Calibrated geometries

by

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I. Introduction

This paper is perhaps best characterized as a foundational essay on the geometries of minimal varieties associated to closed forms. The fundamental observation here is the following: Let X be a riemannian manifold, and suppose φ is a closed exterior p-form with the property that

$$\varphi|_\xi \leq \mathrm{vol}_\xi \tag{1}$$

for all oriented tangent p-planes ξ on X. Then any compact oriented p-dimensional submanifold M of X with the property that

$$\varphi|_M = \mathrm{vol}_M \tag{2}$$

is homologically volume minimizing in X, i.e., $\mathrm{vol}(M) \leq \mathrm{vol}(M')$ for any M' such that $\partial M = \partial M'$ and $[M-M']=0$ in $H_p(X;\mathbf{R})$. To see this, one simply notes that $\mathrm{vol}(M)=\int_M \varphi = \int_{M'} \varphi \leq \mathrm{vol}(M')$. (The first equality follows from (2), and the final inequality follows from (1). The middle equality is a consequence of the homology condition and the fact that $d\varphi=0$.)

Condition (2) enables us to associate to an exterior p-form φ a family of oriented p-dimensional submanifolds in X which we call *φ-submanifolds*. If φ is closed and is normalized to satisfy condition (1), then the argument above proves that each φ-submanifold is homologically mass minimizing in X.

A closed exterior p-form φ satisfying (1) will be called a *calibration* and the riemannian manifold X together with this form will be called a *calibrated manifold*.

As an example, let X be a complex hermitian n-manifold with Kähler form ω, and consider $\varphi=(1/p!)\,\omega^p$ for some p, $1\leq p\leq n$. Then the φ-submanifolds are just the canonically oriented complex submanifolds of dimension p in X. If $d\varphi=0$, i.e., if X is a Kähler manifold, then the complex submanifolds are homologically mass minimizing. This is the classical observation of H. Federer [F_1].

We take a moment here for some reflections on the foundations of geometry. The concept of a geometric structure on a manifold X can be formulated in three rather different ways. One can consider a structure to be given by an atlas of coordinate charts whose coordinate transformations lie in a particular pseudogroup of local diffeomorphisms of $\mathbf{R}^n$, cf. [KN]. One could also consider a geometric structure to be defined by a distinguished family of tensor fields on X. However, it is also possible, and more in the spirit of the classical geometries, to consider a structure to be defined by a distinguished family of subvarieties of X. Each of these approaches can be usefully adopted to study, say, complex or foliated geometry.

One of the main points of this paper is to exhibit and study some beautiful geometries of minimal subvarieties which are really not visible from this first viewpoint.

We shall concentrate primarily on geometries in $\mathbf{R}^n$ associated to forms with constant coefficients. A significant part of the work will be to derive a tractable system of partial differential equations whose solutions represent subvarieties in the given geometry. These systems are in a specific sense generalizations of the Cauchy-Riemann equations.

The first geometry to be studied in depth is associated to the form

$$\varphi = \operatorname{Re}\{dz_1 \wedge \ldots \wedge dz_n\}$$

in $\mathbf{C}^n$, where as usual we shall write $z=x+iy$. It consists of Lagrangian submanifolds of "constant phase", and is therefore called *special Lagrangian geometry*. In fact the only Lagrangian submanifolds which are stationary are special Lagrangian.

We recall that the Lagrangian n-planes in $\mathbf{C}^n$ are exactly the U_n-images of the x-axis, and up to unitary coordinate changes, every Lagrangian submanifold is locally the graph $\{y=f(x)\}$, where $f=\nabla F$ for some scalar potential function $F(x)$. (The function F is arbitrary.) Similarly, the special Lagrangian n-planes in $\mathbf{C}^n$ are exactly the SU_n-images of the x-axis. The special Lagrangian Grassmannian is just the fibre of the "Maslov" map: $\text{Lag}\to S^1$, from the oriented Lagrangian n-planes to the circle, given by the complex determinant. Up to SU_n-coordinate changes, special Lagrangian submanifolds are locally graphs of the form $\{y=(\nabla F)(x)\}$ where $F(x)$ is a scalar potential function satisfying a non-linear elliptic equation. When $n=3$, this equation has the following beautiful form:

$$\Delta F = \det(\operatorname{Hess} F). \tag{3}$$

We conclude that the graph of the gradient of any solution to (3) is an absolutely volume-minimizing 3-fold in $\mathbf{R}^6$. In particular, any C^2 solution of (3) is real analytic.

The equation (3) bears an intimate relation to the work of Hans Lewy on harmonic gradient maps [Ly] and explains the mysterious appearance there of the minimal surface equation. This is discussed in Chapter III.

The geometry of special Lagrangian submanifolds is richly endowed (see Sections III.3 and 4), and constitutes a large new class of minimizing currents in $\mathbf{R}^n$. In particular, we are able to explicitly construct simple minimizing cones which are not real analytic (see Section III.3.A) as well as minimizing cones which are quite complicated topologically (see Section III.3.C).

Chapter IV is devoted to the study of three exceptional geometries. There is a geometry of 3-folds (and a dual geometry of 4-folds) in $\mathbf{R}^7$, which is invariant under the standard representation of G_2. This geometry is associated to the 3-form $\varphi(x,y,z)=\langle x, yz\rangle$ where $x,y,z\in\mathbf{R}^7$ are considered as imaginary Cayley numbers. A 3-manifold $M\subset\mathbf{R}^7=\operatorname{Im}\mathbf{O}$ belongs to this geometry if each of its tangent planes is a (canonically oriented) imaginary part of a quaternion subalgebra of the Cayley numbers $\mathbf{O}$.

The local system of differential equations for this geometry is essentially deduced from the vanishing of the *associator* $[x,y,z]\equiv(xy)z-x(yz)$, and thus the geometry is called *associative*. This system of equations has a striking and elegant form. Write $\operatorname{Im}\mathbf{O}=\operatorname{Im}\mathbf{H}\oplus\mathbf{H}$, where $\mathbf{H}$ denotes the quaternions, and consider a function $f\colon U\subset \operatorname{Im}\mathbf{H}\to\mathbf{H}$. Then the graph of f is an associative submanifold if and only if f satisfies the equation

$$Df=\sigma f \tag{4}$$

where D is the Dirac operator and σ is a certain first-order "Monge-Ampère" operator. (See Section IV.2.) The associative geometry contains a 6-dimensional family of special Lagrangian geometries. Furthermore, the Cartan-Kähler theorem give many solutions to the system (4); (see Sections IV.3 and 4.), so this geometry is also highly non-trivial. (1)

(1) *Added in proof.* It was recently observed by Robert Bryant that the cones in this geometry have links which are exactly the holomorphic curves in S^6 (with the standard, G_2-invariant almost complex structure). He went on to prove that every compact Riemann surface can be realized as such a holomorphic curve in S^6. See [Br].

The dual geometry of 4-folds in $\mathbf{R}^7$ is called *coassociative* and has a local system of differential equations similar to (4). It includes the cone on the Hopf map, presented in [LO] as a Lipschitz solution of the minimal surface system which is not of class C^1. Note that, since it is coassociative, this cone must be absolutely minimizing in $\mathbf{R}^7$.

The most facinating and complex geometry discussed here is the geometry of Cayley 4-folds in $\mathbf{R}^8 \cong \mathbf{O}$. This is the family of subvarieties associated to the 4-form $\psi(x, y, z, w) \equiv \frac{1}{2}\langle x(\bar{y}z) - z(\bar{y}x), w\rangle$. It is invariant under the 8-dimensional representation of Spin_7 and contains the coassociative geometry. It also contains both the (negatively oriented) complex and the special Lagrangian geometries for a 7-dimensional family of complex structures on $\mathbf{R}^8$. In fact for any of these structures, the form ψ can be expressed as

$$\psi = -\tfrac{1}{2}\omega^2 + \mathrm{Re}\{dz\}$$

where ω is the Kähler form and $dz = dz_1 \wedge \ldots \wedge dz_4$ as above.

The associated system of equations for a function $f: U \subset \mathbf{H} \to \mathbf{H}$ (guaranteeing that the graph of f in $\mathbf{O} = \mathbf{H} \oplus \mathbf{H}$ is a Cayley fourfold) is again of the form

$$Df = \sigma f. \tag{5}$$

Here D is the Dirac operator (or more suggestively, the quaternion analogue of the operator $\partial/\partial z$ for functions $f: U \subset \mathbf{C} \to \mathbf{C}$.) The operator σ is a homogeneous cubic expression in the first derivatives of f, constructed using the 3-fold cross product in $\mathbf{H}$. It is, in fact, linear in the 3×3 minors of the Jacobian matrix of f.

The Grassmannian of Cayley 4-planes in $\mathbf{R}^8$ is of codimension four in the full Grassmannian. One actually deduces seven equations which include the four appearing in (5). However, outside a small and explicitly described subvariety of the Grassmannian, the seven equations are implied by these four. (Full details appear in Chapter IV.)

A plethora of solutions of (5) is again constructed using Implicit function theorem techniques and also by using the Cartan-Kähler theorem.

Chapter V contains a number of comments concerning generalizations of the main ideas and results of the paper. These comments include the observation that every Cayley 4-fold naturally carries a 21-dimensional family of anti-self-dual SU_2 Yang-Mills fields.

The table of contents and the introductory paragraphs of each section give an indication of the organization and content of the paper. It should be stated that large portions, if not all of Chapter II can be passed over by the uninterested reader. This chapter contains a number of generalities which are appropriate for this paper but not necessary for the study of the explicit geometries subsequently presented. In particular, the reader unfamiliar with currents and geometric measure theory will find these topics unnecessary in Chapters III, IV and V.

II. Grassmann geometries

II.1. Geometries determined by distinguished families of submanifolds

We will adopt the point of view that a geometric structure on a manifold is determined by a distinguished family of submanifolds. This is in contrast to the customary point of view in differential geometry that a geometric structure on a manifold is determined by a distinguished atlas whose coordinate transformations lie in a particular pseudogroup of local difeomorphisms of $\mathbf{R}^n$. One of the main points of this paper will be to exhibit several natural and interesting geometries of the first type which cannot be realized from the second point of view.

In the geometries which we shall investigate the prefered submanifolds are determined by their first order infinitesimal behavior. More precisely, the geometries can all be defined as follows.

Suppose X is an oriented manifold of class C^1 and dimension m which is countable at infinity. Let $G(p, T_xX)$ denote the collection of oriented p-planes in T_xX, the tangent space to X at x. The corresponding bundle will be denoted $G(p, TX)$. Assume that an arbitrary subset $\mathcal{G}$ of $G(p, TX)$ is given. (Note that $\mathcal{G}_x \equiv \mathcal{G} \cap G(p, T_xX)$ may be empty.)

Definition 1.1. Let S be a p-dimensional, oriented C^1 submanifold with possible boundary in X. If, at each point of S the oriented tangent space of S belongs to $\mathcal{G}$, then S is called a *$\mathcal{G}$-submanifold*. The family of $\mathcal{G}$-submanifolds of X constitutes the *$\mathcal{G}$-geometry* of X. Such $\mathcal{G}$-geometries shall fall under the collective name of *Grassmann geometries*.

It is useful to extend the $\mathcal{G}$-geometry of a space to include manifolds with singularities. The nicest such extensions are provided by Geometric measure theory. We remark to those unfamiliar with this subject that it is not essential to our paper. The remainder of this section may be skipped or passed through lightly on a first reading.

Our first generalization of a $\mathcal{G}$-geometry employs the class $\mathcal{R}_p^{\mathrm{loc}}(X)$ of locally rectifiable p-currents on X (cf. $[F_1]$). This class contains p-dimensional submanifolds with singularities. It is also closed under addition and subtraction, and allows certain infinite sums. In order to recall some of the basic properties of these currents, we shall fix a riemannian metric on X. There is then a natural embedding of the grassmannian $G(p, T_xX)$ into the vector space $\Lambda^p T_xX$ of p-vectors given by

$$G(p, T_xX) = \{\xi \in \Lambda^p T_xX : \xi \text{ is a unit simple } p\text{-vector}\}. \tag{1.2}$$

For each locally rectifiable current $T \in \mathscr{R}_p^{\text{loc}}(X)$ there is an associated "volume" measure denoted $\|T\|$, and for $\|T\|$-a.e. x there is an oriented "tangent" p-plane, denoted $\mathbf{T}(x) \in G(p, T_xX)$. The current T is reconstructed from $\|T\|$ and $\mathbf{T}$ by setting $T \equiv \mathbf{T}\|T\|$; that is, $T(\psi) \equiv \int \langle \mathbf{T}, \psi \rangle d\|T\|$ for each test form ψ. The locally rectifiable currents are generalized integral *chains*. They form, in fact, a natural closure of the integral Lipschitz chains on X.

Definition 1.3. Suppose $T \in \mathscr{R}_p^{\text{loc}}(X)$.

(a) If $\pm\mathbf{T}(x) \in \mathscr{G}$ for $\|T\|$-a.e. x then T is called a *$\mathscr{G}$-chain*. If, in addition, $dT=0$ then T is called a *$\mathscr{G}$-cycle*.

(b) If $\mathbf{T}(x) \in \mathscr{G}$ for $\|T\|$-a.e. x then T is called a *positive $\mathscr{G}$-chain*. If, in addition, $dT=0$ then T is called a *positive $\mathscr{G}$-cycle*.

Remark. Unless the *axiom of positivity*,

$$\mathscr{G} \cap (-\mathscr{G}) = \emptyset$$

is satisfied the word "positive" in Definition 1.3 (b) is inappropriate and the concept of a positive $\mathscr{G}$-chain will be dropped from discussion.

In our context it is also useful to consider the general class $\mathscr{M}_p^{\text{loc}}(X)$ of p-dimensional currents of locally finite mass on X. This is simply the space dual to the continuous p-forms with compact support (with the usual topology). One has that $\mathscr{R}_p^{\text{loc}}(X) \subset \mathscr{M}_p^{\text{loc}}(X)$. Furthermore, after fixing a riemannian metric on X, each $T \in \mathscr{M}_p^{\text{loc}}(X)$ gives rise to a Radon measure $\|T\|$ on X and a generalized "tangent" space $\mathbf{T}(x) \in \Lambda^p T_xX$, defined for $\|T\|$-a.e. $x \in X$. The current T is again expressed as $T = \mathbf{T}(x)\|T\|$. (This is an immediate consequence of the Riesz representation theorem.) Note that in this general case $\mathbf{T}(x)$ need not be a simple vector. However, $\mathbf{T}(x)$ is always of unit length in the *mass norm* on $\Lambda^p T_xX$. This is the norm whose unit ball is the convex hull of the unit simple p-vectors, $G(p, T_xX)$. The function $\mathbf{T}(x)$ is $\|T\|$ measurable and bounded. Consequently, $\mathbf{T}(x)$ has pointwise meaning for x in the Lebesque set, Leb(T), of $\mathbf{T}$ with respect to $\|T\|$. In fact,

$$\langle \mathbf{T}(x), \psi(x) \rangle = \lim_{\varepsilon \to 0^+} \frac{1}{\|T\|(B(x,\varepsilon))} \int_{B(x,\varepsilon)} \langle \mathbf{T}, \psi \rangle \, d\|T\|$$

for each $x \in \text{Leb}(T)$.

Remark. The family of subsets of X of $\|T\|$ measure zero is independent of the choice of riemannian metric and; except for a set of $\|T\|$ measure zero, the tangent p-vector $\mathbf{T}(x)$ is determined (up to positive multiples) independent of the metric.

We now let ch $\mathcal{G}_x$ denote the convex hull of $\mathcal{G}_x$ in $\Lambda^p T_x X$, and set ch $\mathcal{G} = \bigcup_x \text{ch}\,\mathcal{G}_x$.

Definition 1.4. A current $T \in \mathcal{M}_p^{\text{loc}}(X)$ is called a *positive $\mathcal{G}$-current* if $\mathbf{T}_x \in \text{ch}\,\mathcal{G}_x$ for $\|T\|$−a.e. $x \in X$.

Remark. The terminology "positive $\mathcal{G}$-cycle" has already been employed in Definition 1.3(b), where the current T is required to be locally rectifiable. Therefore, we must refer to a positive $\mathcal{G}$-current T with $dT=0$ simply as a *closed positive $\mathcal{G}$-current.*

A discussion of positive $\mathcal{G}$-currents is given in Appendix A of this paper. It is shown that in the cases of interest here, a substantially weaker notion of positive $\mathcal{G}$-current automatically implies the property of locally finite mass. This generalizes known results for positive (1,1)-currents in complex analysis.

Note that the concepts of $\mathcal{G}$-chain, positive $\mathcal{G}$-chain and positive $\mathcal{G}$-current are independent of the choice of riemannian metric on X. (See the remark above.)

Remark. In general, a current T which is both $\mathcal{G}$-positive and locally rectifiable need not be a positive $\mathcal{G}$-chain. However, in the cases of interest, the *completeness axiom* I:

$$\text{ch}\,\mathcal{G} \cap G(p, TX) = \mathcal{G}$$

is satisfied; so that a locally rectifiable positive $\mathcal{G}$-current is a positive $\mathcal{G}$-chain.

Remark. Occassionally, it is conveneient to have still other concepts extending that of a $\mathcal{G}$-submanifold. For example, suppose X is a real-analytic manifold. Then a $\mathcal{G}$-chain T whose support is a p-dimensional, real analytic subvariety of X will be called an *analytic $\mathcal{G}$-chain.* Furthermore, if T is closed, it will be called an *analytic $\mathcal{G}$-cycle.* Most concepts of this type are fairly obvious and will not be mentioned here.

The set of positive $\mathcal{G}$-chains has the following important compactness property. *Suppose $\mathcal{G}$ is closed. Then for any $k>0$ and any compact set $K \subseteq X$, the set of positive $\mathcal{G}$-chains T with* supp$(T) \subseteq K$ *and* $\|T\|(K) \leq k$, *is compact in the weak topology.* This follows easily from fundamental results in [F_1].

II.2. Local structure theorems

In this section we briefly examine the concepts introduced in Section 1 for three classical $\mathcal{G}$-geometries.

Example I: *Complex geometries.* Suppose X is a complex manifold, and let $\mathscr{G}\subset G(2p, TX)$ be the subset of canonically oriented complex p-planes. Obviously a $\mathscr{G}$-submanifold is just a complex submanifold (with possible boundary) in X. By the Structure theorem of J. King (cf. [K] or [H_2]), a positive $\mathscr{G}$-chain T is, in fact, a "positive holomorphic chain". That is, away from the support of its boundary, T is a locally finite sum $T=\Sigma n_j\,[V_j]$ where each n_j is a positive integer and each V_j a pure p-dimensional complex subvariety of X.

By the more general Structure theorem of Harvey-Schiffman (cf. [HS] and [H_2]) a general $\mathscr{G}$-chain T is, correspondingly, a general "holomorphic chain", i.e., away from the support of its boundary, $T=\Sigma n_j\,[V_j]$ as above, where now each n_j is an arbitrary integer. Actually, it was necessary in [HS] to make the "support hypothesis" that the Hausdorff $(2p+1)$-measure of supp T vanish. It still remains conjectural that this hypothesis is unnecessary.

Finally, the local structure of closed positive $\mathscr{G}$-currents on X is intimately related to the theory of plurisubharmonic functions (cf [Lg_1], [H_1], [HP_2]) since a current T of bidegree $(1, 1)$ is a closed positive $\mathscr{G}$-current if and only if T is locally of the form $T=i\partial\bar{\partial}\varphi$ with φ plurisubharmonic. Moreover, let $T\in\mathscr{M}_p^{\mathrm{loc}}(X)$ be such a current, and for each $c>0$ consider the subset $E_c(T)$ consisting of all the points of $\|T\|$-density $\geqslant c$. It is a striking fact that in any region where T has no boundary, $E_c(T)$ is a complex subvariety of dimension $\leqslant p$. Steming from results of Bombieri [Bo], this fact was conjectured by Harvey-King [HK, p. 52] and proved by Siu [Su]. A second shorter proof was given by Lelong [Lg_2].

Example II: *Foliation geometries.* Suppose F is an oriented, p-dimensional foliation of X (cf. [L_3]), and let $\mathscr{G}$ be the set of tangent planes to the leaves. Note that a connected $\mathscr{G}$-submanifold without boundary is just a closed leaf of the foliation.

The local structure theorems for currents in this geometry all follow more or less trivially from standard facts. Let us fix a riemannian metric on X and denote by $\mathbf{F}$ the field of unit simple p-vectors tangent to F. Then any positive $\mathscr{G}$-current T can be expressed as $T=\|T\|\mathbf{F}$. The condition that the current T be closed i.e., $dT=0$ can be understood as follows. Choose local coordinates $(x_1, \ldots, x_p, y_1, \ldots, y_{n-p})$ on X so that the leaves are defined by the equations: $y_j=\text{constant}, j=1, \ldots, n-p$, then a positive $\mathscr{G}$-current T is of the form $T=\mu(\partial/\partial x_1 \wedge \ldots \wedge \partial/\partial x_p)$ where $\mu\geqslant 0$ is a measure. The condition $dT=0$ means precisely that $\partial\mu/\partial x_j=0$ for $j=1, \ldots, p$. Interior regularity for this system of equations is particularly simple; μ must be independent of the variables $x_1, \ldots, x_p$. Thus μ can be written as $\mu=1\otimes\nu$ where 1 denotes Lebesque measure on $\mathbf{R}^p$ and ν is a Radon measure on $\mathbf{R}^{n-p}$. Note: $\mu \equiv f\|T\|$ and $\partial/\partial x_1 \wedge \ldots \wedge \partial/\partial x_p = \mathbf{T}/f$ for some smooth nonvanishing function f.

Suppose now that T is a $\mathcal{G}$-chain without boundary (i.e. *a* $\mathcal{G}$-cycle). In order for $T=(1\otimes\nu)(\partial/\partial x_1 \wedge \ldots \wedge \partial/\partial x_p)$ to be a p-dimensional rectifiable current $\mathbf{R}^n$, ν must be 0-dimensional rectifiable current in $\mathbf{R}^{n-p}$, i.e., a locally finite sum of δ-functions (see [HL_5] for more details). Consequently,

THEOREM 2.1. *Away from the support of its boundary, every $\mathcal{G}$-chain T can be expressed as a locally finite sum $T=\sum n_j [L_j]$ where each n_j is an integer and each L_j is a oriented closed leaf in X-*(supp *dT*)*, with locally finite volume in X. Moreover, T is positive if and only if each n_j is positive.*

Continuing the analogy with the complex case, we let T be a positive $\mathcal{G}$-current without boundary and consider the set $E_c(T)$ of all points of $\|T\|$-density $\geqslant c$. *For each $c>0$, $E_c(T)$ is a locally finite union of closed leaves.* This follows from the fact that the points of density $\geqslant c$ for the 0-dimensional current ν must be isolated.

Example III: *Lagrangian geometries*. Suppose X is a symplectic manifold, that is, a $2n$-manifold equipped with a closed 2-form ω such that ω^n never vanishes. A tangent n-plane P on X is said to be *Lagrangian* if $\omega|_P=0$. We let $\mathcal{G}\subset G(n, TX)$ be the subset of oriented Lagrangian planes. Then a $\mathcal{G}$-submanifold is what is commonly called a "Lagrangian submanifold".

Since in this case $\mathcal{G}_x=-\mathcal{G}_x$, a positive $\mathcal{G}$-chain is no more special than a $\mathcal{G}$-chain. In this geometry the $\mathcal{G}$-chains will be called *Lagrangian chains*. The local structure of *Lagrangian cycles* (Lagrangian chains without boundary) is not understood.

In Chapter 3 we shall study "special Lagrangian geometries" where the Axiom of positivity is satisfied. Here one might expect a rich local structure theory as in the cases above.

II.3. Grassmann geometries determined by a differential form of comass one

The $\mathcal{G}$-geometries of particular interest in this paper are all defined by means of a riemannian metric and a p-form.

The general construction is given as follows. Let X be a riemannian manifold, and let $\varphi\in\Gamma(\Lambda^p T^*X)$ be an exterior p-form on X. Then at each $x\in X$, we define the *comass* of φ_x to be

$$\|\varphi\|_x^* \equiv \sup\{\langle\varphi_x, \xi_x\rangle : \xi_x \text{ is a unit simple } p\text{-vector at } x\}, \tag{3.1}$$

that is, $\|\varphi\|_x^*$ is the supremum of φ restricted to $G(p, T_x X) \subset \Lambda^p T_x X$. Furthermore, if A is any subset of X, we define the *comass of φ on A* to be

$$\|\varphi\|_A^* = \sup_{x \in A} \|\varphi\|_x^*.$$

Note that the comass norm on $\Lambda^p T_x^* X$ is dual to the mass norm on $\Lambda^p T_x X$, whose unit ball is defined to be the convex hull of $G(p, T_x X)$.

Definition 3.2. Suppose φ is a smooth p-form of comass 1 on X. We define $\mathscr{G}(\varphi)$ to be the union of the sets

$$\mathscr{G}_x(\varphi) \equiv \{\xi_x \in G(p, T_x X) : \langle \varphi_x, \xi_x \rangle = 1\}. \tag{3.3}$$

That is, $\mathscr{G}(\varphi)$ is the collection of planes where φ assumes its maximum. The $\mathscr{G}(\varphi)$-geometry will be simply called a *φ-geometry*. Its associated objects will be called *φ-submanifolds, φ-chains, φ-cycles, positive φ-chains, positive φ-cycles, positive φ-currents,* and finally *closed positive φ-currents*.

Note that φ need not have comass one at each point or, equivalently, $\mathscr{G}_x(\varphi)$ may be empty for some x.

Let T be an arbitrary de Rham p-current with compact support on X, i.e., an arbitrary element of the dual space to the smooth p-forms. The *mass* of T is defined to be

$$M(T) = \sup \{T(\psi) : \|\psi\|_X^* \leq 1\}. \tag{3.4}$$

If $M(T) < \infty$, then $T \in \mathscr{M}_p^{\mathrm{cpt}}(X)$ and $M(T) = \|T\|(X)$ (see Section II.2). If T corresponds to integration over an oriented p-dimensional submanifold S of X, then $M(T) = \mathrm{vol}(S)$.

LEMMA 3.5. *Let φ be a smooth p-form of comass 1 on X, and let T be an arbitrary p-current with compact support. Then*

$$T(\varphi) \leq M(T)$$

with equality if and only if T is a positive φ-current.

In particular, if S is a compact oriented p-dimensional submanifold (with possible boundary in X, then

$$\int_S \varphi \leq \mathrm{vol}(S)$$

with equality if and only if S is a φ-submanifold.

Proof. The inequality is trivial. Equality can occur only when $M(T)<\infty$, and we may write $T=\mathbf{T}\|T\|$. Thus $T(\varphi)=\int \varphi(\mathbf{T})\|T\|\leq\int\|T\|=M(T)$ and equality occurs if and only if $\varphi(\mathbf{T})=1$ $\|T\|$−a.e., i.e., if and only if T is a positive φ-current.

If T corresponds to an oriented submanifold S, then $\|T\|$ is Hausdorff p-measure restricted to S and $\mathbf{T}$ is the field of oriented unit tangent p-vectors to S. The second statement is easily seen to be a special case of the first.

Note that any φ-geometry automatically satisfies the Axiom of positivity and the Axiom of completeness I mentioned in section II.1.

We now examine the classical examples from this point of view.

Example I: *Complex geometries.* Suppose X is a complex manifold and let $\mathscr{G}\subset G(2p, TX)$ be the canonically oriented complex p-planes as above. We choose a riemannian metric $\langle\cdot,\cdot\rangle$ on X such that the map $J: TX\to TX$ corresponding to complex multiplication by $\sqrt{-1}$ is orthogonal in each tangent space. (Such metrics exist since one can always average over the finite group generated by J.) The associated "*Kähler*" *2-form* ω is defined by setting

$$\omega(V, W)=\langle JV, W\rangle.$$

Since $J^2=-1$, we have $\omega(V, W)=-\omega(W, V)$, so ω is a smooth exterior 2-form on X. We now set

$$\Omega_p=\frac{1}{p!}\omega^p. \tag{3.4}$$

Then Wirtinger's inequality (cf. [F_1]) says that Ω_p has comass one at each point x and that $\mathscr{G}_x(\Omega_p)$ is just the grassmannian of (positive) complex p-planes. Consequently, this $\mathscr{G}$-geometry is realized as a $\mathscr{G}(\varphi)$-geometry.

Example II: *Foliation geometries.* Suppose F is an oriented p-dimensional foliation on X. Introduce a riemannian metric on X. The metric induces an isomorphism between Λ^pT_xX and $\Lambda^pT^*_xX$. let φ be the unit decomposable p-form corresponding to the field $\mathbf{F}$ of tangent p-planes to F. Then at each point x, φ is of comass one and $\mathscr{G}_x(\varphi)=\{\mathbf{F}_x\}$. Therefore each foliation geometry can be realized as a φ-geometry.

Remark. There exist many other forms φ of comass one at each point which determine the same foliation geometry. In fact, the form φ need not be decomposable in order that $\mathscr{G}_x(\varphi)=\{\mathbf{F}_x\}$. This observation is crucial in [HL_5].

Example III: *Lagrangian geometries.* Since Lagrangian geometries satisfy

$\mathscr{G}_x = -\mathscr{G}_x$, it is impossible to find φ with $\mathscr{G} = \mathscr{G}(\varphi)$. However, for the special Lagrangian geometries of Chapter III, such forms exist.

II.4. Calibrated geometries

The key fact concerning φ-geometries is that for any φ-manifold S, $\operatorname{vol}(S) = \int_S \varphi$. This fact is important whenever the form φ is closed.

Definition 4.1. A smooth p-form φ on a riemannian manifold X is said to be a *calibration* if φ is of comass one on X and $d\varphi = 0$. A riemannian manifold together with a calibration is called a *calibrated manifold.*

The fundamental observation is the following. Recall that the homology of the complex of deRham currents with compact support on X is naturally isomorphic to $H_*(X; \mathbf{R})$.

THEOREM 4.2. *Suppose X is a calibrated manifold with calibration φ, and suppose T is a positive φ-current with compact support. Let T' be any compactly supported current homologous to T (i.e., $T - T'$ is a boundary and in particular $dT = dT'$). Then*

$$M(T) \leq M(T') \tag{4.3}$$

with equality if and only if T' is a positive φ-current.

Proof. Since $T - T' = dS$ where S has compact support, we have $T(\varphi) - T'(\varphi) = (dS)(\varphi) = S(d\varphi) = 0$. Hence, by Lemma 3.5 we have

$$M(T) = T(\varphi) = T'(\varphi) \leq M(T')$$

with equality if and only if T' is a positive φ-current. Briefly, Theorem 4.2 says that if $d\varphi = 0$, then any positive φ-current T is *homologically mass minimizing* among all (real) currents with the same boundary. Furthermore, any homologous current with the same minimum mass must also be a positive φ-current. Of course this means that any φ-submanifold is a minimal submanifold of X.

There are two cases of particular interest here. Suppose X is a riemannian manifold with calibration φ.

COROLLARY 4.4. *Any compactly supported positive φ-cycle on X is a current of least mass in its deRham homology class. Furthermore, every other rectifiable current of least mass in this homology class must also be a positive φ-cycle.*

Federer and Fleming [FF] showed that the homology of the complex of rectifiable currents whose boundaries are also rectifiable on X is naturally isomorphic to $H_*(X;\mathbf{Z})$. Furthermore, if X is compact one can find a rectifiable cycle of least mass in each homology class. Note, however, that a positive φ-cycle T minimizes not only among rectifiable currents, but among all real currents homologous to T. Thus, for example, a positive φ-cycle can never represent a torsion class in $H_p(H;\mathbf{Z})$.

COROLLARY 4.5. *Let T be a compactly supported positive φ-chain with boundary $B=dT$. If $H_p(X;\mathbf{R})=0$, then T is a solution to the Plateau problem for B, i.e., T is a current of least mass among all compactly supported currents with boundary B in X.*

In particular, if φ is a calibration on $\mathbf{R}^n$ with the usual metric, then any positive φ-chain T with compact support is a solution to the Plateau problem for $B=dT$.

COROLLARY 4.6. *Suppose X is of class C^k, $2\leq k\leq\omega$. Each φ-submanifold S without boundary of class C^2 is actually of class C^k.*

Proof. S is a classical minimal submanifold so that S is of class C^k by the basic regularity results in [M].

We now return briefly to our standard examples.

Example I: *Kähler geometry*. Suppose X is a complex manifold with a metric $\langle\cdot,\cdot\rangle$ and 2-form ω as in Section II.3. If $d\omega=0$, then X is said to be a Kähler manifold. Setting $\Omega_p=(1/p!)\,\omega^p$, we recover the well-known result of Federer [F_1] that complex subvarieties of a Kähler manifold are homologically mass minimizing.

Example II: *Foliation geometries*. Let X be a manifold with an oriented, p-dimensional foliation F. Suppose there exists a riemannian metric and a calibration φ on X such that $\mathcal{G}(\varphi)=\{\mathbf{F}\}$. Then every compactly supported positive φ-current (or "foliation current" in the sense of Ruelle-Sullivan [RS]) is homologically mass minimizing. In particular, every d-closed foliation current (compactly supported positive φ-current without boundary) is nontrivial in $H_p(X;\mathbf{R})$. It will be shown in [HL_5] that this last condition is, in fact, sufficient for the existence of such a metric and calibration.

Any fibre bundle with discrete structure group (cf [L_3]) gives a foliation of this type (the foliation transverse to the fibres).

In certain applications it is important to have the following generalized notion of a calibration provided by the theory of flat and coflat currents (cf. Federer [F_2], [F_3]). Let X be a riemannian manifold.

Definition 4.7. Suppose φ is a calibration of degree p on $X-S_\varphi$, where S_φ is a closed subset of X of Hausdorff p-measure zero. Since $\|\varphi(x)\|^*\leq 1$ for all $x\in X-S_\varphi$ the coefficients of φ uniquely determine functions on X that are locally bounded and measurable, i.e. they are in $L^\infty_{\text{loc}}(X)$. The (generalized) p-form φ on X will be called a *coflat calibration on X*.

A key property of a calibration, that it be d-closed, is also valid for coflat calibrations.

LEMMA 4.8. *Suppose φ is a coflat calibration on X, then $d\varphi=0$ on* X.

By definition a current ψ is coflat if both ψ and $d\psi$ have coefficients in L^∞_{loc}. Consequently, a corollary of Lemma 4.8 is that a coflat calibration is coflat; justifying the terminology in Definition 4.7.

Proof. This is just a special case of the following very general "Removable singularity" theorem. (See Theorem 4.1 (b) in Harvey-Polking [HP_1].) Suppose $P(x, D)$ is any linear differential operator of order n on an open set $U\subseteq\mathbf{R}^m$, and suppose S is a closed subset of U with Hausdorff $(m-n)$-measure zero. Then each function $f\in L^\infty_{\text{loc}}(U)$ satisfying $P(x, D)f=0$ on $U-S$ also satisfies $P(x, D)f=0$ on U.

Remark. Note that in order to conclude $d\varphi=0$ on X we only used the fact that S_φ has Hausdorff $(m-1)$-measure zero, where $m=\dim(X)$.

A coflat calibration φ on X can be used to define a $\mathcal{G}(\varphi)$-geometry on X by defining $\mathcal{G}_x(\varphi)$ to be empty for $x\in S_\varphi$. However, in order to maintain the validity of Theorem 4.2 (positive φ-currents are "minimal"), we must restrict attention to positive φ-currents which are also locally flat. We can now state a fundamental result of Federer [F_2] in the following modified form.

THEOREM 4.9. *Let φ be a coflat calibration on a riemannian manifold X, and suppose T is a flat positive φ-current on X. Then*

$$M(T)\leq M(T')$$

for all flat currents T′ which are homologous to T. Furthermore, $M(T)=M(T')$ for such a T′ if and only if T′ is also φ-positive.

In this theorem "homologous" can be taken to mean homologous in the complex of flat currents or in the complex of all compactly supported deRham currents, since the two notions are equivalent [F_1].

A number of interesting coflat calibrations of codimension one in $\mathbf{R}^m$ have essentially been constructed in [L$_2$] by means of certain compact Lie group representations. Each of the resulting geometries contains a mass-minimizing cone of codimension one. Among these is the Simons cone on $S^3 \times S^3 \subset \mathbf{R}^8$, which was originally proved to be mass-minimizing by Bombieri, di Georgi and Giusti.

To pass from the material contained in [L$_2$] to the construction of a coflat calibration is not completely straight forward, so we shall give an outline of the construction here. Let G be a compact connected Lie group acting orthogonally on $\mathbf{R}^m$ and let $\mathcal{R} \subset \mathbf{R}^m$ be the union of the principal orbits of G. Recall (cf. [HsL]) that $\mathcal{R}$ is an open subset, $\pi: \mathcal{R} \to \mathcal{R}/G$ is a smooth fibre bundle whose complement has locally finite $m-2$ Hausdorff measure. We shall construct coflat calibrations of dimension $m-1$ on $\mathbf{R}^m$ which are smooth in $\mathcal{R}$.

To do this we introduce some notation. Let p be the dimension of the principal orbits. Then we let $v: \mathcal{R} \to \mathbf{R}$ denote the volume function of the orbits, and we let Ω_0 denote the unit p-form along the orbits. We also let H denote the mean curvature vector field to the orbits and set H^* equal to the dual 1-form, i.e., $H^*(V) = \langle V, H \rangle$. Suppose $e_1, \ldots, e_n$ are local orthonormal vector fields in $\mathcal{R}$ such that $e_1, \ldots, e_p$ are tangent to the orbit foliation. Let $e_1^*, \ldots, e_n^*$ be the dual 1-forms. Then we have,

$$v(x) = \mathrm{vol}(G \cdot x) = \mathcal{H}^p(G \cdot x)$$

$$\Omega_0 = e_1^* \wedge \ldots \wedge e_p^*$$

$$H^* = \sum_{j=1}^{p} \sum_{\alpha=p+1}^{n} \langle \nabla_{e_j} e_j, e_\alpha \rangle \, e_\alpha^* .$$

Straightforward calculations establish the following:

$$\frac{dv}{v} = -H^*, \tag{4.10}$$

$$d\Omega_0 \equiv -H^* \wedge \Omega_0 \quad (\mathrm{mod}\, I \cdot I) \tag{4.11}$$

where I is the ideal of differential forms which vanish (and thereby define) the orbit foliation. Combining these equations, we see that

$$d\left(\frac{1}{v}\Omega_0\right) \equiv 0 \quad (\mathrm{mod}\, I \cdot I) \tag{4.12}$$

This gives the following result.

THEOREM 4.13. *Let v and Ω_0 be as above. Suppose ω_0 is any closed form of degree $m-p-1$ in $\mathcal{R}$ such that $\|\omega_0\|^* \leq v$. Then the smooth form*

$$\Omega = \frac{1}{\nu}\Omega_0 \wedge \omega_0$$

defines a coflat calibration in $\mathbf{R}^m$.

Note that the form ω_0 can be simply constructed in the orbit space $\mathcal{R}/G$ and then pulled back by π. In [L_2], all the cases where $\dim(\mathcal{R}/G)=2$, were considered. In each case a closed 1-form ω_0 was constructed on $\mathcal{R}/G$ with the property that $\|\pi^*\omega_0\|^* \leq \nu$.

II.5. The importance of the euclidean case—tangent cones

An obvious and appealing class of calibrated geometries is provided by the parallel forms of comass one in Euclidean space. Such geometries are the focus of the deeper parts of this paper.

