SIMPLICES OF MAXIMAL VOLUME IN HYPERBOLIC n-SPACE

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1. Introduction

Consider hyperbolic n-space H^n represented as the Poincaré disk model

$$H^n \sim D^n = \{\mathbf{x} \in \mathbf{R}^n \mid \|\mathbf{x}\| < 1\}$$

with the Riemannian metric

$$ds^2 = \frac{4}{(1-r^2)^2} \sum_{i=1}^{n} (dx_i)^2 \quad \text{where} \quad r^2 = \sum_{i=1}^{n} x_i^2.$$

The geodesics in H^n are the circles orthogonal to the "sphere at infinity"

$$\partial H^n = \{\mathbf{x} \in \mathbf{R}^n \mid \|\mathbf{x}\| = 1\} = S^{n-1}.$$

An *n-simplex* in H^n with vertices $\mathbf{v}_0, ..., \mathbf{v}_n \in H^n \cup \partial H^n$ is the closed subset of H^n bounded by the $n+1$ spheres which contain all the vertices except one and which are orthogonal to S^{n-1}. A simplex is called *ideal* if all the vertices are on the sphere at infinity. It is easy to see that the volume of a hyperbolic n-simplex is finite also if some of the vertices are on the sphere at infinity. A simplex is called *regular* if any permutation of its vertices can be induced by an isometry of H^n. This makes sense also for ideal simplices since any isometry of H^n can be extended continuously to $H^n \cup \partial H^n$. There is, up to isometry, only one ideal regular n-simplex in H^n.

The main result of the present paper is the following theorem which was conjectured by Thurston ([6], section 6.1).

THEOREM 1. *In hyperbolic n-space, for $n \geqslant 2$, a simplex is of maximal volume if and only if it is ideal and regular.*

Since any hyperbolic n-simplex is contained in an ideal one it suffices, when proving Theorem 1, to consider ideal simplices. We shall use the notation $\tau[n]$ for an arbitrary ideal n-simplex in H^n, while $\tau_0[n]$ always denotes a regular $\tau[n]$.

For $n=2$ any $\tau[2]$ is regular and has area equal to π, so in this case the theorem is trivially true.

For $n=3$ one has Lobatcheffsky's volume formula, [1]. For the form of it given below see e.g. Milnor [3]. In any $\tau[3]$ opposite dihedral angles are equal, and if α, β, γ are the three dihedral angles at one vertex then $\alpha+\beta+\gamma=\pi$, and the volume is given by

$$V(\tau[3]) = \Lambda(\alpha)+\Lambda(\beta)+\Lambda(\gamma)$$

where

$$\Lambda(\sigma) = -\int_0^\sigma \log (2 \sin u)\, du.$$

As shown in [3] this formula implies Theorem 1 for $n=3$.

The motivation for the present study is a very elegant proof, due to Gromov, of Mostow's rigidity theorem, [5], for oriented closed hyperbolic 3-manifolds. The theorem states that for $n \geqslant 3$ two oriented, closed, hyperbolic n-manifolds which are homotopy equivalent are automatically isometric. It is clear that Gromov's proof (as presented in Thurston's lecture notes [6], section 6.3) works also for $n>3$ once one knows that ideal simplices of maximal volume in H^n are automatically regular.

For the convenience of the reader we give here a very brief outline of Gromov's argument.

Let $f: M \to N$ be a homotopy equivalence between closed, oriented hyperbolic n-manifolds with $n \geqslant 3$. To prove that M and N are isometric one notes that they are orbit spaces $\Gamma \diagdown H^n$ and $\Theta \diagdown H^n$, respectively, for discrete isometry groups Γ and Θ on hyperbolic n-space H^n. Also, f induces an isomorphism $f_*: \Gamma \to \Theta$ and it lifts to a map $\tilde{f}: H^n \to H^n$ which is equivariant with respect to $f_*: \Gamma \to \Theta$. The first step now consists in showing that $\tilde{f}$ "induces" a continuous map $f^\infty: S^{n-1} \to S^{n-1}$ on the sphere at infinity; f^∞ is also equivariant with respect to f_*. In the second step one utilizes Gromov's norm to prove that f^∞ has the following property:

(1.1) Whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n \in S^{n-1}$ span an ideal hyperbolic simplex of maximal volume then so do $f^\infty(\mathbf{v}_0), f^\infty(\mathbf{v}_1), \ldots, f^\infty(\mathbf{v}_n)$.

