Disjoint spheres, approximation by imaginary quadratic numbers, and the logarithm law for geodesics

by

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§ 0. Introduction

This paper is based on the principle that probabalistic independence of certain sets in Euclidean space is forced by a disjoint collection of spheres in a Euclidean space of one higher dimension. (See Figure 1.)

This principle allows a new proof of (a new variant of) Khintchine's approximation theorem for almost all reals by rationals § 3. The new proof extends naturally to the approximation of almost all complex numbers by ratios of integers $p/q, p, q \in \vartheta(\sqrt{-d})$ in imaginary quadratic fields.

Let $0 \leqslant a(x) < 1, x$ a positive real, be any function so that the size of $a(x)$ up to bounded ratio only depends on the size of x up to bounded ratio. The following theorem is proved in § 7.

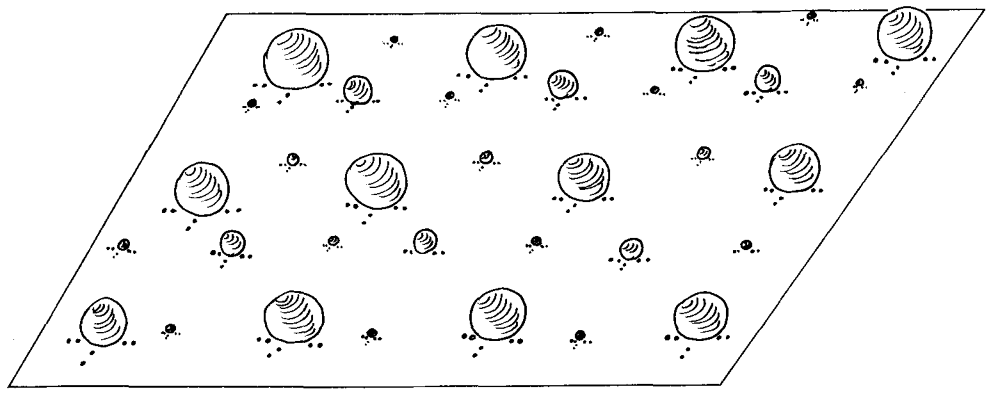


Figure 1.

THEOREM 1 (generalized Khintchine). *For almost all complex numbers z there are infinitely many pairs* $p, q \in \vartheta\sqrt{-d}$ *so that*

$$|z-p/q| \leqslant \frac{a(|q|)}{|q|^2} \quad \text{and ideal } (p,q) = \vartheta(\sqrt{-d})$$

iff

$$\int^{\infty} \frac{a(x)^2}{x}\,dx = \infty.$$

It turns out such approximation results for fixed d are equivalent to the way in which a random geodesic on a certain complete hyperbolic three manifold V_d of finite volume (but non-compact) occasionally ventures out into one of the cuspidal ends.(1) The analogue of these approximation results is proved in the same way for all hyperbolic manifolds V of finite volume. For example, let $\operatorname{dist} v(t)$ denote the distance from a fixed point in V of the point achieved after traveling a time t along the geodesic with initial direction v. Along the random geodesic the function $\operatorname{dist} v(t)$ has a well defined limit superior (the logarithm law) analogous to the law of the iterated logarithm for a random path on the line (another result of Khintchine). (See Figure 2.)

THEOREM 2 (logarithm law for geodesics). *If* $V=\mathbf{H}^{d+1}/\Gamma$ *where* Γ *is a cofinite volume discrete subgroup of hyperbolic isometries which is not cocompact, then for almost all starting directions v of geodesics*

$$\limsup_{t\to\infty} \frac{\operatorname{dist} v(t)}{\log t} = 1/d.$$

(1) The manifold V_d is hyperbolic 3-space modulo the Bianchi group Γ_d consisting of 2×2 matrices with entries in $\vartheta(\sqrt{-d})$ and determinant 1.

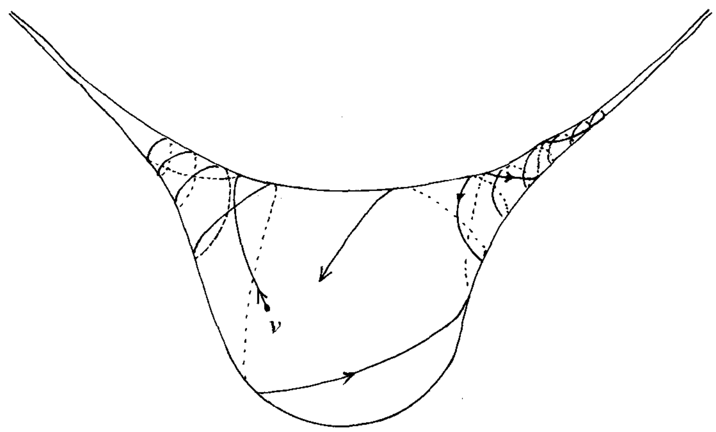


Figure 2.

This theorem is proved in a more precise form in § 9. In each of these two theorems there is also a quantitative assertion that the number of times the desired approximation or the lim sup is achieved is infinitely often as large as the corresponding diverging integral (§ 2).

Finally, in § 10 we discuss briefly an application of this *disjoint sphere method* to the Hausdorff geometry of limit sets of geometrically finite Kleinian groups—the original motivation for this work.

Acknowledgement. The connection between geodesic excursion on $\mathbf{H}^2/PSl(2,\mathbf{Z})$ and Khintchine's metric theory was pointed out to me by David Kahzdan when I conjectured Theorem 2 to him. He also explained how most of the generalization of Khintchine's proof would go. Two other ideas in the proof, §§ 2 and 3, were derived during discussions with Jon Aaronson and Dan Rudolph.

I also benefited from discussions about rational approximation with Wolfgang Schmidt whose quantitative result (1960) in the real case is presently beyond the discussion here of Khintchine's results.

Finally, a literature search beginning with A. L. Schmidt's paper (Acta Math., 1975) on continued fractions of Gaussian integers [ASc] led to the paper of W. J. LeVeque (1952) where he proves the Khintchine metrical theorem (a variant of our Theorem 1) for the gaussian field $\mathbf{Q}(\sqrt{-1})$. LeVeque also makes use of the disjoint spheres, a suggestion of K. Mahler, but in a different part of his paper.

§ 1. Abstract Borel-Cantelli

If $A_1, A_2, \ldots$ is a sequence of subsets of a probability space X and $A_\infty = \{x : x \in A_i$ for infinitely many $i\}$ we want to compare the conditions

(i) $\Sigma_i |A_i| = \infty$, $|A_i| =$ measure A_i

and

(ii) $|A_\infty| > 0$.

The first proposition is very standard. We recall the proof to establish notation.

PROPOSITION 1. *If* $|A_\infty| > 0$, *then* $\Sigma_i |A_i| = \infty$.

Proof. Let $\varphi_N(x) =$ sum of the characteristic functions of A_i for $i \leq N$. Then by definition $A_\infty = \{x: \lim_{N \to \infty} \varphi_N(x) = \infty\}$. Since the φ_N are monotone increasing

$$\lim \int \varphi_N = \int \lim \varphi_N$$

by the Lebesgue monotone convergence theorem. One side is $\Sigma_i |A_i|$ and the other is infinity if $|A_\infty| > 0$.

