Normal forms for real surfaces in $\mathbf{C}^2$ near complex tangents and hyperbolic surface transformations

by

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0. Introduction

It is well known that the complex analytical properties of a real submanifold M in the complex space $\mathbf{C}^n$ are most accessible through consideration of the complex tangents to M. The properties we have in mind are related to the behavior of holomorphic functions on or near M and to the behavior of M under biholomorphic transformation. The case in which M is a real hypersurface is most familiar, while much less is known for higher codimension. In this paper we consider the critical case of a real n-dimensional manifold M in $\mathbf{C}^n$, which we also assume to be real analytic. At a generic point M is locally equivalent to the standard $\mathbf{R}^n$ in $\mathbf{C}^n$. However, near a complex tangent M may aquire a non-trivial local hull of holomorphy and other biholomorphic invariants.

We begin with the simplest non-trivial case, which is a surface $M^2 \subset \mathbf{C}^2$ with an isolated, suitably non-degenerate complex tangent. Here one already encounters a rich structure and non-trivial problems. In coordinates $z_j = x_j + iy_j$, $j = 1, 2$, M may be written locally as

$$R(z, \bar{z}) = -z_2 + q(z_1, \bar{z}_1) + \ldots = 0,$$
$$q = \gamma z_1^2 + z_1 \bar{z}_1 + \gamma \bar{z}_1^2, \quad 0 \leq \gamma < \infty.$$

The z_1-axis is tangent to M at the origin. M, or more precisely, this complex tangent is said to be elliptic if $0 \leq \gamma < 1/2$, hyperbolic if $1/2 < \gamma$, or parabolic if $\gamma = 1/2$. We shall prove the following theorem.

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THEOREM 1. *Let M be a real analytic surface in $\mathbf{C}^2$ with an elliptic complex tangent at a point p with $0<\gamma<1/2$. Then there exists a holomorphic coordinate system (z_1, z_2) in which $p=0$, and M has locally the form*

$$x_2 = z_1 \bar{z}_1 + \Gamma(x_2)(z_1^2+\bar{z}_1^2), \quad y_2 = 0,$$

where $\Gamma = \gamma+\delta x_2^s$, $\delta=\pm 1$, $s \in \mathbf{Z}^+$, or $\Gamma=\gamma$ $(s=\infty)$. The quantities γ, δ, s form a complete system of biholomorphic invariants for M near p.

A consequence of Theorem 1 is that the local hull of holomorphy of M near p is precisely the real analytic 3-manifold-with-boundary $\tilde{M}$: $x_2 \geq z_1 \bar{z}_1 + \Gamma(x_2)(z_1^2+\bar{z}_1^2)$, $y_2=0$. $\tilde{M}$ is the union of a one-parameter family of ellipses, the boundaries of which are the curves on M gotten by setting $x_2=c>0$. Another consequence is that such an M always admits the biholomorphic involution corresponding to $(z_1, z_2) \mapsto (-z_1, z_2)$. It is interesting to note that M is locally equivalent to an *algebraic* surface.

We also have the analogue of Theorem 1 in the n-dimensional case. This theorem will be reduced to a seemingly unrelated problem, namely that of a normal form for a pair of involutions τ_1, τ_2 which are holomorphic mappings in a neighborhood of a common fixed point p in $\mathbf{C}^2$. They are subjected to biholomorphic mappings ψ keeping p fixed, by replacing τ_j by $\psi^{-1}\tau_j\psi$. We ask for a classification of the pairs of (τ_1, τ_2), and more generally of the group generated by the τ_j, under the pseudo group of biholomorphic mappings near p.

Taking ξ, η as coordinates in $\mathbf{C}^2$, p as the origin and the linearized maps $d\tau_j|_0$ as

$$d\tau_j\colon (\xi, \eta) \mapsto (\lambda_j \eta, \lambda_j^{-1}\xi) \quad \text{with} \quad \lambda_1 = \lambda_2^{-1} = \lambda \neq 0$$

we can state our result as follows:

THEOREM 2. *If $|\lambda| \neq 1$ then there exists a biholomorphic mapping ψ near the origin with $\psi(0)=0$, taking the two given holomorphic involutions τ_j into*

$$\psi^{-1}\tau_j\psi\colon (\xi, \eta) \mapsto (\Lambda_j(\xi\eta)\eta, \Lambda_j(\xi\eta)^{-1}\xi)$$

where

$$\Lambda_1 = \Lambda_2^{-1} = \lambda+\delta(\xi\eta)^s, \quad \delta = 1, 0, \ s \geq 1.$$

For our application we will have to consider these holomorphic involutions τ_j in conjunction with an antiholomorphic involution ϱ describing the reality condition and satisfying

$$\tau_1 \varrho = \varrho \tau_2.$$

This leads to a finer classification of (τ_1, τ_2) under the group of biholomorphic mappings ψ which commute with ϱ. In particular, we will have λ real or $|\lambda|=1$ in correspondence to the elliptic or hyperbolic quadric.

In Theorem 2 we had to rule out that λ lies on the unit circle. Actually, if $|\lambda|=1$, λ not a root of unity, one can still find a formal power series expansion for ψ, but in general one has to expect divergence of these series. This is a "small divisor problem" as one encounters it in Celestial Mechanics. The product $\varphi=\tau_1\tau_2$ is the crucial map which has to be normalized. Its linearization at the origin $d\varphi|_0$ has the eigenvalues λ^2, λ^{-2}.

However, in celestial mechanics one restricts oneself to area-preserving mappings, and the analogous equivalence problem was studied by G. D. Birkhoff [3]. In our case the area-preserving property is replaced by the condition

$$\varphi^{-1} = \tau_2\tau_1 = \tau_1^{-1}\varphi\tau_1$$

which corresponds to "reversible" systems of differential equations. Mappings of this nature which can be represented as a product of involutions actually also played a role in Birkhoff's study of the restricted three body problems [2].

In case λ is not on the unit circle one has no difficulty of small divisors but the corresponding convergence proofs are not straightforward. We apply here a refinement of the majorant method as it was developed for area-preserving mappings in [12] and [11].

One may be led to the involutions τ_1, τ_2, ϱ of Theorem 2 by the problem of characterizing intrinsically the trace f on M of a function g holomorphic in a neighborhood of M. The key to this is to complexify M. Replacing $\bar{z}$ by independent complex variables w in the equation for M gives the complex analytic surface

$$\mathfrak{M} = \{(z, w) \in \mathbf{C}^4 : R(z, w) = 0,\ \bar{R}(w, z) = 0\}.$$

If the natural projections $\pi_1(z, w)=z$, $\pi_2(z, w)=w$ are restricted to $\mathfrak{M}$, then f and g are related by $f=g\circ\pi_1$. For $\gamma\neq 0$, π_1 and π_2 are two-fold branched coverings. The covering transformations τ_2, τ_1 are holomorphic involutions on $\mathfrak{M}$ fixing the origin. The condition $f\circ\tau_2=f$ is an intrinsic characterization of the restriction of a function holomorphic in z. It is a discrete analogue of the local characterization [9] by H. Lewy of the restriction of a holomorphic function to a strongly pseudo-convex real hypersurface. τ_2 corresponds to the tangential Cauchy-Riemann operator, and the mapping φ is a discrete version of the Levi-form. In the elliptic case φ can be embedded in a flow φ^t, t

complex. The orbits of this flow intersect M precisely in the curves bounding the above mentioned analytic discs.

If the surface M is elliptic, then λ is real, $\lambda \neq \pm 1$, and the origin is a hyperbolic fixed point of φ in the sense of mappings. In this case we have a satisfactory theory. If M is hyperbolic, φ is elliptic. The subtleties of the theory of elliptic mappings, e.g. small divisors, make the theory of hyperbolic surfaces much more difficult.

Previously it was known to Bishop [4] that γ is a biholomorphic invariant. He also proved in the elliptic case the existence of a one-parameter family of analytic discs with boundaries on $M^2 \subset \mathbf{C}^2$ near the complex tangent p. Hunt [7] investigated further the regularity of $\tilde{M}$, the union of these discs. In [8] it was shown that $\tilde{M}$ is a C^∞ manifold-with-boundary for $0 \leq \gamma < 1/2$, and that the discs are unique. In [1] Bedford and Gaveau consider hulls of holomorphy from a global viewpoint.

In section one of this paper we discuss the connection between surfaces and involutions. In fact, we show the equivalence of certain complex surfaces $\mathfrak{M}$ with suitable pairs of involutions τ_1, τ_2. We dicuss thoroughly in section 2 the quadric surfaces, which correspond to pairs of linear involutions. Here the basic phenomena are clearly revealed. Section 3 deals with pairs of non-linear involutions on a formal level, and section 4 contains the convergence proof for Theorem 2. In section 5 these results are applied to derive the normal form for the manifold M.

In section 6 we discuss hyperbolic surfaces. In particular, we show divergence of the transformation into normal form for an example, using ideas previously developed for area-preserving Cremona transformations [10].

1. Surfaces and involutions

Let M be a smooth real analytic surface in $\mathbf{C}^2$. It may be described locally by two independent real equations or by one complex equation,

$$M: R(z, \bar{z}) = 0, \quad \bar{R}(\bar{z}, z) = 0, \quad dR \wedge d\bar{R} \neq 0, \tag{1.1}$$

where R is a power series in $z=(z_1, z_2)$ and $\bar{z}$. We wish to investigate the local properties of M under the pseudo-group of local biholomorphic transformations

$$z' = f(z), \quad \bar{z}' = \bar{f}(\bar{z}).$$

We assume that the point $z=0$ lies on M. By interchanging R and $\bar{R}$ we may assume that the holomorphic linear term in R is non-zero. After introducing this linear function as a new z_2 variable, we may assume that M has the form $z_2=p\bar{z}_1+q\bar{z}_2+\dots$ and after a further transformation we achieve $q=0$, so that

$$z_2 = p\bar{z}_1+F(z_1\bar{z}_1), \quad F = O(|z_1|^2). \tag{1.2}$$

If $p\neq 0$ we may introduce new coordinates z' by $z_2=pz_2'+F(z_1', z_2')$, $z_1=z_1'$. M then goes over into the totally real plane $z_2'=\bar{z}_1'$. Hence, M has no invariants near such a point. We henceforth assume that $p=0$, so that the z_1-axis is a complex tangent to M at the origin. M is then given by $z_2=az_1^2+bz_1\bar{z}_1+c\bar{z}_1^2+\dots$. We make the non-degeneracy assumption that $b\neq 0$. Then by a quadratic change of z_2 we may achieve $b=1$, $a=\bar{c}=\gamma$. A rotation of z_1 makes $0\leq\gamma$. The surface M is now assumed to have the form

$$M: \begin{array}{l} z_2 = F(z_1, \bar{z}_1) = q(z_1, \bar{z}_1)+H(z_1, \bar{z}_1), \\ q = \gamma z_1^2+z_1\bar{z}_1+\gamma\bar{z}_1^2, \quad 0\leq\gamma<\infty, \; H = O(|z_1|^3). \end{array} \tag{1.3}$$

The non-negative number γ is a biholomorphic invariant of M first considered by Bishop [4]. The complex tangent is *elliptic* if $0\leq\gamma<1/2$, *parabolic* if $\gamma=1/2$, or *hyperbolic* if $1/2<\gamma<\infty$.

For our investigation it will be necessary to characterize those real analytic functions on the surface M which are the restrictions of functions holomorphic in some neighborhood of M. This is facilitated by complexifying M. We replace $\bar{z}$ by independent variables $w=(w_1, w_2)$ in (1.1) and define a smooth complex analytic surface $\mathfrak{M}$ in $\mathbf{C}^4$ by

$$\mathfrak{M}: R(z, w) = 0, \; \bar{R}(w, z) = 0.$$

Complex conjugation $(z, \bar{z})\mapsto(\bar{z}, z)$ goes over into the anti-holomorphic involution

$$\varrho(z, w) = (\bar{w}, \bar{z}).$$

More generally we consider a complex surface

$$\mathfrak{M}: R(z, w) = 0, \quad S(z, w) = 0, \quad dR\wedge dS \neq 0,$$

passing through the origin of $\mathbf{C}^4$ under the wider group of transformations

$$z' = f(z), \quad w' = g(w). \tag{1.4}$$

Such an $\mathfrak{M}$ comes from a real surface $M\subset\mathbf{C}^2$ if and only if $\varrho\mathfrak{M}=\mathfrak{M}$, and such a transformation is induced by a holomorphic mapping of $\mathbf{C}^2$ if and only if $f(z)=\bar{g}(z)$. (The bar indicates complex conjugation of the coefficients only in the series g.)

There are two invariant projections on $\mathbf{C}^4$, $\pi_1(z, w)=z$, $\pi_2(z, w)=w$. They are related by $\pi_2=c\pi_1\varrho$, where c denotes complex conjugation. We denote the restrictions of π_1 and π_2 to $\mathfrak{M}$ by the same symbols. The real and imaginary parts of z_1 may be taken as coordinates on M, and (z_1, w_1) as coordinates on $\mathfrak{M}$. A real analytic function $f=f(z_1, \bar{z}_1)$ on M may be continued locally to a function $f=f(z_1, w_1)$ holomorphic on $\mathfrak{M}$. The original function is the restriction of a holomorphic function if and only if the extended function f satisfies $f=g\circ\pi_1$ for some function $g=g(z)$ holomorphic in z. Similarly f is the restriction of an anti-holomorphic function if and only if the extended f satisfies $f=g\circ\pi_2$, $g=g(w)$. f is real if and only if $f\circ\varrho=cf$.

The possible linear structure of $\mathfrak{M}$ is more varied. To describe it let $P_z=\{w=0\}$ and $P_w=\{z=0\}$ denote the z and w coordinate planes, and P denote the tangent complex two-plane to $\mathfrak{M}$ at the origin. There are four possibilities;

(1) P is totally real: $\dim P\cap P_z=\dim P\cap P_w=0$,
(2) P is partially holomorphic: $\dim P\cap P_z\geq 1$,
(3) P is partially anti-holomorphic: $\dim P\cap P_w\geq 1$,
(4) P is complex: $\dim P\cap P_z=\dim P\cap P_w=1$.

