \sigma \circ h$; $\hat{h}: S^{2N(m-k)-1} \to S^{2N(m-k-1)-1}$. Using (3.7), and writing $\hat{h} = (\hat{h}_{k+2}, \ldots, \hat{h}_m)$, we see that $\hat{h}$ verifies

$$\hat{h}(e^{i\theta}\zeta) = \left(e^{i(k+2)\theta} \hat{h}_{k+2}(\zeta), \ldots, e^{im\theta} \hat{h}_m(\zeta)\right)$$

for all $\zeta \in S^{2N(m-k)-1}$ and $e^{i\theta} \in S^1$. But this is impossible by Proposition 3.2. The proof of Lemma 3.4 is complete. □

Proof of the existence of $\nu(k)$. Since $c_m(k-1) \leq c_m(k)$, it suffices to show that for each k, and each $m \geq k+1$, $c_m(Nk) \leq \nu(Nk)$. By Lemma 3.4 we know that each $A \in \mathscr{A}_m(Nk)$ intersects E^{k+1}. Hence, for all $A \in \mathscr{A}_m(Nk)$

$$\min_{z \in A} \Psi(z) \leq \max_{z \in E^{k+1}} \Psi(z)$$

whence

$$\delta_m(Nk) \leq \max_{z \in E^{k+1}} \Psi(z). \tag{3.8}$$

Now, for $z \in E^{k+1}$, one has

$$A(z) \leq (k+1) \|z\|^2_{L^2(0,2\pi)}.$$

Therefore, for $z \in E^{k+1}$,

$$\Psi(z) \leq ((k+1)/2) \|z\|^2_{L^2(0,2\pi)} - a\|z\|^{p+1}_{L^{p+1}(0,2\pi)}. \tag{3.9}$$

Since the right hand side of (3.9) is bounded from above independently of $z \in E^{k+1}$, we conclude, using (3.4), (3.8) and (3.9) that $c_m(Nk) \leq \nu(Nk)$. Whence,

$$c_m(k) \leq \nu(k) < \infty, \quad \forall k \in \mathbf{N}, \quad \forall m \geq k+1.$$

It is clear from the preceding argument that $\nu(k)$ only depends on p (that is on $\theta \in (0, \frac{1}{2})$), a and b. □

Proof of the existence of $\mu(k)$. We now construct a particular set $A \in \mathscr{A}_m(k)$. This will show incidentally that $\mathscr{A}_m(k)$ is not empty and thus that $c_m(k)$ is well defined. Let $k = Nk_0 - l$ with k_0, $l \in \mathbf{N}$, $0 \leq l < N$. For $\beta = (\beta_1, \dots, \beta_l) \in \mathbf{C}^l$, write $\beta = (\beta_1, \dots, \beta_l, 0, \dots, 0)$ so that $\beta \in \mathbf{C}^N$ and $\mathbf{C}^l \subset \mathbf{C}^N$. For $\beta = (\varrho_1 e^{i\theta_1}, \dots, \varrho_N e^{i\theta_N}) \in \mathbf{C}_N$, $(\varrho_1, \dots, \varrho_N \geq 0)$ and $j \in \mathbf{N}$, we use the notation:

$$(\beta)^j = \left(\varrho_1 e^{ij\theta_1}, \dots, \varrho_N e^{ij\theta_N}\right).$$

An element $\zeta \in S^{2Nm-2k-1}$ will be written as

$$\zeta = \left(\zeta_{k_0}, \zeta_{k_0+1}, \dots, \zeta_m\right) \quad \text{with} \quad \zeta_j \in \mathbf{C}^N \quad \text{for} \quad k_0+1 \leq j \leq N$$

and $\zeta_{k_0} \in \mathbf{C}^l \subset \mathbf{C}^N$, $\Sigma_{j=k_0}^m |\zeta_j|^2 = 1$. Now set

$$h(\zeta)(t) = \left(1/\sqrt{2\pi}\right) \sum_{j=k_0}^{m} (\zeta_j)^j j^{-1/2} e^{ijt}.$$

Thus, $h: S^{2Nm-2k-1} \to E^m_{k_0} \setminus \{0\} \subset E^m \setminus \{0\}$ is a continuous mapping. Since $(e^{i\theta}\zeta)^j = e^{ij\theta}(\zeta)^j$, it is clear that $h(e^{i\theta}\zeta)(t) = h(\zeta)(t+\theta)$, that is h is equivariant. Thus, $h \in \mathscr{H}_m(k)$ and $A = h(S^{2Nm-2k-1})$ verifies $A \in \mathscr{A}_m(k) \neq \varnothing$. Furthermore, by construction, $h(\zeta) \in S = \{z \in E;\, A(z) = 1\}$.

For any $z \in S$, let $\lambda(z)=\{a'(q+1)\int_0^{2\pi}|z|^{q+1}\}^{-1/(q-1)}$. This choice of $\lambda=\lambda(z)$ maximizes the function $\lambda \mapsto \Phi(\lambda z)$ when $A(z)=1$. Therefore,

$$\Phi[\lambda(z)z]=[(q-1)/2(q+1)]\left\{a'(q+1)\int_0^{2\pi}|z|^{q+1}\right\}^{-2/(q-1)} \tag{3.10}$$

Observe that $\lambda(z)$ is invariant under the S^1-action: $\lambda(T_\tau z)=\lambda(z)$. Let us now set $\hat{h}(\zeta)=\lambda[h(\zeta)]\,h(\zeta)$. Clearly, $\hat{h}$ is continuous and equivariant under the S^1-action. Let $\hat{A}=\hat{h}(S^{2Nm-2k-1})$ so that $\hat{A}\in\mathscr{A}_m(k)$. Then,

$$\gamma_m(k)\geq \min_{z\in A}\Phi(z). \tag{3.11}$$

Since $h(\zeta)\in S\cap E^m_{k_0}$, $\forall \zeta\in S^{2Nm-2k-1}$, we derive from (3.11)

$$\gamma_m(k)\geq \min_{z\in S\cap E^m_{k_0}}\Phi(\lambda(z)z).$$

Hence, using (3.10) we obtain

$$\gamma_m(k)\geq\gamma\min_{z\in S_{k_0}}\left\{\int_0^{2\pi}|z|^{q+1}\right\}^{-2/(q-1)} \tag{3.12}$$

for all $m\geq k+1$, $k\in\mathbf{N}^*$; $\gamma>0$ is some constant, and S_{k_0} denotes $S_{k_0}=S\cap(E^{k_0-1})^{\perp}$.

It just suffices to observe that the right hand side in (3.12) goes to $+\infty$ as $k\to+\infty$. By Hölder's inequality, one has

$$\|z\|_{L^{q+1}}\leq\|z\|^{\alpha}_{L^2}\|z\|^{1-\alpha}_{L^r} \tag{3.13}$$

for any r such that $2<q+1<r<\infty$ and where

$$\alpha=(2r-2q-2)(rq+r-2q-2)^{-1}\quad(0<\alpha<1).$$

Furthermore, since for $z\in S\cap E^+=S_1$, one has $A(z)=\|z\|^2_E=1$, and since $E\hookrightarrow L^r(S^1)$, there exists $C_r>0$ such that

$$\|z\|_{L^r}\leq C_r\|z\|_E=C_r,\quad\forall z\in S\cap E^+. \tag{3.14}$$

We also know that

$$\int_0^{2\pi}|z|^2\leq k_0^{-1}A(z)\leq k_0^{-1},\quad\forall z\in S_{k_0}. \tag{3.15}$$

Noticing that $\alpha \to 2(q+1)^{-1}$ as $r \to +\infty$, (3.13)–(3.15) yield that for any $\varepsilon>0$, there exists a constant $C_\varepsilon>0$ such that

$$\int_0^{2\pi} |z|^{q+1} \leq C_\varepsilon k^{-(1-\varepsilon)}, \quad \forall z \in S_{k_0}. \tag{3.16}$$

Therefore, it follows from (3.13) and (3.16) that $\forall \varepsilon>0$, there exists $\gamma_\varepsilon>0$ (γ_ε only depends on $\varepsilon>0$, a' and q) such that

$$c_m(k) \geq \gamma_\varepsilon k^{[2/(q-1)]-\varepsilon} - 2\pi b'. \tag{3.17}$$

Let us choose $\mu(k)=\gamma k^{1/(q-1)}-2\pi b'$ where γ is the constant γ_ε in (3.17) corresponding to $\varepsilon=1/(q-1)$. Then $\mu(k)\to\infty$ as $k\to\infty$ and $c_m(k)\geq\mu(k)$. Moreover, this holds uniformly for any Hamiltonian H satisfying (H4).

The proof of Proposition 3.3 is thereby complete. □

Remark 3.5. It will be seen in section 7 that the estimate (3.17) on the growth of $c_m(k)$ as $k\to+\infty$ can be much improved, at least along a subsequence of indices k. □

By letting $m\to+\infty$, we will not derive from the $c_m(k)$ the existence of critical values of I^*.

THEOREM 3.6. *Suppose* $H \in C^1(\mathbf{R}^{2N}, \mathbf{R})$ *satisfied* (H2) *and* (H4). *Then, problem*

$$\dot{z} = \mathfrak{g} H'(z) \tag{1.2}$$

possesses at least one non-constant 2π*-periodic solution.*

Remark 3.7. As we pointed out in the introduction, Theorem 3.6 is weaker than the result of Rabinowitz [33] where only (H2) is being assumed. We only indicate this theorem here because it is a natural byproduct of our method. Note also that the method we use here is quite different from the ones of [33] and of all the other related works quoted in section 1. □

Proof of Theorem 3.6. We will show the existence of an infinite family $(z_k)_{k\in\mathbf{N}}$ of 2π-periodic solutions of (1.2) such that $I^*(z_k)\to\infty$ as $k\to\infty$. Observing that if z is a constant solution of (1.2), then I^* is bounded from above by some constant, the existence of one non-constant 2π-periodic solution readily follows. Notice that it is not asserted, and actually not true in general, that z_k has 2π as a *minimal* period. Let us also remark that because of the possibility of dividing the period, the statement of Theorem 3.6 obviously implies the existence of infinitely many 2π-periodic solutions.

The proof of Theorem 3.6 will be divided into three steps:
(a) a truncation procedure of Rabinowitz,
(b) passage to the limit as $m \to +\infty$,
(c) conclusion.

(a) *A truncation procedure of Rabinowitz.* We use here the same type of truncation as the one introduced in Rabinowitz [29]. Let $R \geq 1$ be a real number. Consider a function $\omega_R \in C^\infty[0, +\infty)$ such that $0 \leq \omega_R \leq 1$, $\omega_R'(r) \leq 0$ on $\mathbf{R}_+$ and

$$\omega_R(r) = 1 \quad \text{if} \quad r \leq R \quad \text{and} \quad \omega_R(r) = 0 \quad \text{if} \quad r \geq R+1. \tag{3.18}$$

Define

$$H_R(z) = \omega_R(|z|)\, H(z) + (1-\omega_R(|z|))\, (a'|z|^{q+1} + b') \tag{3.19}$$

where a', b' and q are given by (H4). There exists $\hat{\theta}$, $0<\theta\leq\hat{\theta}<\frac{1}{2}$ and $C>0$ such that

$$H_R(z) \leq \hat{\theta} H_R'(z) \cdot z + C, \quad \forall z \in \mathbf{R}^{2N}, \tag{3.20}$$

with $\hat{\theta}$ *and* C *independent of* $R \geq 1$. (A straightforward calculation shows that it suffices to choose $\hat{\theta} = \max(\theta, 1/(q+1))$ with an adequate $C>0$.) Let us now define the truncated functional

$$I_R^*(z) = \tfrac{1}{2} A(z) - \int_0^2 H_R(z).$$

We require the following lemma.

LEMMA 3.8. *For all* $\beta>0$ *there exists* $\gamma=\gamma(\beta)>0$ *such that for all* $R\geq 1$ *and* $z \in E$ *satisfying* $(I_R^*)'(z)=0$, $I_R^*(z)\leq\beta$, *one has* $z \in L^\infty(S^1)$ *and* $\|z\|_{L^\infty} \leq \gamma(\beta)$.

Proof of Lemma 3.8. From $\dot{z} = \mathfrak{g} H_R'(z)$, we know that

$$H_R(z(t)) \text{ is independent of } t \in \mathbf{R}. \tag{3.21}$$

Furthermore, one has

$$\langle \dot{z}, \mathfrak{g} z \rangle = A(z) - \int_0^{2\pi} H_R'(z) \cdot z \tag{3.22}$$

and

$$\tfrac{1}{2} A(z) - \int_0^{2\pi} H_R(z) \leq \beta. \tag{3.23}$$

Using (3.20) in (3.22) and (3.23), we derive

$$\int_0^{2\pi} H_R(z) \leq C\beta.$$

Then, using (3.21) and (3.3), we obtain

$$\|z\|_{L^\infty} \leq \gamma = C\beta^{1/(p+1)} + C. \tag{3.24}$$

Here and henceforth, C will denote generically various positive constants. Thus, $\gamma(\beta)$ is independent of $R \geq 1$. Later on, in section 5, we shall derive similar estimates in a more general situation. □

(b) *Passage to the limit as $m \to +\infty$.* In the appendix, we prove in a more general setting that I_R^* satisfies the following Palais-Smale type condition.

CONDITION (P.S). *For any $b \in \mathbf{R}$ and any sequence $(z_j)_{j \in \mathbf{N}}$, $z_j \in E$, such that $I_R^*(z_j) \leq b$ and $(I_R^*)'(z_j) \xrightarrow{E'} 0$ there exists a convergent subsequence of (z_j) in E.*

For each $m \in \mathbf{N}$, the restriction of I_R^* to E^m also satisfies the condition (P.S) in E^m. Furthermore, I_R^* satisfies the following slightly different condition introduced by A. Bahri [6]:

CONDITION (P.S)*. *Let F_m be the restriction of I_R^* to E^m. For any $b \in \mathbf{R}$ and any sequence $(z_m)_{m \in \mathbf{N}}$ with $z_m \in E^m$, $F_m(z_m) \leq b$, $\|F'_m(z_m)\|_{(E^m)'} \to 0$, there exists a convergent subsequence of (z_m) in E.*

The proofs are given in the appendix. We require the last condition in order to use a "semi-Galerkin" approximation of E by E^m as $m \to +\infty$.

For $R \geq 1$, let us now define

$$c_{m,R}(k) = \sup_{A \in \mathscr{A}_m(k)} \min_{z \in A} I_R^*(z). \tag{3.25}$$

Since the constants $\hat{\theta}$ and C in (3.20) and q, a', b' in (H4) do not depend on $R > 1$, we know by Proposition 3.3 that

$$\mu(k) \leq c_{m,R}(k) \leq \nu(k), \quad \forall R \geq 1, \quad \forall k \in \mathbf{N}^*, \quad \forall m \geq k+1. \tag{3.26}$$

Here, $\nu(k)$ is associated with the constants $\hat{\theta}$ and C in (3.20) rather than with θ and C of (H2). Notice that H also satisfies (3.20) so that $c_m(k) \leq \nu(k)$.

LEMMA 3.9. *For any $k\geq 1$ and $m\geq k+1$ such that $\mu(k)>0$, $c_{m,R}(k)$ is a critical value of F_m, the restriction of I_R^* to E^m, for any $R\geq 1$.*

This fact is derived in very much the same way as in the case of critical point theory for every functionals. The reader is referred to e.g. Krasnosel'skii [24] or Rabinowitz [32] for results in this direction. A "deformation lemma" tailored for a functional which has an S^1-invariance (rather than the $\mathbf{Z}/2\mathbf{Z}$-invariance of even functionals) is detailed in Benci [13] and could easily be adapted to the present framework. When $H\in C^2(\mathbf{R}^{2N}, \mathbf{R})$, this deformation lemma, however, is not needed. Since they are classical, we omit the details of the proofs here. Let us just indicate a few remarks concerning the way to prove Lemma 3.9 when H is of class C^2:

(i) Then, I_R^* is a C^2 functional on E. Notice that without further restrictions on the growth of $H'(z)$ as $|z|\to+\infty$, it is not even known that I^* is of class C^1 on E. Indeed, E is not embedded in $L^\infty(S^1)$. Note however that I^* is of class C^2 on $E\cap L^\infty$.

(ii) The gradient flow generated by I_R^* in E (or by F_m in E^m) is equivariant with respect to the S^1-action on E (or E^m).

