A constructive proof of the Fefferman-Stein decomposition of BMO ($\mathbf{R}^n$)

by

AKIHITO UCHIYAMA [1]

University of Chicago
Chicago, IL, U.S.A.

1. Introduction

In this note, functions considered are complex-valued unless otherwise explicitly stated. For a function $f(x) \in L^1_{\text{loc}}(\mathbf{R}^n)$, let

$$\|f\|_{\text{BMO}} = \sup |I|^{-1} \int_I |f(x) - f_I| \, dx,$$

where the supremum is taken over all cubes I in $\mathbf{R}^n$, with sides parallel to axis, and where $|I|$ denotes the Lebesgue measure of I and

$$f_I = |I|^{-1} \int_I f(x) \, dx.$$

A function $f(x)$ is said to belong to BMO ($\mathbf{R}^n$) if $\|f\|_{\text{BMO}} < +\infty$.

Let R_j ($j=1, \ldots, n$) be the Riesz transforms. That is,

$$R_j f(x) = (-i\xi_j |\xi|^{-1} \hat{f}(\xi))^{\vee}(x),$$

where $i=(-1)^{1/2}$, $\xi=(\xi_1, \ldots, \xi_n)$ and where $\wedge$ and $\vee$ denote the Fourier and the inverse Fourier transforms, respectively. As is well known,

$$R_j f(x) = C_n \,\text{P.V.} \int (x_j - y_j) \, |x-y|^{-n-1} f(y) \, dy$$

for $f(x) \in \bigcup_{1<p<\infty} L^p(\mathbf{R}^n)$. For $f(x) \in L^\infty(\mathbf{R}^n)$, let

---

(1) This work was supported in part by Science Research Foundation of Japan. (General Research (c) 1980.)

$$\tilde{R}_j f(x) = C_n \text{P.V.} \int \{(x_j - y_j)|x-y|^{-n-1} - (-y_j)|-y|^{-n-1}\chi_{\{y:|y|>1\}}(y)\} f(y)\,dy,$$

where $\chi_E(x)$ denotes the characteristic function of a measureable set E.

C. Fefferman and Stein [8] showed

THEOREM A. $|\int h(x)f(x)\,dx| \leq C\|h\|_{H^1}\|f\|_{\text{BMO}}$, *where*

$$\|h\|_{H^1} = \|h\|_{L^1} + \sum_{j=1}^{n} \|R_j h\|_{L^1}.$$

As a corollary of Theorem A they obtained

THEOREM B. *If* $f(x) \in \text{BMO}(\mathbf{R}^n)$, *then there exist* $g_0(x), \ldots, g_n(x) \in L^\infty(\mathbf{R}^n)$ *such that*

$$f = g_0 + \sum_{j=1}^{n} \tilde{R}_j g_j, \quad \textit{(modulo constants)},$$

and

$$\sum_{j=0}^{n} \|g_j\|_{L^\infty} \leq C\|f\|_{\text{BMO}}.$$

The Fefferman-Stein proof of Theorem A used the subharmonicity of

$$\left\{|P_t * h(x)|^2 + \sum_{j=1}^{n} |P_t * R_j h(x)|^2\right\}^{p/2}$$

on

$$\mathbf{R}_+^{n+1} = \{(x, t): x \in \mathbf{R}^n,\ t > 0\}$$

for $p > (n-1)/n$, where $P_t(x)$ denotes the Poisson kernel. Theorem B was obtained from Theorem A by the Hahn-Banach extension theorem. Until now the existence of $g_0, \ldots, g_n \in L^\infty$ had not been obtained constructively, except for the case $n=1$, where P. W. Jones exhibited g_0 and g_1 using complex function theory. (For the martingale case, see Uchiyama [15]). In this note, we prove Theorem B constructively, and since our proof does not use subharmonicity, we obtain Theorem B in a more general form.

Let $\theta_1(\xi), \ldots, \theta_m(\xi) \in C^\infty(S_{n-1})$, where

$$S_{n-1} = \{x \in \mathbf{R}^n: |x| = 1\}.$$

Let

$$K_j f(x) = (\theta_j(\xi/|\xi|)\hat{f}(\xi))^\vee(x), \quad j = 1, \ldots, m.$$

As is well known (see Lemma 2.A), there exist $\alpha_{\theta_j} \in \mathbf{C}$ and $\Omega_{\theta_j}(x) \in C^\infty(S_{n-1})$ such that

$$\int_{|x|=1} \Omega_{\theta_j}(x) = 0$$

and

$$K_j f(x) = \alpha_{\theta_j} f(x) + \text{P.V.} \int \Omega_{\theta_j}((x-y)/|x-y|)\,|x-y|^{-n} f(y)\,dy$$

for $f(x) \in \bigcup_{1<p<\infty} L^p(\mathbf{R}^n)$. For $f(x) \in L^\infty(\mathbf{R}^n)$, let

$$\tilde{K}_j f(x) = \alpha_{\theta_j} f(x) + \text{P.V.} \int \{\Omega_{\theta_j}((x-y)/|x-y|)\,|x-y|^{-n} - \Omega_{\theta_j}(-y/|y|)\,|y|^{-n}\chi_{\{y:\,|y|>1\}}\} f(y)\,dy.$$

Our result is

THEOREM 1. *If*

$$\operatorname{rank}\begin{pmatrix} \theta_1(\xi) & \ldots & \theta_m(\xi) \\ \theta_1(-\xi) & \ldots & \theta_m(-\xi) \end{pmatrix} \equiv 2 \tag{1.1}$$

on S_{n-1}, *then for any* $f(x) \in \mathrm{BMO}(\mathbf{R}^n)$ *with compact support, there exist* $g_1(x), \ldots, g_m(x) \in L^\infty(\mathbf{R}^n)$ *such that*

$$f = \sum_{j=1}^m \tilde{K}_j g_j \quad \textit{(modulo constants)},$$

and

$$\sum_{j=1}^m \|g_j\|_{L^\infty} \leq C_{1.1}(\theta_1, \ldots, \theta_m)\,\|f\|_{\mathrm{BMO}}.$$

Let **1** be the identity operator. Since

$$\operatorname{rank}\begin{pmatrix} 1 & \xi_1 & \ldots & \xi_n \\ 1 & -\xi_1 & \ldots & -\xi_n \end{pmatrix} \equiv 2 \quad \text{on } S_{n-1},$$

the operators $\mathbf{1}, R_1, \ldots, R_n$ satisfy (1.1). By duality, Theorem 1 gives another proof of Theorem A. More generally, we obtain

COROLLARY 1. *If* (1.1) *holds, then*

$$C_{1.2}(\theta_1, \dots, \theta_m)\,\|h\|_{H^1} \leq \sum_{j=1}^{m} \|K_j h\|_{L^1} \leq C_{1.3}(\theta_1, \dots, \theta_m)\,\|h\|_{H^1}. \tag{1.2}$$

Remark 1.1. The second inequality is well known.

In [6], S. Janson showed that if

$$C\|h\|_{H^1} \leq \|h\|_{L^1} + \sum_{j=1}^{m} \|K_j h\|_{L^1} \leq C'\|h\|_{H^1} \tag{1.3}$$

holds, with C and C' independent of $h(x)$, then

$$\sum_{j=1}^{m} |\theta_j(\xi) - \theta_j(-\xi)| \neq 0 \tag{1.4}$$

on S_{n-1}. Our Corollary 1 gives the converse (conjectured by Janson).

COROLLARY 2. *If* (1.4) *holds, then* (1.3) *holds.*

Remark 1.2. Janson's proof of the necessity of the condition (1.4) shows the necessity of the condition (1.1) in our Theorem 1 and Corollary 1.

Another interesting case is

COROLLARY 3. *If*

$$\sum_{j=1}^{m} \theta_j(\xi) \equiv 1 \tag{1.5}$$

on S_{n-1} *and if there exist* $v_1, \dots, v_m \in \mathbf{R}^n \setminus \{0\}$ *such that*

$$\operatorname{supp} \theta_j \subset \{\xi \in S_{n-1} : \xi \cdot v_j \geq 0\}, \tag{1.6}$$

where $\xi \cdot v_j$ *denotes the inner product in* $\mathbf{R}^n$, *then* (1.2) *holds.*

See [3], [6], and [7] for related results.

In proving Theorem 1, we establish the following somewhat more precise result.

MAIN LEMMA. *Assume that* (1.1) *holds and that* $R > C_{1.4}(\theta_1, \dots, \theta_m)$. *If* $\|f\|_{\mathrm{BMO}} \leq 1$ *and if* $\operatorname{supp} f$ *is compact, then there exist* $g_1(x), \dots, g_m(x) \in L^\infty(\mathbf{R}^n)$ *such that*

$$\left\| f - \sum_{j=1}^{m} \tilde{K}_j g_j \right\|_{\mathrm{BMO}} \leq c_{1.5}(\theta_1, \ldots, \theta_m, R), \tag{1.7}$$

$$\left(\sum_{j=1}^{m} |g_j(x)|^2 \right)^{1/2} \equiv R \tag{1.8}$$

and

$$\operatorname{supp}(g_1 - R), \operatorname{supp} g_2, \ldots, \operatorname{supp} g_m \quad \textit{are compact,} \tag{1.9}$$

where

$$\lim_{R \to \infty} c_{1.5}(\theta_1, \ldots, \theta_m, R) = 0 \tag{1.10}$$

Remark 1.3. If α_{θ_j} and $\Omega_{\theta_j}(x)$ $(j=1, \ldots, m)$ are real-valued, that is, if

$$\theta_j(\xi) \equiv \bar{\theta}_j(-\xi) \tag{1.11}$$

on S_{n-1} and if $f(x)$ is real-valued, then we can take $g_1(x), \ldots, g_m(x)$ to be real-valued.