Example. For $\varepsilon > 0$ we place an interval of size $q^{-2-\varepsilon}$ around each reduced rational p/q in the interval $[0, 1]$ and let A_q denote the union of these. Then $|A_q| \leq q \cdot q^{-2-\varepsilon} = q^{-1-\varepsilon}$ so $\Sigma_q |A_q| < \infty$. Thus $|A_\infty| = 0$ by the proposition, and this means $|x - p/q| < q^{-2-\varepsilon}$ has only finitely many solutions for almost all x.

More generally, we see the direct half of *Khintchine's theorem*: if $\Sigma_q a(q) < \infty$, then $|x - p/q| < a(q)/q$ has finitely many solutions for almost all x.

Remark. It is worth noting that the $q^{-2-\varepsilon}$ result is also true for algebraic numbers (the celebrated Roth theorem) and the proof is very difficult. It is unknown whether algebraic numbers also behave like random numbers for the $a(q)/q$ result or for the positive results described below.

Now we turn to the less trivial converse of Proposition 1. It is easy to give examples where $\Sigma_i |A_i| = \infty$, but $|A_\infty| = 0$. For example let A_i be the intervals $[0, 1/i]$. Ironically, we need to control the overlapping to insure $|A_\infty| > 0$. The standard Borel-Cantelli lemma is Proposition 1 and the statement that the converse is true if the A_i are independent in the sense of probability. Independence implies $|A_{j1} \cap A_{j2} \cap \ldots \cap A_{jn}| = |A_{j1}| \cdot |A_{j2}| \ldots |A_{jn}|$. Actually, much less is needed, and this seems to be less well known.

PROPOSITION 2 (quasi-independent Borel-Cantelli). *Suppose* $\Sigma_i |A_i| = \infty$ *and there is a constant* $c \geq 1$ *so that for* $i < j$, $|A_i \cap A_j| \leq c|A_i| \cdot |A_j|$. *Then* $|A_\infty| > 0$, *in fact there is a set of positive measure* $\bar{A}$ *so that for* $x \in \bar{A}$,

$$\limsup_N \frac{\operatorname{card}\{i : x \in A_i, i \leq N\}}{|A_1| + \ldots + |A_N|} > 0.$$

Proof. Consider φ_N as in Proposition 1. Let $|\varphi_N|_2, |\varphi_N|_1$ denote $(\int \varphi_N^2)^{1/2}$ and $\int \varphi_N$ respectively. By Schwarz $|\varphi_N|_1 \leq |\varphi_N|_2$. Conversely, using our hypothesis,

$$\begin{aligned}
\int \varphi_N^2 &= \sum_{i \leq j \leq N} |A_i \cap A_j| \\
&= \sum_{i \leq N} |A_i| + \sum_{i < j \leq N} |A_i \cap A_j| \\
&\leq \sum_{i \leq N} |A_i| + c \sum_{i < j \leq N} |A_i||A_j| \quad \text{(by quasi-independence)} \\
&\leq c \sum_{i \leq j \leq N} |A_i||A_j| \\
&= c\left(\int \varphi_N\right)^2.
\end{aligned}$$

Thus $|\varphi_N|_2 \leq \sqrt{c}\, |\varphi_N|_1$.

Now consider $\psi_N(x) = \varphi_N(x)/|\varphi_N|_1$ and choose a weak limit ψ in the ball of radius $\sqrt{c}$ of square integrable funcitons. Since $(\psi_N, 1) \to (\psi, 1)$ we have $|\psi|_1 = 1$.

Similarly, ψ is non-negative, so ψ is positive on a set of positive measure. If $A = \text{support } \psi$, then for all $x \in A$, $\lim_N \varphi_N(x) = \infty$, because $\lim_N |\varphi_N|_1 \to \infty$. Thus $A_\infty \supset A$ has positive measure.

Now we show there is a set $\bar{A}$ of positive measure in A_∞ so that if $x \in \bar{A}$

$$\limsup_N \frac{\varphi_N(x)}{|\varphi_N|_1} > 0.$$

If for subset of positive measure in A the lim sup is zero, then for a further subset of positive measure the ratios are ≤ 1 for N sufficiently large. By dominated convergence $\int \psi = 0$ on this subset. This is a contradiction. Thus $\bar{A}$ may be taken to have full measure in A.

Remark. In an earlier paper with Jon Aaronson [AS] we made use of an inequality $|A_i \cap A_{i+j}| \leq c|A_i|\,|A_j|$ for all $i,j>0$ to obtain a similar proof that $|A_\infty|>0$ if $\Sigma_i |A_i|=\infty$.

§ 2. Disjoint circles and quasi-independence

Now we develop a geometric condition which implies the inequality $|A_i \cap A_j|<c|A_i|\cdot|A_j|$ needed in Proposition 2. For simplicity we first treat the case when the probability space is a unit interval I on the bundary of the upper half plane H^+.

Figure 3.

The geometric device is a countable collection C of (interior disjoint) circles in H^+ resting on points $x_1, x_2, \ldots$ of I. Fix some number $\varrho<1$ and say two real numbers have the same ϱ-size if they belong to one of the intervals $(\varrho^{n+1}, \varrho^n]$. Group the circles into collections whose diameters have the same ϱ-size.

Because the circles are interior disjoint there can be no more than $\asymp 1/s$ circles of a given size s.(1) Say that a size s is *good* for the collection of circles C if there are at least $\asymp 1/s$ circles of size s. ($f(s) \asymp g(s)$ means the log of the ratio is bounded.)

Now consider $0<a(x)\leq 1$ where $a(x)$ only varies in a bounded ratio for the numbers of a certain size, s. For each size s let A_s denote the union of intervals of length $a(r_i)\, r_i$ centered at $\{x_i\}$ where $\{x_i\}$ is the set of the resting points of circles of size s and $\{r_i\}$ are the corresponding radii. We will write $a(s)\, s$ for $a(r_i)\, r_i$.

PROPOSITION 3. *There is a constant c so that if $s_2<s_1$ are two sizes and s_2 is good then* $|A_{s_1} \cap A_{s_2}|<c|A_{s_1}|\,|A_{s_2}|$.

Proof. We ignore fixed constants. Then we estimate the number of intervals of A_{s_2} contained in one of the larger intervals of A_{s_1}. This estimate is the maximum of 1 and $a(s_1)\, s_1/s_2$ because the smaller intervals are s_2 apart, and the larger interval has size $a(s_1)\, s_1$. We will see below that $a(s_1)\, s_1/s_2 \geq 1$(in fact $a(s_1)^2 s_1/s_2 \geq 1$).Thus

(1) $\ll$, $\gg$, $\asymp$ mean respectively: the ratio of the two related quantities is bounded above, below or both by fixed constants.

$$|A_{s_1} \cap A_{s_2}| \leq \left(\frac{a(s_1)\,s_1}{s_2}\right)(a(s_2)\,s_2) \quad \text{(number of intervals in } A_{s_1})$$
$$= |A_{s_1}|\, a(s_2).$$

Because s_2 is a good size $|A_{s_2}|=a(s_2)$. Thus for some c, $|A_{s_1} \cap A_{s_2}|<c|A_{s_1}||A_{s_2}|$.