(iii) It is required that $\mu(k)>0$ because $I_R^*(0)=0$. (Recall that we always assume $H(0)=0$.) This allows one to consider deformations along gradient lines in $E\setminus\{0\}$ (or $E^m\setminus\{0\}$). □

By (3.26), for any $R\geq 1$, and any fixed $k\in\mathbf{N}^*$ such that $\mu(k)>0$, the sequence $\{c_{m,R}(k)\}_m$ has a convergent subsequence (along a sequence $m=m_j\to+\infty$) to some $\chi_R(k)$ with $0<\mu(k)\leq\chi_R(k)\leq\nu(k)<\infty$. Since by Lemma 3.9, $c_{m,R}(k)$ is a critical value of F_m, it follows at once from (P.S)* that $\chi_R(k)$ is a critical value of I_R^*.

(c) *Conclusion.* We now show that for any $k\in\mathbf{N}^*$ with $\mu(k)>0$, there exists a critical value of I^* in $[\mu(k), \nu(k)]$. This obviously yields Theorem 3.6 since $\mu(k)\to\infty$ as $k\to\infty$. Let us choose $R>0$ such that $R>\gamma(\nu(k))$, where $\gamma(\cdot)$ is given in Lemma 3.8. Let z_k be a critical point of I_R^* associated with the critical value $\chi_R(k)$. Then, $I_R^*(z_k)=\chi_R(k)\leq\nu(k)$ and $(I_R^*)'(z_k)=0$. Hence, by Lemma 3.8, one has $\|z\|_{L^\infty(S^1)}\leq\gamma(\nu(k))<R$. Therefore $I_R^*(z)=I^*(z)$ for z in a neighborhood of z_k in $E\cap L^\infty$. This implies that $(I^*)'(z_k)=0$ and $\chi_R(k)$ is a critical value of I^* (on the space $E\cap L^\infty$) as soon as $R>\gamma(\nu(k))$.

The proof of Theorem 3.6 is thereby complete. □

Remark 3.10. There are two reasons for introducing the truncation I_R^* and reasoning with it rather than directly with I^*. Firstly, as we already said, I^* needs not be in general a C^1-functional on E. Nevertheless, if $z\in L^\infty$, then $(I^*)'(z)$ is clearly well defined. This was for instance the case with $z=z_k$ in the preceding argument. Another reason is that even if I^* were to be a C^1-functional in E, it is not clear in general that I^* satisfies a condition of the type Palais-Smale. The truncated functional I_R^* however is shown in the appendix to verify such conditions. □

4. Stability of a non-trivial homotopy group

Throughout the remaining of the paper, we use the following notations for level sets of a functional $\varphi: E \to \mathbf{R}$, where $a \in \mathbf{R}$ and $m \in \mathbf{N}$:

$$[\varphi \geq a] = \{z \in E;\ \varphi(z) \geq a\}$$

$$[\varphi \geq a]_m = [\varphi \geq a] \cap E^m = \{z \in E^m\};\ \varphi(z) \geq a\}.$$

The purpose of this section is to derive the next result which plays a crucial role in the proof of Theorem 1.

THEOREM 4.1. *Let $\varphi \in C^0(E, \mathbf{R})$ be a continuous functional invariant under the S^1 action on E. Let $b_m(k) \in \mathbf{R}$ be the numbers defined by*

$$b_m(k) = \sup_{A \in \mathcal{A}_m(k)} \min_{z \in A} \varphi(z)$$

where $m, k \in \mathbf{N}^*$, $m \geq k+1$. Suppose that for some $\varepsilon > 0$ and some $m, k, b_m(k-1)+\varepsilon < b_m(k)-\varepsilon < \infty$. Then, for any set $W \subset E_m$ satisfying

$$[\varphi \geq b_m(k-1)+\varepsilon]_m \supset W \supset [\varphi \geq b_m(k)-\varepsilon]_m$$

there exists $x_0 \in W$ such that the homotopy group of order $2Nm-2k-1$ of W with base-point x_0 is not trivial:

$$\Pi_{2Nm-2k-1}(W, x_0) \neq 0.$$

Remark 4.2. The order of the homotopy group in Theorem 4.1 coincides exactly with the dimension of the spheres used for the computation of $b_m(k)$. □

Remark 4.3. Theorem 4.1 is a kind of "stability" property for the critical values $b_m(k)$ in the following heuristic sense. Suppose $\varphi = I^* \in C^2(E, \mathbf{R})$ and H satisfies (H2) and (H4). Then $b_m(k) = c_m(k)$ where $c_m(k)$ is defined in (3.2) and $c_m(k) \geq \mu(k)$ with $\mu(k) \to \infty$ as $k \to \infty$. Hence, inequalities like $c_m(k-1)+\varepsilon < c_m(k)-\varepsilon$, $\varepsilon > 0$, hold for infinitely many indices m and k. Then, if $\hat{I}$ is but a small perturbation from I^*, one expects the level sets of $\hat{I}$ to be "not too far apart" from those of I^*. More precisely, one expects to have inclusions of the type

$$[I^* \geq c_m(k-1)+\varepsilon]_m \supset [\hat{I} \geq a_k]_m \supset [I^* \geq c_m(k)-\varepsilon]_m$$

for some $a_k \in \mathbf{R}$. From Theorem 4.1 we then see that certain homotopy groups of the level sets $[\hat{I} \geq a_k]_m$ are not trivial. This, in turn, will imply the existence of critical values for the perturbed functional $\hat{I}$ as will be seen later on, in sections 7–9. □

Remark 4.4. The result of Theorem 4.1 is to be compared with a related theorem of Krasnosel'skii [24, Chapter VI] in the framework of even functionals on spheres. (See also Bahri–Berestycki [7, Theorem 3.1].) Indeed, let $S \subset H$ be the unit sphere of an infinite dimensional Hilbert space H. Let $J \in C^1(S, \mathbf{R})$ be an even functional which is bounded from below on S. Define

$$\gamma_k = \inf_{A \in M_k} \max_{x \in A} J(x)$$

where $M_k = \{A \subset S;\ A = g(S^{k-1})$ with g odd and continuous$\}$. Then, for $\gamma_{k-1}+\varepsilon < \gamma_k - \varepsilon$ and $W \subset S$ such that

$$\{x \in S : J(x) \leq \gamma_{k-1}+\varepsilon\} \subset W \subset \{x \in S; J(x) \leq \gamma_k - \varepsilon\}$$

one can show that W is not contractible to a point within the set $\{x \in S;\ J(x) \leq \gamma_k - \varepsilon\}$. Actually, one can show that $\Pi_k(W, x_0) \neq 0$ for some $x_0 \in W$. For more details, we refer the reader to [7, Remark 3.5] (see also Conner–Floyd [17]). □

Proof of Theorem 4.1. We argue by contradiction and we suppose that

$$\Pi_{2Nm-2k-1}(W, x_0) = 0, \quad \forall x_0 \in W. \tag{4.1}$$

By definition of $b_m(k)$, there exists $A \in \mathcal{A}_m(k)$ such that $A \subset [\varphi \geq b_m(k) - \varepsilon]_m$. Whence, there exists $h: S^{2Nm-2k-1} \to E^m \setminus \{0\}$, h being continuous and S^1-equivariant, and such that

$$h(S^{2Nm-2k-1}) = A \subset W.$$

Since we assumed (4.1), h is null homotopic in W. There exists a deformation $U: [0,1] \times S^{2Nm-2k-1} \to W$ satisfying

$$U(0, \zeta) = h(\zeta), \quad \forall \zeta \in S^{2Nm-2k-1} \tag{4.2}$$

$$U(1,\zeta)=x_0, \quad \forall\zeta\in S^{2Nm-2k-1} \tag{4.3}$$

$$U(t,\zeta_0)=x_0, \quad \forall t\in[0,1] \tag{4.4}$$

for some (arbitrarily) fixed exelements $x_0\in A$ and $\zeta_0\in S^{2Nm-2k-1}$. Let us now show that one can use such a deformation U to construct a continuous and S^1-equivariant mapping $\tilde{h}\colon S^{2Nm-2(k-1)-1}\to[\varphi\geq b_m(k-1)+\varepsilon]_m$.

An element $\xi\in S^{2Nm-2(k-1)-1}$ will be written as $\xi=(\zeta,\eta)$ with $\zeta\in\mathbf{C}^{Nm-k}$, $\eta\in\mathbf{C}$ and $|\xi|^2+|\eta|^2=1$. Define $\tilde{h}(\zeta,\eta)=T_{\eta/|\eta|}U(|\eta|,\ \bar{\eta}\zeta/|\eta\zeta|)$ if $\zeta\neq0$ and $\eta\neq0$; $\tilde{h}(\zeta,0)=h(\zeta)$ if $\eta=0$, $|\zeta|=1$; and $\tilde{h}(0,\eta)=T_\eta x_0$ for $\zeta=0$, $|\eta|=1$. Then, it is easily checked using (4.2), (4.3) and that h is equivariant under the S^1-action (e.g. $T_{\eta/|\eta|}h(\bar{\eta}\zeta/|\eta\zeta|)=h(\zeta/|\zeta|)$) that $\tilde{h}$ is continuous. From the definition of $\tilde{h}$ it is straightforward as well to see that $h(e^{i\theta}\zeta,e^{i\theta}\eta)=T_\theta\tilde{h}(\zeta,\eta)$. That is, $\tilde{h}$ is S^1-equivariant. Lastly, since $U([0,1]\times S^{2Nm-2k-1})\subset W$, and $W\subset[\varphi\geq b_m(k-1)+\varepsilon]_m$, we obtain:

$$\tilde{h}(S^{2Nm-2k+1})\subset[\varphi\geq b_m(k-1)+\varepsilon]_m. \tag{4.5}$$

Indeed, φ being invariant under the S^1 action, the set on the right hand side of (4.5) is invariant. Now, since $\tilde{h}\in\mathscr{H}_m(k-1)$, the set $\tilde{A}=\tilde{h}(S^{2Nm-2k+1})$ belongs to the class $\mathscr{A}_m(k-1)$. But we know from (4.5) that

$$\min_{\tilde{A}}\varphi>b_m(k-1)$$

which is impossible as it violates the definition of $b_m(k-1)$ as a sup.

The proof of Theorem 4.1 is thereby complete. □

5. Some truncation procedures

Some truncations will be required later on for technical purposes. We explicate these here and we derive some estimates. Throughout this section, we assume that H satisfies (H 1)–(H 3).

First, we require the same truncation procedure of Rabinowitz [29] that we have already recalled in section 3. Let ω_R be as in section 3 and H_R be defined by (3.18) and (3.19) for $R\geq1$. We know that H_R verifies

$$H_R(z)\leq\hat{\theta}H_R'(z)\cdot z+C, \quad \forall z\in\mathbf{R}^{2N} \tag{3.20}$$

where $\hat{\theta}\in(0,\frac{1}{2})$ and $C>0$ do not depend on $R\geq1$. The only motivation for this truncation is that since $H_R(z)=a'|z|^{q+1}+b'$ for $|z|$ large, the truncated functional

$$I_R^*(z)=\tfrac{1}{2}A(z)-\int_0^{2\pi} H_R(z)$$

is of class C^2 in E and verifies the conditions (P.S), $(\mathrm{P.S})_m$ (1) and (P.S)* (see the appendix and section 3 above).

We now introduce a second truncation concerning the term $\langle f, z\rangle$ in $I(z)$. Let χ be a C^∞-function: $\mathbf{R}^*\to\mathbf{R}$ such that $\chi(s)=1$, $\forall s\leq 1$, $\chi(s)=0$, $\forall s\geq 2$ and $\chi'(s)\leq 0$, $\forall s\geq 0$. Set $\tilde\chi_\varrho(s)=\chi(s/\varrho)$ for $\varrho\geq 1$. Then $\tilde\chi_\varrho$ has the following properties:

$$\tilde\chi_\varrho\in C^\infty(\mathbf{R}^+,\mathbf{R}) \tag{5.1}$$

$$0\leq\tilde\chi_\varrho\leq 1,\quad \tilde\chi_\varrho'(s)\leq 0,\quad \forall s\geq 0 \tag{5.2}$$

$$\tilde\chi_\varrho(s)=1,\quad \forall s\leq\varrho \quad\text{and}\quad \tilde\chi_\varrho(s)=0,\quad \forall s\geq 2\varrho. \tag{5.3}$$

$$|\tilde\chi_\varrho'(s)\cdot s|\leq m_0,\quad \forall s\geq 0, \tag{5.4}$$

where $m_0>0$ is a constant independent of $\varrho\geq 1$. Now let

$$\chi_\varrho(z)=\tilde\chi_\varrho\left(\int_0^{2\pi}|z|^{p+1}\right) \tag{5.5}$$

for $\varrho\geq 1$ and $z\in E$, (p is defined in (H3)). For $\varrho\geq 1$, $R\geq 1$, we define the truncated functional $I_{R,\varrho}$: $E\to\mathbf{R}$ by

$$I_{R,\varrho}(z)=\tfrac{1}{2}A(z)-\int_0^{2\pi} H_R(z)-\chi_\varrho(z)\langle f,z\rangle.$$

We also set

$$I_R(z)=\tfrac{1}{2}A(z)-\int_0^{2\pi} H_R(z)-\langle f,z\rangle.$$

Notice that if $\|z\|_{L^\infty}<R$ and $\|z\|_{L^{p+1}}^{p+1}<\varrho$, one has $I_{R,\varrho}(z)=I(z)$. On the other hand, if $\|z\|_{L^{p+1}}^{p+1}>2\varrho$, one has $I_{R,\varrho}(z)=I_R^*(z)$. The reason for introducing the cut-off function $\chi_\varrho(z)$ is to allow one to get certain precise estimates on the size of the perturbation $|I_R^*(z)-I_{R,\varrho}(z)|$ as will be clear in the conclusion of the proof (see sections 8 and 9) (2).

(1) $(\mathrm{P.S})_m$ is the same condition as (P.S) but refers to the restriction to the subspace E^m rather than to the functional in the whole space E.

(2) Observe that $I_R^*-I_R$ is unbounded on E.

In the sequel, for any functional $\varphi: E \to \mathbf{R}$, we will denote by $\varphi^m = \varphi|_{E^m}$ its restriction to E^m. The next three lemmas contain the estimates that will be required later on.

LEMMA 5.1. *$|I_R^*(z) - I_{R,\varrho}(z)| \leq C\varrho^{1/(p+1)}$ where $C>0$ is a constant, for all $R \geq 1$, $\varrho \geq 1$ and $z \in E$.*

LEMMA 5.2. *There exists two constants $\alpha>0$, $\beta \geq 0$ such that for any $\varrho \geq 1$ and for any $R \geq 1$, one has the following property. If z is a critical point of $I_{R,\varrho}$ such that $I_{R,\varrho}(z) \leq \alpha\varrho - \beta$, then z is a critical point of I_R. Similarly, if for $m \geq 1$, $z \in E^m$ is such that $I^m_{R,\varrho}(z) \leq \alpha\varrho - \beta$ and $(I^m_{R,\varrho})'(z) = 0$, then $(I^m_R)'(z) = 0$.*

LEMMA 5.3. *Let $\varrho_0 \geq 1$ be given. There exists $R_0(\varrho_0) \geq 1$ such that for any ϱ, $1 \leq \varrho \leq R_0$, and $R \geq R_0(\varrho_0)$ the following holds, where α and β are given by Lemma 5.2: Any critical point z of $I_{R,\varrho}$ such that $I_{R,\varrho}(z) \leq \alpha\varrho - \beta$, is a critical point of I; $z \in L^\infty$, and $I = I_{R,\varrho}$ on a neighborhood of z in $E \cap L^\infty$.*

The remaining of this section is devoted to the proofs of these three estimates which are essentially technical.