Notation. A dyadic cube is a cube of the form $\Pi_{j=1}^{n}[k_j 2^{-k}, (k_j+1) 2^{-k}]$, where $k_1, \ldots, k_n$ and k are integers. In the following, I and J denote dyadic cubes. $l(I)$ and x_I denote the side length and the center of I, respectively. aI denotes a cube concentric with I, with sides parallel to the axis and with $l(aI) = al(I)$.

$\mathbf{v}$ and $\boldsymbol{\mu}$ denote elements of $\mathbf{C}^m$.
Σ_{2m-1} denotes $\{\mathbf{v} = (v_1, \ldots, v_m) \in \mathbf{C}^m \colon \Sigma_{j=1}^{m} |v_j|^2 = 1\}$.
$V(\mathbf{v})$ denotes $(\operatorname{Re} v_1, \operatorname{Im} v_1, \ldots, \operatorname{Re} v_m, \operatorname{Im} v_m) (\in \mathbf{R}^{2m})$.
$V(\mathbf{v}) \cdot V(\boldsymbol{\mu})$ denotes the inner product of $V(\mathbf{v})$ and $V(\boldsymbol{\mu})$ in $\mathbf{R}^{2m}$.
$\mathbf{g}(x) = (g_1(x), \ldots, g_m(x))$, $\boldsymbol{\varphi}(x)$ and $\mathbf{p}(x)$ denote $\mathbf{C}^m$-valued functions.
$(\mathbf{K} \cdot \mathbf{g})(x)$ and $(\tilde{\mathbf{K}} \cdot \mathbf{g})(x)$ denote $\Sigma_{j=1}^{m} (K_j g_j)(x)$ and $\Sigma_{j=1}^{m} (\tilde{K}_j g_j)(x)$, respectively.
For $\theta(x), \Omega(x) \in C^\infty(S_{n-1})$ and $y \in \mathbf{R}^n \setminus \{0\}$, $\theta(y)$ and $\Omega(y)$ denote $\theta(y/|y|)$ and $\Omega(y/|y|)$, respectively. The letter C denotes various constants that depend only on $\theta_1(\xi), \ldots, \theta_m(\xi)$.

Acknowledgements. I would like to thank Professor John B. Garnett for correcting my bad English and for his kindness and encouragement during my stay at U.C.L.A., 1980–1981, where I prepared this paper.

2. Preliminary I

LEMMA 2.A. *Let $\theta(\xi)\in C^\infty(S_{n-1})$. Then there exist $\alpha_\theta \in \mathbf{C}$ and $\Omega_\theta(x)\in C^\infty(S_{n-1})$ such that*

$$\int_{|x|=1} \Omega_\theta(x) = 0, \tag{2.1}$$

$$\sup\{|D_{x_j} D_{x_k} \Omega_\theta(x)| : j, k \in \{1, \ldots, n\}, |x| = 1\} \leq c_{2.1}(\theta), \tag{2.2}$$

$$|\alpha_\theta| \leq c_{2.1}(\theta) \tag{2.3}$$

and

$$(\theta(\xi)\hat{f}(\xi))^\vee(x) = \alpha_\theta f(x) + \text{P.V.}\int \Omega_\theta(x-y)\,|x-y|^{-n} f(y)\,dy$$

for any $f(x)\in L^2(\mathbf{R}^n)$, where $c_{2.1}(\theta)$ depends only on

$$\sup\{|D^{(\alpha_1,\ldots,\alpha_n)}\theta(\xi)| : |\xi| = 1, \quad \alpha_1 + \ldots + \alpha_n \leq C_{2.2}(n)\}. \tag{2.4}$$

See Stein [14], p. 75.

Remark 2.1. If

$$\text{Re}\,(\theta(\xi)+\theta(-\xi)) \equiv \text{Im}\,(\theta(\xi)-\theta(-\xi)) \equiv 0,$$

then

$$\text{Re}\,\alpha_\theta = 0 \quad \text{and} \quad \text{Re}\,\Omega_\theta(x) \equiv 0.$$

LEMMA 2.1. *Let $\mathbf{v}\in\Sigma_{2m-1}$, $\boldsymbol{\mu}=(\mu_1,\ldots,\mu_m)$, $\boldsymbol{\mu}'=(\mu_1',\ldots,\mu_m')\in\mathbf{C}^m$ and*

$$\text{rank}\begin{pmatrix}\boldsymbol{\mu}\\ \boldsymbol{\mu}'\end{pmatrix} = 2. \tag{2.5}$$

Then there exist

$$k_1, \ldots, k_m, k_1', \ldots, k_m' \in \mathbf{C}$$

such that

$$\sum_{j=1}^m \mu_j k_j = \sum_{j=1}^m \mu_j' k_j' = 1$$

and

$$\text{Re}\left(\sum_{j=1}^m \bar{\nu}_j (k_j + k_j')\right) = \text{Im}\left(\sum_{j=1}^m \bar{\nu}_j (k_j - k_j')\right) = 0.$$

Proof. Set

$$A = \begin{pmatrix} V(\mathbf{v}) & V(\mathbf{v}) \\ V(i\mathbf{v}) & -V(i\mathbf{v}) \\ V(\bar{\boldsymbol{\mu}}) & 0\dots 0 \\ V(i\bar{\boldsymbol{\mu}}) & 0\dots 0 \\ 0\dots 0 & V(\bar{\boldsymbol{\mu}}') \\ 0\dots 0 & V(i\bar{\boldsymbol{\mu}}') \end{pmatrix} = (A_1 A_2).$$

where A_1 and A_2 are $6\times 2m$ real matrices. Let

$$(a_1 \dots a_6)A = (0 \dots 0).$$

Note that (2.5) implies

$$\max\left(\operatorname{rank}\begin{pmatrix}\mathbf{v}\\ \bar{\boldsymbol{\mu}}\end{pmatrix}, \operatorname{rank}\begin{pmatrix}\mathbf{v}\\ \bar{\boldsymbol{\mu}}'\end{pmatrix}\right) = 2.$$

Say,

$$\operatorname{rank}\begin{pmatrix}\mathbf{v}\\ \bar{\boldsymbol{\mu}}\end{pmatrix} = 2.$$

Then, rank $A_1=4$. So, $a_1=a_2=a_3=a_4=0$. By the linear independence of $V(\bar{\boldsymbol{\mu}}')$ and $V(i\bar{\boldsymbol{\mu}}')$, we get $a_5=a_6=0$. Thus,

$$\operatorname{rank} A = 6.$$

So, there exist $x_1, \dots, x_{2m}, x'_1, \dots, x'_{2m} \in \mathbf{R}$ such that

$$A\begin{pmatrix} x_1 \\ \vdots \\ x_{2m} \\ x'_1 \\ \vdots \\ x'_{2m} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Putting

$$k_1 = x_1 + ix_2, \dots, k_m = x_{2m-1} + ix_{2m},$$
$$k'_1 = x'_1 + ix'_2, \dots, k'_m = x'_{2m-1} + ix'_{2m}$$

then gives the result. Q.E.D.

LEMMA 2.2. *Assume that* (1.1) *holds. Then there exist*

$$\Theta_1(\xi, \mathbf{v}), \ldots, \Theta_m(\xi, \mathbf{v}) \in C^\infty(S_{n-1} \times \Sigma_{2m-1})$$

such that

$$\sum_{j=1}^{m} \theta_j(\xi)\,\Theta_j(\xi, \mathbf{v}) \equiv 1, \tag{2.6}$$

$$\operatorname{Re} \sum_{j=1}^{m} \bar{v}_j\{\Theta_j(\xi, \mathbf{v}) + \Theta_j(-\xi, \mathbf{v})\} \equiv \operatorname{Im} \sum_{j=1}^{m} \bar{v}_j\{\Theta_j(\xi, \mathbf{v}) - \Theta_j(-\xi, \mathbf{v})\} \equiv 0, \tag{2.7}$$

$$\sup\{|D_\xi^{(\alpha_1, \ldots, \alpha_n)} \Theta_j(\xi, \mathbf{v})| : |\xi| = |\mathbf{v}| = 1, \alpha_1 + \ldots + \alpha_n \leq C_{2.2}(n)\} \leq C. \tag{2.8}$$