Notice that, after normalization, each non-zero element $\varphi \in \Lambda^p \mathbf{R}^n$ determines a parallel form of comass 1, and therefore a geometry of (absolutely) minimal varieties in $\mathbf{R}^n$. These varieties are in turn always characterized as solutions to certain systems of partial differential equations, such as the Cauchy-Riemann equations in the case of complex geometries.

Notice also that there is a natural splitting

$$\mathcal{G}(\varphi) = \mathbf{R}^n \times G(\varphi)$$

where

$$G(\varphi) = \{\xi \in G(p, \mathbf{R}^n) : \varphi(\xi) = \|\varphi\|^*\}$$

is the *φ-Grassmannian* (and where, as above, $G(p, \mathbf{R}^n)$ is the set of unit simple p-vectors in $\mathbf{R}^n$). It is easy to see that for a generic φ, $G(\varphi)=\{\xi\}$ for some $\xi \in G(p, \mathbf{R}^n)$, and the associated φ-geometry is a flat foliation geometry. However, for special φ the set $G(\varphi)$ can be quite large. It is the understanding of these cases which constitutes the main part of this paper.

There is of course intrinsic interest in understanding the class of minimal varieties so simply attached to a basic linear algebraic object. However, there is a further reason for studying these particular cases. Namely, a "tangent cone" to a positive φ-current, at a point x_0, where φ is an arbitrary calibration on a riemannian manifold, is a "tangent cone" in the euclidean tangent space at the point x_0 which is a positive φ_{x_0} current. We will discuss this in more detail after the following study of tangent cones in $\mathbf{R}^n$.

Let V denote the vector field $x \cdot \partial/\partial x$ on $\mathbf{R}^n$ and let $\varphi_t(x) \equiv e^t x$ denote the corresponding flow. A current T is a *cone* if $(\varphi_t)_*(T)=T$ for all $t>0$, or equivalently, the Lie derivative $L_V(T)=0$. Note that if $\varphi\in\Lambda^p\mathbf{R}^n$ then

$$p\varphi = L_V(\varphi) = d(V \lrcorner \varphi). \tag{5.1}$$

Since

$$\frac{x}{|x|}\cdot\frac{\partial}{\partial x}\lrcorner\left(\frac{x}{|x|}\cdot dx\wedge\varphi\right)=\varphi-\frac{x}{|x|}\cdot dx\wedge\left(\frac{x}{|x|}\cdot\frac{\partial}{\partial x}\lrcorner\varphi\right),$$

if we define

$$\varphi_n\equiv\frac{x}{|x|}\cdot\frac{\partial}{\partial x}\lrcorner\varphi \tag{5.2}$$

to be the *normal part* of φ, and

$$\varphi_t\equiv\frac{x}{|x|}\cdot\frac{\partial}{\partial x}\lrcorner\left(\frac{x}{|x|}\cdot dx\wedge\varphi\right) \tag{5.3}$$

to be the *tangential* part of φ, then

$$\varphi=\varphi_t+\frac{x}{|x|}\cdot dx\wedge\varphi_n. \tag{5.4}$$

Restricting the equations $d\varphi=0$ and (5.1) to the sphere $S^{n-1}\equiv\{x\in\mathbf{R}^n: |x|=1\}$ yields

$$d\varphi_t=0 \quad\text{and}\quad d\varphi_n=p\varphi_t \tag{5.5}$$

Both of these forms φ_t and φ_n have comass no larger than the comass of φ. For instance $\varphi_n(\xi)=\varphi(x\wedge\xi)$ and $\|\xi\|=\|x\wedge\xi\|$ for any simply $p-1$ vector $\xi\in\Lambda^{p-1}x^\perp$. Consequently, the links of φ-cones are just the φ_n submanifolds of the sphere.

THEOREM 5.6. *Suppose $\varphi\in\Lambda^p\mathbf{R}^n$ is a parallel calibration on $\mathbf{R}^n$. A $(p-1)$-dimensional submanifold M of the sphere S^{n-1} is a φ_n-submanifold if and only if the p-dimensional cone $CM\equiv\{tx\in\mathbf{R}^n: x\in M$ and $t>0\}$ is a φ-submanifold of $\mathbf{R}^n$.*

Let $\pi(x)\equiv|x|$.

THEOREM 5.6. *Suppose $\varphi\in\Lambda^p\mathbf{R}^n$ is a parallel calibration on $\mathbf{R}^n$. A $(p-1)$-positive φ-current on $\mathbf{R}^n$ with $dT=0$. Then*

$$\|T\|(B(0,r))=p^{-1}r\int\frac{\left(\mathbf{T},\frac{x}{|x|}\cdot dx\wedge\varphi_n\right)}{\left\|\mathbf{T}\lrcorner\frac{x}{|x|}\cdot dx\right\|}\|\langle T,\pi,r\rangle\|. \tag{5.8}$$

for a.e. r; *and for* $0<s<r$,

$$\frac{1}{r^p}\|T\|(B(0,r))-\frac{1}{s^p}\|T\|(B(0,s)) = \int_{B(0,r)-B(0,s)} |x|^{-p}(\mathbf{T},\varphi_t)\,\|T\|. \tag{5.9}$$

Proof. Let χ_r denote the characteristic function of the ball radius r. Since $\varphi = p^{-1}d(V\lrcorner\varphi)$,

$$\|T\|(B(0,r)) = (\chi_r T,\varphi) = (\chi_r T, p^{-1}d(V\lrcorner\varphi)) = ([\partial B(0,r)]\wedge T, p^{-1}V\lrcorner\varphi). \tag{5.10}$$

The formula (5.8) follows easily and the proof is omitted. Using (5.10), we see that

$$\begin{aligned}\frac{1}{r^p}\|T\|(B(0,r))-\frac{1}{s^p}\|T\|(B(0,s)) &= p^{-1}(([\partial B(0,r)]-[\partial B(0,s)])\wedge T, |x|^{-p}V\lrcorner\varphi)\\ &= p^{-1}([B(0,r)-B(0,s)]\wedge T, d(|x|^{-p}V\lrcorner\varphi)).\end{aligned}$$

However,

$$\begin{aligned} d(|x|^{-p}V\lrcorner\varphi) &= |x|^{-p}d(V\lrcorner\varphi)-p|x|^{-p-1}d|x|\wedge(V\lrcorner\varphi)\\ &= p|x|^{-p}\left(\varphi-\frac{x}{|x|}\cdot dx\wedge\varphi_n\right) = p|x|^{-p}\varphi_t,\end{aligned}$$

completing the proof.

It is useful to have alternate expressions for $(\mathbf{T}, (x/|x|)\cdot dx\wedge\varphi_n)$ and $(\mathbf{T},\varphi_t)$ exhibiting them as non-negative quantities.

LEMMA 5.11. *Suppose* $\mathbf{T}=\sum_{j=1}^N \lambda_j\xi_j$, *where* $0\leqslant\lambda_j\leqslant 1$, $\Sigma\lambda_j=1$, *and where each* ξ_j *is a unit simple p-vector in* $G(\varphi)$. *(In Section 7 this is seen to be the case if* $\mathbf{T}$ *is any p-vector of unit length in the mass norm with* $(\mathbf{T},\varphi)=1$.*) Then*

$$\left(\mathbf{T},\frac{x}{|x|}\cdot dx\wedge\varphi_n\right) = \sum_{j=1}^N \lambda_j\left|\xi_j\lrcorner\frac{x}{|x|}\cdot dx\right|^2 \tag{5.12}$$

$$(\mathbf{T},\varphi_t) = \sum_{j=1}^N \lambda_j\left|\xi_j\wedge\frac{x}{|x|}\cdot\frac{\partial}{\partial x}\right|^2 \tag{5.13}$$

Proof. We may assume $\mathbf{T}$ is a unit simple p-vector. Choose unit vectors $e_1\in\operatorname{span}\mathbf{T}$ and $e\perp\operatorname{span}\mathbf{T}$ such that $x/|x|=\cos\theta e_1+\sin\theta e$, and complete to an orthonormal set $e_1,\dots,e_p,e$ with $\mathbf{T}\equiv e_1\wedge\dots\wedge e_p$. Then, for example, $(\mathbf{T},\varphi_t)=((e_1\wedge\dots\wedge e_p\wedge(\cos\theta e_1+\sin\theta e))\lrcorner(\cos\theta e_1^*+\sin\theta e^*),\varphi)=\sin^2\theta\pm\sin\theta\cos\theta(e_2\wedge\dots\wedge e_p\wedge e,\varphi)$.

However, (see Section 7), since $\varphi(e_1 \wedge \ldots \wedge e_p)=1$ it follows that $\varphi(e_2 \wedge \ldots \wedge e_p \wedge e)=0$. Thus, $(\mathbf{T}, \varphi_t)=\sin^2\theta$ where

$$\mathbf{T} \wedge \frac{x}{|x|} \cdot \frac{\partial}{\partial x} = \sin\theta e_1 \wedge \ldots \wedge e_p \wedge e,$$

completing the proof of (5.13). The proof of (5.12) is similar.

The above formulae contain the basic information about the local behavior of d-closed positive φ-currents T. If T is a minimal submanifold these formulae are classical (see [F$_1$], 5.4.3). If φ is $(1/p!)\omega^p$ (the Kähler case) these formulae can be found, for example, in [H$_2$] Section 1.9. The important facts about "tangent cones" to minimal submanifolds can be extended to d-closed positive φ-currents using these formulae.

If T is a d-closed positive φ-current on $\mathbf{R}^n$ with $\varphi \in \Lambda^p \mathbf{R}^n$ (so the above formulae apply) then

$$\left\|\left(\frac{1}{r}\right)_*(T)\right\|(B(0,1)) = \frac{1}{r^p}\|T\|(B(0,r))$$

is a monotone increasing function of r because of (5.9) and (5.13).

Consequently, $\{\chi_1(1/r)_*(T)\}_{r<\delta}$ is a weakly compact set in the mass topology.

The cluster points $C \equiv \lim(1/r_j)_*(T)$ of this set with $r_j \to 0$ are called *tangent cones to T at the origin*. Obviously each such C is a d-closed positive φ-current.

The terminology tangent "cone" is justified by the next result (cf. [F$_1$]).

PROPOSITION 5.14. *Suppose $\varphi \in \Lambda^p \mathbf{R}^n$ is a parallel calibration on $\mathbf{R}^n$ and that T is a positive φ-current with $dT=0$. Each of the many cluster points $C \equiv \lim_{j\to 0}(1/r_j)_*(T)$* with $r_j \to 0$, *is a cone, with $\Theta^m(\|C\|, 0)=\Theta^m(\|T\|, 0)$. Moreover, each such C is a d-closed positive φ-current.*

Proof.

$$\begin{aligned}\frac{1}{r^p}\|C\|(B(0,r)) &= \lim_{j\to\infty}\frac{1}{r^p}\left\|\left(\frac{1}{r_j}\right)_* T\right\|(B(0,r)) \\ &= \lim_{j\to\infty}\frac{1}{(rr_j)^p}\|T\|(B(0,rr_j)) \\ &= c_p^{-1}\Theta(\|T\|,0)\end{aligned}$$

because $r^{-p}\|T\|(B(0,r))$ is monotone increasing in r. In particular, the left-hand side of (5.9) with T replaced by C vanishes. Thus $(C, \varphi_t)=0$ for $\|C\|$–a.e. points x. Thus, by (5.13),

$$\xi_j \wedge \frac{x}{|x|} \cdot \frac{\partial}{\partial x} = 0, \quad j = 1, \ldots, N$$

for $\|C\|-$ a.e. points x. Thus

$$C \wedge \frac{x}{|x|} \cdot \frac{\partial}{\partial x} = 0$$

and hence $C \wedge V$ vanishes. However, $L_V(C) = \partial(C \wedge V) + \partial C \wedge V = \partial(C \wedge V)$, so that $L_V(C) = 0$ which proves that C is a cone.

Remark. It is not known whether or not the tangent cones C in Proposition 5.14 are unique. That is, does $\lim_{r \to 0} (1/r)_*(T)$ exist? In fact this is not known in the important special cases where

(1) T is a φ-submanifold (or T is a minimal submanifold)

(2) T is a positive current of bidimension p, p in $\mathbf{C}^n$, $\varphi = (1/p)\omega^p$ with ω the standard Kähler form (see $[H_2]$, Conjecture 1.32).

Now consider an arbitrary calibration on a Riemannian manifold X, and let T be a positive φ-current with $dT = 0$. Fix $x \in \operatorname{supp} T$ and let $e: T_x X \to X$ denote the exponential mapping. The tangent cones C to $(e^{-1})_*(T)$ at the origin are called *tangent cones to T at $x \in X$.*

THEOREM 5.15. *Let X be a calibrated manifold with calibration φ, and let T be a positive φ-current with $dT = 0$. Then each tangent cone C to T at x is a positive φ_x-current in $T_x X$ with $dC = 0$. Moreover, C is a cone with density at the origin the same as the density of T at x.*

The proof of Theorem 5.15 is straightforward and we shall omit the details here. Since $T_x X$ with φ_x is just euclidian space with a parallel calibration, this Theorem 5.15 provides the further justification for studying the euclidian case that was mentioned at the beginning of this section.

II.6. Differential systems and boundaries of φ-submanifolds

Classically, $\mathcal{G}$-geometries arise in the following way. Given a collection $\Psi = \{\psi_j\}_{j=1}^N$ of differential p-forms, define

$$\mathcal{I}_x(\Psi) \equiv \{\xi \in G(p, T_x X): \psi_j(\xi) = 0, \quad j = 1, \ldots, N\} \tag{6.1}$$

at each point x. The elements of $\mathcal{I}(\Psi)$ are called integral elements and an $\mathcal{I}(\Psi)$-submanifold is called an integral submanifold of the system Ψ.

An important question is: when does a φ-geometry $\mathcal{G}(\varphi)$ arise from a system $\psi_1=\ldots=\psi_N=0$ as in (6.1)? Of course, if $\psi(\xi)=0$, then $\psi(-\xi)=0$, so a more precise question is this.

Question 6.2. Given a φ-geometry $\mathcal{G}(\varphi)$ in dimension p, when can one find a collection of p-forms $\Psi=\{\psi_1, \ldots, \psi_N\}$ so that

$$\mathcal{G}(\varphi) \cup (-\mathcal{G}(\varphi)) = \mathcal{I}(\Psi)?$$

Let $\Lambda_x^p(\varphi)$ denote the linear span of $\mathcal{G}_x(\varphi)$ in $\Lambda^p T_x X$. At a fixed point x, we can choose $\psi_1, \ldots, \psi_N$ to be a basis for the annihilator $[\Lambda_x^p(\varphi)]^\circ$. Under reasonable assumptions this can be done in a neighborhood, and our question then becomes

Question 6.3. Given a φ-geometry $\mathcal{G}(\varphi)$, when do we have that

$$\mathcal{G}(\varphi) \cup (-\mathcal{G}(\varphi)) = \Lambda^p(\varphi) \cap G(p, TX)?$$

(Note this is a possibly stronger condition than the axiom of completeness.)

This is related to another important question. In any φ-geometry the form φ has comass 1, that is,

$$\varphi(\xi)^2 \leq \|\xi\|^2 \tag{6.4}$$

for all simple p-vectors ξ. Can this inequality be strengthend to an equality? More precisely:

Question 6.5. Given a φ-geometry $\mathcal{G}(\varphi)$ in dimension p, can one find a collection of p-forms $\Psi=\{\psi_1, \ldots, \psi_N\}$ (locally) such that

$$[\varphi(\xi)]^2 + \sum_{j=1}^{N} [\psi_j(\xi)]^2 = \|\xi\|^2 \tag{6.6}$$

for all simple p-vectors ξ?

If so, then the equations $\psi_j(\xi)=0, j=1, \ldots, N,$ are precisely the conditions for equality in the inequality (6.4). When $\|\xi\|=1$, equality in (6.4) is equivalent to having $\pm\xi \in \mathcal{G}(\varphi)$. Consequently, an affirmative answer to Question 6.5 yields an affirmative answer to Questions 6.2 and 6.3.

We shall show that the answer to Question 6.5 is affirmative for each geometry considered in this paper.

Remark. Each simple p-vector ξ at x can be written as $\xi=v_1\wedge\ldots\wedge v_p$ for

$v_1, \dots, v_p \in T_xX \cong \mathbf{R}^m$. Let $F(v_1, \dots, v_p) \equiv \|v_1 \wedge \dots \wedge v_p\|^2 - \varphi(v_1, \dots, v_p)^2$. Note that F is a homogeneous polynomial which is non-negative. Question 6.3 asks for alternating multilinear forms $\psi_j(v_1, \dots, v_p)$ such that

$$F = \Sigma\, \psi_j^2$$

on $\mathbf{R}^m \times \dots \times \mathbf{R}^m$. Thus we see that at x, Question 6.3 is a modified version of Hilbert's seventeenth problem. No counterexample to this modified problem is known.

Suppose now that φ is a p-form of comass one and that the comass inequality (6.4) has been strengthened to the equality (6.6). Consider the differential ideal I in the exterior algebra generated by $\psi_1, \dots, \psi_N$. It is natural to ask whether I satisfies the hypotheses of the Cartan-Kähler theorem. (See Spivak [Sp], for example.) Unfortunately, this is rarely the case for the examples of interest here. Instead we must consider the *completion* $\tilde{I}$ of I defined by

$$\tilde{I} = \{\psi \in \Gamma(\Lambda^* T^* X) : i_\xi^* \psi = 0 \quad \text{for all } \xi \in \mathcal{G}(\varphi)\} \tag{6.7}$$

where for $\xi \in \mathcal{G}_x(\varphi)$, i_ξ denotes the inclusion map i_ξ: span$(\xi) \to T_xX$. Using $\tilde{I}$ we find that, for the φ-geometries studied in this paper, the Cartan-Kähler theorem is applicable. It can be used to prove the local existence of φ-submanifolds.

At the same time this theorem can be used to give a local characterization of the boundaries of φ-submanifolds. Note that any such boundary Γ has the local property that for each $x \in \Gamma$,

$$T_x\Gamma \subset \text{span}(\xi_x) \text{ for some } \xi_x \in \mathcal{G}_x(\varphi). \tag{6.8}$$

A $(p-1)$-dimensional submanifold Γ which satisfies (6.8) at each point is said to be *maximally φ-like*. (When the φ-geometry is the geometry of complex submanifolds of some given dimension, this is equivalent to the notion of *maximally complex*, cf. [HL_1].) In the examples of geometries studied in this paper, the Cartan-Kähler theorem will be used to prove that (locally and in the real analytic case) the boundaries of φ-submanifolds are exactly those $(p-1)$-dimensional submanifolds which are maximally φ-like.

In order that a compact oriented submanifold Γ bound a compact φ-manifold M, it must satisfy a certain *moment condition*, namely: for each $(p-1)$-form ω on X,

$$\int_\Gamma \omega = 0 \quad \text{if } d\omega \in \Gamma([\Lambda^p(\varphi)]^\circ). \tag{6.9}$$

This follows immediately from Stokes' theorem since $d\omega|_M \equiv 0$.

Question 6.10. When are conditions (6.8) and (6.9) sufficient to conclude that Γ bounds a compact φ-submanifold?

We complete this section by discussion of the above questions for the classical examples.

Example I: *Complex geometries*. Suppose X is a complex hermitian manifold with $-\omega$ the imaginary part of the hermitian form. Let $\Omega_p \equiv (1/p!)\,\omega^p$ and consider the Ω_p-geometry of complex p-dimensional submanifolds of X. Fix $x \in X$ and choose a hermitian orthonormal basis for $T_x X$. Then we can replace $T_x X$ by $\mathbf{C}^n$ and adopt standard notation for $\mathbf{C}^n$. In the spirit of Question 6.5 we have the following strengthening of the classical Wirtinger inequality.

THEOREM 6.11. *For each (real) simple $2p$-vector ξ in $\mathbf{C}^n$,*

$$|\Omega_p(\xi)|^2 + \ldots + \sum_{|I|=2k}' |dz^I \wedge \Omega_{p-k}(\xi)|^2 + \ldots = |\xi|^2. \tag{6.18}$$

If $2p \leq n$ the last term on the left-hand side is $\Sigma'_{|I|=2p} |dz^I(\xi)|^2$. If $2p > n$ the last term on the left-hand side is $\Sigma'_{|I|=2(n-p)} |dz^I \wedge \Omega_{2p-n}(\xi)|^2$.

In order to prove Theorem 6.11 we need the following normal form. (See Harvey-Lawson [HL4] for the proof.)

LEMMA 6.13. *Given a unit simple $2p$-vector ξ in $\mathbf{C}^n$ with $2p \leq n$, there exists a unitary basis $e_1, Je_1, \ldots, e_n, Je_n$ for $\mathbf{C}^n$ over $\mathbf{R}$ and angles*

$$0 \leq \theta_1 \leq \theta_2 \leq \ldots \leq \theta_{p-1} \leq \pi/2, \quad \theta_{p-1} \leq \theta_p \leq \pi,$$

such that

$$\begin{aligned}\xi \equiv e_1 &\wedge (Je_1 \cos\theta_1 + e_2 \sin\theta_1) \wedge e_3 \wedge (Je_3 \cos\theta_2 + e_4 \sin\theta_2) \wedge \ldots \wedge e_{2p-1} \\ &\wedge (Je_{2p-1} \cos\theta_p + e_{2p} \sin\theta_p).\end{aligned} \tag{6.14}$$

Remark. If $2p > n$ then the following formula replaces (6.14).

$$\begin{aligned}&\xi \equiv e_1 \wedge (Je_1 \cos\theta_1 + e_2 \sin\theta_1) \wedge e_3 \wedge (Je_3 \cos\theta_2 + e_4 \sin\theta_2) \wedge \ldots \wedge e_{2(n-p)-1} \\ &\wedge (Je_{2(n-p)-1} \cos\theta_{n-p} + e_{2(n-p)} \sin\theta_{n-p}) \wedge e_{2(n-p)+1} \wedge Je_{2(n-p)+1} \wedge \ldots \wedge e_n \wedge Je_n.\end{aligned} \tag{6.14'}$$

Proof of Theorem 6.11. We first assume that $2p \leq n$. Fix ξ and choose a unitary basis for $\mathbf{C}^n$ so that ξ is in normal form. By the unitary invariance of the expression

(6.12) it will suffice to prove the formula in these special coordinates. Expanding out the expression (6.14) we see that a typical term is given by

$$\eta \equiv \cos\theta_1 \ldots \cos\theta_m \sin\theta_{m+1} \ldots \sin\theta_p\, e_1 \wedge Je_1 \wedge \ldots \wedge e_i \wedge Je_i \wedge \ldots \wedge e_{2m-1} \wedge Je_{2m-1} \wedge e_{2m+1} \wedge e_{2m+2} \wedge \ldots \wedge e_{2p}.$$

The general term in (6.12) is

$$\left(dz^I \wedge e_{m_1} \wedge Je_{m_1} \wedge \ldots \wedge e_{m_{p-k}} \wedge Je_{m_{p-k}}\right)(\xi) \tag{6.15}$$

since powers of the Kähler form can be written as a sum of the complex axis planes. Here $|I|=2k$ and $|M|=p-k$. Replacing ξ by the term η, (6.15) will vanish unless $M \equiv \{1,3,\ldots,2m-1\}, m=p-k$, and $I=\{2m+1,2m+2,\ldots,2p\}$. In summary, if ξ is replaced by η there is only one non-zero term in (6.12): $\cos^2\theta_1 \ldots \cos^2\theta_m \sin^2\theta_{m+1} \ldots \sin^2\theta_p$. Consequently, the left hand side of (6.12) is exactly $\cos^2\theta_1 \ldots \cos^2\theta_p + \ldots + \sin^2\theta_1 \ldots \sin^2\theta_p = (\cos^2\theta_1+\sin^2\theta_1) \ldots (\cos^2\theta_p+\sin^2\theta_p) = 1 = \|\xi\|^2$.

If $2p>n$ the proof is similar. Alternatively, the $2p>n$ case can be derived from the $2p\leq n$ case by using the Hodge $*$-operator and the standard formula for $*(\psi \wedge \Omega_m)$, with ψ primitive.

Remark. For each (real) simple $2p+1$ vector ξ in $\mathbf{C}^n$

$$\sum_{j=1}^{n} |dz_j \wedge \Omega_p(\xi)|^2 + \ldots + \sum_{|I|=2k+1}' |dz^I \wedge \Omega_{p-k}(\xi)|^2 + \ldots = |\xi|^2. \tag{6.16}$$

If $2p+1\leq n$ then the last term on the left hand side is $\Sigma'_{|I|=2p+1} |dz^I(\xi)|^2$. If $2p+1>n$ then the last term on the left hand side is $\Sigma'_{|I|=2p+1} |dz^I \wedge \Omega_{2p+1-n}(\xi)|^2$. The proof of (6.16) is omitted.

Let $\psi_1, \ldots, \psi_N$ denote an enumeration of the real and imaginary parts of $dz^I \wedge \Omega_{p-k}$, where $|I|=2k$. It follows directly from (6.12) that for each $\xi \in G(2p, \mathbf{C}^n)$, $\pm\Omega_p(\xi)=1$ if and only if $\psi_1(\xi)=\ldots=\psi_N(\xi)=0$. These conditions are equivalent to the conditions that $\pm\xi$ represent a complex p-dimensional subspace of $\mathbf{C}^n$ (i.e., $\xi \in G_{\mathbf{C}}(p, \mathbf{C}^n)$).

Let I denote the differential ideal generated by $\psi_1, \ldots, \psi_N$. The integral elements of dimension p are the complex p-dimensional subspaces of $\mathbf{C}^n$.

Note that the differential ideal I is not complete, that is, there exist forms $\psi \notin I$ such that $i_\xi^*\psi=0$ for all $\xi \in G_{\mathbf{C}}(p, \mathbf{C})$.

In order to apply the Cartan-Kähler theorem we consider the completed ideal $\tilde{I}$.

One can show that $\tilde{I}$ consists of the real and imaginary parts of all forms of bidegree r, s with either $r>p$ or $s>p$. In other words,

$$\tilde{I} \underset{\mathbf{R}}{\otimes} C = \Sigma \oplus \Lambda^{r,s}.$$

summed over all pairs r, s with either $r>p$ or $s>p$. This expression characterizes $\tilde{I}$ at each point of the complex manifold X. Clearly we have that $d\tilde{I} \subset \tilde{I}$ (since $d=\partial+\bar{\partial}$, etc.).

Associated with each real subspace V of $\mathbf{C}^n$ is the *holomorphic part* $H(V) \equiv V \cap JV$ of V, and the *complex envelope* $E(V) \equiv V+JV$. Note $H(V) \subset V \subset E(V)$ with $H(V)$ the largest complex vector space contained in V and $E(V)$ the smallest complex vector space containing V. It is straightforward to check that the "regular integral elements" of $\tilde{I}$ (see Spivak [Sp] for definitions) are:

(1) arbitrary if $\dim_{\mathbf{R}} V<p$

(2) the subspaces V with $\dim_{\mathbf{C}} E(V)=p$ if $\dim_{\mathbf{R}} V \geq p$.

Note. For a regular integral element V, a larger subspace W is also regular if and only if $E(W)=E(V)$.

Hence the Cartan-Kähler theorem implies the following.

Let N be a real-analytic submanifold of $\mathbf{C}^n$ with $\dim E(T_z N)=p$ *for each* $z \in N$. *Then there exists a unique complex submanifold $\tilde{N}$, of complex dimension p in $\mathbf{C}^n$; containing N. Moreover, the tangent field to $\tilde{N}$ along N is just $E(TN)$.*

In particular, N is maximally complex if and only if N is the boundary of a complex submanifold (the case where $T_z N$ is a real hyperplane in $E(T_z N)$).

Note. There is a brief elementary proof of these statements which avoids the Cartan-Kähler theorem. (This will not be true to the same extent for our later examples.) Consider $0 \in U \subset \mathbf{R}^k$ where U is open and let f: $U \to N \subset \mathbf{C}^n$ be a local real analytic coordinate chart. There exists $r>0$ such that for $|x|<r$, f is represented by a power series $f(x_1, \ldots, x_k)=\Sigma\, a_I x^I$ with complex vector coefficients. Set $F(z_1, \ldots, z_k)=\Sigma\, a_I z^I$ where $z_j=x_j+iy_j$ and $|z|<r$. The hypothesis on N implies that the rank of the complex Jacobian of F is exactly p for $|x|<r$ and z real (i.e. at points of N). Hence this is true for all z near zero. It follows that image(F) is a p dimensional complex submanifold.

In Harvey-Lawson ([HL_1], [HL_2] and [HL_3]) there is a complete discussion of the global question of when N bounds a complex submanifold.

Example II: *Foliation geometries*. Suppose that φ is a decomposable calibration on a Riemannian manifold X (actually we will not use the hypothesis $d\varphi=0$ in the following). Then $\mathcal{G}_x(\varphi)$ consist of one point which we denote ξ^0_x. Locally, we may choose an orthonormal frame $e_1,\ldots,e_n$ of vector fields so that $\varphi\equiv e_1^*\wedge\ldots\wedge e_p^*$. Let $\psi_1,\ldots,\psi_N$ denote the other axis p-forms, i.e., forms of the type $e_{i_1}\wedge\ldots\wedge e_{i_p}$ where $i_1<\ldots<i_p$ and $p<i_p$. Then the comass inequality $\varphi(\xi)\leq\|\xi\|$ is modified into an equality

$$[\varphi(\xi)]^2+\sum[\psi_j(\xi)]^2=\|\xi\|^2 \quad \text{for all } \xi\in G(p,T_xX). \tag{6.17}$$

Let I denote the differential ideal generated by $\psi_1,\ldots,\psi_N$. Then as a consequence of (6.17) we see that I has only one p-dimensional integral element, namely ξ^0_x, at the point x. Exactly as in Example I the Cartan-Kähler cannot be applied since there are no regular integral elements of dimension $p-1$. To remedy this defect consider the completed ideal

$$\tilde{I}\equiv\left\{\psi\in\Gamma(\Lambda^kT^*X): i^*_{\xi^0}\psi=0\right\}$$

of forms vanishing on span ξ^0_x for each x. This ideal $\tilde{I}$ is a Pfaffian system generated by the one forms $\{e^*_{p+1},\ldots,e^*_n\}$. Thus we are lead to the standard Frobenius theorem applied to the Pfaffian system $\tilde{I}$. If $\tilde{I}$ is closed (i.e. $d\tilde{I}\subset\tilde{I}$) then there exists a foliation whose leaves are φ-submanifolds.

II.7. The mass and comass ball

The importance of calibrations on $\mathbf{R}^n$ with constant coefficients was discussed in the last section. In this section we investigate these euclidean calibrations.

First we establish some notation and outline the elementary concepts. Let $\langle\,,\rangle$ denote the standard inner product on $\mathbf{R}^m$ extended to $\Lambda^p\mathbf{R}^n$ and $\Lambda^p(\mathbf{R}^n)^*$, and let $|\;|$ denote the associated norm. As defined above

$$\|\varphi\|^*\equiv\sup\{\varphi(\xi)\colon\ \xi\in G(p,m)\subset\Lambda^p\mathbf{R}^m\}$$

is the *comass norm*. Let K denote the convex hull of $G(p,m)$ in $\Lambda^p\mathbf{R}^m$. The *mass norm* on $\Lambda^p\mathbf{R}^m$, denoted $\|\;\|$, is by definition the norm whose unit ball is K. Obviously, the mass and comass norms are dual to each other (i.e. the unit ball in the comass norm is K^*, the polar of K).

Given $\varphi \in \partial K^*$ (i.e., $\|\varphi\|^*=1$), the set

$$\mathfrak{F}^*(\varphi) \equiv \{\xi \in \partial K: \varphi(\xi) = 1\}$$

is called the *dual facet of* φ. Any subset $F \subset \partial K$ of this form is also called an *exposed facet* of K, and it is the geometry of these exposed facets that is of central importance to this paper. Note incidentally that the hyperplane $\{\xi \in \Lambda^p \mathbf{R}^m: \varphi(\xi)=1\}$ is a *supporting hyperplane* for K since K meets this hyperplane but lies entirely on the side where $\varphi \leq 1$. Thus, exposed facets are simply the convex sets obtained by intersecting K with supporting hyperplanes.

A convex subset $F \subset \partial K$ is said to be an *extreme facet* of K if each line segment in K which has an interior point in F is a subset of F. Any exposed facet is an extreme facet. Note that if E is an extreme facet of F and F is an extreme facet of K, then E is an extreme facet of K. (The corresponding statement for exposed facets is false.) In particular, each of the extreme points of $\mathfrak{F}^*(\varphi)$ is also an extreme point of K, i.e., a simple vector. This proves the following.

LEMMA 7.1. *Given $\varphi \in \Lambda^p(\mathbf{R}^m)^*$ of comass one, the φ-Grassmannian $G(\varphi)$ is precisely the set of extreme points of the exposed facet $\mathfrak{F}^*(\varphi)$. Consequently $\mathfrak{F}^*(\varphi)$ is the convex hull of $G(\varphi)$.*

The *(exposed) facetal hull* of a set $A \subset \partial K$, denoted $\mathfrak{F}(A)$, is the intersection of all the exposed facets of K which contain A. The dual facet of $A \subset \partial K$ is the set $\mathfrak{F}^*(A) = \{\varphi \in \partial K^*: \varphi^*(\xi)=1 \text{ for all } \xi \in A\}$. Obviously, $\mathfrak{F}^*(\mathfrak{F}^*(A)) = \{\xi \in \partial K: \varphi(\xi)=1 \text{ for all } \varphi \in \mathfrak{F}^*(A)\} = \cap\{\mathfrak{F}^*(\varphi): \varphi \in \mathfrak{F}^*(A)\} = \cap\{\mathfrak{F}^*(\varphi): A \subset \mathfrak{F}(\varphi)\} = \mathfrak{F}(A)$. That is,

the double dual facet of a set $A \subset \partial K$ is equal to the facetal hull of A. (7.2)

We now observe the following.

LEMMA 7.3. *The dual facet of any subset $A^* \subset \partial K^*$ is exposed. In particular, the facetal hull of $A \subset \partial K$ is an exposed facet.*

Proof. We may assume A^* is closed an convex without changing $\mathfrak{F}^*(A^*)$. Let φ_0 be an interior point of A^* (in the sense of convex sets). Then

$$\mathfrak{F}^*(A^*) = \mathfrak{F}^*(\varphi_0) \tag{7.4}$$

since the linear function $\varphi \mapsto \varphi(\xi)$ (for some $\xi \in \partial K$) attains its maximum value of one at an interior point φ_0 of A^* if and only if it is constant on A^*. This proves the first statement. The second follows from (7.2).

The exposed facets of K which are the *largest* (i.e., maximal under inclusion) are exposed by extreme points φ of the comass ball K^*. To see this suppose that $\mathfrak{F}^*(\varphi)$ is maximal but that φ is not an extreme point. Then φ is an interior point in some extreme facet $F^*\subset\partial K^*$ of dimension $\geqslant 1$, and $\mathfrak{F}^*(F^*)=\mathfrak{F}^*(\varphi)$ by (7.4). Let ψ be an extreme point of F^* (and therefore also of K^*). Then $\psi\in F^*$ implies $\mathfrak{F}^*(\psi)\supseteq\mathfrak{F}^*(F^*)=\mathfrak{F}^*(\varphi)$, and so $\mathfrak{F}^*(\psi)=\mathfrak{F}^*(\varphi)$ by maximality.

It is of major importance here to determine the extreme points of the comass ball since they lead to the maximal exposed facets of K and therefore to maximal geometries. We begin our general discussion with the following basic lemma.

LEMMA 7.5. (Canonical form of a simple vector with respect to a subspace.) *Suppose $V\subseteq\mathbf{R}^m$ is a linear subspace and $\xi\in G(p,m)$ is a unit simple p-vector. Then there exists a set of orthonormal vectors $f_1,\dots,f_r$ in V, a set of orthonormal vectors $g_1,\dots,g_s$ in $V^\perp$, and angles $0<\theta_j<\pi/2$ for $j=1,\dots,k$ (where $k\leqslant r,s\leqslant p$ and $r+s-k=p$) such that*

$$\xi=(\cos\theta_1 f_1+\sin\theta_1 g_1)\wedge\dots\wedge(\cos\theta_k f_k+\sin\theta_k g_k)\wedge f_{k+1}\wedge\dots\wedge f_r\wedge g_{k+1}\wedge\dots\wedge g_s. \tag{7.6}$$

Proof. Let $\pi:\mathbf{R}^m\to V$ denote orthogonal projection. Consider the symmetric bilinear form on span ξ defined by $B(u,v)\equiv\langle\pi(u),\pi(v)\rangle$. Let $\varepsilon_1,\dots,\varepsilon_p$ and $\lambda_1,\dots,\lambda_p$ denote the eigenvectors and eigenvalues of B respectively. Then $\pm\xi=\varepsilon_1\wedge\dots\wedge\varepsilon_p$. Since $0\leqslant B(u,u)\leqslant|u|^2$, we have $0\leqslant\lambda_j\leqslant 1$ for all j. Rearrange the indices so that: $0<\lambda_j<1$ for $j=1,\dots,k$, $\lambda_{k+1}=\dots=\lambda_r=1$ and $\lambda_{r+1}=\dots=\lambda_p=0$. Choose $0<\theta_j<\pi/2$ so that $\cos^2\theta_j=\lambda_j$ for $j=1,\dots,k$. Now $|\pi(\varepsilon_j)|^2=B(\varepsilon_j,\varepsilon_j)=\lambda_j=\cos^2\theta_j$ for $j=1,\dots,k$. Hence, $\varepsilon_j=\cos\theta_j f_j+\sin\theta_j g_j$ for uniquely determined unit vectors $f_j\in V$ and $g_j\in V^\perp$ and $j=1,\dots,k$. Set $f_{k+1}=\varepsilon_{k+1},\dots,f_r=\varepsilon_r$, $g_{k+1}=\varepsilon_{r+1},\dots,g_s=\varepsilon_p$. Since $B(\varepsilon_i,\varepsilon_j)=0$ for $i\neq j$, $\pi(\varepsilon_i)$ and $\pi(\varepsilon_j)$ are orthogonal for $i\neq j$ which proves that $f_1,\dots,f_r$ is an orthonormal set of vectors in V. Moreover, $\langle\varepsilon_i-\pi(\varepsilon_i),\varepsilon_j-\pi(\varepsilon_j)\rangle=\langle\varepsilon_i,\varepsilon_j\rangle-B(\varepsilon_i,\varepsilon_j)$, which vanishes for $i\neq j$. Therefore $g_1,\dots,g_s$ is an orthonormal set of vectors in $V^\perp$. Replacing f_1 and g_1 by $-f_1$ and $-g_1$ if necessary, the proof is complete.