We shall study $\mathfrak{M}$ only in a neighborhood of a point at which its tangent plane P is complex (type (4)). Generically through such a point there exist a curve C_1 of points at which the tangent plane is of type (2) and a curve C_2 of points at which it is of type (3). Locally $\mathfrak{M}$ is given by

$$\begin{aligned} z_2 &= F(z_1, w_1) = (q+H)(z_1, w_1) \\ w_2 &= G(z_1, w_1) = (p+K)(z_1, w_1). \end{aligned} \tag{1.5}$$

The quadratic terms q and p are both assumed to have a non-zero $z_1 w_1$-term, and so may be put into the form

$$q = az_1^2+z_1 w_1+aw_1^2, \quad p = bz_1^2+z_1 w_1+bw_1^2,$$

via a transformation (1.4). The product ab is invariant under (1.4). If $ab\neq 0$ then by a substitution $(z_1, w_1)\mapsto(\alpha z_1, \alpha^{-1}w_1)$ we may achieve $a=b=\gamma\in\mathbf{C}$. γ is then invariant up to sign.

We now assume that

$$p = q = \gamma z_1^2+z_1 w_1+\gamma w_1^2, \quad \gamma\neq 0. \tag{1.6}$$

In this case, when restricted to $\mathfrak{M}$, the projections

$$\pi_1(z_1, w_1) = (z_1, F(z_1, w_1)), \quad \pi_2(z_1, w_1) = (w_1, G(z_1, w_1))$$

are locally two-fold branched coverings. The branch locus of π_1 is given by

$$z_2 = F(z_1, w_1), \quad F_{w_1}(z_1, w_1) = 0,$$

which is a smooth curve in the (z_1, z_2)-plane since $F_{w_1 w_1}(0, \bar{0}) = 2\gamma \neq 0$. Likewise, the branch locus of π_2 is a non-singular curve in the w-plane. The equation $w = w'$ or

$$q(z_1', w_1) - q(z_1, w_1) = K(z_1, w_1) - K(z_1', w_1)$$

together with (1.5) generally have a unique solution $z' \neq z$. By the implicit function theorem they define a local self-mapping of $\mathfrak{M}$

$$\tau_1 : \begin{array}{l} z_1' = -z_1 - \dfrac{1}{\gamma} w_1 + h_1(z_1, w_1) \\ w_1' = w_1, \end{array} \tag{1.7}$$

which is an involution, $\tau_1^2 = \text{id}$. Similarly $z' = z$ or $F(z_1, w_1') = F(z_1, w_1)$ and (1.5) determines a second involution

$$\tau_2 : \begin{array}{l} z_1' = z_1 \\ w_1' = -\dfrac{1}{\gamma} z_1 - w_1 + h_2(z_1, w_1). \end{array} \tag{1.8}$$

τ_1 and τ_2 are the covering transformations for π_2 and π_1, $\pi_1 \tau_2 = \pi_1$, $\pi_2 \tau_1 = \pi_2$. The fixed point sets of τ_1 and τ_2 are the curves C_1 and C_2 mentioned above. If $\mathfrak{M}$ satisfies the reality condition, then $\tau_1 \varrho = \varrho \tau_2$.

We may now characterize the trace f on $\mathfrak{M}$ of a function $g(z)$ holomorphic in z. Since $f = g \circ \pi_1$, we necessarily have $f \circ \tau_2 = f$. Conversely, suppose $f = f(z_1, w_1)$ is analytic in (z_1, w_1) and invariant under τ_2. There then exists a single-valued function $g = g(z_1, z_2)$, defined and holomorphic on the base of the branched covering π_1 away from its branch locus, satisfying $f = g \circ \pi_1$, or $f(z_1, w_1) = g(z_1, F(z_1, w_1))$. Since g is bounded it extends to be holomorphic in a neighborhood of $z = 0$ by the Riemann extension theorem. We may also say that the functions z_1 and $F(z_1, w_1)$ generate the algebra of τ_2-invariants. The condition $f \circ \tau_1 = f$ characterizes the trace on $\mathfrak{M}$ of a function holomorphic in w (anti-holomorphic in a neighborhood of M).

The mapping $\varphi=\tau_1\tau_2$,

$$\varphi\colon \begin{aligned} z_1' &= -(1-\gamma^{-2})z_1+\gamma^{-1}w_1+\dots \\ w_1' &= -\gamma^{-1}z_1-w_1+\dots \end{aligned} \tag{1.9}$$

is also very important for the study of the surface M. It has the origin as an isolated fixed point. We shall see in section 2 that this fixed point is hyperbolic if M is elliptic and elliptic if M is hyperbolic.

The condition $f\circ\tau_2=f$ is an analogue of the tangential Cauchy-Riemann equations on a real hypersurface in $\mathbf{C}^n$. In fact, such an analytic $M^{2n-1}\subset\mathbf{C}^n$ yields, upon complexification, $\mathfrak{M}^{2n-1}\subset\mathbf{C}^{2n}$. The two projections $\pi_1,\pi_2\colon\mathfrak{M}^{2n-1}\to\mathbf{C}^n$ each have rank n and $(n-1)$-dimensional fibers. The tangents to the fibers of π_1 are spanned by the $n-1$ independent complexified tangential Cauchy-Riemann operators. A function on $\mathfrak{M}^{2n-1}$ annihilated by these operators is constant on the fibers of π_1 and so comes from a function holomorphic in z alone. In the case $n=2$, there is only one independent tangential vector field P of type $(1,0)$. By complexification, $(P,\bar{P})$ goes over to (P,Q) on $\mathfrak{M}^3$. We consider the flows

$$\psi_1^t=\exp(tP),\quad \psi_2^t=\exp(tQ)$$

with t complex. They commute when $[P,Q]=0$, which implies that the Levi-form of M vanishes. Thus the commutator $\varphi^2=\tau_1\tau_2\tau_1^{-1}\tau_2^{-1}$ may be thought of as a discrete analogue of the Levi-form. Under the assumption that the linear part of φ is not nilpotent, we shall derive a normal form for M in section 3. This may be compared to the normal form in [5] for a non-degenerate real hypersurface.

To further emphasize the importance of τ_1 and τ_2, we next wish to show that two suitable such involutions defined and holomorphic in a neighborhood of and fixing the origin of $\mathbf{C}^2$ give rise to a surface $\mathfrak{M}$ in $\mathbf{C}^4$. Let the coordinates of $\mathbf{C}^2$ be denoted by $X=(x,y)^t$, and suppose

$$\begin{aligned} \tau_j\colon X' &= T_jX+h_j(X),\quad h_j=O(|X|^2), \\ T_j^2 &= I,\quad h_j\circ\tau_j=-T_jh_j,\quad j=1,2. \end{aligned} \tag{1.10}$$

For each j we assume that the 2 by 2 matrix T_j has a (-1)-eigenspace of dimension one, and consequently a one-dimensional $(+1)$-eigenspace. Let the eigenvectors be denoted by v_j^+, v_j^-. We further assume that each of the pairs of vectors (v_1^-,v_2^-), (v_1^-,v_2^+), (v_2^-,v_1^+) is linearly independent. After a linear change of coordinates (x,y), we may assume $v_1^-=(1,0)^t$ and $v_2^-=(0,1)^t$, and so

$$T_1 = \begin{pmatrix} -1 & c_1 \\ 0 & 1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 1 & 0 \\ c_2 & -1 \end{pmatrix}. \tag{1.11}$$

The other two independence conditions are equivalent to $c_1 \neq 0$, $c_2 \neq 0$. More explicitly we now have

$$\begin{aligned} \tau_1 &: \begin{array}{l} x' = -x + c_1 y + f_1(x, y) \\ y' = y + g_1(x, y) \end{array} \\ \tau_2 &: \begin{array}{l} x' = x + g_2(x, y) \\ y' = c_2 x - y + f_1(x, y), \end{array} \end{aligned} \tag{1.12}$$

where the f_j, g_j are of second order. We define four new functions by

$$\begin{aligned} z_1 &= y + y \circ \tau_2 = c_2 x + f_2(x, y) \\ z_2 &= y \cdot y \circ \tau_2 = c_2 xy - y^2 + yf_2(x, y) \\ w_1 &= x + x \circ \tau_1 = c_1 y + f_1(x, y) \\ w_2 &= x \cdot x \circ \tau_1 = c_1 xy - x^2 + xf_1(x, y). \end{aligned} \tag{1.13}$$

Clearly, $(x, y) \mapsto (z, w)$ defines a map of rank two. If we use the first and third equation to eliminate x and y, then the image is seen to lie on a surface

$$\begin{aligned} z_2 &= c_1^{-1} z_1 w_1 - c_1^{-2} w_1^2 + \ldots \\ w_2 &= c_2^{-1} z_1 w_1 - c_2^{-2} z_1^2 + \ldots . \end{aligned}$$

This can be put into the form (1.5), (1.6) by a transformation (1.4). It is clear that τ_1 and τ_2, by fixing the functions w and z, respectively, are the involutions induced by this embedding. Since (w_1, w_2) and (z_1, z_2) generate the functions invariant under τ_1 and τ_2, respectively, it follows that any other such regularly embedded surface realizing the τ_j is equivalent to this one via a transformation (1.4). We have proved the following.

PROPOSITION 1.1. *Every analytic surface* (1.5, 1.6) *gives rise to an intrinsic pair of involutions* (1.12). *Conversely, every such pair* (1.12) *are the intrinsic involutions of some surface* (1.5, 1.6) *in* $\mathbf{C}^4$.

Note that the two $(+1)$-eigenvectors (v_1^+, v_2^+) are dependent precisely when $c_1 c_2 = 4$. This is the parabolic case, corresponding to $\gamma = \pm 1/2$ in (1.6).

An anti-holomorphic involution ϱ fixing the origin in $\mathbf{C}^2$ has the form

$$\begin{aligned} \varrho : X' &= P\bar{X} + k(\bar{X}), \quad k = O(|X|^2) \\ P\bar{P} &= I, \quad k(\overline{\varrho(X)}) = -P\bar{k}(X), \end{aligned} \tag{1.14}$$

the last two equations being equivalent to $\varrho^2=\mathrm{id}$. The fixed points of ϱ are the solutions to the system $X=\varrho(X)$, which in view of the conditions (1.14), is equivalent to a single equation (1.2) with $p\neq 0$. If this surface is transformed into $\mathbf{R}^2\subset\mathbf{C}^2$, then ϱ may be put into the form $\varrho(x,y)=(\bar{x},\bar{y})$. The change $(x,y)\mapsto(x+iy,x-iy)$ gives the form $\varrho(x,y)=(\bar{y},\bar{x})$. If the involutions (1.10) also satisfy $\tau_1\varrho=\varrho\tau_2$ for ϱ in this form, then $z\circ\varrho=\bar{w}$, and (1.12) gives rise to a real surface (1.3) in $\mathbf{C}^2$.

Next, we consider a real analytic n-dimensional manifold in $\mathbf{C}^n$. Generically, such a manifold M is totally real and so locally equivalent to the standard $\mathbf{R}^n$ in $\mathbf{C}^n$. We shall, however, study M near a point p at which it has a complex one-dimensional holomorphic tangent space. We use the following coordinate notation

$$\begin{aligned} &z=(z_1,z_\alpha,z_n),\ z_\alpha=x_\alpha+iy_\alpha,\quad 2\leqslant\alpha\leqslant n-1\\ &x=(x_2,\dots,x_{n-1}). \end{aligned}\tag{1.15}$$

The Greek indices α,β,σ will generally have the range from 2 to $n-1$ throughout this paper. These coordinates are initially chosen so that p is the origin, the z_1-axis is the holomorphic tangent space to M at p, the (z_1,x)-space is the real tangent space T_p to M at 0, and $z_n=0$ is the complex envelope, T_p+iT_p, of this real tangent space. We may then express M locally as a graph

$$\begin{aligned} &z_n=F(z_1,\bar{z}_1,x)\\ &y_\alpha=f_\alpha(z_1,\bar{z}_1,x)=\bar{f}_\alpha(\bar{z}_1,z_1,x), \end{aligned}\tag{1.16}$$

where F,f_α begin with quadratic terms. Those in F have the form $q+q_1+q_2$,

$$q\ =az_1^2+bz_1\bar{z}_1+c\bar{z}_1^2,$$

$$q_1=\sum a_\alpha x_\alpha z_1+b_\alpha x_\alpha\bar{z}_1,$$

$$q_2=\sum c_{\alpha\beta}x_\alpha x_\beta.$$

As in the two dimensional case we make the non-degeneracy assumption that $b\neq 0$, and even that $b=1$, $a=c=\gamma$, $0\leqslant\gamma<\infty$.

Further simplification of F,f_α is made as follows. To eliminate the second term in q_1 we make the change $z_1\mapsto z_1+\Sigma A_\alpha z_\alpha$. Consideration of $q(z_1+\Sigma A_\alpha z_\alpha)$ shows that we must solve

$$2\gamma\bar{A}_\alpha+A_\alpha=-b_\alpha$$

$$\bar{A}_\alpha+2\gamma A_\alpha=-\bar{b}_\alpha,$$

which is always possible if $\gamma \neq 1/2$. The remaining terms in q_1 and q_2 are eliminated by the change $z_n \mapsto z_n - \Sigma (a_\alpha z_\alpha z_1 + c_{\alpha\beta} z_\alpha z_\beta)$. Next we eliminate the quadratic terms in f_α, which have the form $q_\alpha + q_{1\alpha} + q_{2\alpha}$,

$$q_\alpha = a_\alpha z_1^2 + b_\alpha z_1 \bar{z}_1 + \bar{a}_\alpha \bar{z}_1^2, \quad b_\alpha = \bar{b}_\alpha,$$

$$q_{1\alpha} = 2\,\mathrm{Re} \sum c_{\alpha\beta} x_\beta z_1,$$

$$q_{2\alpha} = \sum c_{\alpha\beta\sigma} x_\beta x_\sigma, \quad c_{\alpha\beta\sigma} = \bar{c}_{\alpha\beta\sigma}.$$

The $z_1 \bar{z}_1$-term in q_α is removed by $z_\alpha \mapsto z_\alpha + i b_\alpha z_n$. With this term zero, the remaining quadratic terms are removed by $z_\alpha \mapsto z_\alpha + 2i \Sigma c_{\alpha\beta} z_\beta z_1 + i \Sigma c_{\alpha\beta\sigma} z_\beta z_\sigma + 2i a_\alpha zn$.

