QUASICONFORMAL MAPPINGS AND EXTENDABILITY OF FUNCTIONS IN SOBOLEV SPACES

BY

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§ 1. Introduction

Let $\mathcal{D}$ be an open connected domain in $\mathbf{R}^n$, $n \geq 2$. If α is a multi-index, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbf{Z}^n_+$, the length of α, denoted by $|\alpha|$, is the integer $\sum_{j=1}^n \alpha_j$ and $D^\alpha = (\partial/\partial x_1)^{\alpha_1} \ldots (\partial/\partial x_n)^{\alpha_n}$. A locally integrable function f on $\mathcal{D}$ has a weak derivative of order α if there is a locally integrable function (denoted by $D^\alpha f$) such that

$$\int_{\mathcal{D}} f(D^\alpha \varphi)\, dx = (-1)^{|\alpha|} \int_{\mathcal{D}} (D^\alpha f)\, \varphi\, dx$$

for all C^∞ functions φ with compact support in $\mathcal{D}$. For $1 \leq p \leq \infty$, $k \in \mathbf{N}$, $L^p_k(\mathcal{D})$ is the Sobolev space of functions having weak derivatives of all orders α, $|\alpha| \leq k$, and satisfying

$$\|f\|_{L^p_k(\mathcal{D})} = \sum_{0 \leq |\alpha| \leq k} \|D^\alpha f\|_{L^p(\mathcal{D})} < +\infty.$$

An extension operator on $L^p_k(\mathcal{D})$ is a bounded linear operator

$$\Lambda: L^p_k(\mathcal{D}) \to L^p_k(\mathbf{R}^n) \equiv L^p_k$$

such that $\Lambda f|_{\mathcal{D}} = f$ for all $f \in L^p_k(\mathcal{D})$. We say that $\mathcal{D}$ is an extension domain for Sobolev spaces (E.D.S.) if whenever $1 \leq p \leq \infty$, $k \in \mathbf{N}$, there is an extension operator for $L^p_k(\mathcal{D})$.[2] The following theorem is by now well known.

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(2) We do not require Λ to be an extension operator also on $L^p_m(\mathcal{D})$ for $m < k$. In fact, the one which will be constructed does not have that property.

THEOREM A (Calderón–Stein). *Every Lipschitz domain is an* E.D.S.

Theorem A was proved by A. P. Calderón [2] in the case where $1<p<\infty$; E. M. Stein [20] extended Calderón's result to include the endpoints $p=1, \infty$. For earlier results, see [13] and [17].

The purpose of this paper is to discuss to what extent Theorem A may be improved, i.e., what geometric conditions can be imposed on a domain to guarantee that it will be an E.D.S. We will introduce a class of domains, herein called (ε, δ) domains, every member of which is an E.D.S. Lipschitz domains are contained in this class. Our condition is best possible in the following sense: a finitely connected planar domain is an E.D.S. if and only if it is an (ε, δ) domain (Theorem 3). In a related paper [14], D. Jerison and C. Kening show that a large number of potential-theoretic properties, heretofore known to be true for Lipschitz domains, remain valid for (ε, δ) domains. In some sense then, (ε, δ) domains are the worst domains whose classical function-theoretic properties are the same as those of the Euclidean upper half spaces.

Our extension problem for Sobolev spaces is closely related to certain problems in the theory of quasiconformal mappings. Let $E(\mathcal{D})$ denote the space of functions having finite Dirichlet energy, i.e., those functions f having weak derivatives of all orders α, $|\alpha|=1$, and satisfying

$$\|f\|_{E(\mathcal{D})} = \sum_{|\alpha|=1} \|D^\alpha f\|_{L^n(\mathcal{D})} < +\infty.$$

Since constant functions have zero energy, $E(\mathcal{D})$ is actually a Banach space of functions modulo constants. If $\varphi: \mathcal{D}\to\mathcal{D}'$ is K quasiconformal and $f\in E(\mathcal{D}')$, then $f\circ\varphi\in E(\mathcal{D})$ and $\|f\circ\varphi\|_{E(\mathcal{D})}\leqslant K\|f\|_{E(\mathcal{D}')}$. Consequently, φ gives rise to an isomorphism between $E(\mathcal{D})$ and $E(\mathcal{D}')$. A surface S in the Möbius space $\overline{\mathbf{R}^n}$ is said to be a quasisphere (when $n=2$, a quasicircle) if S is the image of the unit sphere $S^{n-1}\subset \mathbf{R}^n$ under some globally quasiconformal homeomorphism of $\overline{\mathbf{R}^n}$ onto $\overline{\mathbf{R}^n}$. Suppose now that S is a quasisphere and $\mathcal{D}_1$ and $\mathcal{D}_2$ are the two components of S^c. Let φ be a (K) quasiconformal homeomorphisms of $\overline{\mathbf{R}^n}$ onto $\overline{\mathbf{R}^n}$ such that $S=\varphi(S^{n-1})$. If $f\in E(\mathcal{D}_1)$, define an extension Λf of f on $\mathcal{D}_2$ by

$$\Lambda f(x) = f\left(\varphi\left(\frac{\varphi^{-1}(x)}{\|\varphi^{-1}(x)\|^2}\right)\right), \quad x\in\mathcal{D}_2.$$

It is easy to check that $\Lambda f\in E(\mathbf{R}^n)\equiv E$ and $\|\Lambda f\|_E\leqslant 2K\|f\|_{E(\mathcal{D}_1)}$. Therefore, every domain bounded by a quasisphere is an extension domain for the Dirichlet energy space (E.D.E.). The following result of [11] indicates that this condition is essentially best possible in dimension 2.