Sometimes it is convenient to eliminate unnecessary variables. Given a form $\varphi\in\Lambda^p(\mathbf{R}^m)^*$, a subspace $V^*\subseteq(\mathbf{R}^m)^*$ is said to *envelope* φ if $\varphi\in\Lambda^pV^*\subseteq\Lambda^p(\mathbf{R}^m)^*$. There is a unique minimial subspace which envelopes φ, namely:

$$\operatorname{span}\varphi\equiv\{\eta\lrcorner\varphi:\eta\in\Lambda^{p-1}\mathbf{R}^m\}. \tag{7.7}$$

This subspace consists of the essential variables for φ. Note that

$$(\operatorname{span}\varphi)^* = \{v\in\mathbf{R}^m : v \lrcorner \varphi = 0\}^\perp. \tag{7.8}$$

PROPOSITION 7.9. *Given $\varphi\in\Lambda^p(\mathbf{R}^m)^*$ of comass one, the dual facet $\mathfrak{F}^*(\varphi)$ is contained in the subspace $\Lambda^p(\operatorname{span}\varphi)^*$ of $\Lambda^p\mathbf{R}^m$.*

Proof. Since $\mathfrak{F}^*(\varphi)$ is the convex hull of the φ-grassmannian $G(\varphi)$ it suffices to show that each $\eta\in G(\varphi)$ is contained in $\Lambda^p(\operatorname{span}\varphi)^*$. We may assume that η is in the canonical form given by Lemma 7.5 with respect to the subspace $V\equiv(\operatorname{span}\varphi)^*$. Since each $g_j\in V^\perp$, we have $g_j\lrcorner\varphi=0$. Consequently, $|\varphi(\eta)|=\cos\theta_1\ldots\cos\theta_k|\varphi(f_1\wedge\ldots\wedge f_p)|\leq\cos\theta_1\ldots\cos\theta_k<1$ unless $k=0$, and hence $\pm\eta=f_1\wedge\ldots\wedge f_p\in\Lambda^pV$.

Remark. Sometimes Proposition 7.9 will be used in the following form. Given $\varphi\in\Lambda^p(\mathbf{R}^m)^*$ the set $\{\xi\in G(p,m): \varphi(\xi)=\|\varphi\|^*\}$ of unit simple p-vectors that maximize $\varphi(\xi)$ is contained in $\Lambda^p(\operatorname{span}\varphi)^*\subseteq\Lambda^p\mathbf{R}^m$.

PROPOSITION 7.10. *Let $\mathbf{R}^m=V\oplus W$ be an orthogonal decomposition. Suppose $\varphi\in\Lambda^{p-q}V^*$ has comass one and $\psi\in\Lambda^qW^*$ is a unit decomposable q-form. Then $\varphi\wedge\psi\in\Lambda^p(\mathbf{R}^m)^*$ has comass one, and*

$$G(\varphi\wedge\psi)=\{\xi\wedge\eta:\xi\in G(\varphi)\}$$

where $\eta\in\Lambda^qW$ is the unit simple q-vector dual to ψ.

Proof. By Proposition 7.9 we may assume $W^*=\operatorname{span}(\psi)$. Let $\xi\in G(p,m)$ be any unit simple p-vector and put ξ in canonical form with respect to V as in Lemma 7.5. Since $g_1,\ldots,g_s$ are orthonormal in $V^\perp=W$ we must have $s\leq q$. Moreover $(\varphi\wedge\psi)(\xi)=0$ unless $s=q$, in which case $\psi(g_1\wedge\ldots\wedge g_q)=\pm1$ (we may assume $+1$) and

$$\begin{aligned}(\varphi\wedge\psi)(\xi)&=\pm\sin\theta_1\ldots\sin\theta_k\,\varphi(f_{k+1}\wedge\ldots\wedge f_r)\\&\leq\sin\theta_1\ldots\sin\theta_k\leq1.\end{aligned}\tag{7.11}$$

Thus $\|\varphi\wedge\psi\|^*=1$. Moreover, the only possibility for equality in (7.11) is when $k=0$ and $\xi=f_1\wedge\ldots\wedge f_{p-q}\wedge g_1\wedge\ldots\wedge g_q$. In this case we have $(\varphi\wedge\psi)(\xi)=1$ if and only if $f_1\wedge\ldots\wedge f_{p-q}\in G(\varphi)$. This completes the proof.

The following is a third application of Lemma 7.5. Case 2 is a result of Federer.

PROPOSITION 7.12. *Let $\varphi=e_1^*\wedge\ldots\wedge e_p^*+e_{p+1}^*\wedge\ldots\wedge e_{2p}^*\in\Lambda^p(\mathbf{R}^m)^*$. Then for $p\geq2$, φ has comass one and $G(\varphi)$ is given as follows.*

Case 1 ($p=2$). *Let* $Je_1=e_2, Je_3=e_4$, *define a complex structure on*

$$\mathbf{C}^2=\text{span}\{e_1, \ldots, e_4\}.$$

Then $G(\varphi)$ *consists of all canonically oriented complex lines in* $\mathbf{C}^2 \subseteq \mathbf{R}^m$.

Case 2 ($p>3$). $G(\varphi)=\{e_1\wedge\ldots\wedge e_p\}\cup\{e_{p+1}\wedge\ldots\wedge e_{2p}\}$.

Proof. By Proposition 7.9 we may assume that $m=2p$. Case 1 is a direct consequence of the Wirtinger inequality (see Theorem 6.11), so we can assume $p\geqslant 3$. Suppose $\xi\in G(p, 2p)$ and put ξ in canonical form with respect to $V=\text{span}\{e_1, \ldots, e_p\}$ as in Lemma 7.5. Then $\varphi(\xi)=\pm\cos\theta_1\ldots\cos\theta_p\pm\sin\theta_1\ldots\sin\theta_p\leqslant[\cos^2\theta_2\ldots\cos^2\theta_p+\sin^2\theta_2\ldots\sin^2\theta_p]^{\frac{1}{2}}\leqslant\cos\theta_2\ldots\cos\theta_p+\sin\theta_2\ldots\sin\theta_p$, which we may assume by induction on p is $\leqslant 1$. The second inequality is an equality if and only if $\cos\theta_2=\ldots=\cos\theta_p=1$ or $\sin\theta_2=\ldots=\sin\theta_p=1$. It then follows easily that $\varphi(\xi)=1$ if and only if either $\xi=e_1\wedge\ldots\wedge e_p$ or $\xi=e_{p+1}\wedge\ldots\wedge e_{2p}$.

We now investigate the largest exposed facets of the comass ball K^*. These are the facets of the form $\mathfrak{F}^*(\xi)$ where $\xi\in G(p, m)$. Given a unit decomposable p-vector ξ, we may choose an orthonormal basis $e_1, \ldots, e_m$ for $\mathbf{R}^m$ so that $\xi\equiv e_1\wedge\ldots\wedge e_p$. Let $e_I^*\equiv e_{i_1}^*\wedge\ldots\wedge e_{i_p}^*$, $i_1<i_2<\ldots<i_p$, denote the Ith axis p-form. The axis p-form $\varphi\equiv e_{1\ldots p}^*$ certainly belongs to $\mathfrak{F}^*(\xi)$. It is clear that ξ is an exposed point of the mass ball, exposed by φ. That is $\mathfrak{F}^*(\varphi)=\{\xi\}$. Consequently, by (7.2) the facetal hull of φ is exposed by ξ. That is,

$$\mathfrak{F}(\varphi)=\mathfrak{F}^*(\xi).$$

Now we shall prove that φ is the center of an axis "diamond" contained in $\mathfrak{F}^*(\xi)$.

PROPOSITION 7.13. *Suppose* $\xi\equiv e_1\wedge\ldots\wedge e_p$ *and* $\varphi\equiv e_1^*\wedge\ldots\wedge e_p^*$. *If* $\psi\equiv e_I^*$ *is an axis* p*-form with* $\dim(\text{span}\,\varphi\cap\text{span}\,\psi)\leqslant p-2$ (*i.e.*, $|I\cap\{1, \ldots, p\}|\leqslant p-2$), *then* $\varphi\pm\psi$ *is on the boundary of the facet* $\mathfrak{F}^*(\xi)$. *Moreover, the diamond D formed by taking the convex hull of the axis p-planes* $\pm e_I^*$ *with* $|I\cap\{1, \ldots, p\}|\leqslant p-2$ *has the following two properties:*

(1) $\varphi+D\subset\mathfrak{F}^*(\xi)$

(2) $\mathfrak{F}^*(\xi)\subset$*affine span of* $\varphi+D$.

By the affine span of a set A we mean the smallest affine subspace containing A.

Remark. It is clear that we may replace D in Proposition 7.13 by the larger set D'

defined to be the convex hull of all unit simple p-forms ψ such that $\dim(\operatorname{span}\varphi \cap \operatorname{span}\psi) \leq p-2$ and $\operatorname{span}\psi = (\operatorname{span}\varphi \cap \operatorname{span}\psi) \oplus ((\operatorname{span}\varphi)^{\perp} \cap \operatorname{span}\psi)$.

Remark. The fact that $\mathfrak{F}^*(\xi) \subset \text{affine span}\,\varphi + D$ provides a useful procedure for discovering interesting forms of comass one. Suppose φ has comass one and $G(\varphi)$ contains an axis p-vector, say e_J (with respect to some orthonormal basis $e_1, \dots, e_m$ for $\mathbf{R}^m$). Then $p(m-p)+1$ of the coefficients of $\varphi \equiv \Sigma'_{|I|=p} \varphi_I e_I^*$ are determined. Namely $\varphi_J = 1$ and $\varphi_I = 0$ for each I with $|I \cap J| = p-1$. Thus, forms φ of comass one such that $G(\varphi)$ contains several axis p-planes are severely limited.

Proof. First we establish the following fact. Suppose φ and ψ are axis p-planes in $(\mathbf{R}^m)^*$ with intersection of dimension $p-2$ or less. Then

$$\|\lambda\varphi + \mu\psi\|^* = \max\{|\lambda|, |\mu|\}. \tag{7.14}$$

Because of Proposition 7.9 the largest value of $(\lambda\varphi+\mu\psi)(\eta)$ for $\eta \in G(p, m)$ is attained when $\eta \in \Lambda^p(\operatorname{span}\varphi + \operatorname{span}\psi)^*$. Thus we may assume that there exists an orthonormal basis $e_1, \dots, e_m$ for $\mathbf{R}^m$ with $\varphi \equiv e_1^* \wedge \dots \wedge e_p^*$ and $\psi \equiv e_1^* \wedge \dots \wedge e_{p-l}^* \wedge e_{p+1}^* \wedge \dots \wedge e_{p+l}^*$ where $p+l=m$ and $l \geq 2$. That is $\lambda\varphi + \mu\psi = e_1^* \wedge \dots \wedge e_{p-l}^* \wedge (\lambda e_{p-l+1}^* \wedge \dots \wedge e_p^* + \mu e_{p+1}^* \wedge \dots \wedge e_{p+l}^*)$. Now $\lambda e_{p-l+1}^* \wedge \dots \wedge e_p^* + \mu e_{p+1}^* \wedge \dots \wedge e_{p+l}^*$ has comass $\max\{|\lambda|, |\mu|\}$ by Proposition 7.12 since $l \geq 2$. Applying Proposition 7.10 completes the proof of (7.14).

In order to prove the first part of Proposition 7.13 we compute the comass of $\varphi + t\psi$ using (7.14) and obtain $\|\varphi + t\psi\|^* = \max\{1, |t|\}$. Consequently $\varphi + t\psi$ is on the boundary of the comass ball if and only if $-1 \leq t \leq 1$. Since $(\varphi+t\psi)(\xi) = 1$ the line segment $\{\varphi + t\psi : -1 \leq t \leq 1\}$ is contained in the dual facet $\mathfrak{F}^*(\xi)$ and $\varphi \pm \psi$ belongs to the boundary of this facet $\mathfrak{F}^*(\xi)$. This also proves that the diamond D has the property that $\varphi + D \subset \mathfrak{F}^*(\xi)$.

It remains to show that $\mathfrak{F}^*(\xi)$ is contained in the affine span of $\varphi + D$. Suppose $\Phi \in \mathfrak{F}^*(\xi)$ with $\Phi = \Sigma'_{|I|=p} \Phi_I e_I^*$. First note that $\Phi_{1\dots p} = \Phi(\xi) = 1$ since $\Phi \in \mathfrak{F}(\xi)$. Now consider I such that $|I \cap \{1, \dots, p\}| = p-1$. Say, for example, $I = (2, \dots, p, p+1)$. We must show $\Phi_I = 0$. Let $\eta \equiv (\cos\theta e_1 + \sin\theta e_{p+1}) \wedge e_2 \wedge \dots \wedge e_p$. Then $\Phi(\eta) = \cos\theta + \Phi_I \sin\theta$ which is $\leq (1+\Phi_I^2)^{\frac{1}{2}}$ with this maximum value obtained for θ with $\cos\theta = (1+\Phi_I^2)^{-\frac{1}{2}}$. Since $\Phi(\eta) \leq 1$, this proves $\Phi_I = 0$. Therefore Φ belongs to the affine span of $\varphi + D$, completing the proof of Proposition 7.13.

A more careful look at the proof of Proposition 7.13 enables us to compute the $\varphi + \psi$ Grassmanian.

PROPOSITION 7.15. *Suppose* $\varphi=e_1^*\wedge\ldots\wedge e_p^*+e_1^*\wedge\ldots\wedge e_{p-l}^*\wedge e_{p+1}^*\wedge\ldots\wedge e_{p+l}^*$ *for* $l\geq 2$. *Then* $\|\varphi\|^*=1$ *and* $G(\varphi)$ *is given as follows.*

Case 1 ($l=2$). *Define a complex structure on* $V\equiv\operatorname{span}\{e_{p-1},\ldots,e_{p+2}\}$ *by setting* $Je_{p-1}=e_p$ *and* $Je_{p+1}=e_{p+2}$. *Then* $G(\varphi)=\{e_1\wedge\ldots\wedge e_{p-2}\wedge\eta:\eta$ *corresponds to a canonically oriented complex line in* $V\subseteq\mathbf{R}^m\}$.

Case 2 ($l\geq 3$). $G(\varphi)=\{e_1\wedge\ldots\wedge e_p\}\cup\{e_1\wedge\ldots\wedge e_{p-l}\wedge e_{p+1}\wedge\ldots\wedge e_{p+l}\}$.

In the special case of $\Lambda^2\mathbf{R}^m$ (and by Hodge duality $\Lambda^{m-2}\mathbf{R}^m$) we can give a complete description of the facets of both the mass and the comass ball.

THEOREM 7.16. *Suppose F is an extreme facet of the mass ball K in* $\Lambda^2\mathbf{R}^m$. *Then there exists a subspace* $\mathbf{R}^{2n}\subset\mathbf{R}^m$ *of even dimension, a (orthogonal) complex structure J on* $\mathbf{R}^{2n}(=\mathbf{C}^n)$, *and a form* $\varphi\equiv e_1^*\wedge Je_1^*+\ldots+e_n^*\wedge Je_n^*$ *(the associated Kähler form) on* $\mathbf{C}^n$ *such that* $F=\mathfrak{F}^*(\varphi)$ *is an exposed facet and* $\mathcal{G}(\varphi)\cong\mathbf{P}^{n-1}(\mathbf{C})$ *is the grassmannian of complex lines in* $\mathbf{C}^n\subseteq\mathbf{R}^m$. *The dual facet* F^* *consists of* $\varphi+\{\psi\in\Lambda^2((\mathbf{R}^{2n})^\perp)^*:\|\psi\|^*\leq 1\}$. *Moreover, each extreme facet of the comass ball* K^* *is of this form.*

The standard proof is omitted.

Remark. Note that, in particular, each extreme facet of the mass or comass ball is also an exposed facet. Also note that the extreme points of the comass ball are just the points $\varphi\equiv e_1^*\wedge Je_1^*+\ldots+e_n^*\wedge Je_n^*$ with $n=[m/2]$.

Similar results hold for the mass and comass ball in $\Lambda^{m-2}\mathbf{R}^m$ and $\Lambda^{m-2}(\mathbf{R}^m)^*$, since $\Lambda^2\mathbf{R}^m\cong\Lambda^{m-2}\mathbf{R}^m$ with $G(2,m)\cong G(m-2,m)$.

This Theorem 7.16 says that we have not overlooked any constant coefficient calibrations φ on $\mathbf{R}^m$ which are of degree $p=2$ or $p=m-2$. For $p=2$ the only possibility is the standard Kähler form ω (with respect to some orthogonal complex structure on $\mathbf{R}^{2n}\subset\mathbf{R}^m$). For $p=m-2$ the only possibilities are

$$\frac{1}{(n-1)!}\omega^{n-1}\quad(\text{for } m=2n \text{ even})$$

and

$$\frac{1}{(n-1)!}\omega^{n-1}\wedge e_m\quad(\text{for } m=2n+1 \text{ odd}).$$

Remark. The Hodge $*$-operator gives an isomorphism of $\Lambda^p(\mathbf{R}^m)^*$ and $\Lambda^{m-p}(\mathbf{R}^m)^*$ which preserves the comass ball. Consequently, one need only consider $p\leq 2$, in examining the facets of the mass ball K or the extreme points of the comass ball K^*.

We conclude this section with an example of how the above "algebraic" results can be used to study φ-chains.

THEOREM 7.17. *Suppose* $\mathbf{R}^m = V \oplus W$ *is an orthogonal decomposition and* $\varphi \in \Lambda^p V^* \subset \Lambda^p \mathbf{R}^m$ *is a parallel calibration on* $\mathbf{R}^m$ *depending only on the variables in* V. *Each positive* φ*-cycle* T *on* $\mathbf{R}^m$ *must be of the form*

$$T \equiv \sum_{j=1}^{\infty} T_j \otimes [a_j]$$

where $\{a_1, a_2, \ldots,\}$ *is a discrete subset of* W *and each* T_j *is a positive* φ*-cycle in* $V \subset \mathbf{R}^m$.

Proof. By hypothesis $T \in \mathscr{R}_p^{\mathrm{loc}}(\mathbf{R}^m)$, $dT=0$ and $\varphi(\mathbf{T})=1$ for $\|T\|$−a.e. $x \in \mathbf{R}^m$. By Proposition 7.9 (also see the remark following this Proposition 7.9), we know that $\mathbf{T} \in \Lambda^p V \subset \Lambda^p \mathbf{R}^m$ for $\|T\|$−a.e. $x \in \mathbf{R}^m$.

Let χ denote the characteristic function of the product of a ball in V with a ball in W, chosen so that χT is an integral current. We may apply the next Theorem to χT, and complete the proof of Theorem 7.17 except for the conclusion that $\{a_1, \ldots\}$ is discrete. This last fact follows immediately from the results of Section 5 which gives a lower bound for the mass of a positive φ-cycle S in a ball of radius r.

THEOREM 7.18. *Suppose* $\mathbf{R}^m = V \oplus W$ *is an orthogonal decomposition. Each integral current* $T \in I_p(\mathbf{R}^m)$ *with* $\mathbf{T}(x) \in \Lambda^p V$ *for* $\|T\|$−a.e. x *is of the form*

$$T = \sum_{j=1}^{\infty} S_j \otimes [a_j]$$

where $S_j \in I_p(V)$ *and*

$$N(T) = \sum_{j=1}^{\infty} N(S_j).$$

Proof. Integral currents R with the property $\mathbf{R}(x) \in \Lambda^p V$ for $\|R\|$−a.e. x will be referred to as being *horizontal*. The decomposition theorem in $[F_1]$, 4.2.25 implies that T can be decomposed into $\Sigma_{j=1}^{\infty} T_j$ with $N(T) = \Sigma N(T_j)$ and each T_j indecomposable. A current $R \in I_p(\mathbf{R}^m)$ is said to be a *piece of* T if $N(T) = N(R) + N(T-R)$. If a current $T \in I_p(\mathbf{R}^m)$ has a piece different from T and 0 then T is said to be *decomposable*. Otherwise T is said to be *indecomposable*. Of course each T_j in the above decomposition is a piece of T. Consequently, we must show:

(1) If $T \in I_p(\mathbf{R}^m)$ is horizontal then each piece R *of* T is horizontal.

(2) If $T\in I_p(\mathbf{R}^m)$ is horizontal and indecomposable then T must be of the form $S\otimes[a]$ for some indecomposable $S\in I_p(V)$ and some $a\in W$.

Proof of (1). Let y denote a linear function on W. A current S can be uniquely expressed as $S_0\wedge dy+S_1$ where S_0 and S_1 can be expressed in terms of one forms orthogonal to dy. The current S is horizontal if and only if $S_1=0$ for each linear function y on W. Consequently $T=T_0\wedge dy$; and if $R=R_0\,dy+R_1$ is a piece of T we must show that $R_1=0$. First note that

$$\begin{aligned} M(R_0\wedge dy) &= \sup\{(R_0\wedge dy)(\psi)\colon dy \lrcorner \psi=0, \|\psi\|^*\leq 1\}\\ &= \sup\{(R_0\wedge dy+R_1)(\psi)\colon dy \lrcorner \psi=0, \|\psi\|\leq 1\}\\ &\leq \sup\{(R_0\wedge dy+R_1)(\psi+\varphi\wedge dy)\colon dy \lrcorner \psi = dy \lrcorner \varphi=0,\\ &\qquad \text{and} \quad \|\psi+\varphi\, dy\|^*\leq 1\}\\ &\leq M(R_0\wedge dy+R_1). \end{aligned}$$

Next we prove that equality holds, i.e. $M(R_0\wedge dy)=M(R)$, if and only if $R_1=0$. To prove this we may choose ψ with $dy\lrcorner\psi=0$ and $\|\psi\|^*=1$ so that $(R_0\wedge dy)(\psi)=M(R_0)=M(R_0\wedge dy)$. Also choose φ with $dy\lrcorner\varphi=0$ and $\|\varphi\|^*=1$ so that $R_1(\varphi\wedge dy)=M(R_1)$.

Then for each θ,

$$\begin{aligned} \cos\theta M(R_0\wedge dy)+\sin\theta M(R_1) &= (R_0\wedge dy+R_1)((\cos\theta)\,\psi+(\sin\theta)\,\varphi\wedge dy)\\ &\leq M(R)\|(\cos\theta)\,\psi+(\sin\theta)\,\varphi\wedge dy\|^*\\ &\leq M(R). \end{aligned}$$

To prove the last inequality note that

$$\|\psi+\varphi\wedge dy\|^*\leq\sqrt{(\|\psi\|^*)^2+(\|\varphi\|^*)^2}.$$

This can be seen as follows.

Let $e^*\equiv dy$ and put $\xi\in G(p,m)$ in canonical form with respect to e. That is $\xi=((\cos\alpha)\,e+(\sin\alpha)\,V)\wedge\eta$ where $e\lrcorner\eta=0$ and $e\perp V$. Then

$$\begin{aligned} (\psi+\varphi\wedge dy)(\xi) &= \cos\alpha(\varphi\wedge dy)(e\wedge\eta)+\sin\alpha\psi(v\wedge\eta)\\ &\leq \cos\alpha\|\varphi\|^*+\sin\alpha\|\psi\|^*\leq\sqrt{(\|\varphi\|^*)^2+(\|\psi\|^*)^2}. \end{aligned}$$

The inequality

$$\cos\theta M(R)+\sin\theta M(R_1)\leq M(R)$$

established above for all θ implies that $M(R_1)=0$ as desired. Now by hypothesis $M(R)+M(T-R)=M(T)$.

Also $M(R_0\wedge dy)\leqslant M(R)$ with equality if and only if $R_1=0$. Similarly $M(T-R_0\wedge dy)\leqslant M(T-R)$ with equality if and only if $R_1=0$. Therefore $M(R_0\wedge dy)+M(T-R_0\wedge dy)\leqslant M(T)$ and hence equality must occur. That is $R_1=0$. This proves part (1) that R is horizontal.

Proof of (2). Suppose T is horizontal. If A is any Borel subset of W, the current $\chi_A T$ is:

(i) integral with boundary $\chi_A \partial T$
(ii) horizontal
(iii) a piece of T.

Proof of (i). Smooth T by smoothing each coefficient. This gives a family of smooth T_ε with $T_\varepsilon\to T$ and each T_ε horizontal. Since $\partial(\chi_A T)=(\partial\chi_A)\wedge T_\varepsilon+\chi_A\partial T_\varepsilon=\chi_A\partial T_\varepsilon$, we have $\partial(\chi_A T)=\chi_A\partial T$.

Proof of (ii). Obvious.

Proof of (iii). $M(\chi_A T)+M((1-\chi_A)T=M(T)$; and $M(\partial(\chi_A T))+M(\partial((1-\chi_A)T))=M(\partial T)$, because of (i).

The measure, $\nu(A)\equiv M(\chi_A T)$ for all $A\subset W$, must be concentrated at a point $a\in W$ or (i), (ii) and (iii) above imply that T is decomposable. Therefore T must be supported in $V\times\{a\}$. Since T is flat it must be of the form $S\otimes[a]$ by a support theorem of Federer [F_1].

Appendix II.A. Positive φ-currents

Suppose φ is a p-form of comass one on a riemannian manifold X, and consider the associated Grassmann φ-geometry. In this appendix we discuss various equivalent definitions of a positive φ-current. It is not necessary to assume φ is closed. In fact, one can axiomatize the properties of $\mathcal{G}=\mathcal{G}(\varphi)$ needed for the discussion. However, we prefer to work with the case of φ-geometries.

Definition A.1. Suppose ψ is a smooth p-form with compact support (i.e. a *test form*) on X. If $\langle\psi_x,\xi_x\rangle\geqslant 0$ for each $\xi_x\in\mathcal{G}(\varphi)$, then ψ is called a *φ-non-negative test form*. If ψ is only continuous and the above condition holds, then ψ is called a *φ-non-negative continuous test form*. Note that the cone of φ-non-negative test forms includes the vector space which annihilates the span of $\mathcal{G}(\varphi)$.

The notion of a positive φ-current can be expressed dually in several different ways.

PROPOSITION A.2. *Let T be any deRham current of dimension p on X. Then the following are equivalent.*

(a) *$\psi \lrcorner T$ is a positive measure for each φ-non-negative test form ψ.*

(b) *$T \in \mathcal{M}_p^{\text{loc}}(X)$ and $\psi \lrcorner T$ is a positive measure for each φ-non-negative continuous test form ψ.*

(c) *T is a positive φ-current on X.*

Remark. Condition (a) may be rewritten as

(a)′ $T(\psi) \geqslant 0$ for each φ-non-negative test form ψ, since $(\psi \lrcorner T)(f) = T(f\psi)$ for each smooth function f.

Similarly, (b) may be rewritten. As an immediate corollary of (a)′ we have that: The cone $\{T \in \mathcal{D}_p'(X) : T \text{ is } \varphi\text{-positive}\}$ is weakly closed in $\mathcal{D}_p'(X)$.

Proof. Clearly (b) implies (a). We show that (c) implies (b). By hypothesis, $T \in \mathcal{M}_p^{\text{loc}}(X)$ and $\mathbf{T} \in \operatorname{ch} \mathcal{G}(\varphi)$, $\|T\|$–a.e. Hence, given a φ-non-negative continuous test form ψ we have $\langle \psi, \mathbf{T} \rangle \geqslant 0$, $\|T\|$–a.e., and so

$$(\psi \lrcorner T)(f) = T(f\psi) = \int_X \langle \psi, \mathbf{T} \rangle f d\|T\| \geqslant 0$$

for any continuous function $f \geqslant 0$. Therefore, the measure $(\psi \lrcorner T)$ is positive as claimed.

It remains to prove that (a) implies (b). We must show that $T \in \mathcal{M}_p^{\text{loc}}(X)$, i.e., that T has measure coefficients. Fix a compact set $K \subset X$. Consider the associated compact set $\mathfrak{J}_K^*(\varphi) \equiv \{\xi \in \Lambda^p T_x X : \langle \varphi_x, \xi_x \rangle = 1, \|\xi_x\| = 1 \text{ and } x \in K\}$. Since $\varphi \equiv 1$ on $\mathfrak{J}_K^*(\varphi)$ we see that any form sufficiently close to φ (in the C^0-topology) will be positive on $\mathfrak{J}_K^*(\varphi)$. It follows easily that given any point $x \in X$, we can find non-negative test forms $\psi_1, \ldots, \psi_N$ giving a local frame field for $\Lambda^p T^* X$ near x. By hypothesis each $\psi_j \lrcorner T$ is a positive measure. Therefore T has measure coefficients.

It remains to show that $\mathbf{T}_x \in \mathfrak{J}_x^*(\varphi) = \operatorname{ch} \mathcal{G}_x(\varphi)$ for each $x \in \operatorname{Leb}(T)$. Recall that for $x \in \operatorname{Leb}(T)$ and any test form ψ,

$$\langle \mathbf{T}_x, \psi_x \rangle = \lim_{\varepsilon \to 0^+} \frac{\displaystyle\int_{B(x,\varepsilon)} \langle \mathbf{T}, \psi \rangle \, d\|T\|}{\displaystyle\int_{B(x,\varepsilon)} d\|T\|}. \tag{A.3}$$

To begin we consider the open set $U=\{x\in X: \|\varphi\|^*<1\}$. Note that for any $x\in U$, we have $\mathfrak{J}_x^*(\varphi)=\emptyset$, and so *any* test form ψ with support in U is automatically non-negative. In particular, $T(\psi)\geq 0$ and $T(-\psi)=T(\psi)\geq 0$. It follows that

$$\|T\|(U)=0,$$

and we need only consider points x where $\|\varphi_x\|^*=1$.

Fix such a point x and consider the cone $C_x(\varphi)$ on $\mathfrak{J}_x^*(\varphi)$ with vertex the origin. By definition, the *dual cone* to $C_x(\varphi)$ is $C_x'(\varphi)\equiv\{\psi\in\Lambda^p T_x^* X: \langle\psi_x, \xi_x\rangle\geq 0 \text{ for all } \xi_x\in C_x(\varphi)\}$. Since $C_x(\varphi)$ is closed and convex, the Bipolar theorem states that $(C_x'(\varphi))'=C_x(\varphi)$. Therefore $\mathbf{T}_x\in C_x(\varphi)$ if and only if

$$\langle \mathbf{T}_x, \psi_x\rangle \geq 0 \tag{A.4}$$

for all $\psi_x\in C_x'(\varphi)$. Now the interior of $C_x'(\varphi)$, which consists of all ψ which are strictly positive on $\mathfrak{J}_x^*(\varphi)$ (e.g. φ itself), is dense in $C_x'(\varphi)$. Hence, it suffices to establish (A.4) for all such ψ_x. Recall that if $K\subset X$ is compact, then $\mathfrak{J}_K^*(\varphi)=\cup_{y\in K}\mathfrak{J}_y^*(\varphi)$ is also compact. Consequently, if ψ_x is strictly positive on $\mathfrak{J}_x^*(\varphi)$, then any continuous p-form ψ extending ψ_x is strictly positive on $\mathfrak{J}_y^*(\varphi)$ for all y near x. Consequently, given any ψ_x which is strictly positive on $\mathfrak{J}_x^*(\varphi)$, we can find a φ-non-negative test form ψ extending ψ_x. Applying (A.3) to ψ we obtain $\langle \mathbf{T}_x, \psi_x\rangle\geq 0$ as desired. This completes the proof.

Remark. Each current of the form $T=\delta_x\xi_x$ with $\xi_x\in C_x(\varphi)$ for some $x\in X$ is a positive φ-current. Let C denote the cone all (finite) convex combinations of such currents. First consider C as a cone in $\mathcal{M}_p^{\text{loc}}(X)$. Then the cone dual to C is exactly the cone of φ-non-negative continuous test forms. The Bipolar theorem states that the double dual of C is the weak closure of C. Consequently, by Proposition A.2, we have

COROLLARY A.5. *The weak closure of C in $\mathcal{M}_p^{\text{loc}}(X)$ is the cone of positive φ-currents.*

In exactly the same way one can prove:

COROLLARY A.6. *The weak closure of C in the space $\mathcal{D}_p'(X)$ of deRham currents, is the cone of positive φ-currents.*

Let $\Lambda_x(\varphi)$ denote the linear span of $\mathcal{G}_x(\varphi)$ in $\Lambda^p T_x X$. Using the riemannian metric to introduce an inner product $(\,,)$ in $\Lambda_x(\varphi)$ we may define the cone

$$C_x^\dagger(\varphi) \equiv \{\eta_x \in \Lambda_x(\varphi): (\eta_x, \xi_x) \geq 0 \quad \text{for all } \xi_x \in C_x(\varphi)\}. \tag{A.7}$$

Since $*(\eta_x, \xi_x) = \eta_x \wedge *\xi_x$, the original cone $C_x(\varphi)$ is contained in the cone $C_x^\dagger(\varphi)$ if and only if $*(\eta_x \wedge *\xi_x) \geq 0$ for all $\xi_x, \eta_x \in C_x(\varphi)$; that is if and only if η_x and $*\xi_x$ are compatibly oriented for every pair η_x, ξ_x of φ planes.

It is natural to consider the $\mathcal{G}^\dagger(\varphi)$-geometry defined by setting

$$\mathcal{G}_x^\dagger(\varphi) \equiv C_x^\dagger(\varphi) \cap G(p, T_x X). \tag{A.8}$$

If $C_x(\varphi) \subset C_x^\dagger(\varphi)$ and

$$\mathcal{G}_x(\varphi) \cup (-\mathcal{G}_x(\varphi)) = \Lambda_x^p(\varphi) \cap G(p, T_x X)$$

(Question 6.3 above) then

$$\mathcal{G}^\dagger(\varphi) = \mathcal{G}(\varphi).$$

Consequently, the positive $\mathcal{G}^\dagger(\varphi)$ chains are just the positive $\mathcal{G}(\varphi)$ chains.

Nevertheless, it is useful to consider the cone of currents $T \in \mathcal{M}_p^{\mathrm{loc}}(X)$ such that

$$\mathbf{T}_x \in C_x^\dagger(\varphi) \quad \text{for } \|T\|\text{-a.a. } x.$$

These currents are called *weakly positive φ-currents.*

Example I: Complex geometries. Here $\Lambda_x(\varphi)$ is just the space $\Lambda^{p,p}T_x x$ of real p, p-vectors. In [HKn], the cone $C_x(\varphi)$ is called the cone of strongly positive p, p-vectors; while $C_x^\dagger(\varphi)$ is called the cone of weakly positive p, p-vectors. Note that $*(\xi \wedge *\eta) \geq 0$ for complex planes ξ, η so that $C(\varphi) \subseteq C^\dagger(\varphi)$ (and hence $\mathcal{G}^\dagger(\varphi) = \mathcal{G}(\varphi)$ are remarked above). If $p=1$ or $n-1$, then $C_x^\dagger(\varphi) = C_x(\varphi)$; however for $1<p<n-1$ $C_x(\varphi)$ is a proper subset of $C_x^\dagger(\varphi)$. (See [HKn] for proofs and further results.)

III. Special Lagrangian geometry

In this chapter we shall study the geometry of n-folds in $\mathbf{R}^{2n} \cong \mathbf{C}^n$ associated to the calibration $\alpha = \mathrm{Re}\{dz_1 \wedge \ldots \wedge dz_n\}$. The submanifolds in this geometry are Lagrangian submanifolds of $\mathbf{C}^n$ which satisfy an additional "determinant" condition. They are therefore called "Special Lagrangian" submanifolds. They, of course, have the property of being absolutely area minimizing.

In the first section we prove that α has comass one and characterize the Grassmannian of Special Lagrangian n-planes. In the second section we derive a differential equation whose solutions correspond to the Special Lagrangian submanifolds. In particular, we consider any such manifold M locally as a graph over its tangent plane. When M is Lagrangian, this graphing function is the gradient of a scalar "potential" function F. We derive a second order differential equation for F which holds exactly when M is Special Lagrangian. For example, when $n=3$, the equation is: $\Delta F = \det(\operatorname{Hess} F)$. This equation is non-linear, but we prove that its linearization at any solution is always elliptic.

We then present a number of explicit constructions of special Lagrangian varieties. For example, we prove that the normal bundle to an "austere" submanifold of $\mathbf{R}^n$ is always special Lagrangian in $T^*\mathbf{R}^n \cong \mathbf{R}^{2n}$. There are two cases of particular interest. A surface in $\mathbf{R}^n$ is austere if and only if it is minimal. Hence, the conormal bundle of any minimal surface is a special Lagrangian variety. The second case consists of cones on austere submanifolds of the sphere. In particular, each compact minimal surface in S^3 gives rise to a topologically complicated, minimizing 4-dimensional cone in $\mathbf{R}^8$. Many new singularity types for minimal currents are constructed in this way.

In the next section we show the relationship of special Lagrangian geometry to the work of Hans Lewy [Ly] on harmonic gradients.

The last section examines boundaries of special Lagrangian varieties and the Cauchy problem for the special Lagrangian differential equation.

As we point out in Chapter V (Section 3), Special Lagrangian geometries are naturally defined on any Ricci-flat Kähler manifold. The existence of such manifolds is established by Yau's recent proof of the Calabi conjecture.

III.1. The special Lagrangian inequality

Let $\mathbf{C}^n$ denote complex euclidean n-space, with coordinates $z=(z_1, \ldots, z_n)$, where $z=x+iy$ with $x=(x_1, \ldots, x_n)$ and $y=(y_1, \ldots, y_n)$. Let $\mathbf{R}^n$ denote the subset of $\mathbf{C}^n$ where $y=0$, with the standard orientation. The form we shall study is

$$\alpha \equiv \operatorname{Re}\{dz_1 \wedge \ldots \wedge dz_n\} \in \Lambda^n \mathbf{C}^n. \tag{1.1}$$

An oriented real n-plane ξ in $\mathbf{C}^n$ is called *totally real* if it contains no complex lines. That is, if $u \in \xi$ implies $Ju \notin \xi$. An oriented real n-plane ξ in $\mathbf{C}^n$ is called *Lagrangian* if the stronger condition,

$$Ju \perp \zeta \quad \text{for all } u \in \zeta, \tag{1.2}$$

is valid.

Let $(\ ,\ ) \equiv \Sigma_{j=1}^{n} dz_j \otimes d\bar{z}_j$ denote the standard hermitian form on $\mathbf{C}^n$, let $\langle\ ,\ \rangle \equiv \Sigma_{j=1}^{n} dx_j^2 + dy_j^2$ denote the standard inner product on $\mathbf{C}^n$, and let $\omega \equiv \Sigma\, (i/2)\, dz_j \wedge d\bar{z}_j$ denote the standard Kähler form on $\mathbf{C}^n$. They are related by the formula

$$(u, v) = \langle u, v \rangle - i\omega(u, v),$$

for all vectors $u, v \in \mathbf{C}^n$.

Therefore $\langle Ju, v \rangle = \mathrm{Re}\,(Ju, v) = \mathrm{Re}\, i(u, v) = -\mathrm{Im}\,(u, v) = \omega(u, v)$. Consequently we may rephrase the definition of Langrangian replacing (1.2) by the following:

$$\omega \text{ restricted to } \zeta \text{ vanishes.} \tag{1.2$'$}$$

Consider the grassmannian $G(n, 2n)$ of oriented real n-planes in $\mathbf{C}^n$, and let Lag denote the subset consisting of the Lagrangian planes. (Note that Lag consist of oriented planes.) One can easily check (using either (1.2) or (1.2)$'$) that the unitary group U_n acts on Lag. Moreover this action is transitive. Suppose that $\zeta \equiv \varepsilon_1 \wedge \ldots \wedge \varepsilon_n$ and $\zeta' \equiv \varepsilon_1' \wedge \ldots \wedge \varepsilon_n'$ are Lagrangian with the epsilons denoting orthonormal bases. Then $\varepsilon_1, \ldots, \varepsilon_n, J\varepsilon_1, \ldots, J\varepsilon_n$ and $\varepsilon_1', \ldots, \varepsilon_n', J\varepsilon_1', \ldots, J\varepsilon_n'$ are both orthonormal basis for $\mathbf{R}^{2n} \cong \mathbf{C}^n$. Consequently the linear map A sending the unprimed basis into the primed basis is unitary and $A\zeta = \zeta'$. The isotropy subgroup of U_n at the point $\zeta_0 \equiv \mathbf{R}^n$ is SO_n acting diagonally on $\mathbf{R}^n \oplus \mathbf{R}^n$. Thus

$$\mathrm{Lag} \cong U_n / SO_n. \tag{1.3}$$

Definition 1.4. An oriented n-plane ζ in $\mathbf{C}^n$ is called *special Lagrangian* if

(1) ζ is Lagrangian
(2) $\zeta = A\zeta_0$, where $A \in U_n$ has the special property $\det A = 1$.