Proof of Lemma 5.1. By (5.3), $\chi_\varrho(z)=0$ whenever $\|z\|_{L^{p+1}} > (2\varrho)^{1/(p+1)}$. Thus, since $\chi_\varrho \leq 1$, one has

$$|I_R^*(z) - I_{R,\varrho}(z)| \leq \chi_\varrho \|f\|_{L^s} \|z\|_{L^{p+1}} \leq C\varrho^{1/(p+1)}$$

in all cases, where $s^{-1} + (p+1)^{-1} = 1$. □

Proof of Lemma 5.2. In all the following, as usual, C denotes generically the various positive constants that will be called. Let z be a critical point of I_R such that $I_{R,\varrho}(z) \leq \delta$. That is,

$$(I_{R,\varrho})'(z) \equiv -\mathrm{g}\dot{z} - H_R'(z) - T_\varrho'(z) = 0, \tag{5.6}$$

where

$$T_\varrho'(z) = \bar{\chi}_\varrho'\left(\|z\|^{p+1}_{L^{p+1}}\right)(p+1)\langle f, z\rangle |z|^{p-1} z + \chi_\varrho(z) f.$$

The expression of $(I_{R,\varrho})'$ in (5.6) means that we have identified $L^2(S^1)^{2N}$ with its own dual space and thus, the duality pairing between E and E' is defined by $\langle\ ,\ \rangle$. Let us estimate $|\langle T_\varrho'(z), z\rangle|$. Using $|\langle f, z\rangle| \leq \|f\|_{L^s} \|z\|_{L^{p+1}}$ and (5.4), we obtain

$$|\langle T_\varrho'(z), z\rangle| \leq C\|z\|^{-(p+1)}_{L^{p+1}} \|z\|_{L^{p+1}} \|z\|^{p+1}_{L^{p+1}} + C\|z\|_{L^{p+1}}.$$

Whence,

$$|\langle T'_\varrho(z), z\rangle| \leq C\|z\|_{L^{p+1}}. \tag{5.7}$$

From $(I_{R,\varrho})'(z)=0$ and $I_{R,\varrho}(z) \leq \delta \leq \alpha\varrho-\beta$, where α and β will be determined later on, we get

$$A(z)-\int_0^{2\pi} H'_R(z)\cdot z - \langle T'_\varrho(z), z\rangle = 0 \tag{5.8}$$

$$\tfrac{1}{2}A(z)-\int_0^{2\pi} H_R(z) \leq \delta + C\|z\|_{L^{p+1}}. \tag{5.9}$$

Subtracting (5.8) (after multiplication by $\frac{1}{2}$) from (5.9) yields:

$$\int_0^{2\pi} \left\{\tfrac{1}{2}H'_R(z)\cdot z - H_R(z)\right\} dt \leq \delta + C\|z\|_{L^{p+1}}.$$

Using (3.20) we deduce

$$[(2\theta)^{-1}-1]\int_0^{2\pi} H_R(z)\, dt \leq \delta + C + C\|z\|_{L^{p+1}}.$$

But since $H_R(z) \geq a|z|^{p+1}-b$ uniformly in R, we have

$$\int_0^{2\pi} |z|^{p+1} \leq C\delta + C\left\{\int_0^{2\pi} |z|^{p+1}\right\}^{1/(p+1)} + C. \tag{5.10}$$

This implies that $\|z\|_{L^{p+1}}$ is bounded. More precisely, (5.10) yields

$$\int_0^{2\pi} |z|^{p+1} \leq C\alpha\varrho - C\beta + C.$$

Therefore, by choosing adequately $\alpha>0$ and $\beta>0$, we obtain

$$\int_0^{2\pi} |z|^{p+1} \leq \varrho - 1.$$

Hence, it follows that $I_R = I_{R,\varrho}$ on a neighborhood of z and E and then, $I'_R(z)=0$. The preceding próof remains valid if one considers the restrictions of I_R and $I_{R,\varrho}$ to E^m, with $z \in E^m$.

Proof of Lemma 5.3. Let $\delta>0$ be a given real and $z \in E$ a critical point of I_R such that $I_R(z) \leqslant \delta$. As before, using $\langle I_R'(z), z\rangle = 0$, we derive

$$\int_0^{2\pi} H_R(z), \quad \int_0^{2\pi} H_R'(z)\cdot z, \quad \int_0^{2\pi} |z|^{p+1} \leqslant C \tag{5.11}$$

where the constant C only depends on δ and does not depend on R. We are now going to derive an L^∞-estimate for z. Thereby we show at the same time that critical points of I_R in E are classical solutions of (1.1) for R large enough.

Let $t_0 \in [0, 2\pi]$ be fixed and $z_0 = z(t_0)$. One has

$$C \geqslant \int_0^{2\pi} H_R(z(t)) = \int_0^{2\pi} \{H_R(z(t)) - H_R(z_0)\}\, dt + 2\pi H_R(z_0).$$

Therefore, since $(d/dt)\, H_R(z) = H_R'(z)\cdot \dot z = \dot z \cdot \mathfrak{g}f$, we derive

$$2\pi H_R(z_0) \leqslant C - \int_0^{2\pi} dt \int_{t_0}^{t} \dot z(\tau)\cdot \mathfrak{g}f(\tau)\, d\tau.$$

An integration by parts yields:

$$2\pi H_R(z_0) \leqslant C - \int_0^{2\pi} dt \{z(\tau)\cdot \mathfrak{g}f(\tau)\}\Big|_{\tau=t_0}^{\tau=t} + \int_0^{2\pi} dt \int_{t_0}^{t} z(\tau)\cdot \mathfrak{g}\dot f(\tau)\, d\tau.$$

Hence, since $f \in C^1[0, 2\pi]$, we obtain

$$H_R(z_0) \leqslant C + C|z_0| + C\|z\|_{L^1}. \tag{5.12}$$

By (5.11), $\|z\|_{L^1}$ is bounded. Using the fact that $H_R(z_0) \geqslant a|z_0|^{p+1} - b$, we derive from (5.12) that $|z_0| \leqslant C$, with C independent of t_0. Therefore, $z \in L^\infty$ and

$$\|z\|_{L^\infty} \leqslant C \tag{5.13}$$

where the constant C is independent of $R \geqslant 1$ and only depends on δ (and on the constants a, b, ϱ ...).

We now choose $\delta = \alpha\varrho_0 - \beta$ with α and β being given in Lemma 5.2. We denote by $C = C(\varrho_0)$ the corresponding constant in (5.13), and choose $R_0(\varrho_0)$ such that $R_0(\varrho_0) > C(\varrho_0)$. By Lemma 5.2 and by (5.13) it then follows that for any $R \geqslant R_0(\varrho_0)$ and for any $z \in E$ such that $I_{R,\varrho}(z) \leqslant \alpha\varrho - \beta$, $1 \leqslant \varrho \leqslant \varrho_0$, $(I_{R,\varrho})'(z) = 0$, one has $I_{R,\varrho}(z) = I(z)$ on a neighborhood of z in $E \cap L^\infty$ and thus, $I'(z) = 0$.

The proofs of Lemmas 5.1–5.3 are thereby complete. □

Remark 5.4. In the proof of Lemma 5.3, we have used the fact that $f \in C^1(\mathbf{R}, \mathbf{R}^{2N})$. This is the only place where this hypothesis plays a role. Actually, in the above proof we have only used that $f \in W^{1,\sigma}(S^1, \mathbf{R}^{2N})$ with $\sigma^{-1}+(p+1)^{-1}=1$. Thus Theorem 1 is valid with $f \in W^{1,\sigma}$. It is natural to conjecture that one should be able to still weaken this assumption. □

An inspection of the preceding proofs (compare (5.13)) shows that we have actually derived the following more precise a priori estimate that will be useful later.

LEMMA 5.5. *Let $\delta \in \mathbf{R}$ be given. For any $\varrho \geq 1$, $R \geq 1$ such that $\alpha\varrho-\beta \geq \delta$ and $R \geq R_0(\varrho)$ and for any $z \in E$ with $(I_{R,\varrho})'(z)=0$ and $I_{R,\varrho}(z) \leq \delta$, one has $z \in L^\infty$ and $\|z\|_{L^\infty}<R$. Moreover, there exists a constant $C=C(\delta)$, independent of ϱ and R, ($\alpha\varrho-\beta \geq \delta$, $R \geq R_0(\varrho)$) such that $\|z\|_{L^\infty} \leq C$.*

6. Homotopy groups of level sets and critical points

In this section, we recall together with their proofs some results of A. Bahri in Morse theory in the way they apply in the present framework. (See A. Bahri [5, 6] for similar results in a more general setting.) The main objective here will be to extend a theorem in J. T. Schwartz [34] (Theorem 7.3, page 183) to situations which may be degenerate in the sense of Morse theory. This result concerns the triviality of certain homotopy groups of level sets. The extension to the degenerate case is made possible by using a version of the powerful "resolution" method of critical points developed by Marino–Prodi [26].

We first require some notations. For a C^1 functional $f\colon E \to \mathbf{R}$ and $\delta \in \mathbf{R}$, we denote

$$Z^\delta(f) = \{z \in E;\ \ f'(z) = 0,\ \ f(z) \leq \delta\}.$$

In the particular case of the functional I, this means

$$Z^\delta(I) = \{z \in E \cap L^\infty(S^1);\ \ I'(z) = 0,\ \ I(z) \leq \delta\}.$$

(Indeed, $I'(z)$ is clearly well defined when $z \in L^\infty(S^1)$; this modification is not required for the truncated functionals which are of class C^1 on E.)

In order to simplify the wording of some statements below, we denote by $\mathcal{F}_\delta$ the family of all truncated functionals $I_{R,\varrho}$ corresponding to $R \geq R_0(\varrho)$ and $\alpha\varrho-\beta \geq \delta$:

$$\mathcal{F}_\delta = \{I_{R,\varrho};\ \ R \geq 1,\ \ \varrho \geq 1,\ \ \alpha\varrho-\beta \geq \delta,\ \ R \geq R_0(\varrho)\}$$

(α, β and R_0 are given by Lemmas 5.2 and 5.3). From section 5 we know that for any $\delta \in \mathbf{R}$, $Z^\delta(G)=Z^\delta(I)$ for any $G \in \mathscr{F}_\delta$. Furthermore, we know that if $z \in Z^\delta(G)$ and $G \in \mathscr{F}_\delta$, one has $G=I$ on a neighborhood of z in $E \cap L^\infty$. Hence, for such a z, the linear operator $G''(z) \in \mathscr{L}(E, E')$ is independent of the particular choice of $G \in \mathscr{F}_\delta$ (that is, independent of ϱ and R). We shall denote it: $G''(z)=I''(z)$. It should be kept in mind that this is only a notational convention since I is not a C^2-functional on E in general. However, this notation is justified by the fact that if $z \in Z^\delta(G)$, then $z \in L^\infty$ and $I''(z)$ can be defined e.g. on $E \cap L^\infty(S^1)$, and $I''(z)$ then coincides with $G''(z)$.

We now define a kind of coindex associated with the critical values below a given level δ.

Definition 6.1. For $\delta \in \mathbf{R}$, $j_0(\delta) \in \mathbf{Z} \cup \{\pm\infty\}$ is defined to be the least integer j such that

$$\langle I''(v)h, h\rangle > 0, \quad \forall h \in (E^j)^\perp \setminus \{0\}, \quad \forall v \in Z^\delta(I).$$

Remark 6.2. In case $Z^\delta(I)=\varnothing$, we set $j_0(\delta)=+\infty$. It will be seen below that $j_0(\delta)>-\infty$, $\forall \delta \in \mathbf{R}$ and $j_0(\delta)<\infty$ provided $Z^\delta(I)\neq\varnothing$. Notice that on E^m, in a non-degenerate situation, $2N(m-j_0(\delta))$ serves as a lower bound for the Morse coindex of any critical point in $Z_\delta(I)$. (If z is a non-degenerate critical point of I, the coindex is the maximum dimension of a subspace on which the bilinear form $(h_1, h_2) \mapsto \langle I''(v)h_1, h_2\rangle$ is positive definite.) □

PROPOSITION 6.3. *For any $\delta \in \mathbf{R}$ such that $Z^\delta(I)\neq\varnothing$, then $-\infty<j_0(\delta)<+\infty$. Furthermore, in this case, there exists $\varepsilon>0$ such that*

$$\langle I''(v)h, h\rangle \geq \varepsilon \|h\|_E^2, \quad \forall h \in (E^{j_0}), \quad \forall v \in Z^\delta(I).$$

Proof. Let $G=I_{R,\varrho} \in \mathscr{F}_\delta$; let $z \in Z^\delta(I)$. By Lemmas 5.2 and 5.3, we know that $\chi_\varrho=1$ on a neighborhood of z in E. Hence,

$$\langle G''(z)h, h\rangle = A(h) - \int_0^{2\pi} H_R''(z)h^2.$$

By Lemma 5.5, $z \in L^\infty$ and $\|z\|_{L^\infty}<R$. Hence,

$$\langle G''(z)h, h\rangle = \langle I''(z)h, h\rangle = A(h) - \int_0^{2\pi} H''(z)h^2, \quad \forall z \in Z^\delta(I). \tag{6.1}$$

Furthermore, since $\|H''(z)\|_{L^\infty} \leq C=C(\delta)$, for any $z \in Z^\delta(I)$, it follows from (6.1) that

$$\langle I''(z)h,h\rangle \geq A(h) - C\int_0^{2\pi} h^2, \tag{6.2}$$

$$\langle I''(z)h,h\rangle \geq A(h) + C\int_0^{2\pi} h^2. \tag{6.3}$$

Now let us recall that

$$A(h) \leq j\int_0^{2\pi} h^2\,dt, \quad \forall h \in E^j, \quad \forall j \in \mathbf{Z}, \quad j \leq 0,$$

and

$$A(h) + j\int_0^{2\pi} h^2\,dt, \quad \forall h \in (E^{j-1})^\perp, \quad j \in \mathbf{Z}, \quad j \geq 0.$$

Thus, from (6.3), it is clear that $j_0(\delta) > -\infty$. Indeed, if $j \leq -C-1$, then for $h \in (E^{j-1})^\perp \cap E^j$, one has $\langle I''(z)h,\, h\rangle \leq -\|h\|^2_{L^2}$. On the other hand, (6.2) shows that if one chooses $j \geq C+1$, then for any $h \in (E^j)^\perp$, one has

$$\langle I''(z)h,h\rangle \geq \|h\|^2_{L^2}.$$

Hence, $j_0(\delta) < +\infty$.

Let us now prove the second point of Proposition 6.3. Let $j_0 = j_0(\delta)$ be as in Definition 6.1. Thus,

$$\langle I''(z)h,h\rangle \geq 0, \quad \forall h \in (E^{j_0})^\perp \setminus \{0\}, \quad \forall z \in Z^\delta(I). \tag{6.4}$$

To conclude we argue by contradiction. Suppose that there exist a sequence $(h_n) \subset (E^{j_0})^\perp$, $\|h_n\|_E = 1$, and a sequence $(z_n) \subset Z^\delta(I)$ such that

$$0 < \langle I''(z_n)h_n, h_n\rangle \leq \varepsilon_n, \quad \forall n \in \mathbf{N}^*, \quad \text{with} \quad \varepsilon_n \to 0 \quad \text{as} \quad n \to +\infty. \tag{6.5}$$

By condition (P.S) (see the appendix), $Z^\delta(I)$ is compact in E, and by Lemma 5.5, $Z^\delta(I)$ is bounded in the L^∞ norm. Therefore, we can extract from (z_n) and (h_n) subsequences, denoted again by (z_n), (h_n) such that

$$z_n \to z \in Z^\delta(I) \quad \text{in } E \text{ (strong topology)}$$

$$H''(z_n) \to H''(z) \quad \text{in the weak * topology of } L^\infty$$

$$h_n \to h \quad \text{in the weak topology of } E$$

$$h_n \to h \quad \text{in } L^2 \quad \text{(strong topology)}.$$

(Recall that the injection $E \hookrightarrow L^2$ is compact.) Thus,

$$\int_0^{2\pi} H''(z_n)h_n^2 \to \int_0^{2\pi} H''(z)h^2, \quad \text{as} \quad n \to +\infty. \tag{6.6}$$

Lastly, let $h_n = h_n^- + h_n^+$ denote the orthogonal decomposition of h_n such that $h_n^- \in (E^{j_0})^\perp \cap (E^- \oplus E^\circ)$ and $h_n^+ \in (E^{j_0})^\perp \cap E^+$. (If $j_0 \geq 1$, then $h_n = h_n^+$.) Because $(E^{j_0})^\perp \cap (E^- \oplus E^\circ)$ is finite dimensional, we have

$$h_n^- \to h^- \quad \text{in } E \text{ (strongly)},$$

while

$$h_n^+ \to h^+ \quad \text{in the weak topology of } E$$

with

$$h = h^- + h^+.$$

Since $A(h_n) = A(h_n^-) + \|h_n^+\|_E^2$, we know that $A(h) \leq \lim_{n \to +\infty} A(h_n)$. From (6.5), (6.6) and the expression of $I''(z_n)$ we derive

$$\langle I''(z)h, h \rangle = A(h) - \int_0^{2\pi} H''(z)h^2 \leq 0. \tag{6.7}$$

But on the other hand, $h \in (E^{j_0})^\perp$ and comparing (6.4) and (6.7) we derive $h=0$. Then, it follows from (6.5) that

$$\lim_{n \to +\infty} A(h_n) = \lim_{n \to +\infty} \int_0^{2\pi} H''(z_n)h_n^2 = 0.$$

Whence, $h_n \to 0$ strongly in E which contradicts the assumption $\|h_n\|_E = 1$.