At this point Theorem 1 enters. It is used simply to translate (1.1) into

(1.2) Whenever $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n \in S^{n-1}$ span a regular, ideal, hyperbolic simplex, then so do $f^\infty(\mathbf{v}_0), f^\infty(\mathbf{v}_1), \ldots, f^\infty(\mathbf{v}_n)$.

The fourth step then consists in proving that any continuous map $f^\infty: S^{n-1} \to S^{n-1}$ satisfying (1.2) is the "restriction" to S^{n-1} of a unique isometry g of H^n (when $n \geqslant 3$). Since this g is still equivariant with respect to $f_*: \Gamma \to \Theta$ it induces the desired isometry $M \to N$.

The proof of Theorem 1 avoids explicit computation of the volumes $V(\tau_0[n])$. Nevertheless the methods involved can be used to give an asymptotic estimate of $V(\tau_0[n])$ for $n\to\infty$. We found that

$$\lim_{n\to\infty}\frac{V(\tau_0[n])}{V(\sigma_0[n])}=\sqrt{e}$$

where $\sigma_0[n]$ is a regular euclidean n-simplex with vertices on the unit sphere. This asymptotic formula has been known to Milnor for some time [4], but since his proof is less direct than ours, we find it worthwhile to present our proof here (cf. section 4).

2. Recollections about hyperbolic n-space

Besides the Poincaré disk model of H^n we shall use two other models, namely the projective model and the half space model. The *projective model* can be obtained from the Poincaré disk model by use of the map

$$p:\mathbf{x}\to\frac{2}{1+\|\mathbf{x}\|^2}\mathbf{x},\quad \|\mathbf{x}\|<1.$$

Note that $p(H^n)=D^n$ and that p can be extended continuously to $H^n\cup\partial H^n$ by putting $p(\mathbf{x})=\mathbf{x}$ for all $\mathbf{x}\in S^{n-1}$. The induced metric on $p(H^n)$ is

$$ds^2=(1-r^2)^{-1}\sum_i(dx_i)^2+(1-r^2)^{-2}\sum_{i,j}x_ix_j\,dx_i\,dx_j,$$

and the associated volume form is

$$dV=(1-r^2)^{-(n+1)/2}dx_1\dots dx_n.$$

The advantage of the projective model is that geodesics become straight lines in the euclidean geometry on D^n. Hence, if $\tau[n]$ is an ideal hyperbolic n-simplex with vertices $\mathbf{v}_0,\dots,\mathbf{v}_n$ on S^{n-1} then $p(\tau[n])$ is simply the euclidean n-simplex with the same vertices. Therefore, the volume of $\tau[n]$ is given by the formula

$$V(\tau[n])=\int_{p(\tau[n])}(1-r^2)^{-(n+1)/2}\,d\mathbf{r}. \tag{2.1}$$

Let $\mathbf{e}_1,\dots,\mathbf{e}_n$ be the standard basis in $\mathbf{R}^n$. The *half space model* of H^n can be obtained from the Poincaré disk model by use of the map

$$h:\mathbf{x}\mapsto\frac{1}{\|\mathbf{x}-\mathbf{e}_n\|^2}(2x_1,2x_2,\dots,2x_{n-1},1-\|\mathbf{x}\|^2),\ \|\mathbf{x}\|<1.$$

Note that $h(H^n)$ is the half space $\{\mathbf{x}\in\mathbf{R}^n \mid x_n>0\}$. Moreover, h can be extended to the sphere at infinity by using the same formula except for $\mathbf{x}=\mathbf{e}_n$ where one puts $h(\mathbf{e}_n)=\infty$. Then $h(\partial H^n)=\mathbf{R}^{n-1}\cup\{\infty\}$ with $\mathbf{R}^{n-1}=\{\mathbf{x}\in\mathbf{R}^n \mid x_n=0\}$. The induced metric on $h(H^n)$ is

$$ds^2 = x_n^{-2}\sum_i (dx_i)^2,$$

and the associated volume form is

$$dV = x_n^{-n}\, dx_1 \dots dx_n.$$

The geodesics in $h(H^n)$ are half circles and half lines orthogonal to $\mathbf{R}^{n-1}$.