Proof. Take any $(\xi, \mathbf{v}) \in S_{n-1} \times \Sigma_{2m-1}$. Then by (1.1) and Lemma 2.1, there exist $\{k_j(\xi, \mathbf{v})\}_{j=1}^m$ and $\{k_j'(\xi, \mathbf{v})\}_{j=1}^m$ such that

$$\sum_{j=1}^{m} \theta_j(\xi)\,k_j(\xi, \mathbf{v}) = \sum_{j=1}^{m} \theta_j(-\xi)\,k_j'(\xi, \mathbf{v}) = 1 \tag{2.9}$$

and

$$\operatorname{Re} \sum_{j=1}^{m} \bar{v}_j(k_j(\xi, \mathbf{v}) + k_j'(\xi, \mathbf{v})) = \operatorname{Im} \sum_{j=1}^{m} \bar{v}_j(k_j(\xi, \mathbf{v}) - k_j'(\xi, \mathbf{v})) = 0. \tag{2.10}$$

Furthermore, we can take $k_j(\xi, \mathbf{v})$ and $k_j'(\xi, \mathbf{v})$ to be C^∞ in some neighborhood of $(\xi, \mathbf{v})$. Then by the compactness of $S_{n-1} \times \Sigma_{2m-1}$, we can define $k_j(\xi, \mathbf{v})$ and $k_j'(\xi, \mathbf{v})$ to be C^∞ on $S_{n-1} \times \Sigma_{2m-1}$ and to satisfy (2.9)–(2.10). Set

$$\Theta_j(\xi, \mathbf{v}) = \{k_j(\xi, \mathbf{v}) + k_j'(-\xi, \mathbf{v})\}/2.$$

Then (2.6)–(2.7) follow from (2.9)–(2.10). Q.E.D.

LEMMA 2.3. *Let* $f(x) \in L^2(\mathbf{R}^n)$. *Let* $\mathbf{v} \in \Sigma_{2m-1}$. *Set*

$$p_j(x) = (\Theta_j(\xi, \mathbf{v})\,\widehat{\operatorname{Re} f}(\xi))^\vee(x) + i(\Theta_j(\xi, i\mathbf{v})\,\widehat{\operatorname{Im} f}(\xi))^\vee(x), \quad j = 1, \ldots, m \tag{2.11}$$

and

$$\mathbf{p}(x) = (p_1(x), \ldots, p_m(x)).$$

Then

$$V(\mathbf{p}(x))\cdot V(\mathbf{v})\equiv 0 \tag{2.12}$$

and

$$(\mathbf{K}\cdot\mathbf{p})(x)=f(x). \tag{2.13}$$

Proof. Applying Lemma 2.A to $\theta(\xi)=\Theta_j(\xi,\mathbf{v})$, we get $\alpha_{\Theta_j(\cdot,\mathbf{v})}$ and $\Omega_{\Theta_j(\cdot,\mathbf{v})}(x)\in C^\infty(S_{n-1})$. Then

$$\begin{aligned}\sum_{j=1}^m \bar{v}_j p_j(x)=&\sum_{j=1}^m \bar{v}_j\alpha_{\Theta_j(\cdot,\mathbf{v})}\operatorname{Re}f(x)\\ &+\text{P.V.}\int\sum_{j=1}^m \bar{v}_j\Omega_{\Theta_j(\cdot,\mathbf{v})}(x-y)\,|x-y|^{-n}\operatorname{Re}f(y)\,dy\\ &+i\sum_{j=1}^m \bar{v}_j\alpha_{\Theta_j(\cdot,i\mathbf{v})}\operatorname{Im}f(x)\\ &+i\,\text{P.V.}\int\sum_{j=1}^m \bar{v}_j\Omega_{\Theta_j(\cdot,i\mathbf{v})}(x-y)\,|x-y|^{-n}\operatorname{Im}f(y)\,dy.\end{aligned} \tag{2.14}$$

By (2.7) and Remark 2.1, we get

$$\operatorname{Re}\sum_{j=1}^m \bar{v}_j\alpha_{\Theta_j(\cdot,\mathbf{v})}\equiv\operatorname{Re}\sum_{j=1}^m \bar{v}_j\Omega_{\Theta_j(\cdot,\mathbf{v})}(x)\equiv 0,$$
$$\operatorname{Re}\sum_{j=1}^m i\bar{v}_j\alpha_{\Theta_j(\cdot,i\mathbf{v})}\equiv\operatorname{Re}\sum_{j=1}^m i\bar{v}_j\Omega_{\Theta_j(\cdot,i\mathbf{v})}(x)\equiv 0.$$

Then the real part of (2.14) is equal to 0, and we get (2.12). (2.13) follows from (2.6). Q.E.D.

Remark 2.2. Let $\mathbf{v}\in\mathbf{R}^m\cap\Sigma_{2m-1}$. Then if $\bar{\boldsymbol{\mu}}'=\boldsymbol{\mu}$ in Lemma 2.1, we can take $k_1,\dots,k'_m$ to satisfy $\bar{k}'_j=k_j$ $(j=1,\dots,m)$. So, if (1.11) holds, we can take $\Theta_j(\xi,\mathbf{v})$ in Lemma 2.2 to satisfy $\bar{\Theta}_j(\xi,\mathbf{v})=\Theta_j(-\xi,\mathbf{v})$. Furthermore, if $f(x)$ is real-valued, then we can take $\mathbf{p}(x)$ in Lemma 2.3 to be $\mathbf{R}^m$-valued.

3. Preliminary II

Definition 3.1. For a measure μ defined on $\mathbf{R}_+^{n+1}$, let

$$\|\mu\|_c=\sup_I|\mu|(Q(I))/|I|,$$

where the supremum is taken over all closed dyadic cubes in $\mathbf{R}^n$ and

$$Q(I)=\{(x,t): x\in I, t\in(0,l(I))\}.$$

If $\|\mu\|_c<+\infty$, μ is said to be a Carleson measure.

Definition 3.2. For $f(x)\in C(\mathbf{R}^n)$, let

$$\|f\|_{\mathrm{Lip}\,1}=\sup_{x\neq y}|f(x)-f(y)|/|x-y|.$$

For $f(x)\in C^1(\mathbf{R}^n)$, let

$$\|f\|_{\mathrm{Lip}\,2}=\sum_{j=1}^{n}\|D_{x_j}f\|_{\mathrm{Lip}\,1}.$$

LEMMA 3.1. *Suppose that $f(x)$ has compact support and $\|f\|_{\mathrm{BMO}}\leq C_{3.1}$. Then there exist functions $\{b_I(x)\}_I$ and complex numbers $\{\lambda_I\}_I$, where I is taken over all dyadic cubes, such that*

$$f(x)=\sum_I \lambda_I b_I(x), \tag{3.1}$$

$$\operatorname{supp} b_I\subset 3I, \tag{3.2}$$

$$\int b_I(x)\,dx=0, \tag{3.3}$$

$$\|b_I\|_{\mathrm{Lip}\,1}\leq l(I)^{-1}, \tag{3.4}$$

$$\left\|\sum_I |\lambda_I|^2|I|\delta_{(x_I,\,l(I))}\right\|_c\leq 1, \tag{3.5}$$

where $\delta_{(x,t)}$ denotes the Dirac measure concentrated at the point $(x,t)\in\mathbf{R}_+^{n+}$.

Proof. Following Chang-R. Fefferman [5] (see also A. Calderón-Torchinsky [2]), let $\varphi(x)\in\mathscr{D}(\mathbf{R}^n)$ be real-valued and such that

$$\operatorname{supp}\varphi\subset\{x\in\mathbf{R}^n: |x|<1\},$$

$$\int\hat{\varphi}(\xi t)^2 t^{-1}\,dt\equiv 1 \quad \text{for any } \xi\in\mathbf{R}^n\setminus\{0\}.$$

Then

$$f(x)=\int_0^\infty(\varphi_t*\varphi_t*f)(x)\,t^{-1}\,dt$$
$$=\sum_I\iint_{T(I)}\varphi_t(x-y)\,(\varphi_t*f)(y)\,t^{-1}\,dt\,dy$$
$$=\sum_I\tilde{b}_I(x),$$

where

$$T(I)=\{(x,t):x\in I,\,t\in(l(I)/2,\,l(I))\}.$$

and $\varphi_t(x)=t^{-n}\varphi(x/t)$. Then,

$$\operatorname{supp}\tilde{b}_I\subset 3I,\qquad\int\tilde{b}_I(x)\,dx=0 \tag{3.6}$$

and

$$|D_{x_j}\tilde{b}_I(x)|=\left|\iint_{T(I)}D_{x_j}\varphi_t(x-y)\,(\varphi_t*f)(y)\,t^{-1}\,dt\,dy\right|\leqslant\lambda_I\,l(I)^{-1}, \tag{3.7}$$

where

$$\lambda_I=C|I|^{-1/2}\left\{\iint_{T(I)}|\varphi_t*f(y)|^2t^{-1}\,dt\,dy\right\}^{1/2} \tag{3.8}$$

Set

$$b_I(x)=\tilde{b}_I(x)/\lambda_I.$$

Then (3.2)–(3.4) follows from (3.6)–(3.8).

