SPECTRAL SYNTHESIS IN SOBOLEV SPACES, AND UNIQUENESS OF SOLUTIONS OF THE DIRICHLET PROBLEM

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1. Introduction

Consider functions f in the Sobolev space $W_m^q(\mathbf{R}^d)$, $1<q<\infty$, i.e. functions such that $\sum_{0\leqslant|\alpha|\leqslant m}\int_{\mathbf{R}^d}|D^\alpha f|^q dx=\|f\|_{m,q}^q<\infty$. For any set E in $\mathbf{R}^d$ one can define the trace on E of f and of its partial derivatives $D^\alpha f$, $|\alpha|\leqslant m-1$, in a natural way. (See Section 2.) We denote these traces by $f|_E$ and $D^\alpha f|_E$. Our main result is the following theorem.

THEOREM 1.1. *Let $f\in W_m^q(\mathbf{R}^d)$ for some $q>2-1/d$, and some positive integer m. Let $K\subset\mathbf{R}^d$ be closed, and suppose that $D^\alpha f|_K=0$ for all α, $0\leqslant|\alpha|\leqslant m-1$. Then $f\in \overset{0}{W}{}_m^q(K^c)$, i.e. there exist functions $\varphi_n\in C_0^\infty$ such that each φ_n vanishes on a neighborhood of K, and $\lim_{n\to\infty}\|f-\varphi_n\|_{m,q}=0$.*

By analogy with the classical spectral synthesis of Beurling (see e.g. [20]) we say that sets K with the approximation property in the theorem admit *(m,q)-synthesis*. Thus, in contrast to the situation in harmonic analysis, the conclusion here is that *all closed sets in $\mathbf{R}^d$ admit (m,q)-synthesis, at least if $q>2-1/d$.*

Among the consequences we mention the following uniqueness theorem for the Dirichlet problem. This is in fact an equivalent formulation of the result in the case $q=2$. By way of illustration we only formulate the theorem in the simplest case. Generalizations to more general elliptic equations are immediate. See T. Kolsrud [21] for an extension to situations where u is defined only in G.

THEOREM 1.2. *Let $G\subset\mathbf{R}^d$ be a bounded open set. Let $u\in W_m^2(\mathbf{R}^d)$ satisfy $\Delta^m u=0$ in G, and $D^\alpha u|_{\partial G}=0$, $0\leqslant|\alpha|\leqslant m-1$. Then $u\equiv 0$ in G.*

That this is a consequence of Theorem 1.1 is obvious, because it is well known that if $\Delta^m u=0$ in G and $u\in \overset{0}{W}{}_m^2(G)$, then $u=0$.

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Consequences for L^p-approximation by solutions, and stability are given below in Section 6.

In the case $m=1$, $q=2$, Theorem 1.1 is due to J. Deny [9; Theorem II: 2, p. 143]. (This reference was unfortunately overlooked in the survey [18].) This result was extended to abstract Dirichlet spaces by A. Beurling and J. Deny [7], [10; p. 168, p. 172]. For $m=1$, $1<q<\infty$, the theorem is due to V. P. Havin [13], and T. Bagby [6]. See also [14; Lemma 4]. All these proofs depend on the fact that W_1^q is closed under truncation.

Theorem 1.2 was proved by S. L. Sobolev [33], [33a; Theorem 3, §§ 14, 15] in the case when G is bounded by a finite union of smooth manifolds (of arbitrary dimension). In the above generality the uniqueness problem was formulated, and its equivalence with the $(m, 2)$-spectral synthesis problem was pointed out by B. Fuglede. (See B.-W. Schulze and G. Wildenhain [31; IX, § 5].) See also [18a].

The (m, q)-synthesis problem was also approached by J. Polking [29] and the author [16], [17], who were motivated by an L^p-approximation problem for harmonic and polyharmonic functions. See also [18].

The present paper is a continuation of [17], and the same technique is used. However, an effort has been made to make the paper readable independently of [17].

The proofs depend on a detailed study of the behavior of functions in W_m^q close to their zeroes. This study depends on the properties of (m, q)-capacities and the corresponding (non-linear for $q\neq 2$) potentials, the theory of which is due mainly to B. Fuglede, Ju. G. Rešetnjak, N. G. Meyers, D. R. Adams, V. G. Maz'ja, and V. P. Havin. See e.g. [12], [30], [27], [3], [25], [26], [14]. See Section 2 below.

I am grateful to V. G. Maz'ja for an enlightening conversation in connection with the crucial Theorem 4.2, and to B. Fuglede for drawing my attention to the equivalence of $(m, 2)$-synthesis and uniqueness for the Dirichlet problem.

I am also indebted to these mathematicians, and to D. R. Adams, T. Bagby, V. P. Havin, and P. W. Jones for pointing out a number of obscurities and inaccuracies in the manuscript, and for many other useful comments.

Our notational conventions are the following: If E is a set, its interior, closure, and complement are denoted respectively E°, $\bar{E}$, and E^c. If G is an open set, $C_0^\infty(G)$ denotes the infinitely differentiable functions with support in G. $\overset{0}{W}{}_m^q(G)$ is the closure in $W_m^q(G)$ of $C_0^\infty(G)$. $\nabla^m f$ is the mth gradient of a function, i.e. $\nabla^m f=\{D^\alpha f;\ |\alpha|=m\}$, where α denotes multiindices, and $|\nabla^m f|=\sum_{|\alpha|=m}|D^\alpha f|$. The ball $\{y\in\mathbf{R}^d;\ |y-x|<\delta\}$ will be denoted $B_\delta(x)$ or B_δ. If $\delta=2^{-n}$ we write $B_n(x)$. The letter A will denote various positive constants, whose value can change from one line to the next.

Added in proof, **Dec. 30, 1981.** T. Wolff has recently proved that Theorem 3.2 below (the Kellogg property) remains true for $q>1$. It follows that Theorem 3.1 and its corollaries, including Theorem 1.1, hold for $q>1$.

2. Preliminaries about capacities and potentials

In this section we give a quick review of some necessary facts. We always assume $1<q<\infty$, and set $1/p=1-1/q$.

If $mq>d$ the elements in $W_m^q(\mathbf{R}^d)$ can be redefined on sets of measure zero to be continuous functions, but for $mq\leqslant d$ this is no longer the case. Then the natural way of measuring the deviation from continuity is by means of (m, q)-capacity, denoted $C_{m,q}$.

For compact sets $K\subset\mathbf{R}^d$ the (m, q)-capacity is defined by

$$C_{m,q}(K)=\inf\{\|\varphi\|_{m,q}^q;\ \varphi\in C_0^\infty,\ \varphi\geqslant 1 \text{ on } K\}.$$

The definition is extended to open sets G by

$$C_{m,q}(G)=\sup\{C_{m,q}(K);\ K\subset G,\ K \text{ compact}\},$$

and to arbitrary sets E by

$$C_{m,q}(E)=\inf\{C_{m,q}(G);\ G\supset E,\ G \text{ open}\}.$$

Thus $C_{m,q}$ is an outer capacity.

If a statement is true for all x except for a set E with $C_{m,q}(E)=0$, we say that the statement is true (m, q)-*quasi everywhere* ((m, q)-q.e.).

We will denote by G_m the Bessel kernel, defined as the inverse Fourier transform of $\hat{G}_m(\xi)=(1+|\xi|^2)^{-m/2}$. Then it is well known (Calderón [8], see also Stein [34]) that a function f is in $W_m^q(\mathbf{R}^d)$, $1<q<\infty$, if and only if it can be represented as a convolution $f=G_m*g$, $g\in L^q$, and that $A^{-1}\|g\|_q\leqslant\|f\|_{m,q}\leqslant A\|g\|_q$.

It is then not hard to show that (m, q)-capacity can be defined equivalently by

$$C_{m,q}(E)=\inf\{\|g\|_q^q;\ g\geqslant 0,\ G_m*g\geqslant 1 \text{ on } E\} \tag{2.1}$$

for arbitrary sets E.

For technical reasons it is sometimes more convenient to use the Riesz kernel $R_m(x)=|x|^{m-d}$, $0<m<d$. Then $A^{-1}G_m(x)\leqslant R_m(x)\leqslant AG_m(x)$ for $|x|\leqslant 1$, and

$$A^{-1}C_{m,q}(E)\leqslant\inf\{\|g\|_q^q;\ g\geqslant 0,\ R_m*g\geqslant 1 \text{ on } E\}\leqslant AC_{m,q}(E)$$

for $E\subset\{|x|\leqslant 1\}$.

If $2m<d$, $(m, 2)$-capacity is equivalent to the classical Riesz capacity with respect to the kernel R_{2m}. Thus $(1, 2)$-capacity is equivalent to Newtonian capacity if $d\geqslant 3$ (and locally to logarithmic capacity in the plane).

The (m, q)-capacity has many nice properties. It is subadditive, i.e.

$$C_{m,q}(E_1 \cup E_2) \leqslant C_{m,q}(E_1) + C_{m,q}(E_2),$$

and one can show that it is left continuous, i.e. for any increasing sequence of sets $\{E_i\}$

$$C_{m,q}\left(\bigcup_1^\infty E_i\right) = \lim_{i\to\infty} C_{m,q}(E_i).$$

(See Fuglede [12], N. G. Meyers [27], Ju. G. Rešetnjak [30], V. G. Maz'ja–V. P. Havin [25].) Thus, Choquet's capacitability theorem applies, so that if E is a Borel or Suslin set,

$$C_{m,q}(E) = \sup\{C_{m,q}(K);\, K \subset E,\, K \text{ compact}\}.$$

The same authors proved that for Suslin sets (m, q)-capacity has a dual definition:

$$C_{m,q}(E)^{1/q} = \sup\{\mu(E);\, \mu \geqslant 0,\, \operatorname{supp}\mu \subset E,\, \|G_m * \mu\|_p \leqslant 1\}. \tag{2.2}$$

It follows that the extremal function g in (2.1) has the form $g = (G_m * \mu)^{p-1}$, where $\operatorname{supp}\mu \subset \bar{E}$, $\mu(\bar{E}) = C_{m,q}(E)$, and $G_m * (G_m * \mu)^{p-1} \geqslant 1$ (m, q)-quasi everywhere on E.

The metric properties of $C_{m,q}$ are very well known, and the most exhaustive results are found in Maz'ja and Havin [25]. Here we content ourselves with the following:

If $mq > d$ then $C_{m,q}(\{a\}) > 0$.
If $mq < d$ then $A^{-1}\delta^{d-mq} \leqslant C_{m,q}(B_\delta) \leqslant A\delta^{d-mq}$, $0 < \delta \leqslant 1$.
If $mq = d$ then $A^{-1}\left(\log \dfrac{2}{\delta}\right)^{1-q} \leqslant C_{m,q}(B_\delta) \leqslant A\left(\log \dfrac{2}{\delta}\right)^{1-q}$, $0 < \delta \leqslant 1$.

If $K \subset \mathbf{R}^d$ is a locally Lipschitz manifold of dimension $d-k$, then

$$C_{m,q}(K) = 0 \quad \text{if} \quad mq \leqslant k,$$

$$C_{m,q}(K) > 0 \quad \text{if} \quad mq > k.$$

This last statement is what is needed in order to see that the theorem of Sobolev quoted in the introduction is a consequence of Theorem 1.2. See [18 a].