The proof of Proposition 6.3 is thereby complete. □

The motivation for introducing the index $j_0(\delta)$ is that it allows one to show the triviality of certain homotopy groups of level sets of functionals in the class $\mathscr{F}_\delta$. The main result of this section is the following.

THEOREM 6.4. *Let $\delta \in \mathbf{R}$ and G be any functional in $\mathscr{F}_\delta$. There exists an integer $m_0 = m_0(\delta, G) \in \mathbf{N}^*$ such that for all $m \geq m_0(\delta, G)$, the homotopy groups of $[G \geq \delta]_m$ are trivial for all orders up to $2N(m - j_0(\delta)) - 1$, provided $Z^\delta(I) \cap [G \geq \delta] = \varnothing$. That is, if δ is* *not a critical value of I, then* $\pi_l([G\geq\delta]_m)=0$, $\forall l\leq 2N(m-j_0(\delta))-1$, $\forall m\geq m_0(\delta, G)$, $\forall G\in\mathscr{F}_\delta$.

Remark 6.5. This theorem is due to A. Bahri. Results in the spirit of Theorem 6.4, but in a more general setting are presented in [5, 6]. As will be seen in the proof, it is related to and extends classical results from Morse theory (compare Theorem 7.3 (page 183) in J. T. Schwartz [34]). Theorem 6.4 will allow us in the next section to derive an estimate from below on the growth of the critical values $c_m(k)$ (as $k\to+\infty$), where the $c_m(k)$ are defined in (3.2). Furthermore, it plays a crucial role in the conclusion of the proof of Theorem 1 in section 9. □

Theorem 6.4 relies on the next lemma, which is an adaptation to the present framework of a method of Marino and Prodi [26]. This method concerns the "resolution" of a compact set of critical points for a given functional into a finite number of non degenerate critical points for functionals which can be arbitrarily near, in an appropriate sense, to the given functional. The precise result of Marino and Prodi that we use will be recalled below: See Proposition 6.8.

LEMMA 6.6. *Let* $\delta\in\mathbf{R}$ *and* $G\in\mathscr{F}_\delta$ *be such that* δ *is not a critical value of* G *(that is,* δ *is not a critical value of* I*). There exists* $m_0=m_0(\delta)$ *such that for any* $m\geq m_0(\delta)$*, there exists* $\tilde G\in C^2(E^m, \mathbf{R})$ *with the following properties:*

(i) $[\tilde G>\delta]_m=[G>\delta]_m$.

(ii) δ *is not a critical value of* $\tilde G$.

(iii) *The critical points of* $\tilde G$ *(if any) which are contained in the set* $\{z\in E^m;\ \tilde G(z)\leq\delta\}$ *are in finite number, are non-degenerate and have a coindex greater than or equal to* $2N(m-j_0(\delta))$.

(iv) $\tilde G$ *satisfies the condition* $(\mathrm{P.S})_m$.

Remark 6.7. In (iv), $(\mathrm{P.S})_m$ refers to the Palais-Smale condition in the space E^m. (See the appendix for the precise definition.) In (iii), the non-degeneracy of a critical point z means that $\tilde G''(z)\in\mathscr{L}(E^m, (E^m)')$ is an invertible operator. The coindex of z is then defined to be the maximum dimension of a subspace of E^m on which the Hessian bilinear form $(h_1, h_2)\mapsto\langle\tilde G''(z)h_1, h_2\rangle$ is positive definite. The index is defined to be the maximum dimension of a subspace on which this form is negative definite. Notice that in the present framework, due to the presence of E^- in E^m, the index is, in general, infinite. □

The proof of the lemma is essentially technical. Its derivation will be done through a sequence of lemmas and will take up the remaining of this section. But before that, we first apply Lemma 6.6 and complete the proof of Theorem 6.4.

Proof of Theorem 6.4. Let $m \geq m_0$ with $m_0 = m_0(\delta)$ being as in Lemma 6.1. Since $[\tilde{G} \geq \delta]_m = [\tilde{G} \geq \delta]_m$, we want to show that $\Pi_l([\tilde{G} \geq \delta]_m) = 0$ for all $l \leq 2N(m - j_0(\delta))$. By Lemma 6.6 we are in the situation where classical Morse theory applies (for $\tilde{G}$ on the set $\{z \in E^m;\ \tilde{G}(z) \leq \delta\}$). Hence, we know (compare J. T. Schwartz [34, Theorem 7.3]) that $\Pi_l(E^m, [\tilde{G} \geq \delta]_m) = 0$ for all $l \leq 2N(m - j_0(\delta))$. Indeed, l is smaller than the coindex of any critical point of $\tilde{G}$ in the set $\{z \in E^m;\ \tilde{G}(z) \leq \delta\}$. Here, $\Pi_l(E^m, [\tilde{G} \geq \delta]_m)$ denotes the *relative* homotopy group of order l. Let us write down the exact sequence for this pair:

$$\ldots \to \Pi_l(E^m, [\tilde{G} \geq \delta]_m) \to \Pi_{l-1}([\tilde{G} \geq \delta]_m, x) \to \Pi_{l-1}(E^m, x) \to \ldots \tag{6.8}$$

where x is any given point in $[\tilde{G} \geq \delta]_m$. Since $\Pi_{l-1}(E^m, \cdot) = 0$, the exact sequence (6.8) reads for all $l \leq 2N(m - j_0(\delta))$:

$$0 \to \Pi_{l-1}([\tilde{G} \geq \delta]_m) \to 0. \tag{6.9}$$

Therefore, $\Pi_l([G \geq \delta]_m) = 0$, $\forall l \leq 2N(m - j_0(\delta)) - 1$. □

The proof of Theorem 6.4 is thereby complete but for the proof of Lemma 6.6 to which we now turn. This lemma is mainly a consequence of the following result due to Marino and Prodi [26] (Theorem 2.2, page 14). We denote by d the distance in a given Hilbert space.

PROPOSITION 6.8. *Let Ω be a C^2 open subset of a Hilbert space H and let $f \in C^2(\Omega, \mathbf{R})$. Assume that f' is a Fredholm operator (of null index) on the critical set $Z(f) = \{u \in \Omega, f'(u) = 0\}$. Suppose furthermore that $Z(f)$ is compact. Then, for any $\varepsilon_0 > 0$ and $\eta_0 > 0$, there exists $g \in C^2(\Omega, \mathbf{R})$ verifying the following properties.*

(g.1) $g(u) = f(u)$, *if* $d(u, Z(f)) \geq \eta_0$.

(g.2) $\|g(u) - f(u)\|_H \leq \varepsilon_0$, $\|g'(u) - f'(u)\|_{H'} \leq \varepsilon_0$, $\forall u \in \Omega$.

(g.3) $\|g''(u) - f''(u)\|_{\mathscr{L}(H, H')} \leq \varepsilon_0$, $\forall u \in \Omega$.

(g.4) *The critical points of g (if any) are in finite number and are non-degenerate.*

(g.5) *If f satisfies* (P.S), *then g can be chosen to satisfy* (P.S) *too.*

For the proof of this result, we refer the reader to Marino and Prodi [26]. There is just one minor modification with respect to Theorem 2.2 in [26]. The statement of Marino and Prodi actually concerns functionals which are defined on a Riemanian manifold modelled on a Hilbert space H and (g.3) does not appear in the statement given in [26]. It is easily checked, however, by an inspection of the proof in [26], that the argument of Marino and Prodi carries, virtually without change, to obtain (g.3) as well. □

To derive Lemma 6.6 from Proposition 6.8, we require several technical lemmas.

LEMMA 6.9. *Let $\delta \in \mathbf{R}$ and G be any functional in the class $\mathcal{F}_\delta$. Then, $G \in C^2(E, \mathbf{R})$ and G satisfies the Palais-Smale type conditions* (P.S), (P.S)$_m$ *and* (P.S)*. (1)

The proof of Lemma 6.9 is given in the appendix. In the following, for a closed set $F \subset E$, and $\eta > 0$, we denote by $N_\eta(F)$ the uniform open η-neighborhood about F: $N_\eta(F) = \{x \in E;\ d(x, F) < \eta\}$. For the restriction G^m of $G \in \mathcal{F}_\delta$ to E^m, we also use the notation

$$Z^c(G^m) = \{z \in E^m;\quad (G^m)'(z) = 0,\quad G^m(z) \leq c\}.$$

LEMMA 6.10. *Let $\delta \in \mathbf{R}$ and $G \in \mathcal{F}_\delta$. Assume that δ is not a critical value of G (or equivalently of I). Then, there exists $\tilde{m}_1 = \tilde{m}_1(\delta) \in \mathbf{N}^*$ such that δ is not a critical value of G^m, $\forall m \geq \tilde{m}_1$. Furthermore, for any $\eta > 0$, there exists $m_1 = m_1(\eta, \delta) \in \mathbf{N}^*$ such that for any $m \geq m_1$, one has*

$$Z_\delta(G^m) \subset N_\eta(Z^\delta(I)).$$

Lemma 6.10 is an easy consequence of Lemma 6.9 (and in particular, of condition (P.S)*). We omit the details of the proof. (Notice that from (P.S) it follows that $Z_\delta(I)$ is compact in E.) □

LEMMA 6.11. *Let $\delta \in \mathbf{R}$ and $G \in \mathcal{F}_\delta$. For any $m \in \mathbf{N}^*$, $(G^m)'\colon E^m \to (E^m)'$ is a Fredholm operator of null index on $Z^\delta(G^m)$.*

The proof of Lemma 6.11 is given in the appendix. Let us just observe here that the content of Lemma 6.11 is to be understood in the sense that $(G^m)'$ is a Fredholm operator of null index, through the identification of E^m and $(E^m)'$ given by the restriction of E^m of the E-scalar product defined above. □

LEMMA 6.12. *Let $\delta \in \mathbf{R}$, $G \in \mathcal{F}_\delta$. There exists $\eta > 0$ (η depends on δ and G), there exists $m_2 = m_2(\delta) \in \mathbf{N}^*$ and there exists $\varepsilon > 0$ such that*

$$\langle (G^m)''(z)h, h\rangle \geq \varepsilon \|h\|_E^2,\quad \forall z \in N_\eta(Z^\delta(G^m)) \cap E^m,\quad \forall h \in [E^{j_0(\delta)}]^\perp \cap E^m.$$

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(1) The precise definitions of these conditions are recalled in the appendix.

Proof. By Proposition 6.6, there exists $\varepsilon>0$ such that:

$$\langle I''(z)h,h\rangle \geqslant 2\varepsilon\|h\|_E^2, \quad \forall h\in[E^{j_0}]^\perp, \quad \forall z\in Z^\delta(I), \tag{6.10}$$

where $j_0=j_0(\delta)$. Recall that $I''(z)=G''(z)$ for $z\in Z^\delta(I)$. Since $G''(z)$ is continuous with respect to z, there exists, for all $z\in Z^\delta(I)$, an open neighborhood $V(z)$ of z in E such that

$$\langle G''(v)h,h\rangle \geqslant \varepsilon\|h\|_E^2, \quad \forall h\in[E^{j_0}]^\perp, \quad \forall v\in V(z), \quad \forall z\in Z^\delta(I). \tag{6.11}$$

By condition (P.S) in Lemma 6.9, $Z^\delta(I)$ is compact. Hence, there exists $\eta'>0$ such that

$$N_{\eta'}(Z^\delta(I))\subset \bigcup_{z\in Z^\delta(I)} V(z). \tag{6.12}$$

Now, let $m_2(\delta)=m_1(\eta',\delta)$, where m_1 is given by Lemma 6.10. For $m\geqslant m_2$ one has

$$Z^\delta(G^m)\subset N_{\eta'}(Z^\delta(I)). \tag{6.13}$$

Lastly, applying Lemma 6.9, it is easily seen that $\bigcup_{m\geqslant m_2} Z^\delta(G^m)\cup Z^\delta(I)$ is compact. Hence, from (6.13), there exists $\eta>0$, with η depending only on G and δ, not on m, such that

$$N_\eta(Z^\delta(G^m))\subset N_{\eta'}(Z^\delta(I)), \quad \forall m\geqslant m_2. \tag{6.14}$$

Combining (6.11), (6.12) and (6.14) yields Lemma 6.12. □

LEMMA 6.13. *Let $\delta\in\mathbf{R}$, $G\in\mathcal{F}_\delta$ and $\eta>0$. Let $m_2\in\mathbf{N}^*$ be as in Lemma 6.12. For all $m\geqslant m_2$, there exists $\varepsilon'>0$ (ε' depends on m and δ) such that*

$$\|(G^m)'(z)\|_{(E^m)'}\geqslant\varepsilon', \quad \forall z\in E^m\setminus N_\eta(Z^\delta(G^m)) \quad \text{and} \quad G(z)\leqslant\delta.$$

Lemma 6.13 is an easy (and classical) consequence of conditions $(\mathrm{P.S})_m$ in Lemma 6.9. □

We are now ready to prove Lemma 6.6, and this will conclude this section.

Proof of Lemma 6.6. Firstly, we define $m_0(\delta)=\max(m_1, m_2)$, where m_1 and m_2 are given in Lemmas 6.10 and 6.12. Thus, by Lemma 6.10, δ is not a critical value of G^m for $m\geqslant m_0$. Let us now apply the result of Marino and Prodi, Proposition 6.8, in the following setting.

Let m be a fixed integer with $m \geq m_0$. We take $H=E^m$, $\Omega=\{z \in E^m;\ G(z)<\delta\}$ and $f=G^m$. Let us observe that the hypotheses of Proposition 6.8 are verified. Indeed, since δ is not a critical value of G^m as $m \geq m_1$ (Lemma 6.10), Ω is a C^2 open subset of E^m. By Lemma 6.9, condition $(\text{P.S})_m$, and using the fact that $Z^\delta(G^m) \subset \Omega$, we see that $Z(f)=Z^\delta(G^m)$ is compact. Lastly, by Lemma 6.11, $(G^m)'$ is a Fredholm operator of null index on $Z^\delta(G^m)$.

Thus, Proposition 6.8 applies. Let $F=E^m \setminus \Omega=\{z \in E^m;\ G^m(z) \geq \delta\}$. For $\delta_1<\delta$, let $F_1=\{z \in E^m;\ G^m(z) \geq \delta_1\}$; $F_1 \supset F$. Since $Z^\delta(G^m)$ is compact and $Z^\delta(G^m) \subset \Omega$, there exists $\delta_1<\delta$, δ_1 sufficiently close to δ so that $Z^\delta(G^m) \cap F_1=\varnothing$. Then, set

$$2\eta' = d(F_1, Z^\delta(G^m)) > 0. \tag{6.15}$$

Thus, $\{z \in E^m;\ d(z, Z^\delta(G^m))>\eta'\}$ is a neighborhood of F_1 and of F in E^m. Notice that if $d(z, Z^\delta(G^m))<\eta'$, one has $G^m(z)<\delta_1$. We now choose $\varepsilon_0>0$ and $\eta_0>0$ in the following way

$$2\varepsilon_0 = \min(\varepsilon, \varepsilon', \delta-\delta_1); \quad 2\eta_0 = \min(\eta, \eta') \tag{6.16}$$

where ε, ε' and η are given in Lemmas 6.12 and 6.13. δ_1 is associated with F_1 as above and η' is defined in (6.15).

Let $g \in C^2(\Omega, \mathbf{R})$ be the functional given by Proposition 6.8. We now define

$$\tilde{G} = g \text{ on } \Omega \quad \text{and} \quad \tilde{G} = G^m \text{ on } F. \tag{6.17}$$

We claim that $\tilde{G}$ satisfies the properties listed in Lemma 6.6. By (6.15)–(6.16), the set $\{z \in E^m;\ d(z, Z^\delta(G^m))>\eta_0\}$ is a neighborhood of $\partial\Omega$ since it contains F_1. Hence, by (g.1) in Proposition 6.8, $g=G^m$ on a neighborhood of $\partial\Omega$; $\tilde{G}$ is a C^2 functional: $E^m \to \mathbf{R}$.