Thus we have singled out the fibre above $1 \in S^1$ in the fibration

$$\mathrm{Lag} \cong U_n / SO_n \xrightarrow{\det_{\mathbf{C}}} S^1. \tag{1.5}$$

This fibre above $1 \in S^1$ consisting of all the special Lagrangian n-planes will be denoted $S(\mathrm{Lag})$.

Remark. Each fibre of the map (1.5) is the Grassmannian associated to a calibration on $\mathbf{R}^{2n}$. In fact, this family of Grassmannians belongs to the family of forms:

$$\alpha_\theta \equiv \mathrm{Re}\,\{e^{i\theta}\cdot dz_1\wedge\ldots\wedge dz_n\}$$

for $0\leqslant\theta<2\pi$. Thus we have an S^1-family of "Special Lagrangian" geometries compatible with the given complex structure. Since these geometries are equivalent under U_n, it will suffice to study the one associated to $\alpha=\alpha_0$.

For notational convenience we set

$$\beta = \mathrm{Im}\,dz = \mathrm{Im}\,\{dz_1\wedge\ldots\wedge dz_n\}$$

so that $dz=dz_1\wedge\ldots\wedge dz_n=\alpha+i\beta$, with α and β real.

THEOREM 1.7. *For any* $\zeta\in G(n,2n)\subset\Lambda^n\mathbf{R}^{2n}$,

$$|dz(\zeta)|^2=\alpha(\zeta)^2+\beta(\zeta)^2=|\zeta\wedge J\zeta|.$$

Proof. Suppose $\varepsilon_1,\ldots,\varepsilon_n$ is an oriented basis for $\zeta\in G(n,2n)$ (not necessarily orthogonal). Let $e_1,\ldots,e_n, Je_1,\ldots,Je_n$ denote the standard basis for $\mathbf{R}^n\oplus\mathbf{R}^n=\mathbf{C}^n$. Let A denote the linear map sending e_j to ε_j and Je_j to $J\varepsilon_j$. In particular A is complex linear, and $\lambda\zeta\equiv\varepsilon_1\wedge\ldots\wedge\varepsilon_n=A(e_1\wedge\ldots\wedge e_n)$, where $\lambda>0$. Now $(dz_1\wedge\ldots\wedge dz_n, A(e_1\wedge\ldots\wedge e_n))=\det_{\mathbf{C}}A$, and since $dz_1\wedge\ldots\wedge dz_n\equiv\alpha+i\beta$ this can be rewritten as

$$\lambda\alpha(\zeta)=\mathrm{Re}\,(\det\nolimits_{\mathbf{C}}A) \tag{1.8 a}$$

$$\lambda\beta(\zeta)=\mathrm{Im}\,(\det\nolimits_{\mathbf{C}}A). \tag{1.8 b}$$

Thus

$$\begin{aligned}\alpha(\lambda\zeta)^2+\beta(\lambda\zeta)^2&=|\det\nolimits_{\mathbf{C}}A|^2=\det\nolimits_{\mathbf{R}}A=|A(e_1\wedge Je_1\wedge\ldots\wedge e_n\wedge Je_n)|\\&=|A(e_1\wedge\ldots\wedge e_n\wedge Je_1\wedge\ldots\wedge Je_n)|=\lambda^2|\zeta\wedge J\zeta|.\end{aligned}$$

This completes the proof.

LEMMA 1.9. $|\zeta\wedge J\zeta|\leqslant|\zeta|^2$, *for all* $\zeta\in G(n,2n)$, *with equality if and only if* ζ *is Lagrangian.*

Proof. Let $\varepsilon_1,\ldots,\varepsilon_n$ denote an oriented orthonormal basis from ζ. Hadamard's inequality implies that

$$|\zeta\wedge J\zeta|=|\varepsilon_1\wedge\ldots\wedge\varepsilon_n\wedge J\varepsilon_1\wedge\ldots\wedge J\varepsilon_n|\leqslant|\varepsilon_1|\ldots|\varepsilon_n|\,|J\varepsilon_1|\ldots|J\varepsilon_n|,$$

with equality if and only if $\varepsilon_1,\ldots,\varepsilon_n, J\varepsilon_1,\ldots,J\varepsilon_n$ is an orthogonal system. Since ζ is

Lagrangian if and only if $\varepsilon_1, \ldots, \varepsilon_n, J\varepsilon_1, \ldots, J\varepsilon_n$ is an orthogonal system, the lemma follows.

THEOREM 1.10. *The form* $\alpha \equiv \operatorname{Re} dz$ *has comass one. In fact,* $\alpha(\zeta) \leq |\zeta|$ *for all* $\zeta \in G(n, 2n)$, *with equality if and only if* ζ *is special Lagrangian.*

Proof. $\alpha^2(\zeta)+\beta^2(\zeta)=|\zeta \wedge J\zeta| \leq |\zeta|^2$, with equality if and only if ζ Lagrangian. Therefore $\alpha(\zeta) \leq |\zeta|$ and equality holds if and only if ζ is Lagrangian and upon writing $\zeta = A(e_1 \wedge \ldots \wedge e_n)$ with A unitary, $\det_{\mathbf{C}} A = \alpha(\zeta)+i\beta(\zeta)=1$.

COROLLARY 1.11. *Suppose* $\zeta \in G(n, 2n)$. *Then either* ζ *or* $-\zeta$ *is special Lagrangian if and only if*

(1) ζ *is Lagrangian,*

and

(2) $\beta(\zeta)=0$.

Moreover, if A is any complex linear map sending $\zeta_0 = e_1 \wedge \ldots \wedge e_n$ *into* $\lambda\zeta$ *with* $\lambda \in \mathbf{R}$, *then* $\lambda\beta(\zeta) = \operatorname{Im} \det_{\mathbf{C}} A$.

Proof. The characterization of $\pm\zeta$ special Lagrangian is an immediate consequence of the proof of Theorem 1.10. The fact that $\lambda\beta(\zeta) = \operatorname{Im} \det_{\mathbf{C}} A$ is just (1.8 b).

The above results imply, in particular, that $\alpha \equiv \operatorname{Re} dz$ is a calibration.

Definition 1.12. The form $\alpha \equiv \operatorname{Re}(dz)$ on $\mathbf{C}^n$ is called the *special Lagrangian calibration on* $\mathbf{C}^n$.

Remark. The form α is left fixed by SU_n but not U_n. Also SU_n acts transitively on $S(\text{Lag})$ with isotropy subgroup at $\zeta_0 \equiv \mathbf{R}^n \subset \mathbf{C}^n$ equal to SO_n; i.e.

$$S(\text{Lag}) = SU_n/SO_n. \tag{1.13}$$

The results presented above contain an alternate definition of a Lagrangian subspace of $\mathbf{C}^n$.

PROPOSITION 1.14. $|dz(\zeta)| \leq |\zeta|$ *for all* $\zeta \in G(n, 2n)$ *with equality if and only if* ζ *is Lagrangian.*

Proof. By Theorem 1.7 and Lemma 1.9

$$|dz_1 \wedge \ldots \wedge dz_n(\zeta)|^2 = |\zeta \wedge J\zeta| \leq |\zeta|^2$$

with equality if and only if ζ is Lagrangian.

This characterization of Lagrangian planes can be used to graph a Lagrangian submanifold over one of the 2^n axis Lagrangian planes.

COROLLARY 1.15. *Suppose* $\zeta \in \mathrm{Lag}$ *is not one of the axis n-planes. Then* ζ *can be written as the graph of a linear map over at least two of the* 2^n *axis Lagrangian planes (cf. Arnold* [A], *Chapter* 8.41 D).

Proof. Expand out

$$(dx_1+idy_1)\wedge\ldots\wedge(dx_n+idy_n)=\sum \zeta_k^*+i\sum \eta_k^*,$$

where the ζ_k^* and the η_k^* are the Lagrangian axis planes, and apply Proposition 1.14.

Remark. An alternate approach to the "special Lagrangian inequality", $\alpha(\zeta)\leq|\zeta|$, is provided by the Kähler equalities derived in Chapter II, Section 6. This approach will not be used in the remainder of this paper. Let Ω_p denote $(1/p\,!)\,\omega^p$. Recall that

Case 1 ($n=2p$ even).

$$|dz_1\wedge\ldots\wedge dz_n(\zeta)|^2+\ldots+\sum_{|I|=2k}{}' |dz^I\wedge\Omega_{p-k}(\zeta)|^2+\ldots+[\Omega_p(\zeta)]^2=|\zeta|^2. \tag{1.16}$$

Case 2 ($n=2p+1$ odd).

$$|dz_1\wedge\ldots\wedge dz_n(\zeta)|^2+\ldots+\sum_{|I|=2k+1}{}' |dz^I\wedge\Omega_{p-k}(\zeta)|^2+\ldots+\sum_{j=1}^{n}|dz_j\wedge\Omega_p(\zeta)|^2=|\zeta|^2. \tag{1.17}$$

Since $[\alpha(\zeta)]^2+[\beta(\zeta)]^2=|dz(\zeta)|^2$, the above two equalities immediately imply $\alpha(\zeta)\leq|\zeta|$. The proof of (1.16) and (1.17) shows more; equality holds if and only if ζ is special Lagrangian.

III.2. The special Lagrangian differential equation

III.2.A. The explicit formulation

Definition 2.1. An n-dimensional oriented submanifold M of $\mathbf{C}^n$ is called a *(special) Lagrangian submanifold of* $\mathbf{C}^n$ if the tangent plane to M, at each point, is (special) Lagrangian.

Suppose that M is a special Lagrangian submanifold of $\mathbf{C}^n$. Locally M can be described explicitly as the graph of a function over a tangent plane. Since all special Lagrangian planes are equivalent, under SU_n, to the axis plane $\zeta_0 \equiv \mathbf{R}^n$, we may consider M to be given as the graph, in $\mathbf{R}^n + i\mathbf{R}^n = \mathbf{C}^n$, of a function $y=f(x)$ where $z=x+iy$.

Recall the classical fact.

LEMMA 2.2. *Suppose $\Omega \subseteq \mathbf{R}^n$ is open and $f\colon \Omega \to \mathbf{R}^n$ is a C^1 mapping. Let M denote the graph of f in $\mathbf{C}^n = \mathbf{R}^n + i\mathbf{R}^n$. Then the graph M is Lagrangian if and only if the Jacobian matrix $((\partial f^i/\partial x_j))$ is symmetric. In particular, if Ω is simply connected, the M is Lagrangian if and only if $f = \nabla F$, is the gradient field of some potential function $F \in C^2(\Omega)$.*

Proof. We replace f by its Jacobian f_* at some fixed point. Then $f_*\colon \mathbf{R}^n \to \mathbf{R}^n$ is linear and its graph is of the form $TM = \{x + if_*(x) \colon x \in \mathbf{R}^n\}$. By definition TM is Lagrangian if and only if $Jv \perp TM$ for all $v \in TM$. Suppose $v = x + if_*(x)$. Then $Jv = -f_*(x) + ix$. Thus TM is Lagrangian if and only if $-f_*(x) + ix$ and $x' + if_*(x')$ are orthogonal for all $x, x' \in \mathbf{R}^n$, i.e., if and only if $-\langle f_*(x), x'\rangle + \langle x, f_*(x')\rangle = 0$ for all $x, x' \in \mathbf{R}^n$. Consequently M is Lagrangian if and only if the Jacobian matrix of f is symmetric at each point of Ω. Since Ω is simply connected, this is equivalent to the existence of a potential function $F\colon \Omega \to \mathbf{R}$ with $\nabla F = f$.

In order for the graph of f to be special Lagrangian it must be Lagrangian and satisfy one other condition. Let

$$\operatorname{Hess} F \equiv \left(\left(\frac{\partial^2 F}{\partial x_i \partial x_j} \right) \right)$$

denote the Hessian matrix of $F\colon \Omega \to \mathbf{R}$ and let $\sigma_j(\operatorname{Hess} F)$ denote the jth elementary symmetric function of its eigenvalues.

THEOREM 2.3. *Suppose $F \in C^2(\Omega)$ with $\Omega^{\text{open}} \subset \mathbf{R}^n$ Let $f \equiv \nabla F$ denote the gradient field, and let M denote the graph of f in $\mathbf{C}^n = \mathbf{R}^n \oplus i\mathbf{R}^n$. Then M (with the correct orientation) is special Lagrangian if and only if*

$$\sum_{k=0}^{[(n-1)/2]} (-1)^k \sigma_{2k+1}(\operatorname{Hess} F) = 0, \tag{2.4}$$

or equivalently,

$$\operatorname{Im}\{\det_{\mathbf{C}}(I + i\operatorname{Hess} F)\} = 0. \tag*{(2.4)$'$}$$

Remark. The special case $n=3$ is worth noting. In this case (2.4) becomes the simple and beautiful equation

$$\Delta F = \det(\operatorname{Hess} F).$$

That is, the Laplacian of F equals the Monge-Ampére of F.

Proof. Replacing M by the tangent space to M at a fixed point, we may assume that $f: \mathbf{R}^n \to \mathbf{R}^n$ is linear and symmetric. In fact, f is simply the linear map associated with the quadratic form $\operatorname{Hess} F$. The graph of f is the image of $\zeta_0 \equiv e_1 \wedge \ldots \wedge e_n = \mathbf{R}^n$ under the complex linear map $A: \mathbf{C}^n \to \mathbf{C}^n$ defined by $A \equiv I + if$. It now follows immediately from Corollary 1.11 that M is special Lagrangian if and only if (2.4)′ holds. It remains only to prove the equivalence of (2.4)′ and (2.4).

Since the action of SO_n on $\mathbf{C}^n$, given by $g(x+iy) = gx + igy$, preserves the set of special Lagrangian n-planes, we may replace f by any linear map of the form $g \circ f \circ g^{-1}$ for $g \in SO_n$. In particular since f is symmetric, we may assume it is diagonal with eigenvalues $\lambda_1, \ldots, \lambda_n$. In this case we have $\operatorname{Im}\{\det_{\mathbf{C}}(I+if)\} = \operatorname{Im}\{\Pi_{j=1}^n (1+i\lambda_j)\} = \Sigma_k (-1)^k \sigma_{2k+1}(f)$. Since the first and last terms are SO_n-invariant, this proves the equivalence of (2.4) and (2.4)′ in general.

Remark 2.6. From equation (1.8 a) and the argument above,

$$\lambda\alpha(\zeta) \equiv \operatorname{Re}\left[\det_{\mathbf{C}}(I + i \operatorname{Hess} F)\right] = \sum_{k=0}^{[n/2]} (-1)^k \sigma_{2k}(\operatorname{Hess} F).$$

(Here $\sigma_0 \equiv 1$ by definition.) The correct orientation for the graph of f in Theorem 2.3 can be determined by computing the sign of $\alpha(\zeta)$ from the equation above. That is, if equation (2.4) is satisfied (so that $\pm\zeta$ is special Lagrangian), then

$$(\operatorname{sign} \alpha(\zeta)) \cdot \zeta \in S(\mathrm{Lag}).$$

One can show that $\{\zeta \in S(\mathrm{Lag}): \zeta \text{ can be graphed over } \mathbf{R}^n \subset \mathbf{C}^n\}$ has n components if n is even and $n-1$ components if n is odd.

From the main Theorem II.3.5 and the discussion above we now have the following result:

THEOREM 2.7. *Suppose* $F \in C^2(\Omega)$ *with* $\Omega^{\text{open}} \subset \mathbf{R}^n$. *If F satisfies the differential equation*

$$\sum_{k=0}^{[(n-1)/2]} (-1)^k \sigma_{2k+1}(\operatorname{Hess} F) = 0 \tag{2.8}$$

on Ω, then the graph of ∇F is an absolutely volume minimizing submanifold of $\mathbf{R}^{2n}$. Consequently, the regularity results of Morrey [M] *imply that any C^2 solution of equation* (2.8) *is real analytic.*

III.2.B. Ellipticity

The differential equation derived in part A above is non-linear, but it has some remarkable and beautiful properties. We shall show here, for example, that the linearization of this equation at any "solution" is an elliptic operator of laplacian type.

Suppose we are given a real-valued function F over a domain $\Omega\subset\mathbf{R}^n$, and consider the gradient map $f=\nabla F\colon \Omega\to\mathbf{R}^n$ with Jacobian matrix

$$f_* = F_{**} \equiv \operatorname{Hess}(F)$$

at each point. Letting $e_1, .., e_n$ denote the canonical basis of $\mathbf{R}^n$, the special Lagrangian condition is that

$$\begin{cases} \alpha((e_1+if_* e_1)\wedge\ldots\wedge(e_n+if_* e_n))>0 \\ \beta((e_1+if_* e_1)\wedge\ldots\wedge(e_n+if_* e_n))=0 \end{cases}$$

which leads immediately to the equations

$$\begin{cases} \operatorname{Re}\{\det_{\mathbf{C}}(I+if_*)\}>0 \\ \operatorname{Im}\{\det_{\mathbf{C}}(I+if_*)\}=0. \end{cases} \tag{2.9}$$

The second line in (2.9) is simply the equation (2.8) derived above. The inequality in (2.9) determines the appropriate orientation.

We now consider a scalar function U on Ω and set $u=\nabla U\colon\Omega\to\mathbf{R}$. As before we write

$$u_* = U_{**} = \operatorname{Hess}(U).$$

Assuming that F is a given solution of (2.9), we want to consider the linearized operator:

$$L_F(U) \equiv \operatorname{Im}\frac{d}{dt}\det_{\mathbf{C}}\{I+i(f_*+tu_*)\}\Big|_{t=0} \tag{2.10}$$

on all such functions U. For simplicity we write

$$A \equiv I+if_*$$

and observe that

$$\det(A+itu_*) = \det(A)\det(I+itA^{-1}u_*).$$

Consequently,

$$\frac{d}{dt}\det(A+itu_*)\Big|_{t=0} = \det(A)\operatorname{tr}(iA^{-1}u_*) = \operatorname{tr}(iA^*u_*)$$

where A^* denotes the transposed matrix of cofactors of A (considered as an $n\times n$ complex matrix). Observe now since u_* a real $n\times n$ matrix, we have that $\operatorname{Im}(\operatorname{tr}(iA^*u_*))=\operatorname{tr}(\operatorname{Im}(iA^*)u_*)=\operatorname{tr}(\operatorname{Re}(A^*)u_*)$. Hence, the linearization can be written as

$$L_F(U) = \operatorname{tr}\{\operatorname{Re}(A^*)\cdot U_{**}\} \tag{2.11}$$

and the inequality from (2.9) is reexpressed as

$$\det(A) > 0. \tag{2.12}$$

Observe now that L_F is elliptic if and only if the matrix $\operatorname{Re}(A^*)$ is positive definite. However, after an appropriate orthogonal change of basis the symmetric matrix f_* becomes diagonal and we can write

$$A = \begin{pmatrix} 1+i\lambda_1 & & 0 \\ & \ddots & \\ 0 & & 1+i\lambda_n \end{pmatrix}.$$

Then

$$A^* = \begin{pmatrix} \frac{1}{1+i\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{1+i\lambda_n} \end{pmatrix} \det A,$$

and by (2.12) we have that

$$\operatorname{Re}(A^*) = \begin{pmatrix} \frac{1}{1+\lambda_1^2} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{1+\lambda_n^2} \end{pmatrix} \det A > 0.$$

Thus we have proved the following:

THEOREM 2.13. *The linearization of the special Lagrangian operator at any solution F of the system* (2.9) *is a homogeneous secondorder elliptic operator*

$$L_F(U)=\sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2 U}{\partial x^i \partial x^j}$$

where $((a^{ij}(x)))$ *is a positive definite symmetric matrix at each point.*

In particular we can conclude the following:

COROLLARY 2.14. *Let* $\Omega\subset\mathbf{R}^n$ *be a bounded, strictly convex domain with smooth boundary, and let* $F_0\in C^\omega(\Omega)\cap C^2(\bar{\Omega})$ *be a solution of the special Lagrangian system* (2.9). *Denote by* $\varphi_0=F_0|_{\partial\Omega}$ *the boundary values of* F_0. *Then there is an open neighborhood* $\mathcal{U}$ *of* φ_0 *in* $C^2(\partial\Omega)$ *such that for each* $\varphi\in\mathcal{U}$ *there exists a solution* $F\in C^\omega(\Omega)\cap C^2(\Omega)$ *of* (2.9) *with* $F|_{\partial\Omega}=\varphi$.

More succinctly this says that the Dirichlet problem is solvable in a C^2 neighborhood of any solution. This corollary follows from standard Implicit function theorem techniques together with the interior regularity theory of Morrey [M] and the boundary regularity theory of Allard [*A*].

Note that the function $F\equiv 0$ is always a solution of (2.9) over any domain. Hence, we have established the existence of enormous families of special Lagrangian submanifolds in each $\mathbf{C}^n$.

III.2.C. The implicit formulation

We recall the following classical fact.

LEMMA 2.15. *Suppose that* $f_1,\dots,f_n$ *are smooth real valued functions on an open set* $\Omega\subset\mathbf{C}^n$ *and suppose that* $df_1,\dots,df_n$ *are linearly independent at points of* $M\equiv\{z\in\Omega: f_1(z)=\dots=f_n(z)=0\}$. *Then the submanifold M is Lagrangian if and only if all the Poisson brackets*

$$\{f_j,f_k\}\equiv\sum_{l=1}^{n}\left(\frac{\partial f_j}{\partial x_l}\frac{\partial f_k}{\partial y_l}-\frac{\partial f_j}{\partial y_l}\frac{\partial f_k}{\partial x_l}\right)\equiv 2i\sum_{l=1}^{n}\left(\frac{\partial f_j}{\partial \bar{z}_l}\frac{\partial f_k}{\partial z_l}-\frac{\partial f_j}{\partial z_l}\frac{\partial f_k}{\partial \bar{z}_l}\right)$$

vanish on M.

The proof is immediate from either the definition (1.2) or (1.2)′ with $\omega\equiv\Sigma_{k=1}^{n}dx_k\wedge dy_k$.

Suppose M is Lagrangian and described implicitly as in Lemma 2.15. Then the normal n-plane to M is spanned (over $\mathbf{R}$) by $\partial f_k/\partial x + i(\partial f_k/\partial y)$, $k=1, \ldots, n$. The operator J sends normal vectors to tangent vectors, and hence the vectors

$$-\frac{\partial f_1}{\partial y}+i\frac{\partial f_1}{\partial x}, \ldots, -\frac{\partial f_n}{\partial y}+i\frac{\partial f_n}{\partial x}$$

span the tangent space to M. The complex $n\times n$ matrix $((2i\partial f_j/\partial \bar{z}_i))$ sends $e_1, \ldots, e_n$ into the above basis for the tangent space for M. Thus the next theorem is an immediate consequence of Corollary 1.11 and Lemma 2.15 above.

THEOREM 2.16. *Suppose* $M \equiv \{z \in \Omega : f_1(z) = \ldots = f_n(z) = 0\}$ *is an implicitly described Lagrangian submanifold of* $\Omega^{\text{open}} \subset \mathbf{C}^n$. *Then M (with the correct orientation) is special Lagrangian if and only if*

(1) $\operatorname{Im}\{\det_{\mathbf{C}}((\partial f_i/\partial \bar{z}_j))\} = 0$ *on M for n even*

or

(2) $\operatorname{Re}\{\det_{\mathbf{C}}((\partial f_i/\partial \bar{z}_j))\} = 0$ *on M for n odd.*

III.2.D. A note

It is interesting that the special Lagrangian submanifolds are just the Lagrangian submanifolds which are minimal. That is;

PROPOSITION 2.17. *A connected submanifold* $M \subset \mathbf{R}^{2n} \cong \mathbf{C}^n$ *is both Lagrangian and stationary if and only if M is special Lagrangian with respect to one of the calibrations* $\alpha_\theta \equiv \operatorname{Re}\{e^{i\theta} dz\}$.

Proof. Of course any special Lagrangian submanifold is minimal. Conversely given a Lagrangian submanifold M we consider the function $\theta : M \to \mathbf{R}/2\pi\mathbf{Z}$ defined by the relations (see Proposition 1.14):

$$dz(\mathbf{M}_p) = e^{i\theta(p)} \tag{2.18}$$

a straightforward calculation shows that for any tangent vector V to M

$$V(\theta) = -\langle H, JV\rangle \tag{2.19}$$

where H is the mean curvature vector of M in $\mathbf{R}^{2n}$ and where J is the almost complex structure on $\mathbf{C}^n$. It follows that M is minimal if and only if θ is constant.

III.3. Examples of special Lagrangian submanifolds

In this section we present a large collection of specific special Lagrangian subvarieties of $\mathbf{C}^n$. The collection includes many new area-minimizing cones. One of our basic constructions is a specialization of the fact that for any submanifold $M^p \subset \mathbf{R}^n$, the normal bundle $N^*(M^p) \subset T^*(\mathbf{R}^n) \cong \mathbf{R}^n \times \mathbf{R}^n \cong \mathbf{C}^n$ is Lagrangian.

III.3.A. Examples invariant under the maximal torus in SU_n

Our first set of examples is invariant under the subgroup $\mathbf{T}^{n-1} = \{\mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}): \theta_1 + \ldots + \theta_n = 0\}$ in SU_n.

THEOREM 3.1. *Let M_c denote the locus in $\mathbf{C}^n$ of the equations:*

$$|z_1|^2 - |z_j|^2 = c_j, \quad j=2, \ldots, n \tag{3.2}$$

and

$$\begin{cases} \mathrm{Re}(z_1 \ldots z_n) = c_1 & \text{if } n \text{ is even,} \\ \mathrm{Im}(z_1 \ldots z_n) = c_1 & \text{if } n \text{ is odd.} \end{cases} \tag{3.3}$$

Then M_c (with the correct orientation) is a special Lagrangian submanifold of $\mathbf{C}^n$.

Remark. This theorem provides us with examples of special Lagrangian cones which are not real analytic, thus providing the simplest examples of absolutely area minimizing cones which are not real analytic.

Suppose n is odd and choose all the constants $c_1, \ldots, c_n$ to be zero. The cone M is the union of two cones M^+ and M^- with vertices at the origin through the $n-1$ dimensional tori T^+ and T^- respectively where

$$T^+ \equiv \{(e^{i\theta_1}, \ldots, e^{i\theta_n}): \theta_1 + \ldots + \theta_n = 0\}$$

$$T^- \equiv \{(e^{i\theta_1}, \ldots, e^{i\theta_n}): \theta_1 + \ldots + \theta_n = \pi\}.$$

Note that T^+ and T^- are disjoint and that $-T^+ = T^-$ so neither M^+ nor M^- is real analytic. However, both the cones M^+ and M^- are special Lagrangian.

Proof. Using the formula

$$\{f, g\} = 2i \sum_{k=1}^{n} \left(\frac{\partial f}{\partial \bar{z}_k} \frac{\partial g}{\partial z_k} - \frac{\partial f}{\partial z_k} \frac{\partial g}{\partial \bar{z}_k} \right) \tag{3.5}$$

one computes that for any pair of the n-functions $f_1(z), \ldots, f_n(z)$ defined by (3.2) and (3.3) the Poisson bracket is zero. Thus by Lemma 2.15, M_c is Lagrangian. If n is even then, with $f_1(z)=2 \operatorname{Re} z_1 \ldots z_n - c_1$,

$$\left(\left(\frac{\partial f_i}{\partial \bar{z}_j}\right)\right) = \begin{pmatrix} \bar{z}_1 \ldots \bar{z}_n/\bar{z}_1 & \ldots & \ldots & \ldots & \bar{z}_1 \ldots \bar{z}_n/\bar{z}_n \\ z_1 & -z_2 & & & \\ z_1 & & -z_3 & & \\ \vdots & & & \ddots & \\ z_1 & & & & -z_n \end{pmatrix}.$$

Expanding in the first row shows that $\det_{\mathbf{C}}((\partial f_i/\partial \bar{z}_j))$ is a sum of terms of the form $\pm|z_1 \ldots z_n|^2/|z_k|^2$. Therefore, $\operatorname{Im}\{\det_{\mathbf{C}}((\partial f_i/\partial \bar{z}_j))\}=0$ and so M_c is special Lagrangian by Theorem 2.16. If n is odd the proof is similar.

Of course the above proof obscures the idea behind the construction of the examples M_c. They were obtained as follows. The maximal torus $\mathbf{T}^{n-1}$ has a commutative Lie algebra generated by $\partial/\partial\theta_1 - \partial/\partial\theta_j, j=2, \ldots, n$. Now in $\mathbf{C}, \partial/\partial\theta \equiv -y(\partial/\partial x)+x(\partial/\partial y)$, and $\partial/\partial\theta \lrcorner dx \wedge dy = -(x\,dx+y\,dy)$. Therefore $\partial/\partial\theta$ is a Hamiltonian vector field with Hamiltonian function $H(z) \equiv \frac{1}{2}|z|^2$, since $dH = x\,dx+y\,dy$.

In particular, we have the $n-1$ Hamiltonian functions $f_j(z)=\frac{1}{2}(|z_1|^2-|z_j|^2)$ with associated Hamiltonian vector fields $\partial/\partial\theta_1-\partial/\partial\theta_j$ on $\mathbf{C}^n$ ($j=2, \ldots, n$). Since these vector fields commute, the Poisson brackets $\{f_i, f_j\}$ vanish. The functions $f_2, \ldots, f_n$ must be constant on M if M is to be Lagrangian and invariant under $\mathbf{T}^{n-1}$. To construct the last function $f_1(z)$, assume n is even. Then f_1 must satisfy $\operatorname{Im}\{\det_{\mathbf{C}}((\partial f_i/\partial \bar{z}_j))\}=0$ by Theorem 2.16. But as noted above, after substituting for $f_2, \ldots, f_n$, $\det_{\mathbf{C}}((\partial f_i/\partial \bar{z}_j))$ is a sum of terms of the form

$$\pm \bar{z}_k \frac{\partial f_1}{\partial \bar{z}_k} z_1 \ldots z_n/|z_k|^2.$$

The equations $\{f_1, f_j\}=0$ for $j=2, \ldots, n$ become

$$\bar{z}_1 \frac{\partial f_1}{\partial \bar{z}_1} - \bar{z}_j \frac{\partial f_1}{\partial \bar{z}_j} = 0, \quad j=2, \ldots, n.$$

Hence, $\operatorname{Im}\{\det_{\mathbf{C}}((\partial f_i/\partial \bar{z}_j))\}=0$ reduces to

$$\operatorname{Im}\left(\bar{z}_1 \frac{\partial f_1}{\partial \bar{z}_1} \cdot z_1 \ldots z_n\right) = 0.$$

These equations are solved by $f_1(z) \equiv \operatorname{Re}(z_1 \ldots z_n)$. The case n odd is similar.

III.3.B. Examples invariant under SO_n

The second family of examples are invariant under the diagonal action of SO_n on $\mathbf{C}^n \cong \mathbf{R}^n \times \mathbf{R}^n$. In order for an n-dimensional submanifold M^n of $\mathbf{C}^n$ to be SO_n invariant it must be the orbit of a curve Γ in the positive quadrant of the first complex axis plane $\mathbf{C}$. That is $M \equiv \{(x, y) \in \mathbf{C}^n: |x|y = |y|x \text{ and } (|x|, |y|) \in \Gamma\}$. Let r denote $|x|$ and ϱ denote $|y|$. Suppose Γ is the graph of a function $\varrho(r)$. Choose $\varphi(r)$ to be a primitive of $\varrho(r)$. Then $\nabla\varphi = \varrho(r)\nabla r = \varrho(r)(x/r)$. Thus M is the graph of $\nabla\varphi$ and is consequently always Lagrangian.

THEOREM 3.5. *Let*

$$M_c \equiv \{(x, y) \in \mathbf{C}^n: |x|y = |y|x \quad \textit{and} \quad \operatorname{Im}(|x| + i|y|)^n = c\}.$$

Then M_c (with the correct orientation) is a special Lagrangian submanifold of $\mathbf{C}^n$.

Proof. As noted above M is Lagrangian. Generically M is the graph of $F(x) \equiv \varrho(|x|)(x/|x|)$. The differential F_* of this map from $\mathbf{R}^n$ to $\mathbf{R}^n$ is given by the matrix $F_* = (h_{ij})$ where

$$h_{ij} = \frac{\partial}{\partial x_i}\left(\frac{\varrho(|x|)}{|x|} x_j\right) = \frac{\varrho(|x|)}{|x|}\delta_{ij} + \frac{d}{dr}\left(\frac{\varrho(r)}{r}\right)\frac{x_i x_j}{|x|}.$$

Thus the linear map $F_*: \mathbf{R}^n \to \mathbf{R}^n$ has the eigenvector x with eigenvalue

$$\frac{\varrho(r)}{r} + r\frac{d}{dr}\left(\frac{\varrho(r)}{r}\right) = \frac{d\varrho}{dr}.$$

Moreover, the hyperplane perpendicular to x is an eigenspace with eigenvalue $\varrho(r)/r$ of multiplicity $n-1$. Let $K: \mathbf{C}^n \to \mathbf{C}^n$ denote the complex linear map defined by mapping e_j to $e_j + iF_*(e_j)$, $j = 1, \ldots, n$. That is, let $K = I + iF_*$. Then K maps $\mathbf{R}^n \subset \mathbf{C}^n$ onto the graph of F_*. Hence the graph of $F*$ (oriented correctly) is special Lagrangian if and only if

$$\operatorname{Im}\{\det_{\mathbf{C}} K\} = 0.$$

Since $\lambda_1 = d\varrho/dr$, $\lambda_j = \varrho(r)/r$, $j = 2, \ldots, n$ are the eigenvalues of F_*,

$$\det_{\mathbf{C}} K = \prod_{j=1}^{n}(1 + i\lambda_j) = \left(1 + i\frac{\varrho(r)}{r}\right)^{n-1}\left(1 + i\frac{d\varrho}{dr}\right).$$

Therefore

$$\operatorname{Im}\{\det_{\mathbf{C}} K\, dr\} = \frac{1}{r^{n-1}}\operatorname{Im}\{(r + i\varrho(r))^{n-1}(dr + id\varrho)\} = 0$$

if and only if

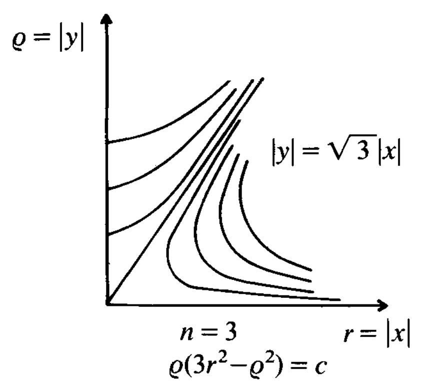
$$\operatorname{Im}[(r+i\varrho)^{n-1}(dr+id\varrho)]=0. \tag{3.6}$$

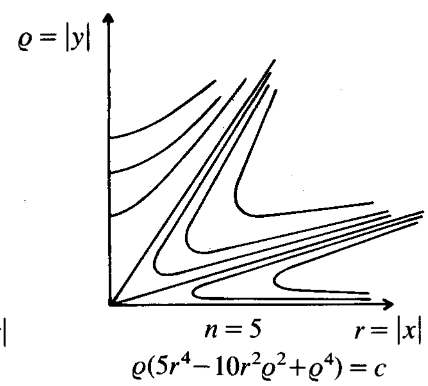
This is an exact equation with potential

$$\operatorname{Im}\left(\frac{1}{n}(r+i\varrho)^n\right),$$

and the proof is complete.

Note that for $c\neq 0$, each component of the manifold M_c is diffeomorphic to $\mathbf{R}\times S^{n-1}$. The variety M_0 is a union of linear subspaces.





III.3.C. Special Lagrangian normal bundles

It is a classical fact that for any submanifold M of a manifold X, the canonical embedding of it's normal bundle $N(M)$ into T^*X is Lagrangian with respect to the natural symplectic structure on T^*X. In this section we shall determine the p-dimensional submanifolds M of $\mathbf{R}^n$ whose normal bundles $N(M)$ are special Lagrangian in $\mathbf{R}^n\oplus\mathbf{R}^n$.

We define the embedding

$$\psi: N(M)\to\mathbf{R}^n\oplus\mathbf{R}^n \tag{3.7}$$

by setting $\psi(\nu_x)=(x,\nu_x)$ where, in the second factor, ν_x is regarded as a vector based at the origin. We shall compute the tangent space to this embedding at ν_{x_0}. Near x_0 we choose an orthonormal tangent frame field $e_1,\dots,e_p$ and a similar normal frame field $\nu_1,\dots,\nu_q$, $p+q=n$, so that $(e_1,\dots,\nu_q)$ is positively oriented. For convenience we assume that the fields ν_k are chosen so that $(\nabla\nu_k)^N_{x_0}=0$ where $(\ )^N$ denotes orthogonal projection onto the normal space $N_{x_0}(M)$. We recall that the *second fundamental form* of M in the normal direction ν is given by

$$A^\nu(V) = (\nabla_V \tilde{\nu})^T \tag{3.8}$$

where $\tilde{\nu}$ is any extension of ν to a local normal field and where $(\ )^T=1-(\ )^N$ is orthogonal projection onto $T_{x_0}M$.

With respect to coordinates (x, t) on $N(M)$, where x is a parameterization of M near x_0 and $t=(t_1, \ldots, t_q) \in \mathbf{R}^q$, the mapping (3.7) can be expressed as

$$\psi(x, t) = (x, \sum_j t_j \nu_j(x)).$$

The tangent space to this embedding at $\nu(x_0)=\Sigma\, c_j \nu_j(x_0)$ is spanned by the vectors

$$E_j \equiv \psi_*(e_j) = (e_j, A^\nu(e_j)), \quad j = 1, \ldots, p$$
$$N_j \equiv \psi_*(\partial/\partial t_j) = (0, \nu_j), \quad j = 1, \ldots, q$$

where the fields e_j, ν_j are evaluated at x_0.

We now consider the complex structure J defined on $\mathbf{C}^n \equiv \mathbf{R}^n \oplus \mathbf{R}^n$ by setting $J(X, Y)=(-Y, X)$. We clearly have that $\langle JN_j, N_k\rangle = \langle JN_j, E_l\rangle = -\langle N_j, JE_l\rangle = 0$ for all j, k, l. Moreover, $\langle JE_j, E_k\rangle = \langle e_k, A^\nu(e_j)\rangle - \langle A^\nu(e_k), e_j\rangle = 0$ from the symmetry of the second fundamental form. Hence, $\psi(N(M))$ is a Lagrangian submanifold of $\mathbf{C}^n \equiv \mathbf{R}^n \oplus \mathbf{R}^n$.