To finish the proof consider the figure

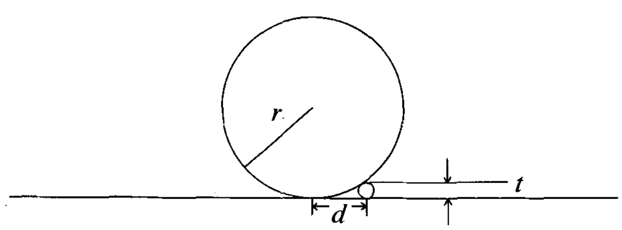


Figure 4.

We have $t\leq(1/r)\,d^2$. In our case if an interval of A_{s_2} intersects one of A_{s_1} we will have $d\leq a(s_1)\,s_1$, $r=s_1$ and $t=s_2$. Thus

$$s_2 \leq (1/s_1)\,(a(s_1)\,s_1)^2$$

or $1\leq a(s_1)^2 s_1/s_2$ which implies $1\leq a(s_1)\,s_1/s_2$.

Remark. The disjoint circles enter our discussion in two ways:

(i) the disjoint circles of *one* size s keep the corresponding intervals of size $a(s)\,s$, a distance s apart;

(ii) the disjoint circles of *different* sizes keep intervals disjoint while the crucial inequality $a(s_1)\,s_1/s_2\geq 1$ is not satisfied. Essentially only this point was missing from the discussion with Kahzdan.

§ 3. Khintchine's metric approximation (a new proof)

For the rational approximation of almost all reals we use the collection of circles (Figure 5) resting on the real axis consisting of circles of diameter $1/q^2$ resting at p/q where p, q are relatively prime.

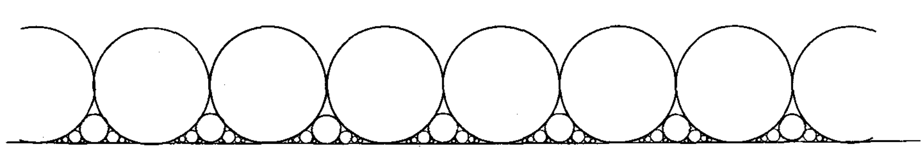


Figure 5.

Using the continued fraction construction, translating, inverting etc., which preserves circles, it is easy to derive the above figure: disjointness, position (p/q), and sizes ($1/q^2$). We offer another proof using discrete groups (in this case $PSl(2, \mathbf{Z})$) in § 7.

The number of $p \leq q$ relatively prime to q is on the average $\geq$constant$\cdot q$ (an easy estimate with the Euler φ-function). The number of circles over the unit interval of a given size (§ 1) $s=(\varrho^{n+1}, \varrho^n)$, is the number of pairs (p, q) with $p \leq q$, p relatively prime to q, and $1/q^2 \in s$.

Thus q varies between a number N and a (constant >1) times N. So for q large we are integrating a quantity on the average as big as q over an interval of size q. We obtain $\asymp q^2$ circles of size $1/q^2$ for every size. (The discrete group proof is in § 6.)

For the collection of disjoint circles in the figure then every size is good. This geometric information about the rationals is the only arithmetic structure used in our proof of Khintchine's theorem.

Now let $0 < a(x) \leq 1$ be a function of $x \in [1, \infty)$ which up to a bounded ratio only depends on the size of x (e.g. $a(x)$ is smooth and $|a'x| <$constant$\cdot a(x)$).

THEOREM 3. *For almost all reals x there are infinitely many solutions*

$$|x - p/q| \leq \frac{a(q)}{q^2}$$

iff $\int_1^\infty (a(x)/x)\,dx$ *diverges.*

Remark. This seems to be a new variant of Khintchine's theorem. It is known some condition on the $a(x)$ is required (besides a divergence condition). In the usual statement one assumes $a(q)/q$ is monotone decreasing. We have merely assumed that the *size* of the desired approximation only depends on the *size* of the denominator q.

The proof is new, with the arithmetical and geometrical parts separated.

Proof. Consider sets A_s defined by placing intervals of the desired approximation size $a(q)/q^2$ about those p/q with $1/q^2$ of size s.

Since all sizes are good (by the above discussion) we have by Proposition 3

$$|A_{s_1} \cap A_{s_2}| \leq c|A_{s_1}||A_{s_2}|.$$

Thus by Proposition 2, A_∞ has positive measure if $\Sigma |A_s| = \infty$.

If q^2 varies in a bounded ratio so does q and therefore also $a(q)$. Thus $\Sigma |A_s| = \infty$ means $\Sigma_i a(x_i) = \infty$ where x_i^{-1} ranges over the sizes ϱ^i. By the regularity property of $a(x)$ this is equivalent to $\int_0^\infty a(\bar{\varrho}^t)\,dt = \infty$. If $x = \bar{\varrho}^t$, this is equivalent to $\int_1^\infty (a(x)/x)\,dx = \infty$.

Since approximation properties are invariant under rational translations the set of positive measure must be of full measure. Q.E.D.

Quantitative form. Let $n(x, N, a)$ denote the number of p/q with $q^2 \leq N$ so that $|x-p/q| \leq a(q)/q^2$.

THEOREM 4. *There exists $c>0$ so that for almost all x,*

$$\limsup_N \frac{n(x, N, a)}{\int_1^N \frac{a(x)}{x} dx} = c.$$

Proof. By Proposition 2 there is a set x of positive measure so that

$$\limsup_k \frac{\operatorname{card}\{s_i : x \in a_{s_i}, i \leq k\}}{|A_{s_1}| + |A_{s_2}| + \ldots + |A_{s_k}|} > 0.$$

But this function of x is constant on orbits of rational translations. So it must be constant a.e.

Applying the definition gives the result. Q.E.D.

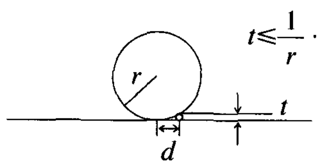
Remark. W. Schmidt (1960) proves a better quantitative result: the lim sup is replaced by a limit and the error is estimated. Schmidt's proof uses more of the arithmetic structure of the situation than the simple properties of Figure 5 used here. Our quantitive result can be added as is to the generalized Khintchine, § 7, and to the logarithm law for geodesics, § 9, because only the simple properties of Figure 5 are needed.

§ 4. Disjoint spheres and Borel-Cantelli with respect to Lebesgue measure

If we have a collection of disjoint spheres resting on a bounded set of the plane and we form sets A_s in the plane which are a union of disks of radius size $a(1/s) \cdot s$ centered on the resting points of spheres of size s we have an exactly analogous discussion to § 1.

Using Lebesgue measure and assuming there are $\asymp 1/s^2$ spheres of size s the result is that A_∞ has positive real measure iff $\int_1^\infty (a^2(x)/x)\,dx = \infty$ ($a(x)$ as in § 3).

The proof goes as before. The Figure 4 argument showing $a(1/s_1)\, s_2/s_1 \geq 1$ if two disks of different sizes intersect has the same force.