The definition (2.1) immediately gives the inequality

$$C_{m,q}(\{x;\, G_m * |g| > \lambda\}) \leqslant \lambda^{-q}\|g\|_q^q.$$

In particular $G_m * g$ is defined (m, q)-q.e.

One can extend this inequality to the Hardy–Littlewood maximal function Mf, defined by

$$Mf(x) = \sup_{\delta>0} \frac{1}{|B_\delta(x)|} \int_{B_\delta(x)} |f(y)|\, dy.$$

In other words, if $f \in W_m^q(\mathbf{R}^d)$, then

$$C_{m,q}(\{x;\, Mf > \lambda\}) \leqslant A\lambda^{-q} \|f\|_{m,q}^q.$$

See D. R. Adams [1]. Then, as Adams observed, if $\{\chi_n\}$ is an approximate identity, i.e. $\chi_n(x) = n^d \chi(nx)$, $\chi \geqslant 0$, $\operatorname{supp} \chi \subset B_0(1)$, $\int \chi\, dx = 1$, one can show in a standard way that $\lim_{n\to\infty} \chi_n * f(x) = \tilde{f}(x)$ exists (m, q)-q.e., and that $\tilde{f}(x) = f(x)$ a.e. Moreover, for any $\varepsilon > 0$ there is an open G such that $C_{m,q}(G) < \varepsilon$, and $\tilde{f}|_{G^c}$ is continuous on G^c. Functions with this property are called *(m, q)-quasicontinuous*. Thus, *every $f \in W_m^q$ has an (m, q)-quasicontinuous representative.* In particular, integrals $G_m * g$, $g \in L^q$, are (m, q)-quasicontinuous. Cf. J. Deny and J.-L. Lions [11], Maz'ja and Havin [25].

It is an important fact that *an element f in W_m^q has a quasicontinuous representative which is essentially unique*, in the sense that if φ and ψ are (m, q)-quasicontinuous, and if $\varphi(x) = \psi(x)$ a.e., then $\varphi(x) = \psi(x)\, (m, q)$-q.e. This result, stated in [11, p. 353], was first proved by H. Wallin [35, Lemma 6] in the classical case $m = 1$, $q = 2$, and was extended to the general situation by Maz'ja and Havin [25], and T. Sjödin [32]. See also Deny [9 a].

We now define the *trace* $f|_E$ of a function f in W_m^q as *the restriction to E of any (m, q)-quasicontinuous representative of f.* Thus $f|_E$ is defined (m, q)-q.e. on E.

Since $D^\alpha f$ belongs to $W_{m-|\alpha|}^q$, the trace $D^\alpha f|_E$ is defined $(m - |\alpha|, q)$-q.e. on E.

3. Results and outline of proof

We shall prove the following stronger variant of Theorem 1.1.

THEOREM 3.1. *Let $f \in W_m^q(\mathbf{R}^d)$ for some $q > 2 - 1/d$. Let K be closed, and suppose that $D^\alpha f|_K = 0$ for all α, $0 \leqslant |\alpha| \leqslant m-1$. Then for any $\varepsilon > 0$ there is a function ω, $0 \leqslant \omega \leqslant 1$, such that $\omega(x) = 1$ on a neighborhood of $K \cup \{\infty\}$, and $\|\omega f\|_{m,q} < \varepsilon$.* Morover, *$\omega$ can either be chosen in C^∞ or so that $(1-\omega) f \in L^\infty$.*

This immediately implies Theorem 1.1. In fact, $(1-\omega)f$ has its support away from K, and approximates f. A suitable convolution gives a C^∞ approximating function.

Remark. The existence of the multiplier ω seems to be new even if f is already known to belong to $\overset{0}{W}{}_m^q(K^c)$. This implies for example the following: Let G be open, let $f \in \overset{0}{W}{}_m^q(G)$, $q > 2 - 1/d$, and suppose $f \geqslant 0$ a.e. on G. Then there exist non-negative functions f_n in $C_0^\infty(G)$

such that $\lim_{n\to\infty} \|f-f_n\|_{m,q}=0$. Another interesting application has recently been given by H. Brézis and F. E. Browder [7a].

Following N. G. Meyers [28] we say that a set E is (k, q)-*thin*, $1<q<\infty$, at a point x, if $mq\leqslant d$, and

$$\int_0^1 \left\{\frac{C_{k,q}(E\cap B_\delta(x))}{\delta^{d-kq}}\right\}^{p-1}\frac{d\delta}{\delta}<\infty,$$

or equivalently,

$$\sum_{n=0}^{\infty}\{2^{n(d-kq)}C_{k,q}(E\cap B_n(x))\}^{p-1}<\infty. \tag{3.1}$$

If a set is not thin it is *thick*. For $k=1$, $q=2$ this agrees with the classical definition. (k, q)-thinness for $q\neq 2$ was first defined by Adams and Meyers [3], and the author [14].

Our proof of Theorem 3.1 depends crucially on the following fact.

THEOREM 3.2. (The Kellogg property.) *Let $E\subset \mathbf{R}^d$. The set of points belonging to E where E is (k, q)-thin has zero (k, q)-capacity, provided $q>2-k/d$.*

For $q=2$, $k=1$ this is classical. For $q=2$, $k\neq 1$ it is due to Fuglede [12a]. For $q\neq 2$ it is Corollary 2 to Theorem 6 in [14]. The proof depends on upper estimates of potentials, due (for $q\neq 2$) to Adams–Meyers [3] and Maz'ja–Havin [25]. These estimates break down for $q\leqslant 2-k/d$, but whether the Kellogg property holds for $1<q\leqslant 2-k/d$ is unknown. This is the only obstacle to proving Theorem 3.1 for all q, $1<q<\infty$. (On the other hand, an extension to "fractional spaces" seems to require different methods.)

Theorem 3.1 will be deduced from the Kellogg property and the following chain of results.

THEOREM 3.3. *Let $K\subset \mathbf{R}^d$. The set of functions f in $W^q_m(\mathbf{R}^d)$ such that $D^\alpha f|_K=0$, $0\leqslant|\alpha|\leqslant m-1$, is a module over C^m. More generally, if f, φ, and $f\varphi$ belong to W^q_m, and $D^\alpha f|_K=0$, $0\leqslant|\alpha|\leqslant m-1$, then $D^\alpha(f\varphi)|_K=0$, $0\leqslant|\alpha|\leqslant m-1$.*

This is an easy consequence of the uniqueness of quasicontinuous representatives. In fact, let $\varphi\in W^q_m$, and let $D^\alpha f|_K=0$, $0\leqslant|\alpha|\leqslant m-1$. Suppose $f\varphi\in W^q_m$. Then $D^\alpha(f\varphi)$ belongs to $W^q_{m-|\alpha|}$, and is therefore $(m-|\alpha|, q)$-quasicontinuous. But $D^\alpha(f\varphi)$ equals the pointwise derivative of $f\varphi$ almost everywhere, so by Leibniz' formula

$$D^\alpha(f\varphi)=\sum_{\beta+\gamma=\alpha}\frac{\alpha!}{\beta!\,\gamma!}D^\beta f\,D^\gamma\varphi.$$

But $|\beta|\leqslant|\alpha|$ and $|\gamma|\leqslant|\alpha|$, so $D^\beta f$ and $D^\gamma\varphi$ belong to $W^q_{m-|\alpha|}$, and they are therefore $(m-|\alpha|, q)$-quasicontinuous. But then the expressions on the left and on the right are both $(m-|\alpha|, q)$-quasicontinuous, and thus equal $(m-|\alpha|, q)$-q.e. The theorem follows.

We can now always assume that K is bounded, and that f has compact support. In fact, by Theorem 3.3 we can multiply f by a cut-off function $\chi \in C_0^\infty$ which is 1 on a ball that is large enough for $\|\chi f - f\|_{m,q}$ to be small. It is thus enough to approximate χf for any such χ, and to replace K by $K \cap \operatorname{supp} \chi$.

THEOREM 3.4. *Let $K \subset \mathbf{R}^d$ be compact, and suppose that K is $(1, q)$-thick at all its points. Let $f \in W_m^q(\mathbf{R}^d)$, $1 < q < \infty$, and suppose that $D^\alpha f|_K = 0$ for all α with $0 \leqslant |\alpha| \leqslant m-1$. Let V be an arbitrary neighborhood of K, and let $\varepsilon > 0$. Then there is a C^∞ function ω with support in V such that $0 \leqslant \omega \leqslant 1$, $\omega = 1$ on a neighborhood of K, and $\|f\omega\|_{m,q} < \varepsilon$.*

This is Theorem 3.1 in [17]. We shall sketch the proof in Section 5 below.

THEOREM 3.5. *Let $K \subset \mathbf{R}^d$ be compact, and suppose either that $C_{m,q}(K) = 0$, or that $C_{m-k,q}(K) = 0$ for some k, $1 \leqslant k \leqslant m-1$, and that K is $(m-k+1, q)$-thick at all its points. Let $f \in W_m^q(\mathbf{R}^d)$, $1 < q < \infty$, and suppose (if $k \geqslant 1$) that $D^\alpha f|_K = 0$ for all α, $0 \leqslant |\alpha| \leqslant k-1$. (Note that $D^\alpha f|_K = 0$ trivially for $k \leqslant |\alpha| \leqslant m-1$.) Then the conclusion of Theorem 3.4 remains true.*

This theorem improves on Theorem 4.1 in [17]. This improvement is the main new contribution of the present paper.

We now sketch the deduction of Theorem 3.1 from Theorems 3.2–3.5. Let K be the given set. As observed above we can assume that K is compact. We assume that $q > 2 - 1/d$, so that the Kellogg property is true for (k, q)-capacity for all $k \geqslant 1$.

Then, by Theorem 3.2, K can be split into $m+1$ disjoint sets, $K = E_0 \cup E_1 \cup \dots \cup E_m$, with the following properties:

(i) K is $(1, q)$-thick everywhere on E_0;
(ii) $C_{k,q}(E_k) = 0$, and E_k (and K) is $(k+1, q)$-thick everywhere on E_k, $k = 1, \dots, m-1$;
(iii) $C_{m,q}(E_m) = 0$.

Suppose the sets E_k are compact. Then, using Theorems 3.4, 3.5, and 3.3, we can successively approximate the given function f by functions f_k, $k = 0, 1, \dots, m$, where $f_0 = (1-\omega_0)f$, $f_k = (1-\omega_k)f_{k-1}$, $0 \leqslant \omega_k \leqslant 1$, $\omega_k \in C_0^\infty(V)$, $\omega_k = 1$ on a neighborhood of E_k, and $\|f_k - f_{k-1}\|_{m,q} = \|\omega_k f_{k-1}\|_{m,q} < \varepsilon 2^{-k-1}$.

It follows that $\|f - f_m\|_{m,q} < \varepsilon$, and $f_m = f(1-\omega)$, where $\omega = 1 - (1-\omega_0) \dots (1-\omega_m)$, $\omega \in C_0^\infty(V)$, $\omega = 1$ on a neighborhood of K.