From this point on we assume that M has the form (1.16) with

$$\begin{aligned} F &= q(z_1, \bar{z}_1) + H(z_1, \bar{z}_1, x), \quad H = O(|z|^3), \\ f_\alpha &= h_\alpha(z_1, \bar{z}_1, x) = \bar{h}_\alpha(\bar{z}_1, z_1, x), \quad h_\alpha = O(|z|^3), \\ q &= \gamma z_1^2 + z_1 \bar{z}_1 + \gamma \bar{z}_1^2, \quad 0 \leq \gamma < \infty, \ \gamma \neq \frac{1}{2}. \end{aligned} \tag{1.17}$$

Before continuing let us examine the locus N of those points near the origin at which M has a complex tangent. We set $r^0 = F - z_n$, $r^\alpha = f_\alpha - y_\alpha$, and

$$\Delta = \frac{\partial(r^0, r^\alpha, \bar{r}^0)}{\partial(z_1, z_\beta, z_n)},$$

the Jacobian determinant. Then N is given by (1.16) together with $\Delta = \bar{\Delta} = 0$. In view of (1.17) $\Delta = \pm (i/2)^{n-2} (\bar{z}_1 + 2\gamma z_1) + \ldots$. The condition $\gamma \neq 1/2$ allows us to solve $\Delta = \bar{\Delta} = 0$ explicitly for z_1 and $\bar{z}_1$: $z_1 = \psi(x)$, $\bar{z}_1 = \bar{\psi}(x)$. Thus if $\gamma \neq 1/2$, then N is a totally real $(n-2)$-dimensional manifold lying on M.

Now we complexify the manifold M by replacing $\bar{z}$ by w in (1.13) to get a complex analytic n-dimensional submanifold $\mathfrak{M}$ in $\mathbf{C}^{2n}$,

$$\begin{aligned} z_n &= F(z_1, w_1, x), \quad 2x_\alpha = z_\alpha + w_\alpha \\ w_n &= \bar{F}(w_1, z_1, x), \\ z_\alpha - w_\alpha &= 2i f_\alpha(z_1, w_1, x) = 2i \bar{f}_\alpha(w_1, z_1, x). \end{aligned} \tag{1.18}$$

We note that these equations imply

$$\begin{aligned} z_\alpha &= x_\alpha + i f_\alpha(z_1, w_1, x) \\ w_\alpha &= x_\alpha - i f_\alpha(z_1, w_1, x). \end{aligned} \tag{1.19}$$

The variables (z_1, w_1, x) will be used as complex coordinates on $\mathfrak{M}$. The two projections $\pi_1(z, w)=z$ and $\pi_2(z, w)=w$, when restricted to $\mathfrak{M}$ have the form

$$\pi_1(z_1, w_1, x) = (z_1, x_\alpha+if_\alpha(z_1, w_1, x),\ F(z_1, w_1, x)),$$
$$\pi_2(z_1, w_1, x) = (w_1, x_\alpha-if_\alpha(z_1, w_1, x),\ \bar{F}(w_1, z_1, x)).$$

Since $\mathfrak{M}$ comes from a real submanifold M, the reflection $\varrho(z, w)=(\bar{w}, \bar{z})$ preserves $\mathfrak{M}$ and induces the anti-holomorphic involution $\varrho(z_1, w_1, x)=(\bar{w}_1, \bar{z}_1, \bar{x})$.

Again the case $\gamma=0$ is exceptional, so we assume that $0<\gamma<\infty$, $\gamma\neq 1/2$. We define a holomorphic involution $\tau_1(z, w)=(z', w')$ on $\mathfrak{M}$ by $w=w'$, which amounts to the equations

$$q(z_1', w_1)-q(z_1, w_1) = \bar{H}(w_1, z_1, x)-\bar{H}(w_1, z_1', x')$$
$$x_\alpha'-ih_\alpha(z_1', w_1, x') = x_\alpha-ih_\alpha(z_1, w_1, x).$$

By the implicit function theorem we get

$$\tau_1:\ \begin{aligned} z_1' &= -z_1-\frac{1}{\gamma}w_1+K+(z_1, w_1, x)\\ w_1' &= w_1\\ x_\alpha' &= x_\alpha+L_\alpha(z_1, w_1, x), \end{aligned} \tag{1.20}$$

for certain functions K, L_α of second order. The condition $\tau_1^2=\mathrm{id}$ gives $K\circ\tau_1=K$, $L_\alpha\circ\tau_1=-L_\alpha$. From $\tau_2=\varrho\tau_1\varrho$, we have

$$\tau_2:\ \begin{aligned} z_1' &= -z_1\\ w_1' &= -\frac{1}{\gamma}z_1-w_1+\bar{K}(w_1, z_1, x)\\ x_\alpha' &= x_\alpha+\bar{L}_\alpha(w_1, z_1, x). \end{aligned} \tag{1.21}$$

π_1 is a two-fold branched covering with covering transformation τ_2. To find the branch locus consider the Jacobian determinant

$$\Delta = \frac{\partial(z_1, z_\alpha, z_n)}{\partial(z_1, x, w_1)},$$

where z_α, z_n are given by the first equations in (1.18) and (1.19). Since

$$\frac{\partial(\Delta, x_\alpha+if_\alpha)}{\partial(w_1, x_\beta)}(0) = \Delta_{w_1}(0) = 2\gamma,$$

$\Delta=0$ and the first equation in (1.19) can be used to eliminate (w_1, x) in the first equation of (1.18). This shows that the branch loci of π_1 and $\pi_2=c\pi_1\circ\varrho$ are smooth analytic hypersurfaces in z- and w-space, respectively. The same argument as in the two dimensional case shows that an analytic function $f=f(z_1, w_1, x)$ is the trace on $\mathfrak{M}$ of a function holomorphic in z if and only if $f\circ\tau_2=f$. Thus the study of the n-manifold M also leads to consideration of a triple of involutions $(\tau_1, \tau_2, \varrho)$.

2. Quadrics and linear involutions

In this section we consider the case in which $M^n\subset\mathbf{C}^n$ is the quadric Q_γ

$$Q_\gamma:\begin{array}{l} z_n = q(z_1,\bar z_1)\equiv q_\gamma(z_1,\bar z_1),\\ y_a=0,\quad 2\leqslant a\leqslant n-1,\end{array}\tag{2.1}$$

$$q_\gamma=\gamma z_1^2+z_1\bar z_1+\gamma\bar z_1^2,\quad 0\leqslant\gamma<\infty$$
$$q_\infty=z_1^2+\bar z_1^2.$$

The coordinates are as in (1.15). This will be a prelude to the study of the general manifolds of section 1. The cases $\gamma=0, 1/2, \infty$ are exceptional and enter the discussion only in a minor way. We also consider the complex quadrics

$$\mathfrak{Q}_\gamma:\begin{array}{l} z_n=w_n=q(z_1,w_1),\\ z_a=w_a,\quad 2x_a=(z_a+w_a),\end{array}\tag{2.2}$$

where $q=q_\gamma$ is of the same form, but with γ complex. $\mathfrak{Q}_\gamma$ may come from a Q_γ by complexification.

The projections $\pi_1(z,w)=z$, $\pi_2(z,w)=w$ restricted to $\mathfrak{Q}_\gamma$ are given by the quadratic mappings

$$\pi_1(z_1,w_1,x)=(z_1,x,q(z_1,w_1)),$$

$$\pi_2(z_1,w_1,x)=(w_1,x,q(z_1,w_1)).$$

If $\gamma=0$, then π_1 collapses the lines $z_1=0$, $x=$const. to points and is otherwise one-to-one. If $\gamma\neq0$, then π_1 and π_2 are two-fold branched coverings having as covering transformations two linear involutions τ_2 and τ_1. The w-planes cut $\mathfrak{Q}_\gamma$ in the point-pairs of the involution τ_1, while the z-planes cut $\mathfrak{Q}_\gamma$ in those of τ_2. Letting $X=(z_1,w_1,x)^t$ be a column coordinate vector we have, as in (1.20, 1.21),

$$\tau_j(X)=T_jX,\ T_j^2=I,\quad j=1,2,\tag{2.3}$$

where

$$T_1 = \begin{bmatrix} -1 & -\gamma^{-1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I \end{bmatrix}, \quad T_2 = \begin{bmatrix} 1 & 0 & 0 \\ -\gamma^{-1} & -1 & 0 \\ 0 & 0 & I \end{bmatrix}, \quad I = I_{n-2}. \tag{2.4}$$

These formulas are also valid for $\gamma=\infty$. Each τ_j has a one-dimensional (-1)-eigenspace, so is a reflection in a hyperplane E_j. The cases $\gamma=\pm 1/2$ correspond to $E_1=E_2$. If $\gamma=\infty$ then a (-1)-eigenvector of τ_1 is a $(+1)$-eigenvector of τ_2, and conversely. Aside from these exceptional cases $E=E_1\cap E_2$, the space of points fixed by both τ_1 and τ_2, has dimension $n-2$, and τ_1 and τ_2 have no other common eigenvectors. The plane $F\colon x=0$ is invariant under both τ_1 and τ_2. If $\mathfrak{Q}_\gamma$ is the complexification of Q_γ, then $\mathfrak{Q}_\gamma$ carries the linear anti-holomorphic involution $\varrho(z_1, w_1, x)=(\bar{w}_1, \bar{z}_1, \bar{x})$ and $\varrho\tau_1=\tau_2\varrho$. ϱ preserves both F and E, and $E=N+iN$ where N is pointwise fixed by ϱ. N is the locus of points at which Q_γ has a complex tangent.

We now turn to the theory of a pair of holomorphic involutions on $\mathbf{C}^n$, which we assume to be given in the form (2.3). First we consider the case $n=2$. The complex 2 by 2 matrices T_j are assumed to satisfy

$$T_j^2 = I, \quad \det T_j + 1 = \operatorname{tr} T_j = 0.$$

Also, we require that T_1 and T_2 have no eigenvectors in common. The mapping $\varphi=\tau_1\tau_2$ has the matrix form

$$\varphi(X) = \Phi X, \quad \Phi = T_1 T_2, \quad \det\Phi = \pm 1. \tag{2.5}$$

LEMMA 2.1. *Let the linear transformations τ_1, τ_2, φ on $\mathbf{C}^2$ be as just described. Then φ is diagonalizable with distinct eigenvalues μ, μ^{-1}, $\mu^2 \neq 1$. If (e_1, e_2) is a basis for which*

$$\varphi(e_1) = \mu e_1, \quad \varphi(e_2) = \mu^{-1} e_2,$$

then

$$\tau_j(e_1) = \lambda_j^{-1} e_2, \quad \tau_j(e_2) = \lambda_j e_1$$

where $\lambda_1\lambda_2^{-1}=\mu$. The eigenvectors (e_1, e_2) may be chosen so that $\lambda_1=\lambda_2^{-1}\equiv\lambda$, $\lambda^4\neq 1$, and are determined up to $(e_1, e_2)\mapsto(\alpha e_1, \pm\alpha e_2)$ or $(e_1, e_2)\mapsto(e_2, e_1)$. $\operatorname{tr}\varphi=\lambda^2+\lambda^{-2}$ is an invariant of τ_1, τ_2.

Proof. Let v be an eigenvector of φ with eigenvalue μ. Then

$$\varphi(v) = \tau_1\tau_2(v) = \mu v, \quad \text{or} \quad \tau_2(v) = \mu\tau_1(v) = \mu\varphi\tau_2(v).$$

Thus $\tau_2(v)$ is also an eigenvector of φ with eigenvalue μ^{-1}. $\tau_2(v)$ and v are independent, since a relation $cv=\tau_2(v)=\mu\tau_1(v)$ would imply that v is a common eigenvector of both τ_1 and τ_2. If $\mu=\mu^{-1}$, then $\varphi=\pm\text{id}$, or $\tau_2=\pm\tau_1$, which again implies a common eigenvector. Relative to the basis $e_1=v$, $e_2=\tau_2(v)$, τ_1 and τ_2 satisfy the above relations with $\lambda_1=\mu$, $\lambda_2=1$. The change of eigenvectors of $\varphi, (e_1,e_2)\mapsto(\alpha e_1,\beta e_2)$, results in $\lambda_j\mapsto\beta\lambda_j\alpha^{-1}$. Hence, we can arrange that $\lambda_1=\lambda_2^{-1}=\lambda$, $\lambda^2=\mu$. We must then restrict to $\alpha=\pm\beta$. Q.E.D.

Now suppose that τ_1 and τ_2 satisfy $\varrho\tau_1=\tau_2\varrho$, for some linear anti-holomorphic involution $\varrho, \varrho^2=\text{id}$,

$$\varrho(X)=P\bar{X},\quad P\bar{P}=I,\quad P\bar{T}_1=T_2P.$$

Again let v be an eigenvector of φ with eigenvalue μ. Then, since $\varphi\varrho\varphi=\tau_1\varrho\tau_2=\varrho$,

$$\varrho(v)=\varphi\varrho(\mu v)=\bar{\mu}\varphi\varrho(v),$$

so that $\varrho(v)$ is an eigenvector of φ with eigenvalue $\bar{\mu}^{-1}$. Hence, either

(i) $\mu=\bar{\mu}$,

or

(ii) $\mu\bar{\mu}=1$.

Suppose μ is real and let e_1,e_2 be eigenvectors of φ as in Lemma 2.1. Since $\bar{\mu}^{-1}$ is the eigenvalue μ^{-1} and $\varrho^2=\text{id}$, we have

$$\varrho(e_1)=ae_2,\quad \varrho(e_2)=\bar{a}^{-1}e_1.$$

From $\varrho\tau_1(e_1)=\tau_2\varrho(e_1)$ we get

$$\lambda_1\bar{\lambda}_2a\bar{a}=1.$$

It follows that

$$\mu=\lambda_1\lambda_2^{-1}=a\bar{a}\lambda_1\bar{\lambda}_1>0. \tag{2.6}$$

The change $(e_1,e_2)\mapsto(\alpha e_1,\beta e_2)$ results in $(a,\lambda_j)\mapsto(\bar{\alpha}a\beta^{-1},\beta\lambda_j\alpha^{-1})$. To make $a=1$, $\lambda_1\lambda_2=1$ by such a change, we require

$$\beta=a\bar{\alpha},\quad \alpha^2\beta^{-2}=\lambda_1\lambda_2.$$

The second condition is $(\alpha/\bar{\alpha})^2=a^2\lambda_1\lambda_2=a\lambda_2(\bar{a}\bar{\lambda}_2)^{-1}$. Since this last term has modulus one, such an α exists. with this normalization, $\lambda_1=\lambda_2^{-1}=\lambda=\bar{\lambda}$. We must now restrict to

$\beta=\bar{a}$, $\alpha^2=\beta^2$. If we arrange that $\lambda>0$, then we must have $\beta=\alpha=\bar{\alpha}$. By interchanging e_1 and e_2 we may take $\lambda>1$.