We are now interested in the conditions under which this manifold is special Lagrangian for some choice of special Lagrangian calibration, $\varphi = \mathrm{Re}\,\{e^{i\theta} dz_1 \wedge \ldots \wedge dz_n\}$. By performing an SO_n change of coordinates on $\mathbf{R}^n$ we may assume that at x_0 the vectors $e_1, \ldots, e_p, \nu_1, \ldots, \nu_q$ give the standard basis of $\mathbf{R}^n$. This change of coordinates is applied simultaneously to the two factors in $\mathbf{R}^n \oplus \mathbf{R}^n$. We denote the standard coordinates in $\mathbf{R}^n \oplus \mathbf{R}^n$ by $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ and set $z_k = x_k + iy_k$. Finally, without loss of generality we can assume that the vectors $e_1, \ldots, e_p$ were chosen to diagonalize the symmetric form A^ν at x_0, i.e., we assume that $A^\nu(e_k) = \lambda_k e_k$ for $k=1, \ldots, p$, at x_0. Consequently, the oriented tangent plane ζ to the embedded normal bundle at ν_{x_0} is given by $\zeta = E_1 \wedge \ldots \wedge E_p \wedge N_1 \wedge \ldots \wedge N_q = (e_1, \lambda_1 e_1) \wedge \ldots \wedge (e_p, \lambda_1 e_p) \wedge (0, \nu_1) \wedge \ldots \wedge (0, \nu_q)$. It follows that

$$(dz_1 \wedge \ldots \wedge dz_n)(\zeta) = i^q \prod_{k=1}^{p} (1 + i\lambda_k). \tag{3.9}$$

In particular, we have $|dz_1 \wedge \ldots \wedge dz_n(\zeta)| = \|\zeta\|$ which gives a second proof that ζ is Lagrangian. We now choose the calibration

$$\varphi = \mathrm{Re}\,\{i^{-q} dz_1 \wedge \ldots \wedge dz_n\} \tag{3.10}$$

and obtain the following result.

THEOREM 3.11. *Let M be any submanifold of dimension p immersed in $\mathbf{R}^n$, and let $\tilde{N}^n = \psi[N(M)]$ be the canonical immersion of its normal bundle in $\mathbf{R}^n \oplus \mathbf{R}^n = \mathbf{C}^n$ given by* (3.7). *Then $\tilde{N}^n$ is special Lagrangian (with respect to the calibration* (3.10)*) if and only if all the invariants of odd order of the second fundamental form at each normal vector to M vanish, i.e.,*

$$\sigma_{2k-1}(A^\nu) = 0 \tag{3.12}$$

for all $k = 1, \ldots, [(p+1)/2]$ and for all ν.

Remark 3.13. This condition on M is equivalent to the condition that for each normal vector ν, the set of eigenvalues of A^ν is invariant under multiplication by -1, i.e., it is of the form

$$(\lambda_1, \ldots, \lambda_p) = (a, -a, b, -b, \ldots, c, -c, 0, 0, \ldots, 0). \tag{3.14}$$

Definition 3.15. Any submanifold of a riemannian manifold whose second fundamental form A satisfies condition (3.12) will be called *austere*.

Note. The total space of the normal bundle to any (not necessarily orientable) submanifold of $\mathbf{R}^n$ inherits an orientation from $\mathbf{R}^n$.

Proof. From (3.9) we see that $\tilde{N}$ will be special Lagrangian if and only if $\Sigma_k (-1)^k \sigma_{2k-1}(\lambda_1, \ldots, \lambda_p) = 0$ and $\Sigma_{k \geq 0} (-1)^k \sigma_{2k}(\lambda_1, \ldots, \lambda_p) > 0$ at each normal vector ν. However, for each real number t, the eigenvalues of $A^{t\nu}$ are $t\lambda_1, \ldots, t\lambda_p$. Hence the first condition is equivalent to the vanishing of the polynomial. $\Sigma_k (-1)^k t^{2k-1} \sigma_{2k-1}(\lambda_1, \ldots, \lambda_p)$, which occurs if and only if $\sigma_{2k-1}(\lambda_1, \ldots, \lambda_p) = 0$ for $k = 1, 3, \ldots, [(p+1)/2]$. This implies condition (3.14) which immediately implies, by (3.9), that $i^{-q}(dz_1 \wedge \ldots \wedge dz_p)(\zeta) = \|\zeta\|^2$. This completes the proof.

Theorem 3.11 leads to a large collection of interesting examples of special Lagrangian varieties. Note that when $p = 2$, the only condition is that $\mathrm{trace}\,(A^\nu) \equiv 0$, so that M^2 is austere if and only if M^2 is minimal.

COROLLARY 3.16. *Let M^2 be any minimal surface properly immersed in $\mathbf{R}^n$. Then the canonical embedding* (3.7) *of its normal bundle in $\mathbf{R}^{2n}$ is an absolutely mass minimizing current in $\mathbf{R}^{2n}$.*

This corollary can be used to construct minimizing 3-folds in $\mathbf{R}^6$ with interesting singularities. Also by choosing M^2 to be a triply periodic minimal surface in $\mathbf{R}^3$, one obtains a minimizing 3-manifold in $\mathbf{R}^6$ invariant under a 3-dimensional lattice of translations.

Another large set of austere submanifolds are the submanifolds of $\mathbf{R}^{2k}$ which are complex analytic with respect to some complex structure on $\mathbf{R}^{2k}$. However, the area minimizing properties of the normal variety are not new in this case.

A particularly nice set of examples comes from considering minimal submanifolds of the unit sphere $S^{n-1}\subset\mathbf{R}^n$. If $M^{p-1}\subset S^{n-1}$ is a minimal submanifold, then the *cone on* M^{p-1},

$$C(M^{p-1})=\{t\cdot x\in\mathbf{R}^n\colon x\in M^{p-1}\quad\text{and}\quad t\in\mathbf{R}\}$$

is a minimal variety in $\mathbf{R}^n$. The normal vectors to M^{p-1} in S^{n-1} at x are exactly the normal vectors to $C(M^{p-1})$ at tx for $t\neq 0$. Furthermore, if the second fundamental form A^ν of M^{p-1} in S^{n-1} has eigenvalues $\lambda_1,\dots,\lambda_{p-1}$, the second fundamental form $\tilde{A}^\nu$ of $C(M^{p-1})$ at tx has eigenvalues $t\lambda_1,\dots,t\lambda_{p-1},0$. It follows that M^{p-1} is an austere submanifold of S^{n-1} if and only if its cone is an austere submanifold of $\mathbf{R}^n$. Hence, we have the following.

THEOREM 3.17. *Let M be any compact austere submanifold of S^{n-1}. Then the "twisted normal cone"*

$$\mathscr{C}\mathscr{N}(M)\equiv\{(tx,s\nu(x))\in\mathbf{R}^n\oplus\mathbf{R}^n\colon x\in M\quad\textit{and}\quad t,s\in\mathbf{R}\},$$

(where $\nu(x)$ ranges over all unit vectors normal to M in S^{n-1} at x) is an n-dimensional cone of least mass in $\mathbf{R}^{2n}$.

Note that $\mathscr{C}\mathscr{N}(M)$ represents a natural "compactification" of the normal variety to $C(M)$. That is, the closure of this normal variety is obtained by adding the cone on the *Gauss image*: $M^*=\{(0,\nu)\in\mathbf{R}^n\oplus\mathbf{R}^n\colon\nu$ a unit normal vector to M in $S^{n-1}\}$ as the "normal space" to the vertex of $C(M)$. The Gauss image M^* can be fruitfully considered as a spherical dual. We note for example that $M^{**}=M$. Furthermore, one can show that (at regular points) M is austere if and only if M^* is austere.

Of course, any minimal surface (of dimension 2) in S^{n-1} is austere. There are two particularly nice cases of this sort.

Example 3.18. Let $\Sigma \subset S^3$ be a compact orientable minimal surface of genus g (cf. $[L_1]$). The dual Σ^* is known classically as the polar surface. The zeros of the function $1-K$, where K is Gauss curvature, form a divisor of degree $4g-4$ on Σ (cf. $[L_1$ Proposition 1.5$]$). These points correspond precisely to the branch points of Σ^*. To each point $x \in \Sigma$, let $x^* = \nu(x)$ denote the unit normal to Σ at x.

COROLLARY 3.19. *Let $\Sigma \subset S^3$ be any compact surface minimally immersed in S^3. Then*

$$\mathscr{C}\mathscr{N}(\Sigma) = \{(tx, sx^*) \in \mathbf{R}^8 : x \in \Sigma, s, t \in \mathbf{R}\}$$

is a minimizing cone of dimension 4 *in* $\mathbf{R}^8$.

The above corollary holds even when Σ is non-orientable. Such examples exist for all topological types but $\mathbf{P}^2(\mathbf{R})$ which is prohibited. The resulting cones have a rather complicated singular structure. Using the examples from $[L_1]$, however, one also obtains cones with large finite symmetry groups.

If Σ is antipodally invariant, then $\mathscr{C}\mathscr{N}(\Sigma)$ is a cone on $M^3 \subset S^7$ where M^3 is the image of a singular map of $\Sigma \times S^1/\mathbf{Z}_2$. ($\mathbf{Z}_2$ is generated by $(-1, -1)$.)

Example 3.20. Another interesting set of minimal surfaces are those of constant positive curvature, given as follows. Let $\{\varphi_1, \ldots, \varphi_{N_k}\}$ be a L^2 orthonormal basis for the spherical harmonics of degree k on S^2. Then (cf. [DW]) the immersion $\Phi_k = (\varphi_1, \ldots, \varphi_{N_k}): S^2 \to S^{N_k - 1}$ is a minimal immersion which commutes with the obvious actions of SO_3. This homogeneity implies that the image has constant Gauss curvature. If k is even, the image is an embedded projective plane. When $k=2$, it is the well known Veronese surface $V \cong \mathbf{P}^2(\mathbf{R}) \subset S^4$. The dual is SO_3-invariant and can be written as $V^* \cong SO_3/\mathbf{Z}_2 \oplus \mathbf{Z}_2$. In spite of the complicated nature of V and V^*, the twisted normal cone is topologically euclidean.

COROLLARY 3.21. *Let V denote the Veronese surface in S^4. Then $\mathscr{C}\mathscr{N}(V)$ is a non-trivial mass-minimizing cone in $\mathbf{R}^{10}$ which is homeomorphic to $\mathbf{R}^5$.*

Proof. The orbit space S^4/SO_3 is homeomorphic to $[-1, 1]$ where the endpoints correspond to V and $-V$. ($V \cap (-V) = \emptyset$.) The remaining orbits are all diffeomorphic to the bundle of unit normal vectors to V in S^4. Of course, each such orbit lies at a fixed distance from V (and $-V$). The orbit corresponding to 0 is exactly V^*. From this information one can easily construct a homeomorphism $\mathscr{CN}(V)\cap S^7 \cong S^4$.

Example 3.22. Consider the minimal submanifold $S^{n-1}\times S^{n-1}=\{(x,y)\in \mathbf{R}^n\times\mathbf{R}^n: |x|^2=|y|^2=\frac{1}{2}\}\subset S^{2n-1}$. This manifold is homogeneous under $SO_n\times SO_n$. The eigenvalues of the second fundamental form at any point are $1,-1,1,-1,\ldots,1,-1$. Hence, this manifold is austere. It follows that

$$\mathscr{CN}(S^{n-1}\times S^{n-1}) = \{(tx, ty, sx, -sy)\in \mathbf{R}^{4n}: |x|=|y|=1 \quad \text{and} \quad s,t\in\mathbf{R}\}$$

is area minimizing. Topologically this is a cone on $S^{n-1}\times S^{n-1}\times S^1/\mathbf{Z}_2$ where $\mathbf{Z}_2$ is generated by $(-1,-1,-1)$.

III.4. Degenerate projections and harmonic gradients

First, in subsection A, we characterize the Lagrangian submanifolds of $T^*\mathbf{R}^*$ which have degenerate projections onto $\mathbf{R}^n$. This places a construction of Hans Lewy [Ly] in an invariant geometric context. In subsection B we discuss the relation of this to his work of harmonic gradient maps.

III.4.A. Degenerate projections

The construction of Lagrangian submanifolds as normal bundles can be somewhat extended. Suppose, that in addition to a p-dimensional submanifold M of $\mathbf{R}^n$, we are given a smooth function h on M. The exterior derivative dh can be used to translate the normal bundle, $N(M)$, in the cotangent bundle $\mathbf{T}^*\mathbf{R}^n|_M$. More precisely, let H denote a smooth extension of h to the ambient space $\mathbf{R}^n$ and define an affine subbundle $A(M,h)$ of $T^*\mathbf{R}^n|_M$ by

$$A_x \equiv N_x(M)+(dH)_x \quad \text{for each } x\in M. \tag{4.1}$$

The exact sequence

$$0\to N_x(M)\to T_x^*\mathbf{R}^n\to T_x^*M\to 0$$

implies that the affine space A_x defined by (4.1) is independent of the extension H of h, and depends only on $(dh)_x\in T_x^*M$.

PROPOSITION 4.2. *The affine bundle $A(M,h)$, constructed as in* (4.1) *above, is a Lagrangian submanifold of $T^*\mathbf{R}^n$.*

Proof. Near $(\mathbf{x},\mathbf{y})\in A$, choose orthonormal coordinates $x\equiv(x',x'')$, $x'\equiv(x_1,\dots,x_p)$ and $y\equiv(y',y'')$ etc. so that the x'-axis is tangent to M at $\mathbf{x}$. Then M can be described explicitly as the graph of $x''=u(x')$. Extend h to a function $H(x')$ depending only on x'. Then A can be parameterized by:

$$(x',y'')\mapsto\left(x',u(x'),-\frac{\partial}{\partial x'}(u(x')\cdot y'')+\frac{\partial H}{\partial x'}(x'),y''\right).$$

Substituting for x'' and y', one finds that $y\cdot dx|_A=dH(x')$, which is d-closed; and hence A is Lagrangian.

The proposition has a converse which provides a structure theorem for Lagrangian submanifolds with degenerate projections.

THEOREM 4.3. *Suppose X is an n-dimensional Lagrangian submanifold of $T^*\mathbf{R}^n$ whose projection $\pi:X\to\mathbf{R}^n$ is degenerate with constant rank p. Then X is an affine subbundle $A(M,h)$* of $T^*\mathbf{R}^n$ *defined as in* (4.1) *with M and h uniquely determined by X.*

Proof. Let M denote $\pi(X)$. By the hypothesis of constant rank p, M is a p-dimensional submanifold of X. Moreover, $T_{\mathbf{x},\mathbf{y}}X=T_{\mathbf{x}}M\times W$ where W is a subspace of $T^*_{\mathbf{x}}\mathbf{R}^n$. In order for X to be Lagrangian we must have $W=(JT_{\mathbf{x}}M)^{\perp}=N_{\mathbf{x}}(M)$, which proves

$$T_{\mathbf{x},\mathbf{y}}X=T_{\mathbf{x}}(M)\times N_{\mathbf{x}}(M).$$

Consequently, if we choose coordinates $x\equiv(x',x'')$ for $\mathbf{R}^n$ with $x'\equiv(x_1,\dots,x_p)$ so that $\partial/\partial x_1,\dots,\partial/\partial x_p$ are tangent to M at $\mathbf{x}$ and let $y=(y',y'')$ denote the corresponding coordinates in $\mathrm{T}^*_x\mathbf{R}^n$, then $\partial/\partial x_1,\dots,\partial/\partial x_p,\partial/\partial y_{p+1},\dots,\partial/\partial y_n$ is a basis for the tangent space to X at $\mathbf{x},\mathbf{y}$. Therefore, X can be expressed as the graph (locally) of a vector valued function $(x''(x',y''),y'(x',y''))$. Moreover, since X is Lagrangian (i.e. $dx\cdot dy|_X\equiv0$) the one form $-y'\cdot dx'+x''\cdot dy''$ is d-closed on M. Consequently this form is exact on M. That is, there exists a generating (or potential) scalar function $F(x',y'')$ with

$$y'(x',y'')\equiv-\frac{\partial F}{\partial x'}\quad\text{and}\quad x''(x',y'')\equiv\frac{\partial F}{\partial y''}. \tag{4.4}$$

Expressing X as a graph over the x',y''-axis yields:

$$X\equiv\left\{\left(x',\frac{\partial F}{\partial y''},-\frac{\partial F}{\partial x'},y''\right)\right\}$$

Since $\pi(X)\equiv M$ and M is the graph of $x''=u(x')$ we must have $\partial F/\partial y''=u(x')$, or

$$F(x', y'')\equiv H(x')+y''\cdot u(x'), \tag{4.5}$$

for some scalar function $H(x')$. This proves:

PROPOSITION 4.6. *Under the hypothesis of Theorem* 4.3 *if X is graphed (locally) over its tangent space $T_{x,y}X\equiv T_x M\times N_x(M)$, the generating function F is affine in the normal variables $y''\in N_x(M)$.*

Now X can be expressed as a graph over the x', y''-axis as follows:

$$X\equiv\left\{\left(x', u(x'), -y''\cdot\frac{\partial u}{\partial x'}+\frac{\partial H}{\partial x'}, y''\right)\right\}. \tag{4.7}$$

Since the normal bundle fiber can be expresesed in coordinates as $N_x(M)=\{(-y''\cdot\partial u/\partial x', y''): y''\in\mathbf{R}^{n-p}\}$ this proves that $\pi^{-1}(x)=N_x(M)+(dH)_x$ as desired.

Remark. We have proved more (locally). Suppose X is a Lagrangian submanifold of $T^*\mathbf{R}^n$ with the property that, first, X can be graphed over the axis n-plane $V\times W$ with $V\subset\mathbf{R}^n$ and $W\subset T_x^*\mathbf{R}^n$ and, second, the generating function is affine in V. Then X is of the form $A(M, h)$.

III.4.B. Harmonic gradients in three variables

If $n=3$ the above considerations are intimately related to the work of Hans Lewy [Ly]. In this case we shall construct special Lagrangian submanifolds M^3 of $\mathbf{C}^3$. Let $z=x+iy$ with $x=(x_1, x_2, x_3)$ and $y=(y_1, y_2, y_3)$. Then given a smooth real-valued function $F(x)$ on an open set $\Omega\subseteq\mathbf{R}^3$, we know from Theorem 2.3 that the graph (correctly oriented) of $\nabla F: \Omega\to\mathbf{R}^3$ is special Lagrangian if and only if

$$\Delta F-\det(\operatorname{Hess} F)\equiv 0. \tag{4.8}$$

Motivated by the discussion above, it is natural to consider the special case where F is affine in x_3. This leads to the following interesting result.

THEOREM 4.9. *Let $F(x_1, x_2, x_3)=h(x_1, x_2)+x_3 u(x_1, x_2)$ where h and u are arbitrary smooth real-valued functions, and let $X=\{(x_1, x_2, x_3, h_1+x_3 u_1, h_2+x_3 u_2, u)\}$ denote the* *graph of* ∇F *in* $\mathbf{R}^3 \oplus \mathbf{R}^3 = \mathbf{C}^3$. *Then X (with the proper orientation) is special Lagrangian if and only if*

$$(1+u_2^2)\,u_{11} - 2u_1\,u_2\,u_{12} + (1+u_1^2)\,u_{22} = 0 \tag{4.10a}$$

and

$$(1+u_2^2)\,h_{11} - 2u_1\,u_2\,h_{12} + (1+u_1^2)\,h_{22} = 0. \tag{4.10b}$$

Remark. The first equation is just the classical minimal surface equation for u and hence is satisfied if and only if the graph of u is a minimal surface S in $\mathbf{R}^3$. The second equation is satisfied if and only if h, considered as a function on the surface S with the induced riemannian metric, is harmonic. If $h \equiv 0$ then Theorem 4.9 is essentially equivalent to Theorem 3.11 with $n=3$.

Proof. Note that $\Delta F = \Delta h + x_3\,\Delta u$ and that

$$\operatorname{Hess} F = \begin{pmatrix} h_{11}+x_3\,u_{11} & h_{12}+x_3\,u_{12} & u_1 \\ h_{21}+x_3\,u_{21} & h_{22}+x_3\,u_{22} & u_2 \\ u_1 & u_2 & 0 \end{pmatrix}$$

A direct calculation shows that $\Delta F - \det(\operatorname{Hess} F) = ax_3 + b$ where a and b are the expressions given in (4.10a) and (4.10b) respectively. This completes the proof.

We continue the discussion of the case $n=3$. Note that X, which is the graph of ∇F, where $F = h + x_3\,u$, has a degenerate projection onto the axis 3-plane with coordinates (x_1, x_2, y_3). This projection is the minimal surface S which is the graph of $y_3 \equiv u(x_1, x_2)$. Note that the orthogonal 3-plane with coordinates y_1, y_2, x_3 is special Lagrangian. Suppose that the projection of X onto this 3-plane is non degenerate and let $G(y_1, y_2, x_3)$ denote the corresponding potential function. The image of $\nabla G\colon \mathbf{R}^3 \to \mathbf{R}^3$ is just the two dimensional minimal surface S determined by $y_3 = u(x_1, x_2)$, and hence $\det(\operatorname{Hess} G)$, the Jacobian determinant of this map ∇G, must vanish identically. Since G must satisfy the special Lagrangian differential equation (4.8), G is harmonic.

Now we turn attention to the converse. A special Lagrangian submanifold X of $\mathbf{C}^3 \equiv \mathbf{R}^3 \oplus i\mathbf{R}^3$ can be expressed locally, after perhaps an SU_3-change of coordinates, as a graph of ∇G, over the axis Lagrangian 3-plane with coordinates (y_1, y_2, x_3). The potential G must satisfy the special Lagrangian differential equation $\Delta G = \det(\operatorname{Hess} G)$. Suppose now that G satisfies the degeneracy condition

$$\operatorname{rank}(\operatorname{Hess} G) \leqslant 2. \tag{4.11}$$

Then the special Lagrangian differential equation becomes the classical equation

$$\Delta G = 0, \tag{4.12}$$

so that G must be harmonic in 3-variables. Under the assumptions (4.11) and (4.12) we have that rank(Hess G) is either 0 or 2 at each point. Assuming it to be 2 everywhere, the image of the map ∇G is a surface S in the (x_1, x_2, y_3)-axis 3-plane. It was originally discovered by Hans Lewy that, in fact, S is a *minimal surface*. He went on to show that S together with a uniquely determined harmonic function h on S, actually characterizes G; that is G could be reconstructed from h and S. This was the first step in his deep work on harmonic gradients in 3-variables. Our discussion above implies that X is the translate of the normal bundle to S by the exterior derivative of a function h on S; and that upon graphing X over the (x_1, x_2, x_3)-axis the generating function $F((x_1, x_2, x_3)$ is of the form $h + x_3 u$. As noted above the special Lagrangian equation for F is equivalent to u satisfying the minimal surface equation and h being a harmonic function on $S \equiv$graph u. Thus the above discussion illustrates that some of Hans Lewy's observations are a natural part of special Lagrangian geometry in $\mathbf{C}^3$.

In summary, this proves the following.

THEOREM 4.13. *Let $X \subset \mathbf{R}^6$ be a connected special Lagrangian submanifold. Then the following are equivalent.*

(i) *X has a degenerate orthogonal projection into $L^\perp$ for some special Lagrangian plane L. This degenerate image is necessarily a branched minimal surface S or a point, and X is necessarily the translate of the normal bundle to S by the exterior derivative of a harmonic function h on S.*

(ii) *X can be locally represented as the graph of a harmonic gradient map $\nabla G: L \to L^\perp$ with degenerate hessian, where L is the same special Lagrangian plane as in* (i).

(iii) *X can be locally represented as the graph of a gradient map $\nabla F: \tilde{L} \to \tilde{L}^\perp$, where $\tilde{L}$ is special Lagrangian and F is as in Theorem* 4.9 *(i.e., F is affine in one of the variables).*

III.5. Boundaries of special Lagrangian submanifolds

The purpose of this section is to give a local characterization of the boundaries of special Lagrangian submanifolds of $\mathbf{C}^n$. As a consequence we will have a local existence theorem which shows that special Lagrangian geometry is quite rich.

The key concept here is that of an *isotropic submanifold* of $\mathbf{C}^n$. This is a submanifold $i: M \to \mathbf{C}^n$ such that $i^*\omega = 0$ where

$$\omega \equiv \frac{i}{2} \sum_k dz_k \wedge d\bar{z}_k$$

is the standard Kähler or symplectic form on $\mathbf{C}^n \cong T^* \mathbf{R}^n$. Of course M is isotropic if and only if each of its tangent spaces is isotropic. Let $I(\omega)$ denote the differential system generated by ω. In terms of differential systems, the isotropic subspaces are just the integral elements of $I(\omega)$ and the isotropic submanifolds are just the integral submanifolds.

Given tangent vectors u, v we say that u, v are *skew orthogonal*, denoted $u \prec v$, if $\omega(u \wedge v) = 0$. Also let $W^{\prec}$ denote the subspace of tangent vectors skew orthogonal to a subspace W. The condition

$$i_W^* \omega = 0 \tag{5.1}$$

that a subspace W be isotropic can be reformulated as

$$W \subset W^{\prec} \quad \text{(i.e. } u \prec v \text{ for all } u, v \in W). \tag{5.2}$$

Of course if W has dimension n then W is isotropic simply means W is Lagrangian. It is a standard fact that

$$W \text{ is isotropic if and only if } W \subset L \text{ where } L \text{ is Lagrangian.} \tag{5.3}$$

In terms of the standard inner product on $\mathbf{C}^n$ we have that

$$W \text{ is isotropic if } Ju \perp W \text{ for each } u \in W. \tag{5.4}$$

A subspace W is *coisotropic* if $W^{\prec}$ is isotropic, that is if $W^{\prec} \subset W$. The standard existence theorem in symplectic geometry states the following.

Suppose N is a k-dimensional isotropic submanifold contained in a coisotropic submanifold P of codimension p. Assume that P is chosen transverse to $(T_x N)^{\prec}$. Then there exists a unique isotropic submanifold M of dimension $k+p$ with $N \subset M \subset P$.

This result is valid in both the C^∞ and the real analytic setting. The proof, from the point of view of differential systems is obtained by noting that the system $I(\omega)|_P$ (restricted to the submanifold P) has Cauchy characteristics $(T_x P)^{\prec}$. Repeated application of this result implies the existence of lots of isotropic submanifolds of dimension $0 \leq k \leq n$.

We now wish to locally characterize the boundaries of special Lagrangian submanifolds. Recall that these submanifolds are much more rigid than the Lagrangian ones, since they are, in particular, minimal submanifolds (and hence real analytic). Of course, the boundary of a Lagrangian submanifold must be an isotropic submanifold. For special Lagrangian submanifolds, the converse is true, at least locally in the real analytic case.

THEOREM 5.5. *Suppose N is a real-analytic isotropic $(n-1)$-dimensional submanifold of $\mathbf{C}^n$. Then there exists a unique special Lagrangian submanifold M containing N.*

Proof. Consider the ideal $I(\omega, \beta)$ generated by the Kähler/symplectic form ω and the n-form $\beta \equiv \operatorname{Im} dz = \operatorname{Im} dz_1 \wedge \ldots \wedge dz_n$. The integral elements of this differential system consist of the isotropic planes of dimension $k<n$ and of the special Lagrangian n-planes. Given an isotropic $(n-1)$-dimensional plane W there exists a unique special Lagrangian plane containing W. To see this we may assume that W is spanned by $e_1, \ldots, e_{n-1}$ where $e_1, \ldots, e_n$ is the standard basis for $\mathbf{R}^n \subset \mathbf{C}^n$.

The Lagrangian planes containing W are of the form $L_\theta = \operatorname{span}\{e_1, \ldots, e_{n-1}, \cos\theta e_n + \sin\theta J e_n\}$. Since $\beta|_L = e_1^* \wedge \ldots \wedge e_{n-1}^* \wedge (\sin\theta)(Je_n)^*$, the unique special Lagrangian plane containing W is spanned by $e_1, \ldots, e_n$. Consequently, each integral element is regular and the Cartan-Kähler theorem applies to yield Theorem 5.5.

Remark 5.6. Note that the above discussion goes through if we replace $\alpha \equiv \operatorname{Re} dz$ by $\alpha_\theta = \operatorname{Re}(e^{i\theta} dz)$. For a given isotropic $(n-1)$-dimensional submanifold N we thereby produce a 1-parameter family M_θ of minimal submanifolds all intersecting transversely in the manifold N.

The Theorem 5.5 may be reformulated by considering N to be the graph of a vector valued function $f: \Sigma \to \mathbf{R}^n$ defined on a hypersurface Σ in $\mathbf{R}^n$ and M to be the graph of a vector valued function F on $\mathbf{R}^n$. Of course M is Lagrangian if and only if $F = \nabla G$ for some scalar function G. The following are equivalent:

$$N \text{ is isotropic.} \tag{5.7 i}$$

$$f \text{ is a compatible 1-jet on } \Sigma. \tag{5.7 ii}$$

$$\text{There exist Cauchy data } g, \frac{\partial g}{\partial n} \text{ on } \Sigma \text{ with } f = \nabla g \text{ on } \Sigma. \tag{5.7 iii}$$

Now Theorem 5.5 can be restated as follows.

THEOREM 5.8. *Suppose Σ is a (real-analytic) hypersurface in $\mathbf{R}^n$ and (real-analytic) Cauchy data g, $\partial g/\partial n$ is given on Σ. Assume that the unique special Lagrangian n-plane containing the isotropic $(n-1)$-plane given by graphing $\nabla g(x_0)$ over $T_{x_0}\Sigma$ projects non-degenerately on $\mathbf{R}^n$. Then there exists a unique solution G to the special Lagrangian differential equation*

$$\sum_{k=0}^{[(n-1)/2]} (-1)^k \sigma_{2k+1}(\mathrm{Hess}\, G) = 0$$

with the prescribed Cauchy data on Σ.

The assumption that the unique special Lagrangian n-plane containing $T_{z_0}N \equiv \mathrm{graph}\, \nabla g(x_0)$ projects non-degenerately onto $\mathbf{R}^n \subset \mathbf{C}^n$ can be seen to be equivalent to Σ being non-characteristic for the special Lagrangian differential equation at the point $z_0 \equiv (x_0, \nabla g(x_0))$. In particular, the Cauchy-Kowalewski theorem implies Theorem 5.8 and hence Theorem 5.5.

IV. The exceptional geometries

In this chapter we shall explore certain fundamental geometries with exceptional automorphism groups. These geometries are all defined in low dimensions and can be presented in a unified way by using the algebra of the Cayley numers. We begin by defining certain calibrations in $\mathbf{R}^7$ and $\mathbf{R}^8$ and proving sharp versions of the comass inequality for them. The forms are, of course, classically known and there is an extensive literature concerning them. However, the sharp inequalities established here are new. In each case, the inequality leads to a first-order system of partial differential equations (analogous to the Cauchy-Riemann equations), with the property that graphs of solutions precisely form the family of subvarieties in our geometry.

Throughout this chapter we shall make extensive use of the octonions (or Cayley numbers) $\mathbf{O}$. A brief discussion of this algebra is given in Appendix A. Certain detailed computations underlying the inequalities are carried out systematically in Appendix B.

IV.1. The fundamental inequalities

We are concerned here with three distinct geometries and our discussion falls accordingly into three parts.

IV.1.A. The associator inequality

Consider on Im $(\mathbf{O})\approx\mathbf{R}^7$ the trilinear form φ given by

$$\varphi(x, y, z) = \langle x, yz\rangle. \tag{1.1}$$

Note that $\varphi(x,x,z)=\langle x,xz\rangle=|x|^2\langle 1,z\rangle=0$ since $z\in\operatorname{Im}(\mathbf{O})$. Similarly, we have $\varphi(x,y,x)=\langle x,yx\rangle=|x|^2\langle 1,y\rangle=0$, and $\varphi(x,y,y)=\langle x,y^2\rangle=-\langle x,y\bar{y}\rangle=-|y|^2\langle x,1\rangle=0$. Hence, the form φ is alternating.

Let $e_1=1$, $e_2=i$, $e_3=j$, $e_4=k$, $e_5=e$, $e_6=ie$, $e_7=je$, $e_8=ke$ denote the standard basis for the octonians $\mathbf{O}$. Let $\omega_1,\dots,\omega_8$ denote the dual basis for $\mathbf{O}^*$. We shall use the notation ω_{pqr} for $\omega_p\wedge\omega_q\wedge\omega_r$. Consulting a multiplication table for $\mathbf{O}$, the form φ is expressed in terms of axis 3-planes as follows.

$$\varphi=\omega_{234}-\omega_{278}-\omega_{638}-\omega_{674}-\omega_{265}-\omega_{375}-\omega_{485}. \tag{1.2}$$

Note, in particular, that $\varphi(\operatorname{Im}\mathbf{H})=1$ where $\operatorname{Im}\mathbf{H}=i\wedge j\wedge k=e_{234}$ is just the canonically oriented imaginary part of the standard quaternion subalgebra $\mathbf{H}$ of $\mathbf{O}$. (Also note $\varphi\wedge\varphi=0$.)

Defintion 1.3. If $\zeta\in G(3,7)\subset\Lambda^3\operatorname{Im}\mathbf{O}$ is the canonically oriented imaginary part of any quarternion subalgebra of $\mathbf{O}$, then the oriented 3-plane ζ is said to be *associative*. The set $G(\varphi)\equiv\{\zeta\in G(3,7)\colon \zeta \text{ is associative}\}$ will be referred to as the *associative grassmannian*.

Now we can state and prove the associator inequality.

THEOREM 1.4. *The form φ has comass one. In fact, $\varphi(\zeta)\leq 1$ for all $\zeta\in G(3,7)\subset\Lambda^3\operatorname{Im}\mathbf{O}$, with equality if and only if ζ is associative.*

Proof. Suppose $\zeta=u_1\wedge u_2\wedge u_3$ where u_1,u_2,u_3 is an oriented orthonormal basis for ζ. Then $\varphi(\zeta)=\langle u_1,u_2u_3\rangle\leq|u_1||u_2||u_3|=1$ by the Schwartz inequality. Moreover, equality holds if and only if the multiplication rules $u_p=u_q u_r$ hold for all cyclic permutations (p,q,r) of $(1,2,3)$. Finally, these rules hold if and only if ζ is the canonically oriented imaginary part of a quaternion subalgebra of $\mathbf{O}$.

Definition 1.5. The 3-form $\varphi\in\Lambda^3(\operatorname{Im}\mathbf{O})^*$, defined by (1.1) is called the *associative calibration on* $\operatorname{Im}\mathbf{O}$.

It is essential for the derivation of the partial differential equations appropriate for "associative geometry" that we strengthen the above inequality into an equality. An extra term is neccessary and it involves the *associator*, defined by

$$[x, y, z] \equiv (xy)z - x(yz) \quad \text{for all } x, y, z \in \mathbf{O}.$$

The basic fact about the associator, which is proved in Lemma A.4, is that it is alternating on $\mathbf{O}$.

THEOREM 1.6. *$\langle x, yz\rangle^2 + \frac{1}{4}|[x, y, z]|^2 = |x \wedge y \wedge z|^2$ for all $x, y, z \in \operatorname{Im}\mathbf{O}$.*

Proof. In Appendix B we prove that the triple cross product $x \times y \times z$ has length $|x \wedge y \wedge z|$, real part $\langle x, yz\rangle$, and imaginary part $\frac{1}{2}[x, y, z]$ (for x, y, z purely imaginary). Theorem 1.6 follows immediately.

This theorem provides an important alternate description of associative 3-planes.

COROLLARY 1.7. *Suppose ζ is an oriented 3-plane in $\operatorname{Im}\mathbf{O}$. Then either ζ or $-\zeta$ is associative if and only if $[x, y, z] = 0$ for $\zeta = x \wedge y \wedge z$.*

Proof. Express $\zeta = x \wedge y \wedge z$, where $|\zeta| = 1$. Then ζ or $-\zeta$ is associative if and only if $\varphi^2(\zeta) = \langle x, yz\rangle^2 = 1$ by Theorem 1.4. Consequently, ζ or $-\zeta$ is associative if and only if $[x, y, z] = 0$ by Theorem 1.6.

Recall now that the group of automorphisms of $\mathbf{O}$ is the Lie group denoted G_2. That is:

$$G_2 \equiv \{g \in GL_8(\mathbf{R}) \colon g(xy) = g(x)g(y), \quad \forall x, y \in \mathbf{O}\}.$$

Suppose $g \in G_2$. Note that $x \in \mathbf{O}$ is imaginary if and only if x^2 is real. Hence $g(x)$ is imaginary if and only if x is imaginary. It then follows from linearity that $g(\bar{x}) = \overline{g(x)}$. Thus g is an isometry. This proves that G_2 is a subgroup of the orthogonal group on $\operatorname{Im}\mathbf{O} = \mathbf{R}^7$. This fact and the above definition of G_2 implies that

$$G_2 = \{g \in O_7 \colon g^*\varphi = \varphi\}.$$

In particular, G_2 acts on the associative grassmannian $G(\varphi)$.

THEOREM 1.8. *The action of G_2 on $G(\varphi)$ is transitive with isotropy subgroup SO_4. Thus $G(\varphi) \cong G_2/SO_4$.*

The action of $SO_4 = Sp_1 \times Sp_1/\mathbf{Z}_2$ on $\operatorname{Im}\mathbf{O}$ (and hence on $G(3, 7)$) is given as follows. We assign to each pair of unit quaternions $(q_1, q_2) \in Sp_1 \times Sp_1$ the map

$$g(a,b)=(q_1 a\bar{q}_1, q_2 b\bar{q}_1). \tag{1.9}$$

One can easily check that each such g belongs to G_2.

Proof. Suppose $\zeta \in G(\varphi)$. Then by definition there exists an orthonormal pair $e_1, e_2 \in \operatorname{Im}\mathbf{O}$ with $\zeta = e_1 \wedge e_2 \wedge e_1 e_2$, and $H_1 \equiv \operatorname{span}(1 \wedge \zeta)$ is a quaternionic subalgebra of $\mathbf{O}$. The isomorphism $g: \mathbf{H} \to \mathbf{H}_1$ sending $i \mapsto e_1, j \mapsto e_2$ extends to an automorphism of $\mathbf{O}$ by Lemma A.15. Thus G_2 acts transitively on $G(\varphi)$.

Let $K_0 \equiv Sp_1 \times Sp_1/\mathbf{Z}_2$ denote the subgroup of G_2 defined by (1.9). Obviously this subgroup leaves $\zeta_0 = i \wedge j \wedge k$ fixed, so that $K_0 \subset K =$ the isotropy subgroup of G_2 at ζ_0. It remains to show that $K_0 = K$.

Clearly every element $g \in K$ can be expressed, with respect to the canonical splitting $\mathbf{O} = \mathbf{H} \oplus \mathbf{H}$, as $g = (g_1, g_2)$ where $g_1 \in \operatorname{Aut}(\mathbf{H}) \cong SO_3$ and $g_2 \in O_4$. The group K_0 is transitive on pairs (F, ε) where F is an oriented orthonormal frame in ζ_0 and ε is a unit vector in $\{0\} \times \mathbf{H}$.

Therefore, after applying an element in K_0 we can assume $g_1 =$ identity and $g_2(1) = 1$. Since g is an automorphism, we have that $g(0,q) = g((q,0)(0,1)) = g(q,0)\,g(0,1) = (q,0)(0,1) = (0,q)$. Hence g_2 is also the identity, and so $g =$ identity. It follows that $K = K_0$ and the proof is complete.

Other G_2 actions that are perhaps more standard are closely related to that in Theorem 1.8. Let $V_{7,2}$ denote the Stiefel manifold of oriented pairs of orthonormal vectors in $\operatorname{Im}\mathbf{O} = \mathbf{R}^7$. Let $V_{7,3}^{\varphi}$ denote the collection of ordered orthonormal triples $e_1, e_2, e_3 \in \operatorname{Im}\mathbf{O}$ with $\varphi(e_1 \wedge e_2 \wedge e_3) = 0$ (i.e. $e_3 \perp e_1 e_2$). Let $\{\varphi = 0\}$ denote the subset of $G(3,7)$ on which φ vanishes. Let S^6 denote the unit sphere in $\operatorname{Im}\mathbf{O}$. Obviously G_2 acts on $V_{7,3}, V_{7,2}, \{\varphi = 0\}$ and S^6.