Let $\tau[n]$ be an ideal simplex with vertices $\mathbf{v}_0, \dots, \mathbf{v}_n$. It is no loss of generality to assume that $\mathbf{v}_0=\mathbf{e}_n$, and hence $h(\mathbf{v}_0)=\infty$. The isometries of $h(H^n)$ fixing ∞ on the boundary form the group generated by (a) translations parallel to $\mathbf{R}^{n-1}$, (b) rotations leaving the x_n-axis pointwise fixed, and (c) multiplications by positive scalars. Hence, by replacing $\tau[n]$ by an isometric n-simplex one can achieve that $h(\mathbf{v}_0)=\infty$ and $h(\mathbf{v}_i)\in S^{n-2}\subseteq\mathbf{R}^{n-1}$ $(i=1, 2, \dots, n)$. Let $\varepsilon(\tau[n])$ be the euclidean $(n-1)$-simplex in $\mathbf{R}^{n-1}$ spanned by $h(\mathbf{v}_1), \dots, h(\mathbf{v}_n)$. Then $h(\tau[n])-\{\infty\}$ consists of those points of $\varepsilon(\tau[n])\times[0, \infty[$ which are outside the unit disk in $\mathbf{R}^n$. Thus, putting $\varrho=(x_1^2+\dots+x_{n-1}^2)^{\frac{1}{2}}$ and $d\boldsymbol{\rho}=dx_1 \dots dx_{n-1}$, one gets

$$\begin{aligned} V(\tau[n]) &= \int_{\varepsilon(\tau[n])}\left(\int_{(1-\varrho^2)^{1/2}}^{\infty} x^{-n}\,dx\right)d\boldsymbol{\rho} \\ &= \frac{1}{n-1}\int_{\varepsilon(\tau[n])}(1-\varrho^2)^{-(n-1)/2}\,d\boldsymbol{\rho}. \end{aligned} \tag{2.2}$$

Let us finally note, that $\tau[n]$ is regular if and only if $\varepsilon(\tau[n])$ is euclidean regular.

3. Proof of Theorem 1

The proof of Theorem 1 relies on an interplay between the formulas (2.1) and (2.2). The fact that (2.1) expresses $V(\tau[n])$ as an integral over an n-dimensional euclidean simplex while (2.2) expresses $V(\tau[n])$ as an integral over an $(n-1)$-dimensional euclidean simplex makes it possible to compare volumes of ideal simplices in H^{n+1} with volumes of ideal simplices in H^n, and finally to prove the main theorem by induction on n.

We start by giving an estimate for the growth of $V(\tau_0[n])$ which will be used in the proof but which is also of interest in itself. Recall that $\tau_0[n]$ denotes a *regular* ideal n-simplex in H^n.

PROPOSITION 2. *For all $n\geqslant 2$ one has*

$$\frac{n-1}{n^2}\leqslant\frac{V(\tau_0[n+1])}{V(\tau_0[n])}\leqslant\frac{1}{n}. \tag{3.1}$$

Remark. The upper bound was noted by Thurston ([6], section 6.1).

Proof. Let $\sigma_0[n]$ be any regular euclidean n-simplex with vertices on S^{n-1}. We shall prove the following three formulas

$$\int_{\sigma_0[n]} (1-r^2)^{-(n+1)/2}\, d\mathbf{r} = V(\tau_0[n]) \tag{3.2}$$

$$\int_{\sigma_0[n]} (1-r^2)^{-n/2}\, d\mathbf{r} = n V(\tau_0[n+1]) \tag{3.3}$$

$$\int_{\sigma_0[n]} (1-r^2)^{-(n-1)/2}\, d\mathbf{r} = \frac{n-1}{n} V(\tau_0[n]). \tag{3.4}$$

Clearly these three formulas imply that

$$\frac{n-1}{n} V(\tau_0[n]) \leqslant n V(\tau_0[n+1]) \leqslant V(\tau_0[n])$$

which is equivalent to (3.1).

Since all ideal, regular n-simplices in H^n are isometric we can assume that $p(\tau_0[n])$ is euclidean regular. Hence (2.1) implies (3.2). Next (2.2) implies (3.3) because regularity of $\tau_0[n+1]$ assures regularity of the euclidean n-simplex $\varepsilon(\tau_0[n+1])$. It remains to prove (3.4).