However, in general the sets E_k are not compact, and the argument has to be modified. We postpone the details to Section 5.

The last statement of Theorem 3.1 is a consequence of the following result.

THEOREM 3.6. *Let $f\in W^q_m(\mathbf{R}^d)$, $1<q<\infty$, and let E be an arbitrary set with $C_{m,q}(E)=0$. Then, for any $\varepsilon>0$ there is a function ω in W^q_m such that $0\leqslant\omega\leqslant 1$, $\omega=1$ on a neighborhood of E, $f(1-\omega)\in L^\infty\cap W^q_m$, and $\|f\omega\|_{m,q}<\varepsilon$.*

This is Lemma 5.2 in [17]. Since the proof is short and relatively self-contained, it will be repeated below in Section 5. J. R. L. Webb [36] has given an interesting application of this theorem to non-linear PDE. See also H. Brézis and F. E. Browder [7a].

4. Tools

Let $K\subset\mathbf{R}^d$ be closed. For any positive integer k and any $q\geqslant 1$ we define the "condenser capacity"

$$C_{k,q}(K\cap B_\delta(x), B_{2\delta}(x))=\inf\left\{\int|\nabla^k\varphi|^q dy;\ \varphi\in C_0^\infty(B_{2\delta}(x)),\ \varphi\geqslant 1 \text{ on } K\cap B_\delta(x)\right\}.$$

We define a "relative capacity" by

$$c_{k,q}(K, B_\delta(x))=\delta^{kq-d}C_{k,q}(K\cap B_\delta(x), B_{2\delta}(x)).$$

Then it is easily seen that $c_{k,q}$ is homogeneous of degree zero, in the sense that

$$c_{k,q}(\delta K, B_\delta(0))=c_{k,q}(K, B_1(0)),\quad \delta>0.$$

Moreover, it is easily seen that

$$c_{k,q}(K, B_\delta)\geqslant A>0 \quad\text{if}\quad kq>d,\quad\text{unless}\quad K\cap B_\delta=\emptyset,$$

and that

$$A^{-1}c_{k,q}(K, B_\delta)\leqslant\delta^{kq-d}C_{k,q}(K\cap B_\delta)\leqslant Ac_{k,q}(K, B_\delta),$$

if $kq<d$.

The proof of Theorem 3.4 depends on the following estimate, which was proved in [17]. (Lemma 2.1, the case $k=1$.)

THEOREM 4.1. *Let $K\subset\mathbf{R}^d$ be closed, let $f\in W^q_m(B_\delta)$, $1<q<\infty$, for some ball B_δ that intersects K, and suppose that $D^\alpha f|_{K\cap B_\delta}=0$ for all α, $0\leqslant|\alpha|\leqslant m-1$. Then*

$$\int_{B_\delta}|f|^q dx\leqslant\frac{A\delta^{mq}}{c_{1,q}(K, B_\delta)}\int_{B_\delta}|\nabla^m f|^q dx.$$

This estimate cannot be used to prove Theorem 3.5, because in that situation $c_{1,q}(K\cap B_\delta)=0$. Theorem 4.1 has to be replaced by the following.

THEOREM 4.2. *Let K and f be as in Theorem 4.1, except that $D^\alpha f|_{K\cap B_\delta}$ is supposed to vanish only for $0\leqslant|\alpha|\leqslant k-1$ for some k, $1\leqslant k\leqslant m-1$. Then there is a polynomial P of degree $\leqslant m-1$ such that*

$$\int_{B_\delta}|f-P|^q dx\leqslant \frac{A\delta^{mq}}{c_{m-k+1,q}(K,B_\delta)}\int_{B_\delta}|\nabla^m f|^q dx,$$

and

$$|P(y)|\leqslant A\sum_{j=k}^{m-1}\delta^{j-d}\int_{B_\delta}|\nabla^j f|\,dx,\quad \textit{for all } y\in B_\delta.$$

What we will actually need is the following consequence.

COROLLARY 4.3. *Let K and f be as in Theorem 4.2. Then*

$$\int_{B_\delta}|f|^q dx\leqslant A\delta^{kq}\int_{B_\delta}|\nabla^k f|^q dx+\frac{A\delta^{mq}}{c_{m-k+1,q}(K,B_\delta)}\int_{B_\delta}|\nabla^m f|^q dx.$$

The corollary follows from the theorem by Minkowski's and Hölder's inequalities, and the well-known inequality

$$\delta^{kq}\int_{B_\delta}|\nabla^k\varphi|^q dx\leqslant A\int_{B_\delta}|\varphi|^q dx+A\delta^{mq}\int_{B_\delta}|\nabla^m\varphi|^q dx. \tag{4.1}$$

See e.g. [24; 1.1.8, p. 23].

Proof of Theorem 4.2. By homogeneity we can assume that $\delta=1$, and that B_δ is the unit ball, which we denote by B. By Hestenes' theorem [19] we can assume that $f\in W_m^q(\mathbf{R}^d)$, and that $\int_{2B}|\nabla^j f|^q dx\leqslant A\int_B|\nabla^j f|^q dx$. Thus, it suffices to construct a polynomial P of degree $m-1$ such that

$$\int_B|f-P|^q dx\leqslant\frac{A}{c_{m-k+1,q}(K,B)}\int_{2B}|\nabla^m f|^q dx,$$

and

$$|P(y)|\leqslant A\sum_{j=k}^{m-1}\int_{2B}|\nabla^j f|\,dx,\quad y\in B.$$

Let $D^\alpha f|_{K\cap B_\delta}=0$, $0\leqslant|\alpha|\leqslant k-1$. Then there is a sequence $\{f_n\}$ of C^∞ functions such that $\lim_{n\to\infty}D^\alpha f_n(x)=0$, $(m-|\alpha|,q)$-q.e. on K, $0\leqslant|\alpha|\leqslant k-1$, uniformly outside a set of arbitrarily small $(m-|\alpha|,q)$-capacity.

We will first consider an arbitrary f in C^∞, then apply the result to f_n, and pass to the limit.

We write the Taylor expansion of f about a point y as

$$f(x) = \sum_{|\alpha|\leqslant m-1} \frac{1}{\alpha!}(x-y)^\alpha D^\alpha f(y) + R_y^{m-1} f(x) = P_y^{m-1} f(x) + R_y^{m-1} f(x).$$

Here

$$R_y^{m-1} f(x) = \frac{1}{(m-1)!}\int_0^{|x-y|} (|x-y|-\tau)^{m-1}(\sigma\cdot\nabla)^m f(y+\tau\sigma)\,d\tau$$

$$= \frac{1}{(m-1)!}\int_0^{|x-y|} u^{m-1}(\sigma\cdot\nabla)^m f(x-u\sigma)\,du,$$

where $\sigma = (x-y)/|x-y|$.

Lemma 4.4. *With the above notation*

$$\int_{|y-x|\leqslant 1} |R_y^{m-1} f(x)|\,dy \leqslant A \int_{|\xi-x|\leqslant 1} |\nabla^m f(\xi)|\,|\xi - x|^{m-d}\,d\xi.$$

Proof. Clearly

$$|R_y^{m-1} f(x)| \leqslant A \int_0^{|x-y|} u^{m-1}|\nabla^m f(x-u\sigma)|\,du = A\int_0^{|x-y|} u^{m-d}|\nabla^m f(x-u\sigma)|\,u^{d-1}\,du.$$

Now set $u\sigma = \xi$, and integrate over the unit sphere.

$$\int_{|\sigma|=1} |R_y^{m-1} f(x)|\,d\sigma \leqslant A\int_{|\xi|\leqslant|x-y|} |\xi|^{m-d}|\nabla^m f(x-\xi)|\,d\xi.$$

Finally

$$\int_{|y-x|\leqslant 1} |R_y^{m-1} f(x)|\,dy = A\int_0^1\int_{|\sigma|=1} |R_y^{m-1} f(x)|\,r^{d-1}\,d\sigma\,dr$$

$$\leqslant A\int_{|\xi|\leqslant 1} |\xi|^{m-d}|\nabla^m f(x-\xi)|\,d\xi, \qquad \text{q.e.d.}$$

Now let $x\in B$ and $z\in 2B$ be arbitrary, and let $y\in B\cap K$. In the end $P_y^{k-1} f(x)$ is going to be small. We will expand $f(x) - P_y^{k-1} f(x)$ as function of x in Taylor series about z. We find

$$f(x) = \sum_{|\alpha|\leqslant m-1} \frac{1}{\alpha!}(x-z)^\alpha D^\alpha f(z) + R_z^{m-1} f(x);$$

$$D^\alpha f(y) = \sum_{|\beta|\leqslant m-|\alpha|-1} \frac{1}{\beta!}(y-z)^\beta D^{\alpha+\beta} f(z) + R_z^{m-|\alpha|-1}(D^\alpha f)(y);$$

$$\begin{aligned}
P_y^{k-1}f(x) &= \sum_{|\alpha|\leqslant k-1} \frac{1}{\alpha!}(x-y)^\alpha D^\alpha f(y) \\
&= \sum_{|\alpha|\leqslant k-1}\ \sum_{|\beta|\leqslant m-|\alpha|-1} \frac{1}{\alpha!\,\beta!}(x-y)^\alpha (y-z)^\beta D^{\alpha+\beta}f(z) \\
&\quad + \sum_{|\alpha|\leqslant k-1} \frac{1}{\alpha!}(x-y)^\alpha R_z^{m-|\alpha|-1}(D^\alpha f)(y) \\
&= \sum_{|\alpha+\beta|\leqslant k-1}\sum\ + \sum_{\substack{|\alpha|\leqslant k-1\\ k\leqslant|\alpha+\beta|\leqslant m-1}}\sum \frac{1}{\alpha!\,\beta!}(x-y)^\alpha (y-z)^\beta D^{\alpha+\beta}f(z) + R \\
&= \sum_{|\gamma|\leqslant k-1} \frac{1}{\gamma!}(x-z)^\gamma D^\gamma f(z) + \sum_{\substack{|\alpha|\leqslant k-1\\ k\leqslant|\alpha+\beta|\leqslant m-1}}\sum \frac{1}{\alpha!\,\beta!}(x-y)^\alpha (y-z)^\beta D^{\alpha+\beta}f(z) + R.
\end{aligned}$$

In fact,

$$(x-z)^\gamma = \sum_{\alpha+\beta=\gamma}\sum \frac{\gamma!}{\alpha!(\gamma-\alpha)!}(x-y)^\alpha (y-z)^{\gamma-\alpha}.$$

Thus

$$\begin{aligned}
f(x)-P_y^{k-1}f(x) &= \sum_{k\leqslant|\alpha|\leqslant m-1} \frac{1}{\alpha!}(x-z)^\alpha D^\alpha f(z) - \sum_{\substack{|\alpha|\leqslant k-1\\ k\leqslant|\alpha+\beta|\leqslant m-1}}\sum \frac{1}{\alpha!\,\beta!}(x-y)^\alpha (y-z)^\beta D^{\alpha+\beta}f(z) \\
&\quad + R_z^{m-1}f(x) - \sum_{|\alpha|\leqslant k-1} \frac{1}{\alpha!}(x-y)^\alpha R_z^{m-|\alpha|-1}(D^\alpha f)(y).
\end{aligned}$$