The spacing argument to show (the number of disks of A_{s_2} in a disk of A_{s_1})$\leq (a(1/s_1)\, s_1/s_2)^2$ now becomes an area argument.

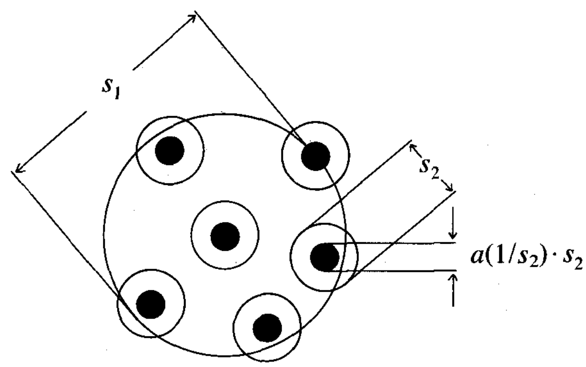


Figure 6.

The boundary effect is treated using the fact that the Lebesgue area is only increased by a factor if the large disk is increased by a factor.

Similarly, we can consider disjoint spheres in $\mathbf{R}^{d+1}_{+}$ *resting on a bounded set of* $\mathbf{R}^d$. *We form sets* $A_s \subset \mathbf{R}^d$ *which are d-balls of radius size* $a(1/s)\cdot$(size of sphere) *centered at resting points, and assuming there are* $\asymp 1/s^d$ *spheres of size s find that Lebesgue measure* $A_\infty > 0$ *iff* $\int_1^\infty (a(x)^d/x)\,dx = \infty$.

§ 5. Disjoint spheres arising from cusps

Let Γ denote any discrete groups of hyperbolic isometries of the upper half space model of $\mathbf{H}^{d+1}$ with boundary $\mathbf{R}^d \cup \infty$. A *cusp* is a conjugacy class of infinite maximal parabolic subgroups and it corresponds to a thin region R in $\mathbf{H}^{d+1}/\Gamma$ with a simple fundamental group generated by short loops. See [T, 5.55].

The inverse image of R in $\mathbf{H}^{d+1}$ (viewed as the upper half space above $\mathbf{R}^d$) consists of a disjoint union of $(d+1)$ balls in $\mathbf{H}^{d+1}$ resting on $\mathbf{R}^d$ plus everything above a plane parallel to $\mathbf{R}^d$ in case ∞ is fixed by a representative subgroup of the cusp. (See Figure 7.)

In the latter case, which can always be arranged, the configuration of disjoint spheres will be invariant by a discrete group of translations of $\mathbf{R}$ having rank $k \leq d$. This group has finite index in the parabolic group fixing infinity and its rank k is called the rank of cusp.

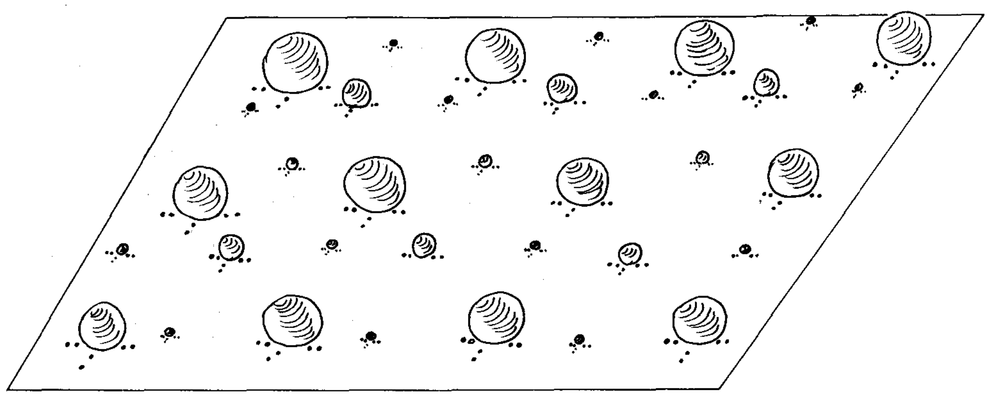


Figure 7.

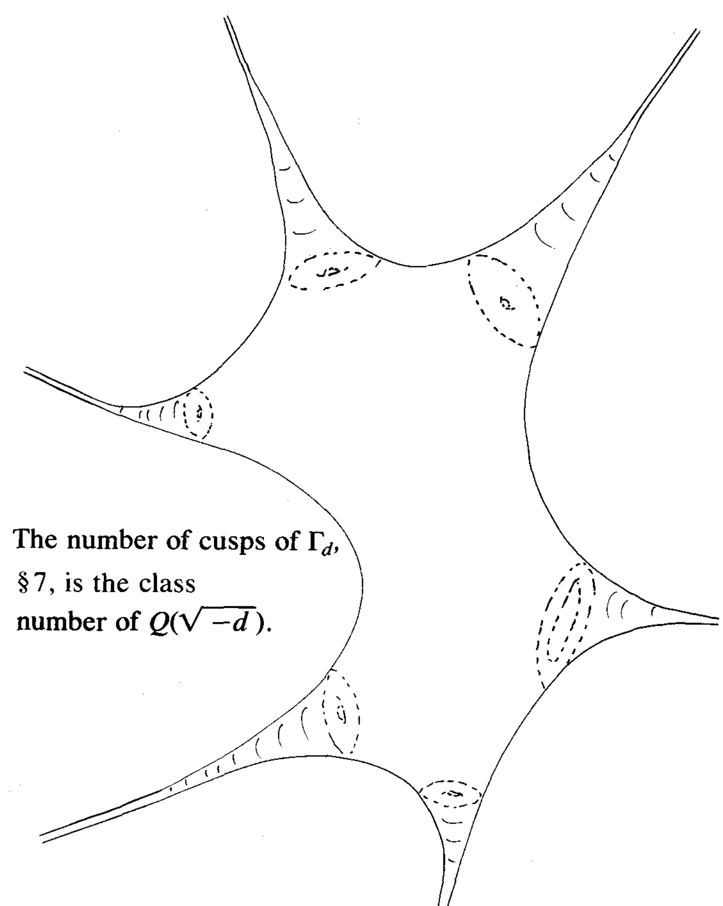


Figure 8.

If $\mathbf{H}^{d+1}/\Gamma$ has finite volume all cusps have rank d and each determines an exponentially thinning end homeomorphic to (d-torus)$\times[0, \infty)$, assuming the parabolic group is torsion free. (See Figure 8.)

§6. Disjoint spheres and the mixing property of the geodesic flow

If Γ is a discrete group of hyperbolic isometries and $\mathbf{H}^{d+1}/\Gamma$ has finite volume, one knows that relative to smooth measure the geodesic flow on the unit tangent is ergodic, preserves a finite invariant measure, and is mixing. If χ denotes the characteristic function of a small ball B in $\mathbf{H}^{d+1}/\Gamma$ lifted to the tangent bundle, then by mixing $\int(\chi\cdot g_t)\cdot(\chi)\to\text{constant}>0$ as $t\to\infty$.

