L^∞ estimates for the $\bar\partial$ problem in a half-plane

by

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§ 1. Introduction

Suppose μ is a σ-finite complex-valued measure on the upper half-plane $\mathbf{R}_+^2=\{z=x+iy: y>0\}$. Then μ is called a Carleson measure if

$$\sup_I \frac{1}{|I|}|\mu|\,(I\times(0,|I|]) = \|\mu\|_C < \infty,$$

where the above supremum is taken over all intervals $I\subset\mathbf{R}$, and where $|\cdot|$ denotes one-dimensional Lebesgue measure. Invoking a fundamental theorem due to Carleson [6], Hörmander [21] showed that the $\bar\partial$ problem $\bar\partial F=\mu$ has a solution F satisfying

$$\|F\|_{L^\infty(\mathbf{R})} \leq C_0\|\mu\|_C$$

where μ is a Carleson measure. (Here and throughout the paper we denote by C_0 various universal constants.) The proof of this was based on the duality between H^1 and L^∞/H^∞ and the fact that

$$\|f^*\|_{L^p} \leq C_0\|f\|_{H^p}$$

where

$$f^*(t) = \sup_{|x-t|<y} |f(x+iy)|.$$

Here H^p, $0<p<\infty$, denotes the classical (holomorphic) Hardy space of functions holomorphic on $\mathbf{R}_+^2$ and satisfying

$$\sup_{y>0}\left(\int_{-\infty}^{\infty} |f(x+iy)|^p\,dx\right)^{1/p} = \|f\|_{H^p} < \infty.$$

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For $p=\infty$ we denote by H^∞ the ring of bounded holomorphic functions on $\mathbf{R}^2_+$ endowed with the supremum norm. We also denote by $\mathcal{H}^p(\mathbf{R}^n)$ the (real variables) Hardy space of all complex-valued harmonic functions on $\mathbf{R}^{n+1}_+=\{(x,y):x\in\mathbf{R}^n, y>0\}$ satisfying $f^*\in L^p(\mathbf{R}^n)$, where

$$f^*(t)=\sup_{|x-t|<y}|f(x,y)|.$$

(Notice that by our definitions, $\mathcal{H}^\infty(\mathbf{R}^n)=L^\infty(\mathbf{R}^n)$.)

L^∞ estimates for the $\bar\partial$ problem play a fundamental role in the H^p theory. They are present, either implicitly or explicitly, in the results of [8], [11], [18], [21], [22], [27], [36]. Our purpose in this paper is to find explicit solution operators for the $\bar\partial$ problem which yield $L^\infty(\mathbf{R})$ solutions from Carleson measure data. It is not hard to see that such solution operators must be nonlinear. Indeed, the solution operators of Theorem 1 are not even continuous. Our solution operators are also highly one-dimensional in form; this reflects the fact that there exist, in the ball in $\mathbf{C}^n$, $\bar\partial$ closed forms satisfying a Carleson condition but not admitting any L^∞ solutions. (See Theorem 3.1.2 of [36].)

The duality approach to finding solutions of $\bar\partial F=\mu$ is sufficient for many problems arising in the H^p theory. In certain situations, however, one would like to obtain more information on the solutions than duality permits. We cite two examples of problems where the classical duality proof does not immediately give satisfactory answers.

(i) Can one infer smoothness or L^p behavior for F from the known properties of μ?

(ii) If $\|\mu\|_C\leq 1$ can one construct a linear operator S solving $\bar\partial(S(w(z)\mu))=w(z)\mu$ such that $\|S(w(z)\mu)\|_{L^\infty(\mathbf{R})}\leq C_0\|w\|_{L^\infty}$?

Our solution operators can be used to answer problems (i) and (ii). It should be pointed out that A. Uchiyama (unpublished) has recently found another method for solving (ii) which uses duality. A constructive proof of solving $\bar\partial F=\mu$ is presented in [22], where the solution F is given as (essentially) a convex combination of Blaschke products. This approach is attractive in certain contexts (e.g., problems related to the Chang-Marshall theorem [11], [27]) but the solutions are quite difficult to compute and give little more regularity than the $L^\infty(\mathbf{R})$ estimate. We remark that problem (ii) above can also be solved by combining Lemma 2.1 of [22] with P. Beurling's interpolation theorem [7]. (P. Beurling's theorem is intimately connected with the construction of our solution operators – this is discussed in section 5.) On the other hand, the construction of our solution operators is extremely simple and flexible and should be useful in situations where neither duality nor the Blaschke product methods of [22] can be used. Using in part the ideas of this paper, Lennart Carleson [10] has recently been able to solve the corona problem for a certain class of planar domains. (His method is necessarily much more complicated than ours.)

For a measure σ on $\mathbf{R}^2_+$ let

$$K(\sigma, z, \zeta) = \frac{1}{z-\zeta} \exp\left\{ \iint_{\operatorname{Im} w \leq \operatorname{Im} \zeta} \left(\frac{-i}{z-\bar w} + \frac{i}{\zeta - \bar w} \right) d|\sigma|(w) \right\}$$

and let

$$K_0(\sigma, z, \zeta) = \frac{2i}{\pi} \left(\frac{\operatorname{Im} \zeta}{z-\bar\zeta} \right) K(\sigma, z, \zeta),$$

$$K_1(\sigma, z, \zeta) = \frac{1}{\pi} \exp\left\{ (i-1) \sqrt{\frac{z - \operatorname{Re} \zeta}{\operatorname{Im} \zeta}} + \sqrt{2} \right\} K(\sigma, z, \zeta).$$