Take any dyadic I. Then,

$$\sum_{J\subset I}|\lambda_J|^2|J|=C\sum\iint_{T(J)}|\varphi_t*f(x)|^2t^{-1}\,dt\,dy$$
$$=C\iint_{Q(I)}|\varphi_t*f(y)|^2t^{-1}\,dt\,dy\leqslant C\|f\|^2_{\mathrm{BMO}}|I|$$

by Fefferman-Stein [5] p. 146. Q.E.D.

Remark 3.1. By almost the same argument, we can show

$$\|b_I\|_{\mathrm{Lip}\,2}\leqslant l(I)^{-2} \tag{3.9}$$

instead of (3.4).

Definition 3.3. For $\{\lambda_I\}_I$ obtained by Lemma 3.1, set

$$\eta_k(x) = \sum_{I:\, l(I)=2^{-k}} |\lambda_I| \{1+2^k|x-x_I|\}^{-n-1},$$

$$\varepsilon_k(x) = \sum_{j=0}^{\infty} (2/3)^j \eta_{k-j}(x).$$

LEMMA 3.2. *With $\eta_k(x)$ and $\varepsilon_k(x)$ defined as above, we have*

$$\eta_k(x) \leq \varepsilon_k(x) \leq C_{3.2}, \tag{3.10}$$

$$\eta_k(x) \leq C_{3.2}(2^k|x-y|+1)^{n+1}\eta_k(y), \tag{3.11}$$

$$\varepsilon_k(x) \leq C_{3.2}(2^k|x-y|+1)^{n+1}\varepsilon_k(y) \tag{3.12}$$

and

$$\left\| \sum_{k=-\infty}^{\infty} \varepsilon_k(x)\,\eta_k(x)\,\delta_{t=2^{-k}} \right\|_{\mathrm{C}} \leq C_{3.2}, \tag{3.13}$$

where $\delta_{t=a}$ denotes the measure induced by n-dimensional Lebesgue measure on the hyperplane $t=a$ in $\mathbf{R}_+^{n+1}$.

Proof. Since (3.10)–(3.12) are easy, we verify only (3.13). Take any dyadic cube I in $\mathbf{R}^n$. Then

$$\iint_{Q(I)} \sum_{k=-\infty}^{\infty} \eta_k(x)^2 \delta_{t=2^{-k}} \leq C \iint_{Q(I)} \sum_{k=-\infty}^{\infty} \sum_{J:\, l(J)=2^{-k}} |\lambda_J|^2 \{1+2^k|x-x_J|\}^{-n-1} \delta_{t=2^{-k}}$$

$$\leq C \sum_{L:\, l(L)=l(I)} \{1+l(I)^{-1}|x_I-x_L|\}^{-n-1} \sum_{J:\, J\subset L} |\lambda_J|^2 |J| \leq C|I|$$

by (3.5). Take any $j \geq 0$. Then by the above and (3.10)

$$\iint_{Q(I)} \sum_{k=-\infty}^{\infty} \eta_{k-j}(x)^2 \delta_{t=2^{-k}} \leq C|I|(1+j).$$

Thus, by the Schwarz inequality,

$$\iint_{Q(I)} \sum_{k=-\infty}^{\infty} \eta_{k-j}(x)\,\eta_k(x)\,\delta_{t=2^{-k}} \leq C|I|(1+j)^{1/2}.$$

So

$$\left\| \sum_{k=-\infty}^{\infty} \varepsilon_k(x)\, \eta_k(x)\, \delta_{t=2^{-k}} \right\|_{\mathrm{c}} \leqslant \sum_{j=0}^{\infty} (2/3)^j (1+j)^{1/2} < +\infty. \qquad \text{Q.E.D.}$$

LEMMA 3.3. *Let j be a positive integer. Let $\{b_I(x)\}_I$ be such that*

$$\operatorname{supp} b_I \subset 2^j I, \tag{3.14}$$

$$\int b_I(x)\, dx = 0, \tag{3.15}$$

$$\|b_I\|_{\mathrm{Lip}\,1} \leqslant (2^j l(I))^{-1}. \tag{3.16}$$

Then for any $\{\lambda_I\}_I \subset \mathbf{C}$ and for any $\beta > \alpha > 0$,

$$\left\| \sum_{I:\, \alpha < l(I) < \beta} \lambda_I\, b_I(x) \right\|_{L^2} \leqslant C_{3.3}\, 2^{jn} \big(\sum_I |\lambda_I|^2 |I|\big)^{1/2}.$$

Proof. By (3.14)–(3.16), we get

$$\left| \int b_I(x)\, \overline{b_J(x)}\, dx \right| \leqslant C 2^{jn} |J|\, l(J)/l(I) \tag{3.17}$$

if $l(J) \leqslant l(I)$. For $k \geqslant 0$, let $\mathscr{G}_k(I)$ be the collection of all dyadic cubes such that

$$l(J) = 2^{-k} l(I) \quad \text{and} \quad 2^j J \cap 2^j I \neq \varnothing.$$

Then,

$$\begin{aligned}
\int \left| \sum_I \lambda_I\, b_I(x) \right|^2 dx &\leqslant C \sum_{k=0}^{\infty} \sum_I |\lambda_I| \sum_{J \in \mathscr{G}_k(I)} |\lambda_J| \left| \int b_J(x) \overline{b_I(x)} dx \right| \\
&\leqslant C \sum_k \sum_I |\lambda_I| \sum_J |\lambda_J|\, 2^{jn} |J|\, l(J)/l(I) \quad \text{by (3.17)} \\
&\leqslant C 2^{jn} \sum_k 2^{-k(n+1)} \left(\sum_I |\lambda_I|^2 |I| \right)^{1/2} \cdot \left(\sum_I \left(\sum_{J \in \mathscr{G}_k(I)} |\lambda_J| \right)^2 |I| \right)^{1/2} \\
&\leqslant C 2^{jn} \sum_k 2^{-k(n+1)} \left(\sum_I |\lambda_I|^2 |I| \right)^{1/2} \cdot \left(\sum_I \sum_J |\lambda_J|^2\, 2^{(j+k)n} |I| \right)^{1/2} \\
&\leqslant C 2^{2jn} \sum_I |\lambda_I|^2 |I|. \qquad \text{Q.E.D.}
\end{aligned}$$

LEMMA 3.4. *Assume that* $\{b_I(x)\}_I$ *and* $\{\lambda_I\}_I \subset \mathbf{C}$ *satisfy* (3.14)–(3.16) *and*

$$\left\| \sum_I |\lambda_I|^2 |I| \, \delta_{(x_I, l(I))} \right\|_{\mathrm{c}} \leq 1. \tag{3.18}$$

Let $\alpha > 0$ *and set*

$$f(x) = \sum_{I:\, l(I) < \alpha} \lambda_I b_I(x).$$

Then

$$\|f\|_{\mathrm{BMO}} \leq C_{3.4} 2^{jn}.$$

Proof. Take any cube I (not necessarily dyadic). Set

$$f_1(x) = \sum_{\{J:\, l(J) < \alpha,\, l(J) < 2^{-j} l(I) \text{ and } 2^j J \cap I \neq \varnothing\}} \lambda_J b_J(x)$$

and

$$f_2(x) = f(x) - f_1(x).$$

Since $|\lambda_J| \leq 1$ by (3.18),

$$|f_2(x) - f_2(y)| \leq C 2^{jn} \tag{3.19}$$

for any $x, y \in I$. By Lemma 3.3 and (3.18),

$$\|f_1\|_{L^2}^2 \leq C 2^{2jn} \sum_{\{J:\, l(J) < 2^{-j} l(I) \text{ and } 2^j J \cap I \neq \varnothing\}} |\lambda_J|^2 |J| \leq C 2^{2jn} |I|.$$

So, by the above estimate and (3.19)

$$\int_I |f(x) - f_2(x_I)|^2 \, dx \leq C 2^{2jn} |I|. \qquad \text{Q.E.D.}$$

LEMMA 3.5. *Let* I *and* $p(x) \in C^1(\mathbf{R}^n)$ *be such that*

$$\int p(x) \, dx = 0, \tag{3.20}$$

$$|p(x)| \leq l(I)^{n+1} / (l(\mathrm{I}) + |x - x_I|)^{n+1} \tag{3.21}$$

$$|D_{x_j} p(x)| \leq l(I)^{n+1} / (l(\mathrm{I}) + |x - x_I|)^{n+2} \tag{3.22}$$

for $j=1,\ldots,n$. *Then there exist* $\{\beta_j(x)\}_{j=0}^{\infty}\subset C^1(\mathbf{R}^n)$ *such that*

$$p(x)=\sum_{j=0}^{\infty}2^{-j(n+1)}\beta_j(x), \tag{3.23}$$

$$\operatorname{supp}\beta_j\subset 2^jI, \tag{3.24}$$

$$\|\beta_j\|_{\mathrm{Lip}\,1}\leqslant C_{3.5}\,2^{-j}l(I)^{-1}, \tag{3.25}$$

$$\int\beta_j(x)\,dx=0. \tag{3.26}$$

Proof. By dilation and translation, we may assume $x_I=0$ and $l(I)=2$. Let $h(t)\in C^\infty(\mathbf{R})$ by such that

$$\operatorname{supp}h\subset(1/4,3/4),\quad \sum_{j=1}^{\infty}h(t/2^j)\equiv 1\quad\text{for } t>1.$$