Now integrate over $\{z;\ |z|\leqslant 2\}$. By Lemma 4.4

$$\begin{aligned}
&\Bigg| f(x)-P_y^{k-1}f(x) - \sum_{k\leqslant|\alpha|\leqslant m-1} \frac{1}{\omega_d\,\alpha!}\int_{|z|\leqslant 2}(x-z)^\alpha D^\alpha f(z)\,dz \\
&\qquad - \sum_{\substack{|\alpha|\leqslant k-1\\ k\leqslant|\alpha+\beta|\leqslant m-1}}\sum \frac{1}{\omega_d\,\alpha!\,\beta!}(x-y)^\alpha \int_{|z|\leqslant 2}(y-z)^\beta D^{\alpha+\beta}f(z)\,dz\Bigg| \\
&\leqslant A\int_{|\xi|\leqslant 2}|\nabla^m f(\xi)|\,|\xi-x|^{m-d}d\xi + A\sum_{|\alpha|\leqslant k-1}\int_{|\xi|\leqslant 2}|\nabla^m f(\xi)|\,|\xi-y|^{m-|\alpha|-d}d\xi \\
&\leqslant A\int_{|\xi|\leqslant 2}|\nabla^m f(\xi)|\,|\xi-x|^{m-d}d\xi + A\int_{|\xi|\leqslant 2}|\nabla^m f(\xi)|\,|\xi-y|^{m-k+1-d}d\xi.
\end{aligned}$$

Now apply (2.2) and let μ be a unit measure with support in K, such that

$$\left\{\int_{|\xi|\leqslant 2}\left(\int|\xi-y|^{m-k+1-d}d\mu(y)\right)^p d\xi\right\}^{1/p} \leqslant A c_{m-k+1,q}(K,B)^{-1/q},$$

and integrate with respect to y. We find

$$\left|f(x)-\int P_y^{k-1}f(x)\,d\mu(y)-P(x)\right|\leqslant A\int_{|\xi|\leqslant 2}|\nabla^m f(\xi)|\,|\xi-x|^{m-d}d\xi + A\int_{|\xi|\leqslant 2}|\nabla^m f(\xi)|\,d\xi\int|\xi-y|^{m-k+1-d}d\mu(y).$$

Here $P(x)$ is a polynomial of degree $\leqslant m-1$ which satisfies

$$|P(x)|\leqslant A\sum_{k\leqslant|\alpha|\leqslant m-1}\int_{|z|\leqslant 2}|D^\alpha f(z)|\,dz,\quad x\in B.$$

We apply this to f_n where $f_n\to f$, and choose μ so that $D^\alpha f_n(y)$ tends to zero uniformly on the support of μ, $0\leqslant|\alpha|\leqslant k-1$. Clearly the corresponding polynomials P_n converge to a polynomial P, such that

$$\begin{aligned}|f(x)-P(x)|&\leqslant A\int_{|\xi|\leqslant 2}|\nabla^m f(\xi)|\,|\xi-x|^{m-d}d\xi+A\int_{|\xi|\leqslant 2}|\nabla^m f(\xi)|\,d\xi\int|\xi-y|^{m-k+1-d}d\mu(y)\\&\leqslant A\int_{|\xi|\leqslant 2}|\nabla^m f(\xi)|\,|\xi-x|^{m-d}d\xi+A\left\{\int_{|\xi|\leqslant 2}|\nabla^m f(\xi)|^q d\xi\right\}^{1/q}\cdot c_{m-k+1,q}(K,B)^{-1/q}.\end{aligned}$$

Integrating over x we finally obtain the desired inequality

$$\int_{|x|\leqslant 1}|f(x)-P(x)|^q dx\leqslant A\int_{|x|\leqslant 2}|\nabla^m f(x)|^q\,dx\cdot c_{m-k+1,q}(K,B)^{-1}.$$

Remark 1. By using the inequality ([15; (3)])

$$\int_{|\xi|\leqslant 2}|\nabla^m f(\xi)|\,|\xi-x|^{m-d}d\xi\leqslant AM(\nabla^m f)(x)^{1-\theta}\cdot\left\{\int|\nabla^m f|^q d\xi\right\}^{\theta/q},$$

$\theta=mq/d$, and the Hardy–Littlewood maximal inequality, we obtain the "Sobolev exponent":

$$\left\{\int_{|x|\leqslant 1}|f(x)-P(x)|^{q^*}dx\right\}^{1/q^*}\leqslant A\left\{\int_{|x|\leqslant 1}|\nabla^m f(x)|^q dx\right\}^{1/q}\cdot c_{m-k+1,q}(K,B)^{-1/q},\qquad \frac{1}{q^*}=\frac{1}{q}-\frac{m}{d}. \tag{4.2}$$

Remark 2. Let K belong to the unit ball B, let $f\in W_m^q(B)$, and let $D^\alpha f|_K=0$, $0\leqslant|\alpha|\leqslant k-1$, for some $k<m$. It follows from the closed graph theorem that there exists a constant

C_K, independent of f, and a polynomial P of degree $\leqslant m-1$ such that $D^\alpha P|_K=0$ for $0\leqslant |\alpha|\leqslant k-1$, and $\int_B |f-P|^q dx \leqslant C_K \int_B |\nabla^m f|^q dx$. In particular $\int_B |f|^q dx \leqslant C_K \int_B |\nabla^m f|^q dx$ if there are no such polynomials. See Maz'ja [22; § 5], where these questions are studied in detail.

However, it is difficult to say anything about C_K that is useful for our purposes, because C_K depends on algebraic properties of K. For example (as pointed out by V. G. Maz'ja in conversation), if $m=q=d=2$, there is a constant $C_K<\infty$ such that $\int_B |f|^2 dx \leqslant C_K \int_B |\nabla^2 f|^2 dx$, if K consists of three non collinear points, but not if K consists of two points. The idea of Theorem 4.2 comes from this observation. It is by allowing a polynomial which does not vanish on K, that we can obtain an estimate in terms of capacity.

Remark 3. Estimates related to Theorem 4.1 were proved independently by N. G. Meyers [28a]. D. R. Adams has pointed out to the author that Meyers' arguments can be modified so that they give alternative proofs of Theorems 4.1 and 4.2.

The following simple lemma will be needed in the proof of Theorem 3.5.

LEMMA 4.5. *Let $f\in W_1^q(B_\delta)$. Then for any ball $B_{\delta/2}\subset B_\delta$,*

$$\left|\left\{\frac{1}{|B_\delta|}\int_{B_\delta}|f|^q dx\right\}^{1/q}-\left\{\frac{1}{|B_{\delta/2}|}\int_{B_{\delta/2}}|f|^q dx\right\}^{1/q}\right|\leqslant A\delta\left\{\frac{1}{|B_\delta|}\int_{B_\delta}|\nabla f|^q\right\}^{1/q}.$$

Proof. As before we can assume that $f\in W_1^q(\mathbf{R}^d)$, and that $\delta=1$. Then the left hand side in the inequality equals

$$\left|\left\{\frac{1}{|B_1|\,|B_{1/2}|}\int_{B_{1/2}}\left(\int_{B_1}|f(x)|^q dx\right)dy\right\}^{1/q}-\left\{\frac{1}{|B_1|\,|B_{1/2}|}\int_{B_1}\left(\int_{B_{1/2}}|f(y)|^q dy\right)dx\right\}^{1/q}\right|$$
$$\leqslant\left\{\frac{1}{|B_1|\,|B_{1/2}|}\int_{B_{1/2}}\int_{B_1}|f(x)-f(y)|^q dx\,dy\right\}^{1/q}.$$

Now, as in Lemma 4.4, for any $y\in B_{1/2}$,

$$\int_{B_1}|f(x)-f(y)|^q dx\leqslant\int_{|x-y|\leqslant 2}|f(x)-f(y)|^q dx\leqslant A\int_{|\xi|\leqslant 2}|\xi|^{1-d}|\nabla f(y-\xi)|^q d\xi,$$

whence

$$\int_{B_{1/2}}\int_{B_1}|f(x)-f(y)|^q dx\,dy\leqslant A\int_{|\xi|\leqslant 5/2}|\nabla f(\xi)|^q d\xi,$$

which proves the lemma.

Repeating the argument we obtain the following lemma, which is a special case of a theorem of Sobolev. (See e.g. [24; 1.1.11, p. 26].)

LEMMA 4.6. *Let $f \in W^q_m(B_\delta)$, and let $B_{\delta/2} \subset B_\delta$. Then*

$$\int_{B_\delta} |f|^q\,dx \leq A\left(\int_{B_{\delta/2}} |f|^q\,dx + \delta^{mq}\int_{B_\delta} |\nabla^m f|^q\,dx\right).$$

Finally, the following interpolation inequality from [15] will have an important part to play.

LEMMA 4.7. *Let $f = R_m * g$, $g \in L^q(\mathbf{R}^d)$, and let $1 \leq j \leq m$. Then*, (j, q)-q.e.

$$|\nabla^{m-j} f| \leq A R_j * |g| \leq A(Mg)^{1-j/m}(R_m * |g|)^{j/m}.$$

(Mg denotes the Hardy–Littlewood maximal function.)

5. Proofs of the main results

In this section we will use the following abbreviated notation for the average over a set B:

$$B(f) = \left\{\frac{1}{|B|}\int_B |f|^q\,dx\right\}^{1/q}.$$

Here B will be either a ball $B_n(x) = B_{2^{-n}}(x)$ or a cube Q, and q is fixed, $1 < q < \infty$.

Let K and f satisfy the assumptions of Theorem 3.4. Then, by Theorem 4.1 and the definition of thinness (3.1), for every $x \in K$ and $B_n = B_n(x)$

$$\sum_{n=1}^{\infty}\left\{\frac{2^{-nm}B_n(\nabla^m f)}{B_n(f)}\right\}^p = \infty. \tag{5.1}$$

Thus Theorem 3.4 is contained in the following result, which was essentially proved in [17; Theorem 3.1].

THEOREM 5.1. *Let K be compact, and let f be a function in $W^q_m(\mathbf{R}^d)$, $1 < q < \infty$, which satisfies* (5.1) *for all $x \in K$. Then, given any $\varepsilon > 0$, and any sufficiently small neighborhood V of K, there is a function ω in $C_0^\infty(V)$ such that $0 \leq \omega \leq 1$, $\omega = 1$ on a neighborhood of K, and $\|f\omega\|_{m,q} \leq A \int_V |\nabla^m f|^q\,dx < \varepsilon$.*

Remark 1. It is clearly enough to assume that $f \in W^q_m(K^c)$, and extend f and its derivatives to $\mathbf{R}^d$ by setting them equal to zero on K. The conclusion is that the extended function belongs to $W^q_m(\mathbf{R}^d)$ and to $\overset{0}{W}{}^q_m(K^c)$.