THEOREM 1. *If μ is a Carleson measure, then*

$$S_0(\mu)(z) = \iint_{\mathbf{R}^2_+} K_0(\mu/\|\mu\|_C, z, \zeta)\, d\mu(\zeta)$$

and

$$S_1(\mu)(z) = \iint_{\mathbf{R}^2_+} K_1(\mu/\|\mu\|_C, z, \zeta)\, d\mu(\zeta)$$

satisfy $S_k(\mu)(z) \in L^1_{\text{loc}}$ on $\mathbf{R}^2_+$ and $\bar\partial S_k(\mu) = \mu$ in the sense of distributions, $k=0,1$. If $x \in \mathbf{R}$, the above integrals converge absolutely and

$$\iint_{\mathbf{R}^2_+} |K_k(\mu/\|\mu\|_C, x, \zeta)|\, d|\mu|(\zeta) \leq C_0 \|\mu\|_C, \quad k = 0, 1.$$

In particular,

$$|S_k(\mu)(x)| \leq C_0 \|\mu\|_C, \quad k = 0, 1.$$

The solution operators S_0 and S_1 differ only in the way that $S_0(\mu)$ and $S_1(\mu)$ decay when μ is compactly supported. In that case $S_0(\mu)$ decays like $|z|^{-2}$, while $S_1(\mu)$ decays faster than any polynomial in $|z|^{-1}$.

Suppose $0<p_0<p<p_1\leq\infty$ and $f\in L^p(\mathbf{R})$. For many purposes in analysis (e.g., Marcinkiewicz-type interpolation) one wants to be able to split f into f_0+f_1, where $f_0 \in L^{p_0}$, $f_1 \in L^{p_1}$, and where f_0 and f_1 have certain good properties. Our solution operator S_1 allows us to obtain a decomposition of Marcinkiewicz type for functions

$f \in H^p$, where $f_0 \in H^{p_0}$ and $f_1 \in H^{p_1}$. Since decompositions of this type are known when $p_1 < \infty$ (see [17]), our results are stated only for $p_1 = \infty$. The proof we give, however, extends to the general case.

THEOREM 2. *Suppose $0<p_0<p<\infty$ and suppose $f \in H^p$. If $\alpha>0$, there is a Marcinkiewicz decomposition of f, $f=F_\alpha+f_\alpha$, where $F_\alpha \in H^{p_0}$, $f_\alpha \in H^\infty$, and such that*

$$\|F_\alpha\|_{H^{p_0}}^{p_0} \leq C_{p_0} \int_{\{f^*>\alpha\}} |f^*|^{p_0}\, dx$$

and

$$\|f_\alpha\|_{H^\infty} \leq C_0 \alpha.$$

Our results on the $\bar\partial$ problem yield some new results on interpolation of operators on Hardy spaces. We consider two methods of interpolation, namely the real method as described in [19], [32] and the complex method as described in Calderón [3]. In the real method the intermediate spaces are denoted by $(\cdot,\cdot)_{\theta,q}$, where $0<\theta<1$ and $0<q\leq\infty$. In the complex method the intermediate spaces are denoted by $(\cdot,\cdot)_\theta$, where $0<\theta<1$. A full account of both of these methods can be found in [2]. When interpolating between H^p spaces where $p<1$ in the complex method, some minor modifications of Calderón's method are needed; these can be found in [25] and [30]. Let $H^{p,q}$ denote the class of all functions f holomorphic on $\mathbf{R}^2_+$ and such that f^* is in the Lorentz space $L^{p,q}(\mathbf{R})$. Also let $\mathcal{H}^{p,q}(\mathbf{R}^n)$ denote the class of all functions f harmonic on $\mathbf{R}^{n+1}_+$ and such that $f^* \in L^{p,q}(\mathbf{R}^n)$. It is known (see [17] and [20]) that

$$\mathcal{H}^{p_0}(\mathbf{R}^n), L^\infty(\mathbf{R}^n)_{\theta,q} = (\mathcal{H}^{p_0}(\mathbf{R}^n), \mathrm{BMO}(\mathbf{R}^n))_{\theta,q} = \mathcal{H}^{p,q}(\mathbf{R}^n), \frac{1}{p} = \frac{(1-\theta)}{p_0}, 0<p_0<\infty.$$

These results imply the relations

$$(H^{p_0}, H^{p_1})_{\theta,q} = H^{p,q}, \quad \frac{1}{p} = \frac{(1-\theta)}{p_0} + \frac{\theta}{p_1}, \quad 0<p_0<p_1<\infty.$$

For the complex method the known results are:

$$(H^{p_0}, H^{p_1})_\theta = H^p, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad 0<p_0<p_1<\infty;$$

$$(\mathcal{H}^{p_0}(\mathbf{R}^n), \mathcal{H}^{p_1}(\mathbf{R}^n))_\theta = \mathcal{H}^p(\mathbf{R}^n), \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad 0<p_0<p_1<\infty;$$

$$(L^{p_0}(\mathbf{R}^n), L^\infty(\mathbf{R}^n))_\theta = (L^{p_0}(\mathbf{R}^n), \mathrm{BMO}(\mathbf{R}^n))_\theta = L^p(\mathbf{R}^n), \quad \frac{1}{p} = \frac{1-\theta}{p_0}, \quad 1<p_0<\infty.$$

These last results can be found in respectively [33], [4], and [18]. Our next two theorems complete the classification of the intermediate spaces (in the real and complex methods) between H^{p_0} and H^{p_1} by allowing H^∞ to be an endpoint space. Applying the reiteration theorem (see e.g. [2]), Theorems 3 and 4 yield as corollaries the one-dimensional versions of the results listed above.