We shall apply Gauss' divergence formula

$$\int_{\sigma_0[n]} \operatorname{div} \mathbf{V}(\mathbf{r})\, d\mathbf{r} = \int_{\partial\sigma_0[n]} \mathbf{V}\cdot\mathbf{n}\, dS \tag{3.5}$$

to the vector field

$$\mathbf{V}(\mathbf{r}) = (1-r^2)^{-(n-1)/2}\mathbf{r}, \quad \|\mathbf{r}\| < 1.$$

Here, of course, $\mathbf{n}$ is the outward pointing normal to the boundary $\partial\sigma_0[n]$. An easy computation shows that

$$\operatorname{div} \mathbf{V}(\mathbf{r}) = (1-r^2)^{-(n-1)/2} + (n-1)(1-r^2)^{-(n+1)/2}.$$

For simplicity put

$$\varphi_n(\alpha) = \int_{\sigma_0[n]} (1-r^2)^{-\alpha}\, d\mathbf{r}. \tag{3.6}$$

Then the left hand side of (3.5) becomes

$$\varphi_n\left(\frac{n-1}{2}\right) + (n-1)\,\varphi_n\left(\frac{n+1}{2}\right).$$

To compute the right hand side of (3.5) we note that $\partial\sigma_0[n]$ consists of $(n+1)$ regular $(n-1)$-simplices $\partial_i\sigma_0[n]$, $i=0, 1, \ldots, n$. On $\partial_i\sigma_0[n]$ one has

$$1-r^2=\varrho_n^2-\varrho^2, \quad \mathbf{r}\in\partial_i\sigma_0[n]$$

$$\mathbf{r}\cdot\mathbf{n}=1/n$$

where $\varrho_n=(1-n^{-2})^{\frac{1}{2}}$ is the radius of the circumscribed $(n-2)$-sphere for $\partial_i\sigma_0[n]$, and ϱ denotes the distance from the center of $\partial_i\sigma_0[n]$ to the point $\mathbf{r}\in\partial_i\sigma_0[n]$. Therefore the right hand side of (3.5) becomes

$$\frac{n+1}{n}\int_{\partial_0\sigma_0[n]}(\varrho_n^2-\varrho^2)^{-(n-1)/2}\,dS.$$

Since $\partial_0\sigma_0[n]$ is isometric to $\varrho_n\cdot\sigma_0[n-1]$ this integral transforms into

$$\frac{n+1}{n}\int_{\sigma_0[n-1]}(\varrho_n^2-\varrho_n^2r^2)^{-(n-1)/2}\varrho_n^{n-1}\,d\mathbf{r}=\frac{n+1}{n}\int_{\sigma_0[n-1]}(1-r^2)^{-(n-1)/2}\,d\mathbf{r}=\frac{n+1}{n}\,\varphi_{n-1}\left(\frac{n-1}{2}\right).$$

Thus we have proved

$$\varphi_n\left(\frac{n-1}{2}\right)+(n-1)\,\varphi_n\left(\frac{n+1}{2}\right)=\frac{n+1}{n}\,\varphi_{n-1}\left(\frac{n-1}{2}\right). \tag{3.7}$$

By (3.2) and (3.3) $\varphi_n((n+1)/2)=V(\tau_0[n])$ and $\varphi_{n-1}((n-1)/2)=(n-1)\,V(\tau_0[n])$. Hence

$$\varphi_n\left(\frac{n-1}{2}\right)=\frac{n-1}{n}\,V(\tau_0[n])$$

which proves (3.4).

LEMMA 3. *Let $f\colon]0, 1]\to\mathbf{R}$ be continuous and concave. Let $\mathbf{c}$ be the center of mass of an arbitrary euclidean n-simplex $\sigma[n]$ with vertices on S^{n-1}, and put $c=\|\mathbf{c}\|$. Then*

$$V(\sigma[n])^{-1}\int_{\sigma[n]}f(1-r^2)\,d\mathbf{r}\leqslant V(\sigma_0[n])^{-1}\int_{\sigma_0[n]}f((1-c^2)(1-r^2))\,d\mathbf{r}$$

whenever both of these improper integrals converge. Moreover, if f is strictly concave then equality holds if and only if $\sigma[n]$ is regular.