Remark 2. It is easily seen that if $f \in W_m^q(\mathbf{R}^d)$ and satisfies (5.1) (m, q)-q.e. on K, then $D^\alpha f|_K = 0$, $0 \leq |\alpha| \leq m-1$. In fact, by a theorem of N. G. Meyers [28; Theorem 2.1]

$$\sum_{n=1}^{\infty} \{2^{-nm} B_n(\nabla^m f)\}^p < \infty$$

for $B_n = B_n(x)$, (m, q)-q.e. x in K. Thus, by (5.1), $\liminf_{n\to\infty} B_n(f) = 0$ for (m, q)-q.e. x in K. But we know that

$$\lim_{n\to\infty} \frac{1}{|B_n|} \int_{B_n} f(y)\, dy = f(x)$$

exists for (m, q)-q.e. x (Section 2 above), and thus $f(x) = 0$, (m, q)-q.e. on K. That $D^\alpha f|_K = 0$ for $1 \leq |\alpha| \leq m-1$ follows in the same way using (4.1) and (5.1).

To prove Theorem 5.1 we decompose $\mathbf{R}^d$ in the standard way into meshes $\mathcal{M}_n$ of closed dyadic cubes. The cubes in $\mathcal{M}_n$ will usually be denoted $\{Q_{ni}\}_{i=1}^{\infty}$ with some arbitrary enumeration. The side of Q_{ni} has length $l(Q_{ni}) = 2^{-n}$. By rQ, $r > 0$, we denote the cube concentric to Q with side $l(rQ) = rl(Q)$. We denote $7Q_{ni} = \tilde{Q}_{ni}$.

For any Q_{ni} we set

$$\lambda_{ni} = \min\left\{1, \left(\frac{2^{-nm}\tilde{Q}_{ni}(\nabla^m f)}{Q_{ni}(f)}\right)^p\right\}. \tag{5.2}$$

Now fix a point $x_0 \in K$ and let $\{Q_{n0}\}_{n=0}^{\infty}$, $Q_{n0} \in \mathcal{M}_n$, be a sequence of nested cubes that contain x_0.

Let $\bar{\lambda}_n = \max\{\lambda_{ni};\, Q_{ni} \subset 3Q_{n0}\}$, and $\underline{\lambda}_n = \min\{\lambda_{ni};\, Q_{ni} \subset 3Q_{n0}\}$. Since $\tilde{Q}_{n0} \supset B_n(x_0)$, it follows from (5.1) that

$$\sum_{n=0}^{\infty} \underline{\lambda}_n = \infty.$$

Let $Q_{n+1,j} \subset 3Q_{n+1,0}$, and $Q_{ni} \subset 3Q_{n0}$. Then a moment of thought shows that $\tilde{Q}_{n+1,j} \subset \tilde{Q}_{ni}$. (In [17] this was stated erroneously with 5 instead of 7.) Moreover by Lemma 4.6,

$$Q_{ni}(f) \leq AQ_{n+1,j}(f) + A2^{-nm}\tilde{Q}_{n+1,j}(\nabla^m f).$$

Thus by (5.2)

$$Q_{ni}(f) \leq A(\lambda_{n+1,j}^{-1/p} + 1)\, 2^{-(n+1)m}\tilde{Q}_{n+1,j}(\nabla^m f) \leq A\lambda_{n+1,j}^{-1/p}\, 2^{-(n+1)m}\tilde{Q}_{ni}(\nabla^m f);$$

$$\lambda_{ni}^{-1/p}\, 2^{-nm}\tilde{Q}_{ni}(\nabla^m f) \leq A\lambda_{n+1,j}^{-1/p}\, 2^{-(n+1)m}\tilde{Q}_{ni}(\nabla^m f);$$

$$\lambda_{ni} \geq M^{-1}\lambda_{n+1,j},$$

and thus

$$\underline{\lambda}_n \geq M^{-1}\bar{\lambda}_{n+1}, \quad \text{for some} \quad M = M(d, m, p) > 0.$$

Once these observations have been made, the following lemma is the same as Lemma 3.2 in [17].

LEMMA 5.2. *Under the above assumptions there exists a C^∞ function ω with the following properties:*

(a) $\omega(x)=0$ *outside an arbitrarily prescribed neighborhood V of K;*

(b) $\omega(x)=1$ *on a neighborhood of K;*

(c) $0\leqslant\omega\leqslant 1$;

(d) *Every x is contained in some Q_{ni} such that*

$$|\nabla^k\omega(x)|\leqslant A\lambda_{ni}2^{nk},\quad k=1,2,\ldots,A=A(k);$$

(e) *There is a constant A, only depending on d, such that for all x*

$$\sum_{n=0}^{\infty}\sum_i \lambda_{ni}\chi(x;\tilde{Q}_{ni})\leqslant A,$$

where the sum is extended over only those indices i for which $\nabla\omega$ is not identically zero on Q_{ni}. ($\chi(\cdot, E)$ denotes the characteristic function of E.)

Since the deduction of Theorem 5.1 from Lemma 5.2 is short we repeat it here.

Let V be a neighborhood of K, and choose ω in Lemma 5.2 so that $\tilde{Q}_{ni}\subset V$ whenever Q_{ni} intersects $\operatorname{supp}\omega$.

It is enough to estimate $\int_{\mathbf{R}^d}|\nabla^k\omega|^q|\nabla^{m-k}f|^q dx$, $0\leqslant k\leqslant m$. By Lemma 5.2 (d) we can decompose $\mathbf{R}^d$ in a disjoint union of sets Q'_{ni}, $(n,i)\in I$, such that $Q'_{ni}\subset Q_{ni}$, and $|\nabla^k\omega(x)|\leqslant A\lambda_{ni}2^{nk}$ on Q'_{ni}.

We observe that by (5.2) and (4.1)

$$\int_{Q_{ni}}|\nabla^{m-k}f|^q dx\leqslant A\lambda_{ni}^{1-q}2^{-nkq}\int_{\tilde{Q}_{ni}}|\nabla^m f|^q dx.$$

Thus, for $1\leqslant k\leqslant m$,

$$\int_{\mathbf{R}^d}|\nabla^k\omega|^q|\nabla^{m-k}f|^q dx\leqslant A\sum_{(n,i)\in I}{}'\lambda_{ni}^q 2^{nkq}\int_{Q'_{ni}}|\nabla^{m-k}f|^q dx\leqslant A\sum_{(n,i)\in I}{}'\lambda_{ni}\int_{\tilde{Q}_{ni}}|\nabla^m f|^q dx.$$

Here $\sum'$ indicates that we sum over only those Q_{ni} where $\nabla\omega$ is not identically zero. Thus, the sum is finite, although each point in K belongs to infinitely many $\tilde{Q}_{ni}$ with $(n,i)\in I$.

It follows from (e) in Lemma 5.2 that

$$\sum_{(n,i)\in I}{}'\lambda_{ni}\int_{\tilde{Q}_{ni}}|\nabla^m f|^q dx=\int_V\sum_{(n,i)\in I}{}'\lambda_{ni}\chi(x,\tilde{Q}_{ni})|\nabla^m f|^q dx\leqslant A\int_V|\nabla^m f|^q dx.$$

For $k=0$ we have

$$\int_{\mathbf{R}^d} |\omega \nabla^m f|^q dx \leqslant \int_V |\nabla^m f|^q dx.$$

We know that $\nabla^{m-1} f(x)=0$ a.e. on K. Since $\nabla^{m-1} f$ is absolutely continuous on almost all lines, and distribution derivatives are a.e. equal to ordinary derivatives, it follows from Fubini's theorem that $\nabla^m f(x)=0$ a.e. on K. Thus $\int_V |\nabla^m f|^q dx < \varepsilon$ if V is small enough. This proves Theorem 5.1.

If $C_{1,q}(K)=0$, (5.1) is no longer true in general. Its place is taken by the following result.

THEOREM 5.3. *Let $f \in W_m^q(\mathbf{R}^d)$, and suppose that $f|_K = \ldots = \nabla^{k-1} f|_K = 0$ for some k, $1 \leqslant k \leqslant m-1$. Suppose that K is $(m-k+1, q)$-thick at a point x. Then, for $B_n = B_n(x)$, either*

$$\sum_{n=1}^{\infty} \left\{ \frac{2^{-nm} B_n(\nabla^m f)}{B_n(f)} \right\}^p = \infty,$$

i.e. (5.1) *holds, or*

$$\limsup_{n\to\infty} \frac{B_n(f)}{2^{-nk} B_n(R_{m-k} * |\nabla^m f|)} \leqslant A < \infty. \tag{5.3}$$

Here A is independent of f, K, and x.

Proof. By Corollary 4.3

$$B_n(f) \leqslant A 2^{-nk} B_n(\nabla^k f) + A 2^{-nm} c_{m-k+1,q}(K, B_n)^{-1/q} B_n(\nabla^m f).$$

Thus, either

$$B_n(f) \leqslant A 2^{-nm} c_{m-k+1,q}(K, B_n)^{-1/q} B_n(\nabla^m f) \tag{5.4}$$

for all sufficiently large n, or else

$$B_n(f) \leqslant A 2^{-nk} B_n(\nabla^k f) \tag{5.5}$$

for a sequence of integers n tending to $+\infty$. In the former case (5.1) follows from the definition of $(m-k+1, q)$-thickness.

Now assume that (5.1) does not hold, so that $\lim_{n\to\infty} 2^{-nm} B_n(\nabla^m f)/B_n(f)=0$. We observe that this implies that

$$\lim_{n\to\infty} 2^{-nm} B_n(\nabla^m f)/B_{n+r}(f) = 0$$

for any fixed positive integer r.

In fact, by Lemma 4.6, $B_n(f) \leqslant AB_{n+r}(f) + A2^{-nm}B_n(\nabla^m f)$, and thus

$$B_{n+r}(f)/2^{-nm}B_n(\nabla^m f) \geqslant A^{-1}B_n(f)/2^{-nm}B_n(\nabla^m f) - A^{-1},$$

which tends to $+\infty$, as $n \to \infty$.

Let n_0 be so large that

$$2^{-nm}B_{n-r}(\nabla^m f) \leqslant \varepsilon B_n(f) \tag{5.6}$$

for all $n \geqslant n_0$. Here $\varepsilon > 0$ and $r > 0$ will be chosen later.

Let $n_1 > n_0$ be an integer such that (5.5) holds for $n = n_1$. By assumption n_1 can be chosen arbitrarily large, so the theorem follows if we can prove that

$$B_\nu(f) \leqslant A2^{-\nu k}B_\nu(R_{m-k} * |\nabla^m f|) \tag{5.7}$$

for $n_0 \leqslant \nu \leqslant n_1$.

We shall prove (5.7) by induction on ν. We know (5.7) is true for $\nu = n_1$. We assume it has been proven for $n_0 < n \leqslant \nu \leqslant n_1$, and we want to prove (5.7) for $\nu = n - 1$.

We write $|\nabla^m f| = g$. By (4.1), (5.7) implies

$$B_\nu(\nabla^{k-j} f) \leqslant A2^{-\nu j}B_\nu(R_{m-k} * g). \tag{5.8}$$

We claim that (5.7) implies

$$B_\nu(R_{m-k} * g) \leqslant \tfrac{3}{2}B_{\nu-1}(R_{m-k} * g), \tag{5.9}$$

if r and ε have been chosen suitably.