THEOREM 3. *If* $0<p_0<\infty$, *then* $(H^{p_0}, H^\infty)_{\theta,q}=H^{p,q}$, $1/p=(1-\theta)/p_0$.

THEOREM 4. *If* $0<p_0<\infty$, *then* $(H^{p_0}, H^\infty)_\theta=H^p$, $1/p=(1-\theta)/p_0$.

The methods of [17], [18], and [33] do not apply in the context of Theorems 3 and 4 for two basic reasons. Firstly, $f\in \operatorname{Re} H^p$, $0<p<\infty$ if and only if $f^*\in L^p$; this fails for Re H^∞. Secondly, the Hilbert transform is bounded on L^p, $1<p<\infty$, while it is not bounded on L^∞. The proof of Theorem 3 follows almost immediately from Theorem 2. (A detailed proof would follow the lines of the argument given at the end of [17].) The proof of Theorem 4 requires a separate argument.

At this point it is perhaps appropriate to comment on an unfortunate typographical error in [18], which was pointed out to this author by E. M. Stein. It is mistakenly stated on page 157 of that paper that $(\mathscr{H}^1(\mathbf{R}^n), L^p(\mathbf{R}^n))_\theta=L^q(\mathbf{R}^n)$, $1/q=1-\theta+\theta/p$, $1<p\leq\infty$. The mistake lies in the statement $1<p\leq\infty$, which should read $1<p<\infty$. In other words, the methods of [18] do not identify (and the authors do not intend to) the intermediate spaces $(\mathscr{H}^1(\mathbf{R}^n), L^\infty(\mathbf{R}^n))_\theta$. The idea of [18] is that if $1<p<\infty$, then by duality, $(\mathscr{H}^1(\mathbf{R}^n), L^p(\mathbf{R}^n))_\theta=L^q(\mathbf{R}^n)$ if $(\mathrm{BMO}(\mathbf{R}^n), L^{p'}(\mathbf{R}^n))_\theta=L^{q'}(\mathbf{R}^n)$, where $1/p+1/p'=1/q+1/q'=1$. Since the # function of [18] sends BMO to L^∞ and $L^{p'}$ to $L^{p'}$, the # function must send $(\mathrm{BMO}(\mathbf{R}^n), L^{p'}(\mathbf{R}^n))_\theta$ to $(L^\infty(\mathbf{R}^n), L^{p'}(\mathbf{R}^n))_\theta=L^{q'}(\mathbf{R}^n)$. An application of Theorem 5 of [18] now yields the result $(\mathrm{BMO}(\mathbf{R}^n), L^{p'}(\mathbf{R}^n))_\theta=L^{q'}(\mathbf{R}^n)$. The typographical error $1<p\leq\infty$ is all the more unfortunate since it seems to have become "well known" and is stated, e.g., in [2] and [31]. Our Theorem 4 rectifies this situation in dimension one.

THEOREM 5. *Suppose* X_0 *equals either* $\mathscr{H}^1(\mathbf{R})$ *or* $L^1(\mathbf{R})$ *and suppose* X_1 *equals either* $H^\infty+\bar H^\infty$, $L^\infty(\mathbf{R})$, *or* $\mathrm{BMO}(\mathbf{R})$. *Then*

$$(X_0, X_1)_\theta=L^p(\mathbf{R}),\quad \frac{1}{p}=1-\theta.$$

Proof. The statement $(L^1(\mathbf{R}), L^\infty(\mathbf{R}))_\theta=L^p(\mathbf{R})$ is classical. By Theorem 4, $L^p(\mathbf{R})= H^p\oplus \bar H^p=(H^1, H^\infty)_\theta\oplus(\bar H^1, \bar H^\infty)_\theta\subset(\mathscr{H}^1(\mathbf{R}), L^\infty(\mathbf{R}))_\theta\subset(L^1(\mathbf{R}), L^\infty(\mathbf{R}))_\theta=L^p(\mathbf{R})$, and consequently $(\mathscr{H}^1(\mathbf{R}), L^\infty(\mathbf{R}))_\theta = L^p(\mathbf{R})$. The same reasoning shows $(\mathscr{H}^1(\mathbf{R}), H^\infty+\bar{H}^\infty)_\theta$ and $(L^1(\mathbf{R}), H^\infty+\bar{H}^\infty)_\theta$ equal $L^p(\mathbf{R})$. To calculate $(\mathscr{H}^1(\mathbf{R}), \mathrm{BMO}(\mathbf{R}))_\theta$, we first observe that $L^p(\mathbf{R}) = (\mathscr{H}^1(\mathbf{R}), L^\infty(\mathbf{R}))_\theta \subset (\mathscr{H}^1(\mathbf{R}), \mathrm{BMO}(\mathbf{R}))_\theta$. Since $\mathscr{H}^1(\mathbf{R})$ functions integrate both functions in $L^\infty(\mathbf{R})$ and $\mathrm{BMO}(\mathbf{R})$, functions in $(\mathscr{H}^1(\mathbf{R}), \mathrm{BMO}(\mathbf{R}))_\theta$ must integrate functions in $(L^\infty(\mathbf{R}), \mathscr{H}^1(\mathbf{R}))_\theta = L^p(\mathbf{R})$. Consequently, $(\mathscr{H}^1(\mathbf{R}), \mathrm{BMO}(\mathbf{R}))_\theta \subset L^p(\mathbf{R})$. The same argument shows $(L^1(\mathbf{R}), \mathrm{BMO}(\mathbf{R}))_\theta = L^p(\mathbf{R})$. We remark that one can use the above reasoning plus Theorem 3.1 of [4] to identify $(\mathscr{H}^{p_0}(\mathbf{R}), X_1)_\theta$ as the appropriate Hardy space when $0<p_0<\infty$. In a future paper the author and S. Janson will give generalizations of Theorem 5 to martingales and $\mathbf{R}^n$. This is done by carefully examining the stopping time argument presented in §4.