The picture in the universal cover is:

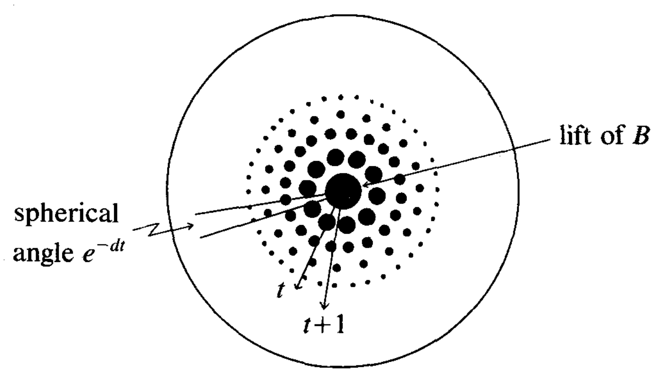


Figure 9.

The integral counts $e^{-dt}\cdot$(number of Γ orbit balls approximately t away from a fixed lift of B). Thus the number of orbit points in fixed width spherical shells is caught between two constants times e^{dt}, t the radius.

Note. This mixing was used over 10 years ago by Margulis to derive this kind of estimate.

On the other hand the orbit of B falls into groups uniformly distributed on the horospheres (spheres in $\mathbf{H}^{d+1}/\Gamma$ tangent to $\mathbf{R}^d$) along the Γ orbit of one cusp:

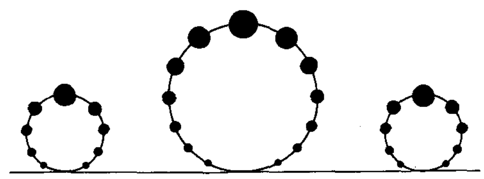


Figure 10.

The sum of $e^{-d(x,\gamma x)}$ over one horosphere is commensurable to the largest term ($(x, y)=$ hyperbolic distance) each being comparable to the solid angle of the horosphere viewed from the fixed lift of B (with center x_0, say). Thus we are close to the proof of

PROPOSITION 4. *There is a $\varrho<1$, so that the number of spheres resting on a compact set of $\mathbf{R}^d$ in a horospherical family of a fixed spherical size $s \in (\varrho^{n+1}, \varrho^n]$ is comparable to $(1/s)^d$.*

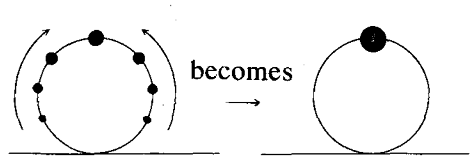
Proof. (1) If a horosphere has Euclidean size s in the unit ball model the closest point to the center is at hyperbolic distance d where $s \asymp e^{-d}$.

(2) For a compact set of $\mathbf{R}^d$, sizes of horospheres resting there are comparable to the corresponding sizes of horospheres in the unit ball which are the image by stereographic projection. Thus we may work in either model.

(3) We refer to the term $e^{-d(x_0,\gamma x_0)}$ as the solid angle of (a unit object at) γx_0 as viewed from x_0.

Now the total solid angle of the part of the orbit inside a ball of radius T about x_0 is at most ce^T. By the mixing argument above the solid angle in a spherical shell of unit width is at least $c'e^T$, for T sufficiently large.

If we recollect the solid angle on each horosphere and move it to the orbit point closest to x_0,



we only increase this solid angle by a definite factor.

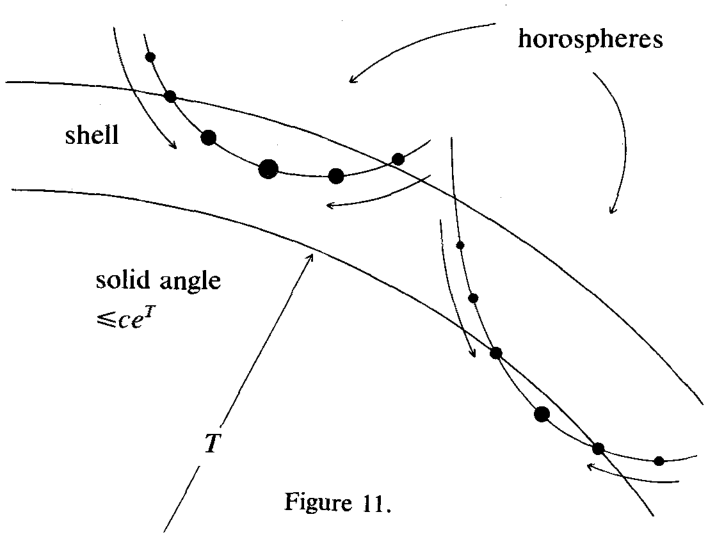


Figure 11.

Thus the recollecting process only loses from the shell an amount of solid angle at most ce^T. By considering shells of width k thick enough we can be sure that at least $c''e^T$ remains in each such thick shell.

This will mean there are $c'''e^T$ (at least) horospheres with their tops in the successive shells $[T, T+k]$. Q.E.D.

§ 7. Disjoint spheres and imaginary quadratic fields

Let Γ_d denote the Bianchi group consisting of 2×2 matrices of determinant one with entries in the ring of integers $\vartheta=\vartheta(d)$ of $\mathbf{Q}(\sqrt{-d})$, where d is a positive integer which is not a perfect square.

Suppose $p, q \in \vartheta(d)$ are relatively prime in the sense that $pr+qs=1$ for $r, s \in \vartheta$. Equivalently, ideal $(p, q)=\vartheta$. Then

$$\gamma = \begin{pmatrix} p & -s \\ q & r \end{pmatrix}$$

belongs to Γ_d and $\gamma(\infty)=p/q$ since $\gamma(z)=(pz-s)/(qz+r)$. The image of a horizontal plane at height one will be a sphere resting on p/q of some diameter $d(p, q)$.

PROPOSITION 5. $d(p, q)=1/|q|^2$.

Proof. If the element γ^{-1} is the composition of an inversion about a sphere with center p/q and radius R followed by a Euclidean reflection, then R is the radius of the circle $|1/(qz+r)|^2=1$, i.e. $R=1/|q|$. Thus γ^{-1} takes a sphere resting at p/q of diameter $1/|q|$ to a horizontal plane at height $1/|q|$. It follows a plane at height 1 is carried by γ to a sphere of diameter $1/|q|^2$ resting at p/q.

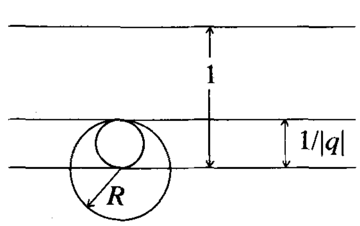


Figure 12.

COROLLARY. *The Γ_d orbit of the horosphere at ∞ consists of disjoint spheres resting on these p/q where ideal $(p, q)=\vartheta$ and having diameters a constant times $1/|q|^2$.*

Since $\mathbf{H}^3/\Gamma_d$ has finite volume the proposition of § 6 implies the number of spheres a certain size s is at least constant $\cdot(1/s)^2$. So we are in a position to generalize Khintchine metric approximation theory to imaginary quadratic fields. Let $0<a(x)\leq 1$ be a function so that the size of $a(x)$ up to bounded ratio only depends on the size of x up to bounded ratio.