Now suppose $\mu\bar{\mu}=1$, so that φ has eigenvalues $\mu=\bar{\mu}^{-1}$ and μ^{-1}. Then $\varrho(e_1)=ae_1$, and similarly, $\varrho(e_2)=be_2$. From $\varrho^2=\mathrm{id}$ we get $a\bar{a}=b\bar{b}=1$. The above change of eigenvectors results in $(a,b)\mapsto(\bar{\alpha}a\alpha^{-1},\bar{\beta}b\beta^{-1})$. Therefore we can choose e_1,e_2 so that $a=b=1$. Now we must restrict to α and β real. The relation $\varrho\tau_1(e_1)=\tau_2\varrho(e_1)$ gives $\bar{\lambda}_1=\lambda_2$. Hence by choice of real α and β, we can make $\lambda_1\lambda_2=1$. Thus, $\lambda_1=\lambda_2^{-1}=\lambda$, $\lambda\bar{\lambda}=1$. We can arrange that $\mathrm{Re}\,\lambda>0$, then we must restrict to $\beta=\alpha=\bar{\alpha}$. By interchanging e_1 and e_2 we can make $0<\arg\lambda<\pi/2$.

We introduce coordinates (ξ,η) by $X=\xi e_1+\eta e_2$, where (e_1,e_2) are the above chosen eigenvectors of φ. We have proved the following.

LEMMA 2.2. *Let τ_1,τ_2,φ be as in Lemma 2.1 and suppose that $\varrho\tau_1=\tau_2\varrho$ for some linear anti-holomorphic involution ϱ. Then there exist linear coordinates (ξ,η) in which*

$$\tau_1(\xi,\eta)=(\lambda\eta,\lambda^{-1}\xi),\quad \tau_2(\xi,\eta)=(\lambda^{-1}\eta,\lambda\xi). \tag{2.7}$$

Also, either

$$\begin{aligned}&\text{(i) } \varrho(\xi,\eta)=(\bar{\eta},\bar{\xi}) \quad \text{and} \quad \lambda=\bar{\lambda}>1, \text{ or}\\ &\text{(ii) } \varrho(\xi,\eta)=(\bar{\xi},\bar{\eta}) \quad \text{and} \quad \lambda\bar{\lambda}=1,\ 0<\arg\lambda<\pi/2.\end{aligned} \tag{2.8}$$

Such coordinates are determined up to $(\xi,\eta)\mapsto(\alpha\xi,\alpha\eta)$, $\alpha=\bar{\alpha}$.

Next we consider two linear holomorphic involutions τ_1,τ_2 on $\mathbf{C}^n$. We assume that each τ_j is a reflection in a hyperplane E_j and that $E_1\neq E_2$. Let $E=E_1\cap E_2$ and v_j be a (-1)-eigenvector of T_j, $j=1,2$. We also assume that E,v_1,v_2 span $\mathbf{C}^n$.

LEMMA 2.3. (a) *Let τ_1,τ_2 be involutions on $\mathbf{C}^n$ as just described. There exist complex linear coordinates $\xi,\eta,\zeta=(\zeta_3,\ldots,\zeta_n)$ in which*

$$\tau_j(\xi,\eta,\zeta)=(\lambda_j\eta,\lambda_j^{-1}\xi,\zeta),\quad j=1,2. \tag{2.9}$$

They may be so chosen that $\lambda_1=\lambda_2^{-1}=\lambda$ and are then determined up to replacement by $(\alpha\xi,\pm\alpha\eta,B\zeta)$, $\alpha\in\mathbf{C}$, $B\in GL(n-2,\mathbf{C})$, or by (η,ξ,ζ).

(b) *If also $\varrho\tau_1=\tau_2\varrho$ for a linear anti-holomorphic involution ϱ, then these coordinates can be further specialized so that either*

$$\begin{aligned}&\text{(i) } \varrho(\xi,\eta,\zeta)=(\bar{\eta},\bar{\xi},\bar{\zeta}) \quad \text{and} \quad \lambda>1, \text{ or}\\ &\text{(ii) } \varrho(\xi,\eta,\zeta)=(\bar{\xi},\bar{\eta},\bar{\zeta}) \quad \text{and} \quad \lambda\bar{\lambda}=1,\ 0<\arg\lambda<\pi/2.\end{aligned} \tag{2.10}$$

They are then determined up to replacement by $(\alpha\xi, \alpha\eta, B\zeta)$, $\alpha \in \mathbf{R}$, $B \in GL(n-2, \mathbf{R})$.

Proof. (a) Let F be the space spanned by v_1 and v_2 so that $\mathbf{C}^n = F \oplus E$. We claim that F is invariant under both τ_1 and τ_2. Since $\tau_1 v_1 = -v_1$, consider

$$\tau_1 v_2 = \alpha v_1 + \beta v_2 + w, \quad w \in E.$$

We must show $w=0$. Since $\tau_1^2 = \mathrm{id}$,

$$v_2 = -\alpha v_1 + \beta \tau_1 v_2 + w = (\beta - 1)\alpha v_1 + \beta^2 v_2 + (\beta + 1) w.$$

If $w \neq 0$, then (v_1, v_2, w) are independent so $\beta = -1$, and $\alpha = 0$. This implies that

$$\tau_1(v_2 - \tfrac{1}{2}w) = -(v_2 - \tfrac{1}{2}w),$$

hence $v_2 - w/2 = cv_1$, which contradicts independence. Hence, $w=0$, and $\tau_1(F)=F$. A similar argument shows that $\tau_2(F)=F$. Let τ_j' be the restriction of τ_j to F. Then $\det \tau_j' = -1$, and by the condition on v_1, v_2 and E, τ_1' and τ_2' can have no common eigenvector. Hence, we may apply Lemma 2.1 to τ_1', τ_2', to get basis vectors e_1, e_2 of F. We let $e_3, \dots, e_n$ be any basis of E, and (ξ, η, ζ) coordinates relative to $e_1, \dots, e_n$.

(b) We first show that ϱ leaves E invariant. If $\tau_j w = w$, then $\tau_j \varrho(w) = \varrho(w)$ follows from $\tau_1 \varrho = \varrho \tau_2$, hence $\varrho(E) = E$. Let N be the totally real fixed point set of ϱ on E, $E = N + iN$. Choose the coordinates ζ on E so that $\varrho: \zeta \mapsto \bar{\zeta}$. We next show that F is invariant under ϱ. To see this note that

$$\tau_1 \varrho(F) = \varrho \tau_2(F) = \varrho(F)$$
$$\tau_2 \varrho(F) = \varrho \tau_1(F) = \varrho(F).$$

Hence, $\varrho(F)$ is invariant under both τ_1 and τ_2. Relative to a basis compatible with the decomposition $\varrho(F) \oplus E = \varrho(F) \oplus \varrho(E) = \mathbf{C}^n$ it is easy to see that $\det \tau_j' = -1$, where $\tau_j' = \tau_j|_{\varrho(F)}$. So τ_j' has a (-1)-eigenvector u_j in $\varrho(F)$. By the assumption made on τ_1, τ_2, we must have $u_j = cv_j$. It follows that u_1, u_2 are independent and $\varrho(F) = F$. We now apply Lemma 2.2 to τ_1', τ_2'. Q.E.D.

Given involutions τ_1, τ_2, ϱ as in (2.7, 2.8) in canonical coordinates (ξ, η, ζ) $(\lambda_1 = \lambda_2^{-1} = \lambda)$, we shall construct a quadric Q_γ. For this we must construct the "holomorphic" coordinates z and the "anti-holomorphic" coordinates w. Linear combinations of the ζ_α are invariant under both τ_1 and τ_2. Aside from these the most general linear functions invariant under τ_2 and τ_1, respectively, are

$$z_1 = b(\lambda\xi+\eta)$$
$$w_1 = a(\xi+\lambda\eta),$$

where a and b are complex constants. We should also choose z_1 and w_1 so that ϱ corresponds to $(z_1, w_1, x)\mapsto(\bar{w}_1, \bar{z}_1, \bar{x})$. If $\lambda=\bar{\lambda}$ we need $a=\bar{b}$, while if $\lambda\bar{\lambda}=1$ we need $a\lambda=\bar{b}$. Thus for the two cases in (2.10) we take

$$\text{(i)}\quad \begin{matrix} z_1 = b(\lambda\xi+\eta) \\ w_1 = \bar{b}(\xi+\lambda\eta) \end{matrix},$$

$$\text{(ii)}\quad \begin{matrix} z_1 = b(\lambda\xi+\eta) \\ w_1 = \bar{b}\bar{\lambda}(\xi+\lambda\eta) \end{matrix}.$$

The quadratic functions invariant under both τ_1 and τ_2 are linear combinations of $\xi\eta$ and $\zeta_\alpha\zeta_\beta$.

We want to choose b so that $q(z_1, w_1)$ is a multiple of $\xi\eta$, for some q of the form (2.1). In case (ii) this requires that

$$b^2\lambda^2+\bar{b}^2\bar{\lambda}^2 = b^2+\bar{b}^2.$$

Taking $b\bar{b}=1$, we get $b^4=\lambda^{-2}$.

Hence, in both cases, we arrive at

$$\begin{aligned} z_1 &= i\lambda^{-1/2}(\lambda\xi+\eta), \\ w_1 &= -i\lambda^{-1/2}(\xi+\lambda\eta), \\ z_\alpha &= w_\alpha = x_\alpha = \zeta_\alpha. \end{aligned} \tag{2.11}$$

It follows that

$$\begin{aligned} &q \equiv z_1 w_1+\gamma(z_1^2+w_1^2) = \gamma^{-1}(1-4\gamma^2)\,\xi\eta, \\ &\gamma = (\lambda+\lambda^{-1})^{-1} > 0. \end{aligned} \tag{2.12}$$

We define

$$z_n = w_n = \gamma^{-1}(1-4\gamma^2)\,\xi\eta. \tag{2.13}$$

τ_1, τ_2, ϱ are the involutions induced on the surface (2.12, 2.13). We may write the relation between γ and λ as

$$\gamma\lambda^2 - \lambda + \gamma = 0. \tag{2.14}$$

If $\lambda \neq \pm 1$ is real, (2.14) has two distinct real roots, λ, λ^{-1}. It follows that $\gamma < 1/2$ and the surface is elliptic. If $\lambda\bar{\lambda} = 1$ and $0 < \operatorname{Re}\lambda < 1$, then $\gamma = (1/2)(\operatorname{Re}\lambda)^{-1} > 1/2$ and the surface is hyperbolic.

Conversely, given τ_1 and τ_2 with the matrices (2.4), we can use (2.11) to define the canonical coordinates (ξ, η, ζ). We only have to find λ. $\lambda^2 = \mu$ is an eigenvalue of $\varphi = \tau_1 \tau_2 = \tau_1 \varrho \tau_1 \varrho$. In terms of matrices $\Phi = T_1 T_2 = T_1 P \bar{T}_1 \bar{P} = (-T_1 P)^2$, where T_i are given by (2.4) and

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The eigenvalues of $-T_1 P$ are given by (2.14) together with $\lambda = -1$.

The mapping (2.11), (2.13) has an interesting geometric interpretation. In the elliptic case the relation $w_1 = \bar{z}_1$ corresponds to $\eta = \bar{\xi}$. Under (2.11) the ellipses $q(z_1, \bar{z}_1) = c > 0$ are mapped to the circles $\xi\bar{\xi} = c' > 0$. In fact (2.11, 2.13) maps Q_γ to Q_0. In the hyperbolic case the relation $w_1 = \bar{z}_1$ corresponds to $\xi = \bar{\xi}$, $\eta = \bar{\eta}$. The hyperbolas $q(z_1, \bar{z}_1) = c$ are mapped to the standard hyperbolas $\xi\eta = c'$. Q_γ is mapped to Q_∞ by (2.11, 2.13). Of course, (2.11, 2.13) is *not* holomorphic in the usual sense.

The linear map $\varphi = \tau_1 \tau_2$,

$$\varphi(\xi, \eta, \zeta) = (\mu\xi, \mu^{-1}\eta, \zeta)$$

leaves fixed the linear space $\xi = \eta = 0$, i.e. $z_1 = w_1 = 0$. As mentioned above this is the complexification of the space of those points on Q_γ having complex tangents. When $n = 2$, φ has an isolated fixed point which is hyperbolic ($\mu > 0$) if Q_γ is elliptic, and elliptic ($\mu\bar{\mu} = 1$) when Q_γ is hyperbolic. φ may be interpolated by the flow

$$\varphi^t(\xi, \eta, \zeta) = (e^{t\nu}\xi, e^{-t\nu}\eta, \zeta), \tag{2.15}$$

where $e^\nu = \mu$, $\varphi^1 = \varphi$, and either

(i) $\nu = \bar{\nu}$

or

(ii) $\nu + \bar{\nu} = 0$.

φ^t preserves the family of complex conics $\xi\eta=\text{const.}$, i.e. $q(z_1, w_1)=c$, on $\mathfrak{Q}_\gamma$. If c is real the complex conic meets Q_γ in a real conic, which may be degenerate. φ does not preserve Q_γ since $\varrho^{-1}\varphi\varrho=\varphi^{-1}\neq\varphi$, if $\gamma\neq\infty$. If we allow complex t, then $\varphi^t\varrho=\varrho\varphi^{-\bar{t}}$ in both cases. Thus φ^t commutes with ϱ precisely when $t+\bar{t}=0$. The orbits of φ^t on Q_γ for $t+\bar{t}=0$ are the real conics. The infinitesimal generator is a vector field on Q_γ tangent to these curves.

As mentioned in section 1, $\varphi^2=[\tau_1, \tau_2]=\text{id}$ is a direct analogue of the vanishing of the Levi-form on a real hypersurface. Among the quadrics Q_γ this happens only when $\gamma=\infty$. Q_∞ is the intersection of *two* Levi-flat hypersurfaces $\operatorname{Re}(z_2-2z_1^2)=\operatorname{Im} z_2=0$. A weaker condition is that φ should be nilpotent. This happens precisely when λ is a root of unity and causes difficulties for the normal form in section 3. The eigenvalues of φ are multiple precisely when $\mu=\pm 1$. It is an interesting fact that φ is diagonalizable ($\varphi=-I$) for $\mu=-1$, i.e. $\gamma=\infty$, while for $\mu=+1$, i.e. $\gamma=1/2$, φ is not diagonalizable.

Finally, we make a remark on the automorphism group of Q_γ, $\gamma\neq 0, 1/2, \infty$. It is clear that the holomorphic map

$$\begin{aligned} z_1' &= a(z_n, z_a)\, z_1, \quad a=\bar{a},\ b_a=\bar{b}_a, \\ z_n' &= a^2(z_n, z_a)\, z_n, \quad a\neq 0,\ \det b'\neq 0, \\ z_a' &= b_a(z_n, z_\beta), \quad \text{for } z=0, \end{aligned} \tag{2.16}$$

preserves Q_γ. Via the mapping (2.11) this corresponds to

$$(\xi, \eta, \zeta)\mapsto(a(\xi\eta, \zeta)\,\xi,\ a(\xi\eta, \zeta)\,\eta,\ b_a(\xi\eta, \zeta)\,\zeta_a),$$

which is an automorphism of the set of involutions τ_1, τ_2, ϱ. In the next section we shall use this to show the most general self transformation of Q_γ is of the form (2.16) where a and b_a are arbitrary real formal power series, if γ is not exceptional.