Assuming (5.9) for the moment, we find by (5.8) and Lemma 4.5,

$$\begin{aligned} B_{n-1}(\nabla^{k-1} f) &= B_{n_1}(\nabla^{k-1} f) + \sum_{\nu=n-1}^{n_1-1} B_\nu(\nabla^{k-1} f) - B_{\nu+1}(\nabla^{k-1} f) \\ &\leqslant A2^{-n_1}B_{n_1}(R_{m-k} * g) + A\sum_{\nu=n-1}^{n_1-1} 2^{-\nu}B_\nu(\nabla^k f) \\ &\leqslant AB_{n-1}(R_{m-k} * g)\sum_{\nu=n-1}^{n_1} 2^{-\nu}\left(\frac{3}{2}\right)^{\nu-n+1} = A2^{-(n-1)}B_{n-1}(R_{m-k} * g). \end{aligned}$$

But then, in the same way,

$$\begin{aligned} B_{n-1}(\nabla^{k-2} f) &= B_{n_1}(\nabla^{k-2} f) + \sum_{\nu=n-1}^{n_1-1} B_\nu(\nabla^{k-2} f) - B_{\nu+1}(\nabla^{k-2} f) \\ &\leqslant B_{n_1}(\nabla^{k-2} f) + A\sum_{\nu=n-1}^{n_1-1} 2^{-\nu}B_\nu(\nabla^{k-1} f) \\ &\leqslant A\sum_{\nu=n-1}^{n_1} 2^{-2\nu}B_\nu(R_{m-k} * g) \\ &\leqslant A\sum_{\nu=n-1}^{n_1} 2^{-2\nu}\left(\frac{3}{2}\right)^{\nu-n+1} B_{n-1}(R_{m-k} * g) = A2^{-2(n-1)}B_{n-1}(R_{m-k} * g), \end{aligned}$$

and so on. Thus

$$B_{n-1}(f) \leqslant A2^{-k(n-1)} B_{n-1}(R_{m-k} \ast g)$$

for an absolute constant A, which proves (5.7).

To prove (5.9) we observe that for any positive integer r

$$\begin{aligned} B_\nu(R_{m-k} \ast g) &= \left\{ \frac{1}{|B_\nu|} \int_{B_\nu} (R_{m-k} \ast g)^q \, dx \right\}^{1/q} \\ &\leqslant \left\{ \frac{1}{|B_\nu|} \int_{B_\nu} \left(\int_{B_{\nu-r}} R_{m-k}(x-y)\, g(y)\, dy \right)^q dx \right\}^{1/q} \\ &\quad + \left\{ \frac{1}{|B_\nu|} \int_{B_\nu} \left(\int_{B^c_{\nu-r}} R_{m-k}(x-y)\, g(y)\, dy \right)^q dx \right\}^{1/q}. \end{aligned}$$

Using the inequalities $\|h \ast g\|_q \leqslant \|h\|_1 \|g\|_q$, and $\int_{|x| \leqslant 2^{-n}} R_{m-k}(x)\, dx \leqslant A2^{-n(m-k)}$ on the first term, we find

$$B_\nu(R_{m-k} \ast g) \leqslant A2^{-\nu(m-k)} B_{\nu-r}(g) + \left\{ \frac{1}{|B_\nu|} \int_{B_\nu} \left(\int_{B^c_{\nu-r}} R_{m-k}(x-y)\, g(y)\, dy \right)^q dx \right\}^{1/q}.$$

Similarly

$$B_{\nu-1}(R_{m-k} \ast g) \geqslant \left\{ \frac{1}{|B_{\nu-1}|} \int_{B_{\nu-1}} \left(\int_{B^c_{\nu-r}} R_{m-k}(x-y)\, g(y)\, dy \right)^q dx \right)^{1/q} - A2^{-\nu(m-k)} B_{\nu-r}(g).$$

But by (5.6)

$$B_{\nu-r}(g) \leqslant \varepsilon 2^{\nu m} B_\nu(f).$$

Thus, if $B_\nu(f) \leqslant A2^{-\nu\kappa} B_\nu(R_{m-k} \ast g)$, we have $2^{-\nu(m-k)} B_{\nu-r}(g) \leqslant \varepsilon B_\nu(R_{m-k} \ast g)$. Furthermore, by choosing r large enough, we can make sure that

$$1 - \varepsilon \leqslant R_{m-k}(y)/R_{m-k}(y-x) \leqslant 1 + \varepsilon$$

for $x \in B_n$, $y \in B^c_{n-r}$, independently of $n \geqslant 0$. It follows easily that

$$B_\nu(R_{m-k} \ast g) \leqslant (1 + A\varepsilon)\, B_{\nu-1}(R_{m-k} \ast g) \leqslant \tfrac{3}{2} B_{\nu-1}(R_{m-k} \ast g),$$

if $\varepsilon < 1/2A$. The theorem follows.

The following theorem is contained in Theorem 4.1 in [17], but we repeat the main steps of the proof.

THEOREM 5.4. *Let K be compact, and suppose that $C_{m-k,q}(K) = 0$ for some k, $0 \leqslant k \leqslant m-1$. Let $f \in W^q_m(\mathbf{R}^d)$, and suppose either that $k = 0$, or that $k \geqslant 1$ and f satisfies (5.3) everywhere on K.*

Then, for any neighborhood V of K, and any $\varepsilon>0$, there is an $\omega \in C_0^\infty(V)$ such that $0 \leqslant \omega \leqslant 1$, $\omega=1$ on a neighborhood of K, and $\|f\omega\|_{m,q}<\varepsilon$.

Remark. As in the first remark following Theorem 5.1 it is enough to assume that $f \in W_m^q(K^c)$. See [18 a], Lemma 2.

Proof. Let (if $k \geqslant 1$) $E_N=\{x \in K; B_n(f) \leqslant N2^{-nk}B_n(R_{m-k} * |\nabla^m f|), \forall n \geqslant N\}$. Then E_N is closed, since $B_n(f)$ and $B_n(R_{m-k} * |\nabla^m f|)$ are continuous in x. By assumption $K=\bigcup_{N=M}^\infty E_N$ for any M.

First let M be so large that V is a 2^{-M}-neighborhood of K, and assume either that $K=E_M$, or that $k=0$.

Let $\{Q\}$ be a Whitney covering of K^c, and let $\tilde{Q}=AQ$ be suitably enlarged cubes so that $\tilde{Q}$ contains a ball centered on K which contains Q. Denote the center of Q by x_Q.

Let $\delta>0$ and let G_δ' be the union of all Q such that $|Q|^{-1}\int_{\tilde{Q}}(R_{m-k} * |\nabla^m f|)^q dx>\delta^{-q}$. By Lemma 4.2 (a) in [17] we have $C_{m-k,q}(G_\delta')<A\delta^q\|f\|_{m,q}^q$. Since $C_{m-k,q}(K)=0$ we can therefore choose a neighborhood G_δ of K such that $G_\delta' \subset G_\delta$, and

$$C_{m-k,q}(G_\delta)<A\delta^q\|f\|_{m,q}^q.$$

We can also assume that $\bar{G}_\delta \setminus K$ is a union of Whitney cubes. Then, for all Whitney cubes Q not in G_δ we have

$$\frac{1}{|Q|}\int_{\tilde{Q}}(R_{m-k} * |\nabla^m f|)^q \leqslant \delta^{-q}. \tag{5.10}$$

As in [17], p. 72, we let ν be a positive measure with support in G_δ, such that $V^\nu=R_{m-k} * (R_{m-k} * \nu)^{p-1} \geqslant 1$ on G_δ, and $\|R_{m-k} * \nu\|_p^p \leqslant AC_{m-k,q}(G_\delta)$.

We then let $\Phi(r)$, $r \geqslant 0$, be an increasing C^∞ function such that $\Phi(r)=0$ for $r \leqslant \frac{1}{2}$, and $\Phi(r)=1$ for $r \geqslant 3/4$, and we set $\omega=\Phi \circ V^\nu$, so that $\omega=1$ on G_δ, and $\omega \in C_0^\infty$.

This can be done in such a way that $\operatorname{supp}\omega \subset V$, and so that (by Lemma 4.5 in [17]) there is a function $h \geqslant 0$ such that

$$\int_{\mathbf{R}^d} h^q dx \leqslant AC_{m-k,q}(G_\delta) \leqslant A\delta^q\|f\|_{m,q}^q; \tag{5.11}$$

(If $mq=d$ the integral should be taken over a fixed ball containing K.)

h has the Harnack property, i.e. for any Whitney cube Q

$$A^{-1}h(y) \leqslant h(x) \leqslant Ah(y), \quad x \in Q, \quad y \in Q; \tag{5.12}$$

$$|\nabla^{m-k}\omega(x)| \leqslant Ah(x); \tag{5.13}$$

$$|\nabla^{m-j}\omega(x)| \leqslant Ah(x)^{(m-j)/(m-k)}, \quad k \leqslant j \leqslant m; \tag{5.14}$$

$$|\nabla^{m-j}\omega(x)| \leqslant Ah(x)l(Q)^{j-k}, \quad j \leqslant k, \quad x \in Q. \tag{5.15}$$

Here Lemmas 4.4, and 4.6 in [17] have been used.

We can now estimate $\int_{\mathbf{R}^d}|\nabla^{m-j}\omega|^q|\nabla^j f|^q dx$, $0 \leqslant j \leqslant m$. Let Q be a Whitney cube where $\nabla\omega$ does not vanish identically.

First consider the case $0 \leqslant j \leqslant k$. We know by (5.3), (4.1), and (5.10) that

$$\int_Q |\nabla^j f|^q dx \leqslant Al(Q)^{(k-j)q}\int_{\tilde{Q}}(R_{m-k} * |\nabla^m f|)^q dx \leqslant Al(Q)^{(k-j)q}|Q|\delta^{-q}.$$

Then, by the Harnack property (5.12), (5.15), and (5.11)

$$\int_Q |\nabla^{m-j}\omega|^q|\nabla^j f|^q dx \leqslant Ah(x_Q)^q l(Q)^{(j-k)q} l(Q)^{(k-j)q}|Q|\delta^{-q} \leqslant A\delta^{-q}\int_Q h^q dx.$$

Thus

$$\int_{\mathbf{R}^d}|\nabla^{m-j}\omega|^q|\nabla^j f|^q dx \leqslant A\delta^{-q}\int_{\mathbf{R}^d} h^q dx \leqslant A\|f\|^q_{m,q}.$$

Now let $k+1 \leqslant j \leqslant m-1$. Write $|\nabla^m f| = g$, and $(m-j)/(m-k) = \theta$. By Lemma 4.7 we have $|\nabla^j f| \leqslant A(Mg)^{1-\theta}(R_{m-k} * g)^\theta$. By (5.14), (5.12), (5.10), and Hölder's inequality

$$\begin{aligned}\int_Q |\nabla^{m-j}\omega|^q|\nabla^j f|^q dx &\leqslant A\int_Q h^{\theta q}(Mg)^{(1-\theta)q}(R_{m-k} * g)^{\theta q} \\ &\leqslant Ah(x_Q)^{\theta q}|Q|^\theta\left\{\int_Q (Mg)^q dx\right\}^{1-\theta}\left\{\frac{1}{|Q|}\int_Q (R_{m-k} * g)^q dx\right\}^\theta \\ &\leqslant A\left\{\int_Q h^q dx\right\}^\theta\left\{\int_Q (Mg)^q dx\right\}^{1-\theta}\delta^{-q\theta}.\end{aligned}$$