By the definition (6.17), we certainly have $[G^m]_\delta \subset [\tilde{G}]_\delta$. Conversely, suppose $z \notin [G^m]_\delta$, that is, $G^m(z)<\delta$. Then, if $G^m(z) \neq \tilde{G}(z)$, one has $d(z, Z^\delta(G^m))<\eta_0<\eta'$. Whence, $G^m(z)<\delta_1$. By (g.2) in Proposition 6.8, we obtain $\tilde{G}(z)=g(z)<\delta_1+\varepsilon_0$, hence, $\tilde{G}(z)<\delta$. Thus, $[G>\delta]_m=[\tilde{G}>\delta]_m$ and $\tilde{G}$ verifies (i) in Lemma 6.6. Since $\tilde{G}=G^m$ on a neighborhood of $\tilde{G}^{-1}(\delta)$, (ii) follows from the fact that δ is not a critical value of G^m.

From the property (g.4) and (i) we derive that the critical points of $\tilde{G}$ (if any) in $\{z \in E^m;\ \tilde{G}(z) \leq \delta\}$ are in finite number and are nondegenerate. Now suppose $z \in E^m$ is a critical point of $\tilde{G}$ with $\tilde{G}(z) \leq \delta$. By (g.2), one has $\|(G^m)'(z)\| \leq \varepsilon_0$. Therefore, we know from (6.16) and Lemma 6.13 that $z \in N_\eta(Z^\delta(G^m)) \cap E^m$. Lemma 6.12 then implies

$$\langle (G^m)''(z)h, h \rangle \geq \varepsilon \|h\|^2_{E^m} \quad \forall h \in [E^{j_0}]^\perp \cap E^m. \tag{6.18}$$

Using (g.3) and the fact that $\varepsilon_0 \leqslant \varepsilon/2$, we derive from (6.18):

$$\langle \tilde{G}''(z)h, h\rangle \geqslant (\varepsilon/2)\|h\|_E^2 \quad \forall h \in [E^{j_0}]^\perp \cap E^m. \tag{6.19}$$

This shows that the coindex of any critical point of $\tilde{G}$ in $\{z \in E^m;\ \tilde{G}(z) \leqslant \delta\}$ is greater than or equal to $2N(m-j_0(\delta))$. Hence, $\tilde{G}$ verifies (iii).

It remains to show that $\tilde{G}$ obeys the requirement (iv), that is, $\tilde{G}$ verifies $(\text{P.S})_m$. But this is an easy consequence from (g.5) and Lemma 6.9. (Notice that by (6.16), g and G^m coincide on the set $\{z \in E^m;\ (\delta_1+\delta)/2 \leqslant g(z) \leqslant \delta\} = \{z \in E^m;\ (\delta_1+\delta)/2 \leqslant G^m(z) \leqslant \delta\}$. This fact is obtained by the preceding argument used to check property (i).)

The proofs of Lemma 6.6 and of Theorem 6.4 are thereby complete. □

7. The growth of the critical values

We shall now use Theorems 4.1 and 6.4 to derive an estimate from below on the growth of the critical values $c_m(k)$ (as $k \to +\infty$) which is required to conclude the proof of Theorem 1. (See A. Bahri [4, 5] for similar results in a more general setting.)

Let us recall from section 3 that $c_m(k)$ are defined by

$$c_m(k) = \sup_{A \in \mathscr{A}_m(k)} \min_{z \in A} I^*(z). \tag{3.2}$$

From Proposition 3.3 we know that

$$0 \leqslant \mu(k) \leqslant c_m(k) \leqslant \nu(k) < \infty, \quad \forall m \geqslant k+1 \tag{7.1}$$

with $\mu(k) \to \infty$ as $k \to \infty$. We set

$$c(k) = \overline{\lim_{m \to +\infty}}\ c_m(k). \tag{7.2}$$

Hence, $c(k) \to \infty$ as $k \to +\infty$.

The main result of this section is the following

PROPOSITION 7.1. *Assume H verifies condition* (H3) *or condition* (H4). *There exists a positive constant γ and a sequence of indices $k_i \in \mathbf{N}^*$, $k_i \uparrow +\infty$ as $i \uparrow +\infty$ such that along this sequence, the $c(k)$ defined in* (7.2) *satisfy*

$$c(k_i) \geqslant \gamma (k_i)^{(q+1)/(q-1)}.$$

Let $\Phi(z)=\frac{1}{2}A(z)-a'\int_0^{2\pi}|z|^{q+1}$. Because of (H 3) or (H 4), one has

$$I^*(z)\geq\Phi(z)-2\pi b'. \tag{7.3}$$

Define

$$b_m(k)=\sup_{A\in\mathcal{A}_m(k)}\min_{z\in A}\Phi(z);\quad b(k)=\overline{\lim_{m\to+\infty}}\, b_m(k).$$

By Proposition 3.3, we also have $\mu(k)\leq b_m(k)\leq\nu(k)$ and $b(k)\to\infty$ as $k\to\infty$. (7.3) implies that

$$c_m(k)\geq b_m(k)-2\pi b' \quad\text{and}\quad c(k)\geq b(k)-2\pi b'. \tag{7.4}$$

Hence, Proposition 7.1 is a direct consequence of the next lemma.

LEMMA 7.2. *There exists $\gamma>0$ and a sequence $k_i\uparrow+\infty$ as $i\uparrow+\infty$ such that $b(k_i)\leq\gamma(k_i)^{(q+1)/(q-1)}$, for all k_i.*

This result will be proved with the aid of three lemmas. We want to estimate (from below) the $b(k)$ which are critical values of Φ. Indeed by the proof of Theorem 3.6, we know that $b_m(k)$ is a critical value of $\Phi^m=\Phi|_{E^m}$ and that for any convergent subsequence of $b_m(k)$ along a sequence of indices m which goes to $+\infty$, the limit is a critical value of Φ. To apply Theorem 3.6, it just suffices to observe that in the case of Φ, the truncation procedure defined in (3.19) does not modify Φ. That is, with the notations of sections 3 and 5, one has $\Phi=\Phi_R$ (or even $\Phi=\Phi_{R,\varrho}$).

Now for the functional Φ one can give an explicit and complete description of the set of critical values and of its critical points.

LEMMA 7.3. *The set of critical values of Φ on E consists of the sequence $\{d_k;\ k\in\mathbf{N}\}$ with $d_k=\chi\{k\}^{(q+1)/(q-1)}$ where $\chi>0$ is a constant. (Explicitly, χ is given by: $\chi=\pi(q-1)\{a'\}^{-2/(q-1)}\{q+1\}^{-(q+1)/(q-1)}$.) The set of critical points of Φ is formed by the functions $z_k=\alpha_k e^{ikt}$ with $\alpha_k\in\mathbf{C}^N$, $|\alpha_k|=\{k/a'(q+1)\}^{1/(q-1)}$ and $k\in\mathbf{N}$. Lastly, $\Phi(z_k)=d_k$.*

The critical points of Φ in E are the 2π-periodic solutions of the autonomous Hamiltonian system

$$\dot z=a'(q+1)\,|z|^{q-1}\mathrm{g}z.$$

Lemma 7.3 is an obvious consequence of the fact that any solution of this system satisfies $|z(t)|=\text{constant}$. □

In view of the preceding lemma it would be tempting to write

$$b(k) = d_{E(k/N)}$$

where $E(k/N) = \min\{j \in \mathbf{N}; j \geq k/N\}$. This of course would yield Lemma 7.2 *for the whole* sequence of indices k. We actually conjecture that $b(k) = d_{E(k/N)}$ is true. But, though natural as it may seem, this fact reveals not to be easy to prove. As a matter of fact, the only relation between the $b(k)$'s and the d_k's we know for certain is that: $\forall k \in \mathbf{N}^*$, $\exists r(k) \in \mathbf{N}^*$ such that

$$b(k) = d_{r(k)}. \tag{7.5}$$

$r(k)$ verifies $k \leq k' \Rightarrow r(k) \leq r(k')$ and $r(k) \to \infty$ as $k \to \infty$. Notice that we know the "multiplicity" of each d_j: The associated set of critical points is a $2N-1$ dimensional sphere. Hence, if we would have a "multiplicity" result for the $b(k)$'s guaranteeing that for $k < k'$, and $k' - k > N$ one has $r(k) < r(k')$, then, again, Lemma 7.2 would immediately follow from Lemma 7.3, and the estimate would hold for the whole sequence of indices $k \in \mathbf{N}^*$. But, in the absence of such a multiplicity result, a priori nothing prevents the $r(k)$'s from remaining constant on large sets of indices k. That is, having equalities (or coincidences) of the type

$$r(k) = r(k+1) = \ldots = r(k+j(k))$$

for arbitrarily large (but finite) values of $j(k)$.

Therefore, to derive Lemma 7.2, we apply an indirect method that uses Theorems 4.1 and 6.4 in the next two lemmas. Thus, somewhat surprisingly, the topological properties associated with the $b(k)$'s allow one to obtain an a priori estimate on their growth.

Remark 7.4. There are two reasons for which one cannot readily give a multiplicity statement for the $b(k)$'s. First, using the sets $\mathcal{A}_m(k)$ for the definition of $b_m(k)$ via a maximum does not lead to a multiplicity result. Or, at least, such a result is not known. If we were using a class of sets defined through the cohomological index of Fadell and Rabinowitz [22] or the geometrical S^1-index of Benci [12, 13], we would indeed have a multiplicity property (compare the results in [12, 13, 22]). But then, on the other hand, we would loose, at least as far as we can see, the stability property of the critical values given by Theorem 4.1. Since this property plays a crucial role in the study of perturbations from I^* and in the proof of Theorem 1, we have to define the $b_m(k)$ using the class $\mathcal{A}_m(k)$. Secondly, even if one had a multiplicity property for the $b_m(k)$ it would not be easy to derive a similar statement for the $b(k)$, since $b(k)$ is defined via $b(k)=\overline{\lim}_{m\to+\infty} b_m(k)$. □

In the following, we denote by $j_0(\delta, \Phi)$ the index of Definition 6.1 but associated with the functional Φ instead of I. That is, $j_0(\delta, \Phi)$ is the minimum integer $j \in \mathbf{Z}$ such that

$$\langle \Phi''(v)h, h\rangle > 0, \quad \forall h \in (E^j)^{\perp}\setminus\{0\}, \quad \forall v \in Z^{\delta}(\Phi), \tag{7.6}$$

where $Z^{\delta}(\Phi)=\{z\in E;\ \Phi'(z)=0,\ \Phi(z)\leqslant\delta\}$. Observe that the results of section 5 and 6 apply to the functional Φ as well. The class of truncated functionals $\mathcal{F}_\delta$ associated with φ is, in this case, simply $\{\Phi\}$. Indeed, the truncation procedure of section 5 does not affect Φ: $\Phi_{R,\varrho}=\Phi$, $\forall R$, $\varrho\geqslant 1$.

LEMMA 7.5. *Suppose that for a certain $k\in\mathbf{N}^*$ there exists a sequence $(m_i)\subset\mathbf{N}^*$, $m_i\geqslant k+1$, $m_i\to+\infty$ as $i\to+\infty$ such that*

$$b_{m_i}(k-1)<\delta<b_{m_i}(k)$$

where $\delta\in\mathbf{R}$ is fixed and δ is not a critical value of Φ. Then $j_0(\delta, \Phi)>k/N$.

Proof of Lemma 7.5. For all i, choose an $\varepsilon_i>0$ such that

$$b_{m_i}(k-1)+\varepsilon_i<\delta<b_{m_i}(k)-\varepsilon_i. \tag{7.7}$$

From (7.7) it follows that

$$[\Phi\geqslant b_{m_i}(k-1)+\varepsilon_i]_{m_i}\supset[\Phi\geqslant\delta]_{m_i}\supset[\Phi\geqslant b_{m_i}(k)-\varepsilon_i]_{m_i}. \tag{7.8}$$

Hence, by Theorem 4.1, we know that for some $x_0\in[\Phi\geqslant\delta]_{m_i}$ one has

$$\Pi_{2Nm_i-2k-1}\big([\Phi\geqslant\delta]_{m_i},x_0\big)\neq 0. \tag{7.9}$$

On the other hand, choosing m_i large enough, we have by Theorem 6.4:

$$\Pi_l([\Phi\geqslant\delta]_{m_i})=0, \quad \forall l\leqslant 2N(m_i-j_0(\delta,\Phi))-1. \tag{7.10}$$

Comparing (7.9) with (7.10) yields

$$2N(m_i-j_0(\delta,\Phi))-1<2Nm_i-2k-1.$$

Hence,

$$j_0(\delta, \Phi) < k/N.$$

LEMMA 7.6. *Let $\delta \in \mathbf{R}$ be such that $d_{n+1} > \delta$ where d_n are the critical values of Φ defined in Lemma* 7.3. *Then $j_0(\delta, \Phi) \leqslant qn$.*

Proof of Lemma 7.6. Since $j_0(\delta, \Phi)$ is non-decreasing with respect to δ, we can assume that $d_n \leqslant \delta < d_{n+1}$. By Lemma 7.3 we have: $Z^\delta(\Phi) = \{z;\ z = \alpha_j e^{ijt},\ \alpha_j \in \mathbf{C}^N, |\alpha_j| = \{j/a'(q+1)\}^{1/(q-1)},\ j = 1, \ldots, n\}$. Let $z = z_j$ be an element in $Z^\delta(\Phi)$ with $1 \leqslant j \leqslant n$. Then, since

$$\langle \Phi''(v)h, h\rangle = A(h) - a'(q+1)q \int_0^{2\pi} |v|^{q-1} h^2,$$

we have

$$\langle \Phi''(z_i)h, h\rangle = A(h) - qj \int_0^{2\pi} h^2. \tag{7.11}$$

Let $j_0 = \max\{l \in \mathbf{N},\ l \leqslant qn\}$ denote the integer part of qn. We know that

$$A(h) \geqslant (j_0+1) \int_0^{2\pi} h^2, \quad \forall h \in (E^{j_0})^\perp. \tag{7.12}$$

Since $j_0 + 1 > qj\ \forall j = 1, \ldots, n$, we derive from (7.11) and (7.12) that

$$\langle \Phi''(z)h, h\rangle > 0, \quad \forall h \in (E^{j_0})^\perp, \quad \forall z \in Z^\delta(\Phi). \tag{7.13}$$

By the definition of $j_0(\delta, \Phi)$ this implies that

$$j_0(\delta, \Phi) \leqslant j_0 \leqslant qn.$$

The proof of Lemma 7.6 is thus complete. □

Proof of Proposition 7.1 *and Lemma* 7.2. We know that $\lim_{k\to+\infty} b(k) = +\infty$. Hence, we can find a sequence of indices $k_i \in \mathbf{N}^*$ such that

$$b(k_i - 1) < b(k_i), \quad \forall i, \quad \lim_{i\to+\infty} k_i = +\infty. \tag{7.14}$$

This defines the sequence k_i of Lemma 7.2.

Since $b(k) = \overline{\lim}_{m\to\infty} b_m(k)$ it is easily seen that, for each i, there exists a sequence $m_j \in \mathbf{N}^*$, $\lim_{j\to+\infty} m_j = +\infty$, and a real δ such that

$$b_{m_j}(k_i - 1) < \delta < b_{m_j}(k_i) \quad \text{and} \quad b(k_i - 1) < \delta < b(k_i). \tag{7.15}$$

Since the set of critical values of Φ is discrete, we can furthermore assume that δ is not a critical value of Φ.

By (7.5), there exists $n=r(k_i)$ such that $b(k_i)=d_n$. Therefore, by Lemma 7.6,

$$j_0(\delta,\Phi)\leq q(n-1). \tag{7.16}$$

On the other hand, by (7.15) and Lemma 7.5, we know that

$$j_0(\delta,\Phi)\geq k_i/N. \tag{7.17}$$

Hence, from (7.16) and (7.17), we derive

$$n\geq (k_i/Nq)+1. \tag{7.18}$$

Since $b(k_i)=d_n=\chi\{n\}^{(q+1)/(q-1)}$ we obtain from (7.18)

$$b(k_i)\geq \gamma\{k_i\}^{(q+1)/(q-1)} \tag{7.19}$$

for all $i\geq i_0$, where γ is a positive constant.