3. The formal theory of a pair of involutions

The considerations of section 1 have led us to a pair of holomorphic involutions τ_1, τ_2 defined in a neighborhood of a fixed point on a complex manifold $\mathfrak{M}$. In this and the following section we assume $\mathfrak{M}=\mathbf{C}^n$, with coordinates $x, y, z=(z_a)$, $2\leq a\leq n-1$, and that the origin is the fixed point. We now ask for a new coordinate system $\xi, \eta, \zeta=(\zeta_a)$ in which τ_1, τ_2 take a particularly simple form, a so-called normal form. We shall first discuss the normal form in the realm of formal power series on a purely algebraic level. Later, in the next section, we discuss the question of convergence. The case in which τ_1 and τ_2 are intertwined by an anti-holomorphic involution ϱ will also be considered.

One may proceed directly with the mappings τ_j; however, we shall base our analysis on the mapping $\varphi=\tau_1\tau_2$. As in the linear case (section 2) we shall normalize φ and then show that this forces a normalization of τ_1 and τ_2.

By the results of section 2 we may take τ_j of the form

$$\tau_j\colon \begin{aligned} x' &= \lambda_j y+p_j(x,y,z) \\ y' &= \lambda_j^{-1}x+q_j(x,y,z), \quad j=1,2. \\ z'_\alpha &= z_\alpha+r_{j\alpha}(x,y,z) \end{aligned} \tag{3.1}$$

Then φ has the form

$$\varphi\colon \begin{aligned} x' &= \mu x+f(x,y,z) \\ y' &= \mu^{-1}y+g(x,y,z) \\ z'_\alpha &= z_\alpha+h_\alpha(x,y,z) \end{aligned} \tag{3.2}$$

where $\mu=\lambda_1\lambda_2^{-1}$. Here p_j, q_j, r_j, f, g, h, are formal power series vanishing to second order at the origin. We subject these mappings to the group $\mathfrak{G}^1$ of formal transformations which agree with the identity to second order. Such a $\psi\in\mathfrak{G}^1$ has the form

$$\psi\colon \begin{aligned} x &= U(\xi,\eta,\zeta)=\xi+u(\xi,\eta,\zeta) \\ y &= V(\xi,\eta,\zeta)=\eta+v(\xi,\eta,\zeta) \\ z &= W(\xi,\eta,\zeta)=\zeta+w(\xi,\eta,\zeta), \end{aligned} \tag{3.3}$$

where z, w, W are $(n-2)$-vector valued and u, v, w begin with quadratic terms. We call ψ *normalized* if the power series u, v, w do not contain terms of the form $\xi^{j+1}\eta^j, \xi^j\eta^{j+1}$, or $\xi^j\eta^j$, respectively, for any $j\in\mathbf{Z}^+$. Any formal power series $p=p(\xi,\eta,\zeta)$ may be decomposed as

$$p=\sum_{s=-\infty}^{\infty} p_s(\xi,\eta,\zeta), \quad p_s(\xi,\eta,\zeta)=\sum_{i-j=s}\sum_{|K|=0}^{\infty} p_{ijK}\,\xi^i\eta^j\zeta^K.$$

We shall say that p_s has *type* s. The normalizing conditions on ψ may be expressed as

$$u_1=0, \quad v_{-1}=0, \quad w_0=0. \tag{3.4}$$

LEMMA 3.1. *Any $\psi\in\mathfrak{G}^1$ can be uniquely factored into*

$$\psi=\psi_0\delta,$$

where ψ_0 is normalized and δ has the form

$$\delta\colon(\xi,\eta,\zeta)\mapsto(\alpha\xi,\beta\eta,\zeta+\gamma),$$

where α, β, γ are power series in ζ and the product $\xi\eta$. If ψ is convergent, so are ψ_0 and δ.

Proof. We may define such α, β, γ by

$$U_1 = \alpha(\xi\eta, \zeta)\,\xi, \quad V_{-1} = \beta(\xi\eta, \zeta)\,\eta, \quad W_0 = \zeta + \gamma(\xi\eta, \zeta), \tag{3.5}$$

and form δ as in the statement of the lemma. Since the transformation $p \mapsto p(\alpha\xi, \beta\eta, \zeta+\gamma)$ commutes with the projections $p \mapsto p_s$ it follows that $\psi_0 = \psi \circ \delta^{-1}$ is normalized. Conversely, any such decomposition ψ_0, δ forces (3.5) so δ and ψ_0 are unique. It is also clear that if ψ converges, so does δ and hence also ψ_0. Q.E.D.

LEMMA 3.2. *Let τ_j, $j=1,2$ be two formal involutions given by* (3.1) *with $\mu = \lambda_1 \lambda_2^{-1}$ not a root of unity. Then there exists a unique normalized transformation ψ as in* (3.3) *such that relative to the coordinates (ξ, η, ζ)*

$$\psi^{-1} \circ \tau_j \circ \psi : \begin{cases} \xi' = \Lambda_j \eta \\ \eta' = \Lambda_j^{-1} \xi, \; j = 1, 2; \\ \zeta' = \zeta \end{cases} \quad \psi^{-1} \varphi \psi : \begin{cases} \xi' = M\xi \\ \eta' = M^{-1}\eta; \\ \zeta' = \zeta \end{cases} \tag{3.6}$$

where $M = \Lambda_1 \Lambda_2^{-1}$ and the $\Lambda_j = \lambda_j + \ldots$ are formal power series in ζ and the product $\xi\eta$.

Proof. We proceed by induction on the homogeneous degree in all variables of the terms in $\psi^{-1}\tau_j\psi$. We assume that τ_j has been transformed so as to have the form (3.6) modulo terms of order m and higher by a unique choice of the terms in ψ of order less than m. It will suffice to show that the term of order m in ψ can be chosen uniquely so that $\psi^{-1}\tau_j\psi$ has the form (3.6) modulo terms of order $m+1$. Thus assume τ_j has the form

$$\tau_j : \begin{cases} x' = \Lambda_j y + p_j + \ldots \\ y' = \Lambda_j^{-1} x + q_j + \ldots, \quad j = 1, 2 \\ z' = z + r_j + \ldots \end{cases} \tag{3.7}$$

where $\Lambda_j = \Lambda_j(xy, z)$ are polynomials of degree $< m-1$, p_j, q_j, r_j are homogeneous polynomials of degree $m \geqslant 2$, and the dots indicate higher order terms. Using $\tau_j^2 = \mathrm{id}$ and noting $\Lambda_j \tau_j = \Lambda_j + O(m)$, we get

$$\begin{aligned} &\lambda_j q_j(x, y, z) + p_j(\lambda_j y, \lambda_j^{-1} x, z) = 0, \quad j = 1, 2, \\ &r_j(x, y, z) + r_j(\lambda_j y, \lambda_j^{-1} x, z) = 0, \quad j = 1, 2. \end{aligned} \tag{3.8}$$

It follows that φ has the form

$$\varphi: \begin{cases} x' = Mx+a+\dots \\ y' = M^{-1}y+b+\dots \\ z' = z+c+\dots \end{cases} \tag{3.9}$$

where $M=\Lambda_1\Lambda_2^{-1}$ and a, b, c are the homogeneous polynomials of degree m given by

$$\begin{aligned} a(x,y,z) &= \lambda_1 q_2(x,y,z)+p_1(\lambda_2 y, \lambda_2^{-1}x, z) \\ b(x,y,z) &= \lambda_1^{-1}p_2(x,y,z)+q_1(\lambda_2 y, \lambda_2^{-1}x, z) \\ c(x,y,z) &= r_2(x,y,z)+r_1(\lambda_2 y, \lambda_2^{-1}x, z). \end{aligned} \tag{3.10}$$

Now let ψ have the form (3.3, 3.4) in which u, v, w are homogeneous polynomials of degree m. We shall choose u, v, w so that $\tilde{\varphi}=\psi^{-1}\varphi\psi$ has the form given in (3.6) modulo terms of order $m+1$, and then show that automatically the $\psi^{-1}\tau_j\psi$ also have the form in (3.6) to the same order. Let $\tilde{\varphi}$ be as in (3.9) with $\tilde{M}=M$ and $\tilde{a}, \tilde{b}, \tilde{c}$ homogeneous of degree m in (ξ, η, ζ). Since $M(xy, z)=M(\xi\eta, \zeta)+O(m)$, comparison of terms of degree m in $\psi\tilde{\varphi}=\varphi\psi$ gives

$$\begin{aligned} u(\mu\xi, \mu^{-1}\eta, \zeta)-\mu u(\xi,\eta,\zeta) &= (a-\tilde{a})(\xi,\eta,\zeta) \\ v(\mu\xi, \mu^{-1}\eta, \zeta)-\mu^{-1} v(\xi,\eta,\zeta) &= (b-\tilde{b})(\xi,\eta,\zeta) \\ w(\mu\xi, \mu^{-1}\eta, \zeta)-w(\xi,\eta,\zeta) &= (c-\tilde{c})(\xi,\eta,\zeta). \end{aligned} \tag{3.11}$$

We wish to make $\tilde{a}_s=0$, for $s\neq 1$, $\tilde{b}_s=0$, for $s\neq -1$, and $\tilde{c}_s=0$, for $s\neq 0$, where s indicates the type. This leads to the equations

$$\begin{aligned} (\mu^s-\mu)\, u_s &= a_s, \quad s\neq 1 \\ (\mu^s-\mu^{-1})\, v_s &= b_s, \quad s\neq -1 \\ (\mu^s-1)\, w_s &= c_s, \quad s\neq 0, \end{aligned}$$

which clearly can be solved for u_s, v_s, w_s since by our assumption no power of μ is unity.

For the exceptions just made the left hand sides vanish, forcing

$$\tilde{a}_1 = a_1, \quad \tilde{b}_{-1} = b_{-1}, \quad \tilde{c}_0 = c_0.$$

The normalization (3.4) makes the solution unique. Hence, we can achieve that

$$a = A(xy, z)\,x, \quad b = B(xy, z)\,y, \quad c = C(xy, z) \tag{3.12}$$

by a unique choice of the terms of order m in ψ, if ψ is normalized.

We next show this actually implies $c=0$ and $r_j=0, j=1,2$. To see this let τ_1, τ_2, φ *temporarily* stand for the linear parts of these mappings. By (3.12) we have $C\tau_1=C\tau_2=C$. The third equation in (3.10) is equivalent to

$$r_2+r_1\tau_2=C, \quad r_1+r_2\tau_2=C.$$

(The second relation follows from an application of τ_2.) The second equation of (3.8) gives respectively for $j=1,2$,

$$r_2\tau_2=-r_2, \quad r_1\varphi=-r_1\tau_2.$$

Hence,

$$r_2-r_1\varphi=r_1-r_2=C$$

or

$$r_1-r_1\varphi=2C.$$

Since μ is not a root of unity, this last relation implies that the terms of type $s\neq 0$ in r_1 vanish. Therefore $r_1\tau_1=r_1$. By (3.8) $r_1=0$. It follows that $r_2=-C$ is of type 0, so must also vanish. We next want to show that

$$p_j(x,y,z)=P_j(xy,z)y, \quad q_j(x,y,z)=Q_j(xy,z)x. \tag{3.13}$$

To see this we write the first two equation of (3.10), taking into account (3.12), in the form

$$\lambda_1 q_2+p_1\tau_2=xA, \quad \lambda_1^{-1}p_2+q_1\tau_2=yB.$$

We also write the first equation in (3.8) as

$$q_1=-\lambda_1^{-1}p_1\tau_1, \quad q_2=-\lambda_2^{-1}p_2\tau_2.$$

Eliminating q_1 and q_2 we get

$$p_1-\mu p_2=\lambda_2 yA, \quad p_2-p_1\varphi=\lambda_1 yB,$$

where we have used $(xA)\tau_2=\lambda_2 yA$. Eliminating first p_2 and then p_1 and using $(xA)\circ\varphi=\mu xA$, we get

$$p_1-\mu p_1\varphi=y(\lambda_2 A+\mu\lambda_1 B)$$
$$p_2-\mu p_2\varphi=y(\lambda_2\mu^{-1}A+\lambda_1 B).$$

Again, since μ is not a root of unity, these equations imply that p_1 and p_2 are of type $s=-1$ in (x, y). Thus the first equation in (3.13) holds. Eliminating p_1 and p_2 via $p_j=-\lambda_j q_j \tau_j$ gives

$$q_1-\mu^{-1}q_1\varphi = x(\mu^{-1}\lambda_1^{-1}A+\lambda_2^{-1}B),$$

$$q_2-\mu^{-1}q_2\varphi = x(\lambda_1^{-1}A+\lambda_2^{-1}\mu B).$$

These imply that q_1 and q_2 are of type $s=+1$. Hence, (3.13) is proved.

Returning τ_j, φ to their original meanings, we may write (3.7) as

$$\tau_j\colon \begin{aligned} x' &= (\Lambda_j+P_j)y+\dots \\ y' &= (\Lambda_j^{-1}+Q_j)x+\dots \\ z' &= z+\dots, \end{aligned}$$

where the dots indicate terms of order $m+1$ and higher. The relations (3.8) and (3.13) give

$$\lambda_j^{-1}P_j+\lambda_j Q_j = 0, \quad j=1,2,$$

which imply that

$$(\Lambda_j+P_j)(\Lambda_j^{-1}+Q_j) = 1+O(m).$$

If we replace Λ_j by Λ_j+P_j, then we have achieved (3.7) with the degree m replaced by $m+1$. By induction we can achieve the form (3.6) for $\psi^{-1}\tau_j\psi$ with a unique normalized ψ. The form of $\psi^{-1}\varphi\psi$ follows, and the lemma is proved. Q.E.D.

In view of the applications we wish to make to surfaces we consider the case in which τ_1 and τ_2 are intertwined by one of the linear anti-holomorphic involutions ϱ

(i) $\varrho(x, y, z) = (\bar{y}, \bar{x}, \bar{z})$,

(ii) $\varrho(x, y, z) = (\bar{x}, \bar{y}, \bar{z})$. (3.14)

We have the following lemma.