THEOREM 5. *Fix an imaginary quadratic field $\mathbf{Q}(\sqrt{-d})$ with ring of integers ϑ. For almost all complex numbers z there are infinitely many pairs $p, q \in \vartheta\times\vartheta$ satisfying*

$$|z-p/q| \leq \frac{a(|q|)}{|q^2|}, \quad \text{ideal } (p, q) = \vartheta$$

iff

$$\int_1^\infty \frac{a(x)^2}{x} dx = \infty.$$

Remark. For $a(|q|)\equiv 1$, this was proved by Swan for all but a certain countable set of z.

Proof. We follow the proof of Khintchine's theorem in § 3. We have calculated the positions and sizes of the disjoint spheres in the proposition above. They are disjoint by the discussion of § 5 and there are enough of them by the discussion of § 6.

We construct disks around the bases of p/q of size $a(s)s$ where $s \in 1/|q|^2$ and we apply the Borel-Cantelli of § 4 to prove the result.

§ 8. Disjoint spheres and geodesic excursions

Consider the Figure 13 (a), in which a geodesic of $\mathbf{H}^{d+1}/\Gamma=V$ viewed in $\mathbf{H}^{d+1}$ heads toward a definite point at infinity entering and leaving a sequence of disjoint horospheres which are those of a cuspidal orbit.

In the quotient V these horospheres project to the cuspidal end and the geodesic of Figure 13 (b) enters the end at time t, reaches a maximum penetration at time t' and leaves the cusp at time t''. (See Figure 14.)

The distance penetrated is comparable to the log of the ratio of diameters d/d'. Also up to an additive constant the time t at which a geodesic reaches a point y away from the boundary satisfies $y=e^{-t}$.

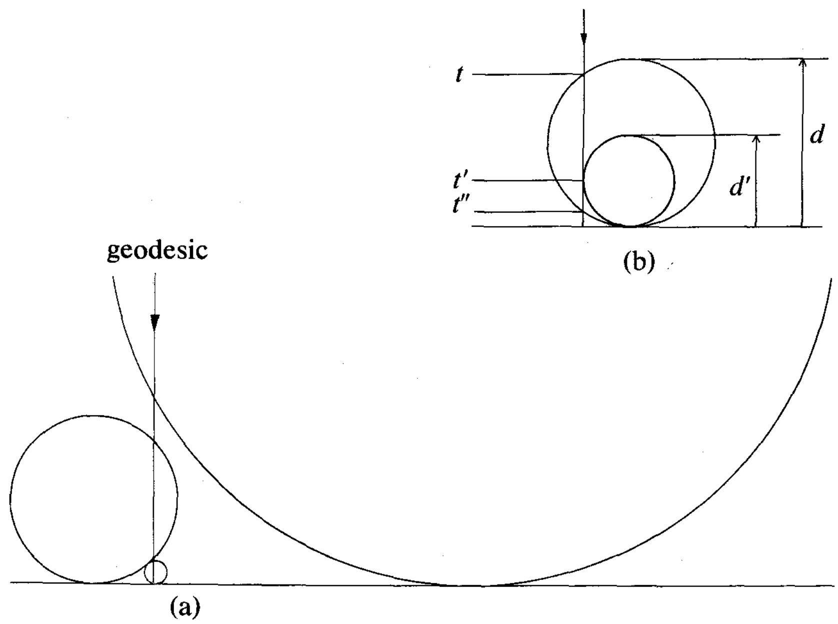


Figure 13.

Thus a geodesic has a sequence of maximal penetrations of distance $d_1, d_2, \ldots$ at times $t_1, t_2, \ldots$ iff the endpoint ξ of the geodesic has a certain sequence of approximations by base points b_i of a sequence of horospheres of radii r_i. Namely up to fixed constants

$$|\xi - b_i| < r_i a_i$$

where $a_i = e^{-d_i}$ and $r_i = e^{-(t_i - d_i)}$.

Thus the excursion pattern of a random geodesic into a cuspidal end is equivalent to the approximation of the random point on the boundary of $\mathbf{H}^{d+1}$ *by the bases of horospheres in that cuspidal orbit.*

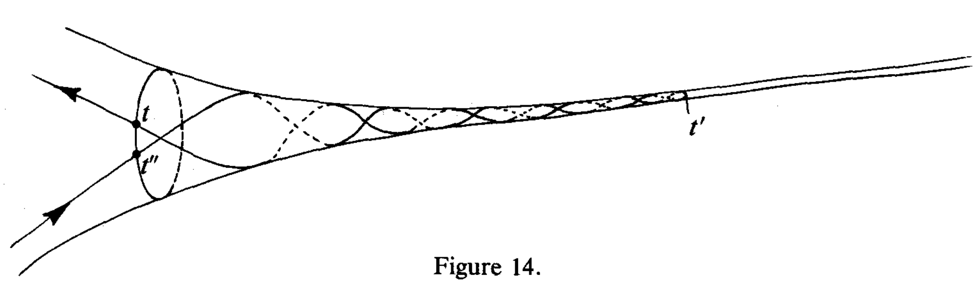


Figure 14.

§ 9. The logarithm law for geodesics

Let Γ be a discrete group of hyperbolic isometries of $\mathbf{H}^{d+1}$ so that $V=\mathbf{H}^{d+1}/\Gamma$ has finite volume. Let $\operatorname{dist} v(t)$ denote the maximum of 1 and the distance from a fixed point in V to the point achieved after traveling time t along the geodesic starting in direction v.

THEOREM 6. *For almost all starting directions v,*

$$\limsup_{t\to\infty} \frac{\operatorname{dist} v(t)}{\log t} = 1/d.$$

Proof. The volume of the part of V where $\operatorname{dist} \geq T$ is $\asymp e^{-dT}$. Thus volume $\{v\colon \operatorname{dist} v(t_i) \geq T_i\} \asymp e^{-dT_i}$ because the geodesic flow is volume preserving (in the unit tangent bundle whose volume fibres over the volume of V with the volume of each fibre constant).

Thus for any ($\varepsilon>0$) if we restrict to integral times $t_1, t_2 \ldots (t_n=n)$ for almost all v the inequalities

$$\operatorname{dist} v(t_n) \geq \left(\frac{1}{d}+\varepsilon\right) \log t_n$$

are true for only finitely many t_n (because $\Sigma_n \exp-(d(1/d+\varepsilon)\log n)<\infty$, Proposition 1). So $\limsup_t \operatorname{dist} v(t)/\log t \leq 1/d$.

The non trivial direction uses the approximation theory by the bases of disjoint horospheres developed above in §§ 2, 4, 5, 6 and 8.