The proofs of Lemma 7.2 and Proposition 7.1 are thereby complete. □

Remark 7.7. Recall that we have defined in section 3:

$$c_{m,R}(k)=\sup_{A\in\mathcal{A}_m(k)}\min_{z\in A} I_R^*(z) \tag{3.25}$$

where $R\geq 1$, $I_R^*(z)=\frac{1}{2}A(z)-\int_0^{2\pi}H_R(z)$ and H_R is the truncated Hamiltonian defined by (3.19). Since $H_R(z)\leq a'|z|^{q+1}+b'$, $\forall R\geq 1$, $\forall z\in\mathbf{R}^{2N}$, we have $I_R^*(z)\geq\Phi(z)-2\pi b'$. Hence, it follows immediately from Lemma 7.1 that

$$c_{m,R}(k_i)\geq\gamma\{k_i\}^{(q+1)/(q-1)} \tag{7.20}$$

where $\gamma>0$ is constant independent of $R\geq 1$. □

To conclude this section we now derive the following consequence of Proposition 7.1 or rather of (7.20). It states the precise result we will use in the next sections.

PROPOSITION 7.8. *Suppose H satisfies* (H 3) *or* (H 4) *and let $p>1$ be such that $q<2p-1$. Let $A>0$ and σ_1, $\sigma_2>0$ be arbitrarily given positive numbers. There exists a real number $M>A$, depending on A, σ_1 and σ_2 with the following property. For all $R\geq 1$, there exists $k\in\mathbf{N}^*$ and a sequence $m_j\to+\infty$ such that*

$$A\leq c_{m_j,R}(k-1)<c_{m_j,R}(k)\leq M$$

and

$$c_{m_j,R}(k)-c_{m_j,R}(k-1)\geq\sigma_1\left\{c_{m_j,R}(k)\right\}^{1/(p+1)}+\sigma_2$$

with $\lim_{m_j\to+\infty} c_{m_j,R}(k)=\overline{\lim}_{m\to+\infty} c_{m,R}(k)\equiv\chi_R(k)$.

For the proof, we require the following simple observation.

LEMMA 7.9. *Let* σ_1, σ_2, $D>0$ *and* $0<\theta<1$ *be given real numbers. Let* k_0, $k_1\in\mathbf{N}^*$ *be also given. Suppose that* (a_k) *is a sequence of real numbers such that:*

(i) $0<a_{k_0}\leq D$

(ii) $0\leq a_k-a_{k-1}\leq\sigma_1(a_k)^\theta+\sigma_2,\quad\forall k,\quad k_0\leq k\leq k_1.$

Then, there exists a constant $C>0$ *depending only on* k_0, D, σ_1, σ_2 *and* θ, *(*C *does not otherwise depend on the sequence* (a_k), *nor does it depend on* k_1*) such that*

$$a_k\leq C(k)^{1/(1-\theta)},\quad\forall k,\quad k_0\leq k\leq k_1.$$

Proof of Lemma 7.9. This result is a more precise version of Lemma 5.1 in Bahri–Berestycki [5]. Let $\tau=(1-\theta)^{-1}$ and set $a_k=k^\tau\alpha_k$. We want to prove that $\alpha_k>0$ is bounded. Inequality (ii) for a_k reads

$$\alpha_k-\{(k-1)/k\}^\tau\alpha_{k-1}\leq\sigma_1 k^{-1}\alpha_k^\theta+\sigma_2 k^{-\tau}.$$

Hence, with $(1-(1/k))^\tau\geq 1-(\tau/k)$, we have

$$\alpha_k-\alpha_{k-1}+(\tau/k)\alpha_{k-1}\leq\sigma_1 k^{-1}\alpha_k^\theta+\sigma_2 k^{-\tau}. \tag{7.21}$$

Let us assume $\alpha_k>\alpha_{k-1}$. From (7.21), it follows that

$$\alpha_{k-1}\leq(\sigma_1/\tau)\alpha_k^\theta+(\sigma_2/\tau)k^{1-\tau}.$$

Again from (7.21) we know that

$$\alpha_k\leq\alpha_{k-1}+\sigma_1 k^{-1}\alpha_k^\theta+\sigma_2 k^{-\tau}.$$

Whence,

$$\alpha_k\leq\sigma_1\{(1/\tau)+(1/k)\}\,\alpha_k^\theta+\sigma_2\{(1/\tau)+(1/k)\}\,k^{1-\tau}$$

or

$$\alpha_k\leq\{1-\theta+(1/k_0)\}\left(\sigma_1\alpha_k^\theta+\sigma_2\{k_0\}^{-\theta/(1-\theta)}\right). \tag{7.22}$$

Now (7.22) obviously implies a bound on α_k: $\alpha_k \leq \mu$, where $\mu > 0$ only depends on the various constants in (7.22), that is on k_0, σ_1, σ_2 and θ. Hence, from (7.21) we deduce, in all cases, that

$$\alpha_k \leq \max(\alpha_{k-1}, \mu).$$

This obviously implies $\alpha_k \leq \max(\alpha k_0, \mu)$ for all k, $k_0 \leq k \leq k_1$. Since $a_k = \alpha_k \{k\}^{1/(1-\theta)}$, we have proved the lemma. □

Proof of Proposition 7.7. It suffices to show the existence of M and $k \in \mathbf{N}^*$, k depending on $R \geq 1$, such that

$$A \leq \chi_R(k-1) < \chi_R(k) \leq M \tag{7.23}$$

and

$$\chi_R(k) - \chi_R(k-1) \geq \sigma_1 \{\chi_R(k)\}^{1/(p+1)} + \sigma_2. \tag{7.24}$$

Indeed, the result of the Proposition clearly follows by taking a sequence $m_j \to +\infty$ such that $\chi_R(k) = \lim_{m_j \to +\infty} c_{m_j,R}(k)$ and observing that $\overline{\lim}_{m_j \to +\infty} c_{m_j,R}(k-1) \leq \chi_R(k-1)$.

To prove the existence of $M > A$ and, for any given $R \geq 1$, of $k \in \mathbf{N}^*$ verifying (7.23) and (7.24), we argue by contradiction. Suppose that $\forall M > A$, $\exists R \geq 1$ such that for all indices k with $A < \chi_R(k) \leq M$, one has

$$\chi_R(k) - \chi_R(k-1) \leq \sigma_1 \{\chi_R(k)\}^{1/(p+1)} + \sigma_2. \tag{7.25}$$

Let $k_0 \in \mathbf{N}^*$ be such that $\chi_R(k_0) > A$, $\forall R \geq 1$. Applying Lemma 7.9 to the sequence $a_k = \chi_R(k)$ and recalling that $\chi_R(k_0) \leq \nu(k_0)$, $\forall R \geq 1$, we obtain

$$\chi_R(k) \leq C k^{(p+1)/p}, \quad \forall k \geq k_0, \quad \forall R \geq 1. \tag{7.26}$$

But since $(p+1)/p < (q+1)/(q-1)$, as $q < 2p+1$, (7.26) is contradictory with (7.20).

The proof of Proposition 7.7 is therefore complete. □

Remark 7.10. We have used here the assumption $q < 2p+1$ of (H 3). It should be emphasized that *this is the only place* in the proof of Theorem 1 where this condition is being employed. □

8. Existence of one forced vibration

We are now ready to conclude the proof of Theorem 1. We first prefer, however, to derive the existence of *one* forced vibration of (1.1). It is hoped that in this way, the argument will be more transparent. In the next section, we give the full proof of Theorem 1 by making the proper modifications in the argument below. The idea here can first be described heuristically in a simple fashion. If I has no critical value at all in E, then for any δ, I^m has no critical value below δ, for large m. This implies that $[I \geq \delta]_m$ or rather $[I_{R,a} \geq \delta]_m$ with an appropriate choice for R and a, in a deformation retract of E^m. Hence, all the homotopy groups of $[I_{R,a} \geq \delta]_m$ are trivial. On the other hand, since $I_{R,a}$ is a perturbation from I_R^*, we shall show that one can apply Theorem 4.1 to establish the nontriviality of a certain homotopy group of such a set. This contradiction will prove the claim.

We now turn to the detailed proof. Thus, to show the existence of one periodic solution of (1.1), we argue by contradiction. Let us assume that I has no critical values at all. Let $C>0$ be as in Lemma 5.1 and $\alpha>0$, $\beta \geq 0$ be given by Lemma 5.2. We apply Proposition 7.8 with the choice $A=\beta$, $\sigma_1 = 2C(2/\alpha)^{1/(p+1)}$, $\sigma_2=2$. Let $M>0$ be the corresponding number whose existence is asserted by Proposition 7.8. Then, from this proposition we know that for each $R \geq 1$, there is an integer $k \in \mathbf{N}^*$ and a sequence $m_j \to +\infty$ such that

$$\beta \leq c_{m_j,R}(k-1) < c_{m_j,R}(k) \leq M. \tag{8.1}$$

$$c_{m_j,R}(k) - c_{m_j,R}(k-1) \leq 2C\left\{\left[c_{m_j,R}(k)+\beta\right]/\alpha\right\}^{1/(p+1)} + 2. \tag{8.2}$$

$$\lim_{m_j \to +\infty} c_{m_j,R}(k) = \chi_R(k). \tag{8.3}$$

Actually, with our choice of constants in Proposition 7.8, the right hand side of (8.2) should read

$$\sigma_1\left\{c_{m_j,R}(k)\right\}^{1/(p+1)} + \sigma_2 = 2C\left\{2c_{m_j,R}(k)/\alpha\right\}^{1/(p+1)} + 2.$$

But using (8.1) we know that $c_{m_j,R}(k) \geq \beta$ and (8.2) obtains.

Let $\varrho_0 = (M+\beta)/\alpha$ and let us prescribe a fixed $R > R_0(\varrho_0)$ where $R_0(\varrho_0)$ is given by Lemma 5.3. Henceforth in order to try to keep the notations simple, we will denote

$$c_{m_j,R}(k) = c_{m_j}(k) \quad \text{and} \quad \chi(k) = \chi_R(k) \equiv \overline{\lim_{m \to +\infty}}\, c_{m,R}(k).$$

Let us set $a = (\chi(k)+\beta)/\alpha$; thus $a \leq \varrho_0$. Hence, by (8.3), for m_j large enough, say $m_j \geq \mu_1$, one derives from (8.2) that

$$c_{m_j}(k) - c_{m_j}(k-1) \geq 2Ca^{1/(p+1)} + 1. \tag{8.4}$$

Suppose $c \leqslant \alpha a - \beta = \chi(k) \leqslant \alpha \varrho_0 - \beta$ is a critical value of $I_{R,a}$. Then, since $R \geqslant R_0(\varrho_0)$, we know by Lemma 5.3 that c is a critical value of I. Therefore, $I_{R,a}$ has no critical value in $(-\infty, \chi(k)]$. Furthermore, by condition (P.S)* (cp. the appendix) this implies that for m large enough, say $m \geqslant \mu_2$, $I^m_{R,a}$ $(=I_{R,a}|_{E^m})$ too has no critical value in $(-\infty, \chi(k)]$. In the sequel, m is fixed such that $m=m_j$ for some j and $m \geqslant \mu_1, \mu_2$.

By Lemma 5.1, we know that $|I^*_R(z) - I_{R,a}(z)| < Ca^{1/(p+1)}$, $\forall z \in E$. Therefore, for all reals d, δ and d_1 which are such that

$$\delta \geqslant d + Ca^{1/(p+1)} \quad \text{and} \quad d_1 \geqslant \delta + Ca^{1/(p+1)}$$

we have

$$[I^*_R \geqslant d]_m \supset [I_{R,a} \geqslant \delta]_m \supset [I^*_R \geqslant d_1]_m. \tag{8.5}$$

We choose $d = c_m(k-1) + \frac{1}{2}$. $\delta = c_m(k-1) + \frac{1}{2} + Ca^{1/(p+1)}$ and $d_1 = c_m(k) - \frac{1}{2}$. Since $\delta \geqslant d + Ca^{1/(p+1)}$ and, by (8.4), $d_1 \geqslant \delta + Ca^{1/(p+1)}$, we derive from (8.5) that:

$$\left[I^*_R \geqslant c_m(k-1) + \tfrac{1}{2}\right]_m \supset [I_{R,a} \geqslant \delta]_m \supset \left[I^*_R \geqslant c_m(k) + \tfrac{1}{2}\right]_m. \tag{8.6}$$

We may now apply Theorem 4.1 and obtain from (8.6):

$$\Pi_{2Nm-2k-1}([I_{R,a} \geqslant \delta]_m, x_0) \neq 0 \tag{8.7}$$

for some x_0. On the other hand, since $\delta < c_m(k)$, the choices of R and m made above guarantee that $I^m_{R,a}$ has no critical value in $(-\infty, \delta]$. As $I^m_{R,a}$ satisfies condition $(\text{P.S})_m$ and is of class C^2, the set $[I^m_{R,a} \geqslant \delta]_m$ is a deformation retract of the whole space E^m. (See e.g. the "non-critical neck principle" in [34] or [7, Lemma 2.2] for related constructions of deformation retracts.) Hence we deduce

$$\Pi_l([I_{R,a} \geqslant \delta]_m) = 0, \quad \forall l \in \mathbf{N}^*. \tag{8.8}$$

We have therefore reached a contradiction with (8.7) when $l = 2Nm - 2k - 1$.

This shows the existence of one periodic solution of (1.1). □

9. Proof of Theorem 1

Let us emphasize that the above argument does not readily extend to obtain the existence of many solutions of (1.1) in spite of (8.6) holding for infinitely many indices k (as will be seen). For as soon as there is one critical value below δ (δ as above), it is impossible to show, and indeed not true in general, that $[I_{R,a} \geqslant \delta]_m$ is a deformation retract of the space E^m. Therefore, to obtain further critical points, we need now to exploit the information pertaining to the *order* of the homotopy group given in Theorem 4.1. This will be made possible by using the precise result of Theorem 6.4.

To prove Theorem 1, we are going to show that the critical values of I are not bounded from above. We argue by contradiction. Suppose that the critical values of I are bounded from above by $A_1>0$. Let $j_0(A_1)$ be the index (associated with I) given by Definition 6.1. Choose an $A>0$ such that $A>\max\{\nu(Nj_0(A_1)), A_1, \beta\}$ where $\nu(j)$ is defined by Proposition 3.3. Hence, if $c_{m,R}(k)>A$ we are sure that $k>Nj_0(A_1)$, since $c_{m,R}(k)\leqslant\nu(k)$, and ν is non-decreasing with respect to k. Let $M>A$ be given by Proposition 7.8 and corresponding to $\sigma_1=2C(2/\alpha)^{1/(p+1)}$ and $\sigma_2=2$. Let $\varrho_0=(m+\beta)/\alpha$ and let R be a fixed real such that $R>R_0(\varrho_0)$ (where $R_0(\varrho_0)$ is defined by Lemma 5.3). We denote $c_m(k)=c_{m,R}(k)$ and $\chi(k)=\chi_R(k)$. Then, by Proposition 7.8 and arguing as in section 8 above, we know that there exists $k\in\mathbf{N}^*$ and a sequence $m_j\to+\infty$ such that $\chi(k)=\lim_{m_j\to+\infty} c_{m_j}(k)$ and

$$A\leqslant c_{m_j}(k-1)<c_{m_j}(k)\leqslant M, \tag{9.1}$$

$$c_{m_j}(k)-c_{m_j}(k-1)<2C\left\{\left[c_{m_j}(k)+\beta\right]/\alpha\right\}^{1/(p+1)}+2. \tag{9.2}$$

As above, set $a=[\chi(k)+\beta]/\alpha$, $\delta=c_m(k-1)+\frac{1}{2}+Ca^{1/(p+1)}$, where $c>0$ is given in Lemma 5.1 and $\alpha>0$, $\beta\geqslant0$ are given by Lemma 5.2. In the following, m will be fixed at a large enough value in such a way that $m=m_j$ for a certain j and

$$c_m(k)-c_m(k-1)\geqslant 2Ca^{1/(p+1)}+1, \tag{9.3}$$

$$m>m_0(\delta), \tag{9.4}$$

where (9.3) follows from (9.2) and $m_0(\delta)$ is the number given by Theorem 6.4. With the notations of section 6 we indeed know that $I_{R,a}\in\mathcal{F}_\delta$ because $\alpha a-\beta=\chi(k)\geqslant\delta$ by (9.2) (where we let $m_j\to+\infty$).

From (9.3) and Lemma 5.1 we derive that

$$\left[I_R^*\geqslant c_m(k-1)+\tfrac{1}{2}\right]_m\supset\left[I_{R,a}\geqslant\delta\right]_m\supset\left[I_R^*\geqslant c_m(k)-\tfrac{1}{2}\right]_m. \tag{9.5}$$

Thus, by Theorem 4.1,

$$\Pi_{2Nm-2k-1}([I_{R,a}\geqslant\delta]_m, x_0)\neq0 \tag{9.6}$$

for some base point x_0.