The organization of the paper is as follows. Theorem 1 is proved in section 2. In section 3 we give two applications of Theorem 1 to the Fefferman-Stein decomposition of $\mathrm{BMO}(\mathbf{R})$. Theorems 2 and 4 are proved in section 4. In section 5 we discuss the relation of H^∞ interpolation to the $\bar{\partial}$ problem.

By conformal equivalence, analogues of all results contained in this paper hold on the unit disk.

§2. Proof of Theorem 1

In this section we prove Theorem 1. Only the last claim of the theorem will be proved; the other two claims follow easily from the proof given below. Let us consider the solution operator S_0. By the form of S_0 it is enough to prove the theorem for the case where $\mu\geq 0$ and $\|\mu\|_C=1$. We first note that if ω, $\zeta\in\mathbf{R}^2_+$ and $\mathrm{Im}\,\omega\leq\mathrm{Im}\,\zeta$, then

$$\mathrm{Re}\left(\frac{i}{\zeta-\bar{\omega}}\right)=\frac{\mathrm{Im}\,\zeta+\mathrm{Im}\,\omega}{|\zeta-\bar{\omega}|^2}\leq\frac{2\,\mathrm{Im}\,\zeta}{|\zeta-\omega|^2}.$$

We also note that the function

$$f(\omega)=\frac{\mathrm{Im}\,\zeta}{(\bar{\zeta}-\omega)^2}$$

is in H^1 and its norm is independent of ζ. Consequently,

$$\begin{aligned}\mathrm{Re}\left\{\iint_{\mathrm{Im}\,\omega\leq\mathrm{Im}\,\zeta}\frac{i}{\zeta-\bar{\omega}}\,d\mu(\omega)\right\}&\leq 2\iint_{\mathbf{R}^2_+}\frac{\mathrm{Im}\,\zeta}{|\bar{\zeta}-\omega|^2}\,d\mu(\omega)\\&\leq C_0\left\|\frac{\mathrm{Im}\,\zeta}{(\bar{\zeta}-\omega)^2}\right\|_{H^1}\|\mu\|_C\\&\leq C_0.\end{aligned}$$

Fix a point $x\in\mathbf{R}$. Since

$$\mathrm{Re}\left(\frac{-i}{x-\bar\omega}\right)=\frac{-\mathrm{Im}\,\omega}{|x-\bar\omega|^2},$$

the proof of Theorem 1 for the operator S_0 will follow immediately from

LEMMA 2.1. *Suppose $\sigma\geq 0$ is a sigma-finite measure on $\mathbf{R}^2_+$, and suppose $x\in\mathbf{R}$. Then,*

$$I_\sigma\equiv\int\!\!\int_{\mathbf{R}^2_+}\frac{1}{|x-\zeta|}\left|\frac{\mathrm{Im}\,\zeta}{x-\bar\zeta}\right|\exp\left\{\int\!\!\int_{\mathrm{Im}\,\omega\leq\mathrm{Im}\,\zeta}\frac{-\mathrm{Im}\,\omega}{|x-\bar\omega|^2}\,d\sigma(\omega)\right\}d\sigma(\zeta)\leq 1.$$

Proof. The lemma follows from comparing I_σ with the integral $\int_0^\infty e^{-t}\,dt$. Suppose for example that $\sigma=\Sigma_{j=1}^N\alpha_j\delta_{\zeta_j}$ is a finite weighted sum of Dirac measures, and suppose $\mathrm{Im}\,\zeta_1\leq\mathrm{Im}\,\zeta_2\leq\ldots\leq\mathrm{Im}\,\zeta_N$. Put $\beta_j=(\alpha_j\,\mathrm{Im}\,\zeta_j/|x-\bar\zeta_j|^2)$. Then since $|x-\zeta|=|x-\bar\zeta|$, $I_\sigma\leq\Sigma_{j=1}^N\beta_j\exp\{-\Sigma_{k=1}^j\beta_k\}<1$, because the last sum is a lower Riemann sum for $\int_0^\infty e^{-t}\,dt$. Standard measure theoretic arguments now complete the proof. The author thanks Professors E. Gorin, S. Hruscev, and S. Vinogradov for pointing out the above argument, which replaces a slightly longer one due to the author.

The proof of Theorem 1 for the operator S_1 now follows from the inequality

$$\left|\exp\left\{(i-1)\sqrt{\frac{x-\mathrm{Re}\,\zeta}{\mathrm{Im}\,\zeta}}+\sqrt{2}\right\}\right|\leq C_0\frac{\mathrm{Im}\,\zeta}{|x-\bar\zeta|},\quad x\in\mathbf{R},\zeta\in\mathbf{R}^2_+.$$

We remark that there is nothing special about the formulae for the kernels K, K_0, and K_1 of Theorem 1. Almost any reasonable kernels which look like K, K_0, K_1 will serve the desired purpose. The reason we have introduced the kernel K_1 is because of the following lemma which will be needed in a later section. For a general box $Q=I\times(0,|I|]$, let x_I denote the center of I.