We want to show for almost all v the inequality

$$\operatorname{dist} v(t) \geq \frac{1}{d} \log t$$

is satisfied for a sequence $t_1, t_2, \ldots$ tending to ∞ (depending on v). By § 8 such a sequence corresponds to approximations of the endpoint $\xi=\xi(v)$ by bases b_i of the horospheres in a cuspidal orbit (there are only finitely cusps in the quotient) of radii r_i,

$$|\xi-b_i| \leq a_i r_i$$

where

$$a_i = e^{-d_i} = \exp-\left(\frac{1}{d}\log t_i\right) = t_i^{-1/d}$$

$$r_i = e^{-t_i-d_i} = \exp-\left(t_i+\frac{1}{d}\log t_i\right) = t_i^{-1/d} e^{-t_i}.$$

Now take $a(x)=(2\log x)^{-1/d}$ in the discussion of §§ 2 and 4 for $x\in[e,\infty)$. Since $\int_e^\infty (a(x)^d/x)\,dx=\infty$ we have by §§ 4 and 6 for almost all ξ infinitely many approximations by bases b of horospheres of size s of the form

$$|\xi-b|\leqslant a(1/s)\,s=(2\log 1/s)^{-1/d}s.$$

For almost any ξ and this sequence $r_i\in s_i$ define t_i by $r_i=t_i^{1/d}e^{-t_i}$. Then

$$(2\log 1/s_i)^{-1/d}=(2\log t_i^{-1/d}e^{t_i})^{-1/d}=(2\log t_i^{-1/d}+2t_i)^{-1/d}.$$

which is eventually $\leqslant t_i^{-1/d}$.

Thus we have found arbitrarily large solutions to the inequality

$$\operatorname{dist} v(t)\geqslant\frac{1}{d}\log t,$$

and the theorem is proved.

Remark. (1) Actually the proof shows we can find for many $\varphi(t)$ arbitrarily large solutions of dist $v(t)\geqslant\varphi(t)$ iff a certain integral diverges. (We leave the formulation to the reader.)

(2) Also the quantitative part of § 2 shows the number of integral times $<N$ that the inequality is satisfied is infinitely often as large as the diverging integral. (Again we leave the formulation to the reader.)

§ 10. Disjoint spheres and the spatial distribution of the canonical geometric measure

Let Γ be a discrete group of hyperbolic isometries (in 3-space say) which has a fundamental domain with finitely many sides. On the limit set of Γ (the set of cluster points in $\partial\mathbf{H}^3=\partial B^3=S^2$ of any orbit of Γ in $\mathbf{H}^3$) there is a canonical geometric measure μ characterized by

$$\gamma^*\mu=|\gamma'|^D\mu,\quad \gamma\in\Gamma.$$

Here D is the Hausdorff dimension of Λ and $|\gamma'|$ is the linear distortion of γ in the Euclidean metric ϱ on the ball model of $\mathbf{H}^3$. (Theorem 1, [S$_2$]).

In this section we study the density function of μ, $\mu(\xi,r)$=the μ mass of an r-disk on the sphere centered at ξ (in the ϱ-metric).

In §§ 4 and 6 of [S_2] the estimate

$$\mu(\xi, r) \asymp r^D \cdot \exp((k(v(t)) - D) \cdot \operatorname{dist} v(t)) \tag{1}$$

was derived, where v points toward $\xi \in \Lambda, r = e^{-t}, v(t)$ and $\operatorname{dist} v(t)$ are as in § 8, and $k(v(t))$ is the rank of the cuspidal end where $v(t)$ is—assuming $\operatorname{dist} v(t)$ is larger than a convenient constant.

(For such geometrical finite groups we work in the convex hull of Λ which after dividing by Γ is compact with cuspidal ends, see [T, §§ 5 and 8] and [S_2, § 2].)

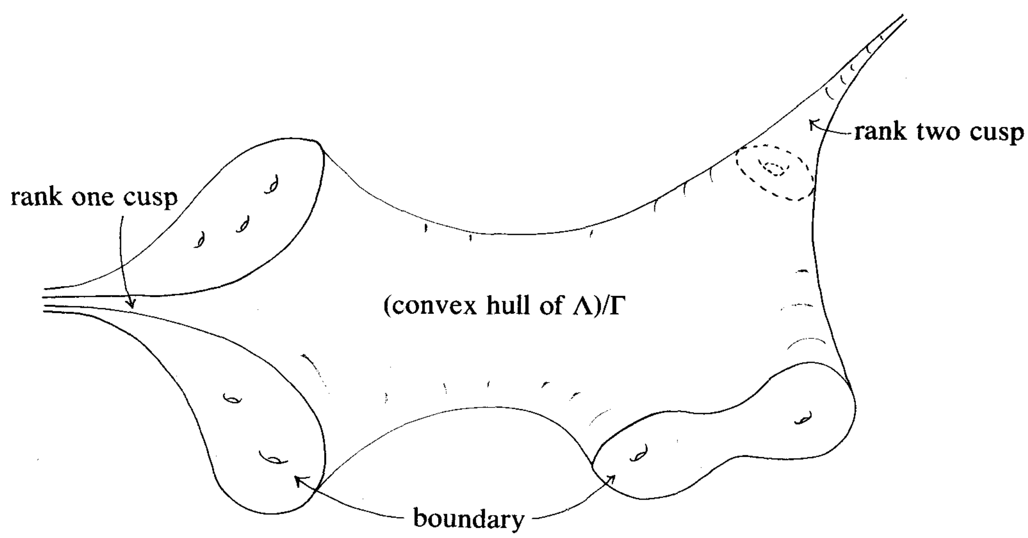


Figure 15.

The measure μ determines a finite invariant measure dm_μ for the geodesic flow which is ergodic [S_2, § 5] and even mixing, see Dan Rudolph [R] and/or note below.

Thus by (1) and the ergodicity of the geodesic flow we can expect the ratio $\mu(\xi, r)/r^D$ for μ almost all ξ to be *arbitrarily large* as $r \to 0$ if the maximal rank k_+ of a cusp is greater than D and to be *arbitrarily small* as $r \to 0$ if the minimal rank k_- of a cusp is less than D. (For Γ operating in $\mathbf{H}^3$, k can be 1 or 2.)

Now the mixing property of the geodesic flow implies, just as in § 6, that the number of disjoint horospheres ((§ 5) in a cuspidal orbit) of size s is comparable to $(1/s)^D$. (The calculation makes use of Proposition 3, § 2 of [S_1].)

Also (1) implies the μ mass of a disk of size $a(1/s)s$ centered at the base of a horosphere of size s in a rank k cuspidal orbit is comparable to $a(1/s)^\sigma s^D$ (using the dictionary of § 8 where $\sigma = 2D - k$).

Then if we form sets A_s, as in §§ 2 and 4, of such disks, the Borel-Cantelli lemma will hold *relative to the measure* μ. Namely for nice functions $a(x), \mu(A_\infty)>0$ iff $\int_1^\infty (a(x)^\sigma/x)\,dx=\infty$, where $2D-\sigma$ is the rank of the cusp. (Note the inequality $a(1/s_1)^2 s_2 s_1 \geq 1$ of Figure 4 is still valid. Thus $a(1/s_1)^{2D}\cdot s_2^D \cdot s_1^D \geq 1$ which implies $a(1/s_1)^\sigma \cdot s_2^D \cdot s_1^D \geq 1$. Note $\sigma>0$ since $D>k/2$, [S$_2$, § 2]. Thus using the obvious μ-measure estimate §§ 2 and 4 carries through—again the boundary effect is taken care of because expanding a disk by a factor only increases its μ mass by at most a factor.)