By Hölder's inequality for sums

$$\begin{aligned}\int_{\mathbf{R}^d}|\nabla^{m-j}\omega|^q|\nabla^j f|^q dx &\leqslant A\left\{\int_{\mathbf{R}^d} h^q dx\right\}^\theta\left\{\int_{\mathbf{R}^d}(Mg)^q dx\right\}^{1-\theta}\delta^{-q\theta} \\ &\leqslant A\delta^{q\theta}\|f\|^{q\theta}_{m,q}\|f\|^{(1-\theta)q}_{m,q}\delta^{-q\theta} \leqslant A\|f\|^q_{m,q}.\end{aligned}$$

Finally for $j = m$, $\int_{\mathbf{R}^d}|\omega|^q|\nabla^m f|^q dx \leqslant \int_{\mathbf{R}^d}|\nabla^m f|^q dx$. Thus, $\|\omega f\|_{m,q}$ is uniformly bounded, independently of δ. On the other hand $\omega(x)f(x) \to 0$, as $\delta \to 0$, for $x \in K^c$. Thus there is a sequence $\{\omega_n\}$, $n \to \infty$, such that $\omega_n f$ tends to zero weakly in W^q_m. By the Banach–Saks theorem there is a sequence of averages tending strongly to zero in $W^q_m(\mathbf{R}^d)$, which finishes the first part of the proof.

Now let K be as in the theorem. We have $K = \bigcup_{N=M}^\infty E_N$, where E_N are compact. Let $0 < \varepsilon < \frac{1}{2}\|f\|_{m,q}$, and let ω_M be the function just constructed, so that supp $\omega_M \subset V$, $0 \leqslant \omega_M \leqslant 1$,

$\omega_M = 1$ on a neighborhood G_M of E_M, and $\|f\omega_M\|_{m,q} < \varepsilon/2$. Set $f_M = f(1-\omega_M)$, so that $\|f - f_M\|_{m,q} < \varepsilon/2$. Then for each $B_n(x)$, $x \in E_{M+1}$, $n \geqslant M+1$, we have

$$B_n(f_M) \leqslant B_n(f) \leqslant (M+1)2^{-nk} B_n(R_{m-k} * |\nabla^m f|),$$

and thus, by (4.1), for $1 \leqslant j \leqslant k$,

$$\begin{aligned} B_n(\nabla^j f_M) &\leqslant A 2^{nj} B_n(f_M) + A 2^{-n(k-j)} B_n(\nabla^k f_M) \\ &\leqslant A(M+1) 2^{-n(k-j)} (B_n(R_{m-k} * |\nabla^m f|) + B_n(R_{m-k} * |\nabla^m f_M|)). \end{aligned}$$

By a slight modification of the first part of the proof, using the fact that $\|f_M\|_{m,q} \leqslant 2\|f\|_{m,q}$, we now construct ω_{M+1} so that $0 \leqslant \omega_{M+1} \leqslant 1$, $\omega_{M+1} = 1$ on a neighborhood G_{M+1} of E_{M+1}, and $\|f_M \omega_{M+1}\|_{m,q} < \varepsilon 2^{-2}$. Set $f_{M+1} = f_M(1-\omega_{M+1})$, so that $\|f_M - f_{M+1}\|_{m,q} < \varepsilon 2^{-2}$, and $f_{M+1} = 0$ on $G_M \cup G_{M+1}$. Proceeding inductively we construct f_N so that $\|f - f_N\|_{m,q} < \varepsilon \sum_1^{N-M} 2^{-n}$, and $f_N = 0$ on $\bigcup_M^N G_n$, $G_n \supset E_n$. But K is compact, so $f_N = 0$ on a neighborhood of K if N is large enough. This proves the theorem, if we set $\omega = 1 - (1-\omega_M) \dots (1-\omega_N)$.

Theorem 3.5 is now an immediate consequence of Theorem 5.3 and the following extension of Theorem 5.1.

THEOREM 5.5. *Let K be compact, let $f \in W_m^q(\mathbf{R}^d)$, and let $0 \leqslant k \leqslant m-1$. Suppose that f satisfies (5.1) everywhere on K, except on a set $E \subset K$ such that $C_{m-k}(E) = 0$, and suppose (if $k \geqslant 1$) that f satisfies (5.3) everywhere on E. Then, for any neighborhood V of K, and any $\varepsilon > 0$, there is an $\omega \in C_0^\infty(V)$ such that $0 \leqslant \omega \leqslant 1$, $\omega = 1$ on a neighborhood of K, and $\|f\omega\|_{m,q} < \varepsilon$.*

Proof. Let V be given so that $\int_V |\nabla^m f|^q dx < \varepsilon^q$, and choose M so large that V is a 2^{-M}-neighborhood of K. Let $B_n = B_n(x)$ and set

$$F_N = \left\{ x \in K;\ \sum_{n=M}^{N} \left(\frac{2^{-nm} B_n(\nabla^m f)}{B_n(f)} \right)^p \geqslant 1 \right\}.$$

We can assume that $B_n(f) \neq 0$ for all n and all $x \in K$, since otherwise $f \equiv 0$ on $B_n(x)$, and K can be replaced by $K \setminus B_n(x)$. Thus the above sum is a continuous function of x, and F_N is compact. Moreover, $K \setminus E \subset \bigcup_M^\infty F_N$.

By the proof of Lemma 5.2 there is a function $\omega_N \in C_0^\infty(V)$ such that $\omega_N = 1$ on a 2^{-N}-neighborhood of F_N, $0 \leqslant \omega_N \leqslant 1$, and $\|\omega_N f\|_{m,q}^q \leqslant A \int_V |\nabla^m f|^q dx$. Moreover, it is easily seen that ω_N can be constructed so that $\omega_N = 1$ on a 2^{-n}-neighborhood of F_n for each $n = M$, $M+1, \dots, N$. ($\{\omega_N\}$ is essentially an increasing sequence.) Thus there is an increasing sequence of open sets G_N, $G_N \supset F_N$, such that $\omega_N = 1$ on G_N, and $\|\omega_N f\|_{m,q}^q \leqslant A \int_V |\nabla^m f|^q dx$.

Set $H = \bigcup_M^\infty G_N$. By weak compactness there is a weakly convergent subsequence of $\{(1-\omega_N)f\}$ that converges to a function f_0, which vanishes on H, and satisfies $\|f - f_0\|_{m,q} <$

$A\varepsilon$. By the Banach–Saks theorem a sequence of averages of $\{(1-\omega_N)f\}$ converges strongly to f_0. We can also assume that these averages are of the form $(1-\tilde{\omega}_n)f$, where $0\leqslant\tilde{\omega}_n\leqslant 1$, and $\tilde{\omega}_n=1$ on some G_{i_n}, with $\lim_{n\to\infty} i_n=\infty$.

Now let $E_N=\{x\in K\setminus H;\ B_n(f)\leqslant N2^{-nk}B_n(R_{m-k}*|\nabla^m f|), \forall n\geqslant N\}$. Then $K\setminus H=\bigcup_M^\infty E_N$ for any M, $K\setminus H$ is compact, and $C_{m-k,q}(K\setminus H)=0$ by assumption. As in the proof of Theorem 5.4 we have for $x\in E_N$ and all $B_n(x)$, $n\geqslant N$, that

$$B_n(f_0)\leqslant B_n(f)\leqslant N2^{-nk}B_n(R_{m-k}*|\nabla^m f|),$$

and

$$B_n(\nabla^j f_0)\leqslant AN2^{-n(k-j)}(B_n(R_{m-k}*|\nabla^m f|)+B_n(R_{m-k}*|\nabla^m f_0|))$$

for $1\leqslant j\leqslant k$.

Thus, as in Theorem 5.4, we can construct a function $\omega'\in C_0^\infty(V)$, $0\leqslant\omega'\leqslant 1$, $\omega'=1$ on a neighborhood H' of $K\setminus H$ so that $\|f_0\omega'\|_{m,q}<\varepsilon$. It follows that $f'=f_0(1-\omega')=0$ on a neighborhood $H\cup H'$ of K, and $\|f-f'\|_{m,q}\leqslant\|f-f_0\|_{m,q}+\|f_0\omega'\|_{m,q}<A\varepsilon$, so $f\in \overset{0}{W}{}^q_m(K^c)$. Moreover, $\|f'-f(1-\tilde{\omega}_n)(1-\omega')\|_{m,q}\leqslant A\max_{|\alpha|\leqslant m}\|D^\alpha(1-\omega')\|_\infty\|f_0-f(1-\tilde{\omega}_n)\|_{m,q}<\varepsilon$ if n is large enough. Since $K\setminus H'$ is compact and $K\setminus H'\subset\bigcup_{n=1}^\infty G_{i_n}$ it follows that $f(1-\tilde{\omega}_n)\times(1-\omega')=0$ on a neighborhood of K if n is large enough. Set $\omega=1-(1-\tilde{\omega}_n)(1-\omega')$. Then ω satisfies all the requirements of the theorem.

Proof of Theorem 3.1. To prove Theorem 3.1 we only need to modify the preceding argument slightly.

Let K be an arbitrary compact set, and let f be the given function. Let V, M, F_N, ω_N, $\tilde{\omega}_n$, f_0, G_{i_n}, and H be as in the preceding proof. Denote H by H_0.

By the Kellogg property and Theorem 4.1 we have $C_{1,q}(H_0\setminus K)=0$. Now let $E_{1N}=\{x\in K\setminus H_0;\ B_n(f)\leqslant A2^{-n(m-1)}B_n(R_1*|\nabla^m f|),\ \forall n\geqslant N\}$. Then E_{1N} is compact, and $C_{1,q}(E_{1N})=0$. Moreover, by the Kellogg property and Theorem 5.3 (applied for $k=m-1$), we have $C_{2,q}(K\setminus(H_0\cup(\bigcup_{N=M}^\infty E_{1N})))=0$.

As in the proof of Theorem 5.4 there are functions $\omega_{1N}\in C_0^\infty(V)$ such that $0\leqslant\omega_{1N}\leqslant 1$, $\omega_{1N}=1$ on a neighborhood H_{1N} of E_{1N}, and functions $f_{1N}=f_0(1-\omega_{1M})\dots(1-\omega_{1N})$, so that $f_{1N}=0$ on $\bigcup_M^N H_{1n}$, and $\|f_0-f_{1N}\|_{m,q}<\varepsilon/2$. Set $H_1=\bigcup_M^\infty H_{1n}$. As in the proof of Theorem 5.5 a sequence of averages of the functions f_{1N} converges strongly to a function f_1, such that $f_1=0$ on $H_0\cup H_1$ and $\|f_0-f_1\|_{m,q}<\varepsilon/2$. We write $f_1=\lim_{n\to\infty}f_0(1-\tilde{\omega}_{1n})$, and assume that $\tilde{\omega}_{1n}=1$ on H_{1i_n}, $\lim_{n\to\infty}i_n=\infty$.