PROPOSITION 1.10. *There are natural diffeomorphisms*:

(a) $V_{7,3}^{\varphi} \cong G_2$

(b) $\{\varphi = 0\} = G_2/SO_3$

(c) $V_{7,2} \cong G_2/SU_2$

(d) $S^6 \cong G_2/SU_3$.

Proof. Part (a) is just a restatement of Lemma A.15 and implies part (b). Part (a) implies that G_2 acts transitively on $V_{7,2}$ and S^6. The computation of the isotropy subgroups in parts (c) and (d) is omitted.

Remark. The unit 3-planes ξ with $\varphi(\xi)=0$ will be referred to as **O**-*generating* 3-*planes* since an orthonormal basis e_1, e_2, e_3 for such a ξ generates **O** by repeated application of the Cayley-Dickson process (see the proof of Lemma A.15). These 3-planes arise in Section 4 below on boundaries.

IV.1.B. The coassociator inequality

The basic 4-form ψ on $\operatorname{Im}\mathbf{O}$ is defined as follows.

Definition 1.11. The 4-form $\psi\in\Lambda^4(\operatorname{Im}\mathbf{O})^*$, defined by

$$\psi(x,y,z,w)\equiv\tfrac{1}{2}\langle x,[y,z,w]\rangle \quad \text{for all } x,y,z,w\in\operatorname{Im}\mathbf{O}$$

is called the *coassociative calibration on* $\operatorname{Im}\mathbf{O}$.

We must prove several results to justify this definition.

LEMMA 1.12. *The form* $\langle x,[y,z,w]\rangle$ *is alternating on* **O**.

Proof. This multilinear form on **O** is alternating in the last three variables y, z, w since the associator is alternating. Moreover, using (A.1) and (A.2)

$$\begin{aligned}2\psi(x,x,z,w) &= \langle x,[x,z,w]\rangle\\ &= \langle x,(xz)w\rangle-\langle x,x(zw)\rangle\\ &= \langle x\bar{w},xz\rangle-|x|^2\langle 1,zw\rangle\\ &= |x|^2(\langle\bar{w},z\rangle-\langle\bar{w},z\rangle)=0,\end{aligned}$$

and so ψ is alternating in all the variables.

Consulting a multiplication table for **O** the above form ψ may be expressed in terms of axis 4-planes.

$$\psi=\omega_{5678}-\omega_{5634}-\omega_{5274}-\omega_{5238}+\omega_{3478}+\omega_{2468}+\omega_{2367}. \tag{1.13}$$

Recall that the $\ast$-operator gives an isometry of the euclidean spaces $\ast\colon\Lambda^3\mathbf{R}^7\to\Lambda^4\mathbf{R}^7$ which maps simple forms to simple forms.

PROPOSITION 1.14. $\psi=\ast\varphi$.

Proof. Compare the formulas expressing φ and ψ in terms of axis planes.

Definition 1.15. An oriented 4-plane $\zeta \in G(4,7) \subset \Lambda^4 \operatorname{Im} \mathbf{O}$ is said to be *coassociative* if the canonically oriented normal 3-plane $*\zeta$ is associative.

Consequently, the associator inequality combined with the fact that $\psi = *\varphi$ immediately proves the following coassociator inequality.

THEOREM 1.16. *The form ψ has comass one. In fact,*

$$\psi(\zeta) \leq 1 \quad \text{for all } \zeta \in G(4,7) \subset \Lambda^4 \operatorname{Im} \mathbf{O}$$

with equality if and only if ζ is coassociative.

Just as with the associator inequality the above inequality can be strengthened into an equality. In Appendix B we define the *coassociator* by,

$$\begin{aligned} \tfrac{1}{2}[x,y,z,w] &= -4 \operatorname{Alt}(\langle y, zw\rangle x) \\ &= -(\langle y, zw\rangle x + \langle z, xw\rangle y + \langle x, yw\rangle z + \langle y, xz\rangle w), \qquad (1.17) \end{aligned}$$

for all $x, y, z, w \in \operatorname{Im} \mathbf{O}$.

Obviously, $[x,y,z,w]$ is alternating.

THEOREM 1.18. $\frac{1}{4}\langle x, [y,z,w]\rangle^2 + \frac{1}{4}|[x,y,z,w]|^2 = |x\wedge y\wedge z\wedge w|^2$, *for all* $x, y, z, w \in \operatorname{Im} \mathbf{O}$.

Proof. In Appendix B we prove that for $x, y, z, w \in \operatorname{Im} \mathbf{O}$, the fourfold cross product $x \times y \times z \times w$ has length $|x\wedge y\wedge z\wedge w|$, real part $\frac{1}{2}\langle x, [y,z,w]\rangle$ and imaginary part $\frac{1}{2}[x,y,z,w]$. Consequently, Theorem 1.18 follows immediately.

COROLLARY 1.19. *Suppose ζ is an oriented 4-plane in* $\operatorname{Im}(\mathbf{O})$. *Then either ζ or $-\zeta$ is coassociative if and only if* $[x,y,z,w]=0$ *for any basis x, y, z, w of ζ.*

Theorem 1.18 provides an alternate description of coassociative 4-planes.

COROLLARY 1.20. *Suppose ζ is an oriented 4-plane in* $\operatorname{Im} \mathbf{O}$. *Then either ζ or $-\zeta$ is coassociative if and only if $xy \in \zeta^\perp$ for all $x, y \in \zeta$.*

Proof. If $x, y \in \zeta$ implies $xy \in \zeta^\perp$, then by (1.17) we see that $[x,y,z,w]=0$ for all $x, y, z, w \in \zeta$, and we apply Corollary 1.19. Conversely, suppose $\pm\zeta$ is coassociative and we are given $z, w \in \zeta$. We must show that $\langle y, zw\rangle = 0$ for any $y \in \zeta$. If y, z, w are linearly dependent, then $\langle y, zw\rangle = \varphi(y,z,w) = 0$ since φ is alternating. If y, z, w are linearly independent, we may choose x so that $\zeta = x\wedge y\wedge z\wedge w$. By Corollary 1.19, we have

$[x, y, z, w]=0$ and, in particular, the coefficient $\langle y, zw\rangle$ of x in expression (1.17) must vanish.

IV.1.C. The Cayley inequality

The basic 4-form on **O** is obtained from the triple cross product (see Appendix B).

Definition 1.21. The four form $\Phi\in\Lambda^4\mathbf{O}^*$ defined by $\Phi(x, y, z, w)\equiv\langle x, y\times z\times w\rangle$, for all $x, y, z, w\in\mathbf{O}$ is called the *Cayley calibration on* **O**.

To justify this definition we must prove several facts.

LEMMA 1.22. *The form* $\langle x, y\times z\times w\rangle$ *is alternating on* **O**.

Proof. Since $y\times z\times w$ is alternating we need only show that $\langle x, x\times z\times w\rangle=0$ for x, z, w orthogonal. In this case $x\times z\times w=x(\bar{z}w)$ by (B.5), and hence

$$\langle x, x\times z\times w\rangle = \langle x, x(\bar{z}w)\rangle = \langle 1, \bar{z}w\rangle|x|^2 = \langle z, w\rangle|x|^2 = 0$$

as desired.

For the moment we postpone geometric characteriaztions of the 4-planes ζ with $\Phi(\zeta)=1$.

Definition 1.23. An oriented 4-plane $\zeta\in G(4,8)\subset\Lambda^4\mathbf{O}$ is called a *Cayley 4-plane* if $\Phi(\zeta)=1$.

THEOREM 1.24. *The form* Φ *has comass one. In fact,* $\Phi(\zeta)\leq 1$ *for all* $\zeta\in G(4,8)\subset\Lambda^4\mathbf{O}$, *with equality if and only if* ζ *is a Cayley 4-plane.*

Proof. Suppose x, y, z, w is an orthonormal basis for ζ. Then $\Phi(\zeta)\equiv\langle x, y\times z\times w\rangle\leq|x|\,|y\times z\times w|=1$ by the Schwartz inequality, since $|y\times z\times w|=|y\wedge z\wedge w|=1$ by Lemma B.4.

Equality holds if and only if $x=y\times z\times w$. Thus for our first characterization of the Cayley 4-planes we have:

PROPOSITION 1.25. *Suppose* $\zeta\in G(4,8)\subset\Lambda^4\mathbf{O}$. *Then either* ζ *or* $-\zeta$ *is a Cayley 4-plane if and only if* $y\times z\times w\in\zeta$ *for all* $y, z, w\in\zeta$.

Proof. One need only consider the case where y, z, w are orthonormal since $y\times z\times w$ is alternating. The proposition then follows easily from the argument given for Theorem 1.24.

This condition on ζ can be described alternatively in terms of a complex structure on $\mathbf{O}$. A distinguished 6-sphere of complex structures on $\mathbf{O}$ is provided by Cayley multiplication.

Let S^6 denote $\{u \in \operatorname{Im}\mathbf{O}: |u|=1\}$; and given $u \in S^6$, let $J_u: \mathbf{O} \to \mathbf{O}$ be defined by $J_u v = vu$ for all $v \in \mathbf{O}$. Since $u^2=-1$ and $(vu)u = v(u^2) = -v$, we have $J_u^2 = -I$. Thus J_u is a complex structure on $\mathbf{O}$.

Given an oriented 2-plane $\alpha \in G(2,8)$, as is well known, one can impose a natural complex structure J on α as follows. If x, y is an oriented orthonormal basis for α then define $Jx=y$ and $Jy\equiv -x$. (This definition of J is independent of the pair x, y.) This complex structure J on α is induced by one of the above distinguished complex structures on $\mathbf{O}$. To prove this we first define a map from $G(2,8)$ to S^6 by sending $\alpha = x \wedge y$ into the cross product $y \times x$. By Lemma B.2 and Lemma B.9, $y \times x \in S^6 \subset \operatorname{Im}\mathbf{O}$ and is independent of the oriented basis x, y for α.

LEMMA 1.26. *Given $\alpha = x \wedge y$, the complex structure $J_{y\times x}$ on $\mathbf{O}$ induces the natural complex structure on the subspace α of $\mathbf{O}$.*

Proof. Suppose x, y is an oriented orthonormal basis for α. Then $y \times x = \bar{x}y$ and hence $J_{y\times x} x = x(\bar{x}y) = |x|^2 y = y$.

Now we can give another characterization of Cayley 4-planes.

PROPOSITION 1.27. *Suppose $\zeta \in G(4,8) \subset \Lambda^4\mathbf{O}$. Then ζ is a Cayley 4-plane if and only if $-\zeta$ is a complex 2-plane with respect to one (all) of the complex structures determined by the two planes α contained in ζ.*

Proof. Suppose ζ is an oriented 4-plane, and z, w is an orthonormal pair in ζ. Let $J_{w\times z}$ denote the complex structure on $\mathbf{O}$ determined by the two plane $\alpha \equiv z \wedge w$. Choose $x, y \in \zeta$ so that x, y, z, w is an oriented orthonormal basis for ζ. Note $J_{w\times z} z = w$ by Lemma 1.26. If ζ is a Cayley 4-plane, then $x = y \times z \times w = y(\bar{z}w) = J_{w\times z} y$ and hence $-\zeta = y \wedge Jy \wedge z \wedge Jz$ is a complex 2-plane. Conversely if $-\zeta$ is complex, then we must have $x = J_{w\times z} y = y(\bar{z}w) = y \times z \times w$ so that ζ is a Cayley 4-plane.

This inequality in Theorem 1.24 can be strengthened into an equality.

THEOREM 1.28. *For all $x, y, z, w \in \mathbf{O}$,*

$$\Phi(x \wedge y \wedge z \wedge w)^2 + |\operatorname{Im} x \times y \times z \times w|^2 = |x \wedge y \wedge z \wedge w|^2.$$

Proof. In Appendix B we prove that $x\times y\times z\times w$ has length $|x\wedge y\wedge z\wedge w|$ and real part $\Phi(x, y, z, w)$. Theorem 1.28 follows immediately.

Remark. Recall from part (3) of Lemma B.15 that

$$2\,\mathrm{Im}\, x\times y\times z\times w = [x, y, z, w]+x_1[y, z, w]+y_1[z, x, w]+z_1[x, y, w]+w_1[y, x, z].$$

COROLLARY 1.29. *Suppose ζ is an oriented 4-plane in $\mathbf{O}$. Then either ζ or $-\zeta$ is a Cayley 4-plane if and only if $\mathrm{Im}\, x\times y\times z\times w=0$ for any basis x, y, z, w for ζ.*

The characterization of Cayley 4-planes given by this corollary will provide us with a means of deriving a system of partial differential equations associated with Cayley 4-planes.

There are several alternate descriptions of the Cayley calibration Φ. First we have:

PROPOSITION 1.30. $\Phi=1^*\wedge\varphi+\psi$.

From (1.2) and (1.13), we then conclude that Φ is the following sum of axis 4-planes.

COROLLARY 1.31.

$$\Phi = \omega_{1234}-\omega_{1278}-\omega_{1638}-\omega_{1674}-\omega_{1265}-\omega_{1375}-\omega_{1485}$$
$$+\omega_{5678}-\omega_{5634}-\omega_{5274}-\omega_{5238}+\omega_{3478}+\omega_{2468}+\omega_{2367}.$$

Proof. By Lemma B.9 and Lemma B.15 we have that

$$y\times z\times w = \langle y', z'w'\rangle+\tfrac{1}{2}[y, z, w]+y_1\tfrac{1}{2}[w, z]+z_1\tfrac{1}{2}[y, w]+w_1\tfrac{1}{2}[z, y].$$

Therefore, since $\Phi(x, y, z, w)\equiv\langle x, y\times z\times w\rangle$, we see that

$$\begin{aligned}\Phi(x, y, z, w) &= \langle y', z'w'\rangle\, x_1+\langle x', w'z'\rangle\, y_1+\langle x', y'w'\rangle\, z_1+\langle x', z'y'\rangle\, w_1+\langle x, \tfrac{1}{2}[y, z, w]\rangle \\ &= (1^*\wedge\varphi)(x, y, z, w)+\psi(x, y, z, w).\end{aligned}$$

Our second description relates Φ to the complex structure associated to $e=(0, 1)\in\mathbf{H}\times\mathbf{H}$.

PROPOSITION 1.32. *Consider the complex structure J_e on $\mathbf{O}\cong\mathbf{C}^4$. Let ω denote the associated Kähler form and identify $\mathbf{H}\subset\mathbf{O}$ with $\mathbf{R}^4\subset\mathbf{C}^4$. Then*

$$\Phi = -\tfrac{1}{2}\omega\wedge\omega+\mathrm{Re}\,(dz).$$

Proof. Express $\omega\wedge\omega$ and $\mathrm{Re}(dz)$ in terms of the standard basis $e_1, \ldots, e_8$ for $\mathbf{O}$, and compare $-\frac{1}{2}\omega\wedge\omega+\mathrm{Re}(dz)$ with Φ using Corollary 1.31.

For any oriented unit 4-plane $\zeta\in G(4,8)$ the restriction $\omega|_\zeta$ can be put in canonical form. Namely, we may choose $\varepsilon_1, \ldots, \varepsilon_4$ to be an oriented orthonormal basis for ζ, and $0\leq\lambda\leq 1$, $-1\leq\mu\leq\lambda$ such that:

$$\omega|_\zeta = \lambda\varepsilon_1^*\wedge\varepsilon_2^*+\mu\varepsilon_3^*\wedge\varepsilon_4^*.$$

The numbers $\mu\leq\lambda$ are called the eigenvalues of $\omega|_\zeta$. If $\mu=\lambda=0$ then ζ is Lagrangian. More generally:

Definition 1.33. If $\lambda=-\mu$ then $\zeta\in G(4,8)$ is *anti-self dual.* If, in addition, $dz(\zeta)\geq 0$ (i.e., is real and positive) then ζ is *special anti-self dual.*

Note that ζ is anti-self dual with eigenvalues $+1$ and -1 if and only if $-\zeta$ is a complex subspace; and in this case $dz(\zeta)=0$ so that ζ is automatically special.

PROPOSITION 1.34. *Consider the complex structure J_e on $\mathbf{O}\cong\mathbf{C}^4$. Let ω denote the associated Kähler form and let $dz\equiv dz_1\wedge\ldots\wedge dz_4$ denote the form associated with the identification of $\mathbf{H}\subset\mathbf{O}$ with $\mathbf{R}^4\subset\mathbf{C}^4$. A 4-plane $\zeta\in G(4,8)$ is Cayley if and only if ζ is special anti-self dual.*

The proof follows easily from the canonical form,

$$\zeta = e_1\wedge(Je_1\cos\theta_1+e_2\sin\theta_1)\wedge e_3\wedge(Je_3\cos\theta_2+e_4\sin\theta_2),$$

where $0\leq\theta_1\leq\pi/2$ with $\lambda=\cos\theta_1, \theta_1\leq\theta_2\leq\pi$ with $\mu=\cos\theta_2$, and $e_1, e_2, e_3, e_4, Je_1, Je_2, Je_3, Je_4$ is an oriented orthonormal basis for $\mathbf{O}\cong\mathbf{C}^4$ over $\mathbf{R}$.

Remark. This point of view relates to the Yang-Mills equation (cf. Chapter V).

We shall now analyze the subgroup of O_8 which preserves the Cayley calibration. We begin with the following formulation of the group Spin_7. For each $u\in\mathbf{O}$, let $R_u\colon\mathbf{O}\to\mathbf{O}$ be defined by $R_u(x)=xu$. Note that for $u\neq 0$, we have $R_u\in GL^+(\mathbf{O})$.

Definition 1.35. Spin_7 is the subgroup of SO_8 generated by $S^6\equiv\{R_u\colon u\in\mathrm{Im}\,\mathbf{O}$ and $|u|=1\}$.

Remark. The Moufang identity Lemma A.16(b) says that, for each $u\in\mathrm{Im}\,\mathbf{O}$,

$$R_u\circ R_v\circ R_u^{-1} = R_{-uvu}.$$

Therefore, $\chi_g(R_v)=g\circ R_v\circ g^{-1}$ defines an action χ of Spin_7 on the vector space $\{R_v: v\in \mathrm{Im}\,\mathbf{O}\}\cong\mathrm{Im}\,\mathbf{O}\cong\mathbf{R}^7$. In fact, Spin_7 is just the subgroup of SO_8 which conjugates the space of endomorphisms $\{R_v: v\in\mathrm{Im}\,\mathbf{O}\}$ into itself. That is, given $g\in SO_8$, then g belongs to Spin_7 if and only if for every $v\in\mathrm{Im}\,\mathbf{O}$, there exists $w\in\mathrm{Im}\,\mathbf{O}$ such that $g\circ R_v\circ g^{-1}=R_w$. Applying this last expression to $1\in\mathbf{O}$, we see that $w=g(g^{-1}(1)\cdot v)$. Hence, if we define $\tilde{\chi}:\mathrm{Spin}_7\to SO(\mathrm{Im}\,\mathbf{O})\cong SO_7$ by $\tilde{\chi}_g(v)=g(g^{-1}(1)\cdot v)$, then $\tilde{\chi}$ is the standard double cover of SO_7 by Spin_7. It follows that $\mathrm{Spin}_7=\{g\in SO_8: g(g^{-1}(y)\,v)=y\tilde{\chi}_g(v)$ for all $v, y\in\mathbf{O}\}$. Setting $u=g^{-1}(y)$ we finally get the useful alternative definition:

$$\mathrm{Spin}_7\equiv\{g\in SO_8: g(uv)=g(u)\tilde{\chi}_g(v)\quad\text{for all } u, v\in\mathbf{O}\}.$$

PROPOSITION 1.36. *The form* Φ *is fixed by the subgroup* Spin_7.

Proof. Suppose $u\in S^6\subset\mathrm{Im}\,\mathbf{O}$. Then, making use of the equivariance in Proposition B.11 (3), $\Phi(xu\wedge yu\wedge zu\wedge wu)=\mathrm{Re}\,xu\times yu\times zu\times wu=-\mathrm{Re}\,u(x\times y\times z\times w)\,u=-u^2\,\mathrm{Re}\,x\times y\times z\times w=\Phi(x, y, z, w)$.

Next we wish to consider one of the distinguished complex structures J_u where u is a unit imaginary quaternion. By Proposition 1.10 (d) there exists an automorphism $g\in G_2$ with $u=g(e)$. Note that $\tilde{\chi}_g(v)=g(v)$ and hence $g\in\mathrm{Spin}_7$ (in fact it is easy to see that $g\in G_2$ if and only if $g\in\mathrm{Spin}_7$ and $g(1)=1$). Therefore $g^*\Phi=\Phi$ by Proposition 1.36. If $\tilde{\omega}$ denotes the Kähler form corresponding to the complex structure J_u, then it is easy to check that $g^*(\tilde{\omega})=\omega$. Finally, let $\tilde{\mathbf{H}}$ denote $g(\mathbf{H})$, and let $d\tilde{z}$ denote $d\tilde{z}_1\wedge\ldots\wedge d\tilde{z}_4$ where $\tilde{\mathbf{H}}$ is identified with $\mathbf{R}^4\subset\mathbf{C}^4$. Then $g^*(d\tilde{z})=dz$.

PROPOSITION 1.37. *Given a quaternion subalgebra* $\tilde{\mathbf{H}}\subset\mathbf{O}$ *and a unit* $u\perp\mathbf{H}$, *Proposition* 1.32 *remains valid with* J_e *replaced by* J_u *and* $\mathbf{H}\cong\mathbf{R}^4$, *replaced by* $\tilde{\mathbf{H}}\cong\mathbf{R}^4$. *That is,* $\Phi=-\frac{1}{2}\tilde{\omega}\wedge\tilde{\omega}+\mathrm{Re}\,d\tilde{z}$.

Proof. The previous discussion yields the proof since $g\in G_2$ can be chosen so that not only $u=g(e)$ but also $\tilde{\mathbf{H}}=g(\mathbf{H})$ (see Proposition 1.10 (a)).

Remark. Consider the transitive action $\chi_g(J_u)=g\circ J_u\circ g^{-1}$ of Spin_7 on $S^6=\{J_u: u\in\mathrm{Im}\,\mathbf{O}$ and $|u|=1\}$ discussed above. The isotropy subgroup of Spin_7 at the point $J_e\in S^6$ is just $H=\mathrm{Spin}_7\cap U_4$ where U_4 is the unitary group with respect to the complex structure J_e. Since H leaves Φ and $\frac{1}{2}\omega\wedge\omega$ fixed, it must also leave $\mathrm{Re}\,(dz)$ fixed. Therefore, $H\subset SU_4$. However, since $S^6=\mathrm{Spin}_7/H$ we see that $\dim(H)=15$ and that H must be connected. Hence $H=SU_4$, and we have the well known diffeomorphism

$$S^6 \cong \mathrm{Spin}_7/SU_4.$$

We conclude this section by describing the grassmannian of Cayley 4-planes as a homogeneous space.

THEOREM 1.38. *The action of* Spin_7 *on* $G(\Phi)$ *is transitive with isotropy subgroup* $K \cong SU_2 \times SU_2 \times SU_2/\mathbf{Z}_2$. *Thus* $G(\Phi) \cong \mathrm{Spin}_7/K$.

Proof. To begin we observe that since $\mathrm{Spin}_7 \supset SU_4$ (see above), Spin_7 acts transitively on $S^7 \subset \mathbf{O}$. The subgroup fixing $1 \in \mathbf{O}$ contains G_2 by Proposition 1.30. By a dimension count we see that Spin_7/G_2 is a covering space of S^7, and hence we have

$$S^7 \cong \mathrm{Spin}_7/G_2.$$

Suppose now that $\zeta \in G(\Phi)$ is a Cayley 4-plane. Let x, y, z, w be an oriented orthonormal basis for ζ. Since Spin_7 is transitive on S^7, we may choose $g_0 \in \mathrm{Spin}_7$ such that $g_0(x)=1$. By Proposition 1.30 we have that $g_0(y \wedge z \wedge w)$ is an associative 3-plane in $\mathrm{Im}\,\mathbf{O}$. Consequently there is a $g_1 \in G_2 \subset \mathrm{Spin}_7$ so that $g_1 g_0(y \wedge z \wedge w) = i \wedge j \wedge k$ (cf. Theorem 1.8). Setting $g = g_1 g_0$, we have $g(\zeta) = 1 \wedge i \wedge j \wedge k$. This shows that Spin_7 is transitive on $G(\Phi)$.

The group K defined above acts on $\mathbf{O}$ as follows. Given $g = (q_1, q_2, q_3) \in Sp_1 \times Sp_1 \times Sp_1$ (where Sp_1 is the unit quaternions), let

$$g(a+be) = q_3 a \bar{q}_1 + (q_2 b \bar{q}_1)\, e \tag{1.39}$$

for all $a, b \in \mathbf{H}$. The subgroup of $Sp_1 \times Sp_1 \times Sp_1$ acting as the identity on $\mathbf{O}$ is $\mathbf{Z}_2$, generated by $(-1, -1, -1)$. Using the alternative definition of Spin_7 given in the remark above, one can easily show that (1.39) gives an embedding $K \hookrightarrow \mathrm{Spin}_7$. Clearly K leaves the $\zeta_0 = 1 \wedge i \wedge j \wedge k$ fixed, and so $K \subseteq K_0 =$ the isotropy subgroup of Spin_7 at ζ_0. Suppose on the other hand that $g \in K_0$ and $g(1) = x \in \zeta_0$. Normalizing by $h = (1, 1, \bar{x})$ we may assume that $g(1) = 1$. Then g belongs to the isotropy subgroup of G_2 at $i \wedge j \wedge k$, and hence $g \in K$ by Theorem 1.8. This completes the proof.

IV.2. The partial differential equations

In this section we derive systems of partial differential equations (somewhat analogous to the Cauchy-Riemann equations) to be satisfied by a function f in order that the graph of f be a submanifold in one of our exceptional geometries.

IV.2.A. The associator equation

The objects of interest here are the following.

Definition 2.1. An oriented 3-dimensional submanifold M of $\mathbf{R}^7 = \operatorname{Im}\mathbf{O}$ is said to be *associative* if the tangent plane to M at each point is associative.

Note that an associative submanifold is one whose tangent space at each point is the imaginary part of a quaternion subalgebra of $\mathbf{O}$.

Because of the associator inequality Theorem 1.4 we may apply the basic theorem.

THEOREM 2.2 *Associative submanifolds are absolutely area minimizing.*

Recall that any two associative 3-planes are equivalent under G_2 acting on $\operatorname{Im}\mathbf{O}$ (Theorem 1.8). Therefore there is no loss in generality in assuming that the tangent plane to an associative manifold M at a particular point is $\operatorname{Im}\mathbf{H} = i \wedge j \wedge k \subset \operatorname{Im}\mathbf{O}$. Let Ω denote an open subset of $\operatorname{Im}\mathbf{H} \subset \operatorname{Im}\mathbf{O}$, and consider a C^1 map $f\colon \Omega \to \mathbf{H}$. Locally M may be considered as the graph of f over its tangent plane $\operatorname{Im}\mathbf{H}$.

Our next objective is to describe a system of equations that f must satisfy in order for the graph of f to be an associative submanifold. Two differential operators are involved. Let $x = x_1 + x_2 i + x_3 j + x_4 k$ denote a point in $\mathbf{H}$.

Definition 2.3. Let $f\colon \Omega \to \mathbf{H}$ be a C^1 map, where Ω is a domain in $\operatorname{Im}\mathbf{H}$. The *Dirac operator* on f is defined by

$$D(f) = -\frac{\partial f}{\partial x_2} i - \frac{\partial f}{\partial x_3} j - \frac{\partial f}{\partial x_4} k.$$

The *first order Monge-Ampère operator* on f is defined by

$$\sigma(f) = \frac{\partial f}{\partial x_2} \times \frac{\partial f}{\partial x_3} \times \frac{\partial f}{\partial x_4}.$$

Note that the definitions of both operators D and σ are independent of the choice of oriented orthonormal basis for $\operatorname{Im}\mathbf{H}$.

THEOREM 2.4. *Let $f\colon \Omega \to \mathbf{H}$ be a C^1 map, where Ω is a domain in* $\operatorname{Im}\mathbf{H}$. *Then the graph of f is an associative submanifold of* $\operatorname{Im}\mathbf{O} = \operatorname{Im}\mathbf{H} \oplus \mathbf{H}$ *if and only if f satisfies the differential equation*

$$D(f) = \sigma(f).$$

Since associative submanifolds are minimal we conclude the following from the regularity results of C. B. Morrey [M].

COROLLARY 2.5. *Any C^1 solution of the equation $D(f)=\sigma(f)$ is real analytic.*

Proof of Theorem 2.4. It suffices to prove the theorem in the case where $f\colon \operatorname{Im}\mathbf{H}\to\mathbf{H}$ is linear. In this case $D(f)=-f(i)\,i-f(j)\,j-f(k)\,k$, and $\sigma(f)=f(i)\times f(j)\times f(k)$. The oriented graph of f is spanned by $x\equiv i+f(i)\,e$, $y\equiv j+f(j)\,e$, $z\equiv k+f(k)\,e$. Proposition B.14 (2) may be used in conjunction with either Lemma B.10 (2) or with the equation (2)′ occuring in the proof of Lemma A.11 to calculate that

$$\operatorname{Im} x\times y\times z=\operatorname{Im}\{(i(f(j)\times f(k))+j(f(k)\times f(i))+k(f(i)\times f(j))\}+(\sigma(f)-D(f))\,e.$$

Let ζ denote the oriented graph of f. If ζ is associative then $\operatorname{Im} x\times y\times z=0$ and hence, in particular, $\sigma(f)=D(f)$. Conversely, suppose $\sigma(f)=D(f)$. Then $\frac{1}{2}[x,y,z]=\operatorname{Im} x\times y\times z\in\operatorname{Im}\mathbf{H}\subset\operatorname{Im}\mathbf{O}$. By Proposition B.16 (4) (ii), $[x,y,z]$ is orthogonal to x, y, and z. However, since $[x,y,z]\in\operatorname{Im}\mathbf{H}$ and the components of x,y,z in $\operatorname{Im}\mathbf{H}$ are i,j,k respectively, this orthogonality implies that $[x,y,z]=0$. It follows that, with the appropriate orientation, ζ is associative.

Note that in general one must choose the appropriate orientation on each component of the graph of f. Of course, if Ω is connected and df is sufficiently small at some point (for example, if we are graphing over a tangent plane at some point), then the orientation naturally induced from $\operatorname{Im}\mathbf{H}$ is the correct one.

IV.2.B. The coassociator equation

Dual to the associative geometry is the following.

Definition 2.6. An oriented 4-dimensional submanifold M of $\mathbf{R}^7=\operatorname{Im}\mathbf{O}$ is said to be *coassociative* if the tangent plane to M at each point is coassociative.

From Theorem 1.16 we have the following.

THEOREM 2.7. *Coassociative submanifolds are absolutely area minimizing.*

Of course the corresponding result holds for "coassociative varieties" and "coassociative currents".

Arguing as above we see that there is no loss of generality in assuming that a coassociative manifold is given locally as the graph of a function g from an open set in $\mathbf{H}$ to $\operatorname{Im}\mathbf{H}$. Let $x=x_1+x_2i+x_3j+x_4k$ denote a point of $\mathbf{H}$.

Definition 2.8. Let $g\colon\Omega\to\operatorname{Im}\mathbf{H}$ be a C^1 map, where Ω is an open domain in $\mathbf{H}$. Let

$g=g^2i+g^3j+g^4k$ denote the components of g and interpret ∇g^p as a quaterion in the obvious way. The (dual) *Dirac operator* on g is defined by

$$\tilde{D}(g) = -\nabla g^2 i - \nabla g^3 j - \nabla g^4 k.$$

The *(dual) first order Monge-Ampère operator* on g is defined by

$$\tilde{\sigma}(g) = \nabla g^2 \times \nabla g^3 \times \nabla g^4.$$

These operators are independent of the choice of oriented orthonormal basis for $\operatorname{Im}\mathbf{H}$.

THEOREM 2.9. *Let $f\colon \Omega\to\operatorname{Im}\mathbf{H}$ be a C^1 map where Ω is a domain in $\mathbf{H}$. Then the graph of f is a coassociative submanifold of* $\operatorname{Im}\mathbf{O}=\operatorname{Im}\mathbf{H}\oplus\mathbf{H}$ *if and only if f satisfies the differential equation*

$$\tilde{D}(f) = \tilde{\sigma}(f).$$

Proof. A direct proof of this can be given using the coassociator equality 1.18. The details of this proof are omitted. Instead this equation is obtained as a consequence of Theorem 2.22 in the next section (see Remark 2.51).

As above we have the following.

COROLLARY 2.10. *Any C^1 solution to the equation $\tilde{D}(f)=\tilde{\sigma}(f)$ is real analytic.*

This result is sharp. In Section 3 we shall construct a Lipschitz solution to this equation which is not C^1.

IV.2.C. The Cayley equation

Definition 2.11. An oriented 4-dimensional submanifold M of $\mathbf{R}^8\cong\mathbf{O}$ is said to be a *Cayley submanifold* if the tangent plane to M at each point is a Cayley 4-plane.

Remark 2.12. The geometry of Cayley submanifolds includes several other geometries.

(a) A submanifold M which lies in $\operatorname{Im}\mathbf{O}\subset\mathbf{O}$ is Cayley if and only if M is coassociative. This follows directly from Proposition B.14, part (3), which shows that any Cayley 4-plane in $\operatorname{Im}\mathbf{O}$ is coassociative.

(b) A submanifold M of $\mathbf{O}$ of the form $\mathbf{R}\times N$, where N is a submanifold of $\operatorname{Im}\mathbf{O}$, is Cayley if and only if N is associative. This follows from the definition of the Cayley calibration Φ and the fact that $\langle 1, y\times z\times w\rangle=\langle y', z'w'\rangle$ (see Lemma B.9 (2)).

(c) Fix a unit imaginary quaternion $u \in S^6 \subset \operatorname{Im} \mathbf{O}$. Consider the complex structure J_u and let $\mathbf{O} \cong \mathbf{C}^4$. Each complex surface in $\mathbf{O}$, with the reverse orientation, is a Cayley submanifold. This is a consequence of Proposition 1.37.

(d) In addition to choosing one of the distinguished complex structures J_u (as in (c)) choose a quaternion subalgebra $\tilde{\mathbf{H}}$ of $\mathbf{O}$ orthogonal to u and identify $\mathbf{R}^4 \subset \mathbf{C}^4$ with $\tilde{\mathbf{H}} \subset \mathbf{O}$. Each special Lagrangian submanifold of $\mathbf{C}^4 \cong \mathbf{O}$ is a Cayley submanifold. This is also a consequence of Proposition 1.37.

THEOREM 2.13. *Cayley submanifolds are absolutely area minimizing.*

The corresponding result holds for "Cayley varieties" and "Cayley currents".

Each Cayley submanifold can be described locally as the graph of a function f over one of its tangent planes. Recall that any two Cayley 4-planes are equivalent under Spin_7 (Theorem 1.36). Therefore, there is no loss in generality to consider the case where the tangent plane is $\zeta \equiv \mathbf{H} \subset \mathbf{O}$. Let Ω denote an open subset of $\mathbf{H} \subset \mathbf{O}$ and consider a C^1 map $f\colon \Omega \to \mathbf{H}$. The purpose of this section is to extract a simple and rather beautiful system of equations for f which is equivalent to the graph of f being a Cayley submanifold. This system of equations should be considered as the appropriate analogue of the Cauchy-Riemann equations for functions from $\mathbf{H}$ to $\mathbf{H}$. To express this system concisely we shall define three distinct first order operators on C^1 functions $f\colon \Omega \to \mathbf{H}$ with $\Omega^{\text{open}} \subset \mathbf{H}$.

The first, called the *Dirac operator*, is given by the formula

$$Df \equiv \frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2} i - \frac{\partial f}{\partial x_3} j - \frac{\partial f}{\partial x_4} k, \tag{2.14}$$

where (x_1, x_2, x_3, x_4) are coordinates with respect to the basis $1, i, j, k$, for $\mathbf{H}$. It is easy to check that

$$Df = \sum_{j=1}^{4} (\nabla_{e_j} f)\, \bar{e}_j \tag{2.14$'$}$$

where $e_1, \ldots, e_4$ is any orthonormal basis for $\mathbf{H}$. This operator D is a first order, linear, elliptic operator. It is the quaternionic analogue of the operator $\partial/\partial z \equiv \frac{1}{2}(\partial/\partial x - i(\partial/\partial y))$, which acts on $g\colon \mathbf{C} \to \mathbf{C}$. With respect to the quaternion inner product $x\bar{y}$ the adjoint of D is the operator $-\bar{D}$ defined by

$$\bar{D}(f) \equiv \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} i + \frac{\partial f}{\partial x_3} j + \frac{\partial f}{\partial x_4} k, \tag{2.15}$$

The operators D and $\bar{D}$ satisfy the relationship

$$D\bar{D} = \bar{D}D = \Delta \cdot I, \tag{2.16}$$

where Δ is the scalar Laplacian and I is the 4×4 identity matrix.

The second operator, called the *first-order Monge-Ampère operator* is defined as follows.

$$\sigma(f) \equiv \left(\frac{\partial f}{\partial x_2}\times\frac{\partial f}{\partial x_3}\times\frac{\partial f}{\partial x_4}\right)+\left(\frac{\partial f}{\partial x_1}\times\frac{\partial f}{\partial x_3}\times\frac{\partial f}{\partial x_4}\right)i-\left(\frac{\partial f}{\partial x_1}\times\frac{\partial f}{\partial x_2}\times\frac{\partial f}{\partial x_4}\right)j+\left(\frac{\partial f}{\partial x_1}\times\frac{\partial f}{\partial x_2}\times\frac{\partial f}{\partial x_3}\right)k. \tag{2.17}$$

It is straightforward to check that the basis $1, i, j, k$ for $\mathbf{H}$ may be replaced by any oriented orthonormal basis $e_1, \ldots, e_4$ for $\mathbf{H}$ without altering the operator σ. In fact, σ can be expressed in the following invariant form.

$$\sigma(f) = \sum_{i_1<i_2<i_3} \left(\nabla_{e_{i_1}} f \times \nabla_{e_{i_2}} f \times \nabla_{e_{i_3}} f\right) \overline{\left(e_{i_1}\times e_{i_2}\times e_{i_3}\right)} \tag{2.17$'$}$$

Remark 2.18. If f is endependent of x_1, then the above definitions of $D(f)$ and $\sigma(f)$ agree with those given in (2.3).

The operator σ is a non-linear first order operator which is homogeneous of degree three. It is, in fact, a linear function of cofactors of the jacobian matrix for f.

There is an auxiliary operator, homogeneous of degree two, which must be studied.

$$\delta(f) \equiv \operatorname{Im}\left[\left(\frac{\partial f}{\partial x_1}\times\frac{\partial f}{\partial x_2}-\frac{\partial f}{\partial x_3}\times\frac{\partial f}{\partial x_4}\right)i+\left(\frac{\partial f}{\partial x_1}\times\frac{\partial f}{\partial x_3}+\frac{\partial f}{\partial x_2}\times\frac{\partial f}{\partial x_4}\right)j + \left(\frac{\partial f}{\partial x_1}\times\frac{\partial f}{\partial x_4}-\frac{\partial f}{\partial x_2}\times\frac{\partial f}{\partial x_3}\right)k\right]. \tag{2.19}$$

Replacing $1, i, j, k$ by any orthonormal basis e_1, e_2, e_3, e_4 for $\mathbf{H}$ does not alter δ.

$$\delta(f) = \operatorname{Im}\sum_{p<q} \left(\nabla_{e_p} f \times \nabla_{e_q} f\right) \overline{e_p\times e_q} \tag{2.19$'$}$$

The main result concerning the above differential operators is the following.