LEMMA 3.3. *Suppose that the τ_1, τ_2 of Lemma 3.2 also satisfy $\varrho\tau_1=\tau_2\varrho$, where ϱ is one of the anti-holomorphic involutions (3.14). Then the transformation ψ satisfies $\psi\varrho=\varrho\psi$, and the factors Λ_1, Λ_2 are related by*

(i) $\bar{\Lambda}_1(\xi\eta, \zeta) = \Lambda_2(\xi\eta, \zeta)^{-1}$,

(ii) $\Lambda_1(\xi\eta, \zeta) = \bar{\Lambda}_2(\xi\eta, \zeta)$.

Proof. Let τ_j^* denote the normal forms (3.6) so that $\psi\tau_j^*=\tau_j\psi$, $j=1,2$. We have $(\varrho\psi\varrho)(\varrho\tau_j^*\varrho)=(\varrho\tau_j\varrho)(\varrho\psi\varrho)$. By the special form (3.14) of ϱ it is easy to see that $\varrho\psi\varrho$ also has the form (3.3) with the normalization (3.4). Also, $\varrho\tau_j^*\varrho$ is of the form (3.6). By the uniqueness part of Lemma 3.2 it follows that $\varrho\psi\varrho=\psi$ and consequently $\varrho\tau_1^*\varrho=\tau_2^*$. This gives the condition on Λ_1 and Λ_2. Q.E.D.

THEOREM 3.4. *Let τ_1 and τ_2 be two involutions as in Lemma 3.2. Then there exists a transformation ψ in $\mathfrak{G}^1$ taking τ_1 and τ_2 into the form*

$$\begin{aligned}\psi^{-1}\tau_1\psi&:(\xi,\eta,\zeta)\mapsto(\Lambda\eta,\Lambda^{-1}\xi,\zeta)\\ \psi^{-1}\tau_2\psi&:(\xi,\eta,\zeta)\mapsto(\Lambda^{-1}\eta,\Lambda\xi,\zeta),\end{aligned}\tag{3.15}$$

where $\Lambda=\lambda+\dots$, $\operatorname{Re}\lambda>0$, is a formal power series in ζ and the product $\xi\eta$. The most general transformation of the τ_j into this normal form is $\psi\circ\sigma$ where

$$\sigma:(\xi,\eta,\zeta)\mapsto(r(\xi\eta,\zeta)\,\xi,\ r(\xi\eta,\zeta)\,\eta;\ f(\xi\eta,\zeta)),\tag{3.16}$$

and $r(0,0)\neq0$ and f is invertible. If in addition $\varrho\tau_1=\tau_2\varrho$, where ϱ is given by (3.14), then $\psi\varrho=\varrho\psi$ and $r(\xi\eta,\zeta)=\bar r(\xi\eta,\zeta)$, $f(\xi\eta,\zeta)=\bar f(\xi\eta,\zeta)$. Λ satisfies

(i) $\Lambda(\xi\eta,\zeta)=\bar\Lambda(\xi\eta,\zeta)$,

(ii) $\Lambda(\xi\eta,\zeta)\bar\Lambda(\xi\eta,\zeta)=1$,

according to the form of ϱ.

Proof. By the linear theory we may assume $\lambda_1=\lambda_2^{-1}=\lambda$, $\operatorname{Re}\lambda>0$. Consider the mapping

$$(\xi,\eta,\zeta)\mapsto(\nu(\xi\eta,\zeta)\,\xi,\ \nu(\xi\eta,\zeta)^{-1}\eta,\zeta),$$

which preserves $\xi\eta$. It commutes with ϱ if

(i) $\nu\bar\nu=1$,

or

(ii) $\nu=\bar\nu$.

Its effect is to preserve the form (3.6) while replacing Λ_j by $\Lambda_j\nu^{-2}$. We can make $\Lambda_1\Lambda_2=1$ by choosing

$$\nu^4=\Lambda_1\Lambda_2.\tag{3.17}$$

If ϱ is given by (3.14 i), then $\Lambda_1 \bar{\Lambda}_2=1$, so there is a fourth root ν satisfying $\nu\bar{\nu}=1$. If ϱ is given by (3.14 ii), then $\Lambda_1=\bar{\Lambda}_1$ and there is a real fourth root. Let $\psi_1 \in \mathfrak{G}^1$ be a transformation of the τ_j into normal form τ_j^*. We factor $\psi_1=\psi_0 \circ \delta$ as in Lemma 3.1. It is easy to check that the transformation δ takes normalized involutions into normalized involutions. So $\psi_0 \delta \tau_j^* = \tau_j \psi_0 \delta$, or $\psi_0(\delta\tau_j^*\delta^{-1})=\tau_j\psi_0$, implies $\psi_0=\psi$ and $\delta\tau_j^*=\tau_j^*\delta$ by the uniqueness statement in Lemma 3.2. This last relation gives

$$\Lambda_j(\alpha\beta\xi\eta, \zeta+\gamma)\beta = \alpha\Lambda_j(\xi\eta, \zeta), \quad j=1,2.$$

Since $\Lambda_1\Lambda_2=1$, we get $\alpha^2=\beta^2$. The restriction $\operatorname{Re}\lambda>0$ forces $\alpha=\beta\equiv r$, and we set $f=\zeta+\gamma(\xi\eta, \zeta)$. If $\psi_1 \notin \mathfrak{G}^1$ its linear part has the form resulting from Lemma 2.3. This proves the theorem. Q.E.D.

Let Q_γ be one of the quadrics (2.1) with $\gamma\neq 0, 1/2, \infty$. We shall say that Q_γ is an *exceptional hyperboloid* if λ given by (2.14) is a root of unity. Necessarily $\gamma>1/2$. If ψ is a formal automorphism of Q_γ, it induces on $\mathfrak{Q}_\gamma$ a mapping ψ satisfying

$$\psi\tau_j=\tau_j\psi, \quad \psi\varrho=\varrho\psi.$$

By the theorem ψ is of the form (3.16) with r and f real. Passing to z, w coordinates via (2.11), (2.13) we get a mapping of the form (2.16). Hence, we have

COROLLARY 3.5. *Suppose that $\gamma\neq 0, 1/2, \infty$ and that Q_γ is not an exceptional hyperboloid. Then the most general formal automorphism of Q_γ is of the form* (2.16).

The normal form for φ lends itself to showing that φ can be embedded in a flow φ^t with $\varphi^1=\varphi$, $\varphi^0=\mathrm{id}$; $\varphi^{t_1+t_2}=\varphi^{t_1}\circ\varphi^{t_2}$. We discuss this question for $n=2$, since the variables ζ are uninteresting for this problem. In the realm of formal power series a mapping

$$\varphi\colon (\xi, \eta)\mapsto(\mu\xi+\ldots, \mu^{-1}\eta+\ldots)$$

with μ not a root of unity can always be embedded in such a flow; moreover if $\varrho\varphi=\varphi^{-1}\varrho$ we have $\varrho\varphi^t=\varphi^{-t}\varrho$ and the embedding is essentially unique. The freedom is determined by the choice of $\log\mu$ alone.

The existence of such an interpolation follows at once from the normal form

$$\varphi\colon (\xi, \eta)\mapsto(M\xi, M^{-1}\eta)$$

by defining the formal power series

$$N(\xi\eta)=\log\mu+\log(\mu^{-1}M)$$

where the second term is a series without constant term. Thus the embedding is given by

$$\varphi^t: (\xi, \eta) \mapsto (e^{tN}\xi, e^{-tN}\eta).$$

If μ is real, and a posteriori positive, we have $M=\bar{M}$ and can define N as a real series. If $|\mu|=1$ we have $M\bar{M}=1$ and we have in N a series with purely imaginary coefficients, i.e. we have in the two cases

(i) $N=\bar{N}$,

(ii) $N+\bar{N}=0$.

This implies $\varrho\varphi^t=\bar{\varphi}^{-t}\varrho$ in both cases.

This flow is generated by the t-independent vectorfield

$$\dot{\xi} = N(\xi\eta)\,\xi, \quad \dot{\eta} = -N(\xi\eta)\,\eta$$

which preserves the function $\xi\eta$.

We note that φ^t is holomorphic if the transformation into the normal form converges. This fact will be of importance in section 5 in the description of the boundaries of analytic discs as orbits of these flows.

To establish that φ^t is uniquely determined by φ and the choice of $\log\mu$ we note that φ^t has to commute with φ and therefore is of the form

$$\varphi^t: (\xi, \eta) \mapsto (\alpha_t(\xi\eta)\,\xi, \beta_t(\xi\eta)\,\eta)$$

by our previous considerations. This corresponds to a differential equation of the form

$$\dot{\xi} = A(\xi\eta)\,\xi, \quad \dot{\eta} = B(\xi\eta)\,\eta. \tag{*}$$

We claim that $A+B=0$. If this were not the case we would have

$$A+B = c(\xi\eta)^s + \ldots, \quad c \neq 0$$

for some $s \geq 1$ and therefore

$$(\xi\eta)^{\cdot} = (A+B)\,\xi\eta = c(\xi\eta)^{s+1+\cdots}.$$

By integration of this formal differential equation we find

$$(\xi\eta)(t) = (\xi\eta) + ct(\xi\eta)^{s+1} + \ldots .$$

For $t=1$ we see that φ would not preserve $\xi\eta$, a contradiction. Hence $A+B=0$ and $\xi\eta$ is a constant for the differential equation (*) which can be integrated to

$$\varphi^t : (\xi, \eta) \mapsto (e^{tA}\xi, e^{-tA}\eta),$$

i.e. $e^A = M$, $A = \log M$, proving our claim.

There is another way to define φ^t which goes back to G. D. Birkhoff [3]. If μ is not a root of unity one shows inductively that the iterates $\varphi^j, j=1,2,\ldots$ of φ can be written in form

$$\varphi^j : (\xi, \eta) \mapsto \left(\mu^j \xi + \sum_{d=2}^{\infty} f_d(\xi, \eta, \mu^j),\ \mu^{-j}\eta + \sum_{d=2}^{\infty} g_d(\xi, \eta, \mu^j) \right)$$

where f_d, g_d, the homogeneous polynomials of degree d in ξ, η, have coefficients which are polynomials in μ^j and μ^{-j} of degree $\leq d$. By replacing μ^j by $\mu^t = e^{t \log \mu}$ one obtains the formal series for φ^t. It was a fundamental observation of Birkhoff that—at least in the case of area preserving mappings—the series for φ^t will in general diverge for non-integer t even if φ and hence φ^j converges. In fact, in the case of area preserving mappings, the convergence of the transformation into normal form occurs precisely if this embedding can be achieved with convergent φ^t. The relation

$$\varrho \circ \varphi^t = \varphi^{-\bar{t}} \circ \varrho$$

shows that φ^t commutes with ϱ precisely if $t+\bar{t}=0$, i.e. if t is purely imaginary. This will imply that φ^t for $t+\bar{t}=0$ gives rise to a flow on the real analytic manifold M^n (see section 5).

4. Convergence

In general the transformation ψ of Lemma 3.2 taking the pair of involutions τ_1, τ_2 into the normal form (3.6) does not converge even if τ_1 and τ_2 are given by convergent series. However, the following result gives a sufficient condition for convergence. It is proved by a majorant argument along the lines of the argument given in [12] and [11] for hyperbolic area preserving mappings.

THEOREM 4.1. *Let the involutions τ_1, τ_2 be given by* (3.1), *where $p_j, q_j, r_j, j=1,2$, are convergent power series. If $|\lambda_1| \neq |\lambda_2|$ then the normalized transformation ψ and the factors Λ_1, Λ_2 of Lemma* 3.2 *are given by convergent power series.*

Proof. We make use of the following notations. Let $f(x), g(x), \ldots, h(y)$ be power series in some variables **x** and **y**. If $f(x)=\Sigma a_I x^I$ (multi-index notation) then $f^*(x)=\Sigma |a_I| x^I$. Also, $f \prec g$ means that g has non-negative coefficients b_I, and $|a_I| \leq b_I$, for all I. Note that if $f_i \prec g_i$ and $h_1 \prec h_2$ then $h_1(f_1, f_2, \ldots) \prec h_2(g_1, g_2, \ldots)$, if f_i, g_i have no constant terms.

Our argument will be based on the fact that ψ, given by (3.3, 3.4), transforms φ, given by (3.2), into the normal form $\tilde{\varphi}$, given in (3.6). The relation $\psi \circ \tilde{\varphi} = \varphi \circ \psi$ gives the functional equations

$$\begin{aligned} &U(M\xi, M^{-1}\eta, \zeta) - \mu U(\xi, \eta, \zeta) = f(U, V, W) \\ &V(M\xi, M^{-1}\eta, \zeta) - \mu^{-1} V(\xi, \eta, \zeta) = g(U, V, W) \\ &W_\alpha(M\xi, M^{-1}\eta, \zeta) - W_\alpha(\xi, \eta, \zeta) = h_\alpha(U, V, W). \end{aligned} \tag{4.1}$$

We decompose these equations by equating terms of the same type s, $-\infty < s < +\infty$, (see the definition before (3.4)),

$$\begin{aligned} &(M^s - \mu)\, U_s = [f(U, V, W)]_s \\ &(M^s - \mu^{-1})\, V_s = [g(U, V, W)]_s \\ &(M^s - 1)(W_\alpha)_s = [h_\alpha(U, V, W)]_s. \end{aligned} \tag{4.2}$$

By interchanging τ_1 and τ_2 if necessary, we may assume that $|\lambda_1| > |\lambda_2|$, i.e. $|\mu| > 1$. M^{-1} is a formal power series with constant term μ^{-1}, $|\mu^{-1}| < 1$. Let $P^0 = (M^{-1} - \mu^{-1})^*$ and $P = |\mu|^{-1} + P^0$, so that $M^{-1} \prec P$.

We next prove the following relations.

$$(M^s - \mu^k)^{-1} \prec \frac{c}{1 - cP^0}, \quad s \in \mathbf{Z},\ k = 0, \pm 1,\ s \neq k, \tag{4.3}$$

where the constant c is independent of s. We first take $c \geq (1 - \mu^k)^{-1}$ and assume $s \neq 0$. We also note that $|\mu|^{k/s} \leq \sqrt{|\mu|}$ whenever $s \neq k$ and $s \neq 0$. For $s \geq 1$, we have

$$\begin{aligned} (M^s - \mu^k)^{-1} &= M^{-s}(1 - \mu^k M^{-s})^{-1} \\ &= M^{-s} \sum_{j=0}^{\infty} (\mu^k M^{-s})^j \\ &\prec P^s \sum (|\mu|^{k/s} P)^{sj} \\ &\prec (\sqrt{|\mu|}\, P)^s \sum (\sqrt{|\mu|}\, P)^{sj} \\ &\prec (1 - \sqrt{|\mu|}\, P)^{-1}. \end{aligned}$$

Thus (4.3) hold for $s \geq 1$ with $c \geq |\mu|(\sqrt{|\mu|}-1)^{-1}$. Now let $s=-t \leq -1$. Then

$$\begin{aligned}(M^s-\mu^k)^{-1} &= -\mu^{-k}(1-\mu^{-k}M^s)^{-1} \\ &= -\mu^{-k}\sum_{j=0}^{\infty}(\mu^{-k}M^s)^j \\ &< |\mu|^{-k}\sum(|\mu|^{-k/t}P)^{tj} \\ &< |\mu|\sum_{\alpha=0}^{\infty}(\sqrt{|\mu|}\,P)^{\alpha} = |\mu|\,(1-\sqrt{\mu}\,P)^{-1}.\end{aligned}$$

Hence, (4.3) holds for $s \leq -1$ if $c \geq |\mu|^{3/2}(|\mu|^{1/2}-1)^{-1}$.