LEMMA 2.2. *If μ is a Carleson measure, $\|\mu\|_C=1$, then*

$$\left|\int\!\!\int_Q K_1(\mu/\|\mu\|_C,x,\zeta)\,d\mu(\zeta)\right|\leq C_0\exp\left\{-\left(\frac{|x-x_I|}{|I|}\right)^{1/2}\right\}$$

for all $x\in\mathbf{R}$.

The proof of Lemma 2.2 follows easily from Theorem 2 and the form of K_1. In the proof of Theorem 4 we will also need

LEMMA 2.3. *If μ is a Carleson measure, $\|\mu\|_C=1$, and w is a bounded function, then*

$$\iint_{\mathbf{R}^2_+} |K_1(\mu, x, \zeta)|\, |w(\zeta)|\, d|\mu|(\zeta) \le C_0 \|w\|_{L^\infty}$$

for all $x \in \mathbf{R}$.

Lemma 2.3 follows immediately from Theorem 1.

§3. Bounded mean oscillation

Theorem 1 can be used to obtain constructively the Fefferman-Stein decomposition [18], [22] of functions in BMO(**R**). (A constructive method was first presented in [22].) Let $\varphi \in \mathrm{BMO}$ be real-valued and have compact support. Carleson [9] and Varopoulos [36] have both found methods of producing a Carleson measure μ and an L^∞ function v, such that $\|\mu\|_C$, $\|v\|_{L^\infty(\mathbf{R})} \le C_0 \|\varphi\|_*$, and such that

$$\int_{-\infty}^{\infty} F(x)\,(\varphi(x)-v(x))\,dx = \iint_{\mathbf{R}^2_+} F(z)\,d\mu(z)$$

for all $F \in H^1 \cap H^\infty$. With μ and v as above, let $u_k(x)=2iS_k(\mu)(x)$. Then $u_k \in L^\infty(\mathbf{R})$, $\|u_k\|_{L^\infty(\mathbf{R})} \le C_0 \|\varphi\|_*$, and $\varphi = \mathrm{Re}\, u_k + H(\mathrm{Im}\, u_k) + v$, $k=0, 1$, where H denotes the Hilbert transform. See [22] for details.

The above approach to the Fefferman-Stein decomposition is a bit more attractive than the approach in [22] because it provides explicit formulae for u and v. We illustrate this with another example. Suppose $e^\varphi : \mathbf{R}^2_+ \to \mathbf{C} \setminus \{0\}$ is a conformal maping onto some planar domain. Baernstein [1] has shown that then $\varphi \in \mathrm{BMO}$ and $\|\varphi\|_* \le C_0$. (That is to say, the boundary values of φ are in BMO.) Theorem 2 gives a formula for the Fefferman-Stein decomposition of φ. A simple argument using the classical distortion estimates for conformal mappings (see [23]) shows that $\varphi' dx dy$ is a Carleson measure, and $\|\varphi' dx\, dy\|_C \le C_0$. Let $\psi = \mathrm{Re}\, \varphi$ and let $u_k = 2i\, S_k(\bar\partial \psi)$, $k=0, 1$. Then there are constants c_k such that $\varphi = \mathrm{Re}\, u_k + iH(\mathrm{Re}\, u_k) + i(\mathrm{Im}\, u_k + iH(\mathrm{Im}\, u_k)) + c_k$, $k=0, 1$. Smoothness in φ is clearly reflected in the smoothness of u_k, $k=0, 1$. This argument also works when e^φ is merely quasiconformal, because it is still the case that $|\nabla \varphi| dx\, dy$ is a Carleson measure. See [23].

§4. Proofs of Theorems 2 and 4

Our proof of Theorem 2 is in much the same spirit as Koosis' proof [24] of the Burkholder-Gundy-Silverstein theorem. For $\alpha>0$, let $O_\alpha=\{x:f^*(x)>\alpha\}$. Being open, O_α may be written as $\bigcup_j I_j^\alpha$ where the I_j^α are disjoint open intervals. For each I_j^α let $T_j^\alpha \subset \mathbf{R}_+^2$ denote the tent over I_j^α, i.e., T_j^α is the open 45° isosceles triangle lying in $\mathbf{R}_+^2$ with base I_j^α. Also let t_j^α denote $\partial(T_j^\alpha)\cap\mathbf{R}_+^2$, i.e., t_j^α consists of the two sides of T_j^α not lying on $\mathbf{R}$. Now put $H_\alpha(z)=f$ inside $\bigcup_j T_j^\alpha$ and put $H_\alpha(z)\equiv 0$ outside of $\bigcup_j T_j^\alpha$. Then $\bar\partial H_\alpha$ exists as a distribution and $\|\bar\partial H_\alpha\|_C \leqslant \sqrt{2}\,\alpha$. This is because $\bar\partial H_\alpha$ is supported on $\bigcup_j t_j^\alpha$ and $|f|\leqslant\alpha$ on $\bigcup_j t_j^\alpha$. Let $G_\alpha=S_1(\bar\partial H_\alpha)$. We treat f as if its boundary values were a locally integrable function, ignoring the (merely technical) problems arising when $f(x)$ is a distribution. Since $|G_\alpha(x)|\leqslant C_0\alpha$, $x\in\mathbf{R}$, the function $f_\alpha\equiv f-H_\alpha+G_\alpha$ satisfies $f_\alpha\in H^\infty$ and $\|f_\alpha\|_{H^\infty}\leqslant C_0\,\alpha$. Since $F_\alpha\equiv H_\alpha-G_\alpha$ is analytic, we need only estimate its L^{p_0} norm. We present the argument only for the case where $p_0\leqslant 1$; the case where $p_0>1$ follows by interpolation between the estimates we give for $\|G_\alpha\|_{L^1(\mathbf{R})}$ and $\|G_\alpha\|_{L^\infty(\mathbf{R})}$. For each interval I_j^α, let

$$g_{\alpha,j}(z)=\iint_{I_j^\alpha\times(0,|I_j^\alpha|)} K_1(\bar\partial H_\alpha/\|\bar\partial H_\alpha\|_C, z, \zeta)\,\bar\partial H_\alpha(\zeta).$$