Set

$$h_0(t)=1-\sum_{j=1}^{\infty}h(t/2^j).$$

Then

$$\begin{aligned}p(x)&=h_0(|x|)\,p(x)+\sum_{j=1}^{\infty}h(2^{-j}|x|)\,p(x)\\&=\left\{h_0(|x|)\,p(x)+h(|x|)\int\sum_{k=1}^{\infty}h(2^{-k}|y|)\,p(y)\,dy\Big/\int h(|y|)\,dy\right\}\\&\quad+\sum_{j=1}^{\infty}\left\{h(2^{-j}|x|)\,p(x)\right.\\&\quad-h(2^{-j+1}|x|)\int\sum_{k=j}^{\infty}h(2^{-k}|y|)\,p(y)\,dy\Big/\int h(2^{-j+1}|y|)\,dy\\&\quad\left.+h(2^{-j}|x|)\int\sum_{k=j+1}^{\infty}h(2^{-k}|y|)\,p(y)\,dy\Big/\int h(2^{-j}|y|)\,dy\right\}\\&=\tilde\beta_0(x)+\sum_{j=1}^{\infty}\tilde\beta_j(x).\end{aligned}$$

Since

$$\|h(2^{-j}|x|)\,p(x)\|_{\mathrm{Lip}\,1} \leq C2^{-(n+2)j}$$

and

$$\int \sum_{k=j}^{\infty} h(2^{-j}|y|)\,p(y)\,dy \leq C2^{-j},$$

$\tilde{\beta}_j(x)$ can be written in the form $2^{-j(n+1)}\beta_j(x)$ where $\beta_j(x)$ satisfies (3.24)–(3.26). Q.E.D.

Remark 3.2. If $\mathbf{p}(x)$ is a $\mathbf{C}^m$-valued function with properties (3.20)–(3.22) and if

$$V(\mathbf{p}(x))\cdot V(\mathbf{v}) \equiv 0$$

for some vector $\mathbf{v}\in\mathbf{C}^m\backslash\{0\}$, then by the same argument as above, we can get $\mathbf{C}^m$-valued functions $\{\boldsymbol{\beta}_j(x)\}_{j=0}^{\infty}$ such that (3.23)–(3.26) hold and such that

$$V(\boldsymbol{\beta}_j(x))\cdot V(\mathbf{v}) \equiv 0. \tag{3.27}$$

LEMMA 3.6. *Let* $\theta(\xi)\in C^\infty(S_{n-1})$. *Let* $b(x)$ *and* I *be such that* (3.2), (3.3) *and* (3.9) *hold. Then,*

$$p(x) = (\theta(\xi)\,\hat{b}(\xi))^\vee(x)$$

satisfies (3.20),

$$|p(x)| \leq c_{3.6}(\theta)\, l(I)^{n+1}/(l(I)+|x-x_I|)^{n+1}, \tag{3.21$'$}$$

and

$$|D_{x_j}p(x)| \leq c_{3.6}(\theta)\, l(I)^{n+1}/(l(I)+|x-x_I|)^{n+2}, \tag{3.22$'$}$$

where $c_{3.6}(\theta)$ *depends only on* (2.4).

Proof. By Lemma 2.A, there exist $\alpha_\theta\in\mathbf{C}$ and $\Omega_\theta(x)\in\mathbf{C}^\infty(S_{n-1})$ such that

$$p(x) = \alpha_\theta\, b(x) + \mathrm{P.V.}\int \Omega_\theta(x-y)\,|x-y|^{-n}b(y)\,dy = p_1(x)+p_2(x).$$

Clearly, $p_1(x)$ satisfies the desired properties.

If $x\notin 3I$, then

$$|p_2(x)| = \left|\int \{\Omega_\theta(x-y)|x-y|^{-n} - \Omega_\theta(x-x_I)\,|x-x_I|^{-n}\}\, b(y)\,dy\right|$$
$$\leq Cc_{2.1}(\theta)\, l(I)^{n+1}|x-x_I|^{-(n+1)}.$$

Similarly,

$$|D_{x_j} p_2(x)| \leq C c_{2.1}(\theta)\, l(I)^{n+1} |x-x_I|^{-(n+2)}.$$

If $x \in 3I$, then

$$|p_2(x)| \leq \left| \int_{|x-y|<10n^{1/2} l(I)} \Omega_\theta(x-y)\, |x-y|^{-n} (b(y)-b(x))\, dy \right| \leq C c_{2.1}(\theta).$$

Similarly,

$$|D_{x_j} p_2(x)| \leq C c_{2.1}(\theta)\, l(I)^{-1}. \qquad \text{Q.E.D.}$$

LEMMA 3.7. *Let* $\mathbf{v} \in \mathbf{C}^m \setminus \{0\}$. *Let* $b(x)$ *and* I *be such that* (3.2), (3.3) and

$$\|b\|_{\mathrm{Lip}\,2} \leq C_{3.7}\, l(I)^{-2} \tag{3.9$'$}$$

hold. Then there exist $\{\boldsymbol{\beta}_j(x)\}_{j=0}^{\infty}$ *such that* (3.24)–(3.27) and

$$\left(\mathbf{K} \cdot \sum_{j=0}^{\infty} 2^{-j(n+1)} \boldsymbol{\beta}_j\right)(x) = b(x) \tag{3.28}$$

hold.

Proof. Firstly applying Lemma 2.3 to $f=b$ and $\mathbf{v}/|\mathbf{v}|$, we obtain $\mathbf{p}(x)$ with (2.12)–(2.13). By Lemma 3.6 and (2.8), $\mathbf{p}(x)$ satisfies (3.20)–(3.22). Then we can apply Remark 3.2 and obtain $\{\boldsymbol{\beta}_j(x)\}_{j=0}^{\infty}$ with the desired properties. Q.E.D.

Remark 3.3. If (1.11) holds, if $\mathbf{v} \in \mathbf{R}^m \cap \Sigma_{2m-1}$, and if $b(x)$ is real-valued, then by Remark 2.2, we can take $\{\boldsymbol{\beta}_j(x)\}_{j=0}^{\infty}$ to be $\mathbf{R}^m$-valued.

4. Proof of the Main lemma

We may assume

$$\operatorname{supp} f \subset \{x : |x| < 1\} \quad \text{and} \quad \|f\|_{\mathrm{BMO}} \leq C_{3.1}\, C_{3.7}. \tag{4.1}$$

Let M be a large positive integer to be determined later and let $R > 2^{M(n+2)}$. By Lemma 3.1, we have

$$f(x) = \sum_I \lambda_I\, b_I(x), \tag{4.2}$$

where $\{b_I(x)\}_I$ and $\{\lambda_I\}_I$ satisfy (3.2)–(3.5) and (3.9). By (4.1) and (3.8),

$$\lambda_I = 0 \quad \text{if } 3I \cap \{x: |x| < I\} = \varnothing \tag{4.3}$$

and

$$\sum_I |\lambda_I|^2 |I| \leqslant C\|f\|_{L^2}^2. \tag{4.4}$$

We inductively construct $\mathbf{C}^m$-valued functions

$$\{\mathbf{g}_k(x)\}_{k=-M-1}^{\infty}, \quad \{\boldsymbol{\varphi}_k(x)\}_{k=-M}^{\infty}, \quad \{\boldsymbol{\beta}_{I,j}(x)\}_{j=0,1,2,\ldots;\, l(I)\leqslant 2^M},$$

such that

$$\operatorname{supp} \boldsymbol{\beta}_{I,j} \subset 2^j I, \quad \int \boldsymbol{\beta}_{I,j}(x)\, dx = 0, \quad \|\boldsymbol{\beta}_{I,j}\|_{\mathrm{Lip}\,1} \leqslant C_{3.5}(2^j l(I))^{-1}, \tag{4.5}$$

$$\left(\mathbf{K} \cdot \sum_{j=0}^{\infty} 2^{-j(n+1)} \boldsymbol{\beta}_{I,j}\right)(x) = b_I(x), \tag{4.6}$$

$$|\boldsymbol{\varphi}_k(x)| \leqslant C_{4.1}^2\, c_{4.2}(M,R)\, \varepsilon_k(x)\, \eta_k(x), \tag{4.7}$$

$$\operatorname{supp} \boldsymbol{\varphi}_k \subset \{x: |x| \leqslant 2n^{1/2} \max(2^{M-k}, 1)\}, \tag{4.8}$$

$$|\boldsymbol{\varphi}_k(x) - \boldsymbol{\varphi}_k(y)| \leqslant C_{4.1}^2\, c_{4.2}(M,R)\, 2^k |x-y| \quad \text{if } |x-y| \leqslant 2^{-k}, \tag{4.9}$$