THEOREM B (Gol'dshtein, Latfullin, Vodop'yanov). *If $\mathcal{D}\subset R^2$ is simply connected, then $\mathcal{D}$ is an* E.D.E. *if and only if $\partial\mathcal{D}$, is a quasicircle.*

One might naturally guess that an analogue of Theorem B holds for Sobolev spaces, though clearly one cannot extend in that case by using the above quasiconformal reflection argument. Our Theorem 4 asserts that this guess is correct.

For a rectifiable arc $\gamma\subset\mathbf{R}^n$, let $l(\gamma)$ denote the Euclidean arclength of γ. Let $|x-y|$ denote the Euclidean distance between $x, y\in\mathbf{R}^n$, and let $d(x)=\inf_{y\in\mathcal{D}^c}|x-y|$ for $x\in\mathcal{D}$. We say that $\mathcal{D}$ is an (ε, δ) domain if whenever $x, y\in\mathcal{D}$ and $|x-y|<\delta$, there is a rectifiable arc $\gamma\subset\mathcal{D}$ joining x to y and satisfying

$$l(\gamma)\leqslant\frac{1}{\varepsilon}|x-y| \tag{1.1}$$

and

$$d(z)\geqslant\frac{\varepsilon|x-z|\,|y-z|}{|x-y|}\quad\text{for all } z \text{ on } \gamma. \tag{1.2}$$

Domains satisfying the (ε, ∞) condition have been studied previously in [14] and [17]—the definitions given in those papers appear to be slightly different, but are equivalent. Fred Gehring [8] is presently writing an expository paper on these domains.

Condition (1.1) says that $\mathcal{D}$ is locally connected in some quantitative sense. Condition (1.2) says there is a "tube" T, $\gamma\subset T\subset\mathcal{D}$; the width of T at a point z is on the order of $\min(|x-z|, |y-z|)$. It is clear that every Lipschitz domain is an (ε, δ) domain for some values of $\varepsilon, \delta>0$. The boundary of an (ε, δ) domain can, however, be highly nonrectifiable and, in general, no regularity condition on $\partial\mathcal{D}$, can be inferred from the (ε, δ) property. The classical snowflake domain of conformal mapping theory has the property that every subarc of the boundary is nonrectifiable; it can be checked by hand that the snowflake domain is an (ε, ∞) domain for some $\varepsilon>0$. In fact the situation is even worse than this example shows. Let H^α denote α dimensional Hausdorff measure. If $n-1\leqslant\alpha<n$, one can construct a domain $\mathcal{D}\subset\mathbf{R}^n$ such that $\mathcal{D}$ is an $(\varepsilon(\alpha), \infty)$ domain and $H^\alpha(\mathcal{U}\cap\partial\mathcal{D})>0$ for all open sets $\mathcal{U}$ satisfying $\mathcal{U}\cap\partial\mathcal{D}\neq\emptyset$. Such domains arise naturally in the theory of quasiconformal mappings. See for example [10] or [16], pages 104, 105.

Our first result is the following extension of Theorem A.

THEOREM 1. *Suppose $k\in N$ and $\mathcal{D}$ is an (ε, δ) domain. Then there is a bounded linear extension operator Λ_k,*

$$\Lambda_k\colon L^p_k(\mathcal{D})\to L^p_k,\quad 1\leqslant p\leqslant\infty.$$

Furthermore, the norm of Λ_k on $L^p_k(\mathcal{D})$ depends only on ε, δ, p, k, and the dimension n.

The Calderón–Stein operators of Theorem A do have some advantages over our operators Λ_k. Stein [20] constructs one extension operator which works for all p and k, while our operators are different for different values of k. Calderón's operators [2] are different for different values of k, but have the property that whenever $f \in L_k^p(\mathcal{D})$ has compact support in $\mathcal{D}$, its extension vanishes identically outside of $\mathcal{D}$. Our operators Λ_k do not have this property. On the other hand, a slight modification of our operator Λ_k can be used to extend functions in $E(\mathcal{D})$. Our next result answers a question of Fred Gehring.

THEOREM 2. *Every* (ε, ∞) *domain is an* E.D.E.

A celebrated theorem of Ahlfors [1] gives a simple geometric condition which characterizes quasicircles. If Γ is a Jordan curve in $\overline{\mathbf{R}^2}$ and $x, y \neq \infty$ are two distinct points on Γ, the complement of $\{x, y\}$ on Γ consists of two disjoint arcs. The arc of smaller Euclidean diameter is called the smaller arc—note that if Γ passes through ∞, one of the arcs has infinite Euclidean diameter. The theorem of Ahlfors asserts that Γ is a quasicircle if and only if there is a constant $M < +\infty$, independent of x, y, and such that

$$|x-z| \leqslant M|x-y| \tag{1.3}$$

for all z on the smaller arc between x and y. The above Ahlfors conditions is connected to the (ε, δ) condition via the following result (see [15] or [18]).

THEOREM C. *Suppose* $\Gamma \subset \mathbf{R}^2$ *is a Jordan curve and suppose* $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *are the two simply connected domains complementary to* Γ. *The following conditions are equivalent:*

(i) Γ *is a quasicircle.*

(ii) *Either* $\mathcal{D}_1$ *or* $\mathcal{D}_2$ *is an* (ε, ∞) *domain for some* $\varepsilon > 0$.

(iii) $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *are* (ε, ∞) *domains for some* $\varepsilon > 0$.

Our next two theorems show to what extent the (ε, δ) condition is necessary to the study of our problem and relate Theorem 1 to Theorem B.