Then $G_\alpha=\Sigma_j g_{\alpha,j}$ and by Lemma 2.2,

$$\begin{aligned}\int_{-\infty}^{\infty}|G_\alpha|^{p_0}\,dx &\leqslant \sum_j\int_{-\infty}^{\infty}|g_{\alpha,j}|^{p_0}\,dx\\ &\leqslant \sum_j C_{p_0}\alpha^{p_0}|I_j^\alpha|\\ &\leqslant C_{p_0}\int_{O_\alpha}|f^*|^{p_0}\,dx.\end{aligned}$$

Consequently,

$$\begin{aligned}\int_{-\infty}^{\infty}|F_\alpha|^{p_0}\,dx &\leqslant \int_{-\infty}^{\infty}(|H_\alpha|^{p_0}+|G_\alpha|^{p_0})\,dx\\ &\leqslant C_{p_0}\int_{O_\alpha}|f^*|^{p_0}\,dx,\end{aligned}$$

and the theorem is proved.

We remark that the above procedure can be used to provide atomic type decompositions of functions in H^p, $0<p\leq 1$. Such atomic decompositions were first caried out by Coifman [12] and since have been studied by many authors. See e.g., [13] and [26]. Let $f\in H^p$, $0<p\leq 1$, and with F_α as in the proof of Theorem 2 write

$$f=\sum_{n=-\infty}^{\infty}(F_{2^n}-F_{2^{n+1}}).$$

Each term in the sum can be easily decomposed into $\Sigma_j\,\lambda_{n,j}\,q_{n,j}$. Each $q_{n,j}\in H^\infty$ and satisfies

$$|q_{n,j}(x)|\leq |I|^{-1/p}\exp\{-|I|^{-1/2}|x-x_I|^{1/2}\},$$

where x_I is the center of some interval I associated to $q_{n,j}$. The constants $\lambda_{n,j}$ satisfy $\Sigma_{n,j}|\lambda_{n,j}|^p\leq C_p\|f\|^p_{H^p}$. As a consequence, one sees that the same "atoms" can be used to build all H^p functions, $0<p\leq 1$. this should be compared with [12] where the definition of atoms is different for different ranges of p. That an atomic decomposition of H^p should follow from Theorems 3–5 becomes clear upon the reading of [12], [17], [26], and [28].

We now turn to the proof of Theorem 4. By the reiteration theorem it is only necessary to treat the case where $0<p_0\leq 1$. We will give the proof for the case when $p_0=1$, the proof for $0<p_0<1$ is virtualy identical. Select $\theta\in(0,1)$, $p\in(1,\infty)$, such that $1/p=1-\theta$, and let $f\in H^p$ be continuous on $\mathbf{R}^2_+$, of rapid decrease at ∞, and have norm $\|f\|_{H^p}=1$. Let N be a large positive number and let $I_0=(-N,N)$, $Q_0=I_0\times(0,2N]$. Suppose σ is a Carleson measure, $\|\sigma\|_C\leq 100$, and suppose $\sigma-\gamma\geq 0$, where γ is arc-length measure on ∂Q_0. (For a set $\Omega\subset\mathbf{R}^2_+$ we define $\partial\Omega=(\partial\Omega)\cap\mathbf{R}^2_+$.) Let $F_0(z)=f(z)\chi_{Q_0}(z)$ and let

$$G_0(z)=\iint K_1(\sigma/\|\sigma\|_C,z,\zeta)\,\bar\partial F_0(\zeta).$$

Then by Theorem 1 and Lemma 2.4, $F_0-G_0\in H^p$ and $\|f-(F_0-G_0)\|_{H^p}<\varepsilon$ if N is large enough. By a translation and a change of scale we may assume the above properties hold for f with $I_0=(0,1)$, $Q_0=I_0\times(0,1]$.

We now run a stopping time argument. For a cube $Q=I\times(0,|I|]$ let $T(Q)=I\times(|I|/2,|I|]$ denote the top half of Q. Let n_0 be the integer satisfying $2^{n_0}<\sup_{T(Q_0)}|f(z)|\leq 2^{n_0+1}$. We retain the notation used in the proof of Theorem 3 with the exception that we use the (equivalent) maximal function $f^*(t)=\sup_{|x-t|<10y}|f(x+iy)|$.

Let $m_0 \geq n_0+1$ be the smallest integer such that

$$|\{x\in I_0: f^*(x)>2^{m_0}\}| = |I_0 \cap O_{2^{m_0}}| \leq \frac{1}{2}|I_0|.$$