From (4.2), (4.3), and (3.4) we get

$$\begin{aligned}U-\xi = \sum_{s \neq 1} U_s &= \sum_{s \neq +1}(M^s-\mu)^{-1}[f(U,V,W)]_s \\ &< \frac{c}{1-cP^0}\sum_s [f(U,V,W)]_s^* \\ &= \frac{c}{1-cP^0}[f(U,V,W)]^*.\end{aligned}$$

This gives the first of the three relations

$$\begin{aligned}u = U-\xi &< \frac{c}{1-cP^0} f^*(U^*,V^*,W^*), \\ v = V-\eta &< \frac{c}{1-cP^0}\, g^*(U^*,V^*,W^*), \\ w_\alpha = W_\alpha-\zeta_\alpha &< \frac{c}{1-cP^0}\, h_\alpha^*(U^*,V^*,W^*).\end{aligned} \tag{4.4}$$

The second two are proved similarly. If we set $s=-1$ in the second equation of (4.2), we get

$$(M^{-1}-\mu^{-1})\eta = [g(U,V,W)]_{-1},$$

from which follows

$$P^0\eta < [g(U,V,W)]^* < g^*(U^*,V^*,W^*). \tag{4.5}$$

Since f, g, h_α converge and begin with quadratic terms we have a relation

$$f, g, h_\alpha < G\left(x+y+\sum_\alpha z_\alpha\right), \quad G(t) = \frac{c_1 t^2}{1-c_1 t} \tag{4.6}$$

for some $c_1>0$. Now we set $\eta=\zeta_\alpha=\xi$ and define the power series $W(\xi)$ by

$$\xi W(\xi) = u^*+v^*+\sum_{\alpha=2}^{n-1} w_\alpha^*+P^0\xi.$$

Note that $W(0)=0$. From (4.4), (4.5), (4.6) we get

$$\xi W \prec \left(\frac{nc}{1-cW}+1\right) G(Y),$$

where

$$\begin{aligned} Y &= U^*+V^*+\sum_\alpha W_\alpha^* \\ &= n\xi+u^*+v^*+\sum w_\alpha^* \\ &\prec \xi(n+W). \end{aligned}$$

Hence,

$$W \prec \frac{c_2}{1-c_2 W}\frac{c_1\xi(n+W)^2}{1-c_1\xi(n+W)},$$

for a suitable constant c_2. It follows that the series $W(\xi)$ is majorized by the solution $X=X(\xi)$, $X(0)=0$, of the cubic equation

$$X(1-c_2X)(1-c_1\xi(n+X)) = c_1c_2\xi(n+X)^2,$$

which is analytic near $\xi=0$. It follows that u, v, w converge when $\xi=\eta=\zeta_\alpha$ have some non-zero value, and hence in a neighborhood of the origin. From the convergence of the map ψ it follows that Λ_1 and Λ_2 converge.

It is clear that the ν given by (3.17) converges if τ_1 and τ_2 are given by convergent power series.

COROLLARY 4.2. *Let the holomorphic involutions τ_1, τ_2 be as in the above theorem. Then the transformation ψ and the factor Λ in* (3.15) *are holomorphic.*

5. Normal form for surfaces

In this section we use the results of section 3, 4 on the normalization of the involutions τ_1, τ_2, ϱ to transform the surface $M^n \subset \mathbf{C}^n$ into a normal form near a suitable complex tangent. Let (ξ_*, η_*, ζ_*) be the normal coordinates on the complexified surface $\mathfrak{M}^n \subset \mathbf{C}^{2n}$. Then we have

$$\tau_1: \begin{cases} \xi'_* = \Lambda_* \eta_* \\ \eta'_* = \Lambda_*^{-1} \xi_*, \\ \zeta'_* = \zeta_* \end{cases} \quad \tau_2: \begin{cases} \xi'_* = \Lambda_*^{-1} \eta_* \\ \eta'_* = \Lambda_* \zeta_* \\ \zeta'_* = \zeta_* \end{cases}$$

where $\Lambda_* = \Lambda_*(\xi_* \eta_*, \zeta_*) = \lambda + \dots$ and in the two cases

(i) $\varrho(\xi_*, \eta_*, \zeta_*) = (\bar{\eta}_*, \bar{\xi}_*, \bar{\zeta}_*), \quad \bar{\Lambda}_* = \Lambda_*$

or

(ii) $\varrho(\xi_*, \eta_*, \zeta_*) = (\bar{\xi}_*, \bar{\eta}_*, \bar{\zeta}_*), \quad \bar{\Lambda}_* \Lambda_* = 1.$

We still have the freedom to replace (ξ_*, η_*, ζ_*) by

$$\xi_* = r\xi, \quad \eta_* = r\eta, \quad \zeta_* = \zeta, \tag{5.1}$$

$r = r(\xi\eta, \zeta)$, leading to $\Lambda(\xi\eta, \zeta) = \Lambda_*(\xi\eta r^2, \zeta)$.

Of course, also ζ_* can be reparametrized, but we will not make use of this fact, and determine r in such a way that the surface is in a simple normal form.

THEOREM 5.1. *Assume that M^n is a real analytic surface in $\mathbf{C}^n$ given by* (1.16, 1.17) *with* $0 < \gamma < 1/2$, *i.e. in the elliptic case. Then there exists a biholomorphic transformation near the origin taking M^n into the implicit form*

$$\begin{aligned} x_n &= z_1 \bar{z}_1 + \Gamma(x_n, x_\alpha)(z_1^2 + \bar{z}_1^2) \\ y_n &= 0 \\ y_\alpha &= 0 \quad (\alpha = 2, 3, \dots n-1) \end{aligned} \tag{5.2}$$

where $\Gamma = \bar{\Gamma} = \gamma + \dots$.

Proof. As in the linear case (section 2) we introduce

$$z_1 = i\Lambda^{-1/2}(\Lambda\xi + \eta)$$
$$w_1 = -i\Lambda^{-1/2}(\xi + \Lambda\eta)$$
$$z_\alpha = w_\alpha = x_\alpha = \zeta_\alpha.$$

These are equations on $\mathfrak{M}$. The third equation means that z_α is the extension of ζ_α holomorphic in the original z's, and w_α is the extension of ζ_α holomorphic in the original w's. Hence, the (z_α, w_α) are independent functions in the ambient space $\mathbf{C}^{2n}$. The same computation as in section 2 gives

$$z_1 w_1 + \Gamma_1(\xi\eta, \zeta)(z_1^2 + w_1^2) = \Delta(\xi\eta, \zeta)$$

where

$$\Gamma_1 = (\Lambda + \Lambda^{-1})^{-1}$$
$$\Delta = \frac{(\Lambda - \Lambda^{-1})^2}{\Lambda + \Lambda^{-1}} \xi\eta = (\Gamma_1^{-1} - 4\Gamma_1)\xi\eta.$$

We set

$$z_n = w_n = \Delta(\xi\eta, \zeta).$$

Note z_j are τ_2-invariant, while w_j are τ_1-invariant and therefore they are holomorphically or antiholomorphically related to the original coordinates z_j^0. Because of the choice of the linear terms we have in

$$z_j = f_j(z^0) = z_j^0 + \dots$$
$$w_j = g_j(w^0) = w_j^0 + \dots$$

a biholomorphic change of coordinates in $\mathbf{C}^{2n}$. Because of the reality conditions we have $w_1 = cz_1 \circ \varrho$ and $g_j(z^0) = \bar{f}_j(z^0)$. Hence $z^0 \mapsto z$ defines a holomorphic coordinate change in $\mathbf{C}^n$.

Now if we eliminate $\xi\eta, \zeta$ from

$$x_n = \Delta(\xi\eta, \zeta); \quad x_\alpha = \zeta_\alpha$$

and set

$$\Gamma(x_n, x_\alpha) = \Gamma_1(\xi\eta, \zeta),$$

we obtain (5.2). Q.E.D.

In the hyperbolic case, if μ is not a root of unity we find the same normal form if we admit coordinate transformations given only by formal series, since, in general, we have to expect divergence.

Assuming convergence we read off several important facts about M. First M admits the holomorphic involution $(z_1, z_\alpha, z_n) \mapsto (-z_1, z_\alpha, z_n)$. Also, M lies in the linear space $\operatorname{Im} z_\alpha = \operatorname{Im} z_n = 0$. An $(n-1)$-real parameter family of complex lines cut M in a family of disjoint real analytic curves. In the *elliptic* case, where convergence is guaranteed, these are closed real curves bounding linear analytic discs on $z_\alpha=c_\alpha=\bar{c}_\alpha$, $z_n=c_n=\bar{c}_n>0$, bounded by the ellipses

$$c_n = z_1\bar{z}_1+\Gamma(c_n, c_\alpha)(z_1^2+\bar{z}_1^2).$$

These discs sweep out a $(n+1)$-dimensional real analytic manifold $\tilde{M}$ with boundary M. This $\tilde{M}$ is the local holomorphic hull of M. There are no other analytic discs $\mathfrak{C}$ in $\mathbf{C}^n$ with boundaries on M near 0. Indeed, the functions $\operatorname{Im} z_\alpha$, $\operatorname{Im} z_n$ vanish on the boundary of $\mathfrak{C}$, hence identically on $\mathfrak{C}$. Thus z_α, z_n are real constants on $\mathfrak{C}$, and so $\mathfrak{C}$ lies on the discs given by $z_\alpha=c_\alpha$, $z_n=c_n$.

In the hyperbolic case, provided we have a convergent transformation ψ into normal form, this argument shows that there exists no analytic disc with boundary on M^n near 0.

Next we make use of the transformation (5.1) to further simplify the factor Γ. We distinguish two cases: In the first Γ_* is independent of $\xi_*\eta_*$, hence Γ is independent of x_n; in this case no further simplification is achieved. In the second case we write

$$\Gamma_*(\xi_*\eta_*, \zeta_*) = \gamma(\zeta_*)+\sum_{k\geq s}\gamma_k(\zeta_*)(\xi_*\eta_*)^k$$

where $\gamma_s(\zeta_*)\not\equiv 0$; $\gamma(0)=\gamma$. The integer s is a biholomorphic invariant. Since there are points ζ_* with $\gamma(\zeta_*)\neq 0$ near 0 we may assume that $\gamma_s(0)\neq 0$ and will choose $r=r(\xi_*\eta_*, \zeta_*)$ in (5.1) so that Γ has the form

$$\Gamma = \gamma(x_\alpha)+\delta x_n^s, \quad \delta = \pm 1. \tag{5.3}$$

Indeed, since $\Gamma(x_\alpha, x_n)$ is obtained by elimination of $\xi\eta$ from

$$x_n = \Delta(\xi\eta, \zeta) = (\Gamma_1^{-1}-4\Gamma_1)\,\xi\eta$$

$$\Gamma(x_\alpha, x_n) = \Gamma_1(\xi\eta, x_\alpha) = \Gamma_*(\xi\eta r^2, x_\alpha)$$

we have to solve the equation

$$\Gamma_*(\xi\eta r^2, x_\alpha) = \gamma(x_\alpha)+\delta\Delta^s$$

or with $\zeta_\alpha=x_\alpha$

$$[\Gamma_*^{-1}(\xi\eta r^2,\zeta)-4\Gamma_*(\xi\eta r^2,\zeta)]^s = \delta^{-1}(\Gamma_*(\xi\eta r^2,\zeta)-\gamma(\zeta)) = \delta^{-1}r^{2s}\sum_{j\geq 0}\gamma_{j+s}r^{2j}(\xi\eta)^j.$$

We choose $\delta=\pm 1$ so that $\delta\gamma^{-s}(1-4\gamma^2)^s\gamma_s(0)^{-1}>0$; recall that the coefficients of Γ_* are real. Taking the $(2s)^{\text{th}}$ root of both sides we can solve for a real r by the implicit function theorem. The function r is determined up to sign. Thus we have achieved the form (5.3) for Γ.

We may still apply a real invertible transformation $\zeta\mapsto f(\zeta)$; $f(\zeta)=\bar{f}(\zeta)$, $f(0)=0$ to simplify $\gamma(\zeta)$. For example, if $\zeta=0$ is a regular point we could achieve $\gamma(\zeta)=\gamma+\zeta_2$, or $\gamma(x_\alpha)=\gamma+x_2$. Thus, generically M^n has the form

$$\begin{aligned} x_n &= z_1\bar{z}_1+(\gamma+x_2\pm x_n^s)(z_1^2+\bar{z}_1^2), \quad y_\alpha=y_n=0, \quad \text{for } n\geq 3 \\ x_2 &= z_1\bar{z}_1+(\gamma\pm x_2^s)(z_1^2+\bar{z}_1^2), \quad y_2=0, \quad \text{for } n=2. \end{aligned} \tag{5.4}$$

For $n=2$ the automorphism group of M^2 consists only of $(z_1,z_2)\mapsto(\pm z_1,z_2)$ provided that λ is not a root of unity and $\gamma_s\neq 0$, i.e. in case that the normal form does not represent a quadric. This follows readily from our formal considerations.

We recall that the involutions τ_j map $\mathfrak{M}$ into itself, hence also φ and φ^t, defined in section 3 map $\mathfrak{M}$ into itself. However, in order that φ^t maps M, the fixed point set of ϱ, into itself, we need that ϱ and φ^t commute. As was shown at the end of section 3 this is the case for purely imaginary $t=i\sigma$. In the elliptic case the orbits of $\varphi^{i\sigma}$, σ real, are the closed curves which bound the analytic discs.

It suffices to prove this in the normal form. Since $\varphi^{i\sigma}$ preserves $\xi\eta$ as well as ζ_α it follows that the orbits lie on

$$\zeta_\alpha=c_\alpha; \quad \xi\eta=c_n,$$

where c_α, c_n are real constants, $c_n>0$. Hence we have

$$z_\alpha=c_\alpha=w_\alpha; \quad z_n=\text{constant}$$

which proves the claim.