Proof. Let the left and right hand side of the inequality be A and B respectively. Let $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n$ be the vertices of $\sigma[n]$. We have the standard n-simplex

$$\Delta[n]=\{(t_0, t_1, \ldots, t_n)\,|\,t_i\geqslant 0, \textstyle\sum t_i=1\}\subseteq\mathbf{R}^{n+1}.$$

Under the homeomorphism $(t_0, t_1, \ldots, t_n) \to \sum t_i \mathbf{v}_i$ of $\Delta[n]$ with $\sigma[n]$ the measure $V(\sigma[n]^{-1})\,d\mathbf{r}$ on $\sigma[n]$ transforms into a measure μ on $\Delta[n]$ which is just the "Lebesgue measure" normalized to have $\mu(\Delta[n])=1$. Hence

$$A = \int_{\Delta[n]} f(1 - \|\sum_i t_i \mathbf{v}_i\|^2)\,d\mu.$$

Since μ is invariant under the transformation $t_i \to t_{\pi(i)}$ for any permutation π of $0, 1, \ldots, n$ we also have

$$A = \int_{\Delta[n]} f(1 - \|\sum_i t_{\pi(i)} \mathbf{v}_i\|^2)\,d\mu, \quad \text{for any } \pi.$$

If E denotes the formation of mean values over all such π, then

$$A = E\left(\int_{\Delta[n]} f(1 - \|\sum_i t_{\pi(i)} \mathbf{v}_i\|^2)\,d\mu\right).$$

The concavity of f then implies that

$$A \leqslant \int_{\Delta[n]} f(E(1 - \|\sum_i t_{\pi(i)} \mathbf{v}_i\|^2))\,d\mu. \tag{3.8}$$

The mean value involved here can easily be computed from the following formulas

$$\|\sum_i t_{\pi(i)} \mathbf{v}_i\|^2 = \sum_{i \neq j} t_{\pi(i)} t_{\pi(j)} (\mathbf{v}_i, \mathbf{v}_j) + \sum_i t_i^2$$

$$E(t_{\pi(i)} t_{\pi(j)}) = \frac{1}{n(n+1)} \sum_{k \neq l} t_k t_l = \frac{1}{n(n+1)} (1 - \sum_i t_i^2), \quad i \neq j$$

$$\sum_{i \neq j} (\mathbf{v}_i, \mathbf{v}_j) = \|\sum_i \mathbf{v}_i\|^2 - \sum_i \|\mathbf{v}_i\|^2 = (n+1)^2 c^2 - (n+1).$$

Here, of course, $(\cdot, \cdot)$ is the euclidean inner product. One gets

$$A \leqslant \int_{\Delta[n]} f\left(\frac{n+1}{n}(1-c^2)(1 - \sum t_i^2)\right) d\mu. \tag{3.9}$$

If $\sigma[n]$ is regular then equality holds in (3.9). Therefore, if one applies (3.9) to $\sigma_0[n]$ and to $g(x) = f((1-c^2)x)$ one gets

$$B = \int_{\Delta[n]} f\left(\frac{n+1}{n}(1-c^2)(1 - \sum t_i^2)\right) d\mu. \tag{3.10}$$

Here we have used that the center of mass for $\sigma_0[n]$ is $\mathbf{0}$. This finishes the proof of $A \leqslant B$.

If equality holds in Lemma 3 then we also have equality in (3.8). In case of strict concavity this is possible only when

$$\| \sum t_{\pi(i)} \mathbf{v}_i \|^2 = \| \sum t_i \mathbf{v}_i \|^2$$

for all $(t_0, t_1, \ldots, t_n) \in \Delta[n]$ and all permutations π. Letting $t_0 = t_1 = \frac{1}{2}$, $t_i = 0$ for $i > 1$ it follows that

$$\|\mathbf{v}_1 + \mathbf{v}_2\| = \|\mathbf{v}_i + \mathbf{v}_j\| \quad \text{for all } i \neq j.$$

Since $\|\mathbf{v}_i - \mathbf{v}_j\|^2 = 4 - \|\mathbf{v}_i + \mathbf{v}_j\|^2$ we see that

$$\|\mathbf{v}_1 - \mathbf{v}_2\| = \|\mathbf{v}_i - \mathbf{v}_j\| \quad \text{for all } i \neq j,$$

and that guarantees the euclidean regularity of $\sigma[n]$.