Then

$$\left|I_0 \cap O_{2^{m_0-1}}\right| > \frac{1}{2}|I_0|.$$

Let $W_{m_0}=\{J_k^{m_0}\}$ be the dyadic Whitney decomposition of $O_{2^{m_0}}$. Then if $J\in W_{m_0}$ either $J\subset I_0$ or $J\cap I_0=\varnothing$. Let $\{I_j^0\}=\{J\in W_{m_0}: J\subset I_0\}$ and let $Q_j^0=I_j^0\times(0,|I_j^0|]$, $\mathcal{R}_0\equiv Q_0\setminus\{\bigcup_j \bar Q_j^0\}$. Because of the way we have defined f^*, $|f(z)|\leq 2^{m_0}$ on $\partial\mathcal{R}_0$ and consequently $|f(z)|\leq 2^{m_0}$, $z\in\mathcal{R}_0$. Let $\mathcal{J}_0=\{Q_j^0\}$ be the cubes so formed at stage zero. At stage one, consider the individual cubes $Q_j\in\mathcal{J}_0$. For such a cube $Q_j=I_j\times(0,|I_j|]$ let n_j be that integer satisfying $2^{n_j}<\sup_{T(Q_j)}|f(z)|\leq 2^{n_j+1}$ and let $m_j\geq n_j+1$ be the smallest integer satisfying $|I_j\cap O_{2^{m_j}}|\leq\frac{1}{2}|I_j|$. Then $|I_j\cap O_{2^{m_j-1}}|>\frac{1}{2}|I_j|$ and since $I_j\subset O_{2^{m_0}}$, it must be that $m_j>m_0$. Let $W_{m_j}=\{J_k^{m_j}\}$ be the dyadic Whitney decomposition of $O_{2^{m_j}}$ and let $\{I_k^j\}=\{J\in W_{m_j}: J\subset I_j\}$. For each such I_k^j let $Q_k^j=I_k^j\times(0,|I_k^j|]$ and let $\mathcal{R}_j\equiv Q_j\setminus\{\bigcup_k \bar Q_k^j\}$. Then $|f(z)|\leq 2^{m_j}$, $z\in\mathcal{R}_j$. Let $\mathcal{J}_1=\{Q_r^s\}$ be the collection of all such cubes formed at stage one. Proceeding in this manner we decompose Q_0 into $\bigcup\mathcal{R}_j$. Each $\mathcal{R}_j$ is contained in a cube $Q_j=I_j\times(0,|I_j|]$, and $|f(z)|\leq 2^{m_j}$, $z\in\mathcal{R}_j$. Furthermore

$$m_k>m_j \quad \text{if} \quad I_k \subsetneqq I_j,$$

and

$$\Big|\bigcup_{I_k\subsetneqq I_j} I_k\Big| \leq \frac{1}{2}|I_j|.$$

Since $\partial\mathcal{R}_j$ has arclength $l(\partial\mathcal{R}_j)\leq 6|I_j|$, our last inequality and an iteration argument show that if σ is arclength measure on $\bigcup_j(\partial\mathcal{R}_j)$, then σ is a Carleson measure and $\|\sigma\|_C\leq 100$. Now let $f_j(z)=f(z)\chi_{\mathcal{R}_j}(z)$. Since it is a telescoping series, $\Sigma_j\bar\partial f_j=\bar\partial F_0$. Let $g_j(z)=\int\int K_1(\sigma/\|\sigma\|_C, z,\zeta)\bar\partial f_j(\zeta)$. Then by Theorem 1 and Lemmas 2.2 and 2.3, $f_j-g_j\in H^1\cap H^\infty$ and $\|f_j-g_j\|_{H^\infty}\leq C_0 2^{m_j}$, $\|f_j-g_j\|_{H^1}\leq C_0 2^{m_j}|I_j|$. Now define the Banach space $(H^1\cap H^\infty)$ valued function $h_{j,\zeta}$ on $S=\{\zeta: 0\leq\operatorname{Re}\zeta\leq 1\}$ by

$$h_{j,\zeta}(z)=(2^{m_j})^{\alpha(\zeta)}(f_j-g_j)(z),$$

where $\alpha(\zeta)=p(1-\zeta)-1$. We consider the (holomorphic) function $H_\zeta=\Sigma_j h_{j,\zeta}$, $\zeta\in S$. If $\zeta=1+it$, then $h_{j,\zeta}\in H^\infty$, $\|h_{j,\zeta}\|_{H^\infty}\le C_0$, and by Lemma 2.3,

$$\|H_\zeta\|_{H^\infty}\le \sup_{x\in\mathbf{R}}\sum_j |h_{j,\zeta}(x)|\le C_0.$$

(This is where we really use the measure σ.) Now consider the case where $\zeta=0+it$. Then for each j, $h_{j,\zeta}\in H^1$ and

$$\begin{aligned}\|h_{j,\zeta}\|_{H_1}&\le C_0 2^{pm_j}|I_j|\\ &\le C_0 2^{pm_j}\left|I_j\cap O_{2^{m_j-1}}\right|\end{aligned}$$

Recall that if $I_j\subsetneqq I_k$, then $m_j>m_k$. Consequently,

$$\begin{aligned}\|H_\zeta\|_{H^1}&\le\sum_j\|h_{j,\zeta}\|_{H^1}\\ &\le\sum_j C_0 2^{pm_j}\left|I_j\cap O_{2^{m_j-1}}\right|\\ &\le C_0\sum_{n=-\infty}^{\infty}2^{pn}|O_n|\\ &\le C_0\int_{-\infty}^{\infty}|f^*|^p dx\\ &\le C_0.\end{aligned}$$

At the point $\zeta=\theta$ we have

$$H_\theta=\sum_j h_j=F_0-\iint K_1(\sigma/\|\sigma\|_C, z,\zeta)\,\bar\partial F_0(\zeta),$$

because $\Sigma_j\bar\partial f_j=\bar\partial F_0$. By a previous comment, $\|f-H_\theta\|_{H^p}<\varepsilon$. Standard arguments now complete the proof of Theorem 4.