THEOREM 3. *If* $\mathcal{D} \subset \mathbf{R}^2$ *is finitely connected, then* $\mathcal{D}$ *is an* E.D.S. *if and only if* $\mathcal{D}$ *is an* (ε, δ) *domain for some values of* $\varepsilon, \delta > 0$.

THEOREM 4. *If* $\mathcal{D} \subset \mathbf{R}^2$ *is bounded and finitely connected, then the following conditions are equivalent:*

(i) $\mathcal{D}$ *is an* E.D.S.

(ii) $\mathcal{D}$ *is an* E.D.E.

(iii) $\mathcal{D}$ *is an* (ε, ∞) *domain for some* $\varepsilon > 0$.

(iv) $\partial \mathcal{D}$ *consists of a finite number of points and quasicircles.*

Theorem C shows that the two equivalent conditions of Theorem 3 are also equivalent to a suitable local variant of condition (iv) in Theorem 4—this will be discussed in a later section. We also note that there is some evidence in the literature to hint at Theorem 4. One of the classical examples of a domain which is not an E.D.S. is $\mathcal{D}=\{(x,y)\in\mathbf{R}^2: y>|x|^\alpha\}$, where $\alpha\in(0,1)$. (The Sobolev embedding theorem fails for $L_1^{2+\varepsilon}(\mathcal{D})$.) $\partial\mathcal{D}$ is also a classical example of a Jordan curve which does not satisfy the Ahlfors conditions (1.3), i.e., is not a quasicircle.

One cannot hope for exact analogues of Theorems 3 or 4 in dimensions $n\geq 3$. There are two general principles which indicate this. First of all, the simple connectivity property is a much weaker condition in higher dimensions than it is in dimension 2; the failure of the Schoenfliess theorem in $\mathbf{R}^3$ is but one example of this phenomenon. For this reason, one might suspect there is a Jordan domain in $\mathbf{R}^3$ which is an E.D.S. but not an (ε,δ) domain for any values of $\varepsilon,\delta>0$. The second reason for doubting the existence of higher dimensional analogues of Theorems 3 and 4 is that $\mathbf{R}^n$ is highly rigid when $n\geq 3$. For this reason there are very few quasiconformal mappings in $\mathbf{R}^n$, $n\geq 3$, when compared to the case of $\mathbf{R}^2$. As an example we cite the fact that every 1 quasiconformal mapping from the unit ball of $\mathbf{R}^3$ to $\mathbf{R}^3$ is the restriction of a Möbius transform. See [5], [7], [9], and [19] for further discussions of this phenomenon. We state without proof the following results.

(1) There is a Jordan E.D.S. in $\mathbf{R}^3$ which is not an (ε,δ) domain for any values of $\varepsilon,\delta>0$.

(2) There is a domain $\mathcal{D}_1\subset\mathbf{R}^3$ such that $\mathcal{D}_1$ and $\mathcal{D}_2=(\mathcal{D}_1^c)^\circ$ are homeomorphic to balls, $\mathcal{D}_1$ and $\mathcal{D}_2$ are (ε,∞) domains, and $\partial\mathcal{D}_1$ is homeomorphic to S^2 but not a quasisphere.

Here E° denotes the interior of a set E. The second example can be obtained by modifying the construction in [6]. We note, however, that if $\mathcal{D}_2=(\mathcal{D}_1^c)^\circ$ and $\partial\mathcal{D}_1$ is a quasisphere, then $\mathcal{D}_1$ and $\mathcal{D}_2$ are both (ε,∞) domains for some $\varepsilon>0$. See [15] or [18].

The method of proof we present for Theorem 1 is as follows. We extend $f\in L_k^p(\mathcal{D})$ to $(\mathcal{D}^c)^\circ$ by selecting appropriate polynomials for all small Whitney cubes in $(\mathcal{D}^c)^\circ$; these polynomials are then pieced together using the standard partition of unity functions. This idea goes back to Whitney's seminal paper [23], and is the same one used to prove the classical extension theorems for Lipschitz spaces. A good reference for this is [20], Chapter VI. For some applications of this method to the theory of Sobolev spaces see e.g. [3] and [4]. To pick the polynomial for a particular Whitney cube $Q\subset(\mathcal{D}^c)^\circ$, we first reflect Q to a certain Whitney cube $Q^*\subset\mathcal{D}$. This reflection technique was introduced in a recent paper of the author [15] and is closely related to quasiconformal reflection. For $f\in L_k^p(\mathcal{D})$ we then select the polynomial $P=P(Q^*)$ of degree $k-1$ which satisfies

$$\int_{Q^*} D^\alpha(f-P)\,dx=0,\quad 0\leq|\alpha|\leq k-1,$$

and continue this polynomial onto Q. It is then shown that the oscillation of $\Lambda_k f$ over Q is well controlled by the oscillation of f near Q^*; this is where our main difficulties lie. Before outlining the contents of the following sections we warn the reader that Theorem 1 will be proved only for the case where radius $(\mathcal{D}) \geq 1$. For the usual reasons, the norms of the operators Λ_k on $L_k^p(\mathcal{D})$ will tend to ∞ if ε, δ, p, k remain fixed and radius $(\mathcal{D})$ tends to zero, unless we renorm our Sobolev spaces so that polynomials of degree $k-1$ have norm zero in $L_k^p(\mathcal{D})$ whenever radius $(\mathcal{D}) < 1$. Since the modifications needed are unpleasant but routine, we do not present them here.