So for each cusp and μ-almost all ξ there are approximations by bases b_i of cuspidal horospheres of size s_i of the form $|\xi-b_i|\leq(1/s_i)\,s_i$, for $a(x)$ as in §§ 2 and 4, iff $\int_1^\infty (a(x)^\sigma/x)\,dx=\infty$. (The ergodicity of Γ with respect to μ is used to go from positive μ-measure to full μ-measure.)

Now assume $r(s)=a(1/s)\cdot s$ is strictly monotone and write $s(r)$ for the inverse function. Then using the dictionary of § 8 and (1) we have shown

THEOREM 7 (Oscillation of the density function around r^D).

(i) *If $k_+>D$, for μ-almost all ξ*

$$\limsup_{r\to 0} \mu(\xi, r)/r^D \alpha(r) > 0$$

where $\alpha(r)=a(1/s(r))^{D-k_+}$ iff

$$\int_1^\infty \frac{a(x)^{\sigma_+}}{x}\,dx = \infty, \quad \sigma_+ = 2D-k_+.$$

(ii) *If $k_-<D$, for μ-almost all ξ*

$$\liminf_{r\to 0} \mu(\xi, r)/r^D \beta(r) < \infty$$

where $\beta(r)=a(1/s(r))^{D-k_-}$ iff

$$\int_1^\infty \frac{a(x)^{\sigma_-}}{x}\,dx = \infty, \quad \sigma_- = 2D-k_-.$$

Example. Take $a(x)=(\log x)^{-1/\sigma}$ ($\sigma=\sigma_+$ or σ_-). Then $r(s)=(\log 1/s)^{-1/\sigma}\cdot s$, $s(r)$ is between r and $r^{1-\varepsilon}$ for every $\varepsilon>0$ eventually, and $a(1/s(r))$ eventually lies between $(\log 1/r)^{-1/\sigma}$ and $(1-\varepsilon)(\log 1/r)^{-1/\sigma}$ for every $\varepsilon>0$. So we have the

COROLLARY. (i) *If $k_+>D$, for μ-almost all ξ*

$$\limsup_{r\to 0} \mu(\xi, r)/r^D (\log 1/r)^{\delta_+} > 0, \quad \delta_+ = (k_+ - D)/(2D - k_+) > 0.$$

(ii) *If $k_-<D$, for μ-almost all ξ*

$$\liminf_{r\to 0} \mu(\xi, r)/r^D (\log 1/r)^{\delta_-} < \infty, \quad \delta_- = (k_- - D)/(2D - k_-) < 0.$$

Note. If we had further assumed that $a(x)$ is monotone decreasing as in the example, then the integral $\int_0^\infty a(\bar{g}^t)\, dt$ over a set of t of positive density still diverges. A slight modification of the Borel-Cantelli discussion, §§ 2 and 4, where only A_s for good sizes are considered (the other $A_s=\emptyset$), gives the same result only assuming the good sizes form a set of positive density on the log (or t) scale.

In the discussion of this section (and § 9) only *weak-mixing* of the geodesic flow implies the good sizes have full density. Now weak mixing is easier to prove (Dan Rudolph). (Not weak-mixing implies there is a uniformly continuous function so that the limit of Birkhoff sums of the time t_0 map of geodesic flow is not constant. On the other hand the limit (lifted to the tangent bundle to hyperbolic space) is constant on equivalence classes generated by expanding and contracting horospheres. A picture shows such a continuous function is constant.)

Closing remark. If $k_+>D$, the function $\varphi(v(t))=\exp \operatorname{dist} v(t)$, by the above example, is infinitely often $\geq t^\alpha$ some α. Using $\mu\{v: \operatorname{dist} v(t)>T\}\leq e^{-\beta T}$ some $\beta>0$ (deducible from [S$_2$]) yields $\varphi(t)$ is for μ-almost all v eventually $\leq t^{\alpha'}$ some α'. It is now an abstract ergodic theory fact that for any function $\bar{\psi}(t)$ satisfying $\bar{\psi}(\varepsilon t)\leq\delta(\varepsilon)\,\bar{\psi}(t)$ where $\delta(\varepsilon)\to 0$ as $\varepsilon\to 0$,

$$\limsup_{t\to\infty} \frac{\varphi(v(t))}{\bar{\psi}(t)}$$

is either zero of infinity for almost all v. (This is a fairly direct application of the ergodic theorem told to me by Aaronson and due to Tanny in a branching process discussion.)

Now write $\psi(r)=\bar{\psi}(\log 1/r)\, r^D$.

COROLLARY. *For all these $\psi(r)$ the canonical geometric measure μ is not $(k_+>D)$ equivalent to Hausdorff measure relative to $\psi(r)$.*

Proof. Using (1) and above we see for μ-almost all ξ, $\limsup_{r\to 0} \mu(\xi, r)/\psi(r)$ is either zero or infinity. But if μ is equivalent to the (covering) Hausdorff ψ-measure the lim sup is the Radon ratio, see [S$_2$] for example.

Note. We have shown in $[S_2]$ however that if all cuspidal ranks are $\geqslant D$, the canonical measure can be described as the Hausdorff r^D measure defined by *packings* rather than *coverings*.

An example is provided by the set suggested by the accompanying figure. The infinite array of circles are inverted into the triangular interstice, these are translated, the inversion is repeated, etc. to construct a limit set of a group Γ with the above Hausdorff geometry. (This Γ is a subgroup of the Bianchi group, Γ_d, where $d=3$, § 7.)

Namely, for all of the above reasonable gauge functions the canonical geometrical measure on this set is not the Hausdorff (covering) measure. However, the canonical geometrical measure can be described as the Hausdorff μ^D-packing measure of $[S_2]$.

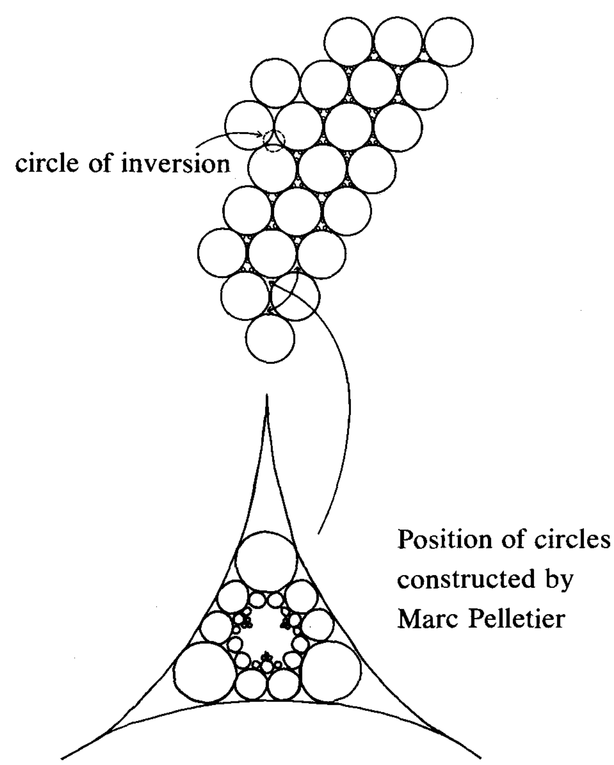


Figure 16.

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