$$\mathbf{g}_{-M-1}(x) \equiv (R, 0, \ldots, 0), \tag{4.10}$$

$$|\mathbf{g}_k(x)| \equiv R, \tag{4.11}$$

$$\mathbf{g}_k(x) - \mathbf{g}_{k-1}(x) = \sum_{I:\, l(I)=2^{-k}} \lambda_I \sum_{j=0}^{M} 2^{-j(n+1)} \boldsymbol{\beta}_{I,j}(x) - \boldsymbol{\varphi}_k(x), \tag{4.12}$$

$$|\mathbf{g}_k(x) - \mathbf{g}_k(y)| \leqslant C_{4.1}\, \varepsilon_k(x)\, 2^k |x-y| \quad \text{if } |x-y| \leqslant 2^{-k}, \tag{4.13}$$

where

$$c_{4.2}(M,R) = 2^{M(n+2)} R^{-1}. \tag{4.14}$$

We temporarily accept this construction. By (4.12), if $k \geqslant -M$, then

$$\mathbf{g}_k(x) - \mathbf{g}_{-M-1}(x) = \sum_{I:\, 2^M \geqslant l(I) \geqslant 2^{-k}} \lambda_I \sum_{j=0}^{M} 2^{-j(n+1)} \boldsymbol{\beta}_{I,j}(x) - \sum_{h=-M}^{k} \boldsymbol{\varphi}_h(x)$$

$$=\sum_{j=0}^{\infty} 2^{-j(n+1)} \sum_{I:\, 2^M \geqslant l(I) \geqslant 2^{-k}} \lambda_I \boldsymbol{\beta}_{I,j} - \sum_{j=M+1}^{\infty} 2^{-j(n+1)} \sum_{I:\, 2^M \geqslant l(I) \geqslant 2^{-k}} \lambda_I \boldsymbol{\beta}_{I,j} - \sum_{h=-M}^{k} \boldsymbol{\varphi}_h$$

$$=(4.15)_k-(4.16)_k-(4.17)_k. \tag{4.18}$$

By (4.4)–(4.5) and Lemma 3.3, $(4.15)_k$ and $(4.16)_k$ converge in L^2 as $k \to \infty$. By (4.7)–(4.8) and (3.13), $(4.17)_k$ converges in L^1 as $k \to \infty$. Since

$$\|\mathbf{g}_k-\mathbf{g}_h\|_{L^\infty} \leqslant 2R,$$

$$\begin{aligned}
\|\mathbf{g}_k-\mathbf{g}_h\|_{L^2}^2 &= \int_0^{2R} 2\alpha |\{x: |\mathbf{g}_k(x)-\mathbf{g}_h(x)|>\alpha\}|\, d\alpha \\
&\leqslant \int_0^{2R} 2\alpha |\{|(4.15)_k-(4.15)_h|>\alpha/3\}|\, d\alpha \\
&\quad + \int_0^{2R} 2\alpha |\{|(4.16)_k-(4.16)_h|>\alpha/3\}|\, d\alpha \\
&\quad + \int_0^{2R} 2\alpha |\{|(4.17)_k-(4.17)_h|>\alpha/3\}|\, d\alpha \\
&\leqslant 9\|(4.15)_k-(4.15)_h\|_{L^2}^2+9\|(4.16)_k-(4.16)_h\|_{L^2}^2+6R\|(4.17)_k-(4.17)_h\|_{L^1}^1 \\
&\to 0, \quad h, k \to \infty.
\end{aligned}$$

Set

$$\mathbf{g}(x)=\mathbf{g}_{-M-1}(x)+\lim_{k\to\infty \text{ in } L^2} (\mathbf{g}_k(x)-\mathbf{g}_{-M-1}(x)).$$

Then (1.8) holds. By (4.3), (4.5), (4.8) and the second formula of (4.18),

$$\operatorname{supp}(\mathbf{g}-(R,0,\ldots,0)) \subset \{x: |x| \leqslant 2n^{1/2}\, 2^{2M}\}.$$

Therefore, (1.9) holds.

By (4.10) and (4.18),

$$\mathbf{g}_k=(R,0,\ldots,0)+(4.15)_k-(4.16)_k-(4.17)_k.$$

Thus, by (4.6),

$$\tilde{\mathbf{K}}\cdot\mathbf{g}_k=\sum_{2^M \geqslant l(I) \geqslant 2^{-k}} \lambda_I b_I - \mathbf{K}\cdot((4.16)_k+(4.17)_k), \quad \text{(modulo constants).}$$

Thus, by (4.2)

$$\tilde{\mathbf{K}}\cdot\mathbf{g}=f-(4.20)-\mathbf{K}\cdot((4.21)+(4.22)) \quad \text{(modulo constants)}, \tag{4.19}$$

where

$$(4.20)=\sum_{l(I)>2^M}\lambda_I b_I(x),$$

$$(4.21)=\sum_{j=M+1}^{\infty}2^{-j(n+1)}\sum_{I:\,2^M\geq l(I)}\lambda_I \boldsymbol{\beta}_{I,j}(x),$$

$$(4.22)=\sum_{j=-M}^{\infty}\boldsymbol{\varphi}_j(x).$$

Further, we temporarily accept the following three inequalities:

$$\|(4.20)\|_{\mathrm{BMO}}\leq C2^{-Mn}, \tag{4.23}$$

$$\|(4.21)\|_{\mathrm{BMO}}\leq C2^{-M}, \tag{4.24}$$

$$\|(4.22)\|_{\mathrm{BMO}}\leq CC_{4.1}^2\, c_{4.2}(M,R), \tag{4.25}$$

where the BMO-norm of $\mathbf{C}^m$-valued function is the sum of the BMO-norms of its components, and we conclude the proof of (1.7).

Take any $\varepsilon>0$. By taking M large enough, we get

$$\|(4.20)\|_{\mathrm{BMO}},\ \|(4.21)\|_{\mathrm{BMO}}<\varepsilon.$$

Taking R large enough depending on ε and M, we get

$$\|(4.22)\|_{\mathrm{BMO}}<\varepsilon$$

by (4.14). By the boundedness of $K_1,\ldots,K_m$ on $\mathrm{BMO}(\mathbf{R}^n)$ and by (4.19),

$$\|f-\tilde{\mathbf{K}}\cdot\mathbf{g}\|_{\mathrm{BMO}}=\|(4.20)+\mathbf{K}\cdot((4.21)+(4.22))\|_{\mathrm{BMO}}\leq C\varepsilon.$$

Therefore (1.7) holds.

The construction of $\{\mathbf{g}_k(x)\}$, $\{\boldsymbol{\varphi}_k(x)\}$ *and* $\{\boldsymbol{\beta}_{I,j}(x)\}$. We construct these function by induction. Define $\mathbf{g}_{-M-1}$ by (4.10). Assume that $\{\mathbf{g}_h\}_{h=-M-1,-M,\ldots,k-1}$, $\{\boldsymbol{\varphi}_h\}_{h=-M,\ldots,k-1}$

and $\{\boldsymbol{\beta}_{I,j}\}_{2^M\geq l(I)\geq 2^{-k+1};\, j=0,1,2,\ldots}$ have been constructed so that (4.5)–(4.14) hold with some sufficiently large $C_{4.1}$ and R. In particular, $\mathbf{g}_{k-1}(x)$ satisfies

$$|\mathbf{g}_{k-1}(x)| \equiv R, \tag{4.11$'$}$$

$$|\mathbf{g}_{k-1}(x)-\mathbf{g}_{k-1}(y)| \leq C_{4.1}\,\varepsilon_{k-1}(x)\,2^{k-1}|x-y| \quad \text{if } |x-y|\leq 2^{-k+1}. \tag{4.13$'$}$$

Applying Lemma 3.7 to $\mathbf{v}=\mathbf{g}_{k-1}(x_I)$ and $b_I(x)$ for each I with $l(I)=2^{-k}$, we get $\{\boldsymbol{\beta}_{I,j}(x)\}_{j=0}^{\infty}$ such that (3.24)–(3.28) hold. Consequently, we have (4.5)–(4.6) and as well

$$V(\boldsymbol{\beta}_{I,j}(x))\cdot V(\mathbf{v}) \equiv 0. \tag{4.26}$$

Note that

$$\begin{aligned}\sum_{I:\, l(I)=2^{-k}} |\lambda_I| \sum_{j=0}^{M} 2^{-j(n+1)}|\boldsymbol{\beta}_{I,j}(x)| &\leq C\sum_{j=0}^{M} 2^{-j(n+1)} \sum_{I:\, l(I)=2^{-k},\, \operatorname{dist}(x,I)\leq 2^{j-k}} |\lambda_I| \\ &\leq C\eta_k(x)\end{aligned} \tag{4.27}$$

and that if $|x-y|<2^{-k}$,

$$\begin{aligned}\sum_{I:\, l(I)=2^{-k}} |\lambda_I| \sum_{j=0}^{M} 2^{-j(n+1)}|\boldsymbol{\beta}_{I,j}(x)-\boldsymbol{\beta}_{I,j}(y)| &\leq C\sum_{j=0}^{M} 2^{-j(n+1)} \sum_{I:\, l(I)=2^{-k},\, \operatorname{dist}(x,I)\leq 2^{1+j-k}} |\lambda_I|\,|x-y|\,2^{k-j} \\ &\leq C\eta_k(x)\,2^k|x-y|.\end{aligned} \tag{4.28}$$

If $0\leq j\leq M$ and if $\boldsymbol{\beta}_{I,j}(x)\neq \mathbf{0}$, then by (4.13)$'$ and (3.12),

$$|\mathbf{g}_{k-1}(x)-\mathbf{g}_{k-1}(x_I)| \leq CC_{4.1}\,2^{M(n+1)}\varepsilon_{k-1}(x)\,2^{k-1}|x-x_I|$$

for these x and I. So, by (4.11)$'$ and (4.26),

$$|V(\mathbf{g}_{k-1}(x)/R)\cdot V(\boldsymbol{\beta}_{I,j}(x)/|\boldsymbol{\beta}_{I,j}(x)|)| \leq CC_{4.1}\,2^{M(n+2)}R^{-1}\varepsilon_{k-1}(x) \tag{4.29}$$

for these j, x and I.