In section 2 we record several lemmas necessary to the proof of Theorems 1–4. The reflection technique $Q \to Q^*$ is also discussed there. For the usual technical reasons we need to know that functions C^∞ on $\mathbf{R}^n$ are dense in $L_k^p(\mathcal{D})$, $1 \leq p < \infty$. To maintain the flow of ideas, this chore is postponed until section 4. In section 3 we construct the operators Λ_k of Theorem 1 and prove (modulo the results of section 4) they are bounded on $L_k^p(\mathcal{D})$. Theorem 2 is proved in section 5. In section 6 we construct a counter-example which proves the converse direction of Theorem B. This counter-example is then used to finish off the proofs of Theorems 3 and 4. We also discuss the connection between the equivalent conditions of Theorem 3 and condition (iv) of Theorem 4.

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§ 2. Some lemmas

In this section we collect several lemmas necessary to the proof of Theorems 1–4. We denote by ∇ the vector $(\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_n)$ and for $m \in \mathbf{Z}^+$ we denote by ∇^m the vector of all possible mth order differentials. Throughout the paper, C denotes various constants depending only on ε, δ, p, k, and the dimension n, and $C(\alpha, \beta, \ldots)$ denotes various constants which also depend on $\alpha, \beta, \ldots$. These constants may differ even in the same string of estimates. Our first lemma follows from the fact that any two norms on a finite dimensional Banach space are equivalent. Since this lemma will be used so often, we will not state it every time it is invoked.

LEMMA 2.1. *Suppose Q is a cube and E, $F \subseteq Q$ are two measurable subsets satisfying $|E|$, $|F| \geq \gamma |Q|$ for some $\gamma > 0$. If P is a polynomial of degree m then*

$$\|P\|_{L^p(E)} \leq C(\gamma, m) \|P\|_{L^p(F)}$$

whenever $1 \leq p \leq \infty$.

If $Q \subset \mathbf{R}^n$ is a cube, let $l(Q)$ denote the edgelength of Q. We say that two cubes touch if a face of one cube is contained in a face of the other. Our next lemma is a variant of the classical Poincaré–Sobolev lemma.

LEMMA 2.2. *Suppose Q_1 and Q_2 are two touching cubes satisfying $\frac{1}{4} \leqslant l(Q_1)/l(Q_2) \leqslant 4$. If $f \in C^\infty$ satisfies*

$$\int_{Q_1 \cup Q_2} D^\alpha f\, dx = 0, \quad 0 \leqslant |\alpha| \leqslant m,$$

then

$$\|f\|_{L^p(Q_1 \cup Q_2)} \leqslant C(m)\, l(Q_1)^{m+1} \|\nabla^{m+1} f\|_{L^p(Q_1 \cup Q_2)}$$

whenever $1 \leqslant p \leqslant \infty$.

For the rest of sections 2–4 we fix an (ε, δ) domain with radius $(\mathcal{D}) \geqslant 1$. We also assume that $\delta \leqslant 1$ since that is the only estimate we will use. Our next lemma says that $\Lambda_k f$ will be defined almost everywhere as soon as it is defined on $(\mathcal{D}^c)^\circ$.

LEMMA 2.3. $|\partial \mathcal{D}| = 0$.

Proof. Fix $x_0 \in \partial \mathcal{D}$ and $y \in \mathcal{D}$. Let Q be a cube centered at x_0 and satisfying $l(Q) \leqslant \frac{1}{2}|x_0 - y|$. Let $x \in \mathcal{D}$ satisfy $|x - x_0| \leqslant \frac{1}{8} l(Q)$ and let γ be the curve guaranteed by (1.1) and (1.2). If $z \in \gamma$ satisfies $|x - z| = \frac{1}{8} l(Q)$ then $d(z) \geqslant (\varepsilon/100)\, l(Q)$. Therefore $|\mathcal{D} \cap Q| \geqslant C\varepsilon^n |Q|$, and by Lebesgue's theorem on differentiation of the indefinite integral, $|\partial \mathcal{D}| = 0$.

Let Ω be an open set in $\mathbf{R}^n$. Then Ω admits a Whitney decomposition, $\Omega = \bigcup_k S_k$. Each S_k is a closed dyadic cube and

$$1 \leqslant \frac{\text{dist}\,(S_k, \partial\Omega)}{l(S_k)} \leqslant 4\sqrt{n}, \quad \text{for all } k, \tag{2.1}$$

$$S_j^\circ \cap S_k^\circ = \varnothing \quad \text{if} \quad j \neq k, \tag{2.2}$$

and

$$\frac{1}{4} \leqslant \frac{l(S_j)}{l(S_k)} \leqslant 4 \quad \text{if} \quad S_j \cap S_k \neq \varnothing. \tag{2.3}$$

See [20], chapter VI for a construction of the Whitney decomposition. Let $\{S_k\} = W_1$ and $\{Q_j\} = W_2$ be the Whitney decompositions of $\mathcal{D}$ and $(\mathcal{D}^c)^\circ$, respectively. Put $W_3 = \{Q_j \in W_2 \colon l(Q_j) \leqslant \varepsilon\delta/16n\}$. For each $Q_3 \in W_3$ we now pick a reflected cube $Q_j^* = S_k \in W_1$.