THEOREM 2.20. *Suppose* $f\colon \Omega \to \mathbf{H}$, *with* $\Omega^{\text{open}} \subset \mathbf{H}$, *is of class* C^1. *Let* $M \subset \mathbf{H}\times\mathbf{H} = \mathbf{H}+\mathbf{H}e = \mathbf{O}$ *denote the graph of* f. *Then* M *is a Cayley submanifold if and only if*

(1) $$D(f) = \sigma(f),$$

and

(2) $$\delta(f) = 0.$$

Moreover, if the jacobian determinant of f is never 1 *then equation* (2) *is an automatic consequence of equation* (1).

It follows immediately from Theorem 2.20 that there exist many interesting Cayley submanifolds. As in the special Lagrangian case, we can retreat to a potential, i.e., we consider $f=\bar{D}g$ where $g: \Omega \to \mathbf{H}$ is a C^2-map and Ω is a bounded domain in $\mathbf{H}$ with smooth boundary. The first equation in Theorem 2.20 now becomes

(1)′ $$\Delta g = \sigma(\bar{D}g).$$

The linearization of this at $g=0$ is simply the Laplace equation $\Delta g_0=0$. Hence, by implicit function techniques, we can solve the Dirichlet problem for (1)′ in a C^2 neighborhood of zero. We can assume that the resulting functions $f=\bar{D}g$ will not have Jacobian determinant identically equal to 1. Hence, the graph of each such f is a Cayley 4-fold.

Proof. It suffices to prove the theorem in the case where $f: \mathbf{H} \to \mathbf{H}$ is $\mathbf{R}$-linear. Let

$$x \equiv 1+f(1)\,e, \quad y \equiv i+f(i)\,e, \quad z \equiv j+f(j)\,e, \quad w \equiv k+f(k)\,e. \tag{2.21}$$

Then the graph of f is the oriented 4-plane spanned by x, y, z, w. Consequently, by Corollary 1.29, the graph of f is a Cayley 4-plane if and only if

$$\operatorname{Im} x \times y \times z \times w = 0.$$

Notice that

$$D(f) = f(1) - f(i)\,i - f(j)\,j - f(k)\,k,$$
$$\sigma(f) = f(i) \times f(j) \times f(k) + (f(1) \times f(j) \times f(k))\,i - (f(1) \times f(i) \times f(k))\,j + (f(1) \times f(i) \times f(j))\,k,$$
$$\delta(f) = \operatorname{Im}\,[(f(1) \times f(i) - f(j) \times f(k))\,i + (f(1) \times f(j) + f(i) \times f(k))\,j + (f(1) \times f(k) - f(i) \times f(j))\,k].$$

It follows directly from Lemma B.10 (3) that

$$\operatorname{Im} x \times y \times z \times w = \delta(f) + (\sigma(f) - D(f))\,e. \tag{2.22}$$

where $x, y, z,$ and w are defined by (2.21).

This proves the first half of Theorem 2.20. Before giving the proof of the second half of Theorem 2.20 we study the equivariance properties of the operators D, σ and δ.

LEMMA 2.23. *Suppose a is a unit quaternion, and x, y, z are arbitrary quaternions. Then,*

(1) $(xa)\times(ya)=\bar{a}(x\times y)\,a$

(2) $(ax)\times(ay)=x\times y$

(3) $(xa)\times(ya)\times(za)=(x\times y\times z)\,a$

(4) $(ax)\times(ay)\times(az)=a(x\times y\times z)$.

Proof.

$$(ax)\times(ay)\times(az)=\tfrac{1}{2}[ax\overline{ay}az-az\overline{ay}ax]=\tfrac{1}{2}[ax\bar{y}z-az\bar{y}x]=a(x\times y\times z)$$

proves (4). The proofs of (1)–(3) are similar.

Remark. Note that a is not required to be purely imaginary, as it is in the analogous Proposition B.11 for the cross product of Cayley numbers.

PROPOSITION 2.24. *Suppose that $f\colon \mathbf{H}\to\mathbf{H}$ is $\mathbf{R}$-linear and that a, b and c are unit quaternions. Then*

(1) $D(L_a\circ f\circ L_b)=aD(f)\,b$ *and* $D(f\circ R_c)=D(R_c\circ f)$

(2) $\sigma(L_a\circ f\circ L_b)=a\sigma(f)\,b$ *and* $\sigma(f\circ R_c)=\sigma(R_c\circ f)$

(3) $\delta(L_a\circ f\circ L_b)=\delta(f)$ *and* $\delta(f\circ R_c)=c\delta(R_c\circ f)\,\bar{c}$.

Proof. Let $\{e_1,\dots, e_4\}$ be an orthonormal basis of $\mathbf{H}$. Then for any unit quaternion x, $\{xe_1,\dots,xe_4\}$ is also an orthonormal basis. Hence,

$$D(L_a\circ f\circ L_b)=\sum af(be_i)\,\bar{e}_i=a\sum f(be_i)\;\overline{(be_i)}\;\;b=aD(f)\,b,$$

and also

$$D(f\circ R_c)=\sum f(e_ic)\,\bar{e}_i=\sum f(e_ic)\,c\;\overline{(e_ic)}\;=D(R_c\circ f).$$

Similarly, by using Lemma 2.23 (4), we see that

$$\sigma(L_a\circ f\circ L_b)=\sum\left(af(e_{i_1})\times af(e_{i_2})\times af(e_{i_3})\right)\overline{\left((be_{i_1})\times(be_{i_2})\times(be_{i_3})\right)}=a\sigma(f)\,b,$$

and that

$$\sigma(f\circ R_c)=\sum(f(e_{i_1})\times f(e_{i_2})\times f(e_{i_3}))\;\overline{((e_{i_1}\bar{c})\times(e_{i_2}\bar{c})\times(e_{i_3}\bar{c}))}=\sigma(R_c\circ f).$$

Finally, using Lemma 2.23 (2), we see that

$$\delta(L_a\circ f\circ L_b)=\sum(af(e_p))\times(af(e_q))\overline{(\bar{b}e_p)\times(\bar{b}e_q)}=\delta(f).$$

The argument for $\delta(f\circ R_c)$ is similar.

In the following we will refer to a Cayley 4-plane in $\mathbf{O}=\mathbf{H}\times\mathbf{H}e$ which can be graphed over $\mathbf{H}$ as a *Cayley 4-graph*. The isotropy subgroup of Spin_7 acting on $G(\Phi)$ at the point $\zeta_0\equiv 1\wedge i\wedge j\wedge k=\mathbf{H}$ is $K\equiv Sp_1\times Sp_1\times Sp_1/\mathbf{Z}_2$ (see Theorem 1.38). Suppose ζ is a Cayley 4-graph of $f\colon\mathbf{H}\to\mathbf{H}$ and $g\equiv(q_1,q_2,q_3)\in Sp_1\times Sp_1\times Sp_1$. Then $g(\zeta)$ is also a Cayley 4-graph since g leaves $\mathbf{H}$ and $\mathbf{H}e$ fixed. Recalling the action (1.39) of g on $\mathbf{O}$, one obtains that $g(\zeta)$ is the graph of $h\colon\mathbf{H}\to\mathbf{H}$ with

$$h(a)\equiv q_2f(\bar{q}_3aq_1)\bar{q}_1. \tag{2.25}$$

Under this action of $Sp_1\times Sp_1\times Sp_1$ on 4×4 matrices we have the following canonical form.

LEMMA 2.26. *Given an R-linear map $f\colon\mathbf{H}\to\mathbf{H}$, with $\det f\geq 0$, there exists $g\in Sp_1\times Sp_1\times Sp_1$ such that the h defined by* (2.25) *has the canonical form*

$$h=R_q\circ P.$$

Here $q\in Sp_1\subset\mathbf{H}$ and P is a semipositive diagonal matrix.

Proof. Recall that by the polar decomposition theorem and the diagonalization of semipositive matrices, there exist $k_1,k_2\in SO_4$ and P semi-positive diagonal so that

$$f=k_1\circ P\circ k_2. \tag{2.27}$$

Recall also that $SO_4=Sp_1\times Sp_1/\mathbf{Z}_2$ where $(a,b)\in Sp_1\times Sp_1$ acts on $\mathbf{R}^4=\mathbf{H}$ by the orthogonal transformation $k(x)=ax\bar{b}=L_a\circ R_{\bar{b}}(x)$. Hence, we may rewrite (2.27) above in the form

$$f=L_{a_1}\circ R_{\bar{b}_1}\circ P\circ L_{a_2}\circ R_{\bar{b}_2}. \tag{2.27$'$}$$

Setting $q_1=b_2$, $q_2=\bar{a}_1$, $q_3=a_2$ and $q=\bar{b}_1\bar{b}_2$, we have that $R_q\circ P=h$ as desired.

Remark 2.28. If $\det f<0$, then Lemma 2.26 remains valid with P replaced by a diagonal matrix with first eigenvalue negative and all other eigenvalues positive.

To complete the proof of Theorem 2.20 we assume that $f=R_q\circ P$ is in canonical form with $q=q_1+q_2i+q_3j+q_4k\in Sp_1\subset\mathbf{H}$, and with the diagonal matrix P having eigenvalues $\lambda_1,\lambda_2,\lambda_3,\lambda_4$. Since $f(1)=\lambda_1 q$, $f(i)=\lambda_2 iq$, $f(j)=\lambda_3 jq$ and $f(k)=\lambda_3 kq$, by definition and Lemma 2.23 we have

$$Df=\lambda_1 q-\lambda_2 iqi-\lambda_3 jqj-\lambda_4 kqk. \tag{2.29}$$

$$\sigma(f)=\lambda_{234}q-\lambda_{134}iqi-\lambda_{124}jqj-\lambda_{123}kqk. \tag{2.30}$$

$$\delta(f)=(\lambda_{12}-\lambda_{34})\operatorname{Im}\bar{q}iqi+(\lambda_{13}-\lambda_{24})\operatorname{Im}\bar{q}jqj+(\lambda_{14}-\lambda_{23})\operatorname{Im}\bar{q}kqk. \tag{2.31}$$

where $\lambda_{ij}=\lambda_i\lambda_j$ and $\lambda_{ijk}=\lambda_i\lambda_j\lambda_k$. Direct calculation yields

$$\begin{aligned}
D(f)-\sigma(f) = {} & [(\lambda_1+\lambda_2+\lambda_3+\lambda_4)-(\lambda_{234}+\lambda_{134}+\lambda_{124}+\lambda_{123})]\,q_1\\
&+[(\lambda_1+\lambda_2-\lambda_3-\lambda_4)-(\lambda_{234}+\lambda_{134}-\lambda_{124}-\lambda_{123})]\,q_2 i\\
&+[(\lambda_1-\lambda_2+\lambda_3-\lambda_4)-(\lambda_{234}-\lambda_{134}+\lambda_{124}-\lambda_{123})]\,q_3 j\\
&+[(\lambda_1-\lambda_2-\lambda_3+\lambda_4)-(\lambda_{234}-\lambda_{134}-\lambda_{124}+\lambda_{123})]\,q_4 k,\\
\delta(f) = {} & [(\lambda_{13}+\lambda_{24}+\lambda_{14}+\lambda_{23})\,q_1q_2+(-\lambda_{13}-\lambda_{24}+\lambda_{14}+\lambda_{23})\,q_3q_4]\,i\\
&+[(\lambda_{12}+\lambda_{34}+\lambda_{14}+\lambda_{23})\,q_1q_3+(\lambda_{12}+\lambda_{34}-\lambda_{14}-\lambda_{23})\,q_2q_4]\,j\\
&+[(\lambda_{12}+\lambda_{34}+\lambda_{13}+\lambda_{24})\,q_1q_4+(-\lambda_{12}-\lambda_{34}+\lambda_{13}+\lambda_{24})\,q_2q_3]\,k
\end{aligned}$$

where $\lambda_{ij}=\lambda_i\lambda_j$ and $\lambda_{ijk}=\lambda_i\lambda_j\lambda_k$. A straightforward but somewhat tedious analysis now shows that if $D(f)=\sigma(f)$, then either $\delta(f)=0$ or $\det(f)=1$. Cases where $\delta(f)\neq 0$ can arise. The simplest example is where $f=R_q$ and no component of q is zero. A generic example is where $q=q_1+q_2 i$ and $(\lambda_1,\ldots,\lambda_4)=(\mu, 1/\mu, \nu, 1/\nu)$ for $\mu,\nu>0$. We leave these final details of the proof of Theorem 2.20 to the reader.

Suppose $f\colon\Omega\to\mathbf{H}$ is of class C^1, with Ω an open subset of $\mathbf{H}$. Let $f\equiv f^1+f^2i+f^3j+f^4k$ express the components of f. Using the rows $\nabla f^1,\nabla f^2,\nabla f^3,\nabla f^4$, instead of the columns $\partial f/\partial x_1,\partial f/\partial x_2,\partial f/\partial x_3,\partial f/\partial x_4$, of the jacobian matrix J_f for f, we may define operators $\tilde{D}$, $\tilde{\sigma}$, and $\tilde{\delta}$:

$$\tilde{D}(f)\equiv\nabla f^1-\nabla f^2 i-\nabla f^3 j-\nabla f^4 k \tag{2.32}$$

$$\tilde{\delta}(f)\equiv\operatorname{Im}[(\nabla f^1\times\nabla f^2-\nabla f^3\times\nabla f^4)i+(\nabla f^1\times\nabla f^3+\nabla f^2\times\nabla f^4)j+(\nabla f^1\times\nabla f^4-\nabla f^2\times\nabla f^3)k]. \tag{2.33}$$

$$\tilde{\sigma}(f) \equiv (\nabla f^2 \times \nabla f^3 \times \nabla f^4) + (\nabla f^1 \times \nabla f^3 \times \nabla f^4)\, i - (\nabla f^1 \times \nabla f^2 \times \nabla f^4)\, j + (\nabla f^1 \times \nabla f^2 \times \nabla f^3)\, k. \tag{2.34}$$

Recall the special cases (Definition 2.8) where $f^1 \equiv 0$. One can easily check that the standard basis $1, i, j, k$ can be replaced by any orthonormal basis $e_1, \ldots, e_4$ as was done in (2.14)′, (2.17)′, and (2.19)′ thereby obtaining invariant forms for $\tilde{D}$, $\tilde{\delta}$, and $\tilde{\sigma}$. Of course, if f is **R**-linear then

$$\tilde{D}(f) = D({}^tf), \quad \tilde{\delta}(f) = \delta({}^tf) \quad \text{and} \quad \tilde{\sigma}(f) = \sigma({}^tf). \tag{2.35}$$

PROPOSITION 2.36. *Suppose $f\colon \Omega \to \mathbf{H}$ is of class C^1 with Ω an open subset of* **H**. *Then,*

(1) $\tilde{D}(f) = \overline{D(f)}$
(2) $\tilde{\sigma}(f) = \overline{\sigma(f)}$
(3) $\tilde{\delta}(f) = \overline{\delta(f)}$

Proof. We may assume $f\colon \mathbf{H} \to \mathbf{H}$ is **R**-linear. Furthermore, we may assume (cf. (2.27)) that $f = L_{a_1} \circ R_{\bar{b}_1} \circ P \circ L_{a_2} \circ R_{\bar{b}_2}$ where P is diagonal with respect to the standard basis $1, i, j, k$ and has eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. Note that ${}^tf = L_{\bar{a}_2} \circ R_{b_2} \circ P \circ L_{\bar{a}_1} \circ R_{b_1}$. Thus, by Proposition 2.24

$$\tilde{D}f = D^tf = D(L_{\bar{a}_2} \circ R_{b_2} \circ P \circ L_{\bar{a}_1} \circ R_{b_1}) = \bar{a}_2 D(R_{b_2 b_1} \circ P)\, \bar{a}_1,$$

and

$$\overline{Df} = \overline{D(L_{a_1} \circ R_{\bar{b}_1} \circ P \circ L_{a_2} \circ R_{\bar{b}_2})} = \overline{a_1 D(R_{\bar{b}_1 \bar{b}_2} \circ P)\, \bar{a}_2} = \bar{a}_2\ \overline{D(R_{\overline{b_2 b_1}} \circ P)}\ \bar{a}_1.$$

To complete the proof of (1) we must show that $D(R_q \circ P) = \overline{D(R_{\bar{q}} \circ P)}$, which follows immediately from (2.29). The proofs of parts (2) and (3) are entirely similar.

Suppose now that $f\colon \mathbf{O} \to \mathbf{H}$ where Ω is an open subset of **H**. Let $M(f) \equiv \{x + f(x)e : x \in \Omega\}$ denote the graph of f over $\Omega \subset \mathbf{H} \subset \mathbf{O}$, and let $\tilde{M}(f) \equiv \{f(x) + xe : x \in \Omega\}$ denote the graph of f over $\Omega e \subset \mathbf{H}e \subset \mathbf{O}$. Note that:

$$\tilde{M}(f) = M(-f)\, e, \tag{2.37}$$

the image of $M(-f)$ under right multiplication R_e by e. Because of Theorem 2.20, $M(f)$ is Cayley if and only if $M(-f)$ is Cayley. Proposition 1.36 implies that $M(-f)$ is Cayley if and only if $M(-f)\, e$ is Cayley. This proves the next result.

PROPOSITION 2.38. *Given $f: \Omega \to \mathbf{H}$ of class C^1 with Ω an open subset of $\mathbf{H}$, the graph of f over $\Omega \subset \mathbf{H} \subset \mathbf{O}$ is Cayley if and only if the graph of f over $\Omega e \subset \mathbf{H}e \subset \mathbf{O}$ is Cayley.*

This Proposition 2.38 is also an immediate consequence of Theorem 2.20 and the next equation (for g $\mathbf{R}$-linear):

$$\operatorname{Im}\{(g(1)+e)\times(g(i)+ie)\times(g(j)+je)\times(g(k)+ke)\} = -\delta(g) - \overline{(\sigma(g)-D(g))}\,e. \tag{2.39}$$

One cay verify this exactly as in (2.22) by using Lemma B.10 (3). Alternately, the equivarience Lemma B.11 (3) implies that the left hand side of (2.39) equals

$$-e\operatorname{Im}\{(1-g(1)\,e)\times(i-g(i)\,e)\times(j-g(j)\,e)\times(k-g(k)\,e)\}\,e.$$

Using (2.22) this equals

$$-e(\delta(-g)-(\sigma(-g)-D(-g))\,e)\,e = \overline{\delta(g)} - \overline{(\sigma(g)-D(g))}\,e.$$

Of course, using Proposition 2.36 we have

$$\operatorname{Im}\{(g(1)+e)\times(g(i)+ie)\times(g(j)+je)\times(g(k)+ke)\} = \tilde{\delta}(g)-(\tilde{\sigma}(g)-\tilde{D}(g))\,e. \tag{2.40}$$

THEOREM 2.50. *Suppose $f: \Omega \to \mathbf{H}$ is a function of class C^1 on an open subset Ω of $\mathbf{H}$. Let $\tilde{M}(f) \equiv \{f(x)+xe : x \in \Omega\}$ denote the graph of f over $\Omega e \subset \mathbf{H}e$ in $\mathbf{O}$. Then $\tilde{M}$ is Cayley if and only if*

(1) $\tilde{D}(f) = \tilde{\sigma}(f)$

and

(2) $\tilde{\delta}(f) = 0$.

Moreover, if the Jacobian determinant of f is never 1, then equation (2) is an immediate consequence of equation (1).

Proof. There are two options. The first is to combine Theorem 2.20, Proposition 2.38, and Proposition 2.36. The second is to the use equation (2.40) directly for the first part of the theorem.

Remark 2.51. If $\operatorname{Re} f = f^1$ vanishes then $\tilde{M}(f) \subset \operatorname{Im}\mathbf{O}$ and in this case $\tilde{M}$ is Cayley if and only if it is coassociative (Remark 2.12(a)). Moreover, $f^1 \equiv 0$ implies that the Jacobian determinant of f vanishes. Consequently, Theorem 2.9 concerning coassociative submanifolds is a corollary of Theorem 2.50.

Finally we conclude this section by establishing the relationships between the operators D, σ, δ on f and on f^{-1}.

PROPOSITION 2.52. *Suppose f is a class C^1 diffeomorphism of an open subset of* $\mathbf{H}$ *to an open subset of* $\mathbf{H}$. *Then setting $y=f(x)$ we have that:*

(1) $D(f^{-1})_y=(\det J_f)^{-1}\,\overline{\sigma(f)_x}$

(2) $\sigma(f^{-1})_y=(\det J_f)^{-1}\,\overline{D(f)_x}$

(3) $\delta(f^{-1})_y=(\det J_f)^{-1}\,\overline{\delta(f)_x}$

In particular, $D(f)=\sigma(f)$ if and only if $D(f^{-1})=\sigma(f^{-1})$.

Proof. Similar to that of Proposition 3.36 and hence omitted.

IV.3. Coassociative fourfolds invariant under Sp_1

One of the most interesting examples of a coassociative submanifold is found by looking for symmetric solutions. The group $Sp_1=S^3\subset\mathbf{H}$ acts on $\mathbf{O}$ in several ways. The action we wish to consider is given by

$$a+be\mapsto qa\bar{q}+b\bar{q}e, \tag{3.1}$$

for each $q\in Sp_1\subset\mathbf{H}$.

Recall from (1.9) that this embeds Sp_1 into G_2, and therefore this action preserves the coassociative calibration ψ. Note that the orbit of any point $a+be\in\mathbf{O}$, with $b\neq 0$ is diffeomorphic to S^3.

To obtain a coassociative fourfold invariant under Sp_1 we choose a fixed unit vector $\varepsilon\in\operatorname{Im}\mathbf{H}$ and seek a curve in the half plane $\mathbf{R}\times\mathbf{R}^+\cong\mathbf{R}\varepsilon\oplus\mathbf{R}^+e\subset\operatorname{Im}\mathbf{O}$ which is swept out into a coassociative submanifold (of dimension 4) under the action of Sp_1. Let (s,r) denote coordinates in $\mathbf{R}\times\mathbf{R}^+$.

THEOREM 3.2. *Suppose $\varepsilon\in\operatorname{Im}\mathbf{H}$ is a fixed unit vector and $c\in\mathbf{R}$. Then*

$$M_c\equiv\{sq\varepsilon\bar{q}+r\bar{q}e\colon q\in Sp_1 \quad and \quad s(4s^2-5r^2)^2=c\}$$

is a coassociative submanifold invariant under Sp_1. In particular, if $c=0$ and $s=(\sqrt{5}/2)r$ then

$$\mathrm{M}_0\equiv\left\{r\left(\frac{\sqrt{5}}{2}q\varepsilon\bar{q}+\bar{q}e\right)\colon q\in\mathrm{Sp}_1=\mathrm{S}^3 \quad\text{and}\quad r\in R^+\right\}$$

which is the cone on the graph of the Hopf map $\eta: S^3 \to S^2$ *defined by* $\eta(q) \equiv \frac{1}{2}\sqrt{5}\,\bar{q}\varepsilon q$, *is a coassociative 4-fold.*

Remark. The cone M_0 above is the graph of the function $\eta: \mathbf{H} \to \operatorname{Im} \mathbf{H}$ defined by

$$\eta(x) = \frac{\sqrt{5}}{2|x|}\bar{x}\varepsilon x \quad \text{for all } x \in \mathbf{H}. \tag{3.3}$$

This graph was first discovered by the second author and R. Osserman [LO]. They observed that η represents a Lipschitz solution to the non-parametric minimal surface system which is not C^1. We have now shown that the graph of η is absolutely area minimizing. Furthermore, we have the following.

THEOREM 3.4. *The function* η *given by* (3.3) *is a Lipschitz solution to* $D(\eta) = \sigma(\eta)$ *which is not* C^1.

This makes sharp the regularity result that C^1 solutions to $D(f) - \sigma(f) = 0$ are real-analytic.

Before proving Theorem 3.2 we establish two lemmas.

LEMMA 3.5. *Suppose that* $M = \{g(x) + x \cdot e : x \in \mathbf{H}\} \subset \operatorname{Im} \mathbf{O}$ *is the graph of a function* $g: \mathbf{H} \to \operatorname{Im} \mathbf{H}$. *Then* M *is invariant under the action* (3.1) *of* Sp_1 *if and only if*

$$g(x) = \frac{1}{|x|^2}\bar{x} g(|x|)\, x \tag{3.6}$$

for all $x \in \mathbf{H}$.

Proof. If M is Sp_1-invariant, then for each $q \in Sp_1 \subset \mathbf{H}$ and each $x \in \mathbf{H}$, the vector $qg(\mathrm{x})\,\bar{q} + x\bar{q}e$ also belongs to M. Consequently,

$$g(x\bar{q}) = qg(x)\,\bar{q}$$

for all $q \in Sp_1$ and all $x \in \mathbf{H}$. Replace x by $|x|$ and q by $\bar{x}/|x|$ to obtain (3.6).

LEMMA 3.7. *Suppose* $\varepsilon \in \operatorname{Im} \mathbf{H}$ *is a fixed unit vector, and* $\varphi: \mathbf{R}^+ \to \mathbf{R}$, *is a given function,* $s = \varphi(r)$. *Define* $g: \mathbf{H} \to \operatorname{Im} \mathbf{H}$

$$g(x) = \frac{\bar{x}\varepsilon x}{|x|^2}\varphi(|x|)$$

so that $M \equiv \operatorname{graph}(g)$ *is* Sp_1*-invariant. Then* M *is coassociative if and only if*

$$\varphi'(r) = \frac{4r\varphi(r)}{4\varphi^2(r) - r^2}. \tag{3.8}$$

Since the right-hand side of (3.8) is homogeneous of degree zero in r and $s=\varphi(r)$, the standard substitution $z=r/s$ yields the integral curves

$$s(4s^2-5r^2)^2=c,$$

thereby completing the proof of Theorem 3.2.

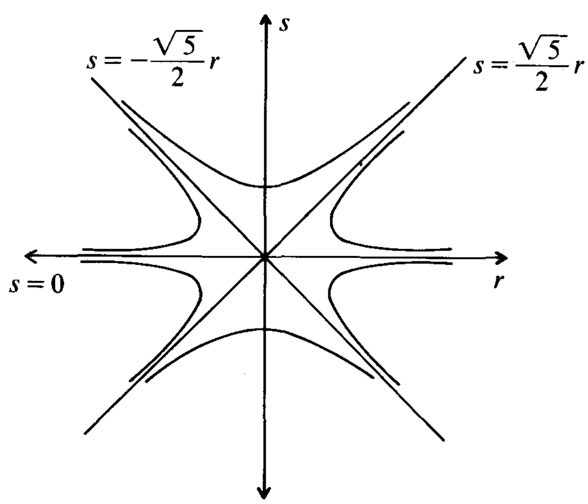


Fig. 3.10

Proof of Lemma 3.7. We must compute $Dg-\sigma g$. The fact that $g(x\bar{q})=qg(x)\bar{q}$ for $q\in Sp_1$, implies that $dg_{x\bar{q}}(u)=qdg_x(uq)\bar{q}$, where dg_x denotes the differential of g at x. From Proposition 2.24 we have that $(Dg)(x\bar{q})=q(Dg)(x)$ and $(\sigma g)(x\bar{q})=q(\sigma g)(x)$. Consequently, it suffices to compute $Dg-\sigma g$ at $x=|x|=r\in\mathbf{R}^+$. For convenience, and without loss of generality, we set $\varepsilon=i$ so that

$$g(x)=\bar{x}ix\frac{1}{r^2}\varphi(r).$$

Straightforward calculation then shows that

$$\frac{\partial g}{\partial x_1}(r)=\varphi'(r)\,i,\quad \frac{\partial g}{\partial x_2}(r)=0,\quad \frac{\partial g}{\partial x_3}(r)=\frac{2}{r}\varphi(r)\,k,\quad \frac{\partial g}{\partial x_4}(r)=-\frac{2}{r}\varphi(r)j.$$

Computing directly from the definitions of D and σ (See (2.14)), we find that

$$(Dg)(r)=\left(\frac{4}{r}\varphi(r)+\varphi'(r)\right)i$$

and

$$(\sigma g)(r) = \frac{4}{r^2}\varphi^2(r)\,\varphi'(r)\,i.$$

Therefore, we have that

$$(Dg-\sigma g)(r\bar{q}) = \frac{1}{r^2}(4r\varphi+(r^2-4\varphi^2)\,\varphi')\,\bar{q}i, \tag{3.11}$$

and the proof is complete.

IV.4. Boundaries of exceptional submanifolds

The purpose of this section is to give a local characterization of the boundaries of associative, coassociative and Cayley submanifolds, at least in the real analytic case. We consider the boundary as a set of initial data and apply the Cartan-Kähler theorem according to the procedure outlined in Section 6 of Chapter II.

We begin with the case of associative submanifolds.

THEOREM 4.1. *Suppose N is a two dimensional, real analytic submanifold of* $\mathrm{Im}\,\mathbf{O}\cong\mathbf{R}^7$. *Then there exists a unique real analytic associative submanifold M of* $\mathrm{Im}\,\mathbf{O}$ *which contains N.*

Proof. Let I denote the ideal generated by the forms $\psi_1,\dots,\psi_7\in\Lambda^3(\mathrm{Im}\,\mathbf{O})^*$, obtained by taking the components of the $\mathrm{Im}\,\mathbf{O}$-valued alternating 3-form $[x,y,z]$ on $\mathrm{Im}\,\mathbf{O}$. By Theorem 1.6 we know that the 3-dimensional integral elements of I are exactly the associative 3-planes, up to orientation. Since I contains no forms of degree <3, any 1 or 2 dimensional plane is a regular integral element. We now make an elementary observation.

LEMMA 4.2. *Given any 2-plane η in* $\mathrm{Im}\,\mathbf{O}$, *there exists a unique associative 3-plane ξ with $\eta\subset\xi$.*

Proof. Choose an orthonormal basis $\{u,v\}$ for η. Then $u\times v$ is a unit vector in $\mathrm{Im}\,\mathbf{O}$ orthogonal to η. Let $\xi\cong u\wedge v\wedge(u\times v)$. Then $\varphi(\xi)=1$ and so ξ is associative. Finally, if ξ' is any associative 3-plane containing η, then $\xi'\cong u\wedge v\wedge w$ for some unit vector w orthogonal to η. Since $1=\varphi(\xi')=\langle w, u\cdot v\rangle$, we must have $w=u\cdot v=u\times v$ and the uniqueness is proved.

Theorem 4.1 now follows from the Cartan-Kähler theorem.

We now consider boundaries of Cayley submanifolds.

THEOREM 4.3. *Suppose N is a three-dimensional, real analytic submanifold of* $\mathbf{O}$. *Then there exists a unique real-analytic Cayley submanifold M of* $\mathbf{O}$ *which contains N.*

Proof. Let I denote the ideal generated by the forms $\psi_1, \ldots, \psi_7 \in \Lambda^4 \mathbf{O}^*$ which are the axis components of the vector-valued alternating form $\operatorname{Im} x \times y \times x \times w$ on $\mathbf{O}$. From Theorem 1.28 we know that a 4-plane is an integral element of I if and only if it is a Cayley 4-plane. Any k-plane with $k<4$ is trivially an integral element, and all integral elements are regular. In fact, we have the following.

LEMMA 4.4. *Given any 3-plane η in* $\mathbf{O}$, *there is a unique Cayley 4-plane ξ with $\eta \subset \xi$.*

Proof. The plane ξ is given by $\xi \cong (x \times y \times z) \wedge x \wedge y \wedge z$ where $\{x, y, z\}$ is any orthonormal basis of η. The Lemma is a direct consequence of the formula

$$\Phi(w, x, y, z) = \langle w, x \times y \times z \rangle.$$

Theorem 4.3 now follows directly from the Cartan-Kähler theorem.

Finally, we consider the boundaries of coassociative submanifolds. In this case there is a necessary condition.

PROPOSITION 4.5. *If N is the boundary of a coassociative submanifold M, then the unit tangent space η_x to N at each point x must be an* $\mathbf{O}$ *generating 3-plane; that is η_x must satisfy the equation*

$$\varphi(\eta_x) \equiv 0,$$

where φ is the associative calibration.

Proof. Choose an orthonormal basis x, y, z for the tangent space to N at a particular point $p \in N$. Then by Lemma 4.4 the tangent space to the coassociative submanifold M at p must be spanned by x, y, z and $x \times y \times z$. Now Re $x \times y \times z = \varphi(x, y, z)$ must vanish since $M \subset \operatorname{Im} \mathbf{O}$.

This necessary condition is also sufficient.

THEOREM 4.6. *Suppose N is an* $\mathbf{O}$ *generating 3-dimensional, real analytic submanifold of* $\operatorname{Im} \mathbf{O}$. *Then there exists a unique real analytic, coassociative submanifold M of* $\operatorname{Im} \mathbf{O}$ *which contains N.*

Proof. Let $\tilde{I}$ denote the ideal generated by $\varphi \in \Lambda^3 (\operatorname{Im} \mathbf{O})^*$ and by

$\psi_1, \dots, \psi_7 \in \Lambda^4(\mathrm{Im}\,\mathbf{O})^*$, the coordinates of $[x, y, z, w]$ on $\mathrm{Im}\,\mathbf{O}$. In the proof of Proposition 4.5 we have shown that

$$i_\xi^* \varphi = 0 \quad \text{for each coassociative 4-plane } \xi. \tag{4.7}$$

Therefore, Theorem 1.18 (the coassociative equality) implies that the 4-dimensional integral elements of $\tilde{I}$ are the coassociative 4-planes. Obviously the 3-dimensional integral elements of $\tilde{I}$ are the elements η satisfying $\varphi(\eta)=0$. These integral elements are regular since each such η is contained in a unique coassociative 4-plane. Theorem 4.6 now follows from the Cartan-Kähler theorem.

Appendix IV.A. The Cayley-Dickson process

In this appendix we collect together the basic facts about the normed algebras, **R**, **C**, **H**, and **O** (cf. Curtis [Cu]). By a *normed algebra* we mean a (not necessarily associative) finite dimensional algebra over **R** with multiplicative unit 1, and equipped with an inner product $\langle\ ,\ \rangle$ whose associated norm $|\ |$ satisfies

$$|xy| = |x|\,|y| \quad \text{for all } x, y.$$

Given a normed algebra B we will adopt the following notational conventions. Let $\mathrm{Re}\,B$ denote the span of $1 \in B$, and let $\mathrm{Im}\,B$ denote the orthogonal complement of $\mathrm{Re}\,B$. Then each $x \in B$ has a unique orthogonal decomposition

$$x = x_1 + x',$$

with $x_1 \in \mathrm{Re}\,B$ and $x' \in \mathrm{Im}\,B$. (Occasionally we let $\mathrm{Re}\,x$ denote x_1 and $\mathrm{Im}\,x$ denote x'.) *Conjugation* is defined by

$$\bar{x} \equiv x_1 - x'.$$

Thus

$$x_1 = \tfrac{1}{2}(x+\bar{x}) \quad x' = \tfrac{1}{2}(x-\bar{x}).$$

Given $w \in B$, let R_w denote the linear operator right multiplication by w. Similarly let L_w denote the linear operator left multiplication by w.

The elementary facts concerning R_w, L_w, and conjugation are:

$$\langle R_w x, R_w y\rangle = \langle x, y\rangle |w|^2, \quad \langle L_w x, L_w y\rangle = \langle x, y\rangle |w|^2 \tag{A.1}$$

$$^tR_w = R_{\bar{w}}, \quad ^tL_w = L_{\bar{w}}. \tag{A.2}$$

$$\bar{\bar{x}} = x, \quad \overline{xy} = \bar{y}\bar{x}, \quad \langle x, y\rangle = \operatorname{Re} x\bar{y} = \tfrac{1}{2}(x\bar{y}+y\bar{x}), \quad x\bar{x} = |x|^2. \tag{A.3}$$

Polarizing the identity $|xw|^2=|x|^2|w|^2$ in x yields the first half of (A.1). To prove (A.2) we may assume $\operatorname{Re} w=0$. Repeated use of (A.1) yields:

$$\langle x, y\rangle (1+|w|^2) = \langle x(1+w), y(1+w)\rangle = \langle x, y\rangle (1+|w|^2)+\langle x, yw\rangle + \langle xw, y\rangle,$$

which proves the first half of (A.2). Repeated use of (A.2) establishes $\overline{xy}=\bar{y}\bar{x}$.

An object of fundamental interest on a normed algebra B is the *associator* defined by

$$[x, y, z] = (xy)z - x(yz)$$

for $x, y, z \in B$. The following weak form of associativity always holds on a normed algebra.

LEMMA A.4. *The trilinear form $[x, y, z]$ on B is alternating.*

This means that the associator vanishes whenever two of its arguments are equal. An algebra on which the associator is alternating is called *alternative*. The lemma states that any normed algebra is alternative.

Proof. Note that the associator vanishes if one of its variables is real. Hence, it suffices to show that the associator vanishes whenever two of its variables are set equal to $w \in \operatorname{Im} B$. Now we prove that $[x, w, \bar{w}]=0$. Since $w\bar{w}=|w|^2$, we must show that $(xw)\bar{w}=x|w|^2$. Note that by (A.2) $\langle (xw)\bar{w}, y\rangle=\langle xw, yw\rangle$ which by (A.1) equals $\langle x, y\rangle |w|^2$. Since $[x, w, \bar{w}]=0$, $[w, y, w]=-[w, w, y]$. Thus it remains to prove that $[w, \bar{w}, z]=0$ which follows in the same way as $[x, w, \bar{w}]=0$.

As an immediate consequence of the Lemma A.4 and the fact that $x\bar{x}=|x|^2$ we have that:

LEMMA A.5. (a) *Each nonzero element $x \in B$ has a unique left and right inverse $x^{-1}=\bar{x}/|x|^2$.*

(b) *Given elements $x, y \in B$ with $x \neq 0$, the equations $xw=y$ and $wx=y$ can be (uniquely) solved for w with $w=\bar{x}y/|x|^2$ and $w=y\bar{x}/|x|^2$ respectively.*

Note that (a) alone without the weak associativity (Lemma A.4) does not imply (b).

From the equation $2\langle x, y\rangle=x\bar{y}+y\bar{x}$, we have that

$$2\langle x, y\rangle w - x(\bar{y}w) - y(\bar{x}w) = [x, \bar{y}, w]+[y, \bar{x}, w].$$

Consequently Lemma A.4 can be reformulated as

$$x(\bar{y}w)+y(\bar{x}w)=2\langle x,y\rangle w, \tag{A.6a}$$

and (proved similarly),

$$(w\bar{y})x+(w\bar{x})y=2\langle x,y\rangle w. \tag{A.6b}$$

In particular, if x and $y\in B$ are orthogonal and $w\in B$ is arbitrary, then

$$x\bar{y}=-\bar{y}x \tag{A.7a}$$

$$x(\bar{y}w)=-y(\bar{x}w) \tag{A.7b}$$

$$(w\bar{y})x=-(w\bar{x})y. \tag{A.7c}$$

The equations in (A.7), which enable one to interchange x and y if they are orthogonal, can be used to motivate the Cayley-Dickson process.

LEMMA A.8. *Suppose that A is a subalgebra (with* $1\in A$*) of the normed algebra B and that* $\varepsilon\in A^{\perp}$ *with* $|\varepsilon|=1$. *Then* $A\varepsilon$ *is orthogonal to A and* $(a+b\varepsilon)(c+d\varepsilon)=(ac-\bar{d}b)+(da+b\bar{c})\varepsilon$ *for all* $a,b,c,d\in A$.