Set

$$\mathbf{h}(x) = \sum_{I:\, l(I)=2^{-k}} \lambda_I \sum_{j=0}^{M} 2^{-j(n+1)}\boldsymbol{\beta}_{I,j}(x)$$

and

$$\mathbf{k}(x) = \mathbf{g}_{k-1}(x)+\mathbf{h}(x).$$

By (4.27)–(4.28),

$$|\mathbf{h}(x)| \leqslant C\eta_k(x), \tag{4.30}$$

$$|\mathbf{h}(x)-\mathbf{h}(y)| \leqslant C\eta_k(x)\, 2^k|x-y| \tag{4.31}$$

if $|x-y|<2^{-k}$. Thus by (4.13)′

$$\begin{aligned}|\mathbf{k}(x)-\mathbf{k}(y)| &\leqslant |\mathbf{g}_{k-1}(x)-\mathbf{g}_{k-1}(y)|+|\mathbf{h}(x)-\mathbf{h}(y)| \\ &\leqslant \{C_{4.1}\, 2^{-1}\varepsilon_{k-1}(x)+C\eta_k(x)\}\, 2^k|x-y| \\ &\leqslant (3/4)\, C_{4.1}\, \varepsilon_k(x)\, 2^k|x-y| \end{aligned} \tag{4.32}$$

if $|x-y|\leqslant 2^{-k}$ and if $C_{4.1}$ is large enough. By (4.30) and (3.10),

$$||\mathbf{k}(x)|-R| \leqslant C. \tag{4.33}$$

Since $\{\lambda_I\}_I$ in (3.8) are real, by (4.27) and (4.29),

$$\begin{aligned}||\mathbf{k}(x)|-R| &\leqslant CR^{-1}\{\eta_k(x)+C_{4.1}\, 2^{M(n+2)}\varepsilon_{k-1}(x)\}\, \eta_k(x) \\ &\leqslant CC_{4.1}\, 2^{M(n+2)}R^{-1}\varepsilon_k(x)\, \eta_k(x). \end{aligned} \tag{4.34}$$

Set

$$\mathbf{g}_k(x) = R\mathbf{k}(x)/|\mathbf{k}(x)|.$$

Then (4.11) is clear. If R is large enough, then by (4.32)–(4.33),

$$|\mathbf{g}_k(x)-\mathbf{g}_k(y)| \leqslant (4/3)\, |\mathbf{k}(x)-\mathbf{k}(y)| \leqslant C_{4.1}\, \varepsilon_k(x)\, 2^k|x-y|$$

provided $|x-y|\leqslant 2^{-k}$. Thus (4.13) holds.

Set

$$\boldsymbol{\varphi}_k(x) = \mathbf{k}(x)-\mathbf{g}_k(x).$$

Then (4.12) is clear. (4.8) follows from (4.3). (4.7) follows from $|\boldsymbol{\varphi}_k(x)|=||\mathbf{k}(x)|-R|$ and (4.34).

Let

$$|x-y| \leqslant 2^{-k}.$$

Then

$$\begin{aligned}\boldsymbol{\varphi}_k(x)-\boldsymbol{\varphi}_k(y) &= |\mathbf{k}(x)|^{-1}(|\mathbf{k}(x)|-\mathrm{R})\,(\mathbf{k}(x)-\mathbf{k}(y))+\mathbf{k}(y)\, R(|\mathbf{k}(x)|-|\mathbf{k}(y)|)/|\mathbf{k}(x)|\cdot|\mathbf{k}(y)| \\ &= (4.35)+(4.36).\end{aligned}$$

By (4.32)–(4.33),

$$|(4.35)| \leqslant CR^{-1}C_{4.1}2^k|x-y|.$$

On the other hand,

$$\begin{aligned}|(4.36)| &\leqslant 2\,\big|\,|\mathbf{k}(x)|-|\mathbf{k}(y)|\,\big| \\ &\leqslant 2\,\big|\,|\mathbf{g}_{k-1}(x)+\mathbf{h}(x)|-|\mathbf{g}_{k-1}(y)+\mathbf{h}(x)|\,\big| + 2\,\big|\,|\mathbf{g}_{k-1}(y)+\mathbf{h}(x)|-|\mathbf{g}_{k-1}(y)+\mathbf{h}(y)|\,\big| \\ &= (4.37)+(4.38).\end{aligned}$$

By (4.13)′ and (4.30)

$$|(4.37)| \leqslant |\mathbf{h}(x)|\,CC_{4.1}2^{k-1}|x-y|\,R^{-1} \leqslant CC_{4.1}R^{-1}2^k|x-y|.$$

By the same reason as the estimate of (4.29)

$$|V((\mathbf{g}_{k-1}(y)+\mathbf{h}(y))/|\mathbf{g}_{k-1}(y)+\mathbf{h}(y)|)\cdot V((\boldsymbol{\beta}_{I,j}(x)-\boldsymbol{\beta}_{I,j}(y))/|\boldsymbol{\beta}_{I,j}(x)-\boldsymbol{\beta}_{I,j}(y)|)| \leqslant CC_{4.1}2^{M(n+2)}R^{-1} \tag{4.39}$$

if $\boldsymbol{\beta}_{I,j}(x)-\boldsymbol{\beta}_{I,j}(y)\neq \mathbf{0}$ and if $0\leqslant j\leqslant M$. Since $\{\lambda_I\}_I$ in (3.8) are real,

$$\begin{aligned}|(4.38)| &\leqslant C\{|V(\mathbf{h}(x)-\mathbf{h}(y))\cdot V((\mathbf{g}_{k-1}(y)+\mathbf{h}(y))/|\mathbf{g}_{k-1}(y)+\mathbf{h}(y)|)|+|\mathbf{h}(x)-\mathbf{h}(y)|^2/R\} \\ &\leqslant CC_{4.1}2^{M(n+2)}R^{-1}2^k|x-y|\end{aligned}$$

by (4.28), (4.31) and (4.39). Thus (4.9) holds if $C_{4.1}$ is large enough, and the induction is completed.

Proof of (4.23). Since

$$|\lambda_I| \leqslant C|I|^{-1}$$

By (4.1) and (3.8), (4.23) is clear.

Proof of (4.24).

$$\|(4.21)\|_{\mathrm{BMO}} \leqslant \sum_{j=M+1}^{\infty} 2^{-j(n+1)}\left\|\sum_{l(I)\leqslant 2^M}\lambda_I\boldsymbol{\beta}_{I,j}\right\|_{\mathrm{BMO}} \leqslant C\sum_{j=M+1}^{\infty}2^{-j}$$

by (3.5) and Lemma 3.4.

Proof of (4.25). By (4.7) and (3.13), for any cube I, (not necessarily dyadic),

$$|I|^{-1}\int_I \left|\sum_{k\geq -\log_2 l(I)} \varphi_k(x)\right| dx \leq C_{4.1}^2\, c_{4.2}(M,R)\,|I|^{-1}\int_I \sum_{k\geq -\log_2 l(I)} \varepsilon_k(x)\,\eta_k(x)\,dx$$

$$\leq CC_{4.1}^2\, c_{4.2}(M,R). \tag{4.40}$$

By (4.9), if $x, y \in I$, then

$$\sum_{k\leq -\log_2 l(I)} |\varphi_k(x)-\varphi_k(y)| \leq CC_{4.1}^2\, c_{4.2}(M,R). \tag{4.41}$$

Thus, (4.25) follows from (4.40)–(4.41).

Proof of Remark 1.3. By (4.10), $\mathbf{g}_{-M-1}(x)$ is $\mathbf{R}^m$-valued. Assume that $\mathbf{g}_{k-1}(x)$ is $\mathbf{R}^m$-valued. Since $f(x)$ is real-valued, $\{\lambda_I\}_I$ and $\{b_I(x)\}_I$ are real-valued. Then by Remark 3.3, for each I with $l(I)=2^{-k}$, we can get $\mathbf{R}^m$-valued $\{\boldsymbol{\beta}_{I,j}(x)\}_{j=0}^{\infty}$ that satisfy (4.5)–(4.6) and (4.26). Then, from its construction, we see that $\mathbf{g}_k(x)$ is also $\mathbf{R}^m$-valued.