LEMMA 2.4. *If $Q_j \in W_3$, there is $S_k \in W_1$ satisfying*

$$1 \leqslant \frac{l(S_k)}{l(Q_j)} \leqslant 4$$

and

$$\text{dist}\,(Q_j, S_k) \leqslant C l(Q_j).$$

Proof. By (2.1) there is $x_0 \in \mathcal{D}$ satisfying dist $(x_0, Q_j) \leqslant 5\sqrt{n}\, l(Q_j)$. Let $y_0 \in \mathcal{D}$ satisfy $|x_0 - y_0| = (8n/\varepsilon)\, l(Q_j)$. Then by (1.1) and (1.2) there is $z_0 \in \mathcal{D}$ satisfying $d(z_0) \geqslant (\varepsilon/2)\,|x_0 - y_0| = 4n\, l(Q_j)$ and $|x_0 - z_0| \leqslant (1/\varepsilon)\,|x_0 - y_0| = (8n/\varepsilon^2)\, l(Q_j)$. If $S_0 \in W_1$ contains z_0, then by (2.1), $l(S_0) \geqslant l(Q_j)$. Let $S_k \in W_1$ satisfy $l(S_k) \geqslant l(Q_j)$ and minimize dist (Q_j, S_k). Then

$$\operatorname{dist}(Q_j, S_k) \leqslant 5\sqrt{n}\, l(Q_j) + \frac{8n}{\varepsilon^2}\, l(Q_j)$$

and by (2.3), $1 \leqslant l(S_k)/l(Q_j) \leqslant 4$.

For each $Q_j \in W_3$ fix a cube $S_k \in W_1$ satisfying the conclusions of Lemma 2.4, and call $S_k = Q_j^*$. There may be more than one way to pick Q_j^* for a given $Q_j \in W_3$. The next three lemmas tell us that no matter how we pick the cubes Q_j^*, the correspondence $Q_j \to Q_j^*$ looks roughly like quasiconformal reflection. The proofs of these lemmas are almost immediate.

LEMMA 2.5. *If $Q_j \in W_3$ and $S_1, S_2 \in W_1$ satisfy the conclusions of Lemma 2.4, then*

$$\operatorname{dist}(S_1, S_2) \leqslant C l(Q_j).$$

LEMMA 2.6. *If $S_k \in W_1$ there are at most C cubes $Q_j \in W_3$ such that $Q_j^* = S_k$.*

LEMMA 2.7. *If $Q_j, Q_k \in W_3$ and $Q_j \cap Q_k \neq \varnothing$, then*

$$\operatorname{dist}(Q_j^*, Q_k^*) \leqslant C l(Q_j).$$

The following figure illustrates the correspondence $Q_j \to Q_j^*$. Q_0 and Q_1 are in W_3 and $Q_0 \cap Q_1 \neq \varnothing$. On the other hand, $Q_0^* \cap Q_1^* = \varnothing$. The property we will use repeatedly is not just that dist $(Q_0, Q_1^*) \leqslant C l(Q_0)$, but that $\varrho(Q_0^*, Q_1^*) \leqslant C$, where ϱ is the (hyperbolic) metric on $\mathcal{D}$ induced by $(\sum_{j=1}^n dx_j^2)/(d(z))^2$. See [15] for a discussion of the hyperbolic metric on (ε, ∞) domains.

Suppose $Q_1, Q_2, \ldots, Q_m$ are cubes such that Q_j and Q_{j+1} touch and $\frac{1}{4} \leqslant l(Q_j)/l(Q_{j+1}) \leqslant 4$ for all j, $1 \leqslant j \leqslant m-1$. We say then that $\{Q_1, Q_2, \ldots, Q_m\}$ is a *chain* connecting Q_1 to Q_m, and define the length of that chain to be the integer m.

LEMMA 2.8. *If $Q_j, Q_k \in W_3$ and $Q_j \cap Q_k \neq \varnothing$, there is a chain $F_{j,k} = \{Q_j^* = S_1, S_2, \ldots, S_m = Q_k^*\}$ of cubes in W_1, connecting Q_j^* to Q_k^* and with $m \leqslant C$.*

Proof. Let γ be the arc connecting Q_j^* and Q_k^* satisfying (1.1) and (1.2). Let $F = \{S_\alpha \in W_1 : S_\alpha \cap \gamma \neq \varnothing\}$. By Lemma 2.7, dist $(Q_j^*, Q_k^*) \leqslant C l(Q_j)$. Since $l(Q_j^*), l(Q_k^*) \geqslant \frac{1}{4} l(Q_j)$, condition (1.2) assures that $d(z) \geqslant C l(Q_j)$ for all z on γ. Since $l(\gamma) \leqslant C l(Q_j)$ there are at most C cubes in F. A suitable subset of F now provides the chain $F_{j,k}$ whose existence was claimed.

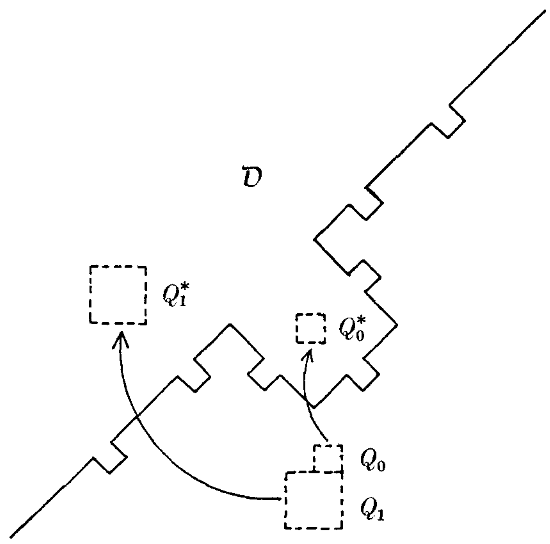

Fig. 1

§ 3. The extension operators

Fix $k \in \mathbf{N}$ and a value of p, $1 \leqslant p \leqslant \infty$. In this section we construct the operator Λ_k and prove (modulo the results of section 4) that it is bounded on $L_k^p(\mathcal{D})$. For each $Q_j \in W_3$ build $\varphi_j \in C^\infty(\mathbf{R}^n)$ such that φ_j is supported on $(17/16)Q_j$, $0 \leqslant \varphi_j \leqslant 1$, $\sum_{Q_j \in W_3} \varphi_j \equiv 1$ on $\bigcup_{Q_j \in W_3} Q_j$, and

$$|D^\alpha \varphi_j| \leqslant C(|\alpha|)\, l(Q_j)^{-|\alpha|} \quad \text{for all} \quad j \text{ and } \alpha.$$