End of proof of Theorem 1. Assume inductively that the theorem holds for some $n \geqslant 3$, and consider an arbitrary $\tau[n+1]$. Put

$$f(t) = t^{-n/2} - K_n t^{-(n+1)/2}, \quad 0 < t \leqslant 1$$

where $K_n = nV(\tau_0[n+1])/V(\tau_0[n])$. An elementary computation shows that f is strictly concave on $]0, 1]$ if and only if $K_n \geqslant n(n+2)/(n+1)(n+3)$. On the other hand Proposition 2 guarantees that $K_n \geqslant (n-1)/n$ which exceeds $n(n+2)/(n+1)(n+3)$ for $n \geqslant 3$. Lemma 3 can, therefore, be applied to f and the euclidean n-simplex $\sigma[n] = \varepsilon(\tau[n+1])$ (cf. section 2). Using also (2.2) (for $n+1$) and (2.1) and letting $\tau[n] = p^{-1}(\sigma[n])$ one gets

$$\begin{aligned} nV(\tau[n+1]) - K_n V(\tau[n]) &\leqslant \int_{\sigma_0[n]} f((1-c^2)(1-r^2))\, d\mathbf{r} \\ &= (1-c^2)^{-n/2} nV(\tau_0[n+1]) - K_n (1-c^2)^{-(n+1)/2} V(\tau_0[n]) \\ &\leqslant (1-c^2)^{-n/2} (nV(\tau_0[n+1]) - K_n V(\tau_0[n])) \\ &= 0. \end{aligned} \tag{3.11}$$

By the inductive hypothesis $V(\tau[n]) \leqslant V(\tau_0[n])$ so (3.11) implies

$$nV(\tau[n+1]) \leqslant K_n V(\tau_0[n]) = nV(\tau_0[n+1]) \tag{3.12}$$

which shows that $V(\tau_0[n+1])$ *is* maximal.

If equality holds in (3.12) then also in (3.11). By Lemma 3 this implies that $\varepsilon(\tau[n+1])$ is euclidean regular. But then $\tau[n+1]$ is hyperbolically regular.

4. An asymptotic formula for $V(\tau_0[n])$

From section 1 we have

$$V(\tau_0[2]) = \pi = 3.14159\ldots$$

$$V(\tau_0[3]) = 3\int_0^{\pi/3} -\log(2\sin\theta)\,d\theta = 1.01494\ldots.$$

We mention, without giving details, that it is possible to compute $V(\tau_0[4])$ using the generalized Gauss formula (cf. Klein [1], p. 205). We found

$$V(\tau_0[4]) = \frac{10\pi}{3}\arcsin\frac{1}{3} - \frac{\pi^2}{3} = 0.26889\ldots.$$

It seems to be very difficult to obtain simple expressions for $V(\tau_0[n])$ when $n \geqslant 5$. However, we have the following asymptotic formula for $V(\tau_0[n])$ (recall that $\sigma_0[n]$ is a regular euclidean n-simplex with vertices on S^{n-1}).

THEOREM 4.

$$\lim_{n\to\infty}\frac{V(\tau_0[n])}{V(\sigma_0[n])} = \sqrt{e}. \tag{4.1}$$

The proof of Theorem 4 relies on an investigation of the functions

$$\varphi_n(\alpha) = \int_{\sigma_0[n]} (1-r^2)^{-\alpha}\,d\mathbf{r}, \quad n = 1, 2, \ldots.$$

When $n \geqslant 2$ $\varphi_n(\alpha)$ is defined for $\alpha \leqslant (n+1)/2$ (in fact $\varphi_n(\alpha) < \infty$ if and only if $\alpha < n$ but we shall not need this). Moreover φ_n is monotonically increasing, and being an integral of a logarithmically convex function φ_n is itself logarithmically convex, i.e. $\alpha \to \log\varphi_n(\alpha)$ is a convex function.

LEMMA 5.

$$\varphi_n(-1)/\varphi_n(0) = \frac{n+1}{n+2}, \quad n \geqslant 1 \tag{4.2}$$

$$\varphi_n\left(\frac{n-1}{2}\right)\Big/\varphi_n\left(\frac{n+1}{2}\right) = \frac{n-1}{n}, \quad n \geqslant 2. \tag{4.3}$$

Proof. Formula (4.3) is an immediate consequence of (3.2) and (3.4). To prove (4.2) consider a regular euclidean simplex $\sigma_0[n]$ with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n$ on the unit sphere.