Since $\delta > c_m(k-1) \geq A \geq A_1$, δ is not a critical value of I. Furthermore, I has no critical values in $[A_1, \delta]$ and therefore (using the notations of section 6), $Z^\delta(I) = Z^{A_1}(I)$. This implies that the index j_0 (of Definition 6.1) associated with I is such that $j_0(\delta) = j_0(A_1)$. From $c_m(k-1) \geq A > \nu(Nj_0(A_1))$ we infer that $k \geq k-1 \geq Nj_0(\delta)$. Therefore, $2Nm-2k-1 \leq 2N(m-j_0(\delta))-1$. Whence, by Theorem 6.4 we obtain:

$$\Pi_{2Nm-2k-1}([I_{R,a} \geq \delta]_m, \cdot) = 0. \tag{9.7}$$

The contradiction between (9.6) and (9.7) shows that the set of critical values of I cannot be bounded from above.

The proof of Theorem 1 is thus complete. □

10. Bounded perturbations from an autonomous Hamiltonian system and open problems

We first state a result concerning forced vibrations for Hamiltonian systems which are bounded perturbations from autonomous Hamiltonian systems. Consider the system

$$\dot{z} = \mathfrak{g}\hat{H}'_z(t, z). \tag{10.1}$$

Suppose that $\hat{H}$ verifies:

($\hat{\mathrm{H}}$1) $\hat{H} \in C^2(\mathbf{R} \times \mathbf{R}^{2N}, \mathbf{R})$.

($\hat{\mathrm{H}}$2) $\hat{H}$ is T-periodic with respect to $t \in \mathbf{R}$.

($\hat{\mathrm{H}}$3) There exists $H \in C^2(\mathbf{R}^{2N}, \mathbf{R})$ satisfying (H2) and (H4) and such that

$$\|\hat{H} - H\|_{C^1(\mathbf{R} \times \mathbf{R}^{2N}, \mathbf{R})} < \infty.$$

THEOREM 10.1. *Under the hypotheses* ($\hat{\mathrm{H}}$1)–($\hat{\mathrm{H}}$3), *the system* (10.1) *possesses infinitely many T-periodic solutions.*

Remark 10.2. For this result, the condition $q < 2p+1$ in (H3) is not required any longer. In hypothesis ($\hat{\mathrm{H}}$3), we only need to assume that

$$|\hat{H}(t,z) - H(z)| \leq C, \quad |\hat{H}'_z(t,z) - H'(z)| \leq C, \quad \forall t \in \mathbf{R}, \quad \forall z \in \mathbf{R}^{2N}. \qquad \square$$

Remark 10.3. Theorem 10.1 is an extension of a result of Rabinowitz [29]. He proved the existence of *one* periodic solution of (10.1) under (H2), ($\hat{\mathrm{H}}$2), ($\hat{\mathrm{H}}$3) and a different hypothesis instead of (H4). Other results concerning non-autonomous systems of the kind (10.1) are given in Rabinowitz [29, 30]. The case of even Hamiltonians, i.e. $\hat{H}(t, -z)=\hat{H}(t, z)$ is considered in a recent work of Benci [11]. (Notice that this hypothesis does not allow one to study forced systems of the type (1.1) with $f\not\equiv 0$.) □

Proof of Theorem 10.1. Since the proof closely follows the lines of the proof of Theorem 1, we only sketch here the main idea. The arguments are technically simpler here because the truncation with the cut-off χ_ϱ is not needed (due to condition ($\hat{\mathrm{H}}$3)). The truncation in R, however, is still required.

We assume $T=2\pi$ and we continue to use the notations and results from the previous sections concerning H. Let

$$J(z)=\tfrac{1}{2}A(z)-\int_0^{2\pi}\hat{H}(t,z(t)).$$

The 2π-periodic solutions of (10.1) are the critical points of the functional J on E. For $R\geq 1$, let ω_R be the function defined in (3.18). Let

$$\hat{H}_R(t,z)=\omega_R(|z|)\,\hat{H}(t,z)+(1-\omega_R(|z|))(a'|z|^{q+1}+b')$$

and

$$J_R(z)=\tfrac{1}{2}A(z)-\int_0^{2\pi}\hat{H}_R(t,z).$$

It is easily checked that

$$|\hat{H}_R(t,z)-H_R(z)|\leq C<\infty,\quad \text{for all } t,\ z \text{ and } R, \tag{10.2}$$

$$|\hat{H}'_{Rz}(t,z)-H'_R(z)|\leq C(R)<\infty,\quad \text{for all } t \text{ and } z. \tag{10.3}$$

From (10.2) we see that

$$|J_R(z)-I^*_R(z)|\leq C,\quad \forall z\in E,\quad \forall R\geq 1, \tag{10.4}$$

where $C>0$ is a constant which is independent of $R\geq 1$. Hence, by (10.4), we have

$$[I^*_R\geq d]_m\supset[J_R\geq d+C]_m\supset[I^*_R\geq d+2C]_m \tag{10.5}$$

for all $R\geq 1$, $d\in\mathbf{R}$ and $m\in\mathbf{N}^*$. Using the estimate (7.20) and (10.5) it is then straightforward to see that there exist infinitely many indices $k\in\mathbf{N}^*$ with the following property. For each such k there exists a sequence $m_j\to+\infty$ such that

$$\left[I^*_R\geq c_{m_j}(k-1)+\tfrac{1}{2}\right]_{m_j}\supset[J_R\geq a_k]_{m_j}\supset\left[I^*_R\geq c_{m_j}(k)-\tfrac{1}{2}\right]_{m_j} \tag{10.6}$$

where a_k is fixed and $a_k\leq\chi_R(k)\leq\nu(k)$.

Notice that to obtain (10.6), we do not require $q<2p+1$, but rather $q>1$ in (7.20). (Indeed, (7.20) implies that $\chi(k)-\chi(k-1)$ cannot remain bounded.) It is not difficult, then, to repeat the arguments of section 8 or 9 in order to derive from (10.6) the existence of a sequence of critical values for J_R which is unbounded from above. Lastly, using a priori estimates of the same kind as those derived in section 5, one can show the existence of a sequence of critical values for J which is unbounded from above. □

It is clear from this argument that Theorem 10.1 remains valid (with the same method of proof) under alternative sets of hypotheses. In particular, in order to derive inclusions of the type (10.6), all that we have used from ($\hat{\mathrm{H}}$3) was the weaker property:

($\hat{\mathrm{H}}$4) $\|\hat{H}-H\|_{C^0(\mathbf{R}\times\mathbf{R}^{2N},\mathbf{R})}<+\infty$.

The hypothesis that $\hat{H}'_z-H'$ is bounded on $\mathbf{R}\times\mathbf{R}^{2N}$ only served to obtain a priori estimates. It can thus be replaced by any other assumption playing this role. For instance

($\hat{\mathrm{H}}$5) $0<\hat{H}(t,z)\leq\theta\hat{H}'_z(t,z)\cdot z,\quad \forall z\in\mathbf{R}^{2N},\quad |z|\geq R,\quad \forall t\in\mathbf{R}$,

where $0<\theta<1$ and $R>0$ are constants. Using the same type of method as in section 5 we obtain the next result that we state without proof.

THEOREM 10.4. *Suppose $\hat{H}$ verifies* ($\hat{\mathrm{H}}$1), ($\hat{\mathrm{H}}$2), ($\hat{\mathrm{H}}$4), ($\hat{\mathrm{H}}$5) *and H verifies* (H1), (H2) *and* (H4). *Then, the system* (10.1) *possesses infinitely many T-periodic solutions.*

As a conclusion we would like to indicate a few open problems in connection with the results presented here.

(1) We conjecture that Theorem 1 remains true under weaker assumptions. Namely, $H\in C^1(\mathbf{R}^{2N},\mathbf{R})$, $f\in C^0(\mathbf{R},\mathbf{R}^{2N})$, f is T-periodic and H verifies (H2). Recall that when $f\equiv 0$, those hypotheses suffice to prove the existence of non-constant free vibrations of (1.2) (see Rabinowitz [33]).

(2) More generally, we may think that non-autonomous systems of the type (10.1) always possess infinitely many T-periodic solutions provided $\hat{H}\in C^1(\mathbf{R}\times\mathbf{R}^{2N},\mathbf{R})$ satisfies ($\hat{\mathrm{H}}$2), ($\hat{\mathrm{H}}$4).

(3) It should be observed that in the framework of problem (1.1) all the periodic solutions have, in general, T as a minimal period. Indeed if f has T as a minimal period, then any z, which is a T-periodic solution of (1.1) has T as a minimal period. This is opposite to the situation for the autonomous case (1.2). For instance if $H(z)=|z|^{p+1}$, there is exactly one solution of (1.2) with minimal period T. It is by and large an open

question to know whether one can give a more precise description of the geometry of the solutions to (1.1) or (1.2). Consider again the case with $H(z)=(1/(p+1))|z|^{p+1}$. Then, the solutions of (1.2) are of the form $z_k=\alpha_k e^{ikt}$ with $\alpha_k \in \mathbf{C}^N$ and $|\alpha_k|=k^{1/(p-1)}$. Now consider the problem (1.1) with f replaced by εf. As $\varepsilon \to 0$, does there exist a family of solutions of (1.1) which approximate z_k, for all $k \in \mathbf{N}^*$?

(4) Lastly, it is a question which arises naturally to know whether a result analogous to Theorem 1 holds for a nonlinear hyperbolic equation of the type

$$\begin{aligned} &u_{tt}-u_{xx}=g(u)+h(t,x), \quad t\in\mathbf{R}, \quad 0<x<2\pi. \\ &u(t,x)=u(t+2\pi,x) \\ &u(t,0)=u(t,2\pi)=0. \end{aligned} \tag{10.7}$$

Assuming that g satisfies the same type of hypotheses as in [15, 28], does (1.3) possess infinitely many solutions for all h?

Remark 10.5. Using the same type of method as the one we have developed here, one could slightly sharpen the results we have presented in [7, 8] for problems of the type

$$\begin{aligned} &-\Delta u=g(x,u)+h(x) \quad \text{in } \Omega, \\ &u=0 \quad \text{on } \partial\Omega, \end{aligned} \tag{10.8}$$

where g is super-linear and odd with respect to u and $\Omega\subset\mathbf{R}^N$ is a bounded domain. In particular with this method, one does not need to work with functionals defined on the unit sphere of $H_0^1(\Omega)$. Rather, one could directly work on the whole space $H_0^1(\Omega)$ with the functionals

$$J(u)=\tfrac{1}{2}\int_\Omega |\nabla u|^2\,dx-\int_\Omega G(x,u)\,dx-\int_\Omega h(x)u\,dx$$

and

$$J^*(u)=\tfrac{1}{2}\int_\Omega |\nabla u|^2\,dx-\int_\Omega G(x,u)\,dx,$$

where $G(x,u)=\int_0^u g(x,s)\,ds$. For instance in [7, Theorem 6.1], hypothesis (6.3) could be eliminated and replaced by (6.5) with this approach. □

Appendix: The Palais-Smale and related conditions

Consider the truncated functionals I_R^* and $I_{R,a}$ defined in sections 3 and 5. We recall that

$$I_R^*(z) = \tfrac{1}{2}A(z) - \int_0^{2\pi} H_R(z),$$

$$I_{R,a}(z) = \tfrac{1}{2}A(z) - \int_0^{2\pi} H_R(z) - \chi_a(z)\langle f, z\rangle$$

where H_R is defined in (3.19) and χ_a is defined through (5.1)–(5.5). On several instances in this paper, we have used the fact that $J = I_R^*$ or $J = I_{R,a}^*$ satisfy the following three conditions:

(P.S) $\forall (z_n) \subset E$, $J(z_n) \leq C$, $J'(z_n) \overset{E'}{\to} 0$, imply (z_n) is precompact in E.

$(\text{P.S})_m$ $\forall (z_n) \subset E^m$, $J(z_n) \leq C$, $(J^m)'(z_n) \xrightarrow{(E^m)'} 0$, imply (z^n) is precompact in E^m.

$(\text{P.S})^*$ $\forall (z_m) \subset E$; $z_m \in E^m$, $J(z_m) \leq C$, $\|(J^m)'(z_m)\|_{(E^m)'} \to 0$, imply (z_m) is precompact in E.

Here, and in the following, C designates various positive constants.

Since R is fixed, we set $H = H_R$. The above three Palais-Smale type properties for I_R^* or $I_{R,a}$ hinge on the following conditions satisfied by H:

$$H \in C^1(\mathbf{R}^{2N}, \mathbf{R}). \tag{A 1}$$

$$H(z) < \theta H'(z)\cdot z + C, \quad \forall z \in \mathbf{R}^{2N}, \quad \text{with } 0 < \theta < \tfrac{1}{2}, \quad C > 0. \ \lim_{|z| \mapsto +\infty} H(z) = +\infty. \tag{A 2}$$

$$|H'(z)|^\gamma \leq aH'(z)\cdot z + b, \quad \forall z \in \mathbf{R}^{2N} \quad \text{where} \quad \gamma > 1, a > 0, b \geq 0 \text{ are constants.} \tag{A 3}$$

Since $H_R(z) = a'|z|^{q+1} + b'$, for $|z| \geq R+1$, with $q > 1$ (where H_R is defined in (3.19)), it is obvious that $H = H_R$ satisfy (A 1)–(A 3). In (A 3) for instance, γ could be chosen to be $\gamma = (q+1)/q$.

Let $J(z) = \frac{1}{2}A(z) - \int_0^{2\pi} H(z)$ and $K(z) = J(z) - \chi_a(z)\langle f, z\rangle$ where a is fixed. The purpose of this appendix is to show the following:

PROPOSITION A. *Suppose H verifies* (A 1)–(A 3) *then J and K satisfy the conditions* (P.S), $(\text{P.S})_m$ *(for all m) and* (P.S)*.

We start with:

Condition (P.S) *for J.* Let $(z_n)\subset E$ be a sequence satisfying

$$\tfrac{1}{2}A(z_n)-\int_0^{2\pi} H(z_n)\,dt\leq C \tag{A 4}$$

$$-\mathfrak{g}\dot{z}_n-H'(z_n)=\varepsilon_n\to 0 \quad \text{in } E'. \tag{A 5}$$

Multiply (A5) by z_n (in the sense of $\langle\ ,\ \rangle$) to obtain:

$$|A(z_n)-\int_0^{2\pi} H'(z_n)\cdot z_n\,dt| = |\langle \varepsilon_n, z_n\rangle_{E',E}|\leq \bar{\varepsilon}_n\|z_n\|_E \tag{A 6}$$

where $\bar{\varepsilon}_n=\|\varepsilon_n\|_{E'}\to 0$ as $n\to+\infty$. Comparing (A 6) with (A 4) and using (A 2) yields

$$\left(\tfrac{1}{2}-\theta\right)\int_0^{2\pi} H'(z_n)\cdot z_n\,dt\leq \tfrac{1}{2}\bar{\varepsilon}_n\|z_n\|_E+C. \tag{A 7}$$

Thus, we obtain

$$\int_0^{2\pi} H'(z_n)\cdot z_n\leq C\bar{\varepsilon}_n\|z_n\|_E+C, \tag{A 8}$$

$$\int_0^{2\pi} H(z_n)\leq C\bar{\varepsilon}_n\|z_n\|_E+C, \tag{A 9}$$

$$\int_0^{2\pi} |z_n|^{p+1}\leq C\bar{\varepsilon}_n\|z_n\|_E+C, \tag{A 10}$$

$$\int_0^{2\pi} |H'(z_n)|^{\gamma}\leq C\bar{\varepsilon}_n\|z_n\|_E+C. \tag{A 11}$$

(A 9) follows from (A 8) by (A 2). In (A 10), $p+1=1/\theta>2$ and (A 10) follows from the observation that (A 2) implies

$$a|z|^{p+1}-b\leq H(z), \quad \forall z\in \mathbf{R}^{2N} \tag{A 12}$$

with $a>0$, $b\geq 0$. Lastly, (A 11) is a consequence of (A 3).