Here λQ denotes the cube concentric with Q, with sides parallel to the axes, and with length $l(\lambda Q) = \lambda l(Q)$. Note that any point lies in the support of at most C functions φ_j. Fix $f \in L_k^p(\mathcal{D})$. For a set $S \subset \mathcal{D}$ of positive measure, let $P(S)$ be the (unique) polynomial in $x_1, x_2, ..., x_n$ of degree $k-1$ satisfying

$$\int_S D^\alpha(f - P(S))\, dx = 0, \quad 0 \leqslant |\alpha| \leqslant k-1.$$

We say that $P(S)$ is the polynomial *fitted* to S. For $Q_j \in W_3$, let $P_j = P(Q_j^*)$ be the polynomial fitted to Q_j^*. The operator Λ_k is defined by setting

$$\Lambda_k f = \sum_{Q_j \in W_3} P_j \varphi_j$$

on $(\mathcal{D}^c)^\circ$. Notice that Λ_k is linear and its definition does not depend on the value of p. By Lemma 2.3, $\Lambda_k f$ is defined almost everywhere on $\mathbf{R}^n$. We first show that $\|\Lambda_k f\|_{L_k^p((\mathcal{D}^c)^\circ)} \leqslant$

$C\|f\|_{L^p_k(D)}$. That of course does not prove Theorem 1, but the rest of the proof consists only of verifying some technical details.

LEMMA 3.1. *Let $F=\{S_1, S_2, ..., S_m\}$ be a chain of cubes in W_1. Then if $0 \leqslant |\beta| \leqslant k$,*

$$\|D^\beta(P(S_1)-P(S_m))\|_{L^p(S_1)} \leqslant C(m)\, l(S_1)^{k-|\beta|}\|\nabla^k f\|_{L^p(\cup F)}.$$

Proof. We first pause to notice that the quantity to be estimated is zero if $|\beta|=k$. By Lemmas 2.1 and 2.2,

$$\begin{aligned}\|D^\beta(P(S_1)-P(S_m))\|_{L^p(S_1)} &\leqslant \sum_{r=1}^{m-1}\|D^\beta(P(S_r)-P(S_{r+1}))\|_{L^p(S_1)}\\ &\leqslant C(m)\sum_{r=1}^{m-1}\|D^\beta(P(S_r)-P(S_{r+1}))\|_{L^p(S_r)}\\ &\leqslant C(m)\sum_{r=1}^{m-1}\{\|D^\beta(P(S_r)-P(S_r\cup S_{r+1}))\|_{L^p(S_r)}\\ &\quad+\|D^\beta(P(S_{r+1})-P(S_r\cup S_{r+1}))\|_{L^p(S_{r+1})}\}\\ &\leqslant C(m)\sum_{r=1}^{m-1}\{\|D^\beta(f-P(S_r))\|_{L^p(S_r)}+\|D^\beta(f-P(S_{r+1}))\|_{L^p(S_{r+1})}\\ &\quad+\|D^\beta(f-P(S_r\cup S_{r+1}))\|_{L^p(S_r\cup S_{r+1})}\}\\ &\leqslant C(m)\sum_{r=1}^{m-1} l(S_r)^{k-|\beta|}\|\nabla^k f\|_{L^p(S_r\cup S_{r+1})}\leqslant C(m)\, l(S_1)^{k-|\beta|}\|\nabla^k f\|_{L^p(\cup F)}.\end{aligned}$$

In the above estimates we have repeatedly made use of property (2.3) of the Whitney decomposition.

For each $Q_j, Q_k \in W_3$ such that $Q_j \cap Q_k \neq \emptyset$, fix a chain $F_{j,k}$ as in Lemma 2.8 and let

$$F(Q_j)=\bigcup_{\substack{Q_k\in W_3\\ Q_j\cap Q_k\neq 0}} F_{j,k}.$$

By Lemma 2.8,

$$\Big\|\sum_{\substack{Q_k\in W_3\\ Q_j\cap Q_k\neq\emptyset}}\chi_{\cup F_{j,k}}\Big\|_{L^\infty}\leqslant C \quad \text{for all } Q_j\in W_3 \tag{3.1}$$

and

$$\Big\|\sum_{Q_j\in W_3}\chi_{\cup F(Q_j)}\Big\|_{L^\infty}\leqslant C. \tag{3.2}$$

LEMMA 3.2. *If $Q_0 \in W_3$ and $0 \leqslant |\alpha| \leqslant k$, then*

$$\|D^\alpha \Lambda_k f\|_{L^p(Q_0)}\leqslant C\|D^\alpha f\|_{L^p(Q_0^*)}+Cl(Q_0)^{k-|\alpha|}\|\nabla^k f\|_{L^p(\cup F(Q_0))}.$$

Proof. On Q_0, $\Lambda_k f$ has the form $\sum_{Q_j\in W_3} P_j\varphi_j$ and $\sum_{Q_j\in W_3}\varphi_j\equiv 1$ on Q_0. Consequently,

$$\|D^\alpha\sum P_j\varphi_j\|_{L^p(Q_0)}\leqslant\|D^\alpha P_0\|_{L^p(Q_0)}+\|D^\alpha\sum(P_0-P_j)\varphi_j\|_{L^p(Q_0)}=\mathrm{I}+\mathrm{II}.$$