Proof. Note $x\in A$ if and only if $\bar{x}\in A$ since $1\in A$. Then $a,b\in A$ implies $\bar{b}a\in A$ and hence $\langle a,b\varepsilon\rangle=\langle\bar{b}a,\varepsilon\rangle=0$ $\forall a,b\in A$. This proves $A\perp A\varepsilon$. Note that $\varepsilon\perp A$ implies $\varepsilon\in\operatorname{Im}B$ and hence $1=\|\varepsilon\|^2=\varepsilon\bar{\varepsilon}=-\varepsilon^2$ so that $\varepsilon^2=-1$. Expanding we have $(a+b\varepsilon)(c+d\varepsilon)=ac+(b\varepsilon)(d\varepsilon)+a(d\varepsilon)+(b\varepsilon)c$. Next we make repeated use of (A.7) above to rewrite the last three terms.

$$(b\varepsilon)(d\varepsilon)=-\bar{d}((\overline{b\varepsilon})\varepsilon)=\bar{d}((\varepsilon\bar{b})\varepsilon)=-\bar{d}((\varepsilon\bar{\varepsilon})b)=-\bar{d}b$$

$$a(d\varepsilon)=a(\varepsilon\bar{d})=\varepsilon(\bar{a}\bar{d})=\overline{(\bar{a}\bar{d})}\varepsilon=(da)\varepsilon$$

$$(b\varepsilon)c=(b\bar{c})\varepsilon.$$

This completes the proof.

Definition A.9. Suppose A is an algebra. Motivated by lemma A.8 we define a product on $A\oplus A$ by:

$$(a,b)(c,d)\equiv(ac-\bar{d}b,\ da+b\bar{c}).$$

The new algebra $B\equiv A\oplus A$ is said to be obtained from A via the *Caley-Dickson process*.

Remark. Under the hypothesis of Lemma A.8 the conclusion may be restated as follows. $A+A\varepsilon$ is a subalgebra of B isomorphic to the algebra $A\oplus A$ obtained from A via the Caley-Dickson process.

Definition A.10. $\mathbf{C}\equiv\mathbf{R}\oplus\mathbf{R}$, $\mathbf{H}\equiv\mathbf{C}\oplus\mathbf{C}$ and $\mathbf{O}\equiv\mathbf{H}\oplus\mathbf{H}$ via the Caley-Dickson process. With the standard basis $1, i, j, k, e, ie, je, ke$ for $\mathbf{O}$ we have $\mathbf{C}=\mathbf{R}+\mathbf{R}i$, $\mathbf{H}=\mathbf{C}+\mathbf{C}j$, $\mathbf{O}=\mathbf{H}+\mathbf{H}e$.

LEMMA A.11. *Suppose $B=A\oplus A$ is obtained from A via the Caley-Dickson process then*

(1) *B is commutative if and only if $A=\mathbf{R}$.*
(2) *B is associative if and only if A is commutative.*
(3) *B is alternative if and only if A is associative.*

Proof. Suppose $x=(a,\alpha)=a+\alpha\varepsilon$, $y=(b,\beta)=b+\beta\varepsilon$, and $z=(c,\gamma)=c+\gamma\varepsilon$. One can easily verify the following formulas.

(1)′ $\frac{1}{2}[x,y]=\frac{1}{2}[a,b]+\operatorname{Im}\bar{\alpha}\beta+(\beta\operatorname{Im}a-\alpha\operatorname{Im}b)\,\varepsilon$.

(2)′ $[x,y,z] = [a,\bar{\gamma}\beta]+[b,\bar{\alpha}\gamma]+[c,\bar{\beta}\alpha]+(\alpha[b,c]-\beta[a,c]+\gamma[a,b]+\alpha[\bar{\beta},\gamma]+[a,\gamma]\bar{\beta}+\gamma[\alpha,\bar{\beta}])\,\varepsilon$, assuming A is associative.

(3)′ $[x,\bar{x},y]=[a,\bar{\beta},a]+[\alpha,\bar{b},a]\,\varepsilon$.

The lemma now follows easily.

Hence, $\mathbf{C}$ is commutative and associative, $\mathbf{H}$ is associative but not commutative, $\mathbf{O}$ is alternative but not commutative and not associative, and $B=\mathbf{O}\oplus\mathbf{O}$ is not an alternative algebra. In particular $B=\mathbf{O}\oplus\mathbf{O}$ is not a normed algebra, although one can check that $x\bar{x}=|x|^2$ is still valid, so that each non-zero element has a unique left and right inverse.

THEOREM A.12. (Hurwitz). *The only normed algebras over* $\mathbf{R}$ *are* $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$ *and* $\mathbf{O}$.

Proof. Suppose B is a normed algebra. Let $A_1=\operatorname{Re}B\,(\cong\mathbf{R})$. If $A_1=B$ we are finished. If not, choose $\varepsilon_1\in A_1^\perp$ with $|\varepsilon_1|=1$, and let $A_2=A_1+A_1\,\varepsilon_1\,(\cong\mathbf{C})$. By Lemma A.8, A_2 is a normed subalgebra of B isomorphic to $\mathbf{C}$. If $A_2=B$ we are finished. If not choose $\varepsilon_2\in A_2^\perp$ with $|\varepsilon_2|=1$ and let $A_3=A_2+A_2\,\varepsilon_2$. Again by Lemma A.8, $A_3\cong\mathbf{H}$. If $A_3=B$ we are finished. If not choose $\varepsilon_2\in A_3^\perp$ with $|\varepsilon_2|=1$ and let $A_4=A_3+A_3\,\varepsilon$. By Lemma A.8, $A_4\cong\mathbf{O}$.

Finally, we must show $A_4=\mathbf{O}=B$. If not choose $\varepsilon\in A_4^\perp$ with $|\varepsilon|=1$ and let

$A_5 = A_4 + A_4\varepsilon$. By Lemma A.8, $A_5 \cong \mathbf{O} \oplus \mathbf{O}$ which is not a normed algebra. This is a contradiction since $A_5 \subset B$ and B is a normed algebra.

The weak form of associativity, corresponding to the fact that the associator is alternating, can be strengthened.

THEOREM A.13 (Artin). *The subalgebra A with unit generated by any two elements of $\mathbf{O}$ is associative (in fact, A is contained in a quaternion subalgebra of $\mathbf{O}$).*

Proof. Suppose $x, y \in \mathbf{O}$ are generators. If $A \cong \mathbf{R}$ we are finished. If not we may assume $\operatorname{Im} x \neq 0$ and define $\varepsilon_1 \equiv \operatorname{Im} x/|\operatorname{Im} x|$. Then by Lemma A.8, $A_1 \equiv \mathbf{R} + \mathbf{R}\varepsilon_1 \cong \mathbf{C}$ is associative. If $y \in A_1$, then $A = A_1$ and we are finished. If not, write $y = y_1 + y_2$ with $y_1 \in A_1$, $y_2 \in A_1^\perp$ and $y_2 \neq 0$. Now let $\varepsilon_2 = y_2/|y_2|$. Then $A_2 \equiv A_1 + A_1\varepsilon_2 \cong \mathbf{H}$ is associative. Also $x, y \in A_2$ and hence $A = A_2 \cong \mathbf{H}$.

PROPOSITION A.14. *$\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$ and $\mathbf{O}$ are normed.*

Proof. It suffices to show that $\mathbf{O}$ is normed. Suppose $x, y \in \mathbf{O}$. Then by Theorem A.13 the subalgebra with unit generated by x, y is associative. Therefore $|xy|^2 = (xy)\overline{(xy)} = (xy)(\bar{y}\bar{x}) = xy\bar{y}\bar{x} = |x|^2|y|^2$

The Cayley-Dickson process can be used to generate the automorphisms of $\mathbf{O}$.

LEMMA A.15. *Suppose e_1, e_2, e_3 is an orthonormal triple in $\operatorname{Im}\mathbf{O}$ with $e_3 \perp e_1 e_2$. Then there exists a (unique) automorphism g of $\mathbf{O}$ sending $i \to e_1$, $j \to e_2$, and $e \to e_3$.*

Proof. Applying Lemma A.8 succesively we have

$$g\colon \mathbf{C} \xrightarrow{\cong} \mathbf{R} + \mathbf{R}e_1 \equiv A_1, \quad g\colon \mathbf{H} \xrightarrow{\cong} A_1 + A_1 e_2 = A_2,$$

and

$$g\colon \mathbf{O} \xrightarrow{\cong} A_2 + A_2 e_3 \equiv \mathbf{O}.$$

Finally we shall need the Moufang identities:

LEMMA A.16.

(a) $$(xyx)z = x(y(xz))$$

(b) $$z(xyx) = ((zx)y)x$$

(c) $$(xy)(zx) = x(yz)x.$$

Proof. Consider the differences of the left and right hand sides above. They vanish if any two of the variables are equal by Theorem A.13. Since they are linear in y and z, we may assume x, y, z are orthogonal. By repeated use of (A.6) we have that: both sides in (a) equal $-|x|^2\bar{y}z$, both sides in (b) equal $-|x|^2\bar{z}y$, and both sides in (c) equal $-|x|^2\bar{z}\bar{y}+2\langle x, yz\rangle x$.

Appendix IV.B. Multiple cross products of Cayley numbers

First we consider the cross product of two octonions. This is well known but is included to motivate the cross product of three and four octonions.

Definition B.1. Let $x\times y\equiv-\frac{1}{2}(\bar{x}y-\bar{y}x)=\operatorname{Im}\bar{y}x$, for all $x, y\in\mathbf{O}$. This product will be called the *cross product* of x and y.

We consider the next Lemma to justify the terminology "cross product".

LEMMA B.2.

(1) *$x\times y$ is alternating*

(2) $|x\times y|=|x\wedge y|$.

Proof. (1) The cross product is alternating since $x\times x=0$.

(2) Consequently, it suffices to compute $|x\times y|$ in the special case where x and y are orthogonal. By condition (A.7a), if x and y are orthogonal than $x\times y=\bar{y}x$. Therefore $|x\times y|=|\bar{y}x|=|x|\,|y|=|x\wedge y|$ as desired.

Remark. The proof that $|x\times y|=|x\wedge y|$ explains why we needed the conjugates in the definition of $x\times y$; namely we wanted the vanishing of $\langle x, y\rangle$ to imply that $x\times y$ could be expressed as one term $\bar{y}x$.

Remark. If $x, y\in\operatorname{Im}\mathbf{H}\subset\mathbf{O}$ then $x\times y$ can be expressed in many ways

$$x\times y=\tfrac{1}{2}[x, y]=xy-\tfrac{1}{2}(xy+yx)=xy+\langle x, y\rangle=xy-\operatorname{Re}xy=\operatorname{Im}xy;$$

and is just the usual cross product on $\mathbf{R}^3\cong\operatorname{Im}\mathbf{H}$.

The natural extension of the cross product to three vectors is the alternation of $x(\bar{y}z)$. (Note that this is not $x\times(y\times z)$!) Since $y\times x$ is already alternating this equals the alternation of $-x(y\times z)$ which is just $-\frac{1}{3}(x(y\times x)+y(z\times x)+z(x\times y))$. However, this expression can be simplified by repeated use of (A.7). Therefore we adopt as our definition of the *triple cross product*:

Definition B.3. Let $x\times y\times z\equiv\frac{1}{2}(x(\bar{y}z)-z(\bar{y}x))$, for all $x, y, z\in\mathbf{O}$.

LEMMA B.4.

(1) *$x\times y\times z$ is alternating on* $\mathbf{O}$

(2) $|x\times y\times z|=|x\wedge y\wedge z|$ *for all* $x, y, z\in\mathbf{O}$.

Proof. (1) Since the subalgebra generated by any two elements is associative.

$$x\times x\times z=\tfrac{1}{2}(x(\bar{x}z)-z|x|^2)=0$$

$$x\times y\times y=\tfrac{1}{2}(x|y|^2-y(\bar{y}x))=0.$$

Obviously $x\times y\times x=0$, and so $x\times y\times z$ is alternating.

(2) As a consequence of (1) we may assume that x, y, z are orthogonal. Then repeated use of (A.7) shows that $-z(\bar{y}x)=x(\bar{y}z)$ and hence that

$$x\times y\times z=x(\bar{y}z) \quad \text{if } x, y, z \text{ are orthogonal.} \tag{B.5}$$

Therefore $|x\times y\times z|=|x(\bar{y}z)|=|x|\,|y|\,|z|=|x\wedge y\wedge z|$.

Now we consider a *fourfold cross product*. This is defined as the alternation of $\bar{x}(y\times z\times w)$.

Definition B.6. Let

$$x\times y\times z\times w\equiv\tfrac{1}{4}(\bar{x}(y\times z\times w)+\bar{y}(z\times x\times w)+\bar{z}(x\times y\times w)+\bar{w}(y\times x\times z)),$$

for all $x, y, z, w\in\mathbf{Q}$.

LEMMA B.7.

(1) *$x\times y\times z\times w$ is alternating*

(2) $|x\times y\times z\times w|=|x\wedge y\wedge z\wedge w|$.

Proof. (1) Since $y\times z\times w$ is alternating, it is obvious from the definition that $x\times y\times z\times w$ is also alternating.

Since both sides of (2) are alternating we may assume for the remainder of the proof that x, y, z, w are mutually orthogonal.

$$\text{If } x, y, z, w\in\mathbf{Q} \text{ are orthogonal, then } x\times y\times z\times w=\bar{x}(y(\bar{z}w)). \tag{B.8}$$

As in the proof of (B.5), repeated use of (A.7) can be used to verify (B.8). The details are omitted.

(2) This follows immediately from (B.8) since $\mathbf{Q}$ is a normed algebra.

The real part of a cross product is an alternating scalar valued form on **Q** and thus a candidate for a calibration on **Q**. Given $x \in \mathbf{Q}$, we let $x_1 \equiv \operatorname{Re} x$ and $x' \equiv \operatorname{Im} x$.

LEMMA B.9.

(1) $\operatorname{Re} x \times y = 0$

(2) $\operatorname{Re} x \times y \times z = \langle x', y'z' \rangle$

(3) $\operatorname{Re} x \times y \times z \times w = \langle x, y \times z \times w \rangle$.

Proof. (1) $2 \operatorname{Re} x \times y = -\frac{1}{2}(\bar{x}y - \bar{y}x) - \frac{1}{2}\overline{(\bar{x}y - \bar{y}x)} = 0$.

(2) First note that $1 \times y \times z = -y \times z$ which has no real part by (1). Therefore $\operatorname{Re} x \times y \times z = \operatorname{Re} x' \times y' \times z'$. In (1.1) it was shown that $\langle x', y'z' \rangle$ is alternating. Therefore we need only prove (2) for x, y, z purely imaginary and orthogonal. In this case we may apply (B.5).

$$\operatorname{Re} x \times y \times z = \operatorname{Re} x(\bar{y}z) = -\operatorname{Re} x(yz) = \langle x, yz \rangle.$$

(3) In Lemma 1.22 we have shown that $\langle x, y \times z \times w \rangle$ is alternating. Therefore we need only prove (3) for x, y, z, w pairwise orthogonal. In this case we can apply (B.5) and (B.7). $\operatorname{Re} x \times y \times z \times w = \operatorname{Re} \bar{x}(y(\bar{z}w)) = \langle x, y(\bar{z}w) \rangle = \langle x, y \times z \times w \rangle$ as desired.

Remark. The fact that

$$\langle x, y \times z \times w \rangle \text{ is alternating in } x, y, z, w \in \mathbf{O},$$

can be reformulated as

$$x \times y \times z \text{ is orthogonal to } x, y, \text{ and } z \in \mathbf{O}.$$

The various cross products can be expressed in a very elegant way in terms of ordered pairs of quaternions. The next lemma is used to derive the Cayley partial differential equation.

LEMMA B.10. *Let x, y, z, w denote the octonions $a + \alpha e$, $b + \beta e$, $c + \gamma e$, $d + \delta e$ respectively, where $a, b, c, d, \alpha, \beta, \gamma, \delta$ are quaternions. Then*

(1) $$x \times y = a \times b - \alpha \times \beta + (a\bar{b} - \beta\bar{a})e$$

(2) $$x \times y \times z = a \times b \times c + a(\beta \times \gamma) + b(\gamma \times \alpha) + c(\alpha \times \beta) + (\alpha \times \beta \times \gamma + \alpha(b \times c) + \beta(c \times a) + \gamma(a \times b))e$$

$$(3)\qquad \begin{aligned} x\times y\times z\times w &= \langle a, b\times c\times d\rangle + \langle a, \beta\times\gamma\times\delta\rangle \\ &\quad +(\alpha\times\beta)(c\times d)-(\alpha\times\gamma)(b\times d)+(\alpha\times\delta)(b\times c) \\ &\quad +(\beta\times\gamma)(a\times d)-(\beta\times\delta)(a\times c)+(\gamma\times\delta)(a\times b) \\ &\quad +[(\beta\times\gamma\times\delta)\bar{a}+(\gamma\times\alpha\times\delta)\bar{b}+(\alpha\times\beta\times\delta)\bar{c}+(\beta\times\alpha\times\gamma)\bar{d} \\ &\quad -\alpha\overline{(b\times c\times d)}-\beta\overline{(c\times a\times d)}-\gamma\overline{(a\times b\times d)}-\delta\overline{(b\times a\times c)}]\,e. \end{aligned}$$

Remark. Notice that these formulas are more elegant than the corresponding standard formulas for $[x, y]$ and $[x, y, z]$ (given in the proof of Lemma A.11); providing additional evidence of the naturality of the cross products vis-à-vis the Cayley-Dickson process.

Proof. The proofs of these formulas are direct calculations, using the definition of the cross products. For example,

$$\bar{x}y = (\bar{a}-\alpha e)(b+\beta e) = \bar{a}b+\bar{\beta}\alpha+(\beta\bar{a}-\alpha\bar{b})\,e$$

$$\bar{y}x = (\bar{b}-\beta e)(a+\alpha e) = \bar{b}a+\bar{\alpha}\beta+(\alpha\bar{b}-\beta\bar{a})\,e,$$

and hence

$$\begin{aligned} x\times y \equiv \tfrac{1}{2}(\bar{y}x-\bar{x}y) &= \tfrac{1}{2}(\bar{b}a-\bar{a}b)-\tfrac{1}{2}(\bar{\beta}\alpha-\bar{\alpha}\beta)+(\alpha\bar{b}-\beta\bar{a})\,e \\ &= a\times b-\alpha\times\beta+(\alpha\bar{b}-\beta\bar{a})\,e, \end{aligned}$$

proving (1).

Next we discuss the equivariance properties of the cross products.

PROPOSITION B.11. *Suppose* $u\in \operatorname{Im}\mathbf{O}$ *and* $|u|=1$ *(hence* $\bar{u}=-u$*). Then for all* $x, y, z, w\in\mathbf{O}$:

(1) $(xu)\times(yu)=u(x\times y)\,\bar{u}$
(2) $(xu)\times(yu)\times(zu)=(x\times y\times z)\,u$
(3) $(xu)\times(yu)\times(zu)\times(wu)=u(x\times y\times z\times w)\,\bar{u}$.

Proof. We may assume that x, y, z, w are pairwise orthogonal. Then

(1) $(xu)\times(yu)=\overline{(yu)}(xu)=-(uy)(xu)=-u(\bar{y}x)\,u=u(z\times y)\,\bar{u}$ by the Moufang identity Lemma A.16 (c).

(2) $(xu)\times(yu)\times(zu)=(xu)\,(\overline{(yu)}(zu))$, which by the proof of (1) equals $-(xu)\,(u(\bar{y}z)\,u)$. Now by the Moufang identity Lemma A.16 (b) the above equals

$$-(((xu)\,u)\,(\bar{y}z))\,u = (x(\bar{y}z))\,u = (x\times y\times z)\,u.$$

(3) By (B.8) and part (2), $(xu)\times(yu)\times(zu)\times(wu)=(\overline{xu})((y(\bar{z}w))u)=-(u\bar{x})((y(\bar{z}w))u)$. This equals $-u(\bar{x}(y(\bar{z}w)))u=u(x\times y\times z\times w)\bar{u}$ by Lemma A.16 (c).

Recall, Definition 1.33, that Spin_7 acts on $\mathbf{O}$ and is generated by $\{R_u : u\in S^6\subset \mathrm{Im}\,\mathbf{O}\}$. Also recall that $\chi_g(v)\equiv\chi_g(R_v)\equiv g\circ R_v\circ g^{-1}=R_w$ (where $w\equiv g(g^{-1}(1)v)$) defines an action of Spin_7 on $\mathbf{R}^7\cong\{R_v : v\in\mathrm{Im}\,\mathbf{O}\}$ and that $\chi:\mathrm{Spin}_7\to SO_7$ is the standard double cover.

COROLLARY B.12. *For each* $g\in\mathrm{Spin}_7$

(1) $(gx)\times(gy)=\chi_g(x\times y)$

(2) $(gx)\times(gy)\times(gz)=g(x\times y\times z)$

(3) $(gx)\times(gy)\times(gz)\times(gw)=\chi_g(x\times y\times z\times w)$.

The cross products can be related to the commutator $[x,y]$, the associator $[x,y,z]$, and a third object, the "coassociator" $[x,y,z,w]$.

Definition B.13. The *coassociator* is defined by

$$\tfrac{1}{2}[x,y,z,w]\equiv-(\langle y',z'w'\rangle x'+\langle z',x'w'\rangle y'+\langle x',y'w'\rangle z'+\langle y',x'z'\rangle w'),$$

for all $x,y,z,w\in\mathbf{O}$.

This definition is consistent with that of the commutator and associator because of the next result.

PROPOSITION B.14. *For all* $x,y,z,w\in\mathrm{Im}\,\mathbf{O}$ *purely imaginary:*

(1) $\mathrm{Im}\,x\times y=\frac{1}{2}[x,y]$

(2) $\mathrm{Im}\,x\times y\times z=\frac{1}{2}[x,y,z]$

(3) $\mathrm{Im}\,x\times y\times z\times w=\frac{1}{2}[x,y,z,w]$

Remark. One can show that

$$2[x,y,z,w]=[wx,y,z]-x[w,y,z]-[x,y,z]w.$$

The right hand side is a function considered by Kleinfeld [Kl].

Proof. (1) is immediate. Since both sides of (2) and (3) are alternating we may assume $x,y,z,w\in\mathrm{Im}\,\mathbf{O}$ are pairwise orthogonal. Then, by (B.5), $\mathrm{Im}\,x\times y\times z=-\mathrm{Im}\,x(yz)=-\frac{1}{2}(x(yz)+(zy)x)$. Repeated use of (A.7) shows that $-(zy)x=(xy)z$, which proves (2).

Orthogonality implies, by (B.8), that $x\times y\times z\times w=x(y(zw))$ and hence

$$\begin{aligned}\langle x\times y\times z\times w, x\rangle &= \langle x(y(zw)), x\rangle = \langle y(zw), 1\rangle |x|^2\\ &= -\operatorname{Re}(y\times z\times w)|x|^2 = -\langle y, zw\rangle |x|^2.\end{aligned}$$

Consequently, $x\times y\times z\times w$ and $\frac{1}{2}[x,y,z,w]$ differ (at most) by an octonian $a\in\mathbf{O}$ orthogonal to $x,y,z,$ and w. Since $[x,y,z,w]\in\operatorname{Im}\mathbf{O}$, we can write $\operatorname{Im} x\times y\times z\times w=\frac{1}{2}[x,y,z,w]+a$ where $a\in\operatorname{Im}\mathbf{O}$ is orthogonal to x,y,z and w. It remains to show that $\langle\operatorname{Im} x\times y\times z\times w, a\rangle=0$. Since $a\in\operatorname{Im}\mathbf{O}$, this is equal to

$$\langle x\times y\times z\times w, a\rangle = \langle x(y(zw)), a\rangle = -\langle a(x(y(zw))), 1\rangle.$$

Using (A.7) to make four permutations of adjacent symbols, this becomes $-\langle x(y(z(wa))), 1\rangle$, which equals $-\langle a, w(z(yx))\rangle$. Using (A.7) to make 6 permutations of adjacent symbols yields $\langle x\times y\times z\times w, a\rangle=\langle x(y(zw)), a\rangle=-\langle a, x(y(zw))\rangle$. Thus $x(y(zw))$ is orthogonal to a, completing the proof of Proposition B.14.

This Proposition B.14 is a special case of the next result.

PROPOSITION B.15. *For all* $x,y,z,w\in\mathbf{O}$.

(1) $\operatorname{Im} x\times y=\frac{1}{2}[x,y]-(x_1 y'-y_1 x')$

(2) $\operatorname{Im} x\times y\times z=\frac{1}{2}[x,y,z]+\frac{1}{2}x_1[z,y]+\frac{1}{2}y_1[x,z]+\frac{1}{2}z_1[y,x]$

(3) $\operatorname{Im} x\times y\times z\times w=\frac{1}{2}[x,y,z,w]+\frac{1}{2}x_1[y,z,w]+\frac{1}{2}y_1[z,x,w]+\frac{1}{2}z_1[x,y,w]+\frac{1}{2}w_1[y,x,z]$.

Proof. Because all the expressions in the proposition are alternating; and have been verified (Proposition B.14) when x,y,z,w are purely imaginary, it suffices to prove (1)–(3) when $x=1$ and when $y,z,w\in\operatorname{Im}\mathbf{O}$ are purely imaginary and pairwise orthogonal.

In this case,

(1) $1\times y=-\frac{1}{2}(y-\bar{y})=-y'$

(2) Note $y\times z=\bar{y}z$ since $\langle y,z\rangle=0$. Now by (B.5), $1\times y\times z=\bar{y}z=y\times z$. Therefore, using Proposition B.14, $\operatorname{Im} 1\times y\times z=\operatorname{Im} y\times z=\frac{1}{2}[y,z]$.

(3) By (B.8) and (B.5), $\operatorname{Im} 1\times y\times z\times w=\operatorname{Im} y(\bar{z}w)=\operatorname{Im} y\times z\times w$, which equals $\frac{1}{2}[y,z,w]$ by Proposition B.14.

The commutator $[x,y]$, the associator $[x,y,z]$, and the new object $[x,y,z,w]$ share common properties. For the sake of completeness we list them for the commutator and associator as well as for $[x,y,z,w]$.

PROPOSITION B.16. *$[x,y]$, $[x,y,z]$, and $[x,y,z,w]$ are*

(1) *alternating*

(2) *imaginary valued (i.e. the real part vanishes)*

(3) *depend only on the imaginary parts of x,y,z,w (for example $[x,y,z]=[x',y',z']$).*

(4) (i) *$[x,y]$ is orthogonal to x and y*

(ii) *$[x,y,z]$ is orthogonal to x,y,z and $[a,b]$ for each subset $\{a,b\}$ of $\{x,y,z\}$*

(iii) *$[x,y,z,w]$ is orthogonal to $[a,b]$ and $[a,b,c]$ for each subset $\{a,b,c\}$ of $\{x,y,z\}$.*

The proof is omitted.

Remark. The orthogonality (4) has useful implications for the decompositions of $\text{Im}\, x\times y$, $\text{Im}\, x\times y\times z$, and $\text{Im}\, x\times y\times z\times w$ given in Proposition B.15.

V. Some observations and questions

The ideas developed here for the euclidean case are equally interesting for a much broader class of manifolds.

V.1. Grassmann geometries in locally symmetric spaces

Let $X=G/K$ be a riemannian symmetric space (cf. [He]) and let $\Lambda_G^*(X)$ denote the algebra of G-invariant differential forms on X. (Recall that $\Lambda_G^p(X)$ is naturally isomorphic to the fixed point set of K acting on $\Lambda^p T_0 X$, where "0" denotes the identity coset in G/K). Each element $\varphi\in\Lambda_G^*(X)$ is parallel in the riemannian connection on X. In particular, $d\varphi=0$ and $\|\varphi\|_x^*$ is independent of the point $x\in X$. Hence, φ gives rise to a G-invariant Grassmann geometry on X.

There are two fundamental cases of interest here: The irreducible symmetric spaces of compact and non-compact type. If G is compact, then there is a natural isomorphism $\Lambda_G^*(X)\cong H^*(X;\mathbf{R})$. Thus to every cohomology class of $X=G/K$ there is associated an interesting geometry of minimal varieties in X. If, for example, we consider X to be one of the Grassmann manifolds, then the cohomology of X consists essentially of the universal characteristic calsses. The associated geometries might be called "characteristic class geometries". For example, let $X=SU_{n+m}/S(U_n\times U_m)$ be the Grassmannian of complex n-planes in $\mathbf{C}^{n+m}$. Then $H^2(X;\mathbf{R})\cong\mathbf{R}$ is generated by the first

Chern class c_1 which, in this case, is also the Kähler form of X. Thus c_1^p-geometry is just the complex geometry of X in dimension p.

Question 1.1. What is the structure of c_k-submanifolds of the complex Grassmannian X, where $c_k \in H^{2k}(X; \mathbf{R})$ denotes the kth universal Chern class?

Question 1.2. What is the structure of p_k-submanifolds of the real Grassmannian X, where $p_k \in H^{4k}(X; \mathbf{R})$ is the kth universal Pontrjagin class?

Of course this question can be asked for any $\varphi \in H^k(X; \mathbf{R})$ where X is a riemannian symmetric space of compact type. The answers are sometimes available. Consider, for example, the quaternionic projective space $X = \mathbf{P}^n(\mathbf{H})$ with its Sp_{n+1}-invariant metric. Let $\varphi \in \Lambda^4_{Sp_{n+1}}(X) \cong H^4(X; \mathbf{R}) \cong \mathbf{R}$ be the parallel 4-form of comass 1. Here one can prove that *any connected φ-submanifold of* $\mathbf{P}^n(\mathbf{H})$ *is an open subset of some projective line* $\mathbf{P}^1(\mathbf{H}) \subset \mathbf{P}^n(\mathbf{H})$. To see this one first observes that at any point $x \in X$, $\mathcal{G}_x(\varphi)$ consists of "quaternion lines" in T_xX, i.e., 4-dimensional subspaces invariant under the local "I, J, K"-endomorphisms. One now applies the elementary fact that if U is a domain in $\mathbf{H}$ then any C^1 map $f: U \to \mathbf{H}^n$ whose differential is $\mathbf{H}$-linear at each point, is a linear map up to constants.

Similar questions can of course be asked for any riemannian symmetric space $X = G/K$ of non-compact type. The question is particularly interesting when one passes to a compact quotient $X' = \Gamma \backslash X$ where Γ is a discrete subgroup of G acting freely and properly discontinuously on X. Any G-invariant form φ on X descends to a parallel form on X'. The φ-cycles on X' correspond to the Γ-invariant φ-cycles on X.

As a simple example let $X = \mathbf{R}^n$ and consider a flat torus $T^n = \mathbf{R}^n/\Lambda$ where Λ is a lattice in $\mathbf{R}^n$. Any φ-cycle in T^n lifts to a Λ-periodic γ-cycle in $\mathbf{R}^n$.

Question 1.3. Do there exist periodic submanifolds of $\mathbf{R}^n$ belonging to the geometries discussed in Chapters 3 and 4?

We note that in complex geometries, such periodic cycles can exist. However the lattices must be special for the complex structure (in order that the quotient $\mathbf{C}^n/\Lambda$ carry integral (p, p)-cohomology classes).

V.2. Locally homogeneous spaces

The discussion above can be essentially carried over to any riemannian homogeneous space $X = G/H$. Each G-invariant form φ gives rise to an interesting φ-geometry on X, although in general φ may not be closed. When G is compact, the complex $(\Lambda^*_G(X), d)$

of G-invariant forms has cohomology isomorphic to the cohomology of X. However, d is not, in general, zero.

A good example is provided by the manifold $S^7=\mathrm{Spin}_7/G_2$ (cf. the discussion in II.5). There are Spin_7-invariant forms in each dimension 0, 3, 4 and 7, corresponding to the forms on $\mathbf{R}^7$ fixed by G_2. Let φ be the invariant 3-form of comass 1. Then $\psi=*\varphi$ is the invariant 4-form of comass 1. Clearly, $d\psi=0$ and $d\varphi=c\cdot\psi$ for some $c\neq 0$. Since $d\psi=0$ and $H_4(S^7)=0$, we see that there are no positive ψ-currents without boundary in S^7. In particular, there are no compact ψ-manifolds without boundary. However, there do exist 3-dimensional φ-manifolds. In fact, we have the following correspondence. Consider $S^7\subset\mathbf{R}^8\cong\mathbf{Q}$ with the action of Spin_7 as given above.

PROPOSITION 2.1. *A 3-manifold $M\subset S^7$ is a φ-submanifold if and only if the 4-dimensional cone $CM=\{tx\in\mathbf{R}^8: x\in M$ and $t>0\}$ is a Cayley submanifold of $\mathbf{R}^8$.*

Proof. Let Ψ denote the Spin_7-invariant 4-form of comass 1 in $\mathbf{R}^8$, and define a 3-form φ on S^7 by setting

$$\varphi_x=x \llcorner \Psi$$

for $x\in S^7$. This form is Spin_7-invariant, since for $x\in S^7$, $V_1, V_2, V_3\in T_xS^7\cong x^\perp$ and $g\in\mathrm{Spin}_7$ we have: $(g^*\varphi)_x(V_1, V_2, V_3)=(\varphi)_{gx}(gV_1, gV_2, gV_3)\equiv\Psi(gx, gV_1, gV_2, gV_3)=(g^*\Psi)(x, V_1, V_2, V_3)=\Psi(x, V_1, V_2, V_3)\equiv\varphi_x(V_1, V_2, V_3)$. Clearly, φ has comass one, since

$$\varphi_x(\xi)=\Psi(x\wedge\xi) \tag{2.2}$$

and $\|\xi\|=\|x\wedge\xi\|$ for any simple 3-vector $\xi\in\Lambda^3(x^\perp)$. Hence, φ is a generator of $\Lambda^3_{\mathrm{Spin}_7}(S^7)$ of comass 1. The remainder of the proof now follows immediately from (2.2).

Let $C^4\subset\mathbf{R}^7\subset\mathbf{R}^8$ be the cone on the Hopf map discussed in Section IV.3. Since C^4 is coassociative, it is in particular a Cayley submanifold of $\mathbf{R}^8$. Let $M=C^4\cap S^7$. Then M is a φ-submanifold of S^7 which is homeomorphic to S^3.

Using the complex geometries contained in the Cayley geometry of $\mathbf{R}^8$, one obtains many more examples of φ-cycles in S^7.

V.3. Special Lagrangian geometries on complex manifolds with trivial canonical bundle

We point out here that Special Lagrangian geometries can be defined on any complex manifold X with trivial canonical bundle. Let Ω denote a never vanishing holomorphic n form on X. Given a hermitian metric on X this metric can be normalized by a conformal factor so that $|\Omega|\equiv 1$ on X. Then

$$\varphi \equiv \mathrm{Re}\,(\sigma)$$

is a calibration on X (d-closed and, in fact, of comass 1 at each point of X).

If the hermitian structure provides a Kähler/symplectic structure then the φ-submanifolds are a distinguished subclass of the Lagrangian submanifolds of X.

For each unit complex number $e^{i\theta}$, we have a corresponding form $\varphi_\theta=\mathrm{Re}\,(e^{i\theta}\,\Omega)$ with its associated geometry. Thus, as in the case of $\mathbf{C}^n$, we obtain an S^1-family of Special Lagrangian geometries on X.

Note. In the case $n=2$, the Special Lagrangian geometries are actually again complex geometries. Each non-zero element in span $\{\omega, \varphi_0, \varphi_{\pi/2}\}$ determines an orthogonal complex structure parallel in the given metric.

In the case that $n=4$, we can also define an S^1-family of *Cayley geometries* on X (if X is Kähler), by setting

$$\Phi_\theta = \tfrac{1}{2}\omega^2+\mathrm{Re}\,(e^{i\theta}\,\Omega)$$

where ω is the Kähler form on X.

The Kähler form can be chosen so that Φ_θ is parallel since it has now been proved by Calabi and Yau (see Yau [Y]) that any compact Kähler manifold X with $c_1(X)=0$ carries a Ricci flat Kähler metric. Such manifolds include hypersurfaces of degree $n+2$ in $\mathbf{P}^{n+1}(\mathbf{C})$.

V.4. A remark concerning holonomy groups

If X is a connected riemannian n-manifold with holonomy group $G\subseteq O_n$, then any G-invariant p-form on $\mathbf{R}^n$ extends to a parallel p-form φ on X. A φ-geometry is then defined. The available holonomy groups of an irreducible riemannian manifold are discussed in [S], [B_1]. It is still unknown whether there exists a riemannian 7-manifold with holonomy G_2 or an 8-manifold with holonomy Spin_7.

V.5. A remark concerning Yang-Mills equations

It is a consequence of our discussion in Chapter IV (See Propostion 1.3.4.) that a Cayley submanifold $M^4\hookrightarrow\mathbf{R}^8\cong\mathbf{O}$ has the following property: *The Kähler 2-form* ω_e

associated to each of the complex structures J_e, $e \in S^6$, *is anti-self-dual when restricted to* M^4.

We now choose e_1, e_2 and $e_3 = e_1 \cdot e_2$ to be an orthonormal basis of $\operatorname{Im} H$ where H is any quaternion subalgebra of **O**, and we consider the $\operatorname{Im} H$-valued 2-form

$$\Omega_H \equiv \sum \omega_{e_k} \otimes e_k.$$

This form is exactly the curvature 2-form of the canonical Sp_1-bundle over $\mathbf{P}(H \oplus H) \cong S^4$, pulled back to $H \oplus H \cong \mathbf{O}$. We denote this bundle by P_H. From our observation above we have the following.

THEOREM 5.1. *Let* M^4 *be a Cayley submanifold of* $\mathbf{R}^8 \cong \mathbf{O}$. *Then for each quaternion subalgebra* $H \subset \mathbf{O}$, *the associated* Sp_1*-bundle* P_H *with its standard connection has an anti-self-dual curvature tensor when restricted to* M^4. *In particular, each such connection gives a global solution to the Yang-Mills field equations over* M^4.

Note. It should be emphasized here that the Yang-Mills equations hold with respect to the conformal structure on M^4 *induced* from the immersion $M^4 \hookrightarrow \mathbf{R}^8$.

The above result leads to a kind of "transform method" for constructing Yang-Mills fields. Let i, j, k be the standard basis for **H**. Consider $\mathbf{H} = \mathbf{C}^2$ with complex structure i. Let $\mathbf{O} = \mathbf{H} \oplus \mathbf{H}$ as above and set $\Omega = \Omega_H$ and $P = P_H$. Then for each holomorphic mapping

$$h: \mathbf{C}^2 \to \mathbf{C}^2$$

we let $M^4 = \operatorname{graph}(h)$ be the corresponding Cayley submanifold. Then for each $g \in \mathrm{Spin}_7$, we have that the bundle g^*P restricted to M has a Yang-Mills connection with anti-self-dual field $g^*\Omega$.

Analogously, we could consider a real valued potential function $F: \mathbf{R}^4 \to \mathbf{R}$ which satisfies the special Lagrangian equation (Theorem III.2.3), and set $M^4 = \operatorname{graph}(\operatorname{grad} F)$. Again for each $g \in \mathrm{Spin}_7$, the form $g^*\Omega|_{M^4}$ is a Yang-Mills field.

To construct anti-self-dual $Sp_1 (= SU_2)$ fields over the riemannian 4-sphere S^4, it suffices to find, say, polynomial maps $\mathscr{P}: \mathbf{C}^2 \to \mathbf{C}^2$ so that the metric induced on $\operatorname{graph}(\mathscr{P})$ is conformally flat.

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