Regularity implies that the inner product $(\mathbf{v}_i, \mathbf{v}_j)$ equals $-1/n$ for $i \neq j$. Let $d\mu$ be the normalized "Lebesgue measure" on $\Delta[n] = \{(t_0, t_1, \ldots, t_n) \mid t_i \geq 0, \sum t_i = 1\} \subseteq \mathbf{R}^{n+1}$. Arguing as in the proof of Lemma 3 we get

$$\begin{aligned}
\varphi_n(-1)/\varphi_n(0) &= V(\sigma_0[n])^{-1} \int_{\sigma_0[n]} (1 - r^2)\, d\mathbf{r} \\
&= \int_{\Delta[n]} (1 - \|\sum_i t_i \mathbf{v}_i\|^2)\, d\mu \\
&= \int_{\Delta[n]} (1 - \sum_i t_i^2 \|\mathbf{v}_i\|^2 - \sum_{i \neq j} t_i t_j (\mathbf{v}_i, \mathbf{v}_j))\, d\mu \\
&= \int_{\Delta[n]} \left(1 - \sum_i t_i^2 + \frac{1}{n} \sum_{i \neq j} t_i t_j\right) d\mu \\
&= \frac{n+1}{n} \int_{\Delta[n]} (1 - \sum_i t_i^2)\, d\mu.
\end{aligned}$$

Since $\mu(\Delta[n]) = 1$ and since $\int_{\Delta[n]} t_i^2\, d\mu$ is independent of $i = 0, 1, \ldots, n$ we get

$$\int_{\Delta[n]} (1 - \sum_i t_i^2)\, d\mu = 1 - (n+1) \int_{\Delta[n]} t_n^2\, d\mu.$$

The map $(t_0, \ldots, t_n) \to (t_1, \ldots, t_n)$ is an affine isomorphism of $\Delta[n]$ onto $\{\mathbf{t} \in \mathbf{R}^n \mid t_i \geq 0, \sum_{i=1}^n t_i \leq 1\}$ which transforms $d\mu$ into the measure $n!\, dt_1 \ldots dt_n$.

Hence,

$$\begin{aligned}
\int_{\Delta[n]} t_n^2\, d\mu &= n! \int_{t_i \geq 0,\ \sum_{i=1}^n t_i \leq 1} t_n^2\, dt_1 \ldots dt_n \\
&= n! \int_0^1 \left(\int_{t_i \geq 0,\ \sum_{i=1}^{n-1} t_i \leq 1 - t_n} dt_1 \ldots dt_{n-1} \right) t_n^2\, dt_n \\
&= n! \int_0^1 \frac{1}{(n-1)!} (1 - t_n)^{n-1} t_n^2\, dt_n \\
&= \frac{2}{(n+1)(n+2)}.
\end{aligned}$$

And thus

$$\varphi_n(-1)/\varphi_n(0) = \frac{n+1}{n} \left(1 - (n+1) \int_{\Delta[n]} t_n^2\, d\mu\right) = \frac{n+1}{n+2}.$$

Proof of Theorem 4. Let $n \geq 2$. Using the logarithmic convexity of φ_n one gets

$$\left(\frac{\varphi_n(0)}{\varphi_n(-1)}\right)^{(n-1)/2} \leq \left(\frac{\varphi_n((n-1)/2)}{\varphi_n(0)}\right) \leq \left(\frac{\varphi_n((n+1)/2)}{\varphi_n((n-1)/2)}\right)^{(n-1)/2}.$$

Since $\varphi_n(0) = V(\sigma_0[n])$ and $\varphi_n((n-1)/2) = ((n-1)/n)V(\tau_0[n])$ (by (3.4)) we get, by applying Lemma 5, that

$$\left(\frac{n+2}{n+1}\right)^{(n-1)/2} \leqslant \frac{n-1}{n}\,\frac{V(\tau_0[n])}{V(\sigma_0[n])} \leqslant \left(\frac{n}{n-1}\right)^{(n-1)/2}.$$

Since

$$\lim_{n\to\infty}\left(\frac{n+2}{n+1}\right)^{(n-1)/2} = \lim_{n\to\infty}\left(\frac{n}{n-1}\right)^{(n-1)/2} = \sqrt{e}$$

this proves Theorem 4.

Remark. Using the fact that the edgelength of $\sigma_0[n]$ is $(2(1+1/n))^{1/2}$ the volume of $\sigma_0[n]$ can easily be computed to be

$$V(\sigma_0[n]) = \frac{\sqrt{n+1}}{n!}\left(1+\frac{1}{n}\right)^{n/2}$$

which is asymptotically equal to $\sqrt{n}/n!\sqrt{e}$ for $n\to\infty$. Hence, by Theorem 4

$$V(\tau_0[n]) \sim \frac{\sqrt{n}}{n!}\,e.$$

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