By Lemma 2.2,

$$\begin{aligned} \mathrm{I} &\leqslant C\|D^\alpha P_0\|_{L^p(Q_0^*)} \leqslant C\|D^\alpha f\|_{L^p(Q_0^*)} + C\|D^\alpha(f-P_0)\|_{L^p(Q_0^*)} \\ &\leqslant C\|D^\alpha f\|_{L^p(Q_0^*)} + Cl(Q_0)^{k-|\alpha|}\|\nabla^k f\|_{L^p(Q_0^*)}. \end{aligned}$$

Now write

$$D^\alpha \sum_{Q_j\in W_3} (P_0-P_j)\varphi_j = \sum_{Q_j\in W_3}\sum_{\alpha\leqslant\beta} C_{\alpha,\beta}(D^{\alpha-\beta}\varphi_j)(D^\beta(P_0-P_j)).$$

To bound II we need only bound the expression $\|(D^{\alpha-\beta}\varphi_j)(D^\beta(P_0-P_j))\|_{L^p(Q_0)}$. There are at most C cubes $Q_j\in W_3$ such that $\varphi_j\not\equiv 0$ on Q_0 and for these Q_j, $Q_j\cap Q_0\neq\varnothing$ and $l(Q_j)\geqslant\frac{1}{4}l(Q_0)$. Consequently, $|D^{\alpha-\beta}\varphi_j|\leqslant Cl(Q_0)^{-|\alpha-\beta|}$ if $\varphi_j\not\equiv 0$ on Q_0. For these indices j we thus obtain the estimate

$$\begin{aligned} \|(D^{\alpha-\beta}\varphi_j)(D^\beta(P_0-P_j))\|_{L^p(Q_0)} &\leqslant Cl(Q_0)^{-|\alpha-\beta|}\|D^\beta(P_0-P_j)\|_{L^p(Q_0)} \\ &\leqslant Cl(Q_0)^{-|\alpha-\beta|}\|D^\beta(P_0-P_j)\|_{L^p(Q_0^*)} \\ &\leqslant Cl(Q_0)^{-|\alpha-\beta|}l(Q_0^*)^{k-|\beta|}\|\nabla^k f\|_{L^p(\cup F_{0,j})} \\ &\leqslant Cl(Q_0)^{k-|\alpha|}\|\nabla^k f\|_{L^p(\cup F_{0,j})}. \end{aligned}$$

The penultimate inequality above follows from Lemmas 2.8 and 3.1. Summing on j and invoking (3.1) we obtain the estimate

$$\mathrm{II}\leqslant Cl(Q_0)^{k-|\alpha|}\|\nabla^k f\|_{L^p(\cup F(Q_0))}.$$

LEMMA 3.3. *If $Q_0\in W_2\setminus W_3$ and $0\leqslant|\alpha|\leqslant k$, then*

$$\|D^\alpha\Lambda_k f\|_{L^p(Q_0)}\leqslant C\sum_{\substack{Q_j\in W_3\\ Q_0\cap Q_j\neq\varnothing}}\{\|\nabla^k f\|_{L^p(Q_j^*)}+\sum_{\beta\leqslant\alpha}\|D^\beta f\|_{L^p(Q_j^*)}\}.$$

Proof. If $\varphi_j\not\equiv 0$ on Q_0, then $Q_0\cap Q_j\neq\varnothing$ and $l(Q_j)\geqslant\frac{1}{4}l(Q_0)\geqslant\varepsilon\delta/64n$. Consequently, on Q_0 we have

$$\begin{aligned} |D^\alpha\Lambda_k f| &= \Big|\sum_{\substack{Q_j\in W_3\\ Q_0\cap Q_j}}\sum_{\beta\leqslant\alpha} C_{\alpha,\beta}(D^{\alpha-\beta}\varphi_j)(D^\beta P_j)\Big| \\ &\leqslant C\sum_{\substack{Q_j\in W_3\\ Q_0\cap Q_j\neq\varnothing}}\sum_{\beta\leqslant\alpha}|D^\beta P_j|. \end{aligned}$$

If $Q_0\cap Q_j\neq\varnothing$, then by Lemma 2.2,

$$\begin{aligned} \|D^\beta P_j\|_{L^p(Q_0)} &\leqslant C\|D^\beta P_j\|_{L^p(Q_j^*)} \\ &\leqslant C\|D^\beta f\|_{L^p(Q_j^*)} + C\|D^\beta(f-P_j)\|_{L^p(Q_j^*)} \\ &\leqslant C\|D^\beta f\|_{L^p(Q_j^*)} + C\|\nabla^k f\|_{L^p(Q_j^*)} \end{aligned}$$

because $l(Q_j^*)\leqslant 1$. Summing on j and β, the lemma is proved.

A simple geometric argument shows

$$\Big\| \sum_{Q_j \in W_2 \setminus W_3} \sum_{\substack{Q_k \in W_3 \\ Q_j \cap Q_k \neq \emptyset}} \chi_{Q_k^*} \Big\|_{L^\infty} \leqslant C. \tag{3.4}$$

Combining Lemmas 3.2 and 3.3 with (3.2) and (3.4) we obtain the following

PROPOSITION 3.4. $\|\Lambda_k f\|_{L_k^p((\mathcal{D}^c)^\circ)} \leqslant C\|f\|_{L_k^p(\mathcal{D})}$.