Now let $E_{2N}=\{x\in K\setminus(H_0\cup H_1);\ B_n(f)\leqslant A2^{-n(m-2)}B_n(R_2*|\nabla^m f|),\ \forall n\geqslant N\}$. Then $C_{2,q}(E_{2N})=0$ for each N, E_{2N} is compact, and $C_{3,q}(K\setminus(H_0\cup H_1\cup(\bigcup_{N=M}^\infty E_{2N})))=0$. We construct functions ω_{2N}, $\omega_{2N}=1$ on neighborhoods H_{2N} of E_{2N}, $f_{2N}=f_1(1-\omega_{2M})\dots(1-\omega_{2N})$, and a function $f_2=\lim f_1(1-\tilde{\omega}_{2n})$, so that $f_2=0$ on $H_2=\bigcup H_{2N}$ as before, and $\|f_2-f_1\|_{m,q}<\varepsilon 2^{-2}$.

We proceed inductively until we have constructed $f_{m-1}=\lim_{n\to\infty} f_{m-2}(1-\tilde{\omega}_{m-1,n})$ so that $\|f_{m-1}-f_{m-2}\|_{m,q}<\varepsilon 2^{-m+1}$, $f_{m-1}=0$ on $\bigcup_{k=0}^{m-1} H_k$, $H_k=\bigcup_{N=M}^{\infty} H_{kN}$, H_{kN} is a neighborhood of

$$E_{kN}=\left\{x\in K\setminus\left(\bigcup_0^{k-1} H_j\right);\ B_n(f)\leqslant A2^{-n(m-k)}B_n(R_k * |\nabla^m f|),\ \forall n\geqslant N\right\}.$$

Then $C_{m,q}(K\setminus(\bigcup_0^{m-1} H_k))=0$, and $K\setminus(\bigcup_0^{m-1} H_k)$ is compact. Thus there is an $\omega_m\in C_0^\infty(V)$ such that $\omega_m=1$ on a neighborhood H_m of $K\setminus(\bigcup_0^{m-1} H_k)$, and $f_m=(1-\omega_m)f_{m-1}$ satisfies $\|f_m-f_{m-1}\|_{m,q}<\varepsilon 2^{-m}$. It follows that $\|f_m-f\|_{m,q}<\varepsilon$, and that $f_m=0$ on $\bigcup_0^m H_k$, which is a neighborhood of K, and thus $f\in \overset{0}{W}{}_m^q(K^c)$.

Finally, if $\lim_{n\to\infty}\|f_{m-1}-(1-\tilde{\omega}_{m-1,n})f_{m-2}\|_{m,q}=0$, it follows as before that

$$\begin{aligned}\|f_m-(1-\omega_m)(1-\tilde{\omega}_{m-1,n})f_{m-2}\|_{m,q}&=\|(1-\omega_m)(f_{m-1}-(1-\tilde{\omega}_{m-1,n})f_{m-2})\|_{m,q}\\&\leqslant A\ \max_{|\alpha|\leqslant m}\|D^\alpha(1-\omega_m)\|_\infty\|f_{m-1}-(1-\tilde{\omega}_{m-1,n})f_{m-2}\|_{m,q}\to 0,\quad \text{as } n\to\infty.\end{aligned}$$

Moreover, by compactness $(1-\omega_m)(1-\tilde{\omega}_{m-1,n})=0$ on a neighborhood of $K\setminus(\bigcup_0^{m-2}H_k)$ if n is large enough. Continuing backwards step by step we find that there are functions $\tilde{\omega}_{k,n_k}$, $k=0,1,\dots,m-1$, so that $\omega=1-(1-\tilde{\omega}_{0,n_0})(1-\tilde{\omega}_{1,n_1})\dots(1-\tilde{\omega}_{m-1,n_{m-1}})(1-\omega_m)=1$ on a neighborhood of K, and $\|f\omega\|_{m,q}<\varepsilon$. This proves Theorem 3.1, except for the last statement. But this follows directly from Theorem 3.6, which we now prove.

Proof of Theorem 3.6. See also [36], where a somewhat simpler proof is given for the case $E=\emptyset$.

Let $f\in W_m^q(\mathbf{R}^d)$, and suppose that $C_{m,q}(E)=0$. As always we can assume that f has compact support, and that E is bounded.

We have $|f(x)|\leqslant A\int|x-y|^{m-d}|\nabla^m f(y)|\,dy=AR_m*g(x)$, $g=|\nabla^m f|$. We can assume that $\|g\|_q=1$.

Let $G_n'=\{x;\ R_m*g(x)>n\}$. Then G_n' is open and $C_{m,q}(G_n')<n^{-q}$. We can choose open G_n so that $E\cup G_n'\subset G_n$ and $C_{m,q}(G_n)<n^{-q}$.

By a now well-known result of V. G. Maz'ja [23; Theorem 3.3] (see also Adams–Polking [4]) there is a function ω such that $\omega(x)=1$ on G_n, $0\leqslant\omega\leqslant 1$, and $\|\omega\|_{m,q}\leqslant An^{-1}$.

As has been pointed out by D. R. Adams [2; Proposition 1], Maz'ja's theorem can now be proved rather easily by means of the interpolation inequality in Lemma 4.7.

Let in fact $\varphi\geqslant 0$ be any function such that $R_m*\varphi\geqslant 1$ on G_n and $\|\varphi\|_q\leqslant n^{-1}$, and let again $\Phi(r)$, $r\geqslant 0$, be an increasing C^∞ function with $\Phi(r)=0$ for $r\leqslant\frac{1}{2}$ and $\Phi(r)=1$ for $r\geqslant 1$.

Then set $\omega=\Phi\circ(R_m*\varphi)$, and estimate the derivatives of ω by means of the chain rule and Lemma 4.7, keeping in mind that $\nabla^j\omega=0$ on G_n. A computation which we omit (see [2], [17; Lemma 4.5] or [36; p. 131]) gives

$$|\nabla^j \omega| \leqslant A(M\varphi)^{j/m}, \quad 1 < j < m,$$

and

$$|\nabla^m \omega| \leqslant AM\varphi + A|\nabla^m(R_m * \varphi)|.$$

It follows from the last inequality that $\int |\nabla^m \omega|^q dx \leqslant A \int \varphi^q dx \leqslant An^{-1}$, i.e. Maz'ja's theorem.

We now want to estimate $\|f\omega\|_{m,q}$. It is easily seen that $\int |\omega|^q |\nabla^m f|^q dx$ is arbitrarily small, so it is enough to estimate $\int |\nabla^j \omega|^q |\nabla^{m-j} f|^q dx$ for $1 \leqslant j \leqslant m$. Applying Lemma 4.7 again we find

$$|\nabla^{m-j} f| \leqslant AR_j * g \leqslant A(Mg)^{1-j/m}(R_m * g)^{j/m}.$$

Moreover, $R_m * g \leqslant n$ off G_n, so we obtain

$$|\nabla^j \omega|\,|\nabla^{m-j} f| \leqslant A(M\varphi)^{j/m}(Mg)^{1-j/m} n^{j/m}, \quad 1 \leqslant j \leqslant m-1.$$

By Hölder's inequality, and the Maximal theorem,

$$\|(M\varphi)^{j/m}(Mg)^{1-j/m}\|_q \leqslant \|\varphi\|_q^{j/m} \|g\|_q^{1-j/m} \leqslant n^{-j/m} \|g\|_q^{1-j/m} = n^{-j/m}.$$

For $j = m$ we have $|\nabla^m \omega|\,|f| \leqslant |\nabla^m \omega| n$, so for all j, $1 \leqslant j \leqslant m$, we find

$$\int |\nabla^j \omega|^q |\nabla^{m-j} f|^q dx \leqslant A, \quad \text{independently of } n.$$

But as $n \to \infty$, $\omega(x) \to 0$ a.e., so the theorem follows by weak compactness, and the Banach–Saks theorem.

6. Applications

We give a few applications and equivalent formulations of Theorem 1.1, in addition to the application to the Dirichlet problem given in Theorem 1.2. Proofs are found in [17]. See also the survey [18].

Let $W^p_{-m}(\mathbf{R}^d)$ denote the dual to $W^q_m(\mathbf{R}^d)$.

THEOREM 6.1. *Let T be a distribution in $W^p_{-m}(\mathbf{R}^d)$. Suppose $1 < p < 2 + 1/(d-1)$. Then for any $\varepsilon > 0$ there are measures μ_α, α multiindices with $0 \leqslant |\alpha| \leqslant m-1$, such that* $\operatorname{supp} \mu_\alpha \subset \operatorname{supp} T$, *and* $\|T - \sum D^\alpha \mu_\alpha\|_{-m,p} < \varepsilon$.

THEOREM 6.2. *Let $G \subset \mathbf{R}^d$ be bounded and open. Suppose $1 < p < 2 + 1/(d-1)$. Let $\Delta^m u = 0$ in G and suppose $u \in L^p(G)$. Then for any $\varepsilon > 0$ there are measures μ_α, $0 \leqslant |\alpha| \leqslant m-1$,* $\operatorname{supp} \mu_\alpha \subset \partial G$, *such that*

$$\left\| u(x) - \sum_{0 \leqslant |\alpha| \leqslant m-1} \int D^\alpha E_m(x-y)\, d\mu_\alpha(y) \right\|_{L^p(G)} < \varepsilon.$$

Here E_m is the fundamental solution to Δ^m, i.e. $\Delta^m E_m = E_m \Delta^m = \delta$.

We say that a compact set $K \subset \mathbf{R}^d$ is *(m, q)-stable* if every f in $W_m^q(\mathbf{R}^d)$ such that f vanishes on K^c actually belongs to $\overset{0}{W}{}_m^q(K^\circ)$. See Babuška [5], Schulze–Wildenhain [31], and [18].

It is well known that K is $(2m, q)$-stable if and only if every f in $L^p(K)$ which satisfies $\Delta^m f = 0$ in K° can be approximated in $L^p(K)$ by functions which satisfy the equation on neighborhoods of K. See [5], [17] and [29].

THEOREM 6.3. *K is (m, q)-stable for $q > 2 - 1/d$ if and only if every f in $W_m^q(\mathbf{R}^d)$ such that $f|_{K^c} = 0$ satisfies $D^\alpha f|_{\partial K} = 0$ for all α, $0 \leqslant |\alpha| \leqslant m-1$.*

THEOREM 6.4. *K is (m, q)-stable for $q > 2 - 1/d$ if K^c is (k, q)-thick (k, q)-*q.e. *on ∂K for $k = 1, 2, \ldots, m$.*

THEOREM 6.5. *K is (m, q)-stable for $q > 2 - 1/d$ if there is an $\eta > 0$ such that $C_{k,q}(U \setminus K) \geqslant \eta C_{k,q}(U \setminus K^\circ)$ for $k = 1, 2, \ldots, m$ and all open sets U.*

Remark. T. Bagby has recently given a necessary and sufficient condition for L^p-approximability by solutions of elliptic equations. See [6a].

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