5. Proof of Theorem 1

Take $R>C_{1.4}(\theta_1,\dots,\theta_m)$ such that $c_{1.5}(\theta_1,\dots,\theta_m,R)<1/10$. Let

$$\|f\|_{\mathrm{BMO}}\leq 1$$

and let supp$f$ be compact. Then by the Main lemma there exists $\mathbf{g}^1(x)$ such that

$$\|f-\tilde{\mathbf{K}}\cdot\mathbf{g}^1\|_{\mathrm{BMO}}\leq 1/5,$$

$$\|\mathbf{g}^1\|_{L^\infty}=R$$

and such that supp$(\mathbf{g}^1-(R,0,\dots,0))$ is compact.

Since

$$\lim_{x\to\infty}\mathbf{K}\cdot(\mathbf{g}^1-(R,0,\dots,0))(x)=0,$$

$\lim_{x\to\infty}\tilde{\mathbf{K}}\cdot\mathbf{g}(x)$ exists. Therefore, there exists $f_1(x)\in \mathrm{BMO}(\mathbf{R}^n)$ such that

$$\|f-\tilde{\mathbf{K}}\cdot\mathbf{g}^1-f_1\|_{\mathrm{BMO}}\leq(1/4)\cdot(1/5)=1/20$$

and such that supp f_1 is compact. Then

$$\|f_1\|_{\mathrm{BMO}}\leq 1/5+1/20=1/4.$$

By applying the above argument to $4f_1$, we get $\mathbf{g}^2(x)$ such that

$$\|f_1-\tilde{\mathbf{K}}\cdot\mathbf{g}^2\|_{\mathrm{BMO}}\leqslant(1/5)\cdot(1/4)=1/20,$$

$$\|\mathbf{g}^2\|_{L^\infty}=R/4,$$

and such that $\mathrm{supp}(\mathbf{g}^2-(R/4,0,\dots,0))$ is compact. Then,

$$\|f-\tilde{\mathbf{K}}\cdot(\mathbf{g}^1+\mathbf{g}^2)\|_{\mathrm{BMO}}\leqslant\|f-\tilde{\mathbf{K}}\cdot\mathbf{g}^1-f_1\|_{\mathrm{BMO}}+\|f_1-\tilde{\mathbf{K}}\cdot\mathbf{g}^2\|_{\mathrm{BMO}}$$
$$\leqslant 1/20+1/20=1/10.$$

By repeating this argument, we get $\{\mathbf{g}^k(x)\}_{k=1}^\infty$ such that

$$f=\tilde{\mathbf{K}}\cdot\sum_{k=1}^\infty\mathbf{g}^k\quad\text{(modulo constants)},$$

and

$$\sum_{k=1}^\infty\|\mathbf{g}^k\|_{L^\infty}\leqslant R+R/4+R/8+R/16+\dots=3R/2.$$

6. Proof of Corollary 1

Since $\theta_1(\xi),\dots,\theta_m(\xi)$ satisfy (1.1), $\bar{\theta}_1(\xi),\dots,\bar{\theta}_m(\xi)$ also satisfy (1.1). Set

$$K_j^*f(x)=(\bar{\theta}_j(\xi)\hat{f}(\xi))^\vee(x).$$

By Theorem 1, for any $f(x)\in\mathrm{BMO}(\mathbf{R}^n)$ with compact support, there exist $g_1(x),\dots,g_m(x)\in L^\infty(\mathbf{R}^n)$ such that

$$f=\sum_{j=1}^m\tilde{K}_j^*g_j\quad\text{(modulo constants)},\tag{6.1}$$

and

$$\sum_{j=1}^m\|g_j\|_{L^\infty}\leqslant C_{1.1}(\bar{\theta}_1,\dots,\bar{\theta}_m)\,\|f\|_{\mathrm{BMO}}.\tag{6.2}$$

For $h(x)\in\mathscr{S}_0(\mathbf{R}^n)$, set

$$u(h)=\sup_{\|f\|_{\mathrm{BMO}}\leqslant 1}\left|\int h(x)\overline{f(x)}\,dx\right|$$

and

$$v(h)=\sum_{j=1}^{m}\|K_j h\|_{L^1},$$

where

$$\mathcal{S}_0(\mathbf{R}^n)=\{h(x)\in\mathcal{S}(\mathbf{R}^n):\hat{h}(\xi)=0 \quad \text{near } 0\}.$$

Then,

$$\begin{aligned}
u(h)&\leqslant C\sup_{\|f\|_{\mathrm{BMO}}\leqslant 1,\, f\in L^\infty,\ \mathrm{supp} f:\ \mathrm{compact}}\left|\int h(x)\overline{f(x)}\,dx\right|\\
&=C\sup_f\left|\int h(x)\overline{\left\{\sum_{j=1}^{m}\tilde{K}_j^* g_j(x)\right\}}\,dx\right| \quad (\text{by } (6.1))\\
&=C\sup_f\left|\int\sum K_j h(x)\overline{g_j(x)}\,dx\right|\\
&\leqslant C\cdot v(h) \quad (\text{by } (6.2)).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
v(h)&=\sup_{\|u_j\|_{L^\infty}\leqslant 1\ (j=1,\dots,m)}\left|\int\sum_{j=1}^{m}K_j h(x)\overline{u_j(x)}\,dx\right|\\
&=\sup\left|\int h(x)\left\{\sum_{j=1}^{m}\overline{\tilde{K}_j^* u_j(x)}\right\}dx\right|\\
&\leqslant Cu(h),
\end{aligned}$$

since $\tilde{K}_j^*$ is a bounded operator from L^∞ to BMO. Thus, we get

$$u(h)\approx v(h) \tag{6.3}$$

for any $h\in\mathcal{S}_0(\mathbf{R}^n)$. In particular, we get

$$v(h)\geqslant C\|h\|_{L^1}.$$

Following the argument of [14] pp. 230–231, we can show that the Banach space $B=\{h(x)\in L^1(\mathbf{R}^n): v(h)<+\infty\}$ equipped with the norm v, is the completion of $\mathcal{S}_0(\mathbf{R}^n)$ with respect to the norm v. If we substitute $m=n+1, K_j=R_j$ $(j=1,\dots,n)$ and $K_{n+1}=\mathbf{1}$, then $B=H^1(\mathbf{R}^n)$. On the other hand, (6.3) tells us that if $\theta_1,\dots,\theta_m$ satisfy (1.1), then the

Banach space B is independent of the choice of $\theta_1, \ldots, \theta_m$. Consequently, if (1.1) holds, then

$$B = H^1(\mathbf{R}^n)$$

and

$$v(h) \approx \|h\|_{H^1(\mathbf{R}^n)}$$

for any $h \in B$.

References

[1] CALDERÓN, A. P., An atomic decomposition of distributions in parabolic H^p spaces. *Adv. in Math.*, 25 (1977), 216–225.

[2] CALDERÓN, A. P. & TORCHINSKY, A., Parabolic maximal functions associated with a distribution. *Adv. in Math.*, 16 (1975), 1–63.

[3] CARLESON, L., Two remarks on H^1 and BMO. *Adv. in Math.*, 22 (1976), 269–277.

[4] — An explicit unconditional basis in H^1. *Bull. Sci. Math.*, 104 (1980), 405–416.

[5] CHANG, S.-Y. & FEFFERMAN, R., A continuous version of duality of H^1 and BMO on the bidisc. *Ann. of Math.*, 112 (1980), 179–201.

[6] COIFMAN, R. & DAHLBERG, B., Singular integral charcterization of nonisotropic H^p spaces and the F. and M. Riesz theorem. *Proc. Symp. Pure Math.*, 35 (1979), 231–234.

[7] COIFMAN, R. & WEISS, G., Extensions of Hardy spaces and their use in analysis. *Bull Amer. Math. Soc.*, 83 (1977), 569–645.

[8] FEFFERMAN, C. & STEIN, E. M., H^p spaces of several variables. *Acta Math.*, 129 (1972), 137–193.

[9] GANDULFO, A., GARCIA-CUERVA, J. & TAIBLESON, M., Conjugate system characterization of H^1: counter examples for the Euclidean plane and local fields. *Bull. Amer. Math. Soc.*, 82 (1976), 83–85.

[10] JANSON, S., Characterization of H^1 by singular integral transforms on martingales and $\mathbf{R}^n$. *Math. Scand.*, 41 (1977), 140–152.

[11] JONES, P. W., *Constructions with functions of bounded mean oscillation*. Ph.D. Thesis. University of California, 1978.

[12] — Carleson measures and the Fefferman-Stein decomposition of BMO(**R**). *Ann. of Math.*, 111 (1980), 197–208.

[13] — L^∞ estimates for the $\bar{\partial}$ problem in a half-plane. To appear in Acta Math.

[14] STEIN, E. M., *Singular integrals and differentiability properties of functions*. Princeton, 1970.

[15] UCHIYAMA, A., A constructive proof of the Fefferman-Stein decomposition of BMO on simple martingales. To appear in the *Proceedings of the conference in honor of Antoni Zygmund*, held at the University of Chicago, 1981.

Received November 20, 1981