Let $z_n=z_n^+ + z_n^- + z_n^\circ$ denote the orthogonal decomposition of z_n along $E=E^+\oplus$

$E^{-}\oplus E^{\circ}$. We recall that $\|z\|_E^2=A(z_n^+)-A(z_n^-)+|z_n^\circ|_{\mathbf{R}^{2N}}$. Since $z_n^\circ=(1/2\pi)\int_0^{2\pi} z_n$, we have by (A 10)

$$|z_n^\circ|_{\mathbf{R}^{2N}}\leqslant\|z_n\|_{L^{p+1}}\leqslant C(\bar{\varepsilon}_n\|z_n\|_E)^{1/(p+1)}+C. \tag{A 13}$$

Notice that $\langle -\mathfrak{g}\dot{z}_n, z_n^+\rangle=A(z_n^+)$ and $\langle -\mathfrak{g}\dot{z}_n, z_n^-\rangle=A(z_n^-)$. Multiplying (A 5) by z_n^+ yields

$$A(z_n^+)=\|z_n^+\|_E^2=\int_0^{2\pi} H'(z_n)\cdot z_n^+ + \langle \varepsilon_n, z_n^+\rangle.$$

Hence, by Hölder's inequality, with $(1/\gamma)+(1/\gamma')=1$, we have

$$\|z_n^+\|_E^2\leqslant \|H'(z_n)\|_{L^\gamma}\|z_n^+\|_{L^{\gamma'}}+\bar{\varepsilon}_n\|z_n^+\|_E. \tag{A 14}$$

From (A 11), we know that $\|H'(z_n)\|_{L^\gamma}\leqslant C(\bar{\varepsilon}_n\|z_n\|_E)^{1/\gamma}+C$. By the injection $E\hookrightarrow L^{\gamma'}$, there exists a constant C such that $\|z_n^+\|_{L^{\gamma'}}\leqslant C\|z_n^+\|_E<C\|z_n\|_E$. Therefore, (A 14) yields

$$\|z_n^+\|_E^2\leqslant C(\bar{\varepsilon}_n)^{1/\gamma}\|z_n\|_E^{1+(1/\gamma)}+\bar{\varepsilon}_n\|z_n\|_E+C. \tag{A 15}$$

And similarly, multiplying (A 5) by z_n^-, we get

$$\|z_n^-\|_E^2\leqslant C(\bar{\varepsilon}_n)^{1/\gamma}\|z_n\|_E^{1+(1/\gamma)}+\bar{\varepsilon}_n\|z_n\|+C. \tag{A 16}$$

Now, using the fact that $\bar{\varepsilon}_n$ is bounded ($\bar{\varepsilon}_n\to 0$), and that $\|z_n\|_E^2=\|z_n^+\|_E^2+\|z_n^-\|_E^2+|z^\circ|_{\mathbf{R}^{2N}}^2$, we derive, by adding up (A 13), (A 15)

$$\|z_n\|_E^2\leqslant C\|z_n\|_E^\sigma+C \tag{A 17}$$

where $\sigma<2$. Thus, from (A 17) we have a priori estimate on z_n in E:

$$\|z_n\|_E\leqslant C. \tag{A 18}$$

One can therefore extract a subsequence of (z_n), which we denote again by (z_n), such that

$$z_n\to z \text{ weakly in } E,\quad z_n\to z \text{ strongly in } L^r,\quad \forall r<\infty,\quad r\geqslant 1. \tag{A 19}$$

(Recall that the injection $E\hookrightarrow L^r$ is compact, for any finite $r\geqslant 1$.) Condition (A 3) obviously implies

$$|H'(z)|\leqslant C|z|^s+C,\quad \forall z\in\mathbf{R}^{2N} \tag{A 20}$$

for some s ($s=(1/(\gamma-1))>1$ by (A 12)). (A 20) implies that the mapping $z\mapsto H'(z)$ is continuous from L^{2s} into L^2. Hence, it follows from $z_n\to z$ strongly in L^{2s} that $H'(z_n)\mapsto H'(z)$ strongly in L^2. Thus, *a fortiori*, $H'(z_n)\mapsto H'(z)$ strongly in E'. Now by (A 5) one has

$$-\mathfrak{g}\dot{z}_n = H'(z_n)+\varepsilon_n \to H'(z) \text{ strongly in } E'. \tag{A 21}$$

The last step needed to conclude is the observation that

$$\|z\|_E^2 = \|\dot{z}\|_{E'}^2 + |z^\circ|_{\mathbf{R}^{2N}}^2, \quad \forall z\in E. \tag{A 22}$$

We now prove (A 22). Let $(\cdot,\ \cdot)_E$ denote the scalar product associated with the Hilbert norm $\|\ \|_E$. That is, $(\varphi,\ \psi)_E = \langle \dot{\varphi}^+,\ \mathfrak{g}\varphi^+\rangle - \langle \dot{\varphi}^-,\ \mathfrak{g}\varphi^-\rangle + \varphi^\circ\cdot\psi^\circ$. ($\varphi^\circ\cdot\psi^\circ$ is the usual product in $\mathbf{R}^{2N}$.) Let us denote $z=u+z^\circ$ where $u=z^+ + z^- \in (E^\circ)^\perp$ and $z^\circ\in E^\circ$. One has

$$\|\dot{z}\|_{E'} = \|\dot{u}\|_{E'} = \max_{\substack{\varphi\in E\\ \|\varphi\|_E=1}} \langle \dot{u},\varphi\rangle. \tag{A 23}$$

For any $\psi\in E$, denote $\varphi=\mathfrak{g}\psi^+ - \mathfrak{g}\psi^- + \psi^\circ$ and observe that $\langle \dot{u},\ \varphi\rangle = (u,\ \psi)_E$. Since the transformation $\psi\mapsto\varphi$ an isometry: $E\to E$, it is straightforward from (A 23) that

$$\|\dot{z}\|_{E'} = \max_{\substack{\psi\in E\\ \|\psi\|_E=1}} (u,\psi)_E = \|u\|_E. \tag{A 24}$$

Since $\|z\|_E^2 = \|u\|_E^2 + |z^\circ|_{\mathbf{R}^{2N}}$, we obtain (A 22).

We can now conclude. By (A 19), $z_n\to z$ in L^1 and therefore $z_n^\circ = (1/2\pi)\int_0^{2\pi} z_n \to z^\circ$ in $\mathbf{R}^{2N}$. By (A 21), $\dot{z}_n\to\dot{z}$ strongly in E'. Therefore, by (A 22), $z_n\to z$ strongly in E. We have thus proved that J satisfies condition (P.S). □

Condition (P.S)$_m$ *for* J. Let $(z_n)\subset E^m$ be a sequence satisfying

$$J(z_n)\le C, \quad (J^m)'(z_n)\to 0 \quad \text{in } (E^m)'. \tag{A 25}$$

Thus, (z_n) satisfies (A 4), while (A 5) is replaced by

$$-\mathfrak{g}z_n - P^m H'(z_n) = \varepsilon_n \to 0 \quad \text{in } (E^m)' \tag{A 26}$$

where P^m denotes the orthogonal projection onto E^m. Indeed, one obviously has $(J^m)'(z) = -\mathfrak{g}\dot{z} - P^m H'(z)$, when identifying the dual space as $(E^m)'$, that is with respect to the duality pairing $\langle\ ,\ \rangle$. By inspection of the preceding argument, one can see that in order to derive the a priori estimate (A 18), (A 5) has only been used via multiplications by z_n, z_n^+ or z_n^-. Since those elements are in E^m, one obtains the very same results by using (A 26). Hence (z_n) is bounded in E. As above one proves using (A 26) that, for a subsequence denoted by z_n, $z_n \overset{E}{\to} z$ weakly, $H'(z_n) \overset{L^2}{\to} H'(z)$, $z_n^\circ \to z^\circ$ and $\dot{z}_n \to \dot{z}$ strongly in $(E^m)'$. Then, using the analogous relation of (A 22) for E^m:

$$\|z\|^2_{E^m} = \|\dot{z}\|^2_{(E^m)'} + |z^\circ|^2_{\mathbf{R}^{2N}}, \quad \forall z \in E^m, \quad \forall m \geq 0 \tag{A 27}$$

(which is derived in the same fashion as (A 22)), we have reached the same conclusion: $z_n \to z$ in E^m. That is, J satisfies $(\text{P.S})_m$. □

Condition (P.S)* *for J*. Let $(z_m) \subset E$ be a sequence such that $z_m \in E^m$, $J(z_m) \leq C$ and

$$\|(J^m)'(z_m)\|_{(E^m)'} \to 0,$$

that is,

$$-g\dot{z}_m - P^m H'(z_m) = \varepsilon_m, \quad \|\varepsilon_m\|_{(E^m)'} \to 0. \tag{A 28}$$

Again, as for (P.S) or $(\text{P.S})_m$, this leads to an a priori estimate of the type: $\|z_m\|_E = \|z_m\|_{E^m} \leq C$. Hence, for a subsequence denoted again by (z_m), one has $z_m \to z$ weakly in E, $z_m \to z$ strongly in L^r, $\forall r$, $1 \leq r < \infty$ and $z_m^\circ \to z^\circ$ in $\mathbf{R}^{2N}$. In view of (A 20), we also know that $H'(z_m) \to H'(z)$ strongly in L^2. Consequently, $P^m H'(z_m) \to H'(z)$ (as $m \to +\infty$), strongly in L^2 and, a fortiori, strongly in E'. Thus we are in the following situation

$$\dot{z}_m = h_m + \varepsilon_m \tag{A 29}$$

with $h_m \overset{L^2}{\to} h$, $z_m^\circ \to z^\circ$, $z_m \overset{E}{\to} z$ weakly and $\|\varepsilon_m\|_{(E^m)'} \to 0$. This implies in particular that $\dot{z} = h$. By (A 27) for all $m \geq 1$, we have

$$\|z_m - P^m z\|^2_{E^m} = \|\dot{z}_m - P^m \dot{z}\|^2_{(E^m)'} + |z_m^\circ - z^\circ|^2_{\mathbf{R}^{2N}}$$

(where we have used the fact that $(P^m z)^{\cdot} = P^m \dot{z}$). Whence,

$$\|z_m - P^m z\|^2_{E^m} \leq \left\{ \|h_m - P^m h\|_{(E^m)'} + \|\varepsilon_m\|_{(E^m)'} \right\}^2 + |z_m^\circ - z^\circ|^2_{\mathbf{R}^{2N}}. \tag{A 30}$$

Since $\|h_m - P^m h\|_{(E^m)'} \leq C \|h_m - P^m h\|_{L^2} \to 0$, as $m \to +\infty$, it follows from (A 30) that

$\|z_m-P^m z\|_{E^m}=\|z_m-P^m z\|_E$ converges to 0 as $m\to+\infty$. This shows that $z_m\to z$ in E. We have proved thereby that J satisfy (P.S)*.

Palais-Smale conditions for K. The arguments to show that K verifies (P.S), $(\text{P.S})_m$ and (P.S)* are very similar to the preceding ones. Therefore, as an example, we just sketch the proof of (P.S).

Let (z_n) be a sequence in E such that $K(z_n)\leq C$ and $K'(z_n)=\varepsilon_n\to 0$ in E'. We denote $K(z)=J(z)-T(z)$ with $T(z)=\chi_z(z)\ \langle f, z\rangle$ for some fixed $a\geq 1$. As in (5.6), we have

$$T'(z)=\tilde{\chi}_a(\|z\|^{p+1}_{L^{p+1}})(p+1)\langle f,z\rangle |z|^{p-1}z+\chi_a(x)f. \tag{A 31}$$

We have the following estimate

$$\|T'(z)\|_{L^{(p+1)/p}}\leq C, \quad \forall z\in E. \tag{A 32}$$

Indeed, $\||z|^{p-1}z\|_{L^{(p+1)/p}}=\left\{\int_0^{2\pi}|z|^{p+1}\right\}^{p/(p+1)}$ and $|\langle f, z\rangle|\leq\|z\|_{L^{p+1}}$ show that

$$\|T'(z)\|_{L^{(p+1)/p}}\leq C\tilde{\chi}_a\left(\|z\|^{p+1}_{L^{p+1}}\right)\|z\|^{p+1}_{L^{p+1}}+C\chi_a(z).$$

Using (5.2), (5.4) and (5.5), we derive (A 32). From (A 32) we know that

$$|\langle T'(z_n),z_n\rangle|, |\langle T'(z_n),z_n^+\rangle|, |\langle T'(z_n),z_n^-\rangle|\leq C\|z_n\|_E. \tag{A 33}$$

This in turn allows one to repeat the argument used in (A 6)–(A 11). Indeed, since $|T(z_n)|\leq C\|z_n\|_E$, we know that

$$\tfrac{1}{2}A(z_n)-\int_0^{2\pi}H(z_n)\leq C+C\|z_n\|_E \tag{A 34}$$

$$-\mathrm{g}\dot{z}_n-H'(z_n)-T'(z_n)=\varepsilon_n\to 0 \quad \text{in } E'. \tag{A 35}$$

Thus, multiplying (A 35) by z_n and using (A 34) yield

$$\int_0^{2\pi}H'(z_n)\cdot z_n,\quad \int_0^{2\pi}H(z_n),\quad \int_0^{2\pi}|z_n|^{p+1},\quad \int_0^{2\pi}|H'(z_n)|^{\gamma}\leq C+C\|z_n\|_E. \tag{A 36}$$

Then, multiplying (A 35) by z_n^+ and z_n^- and using (A 33) lead to the same conclusions as before, that is, $\|z_n\|_E$ is bounded. Lastly, when $z_n\to z$ weakly in E, it is easily seen that $T'(z_n)\to T'(z)$ in L^2, whence in E'. Thus, one derives the strong convergence of z_n in E from (A 35) by using (A 22) in the same way as we did for J.

The proof of Proposition A is thereby complete. □

Proof of Lemma 6.11. With the notations introduced in the appendix, Lemma 6.11 reads as follows:

$$(K|_{E_m})' : E^m \to (E^m)' \text{ is a Fredholm operator of null index}$$

where

$$K(z) = \tfrac{1}{2}A(z) - \int_0^{2\pi} H(z)\,dt - \chi_a(z)\,\langle f, z\rangle .$$

H satisfies (A 1)–(A 3). Furthermore H satisfies:

$$H \in C^2(\mathbf{R}^{2n}, \mathbf{R}). \quad \exists s>0 \quad \text{and} \quad C, C' \text{ such that } |H''(z)| \leq C|s|^s + C'. \tag{A 3'}$$

In order to prove Lemma 6.11, we compute the second derivative of $K^m = K|_{E_m}$: Let $(h, w) \in E^m$. Then we find:

$$\begin{aligned}((K^m)''h, w) = &-\int^{2\pi} \mathfrak{g}h\cdot w\,dt - \int^{2\pi} H''(z)h\cdot w\,dt \\ &+p(p+1)\tilde{\chi}_a\left(\int_0^{2\pi} |z|^{p+1}dt\right)\int_0^{2\pi} |z|^{p-1}h\cdot w\,dt \\ &+(p+1)^2\tilde{\chi}''_a\left(\int_0^{2\pi} |z|^{p+1}dt\right)\left(\int_0^{2\pi} |z|^{p-1}z\cdot h\,dt\right)\left(\int_0^{2\pi} |z|^{p-1}z\cdot w\,dt\right).\end{aligned} \tag{A 37}$$

Let

$$\mathscr{L}(z)(h, w) = -\int_0^{2\pi} \mathfrak{g}\dot{h}\cdot w\,dt + h^\circ\cdot w^\circ \tag{A 38}$$

where h° and w° are the orthogonal projections of h and w on E° and $h^\circ\cdot w^\circ$ denotes their scalar product in $\mathbf{R}^{2n}$, and let

$$\mathscr{R}(z)(h, w) = (K^m)''h\cdot w - \mathscr{L}(z)(h, w). \tag{A 39}$$

Both $\mathscr{L}(z)$ and $\mathscr{R}(z)$ are continuous bilinear and symmetric forms on E^m. Hence, they can be written in the E^m-scalar product, as follows:

$$\mathscr{L}(z)(h, w) = (Lh,\, w)_E \tag{A 40}$$

where

$$Lh = h^+ - h^- + h^\circ \tag{A 41}$$

(h^+, h^- and h° are the orthogonal projections of h onto E^+, E^-, E° respectively) and

$$\mathcal{R}(z)(h, w) = (Rh, w)_E. \tag{A 42}$$

It is not difficult, but technical, to see that the linear operator $R: E^m \to E^m$ is compact. This fact is due to (A 3′) and to the compact embedding of E into $(L^r(S^1))^{2N}$ for any $r \geq 1$. Furthermore, the operator L is obviously invertible, with a continuous linear inverse. Hence, the operator

$$E^m \xrightarrow{L+R} E^m$$

$$h \to Lh + Rh$$

is Fredholm of null index (notice that L and R are self-adjoint operators). But this operator is exactly the second derivative of K^m expressed in the E^m-scalar product. Hence, Lemma 6.11 is proved. □

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