6. Further remarks

(a) *Exceptional hyperbolic surfaces*. We consider a surface M in $\mathbf{C}^2$ given by (1.3) with $\gamma>1/2, H=h+ik$. We assume that the associated mapping φ on $\mathfrak{M}$ is such that $\varphi'(0)$ is nilpotent,

$$\lambda^{2m}=1, \lambda^{2j}\neq 1, \quad j<m.$$

We shall show that M can be holomorphically flattenend to order m and in general to no higher order. For this it suffices to consider transformations of the form

$$\tilde{z}_1 = z_1, \quad \tilde{z}_2 = z_2 + B(z_1, z_2), \tag{6.1}$$

where $B(z_1, z_2)$ is polynomial without constant or linear terms. Restriction to M yields

$$\tilde{z}_2 = q + H + B(z_1, q+H) \equiv q + \tilde{H},$$

so that $\operatorname{Im} B(z_1, q+H) = \tilde{k} - k$. Suppose that k and $\tilde{k}$ begin with terms of degree $n<m$. We shall choose a holomorphic polynomial $B(z_1, z_2)$ of weight n (weight of $z_j = j$, $j=1,2$) so as to annihilate the terms of degree n in $\tilde{k}$. This amounts to solving an equation of the form

$$\operatorname{Im} B(z_1, q) = k, \tag{6.2}$$

where k is a real homogeneous polynomial of degree n. This is a problem on the quadric Q_γ.

Complexifying gives

$$\frac{1}{2i} B(z_1, q(z_1, w_1)) - \frac{1}{2i} \bar{B}(w_1, q(z_1, w_1)) = k(z_1, w_1),$$

which implies that k can be decomposed into the sum of a homogeneous polynomial of degree n invariant under τ_2 and one invariant under τ_1. We pass to the (ξ, η)-coordinate system by the linear change (2.11). The most general such polynomials invariant under τ_1 and τ_2 are

$$f_1 = \sum_{j \le n/2} a_j (\xi^{n-j}\eta^j + \lambda^{n-2j}\xi^j\eta^{n-j}),$$

and

$$f_2 = \sum_{j \le n/2} b_j (\xi^{n-j}\eta^j + \lambda^{2j-n}\xi^j\eta^{n-j}),$$

respectively. The real polynomial k has the form

$$k = \sum_{j=0}^{n} c_j \xi^{n-j}\eta^j, \quad c_j = \bar{c}_j,$$

so that $f_1+f_2=k$ reduces to

$$\begin{aligned} a_j+b_j &= c_j, \quad 0\leq j\leq n/2, \\ \lambda^{n-2j}a_j+\lambda^{2j-n}b_j &= c_{n-j}, \quad 0\leq j\leq n/2. \end{aligned} \tag{6.3}$$

For $2j<n$ these equations have a unique solution a_j, b_j since the determinant does not vanish because of $\lambda^{2(n-2j)}\neq 1$. Conjugating (6.3) we find $b_j=\bar{a}_j$ since $c_j=\bar{c}_j$, $\lambda\bar{\lambda}=1$. For $2j=n$ the two equations agree with the single equation

$$a_j+b_j=c_j.$$

We choose the solution

$$a_j=b_j=\frac{1}{2}c_j$$

so that again $b_j=\bar{a}_j$ holds, since c_j is real. Thus $f_1=\bar{f}_2$ is the trace of a function $(1/2i)B$ holomorphic in z_1, z_2 which gives the solution of (6.2).

The first instance in which (6.3) may not be solvable is $n=m$, $j=0$. If $\lambda^m=-1$, we have the compatibility condition $c_m=-c_0$. For example, $k=\overline{k\circ\varrho}=\xi^m+\eta^m$ cannot be written as such a sum f_1+f_2. We set $\lambda_m=e^{i\pi/m}$, and by (2.14) $\gamma_m=(1/2)\sec(\pi/m)$. In particular $\gamma_3=1$, $\gamma_4=1/\sqrt{2}$. The corresponding surface is (via (2.11))

$$z_2=\gamma_m z_1^2+z_1\bar{z}_1+\gamma_m\bar{z}_1^2+i^{m+1}(\lambda_m)^{m/2}(\lambda_m^2-1)^{-m}\{(-\lambda_m z_1-\bar{z}_1)^m+(z_1+\lambda_m\bar{z}_1)^m\}. \tag{6.4}$$

The imaginary part of the right hand side cannot be made to vanish to higher order.

If $\lambda^m=+1$, we may take $k=\xi^m-\eta^m$.

For $\gamma=\gamma_3=1$, a simpler example of a surface which cannot be flattened to third order is

$$z_2=z_1^2+z_1\bar{z}_1+\bar{z}_1^2+z_1\bar{z}_1(z_1-\bar{z}_1). \tag{6.5}$$

This can be seen by examining (6.2) directly in $(z_1, \bar{z}_1)$-coordinates.

(b) *Divergence in the normal form*. When $\gamma>1/2$ is not exceptional, the results of section 5 show that the surface M can be formally transformed into a real hyperplane. However, as mentioned before the transformation will in general diverge. Rather than prove a general theorem to this effect, we shall give an example of a surface in $\mathbf{C}^2$ which cannot be holomorphically flattened. This surface will be of the form

$$M: \begin{array}{l} z_2 = (k(z_1)+\gamma\bar{z}_1)\,\bar{z}_1 \\ k(z_1) = z_1 + k_0(z_1), \end{array} \tag{6.6}$$

where k_0 is a holomorphic polynomial in z_1 beginning with a term of order ≥ 2.

If M could be transformed into the hyperplane $\operatorname{Im}\bar{z}_2=0$, then this could be accomplished by means of a transformation of the form (6.1) with $B=\gamma z_1^2+O(|z|^3)$. Let $G(z_1,\bar{z}_1)=q(z_1,\bar{z}_1)+\dots$ be the restriction of z_2+B to M. Then $G(z_1,\bar{z}_1)$ is also the restriction to M of $\bar{z}_2+\bar{B}$. Consequently, the complex function $G=G(z_1,w_1)$ on $\mathfrak{M}$ is invariant under both τ_1 and τ_2, and so $G\circ\varphi=G$. Furthermore,

$$dG = q_{z_1}\,dz_1 + q_{w_1}\,dw_1 + \dots$$

is non-zero in a deleted neighborhood of $z_1=w_1=0$. We shall show that if $1/2<\gamma<\infty$, $\gamma\neq 1/\sqrt{2}$, then k can be chosen so that ϱ admits no such (non-trivial) invariant function G in any neighborhood of the origin.

One readily sees that

$$\tau_1\colon (z_1,w_1)\mapsto(-z_1-\gamma^{-1}\bar{k}(w_1),w_1),$$
$$\tau_2\colon (z_1,w_1)\mapsto(z_1,-w_1-\gamma^{-1}k(z_1)),$$

$$\varphi: \begin{array}{l} z_1' = -z_1-\gamma^{-1}\bar{k}(w_1') \\ w_1' = -w_1-\gamma^{-1}k(z_1) \end{array}, \quad \varphi^{-1}: \begin{array}{l} z_1 = -z_1'-\gamma^{-1}\bar{k}(w_1') \\ w_1 = -w_1'-\gamma^{-1}k(z_1) \end{array}. \tag{6.7}$$

In particular both φ and φ^{-1} are polynomial mappings (so called Cremona transformations) of the form

$$\varphi: \begin{cases} z_1' = cz_1^{2\delta}+\dots \\ w_1' = 0+\dots \end{cases}, \quad \varphi^{-1}: \begin{cases} z_1 = 0+\dots \\ w_1 = \bar{c}w_1'^{2\delta}+\dots \end{cases},$$

where $\delta=\deg k$ and the dots indicate terms of lower degree in (z_1,w_1) or (z_1',w_1'). The n-fold iterates of φ and φ^{-1} are of the form

$$\varphi^n: \begin{cases} z_1' = c_n z_1^{(2\delta)^n}+\dots \\ w_1' = 0+\dots \end{cases}, \quad \varphi^{-n}: \begin{cases} z_1 = 0+\dots \\ w_1 = \bar{c}_n w_1^{(2\delta)^n}+\dots \end{cases},$$

where $c_n\neq 0$. A fixed point p of φ^{2n} satisfies $\varphi^n(p)=\varphi^{-n}(p)$, so is a solution to the pair of polynomial equation

$$c_n z_1^{(2\delta)^n}+\dots = \bar{c}_n w_1^{(2\delta)^n}+\dots = 0.$$

The leading terms show that these two polynomials have no common factor. By Bezout's theorem φ^{2n} can have at most $(2\delta)^{2n}$ fixed points.

From (6.7) we see that $dz_1'\wedge dw_1'=-dz_1\wedge dw_1'=dz_1\wedge dw_1$, so that the Jacobian determinant of φ is identically one. Consider the holomorphic vector field

$$X_G=G_{w_1}\frac{\partial}{\partial z_1}-G_{z_1}\frac{\partial}{\partial w_1}.$$

If $G\circ\varphi=G$, then X_G is invariant under φ: $d\varphi(X_G)=X_G$. This follows from the chain rule and the fact that φ is area-preserving. Therefore, if p is a fixed point of φ^{2n} then so is every point on the orbit through p of the flow $\exp tX_G$. If $dG(p)\neq 0$, then this orbit is locally a smooth holomorphic curve. φ^{2n} would then have a continuum of fixed points, which is impossible. It follows that if we can choose $k(z_1)$ so that every deleted neighborhood of the origin contains a fixed point of φ^n, for some n, then M cannot be holomorphically flattened.

To achieve this last property we shall appeal to the Birkhoff fixed point theorem. This theorem applies to area-preserving transformations of the real plane. We must therefore choose a k with real coefficients so that φ leaves invariant the plane of real z_1, w_1. We shall, in fact, take $k(z_1)=z_1+\gamma z_1^3+\dots$ a polynomial with real coefficients so that

$$\varphi:\begin{aligned}z_1'&=(\gamma^{-2}-1)z_1+\gamma^{-1}w+f+O(4)\\ w_1'&=-\gamma^{-1}z_1-w_1+g,\end{aligned}$$

where

$$f=\gamma^{-1}z_1^3+(w_1+\gamma^{-1}z_1)^3,\quad g=-z_1^3.$$

If we subject this to the coordinate change (2.11), we get

$$\varphi:\begin{aligned}\xi'&=\lambda^2\xi-i\sqrt{\lambda}\,(\lambda^2-1)^{-1}(\lambda f+g)+O(4)\\ \eta'&=\lambda^{-2}\eta+i\sqrt{\lambda}\,(\lambda^2-1)^{-1}(f+\lambda g)+O(4),\end{aligned}$$

where

$$\begin{aligned}\lambda f+g&=6\lambda^4(i/\sqrt{\lambda})^3\xi^2\eta+\dots\\ f+\lambda g&=6(i/\sqrt{\lambda})^3\xi\eta^2+\dots.\end{aligned}\tag{6.8}$$

Here the dots indicate the other cubic terms. If $\lambda^2\neq\pm 1,\ \pm i$, i.e. $\gamma\neq 1/2,\ 1/\sqrt{2},\ \infty$, then these other terms can be removed by a further coordinate change which does not alter the two coefficients shown in (6.8). (See [11] § 23 (18), p. 158.) Hence, φ can be transformed into

$$\varphi\colon \begin{array}{l} \xi' = \lambda^2(1+ia\xi\eta)\,\xi+O(4) \\ \eta' = \lambda^{-2}(1-ia\xi\eta)\,\eta+O(4) \end{array}, \quad a = \frac{6\gamma}{\sqrt{4\gamma^2-1}} > 0.$$

Actually, the case $\lambda^2 = \pm i$ i.e. $\gamma = 1/\sqrt{2}$ does not have to be excluded since the relevant terms η^3, ξ^3 in $\lambda f+g$, $f+\lambda g$, respectively, have zero coefficients.

By Birkhoff's theorem [11], p. 174, for each sufficiently small neighborhood U of the origin in the plane of real z_1, w_1 there exists an integer n such that φ^{2n} fixes a point different from O in U. In particular, we have proved the following proposition.

PROPOSITION 6.1. *If* $1/2 < \gamma < \infty$, *then the hyperbolic surface*

$$z_2 = z_1\bar{z}_1 + \gamma\bar{z}_1^2 + \gamma z_1^3\bar{z}_1$$

cannot be transformed into a real hyperplane by means of a (convergent) biholomorphic transformation.

This shows that the formal transformations, which for λ not a root of unity, $|\lambda| = 1$, exist, must be divergent. This example shows also that divergence can not be avoided by inequalities of the type $|\lambda^j - 1| \geq c|j|^{-\nu}$ for all $j \geq 1$. Incidentally, the periodic orbits of φ, as well as its invariant curves, do not have any geometrically significance since they do not lie on M but only on its complexification $\mathfrak{M}$.

(c) *The case* $\gamma = 0$. Here one can apply the formal theory of [5], section 2. We state without proof some of the results of this theory for surfaces in $\mathbf{C}^2$. M has the form

$$M\colon \begin{array}{l} z_2 = z_1\bar{z}_1 + \operatorname{Re} h(z_1) + z_1\bar{z}_1 H(z_1, \bar{z}_1) \\ h(z_1) = \sum_{j \geq k} c_j z_1^j, \quad k = 3, 4, 5, \ldots, \infty. \end{array}$$

The integer k, which is the degree of the lowest pure $\bar{z}_1$-term, is a biholomorphic invariant. M is formally equivalent to the quadric $Q_0\colon z_2 = z_1\bar{z}_1$ if and only if $k = \infty$. Otherwise, M may be formally transformed into the form

$$z_2 = z_1\bar{z}_1 + z_1^k + \bar{z}_1^k + \operatorname{Re}\sum_{j>k} a_j z^j.$$

The complex numbers a_j are not absolute invariants, since this normal form is still subject to the action of the (formal) automorphism group of Q_0. This group can be shown to be made up precisely of those transformations of the form

$$z_1' = G(x_2)\frac{z_1 - x_2 b(x_2)}{1 - \bar{b}(x_2) z_1}$$

$$x_2' = G(x_2)\bar{G}(x_2) x_2,$$

where G and b are arbitrary complex formal power series with $G(0) \neq 0$. There are probably infinitely many real valued invariants in the case $\gamma = 0$.

One still has the projections π_1 and π_2 on the complexified surface $\mathfrak{M}$. The case in which M is formally equivalent to Q_0 is characterized by π_1 (or π_2) being locally one-to-one except for collapsing an analytic curve to a point. Otherwise ($k < \infty$) each π_i is a k-fold branched covering.

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