§5. H^∞ interpolation

Our Theorem 1 is closely related to the study of H^∞ interpolating sequences. A sequence $\{z_j\}$ of points in $\mathbf{R}^2_+$ is called an (H^∞) interpolating sequence if whenever

$\{a_j\}\in l^\infty$ there is $F\in H^\infty$ such that $F(z_j)=a_j$. By a theorem due to Carleson [5], $\{z_j\}$ is an interpolating sequence if and only if

$$\inf_j \prod_{\substack{k\\ k\neq j}} \left|\frac{z_j-z_k}{z_j-\bar z_k}\right| = \delta > 0. \tag{5.1}$$

In [22] the interplay between Carleson measures and interpolating sequences is exploited to find $L^\infty(\mathbf{R})$ solutions of $\bar\partial F=\mu$. The purpose of this section is to show that our Theorem 1 is equivalent to finding explicit solutions for the H^∞ interpolation problem. To demonstrate this we first fix our attention on a remarkable result due to P. Beurling [7]. For an interpolating sequence $\{z_j\}$ let

$$M=\sup_{\|\{a_j\}\|_{l^\infty}\leq 1}\ \inf_{F\in H^\infty}\{\|F\|_{H^\infty}: F(z_j)=a_j\}.$$

P. Beurling has shown that for $\{z_j\}$ and M as above there are functions $F_j\in H^\infty$ such that $F_j(z_k)=\delta_{j,k}$ and $\Sigma_j|F_j(z)|\leq M$ for all $z\in\mathbf{R}^2_+$. Here $\delta_{j,k}$ denotes the Kronecker delta. Our next result is an explicit formula for P. Beurling type functions. Let

$$B(z)=\prod_j \alpha_j\frac{z-z_j}{z-\bar z_j}$$

be the Blaschke product with simple zeros at $\{z_j\}$ and let

$$B_j(z)=\prod_{\substack{k\\ k\neq j}} \alpha_k\frac{z-z_k}{z-\bar z_k}$$

be the Blaschke product with simple zeros at $\{z_k: k\neq j\}$. The α_k are unimodular coefficients chosen so as to make the products converge.

THEOREM 6. *Suppose* $\{z_j\}$ *satisfies* (5.1). *Let*

$$F_j(z)=c_jB_j(z)\left(\frac{y_j}{z-\bar z_j}\right)^2\exp\left\{\frac{-i}{\log 2/\delta}\sum_{y_k\leq y_j}\frac{y_k}{z-\bar z_k}\right\}$$

where

$$c_j=-4(B_j(z_j))^{-1}\exp\left\{\frac{i}{\log 2/\delta}\sum_{y_k\leq y_j}\frac{y_k}{z_j-\bar z_k}\right\}.$$

Then $F_j(z_k)=\delta_{j,k}$ *and*

$$\sum_j |F_j(z)| \leq C_0 \frac{\log 2/\delta}{\delta}$$

for all $z \in \mathbf{R}^2_+$.

Theorem 6 provides for the first time a formula for H^∞ interpolation. For other proofs of Beurling's theorem see Earl [16] and Varopoulos' proof on page 298 of [19]. (Varopoulos' theorem yields a slightly weaker estimate, but the result holds in a more general setting.) The bound $C_0\delta^{-1}\log(2/\delta)$ in Theorem 6 is known to be of optimal order—see page 293 of [19]. Before proceeding to its proof, we observe that Theorem 6 is also related to Theorems 3 and 4. Suppose that $\{\beta_j\} \in l^1$ and put $F(z)=\Sigma\beta_j y_j^{-1} F_j(z)$, where the F_j are as in Theorem 6. Then $F(z_j)=\beta_j y_j^{-1}$, $F \in H^1$, and $\|F\|_{H^1} \leq \Sigma|\beta_j| y_j^{-1}\|F_j\|_{H^1} \leq C(\delta)\Sigma|\beta_j|$. By interpolating between the case where $p=1$ (above) and $p=\infty$ (Theorem 6) we see that if $1 \leq p < \infty$ and $\{\beta_j\} \in l^p$, then for $F(z)=\Sigma\beta_j y_j^{-1/p} F_j(z)$ one has $F(z_j)=\beta_j y_j^{-1/p}$, $F \in H^p$, and $\|F\|_{H^p} \leq C(p,\delta)\|\{\beta_j\}\|_p$. This is the interpolation theorem of Shapiro and Shields [34].

Proof of Theorem 6. We first estimate the quantities c_j. By condition (5.1), $|B_j(z_j)|^{-1} \leq \delta^{-1}$. Since $\|\Sigma y_j \delta_{z_j}\|_C \leq C_0 \log 2/\delta$ (see pp. 287–290 of [19]), the proof of inequality (2.1) shows

$$\operatorname{Re}\left\{\sum_{y_k \leq y_j} \frac{iy_k}{z_j-\bar{z}_k}\right\} \leq C_0 \log\frac{2}{\delta}.$$

Consequently, $c_j \leq C_0\delta^{-1}$. To finish the proof of Theorem 6 we need therefore only demonstrate that for $x \in \mathbf{R}$,

$$\sum_j \left|\frac{y_j}{x-\bar{z}_j}\right|^2 \exp\left\{\frac{-1}{\log 2/\delta} \sum_{y_k \leq y_j} \frac{y_k^2}{|x-\bar{z}_k|^2}\right\} \leq C_0 \log\frac{2}{\delta},$$

because then the maximum principle can be invoked. This last inequality follows from Lemma 2.1 with

$$\sigma = \left(\log\frac{2}{\delta}\right)^{-1} \sum y_j \delta_{z_j}.$$

As a final remark, it should be noted that the author was led to Theorem 1 by first proving Theorem 6.

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