We now show that $\Lambda_k f$ has weak derivatives of all orders α, $0 \leqslant |\alpha| \leqslant k$. By the result of section 4 we may assume f is the restriction to $\mathcal{D}$ of a function $f \in C^\infty(\mathbf{R}^n)$ satisfying $\|D^\alpha f\|_{L^\infty} \leqslant M$, $0 \leqslant |\alpha| \leqslant k$, for some value of $M < \infty$. Since $|\partial\mathcal{D}| = 0$, it is sufficient to show that whenever $0 \leqslant |\alpha| \leqslant k-1$, $(D^\alpha f)\chi_{\overline{\mathcal{D}}} + (D^\alpha \Lambda_k f)\chi_{(\mathcal{D}^c)^\circ}$ is Lipschitz. For then $\Lambda_k f \in L_k^p$ and by Proposition 3.4, $\|\Lambda_k f\|_{L_k^p} \leqslant C\|f\|_{L_k^p(\mathcal{D})}$. Fix a multi-index α, $0 \leqslant |\alpha| \leqslant k-1$, and write

$$D^\alpha \Lambda_k f = (D^\alpha f)\chi_{\overline{\mathcal{D}}} + (D^\alpha \Lambda_k f)\chi_{(\mathcal{D}^c)^\circ}.$$

LEMMA 3.5. *$D^\alpha \Lambda_k f$ is Lipschitz.*

Proof. Fix r, $1 \leqslant r \leqslant n$, and set $\partial/\partial x_r D^\alpha = D^\gamma$. Then by hypothesis, $\|D^\gamma f\|_{L^\infty(\mathcal{D})} \leqslant M$. After setting $p = \infty$, Lemmas 3.2 and 3.3 yield $\|D^\gamma \Lambda_k f\|_{L^\infty((\mathcal{D}^c)^\circ)} \leqslant CM$. Since $\overline{\mathcal{D}}$ is closed and $(\mathcal{D}^c)^\circ$ is open, the lemma will be proved once we know that $D^\alpha \Lambda_k f$ is continuous. To this end, let

$$g_j = \frac{1}{|Q_j^*|}\int_{Q_j^*} D^\alpha f\, dx, \quad \text{for } Q_j \in W_3.$$

It is sufficient to show that for $Q_j \in W_3$,

$$\|D^\alpha \Lambda_k f - g_j\|_{L^\infty(Q_j)} \to 0 \quad \text{as} \quad l(Q_j) \to 0. \tag{3.5}$$

By the estimate for term II in the proof of Lemma 3.2,

$$\Big\|D^\alpha \sum_k (P_j - P_k)\varphi_k\Big\|_{L^\infty(Q_j)} \leqslant Cl(Q_j)^{k-|\alpha|}\|\nabla^k f\|_{L^\infty(\cup F(Q_j))}$$

whenever $Q_j \in W_3$. Consequently,

$$\begin{aligned}\|D^\alpha \Lambda_k f - g_j\|_{L^\infty(Q_j)} &\leqslant \|D^\alpha P_j - g_j\|_{L^\infty(Q_j)} + \Big\|D^\alpha \sum (P_j - P_k)\varphi_k\Big\|_{L^\infty(Q_j)} \\ &\leqslant C\|D^\alpha P_j - g_j\|_{L^\infty(Q_j^*)} + Cl(Q_j)^{k-|\alpha|}\|\nabla^k f\|_{L^\infty(\cup F(Q_j))} \\ &\leqslant Cl(Q_j)\|\nabla D^\alpha f\|_{L^\infty(Q_j^*)} + Cl(Q_j)^{k-|\alpha|}\|\nabla^k f\|_{L^\infty(\cup F(Q_j))} \\ &\leqslant CMl(Q_j) \to 0 \quad \text{as} \quad l(Q_j) \to 0.\end{aligned}$$

The proof of Theorem 1 is now complete, modulo the results of section 4.

§ 4. Approximation by C^∞ functions

Fix $\eta>0$, $k\in\mathbf{Z}_+$, a value of p, $1\leqslant p<\infty$, and $f\in L^p_k(\mathcal{D})$. In this section we construct $g\in C^\infty(\mathbf{R}^n)$ such that $\|f-g\|_{L^p_k(\mathcal{D})}\leqslant C\eta$ and $|D^\alpha g|\leqslant M$, $0\leqslant|\alpha|\leqslant k$, for some value of M. If $\mathcal{D}$ is a Lipschitz domain, an easy convolution argument (see [20], chapter VI) can be used to produce g. In (ε,δ) domains this argument fails rather badly; we use here a polynomial approximation scheme similar to that of section 3.

Let $\varrho=2^{-r}$ be a small number whose value will be fixed later, and let $\{R_j\}=\mathcal{R}$ be the collection of all dyadic cubes R satisfying $l(R)=\varrho$ and $R\subset\mathcal{D}$. Put $\mathcal{R}'=\{R_j\in R\colon R_j\subset S_k$ for some $S_k\in W_1$, $l(S_k)\geqslant(32n^3/\varepsilon)\varrho\}$. For $R_j\in\mathcal{R}'$ let $\tilde R_j$ (resp. $\tilde{\tilde R}_j$) be the cube concentric with R_j, with sides parallel to the axes, and with length $l(\tilde R_j)=(500n^4/\varepsilon^2)\varrho$ (resp. $l(\tilde{\tilde R}_j)=(1\,000n^4/\varepsilon)\varrho$). Conditions (1.1) and (1.2) show $\mathcal{D}\subset\bigcup_{R_j\in\mathcal{R}'}\tilde R_j$ if ϱ is small enough.

Lemma 4.1. *If $R_j, R_k\in\mathcal{R}'$ and $\tilde{\tilde R}_j\cap\tilde{\tilde R}_k\neq\emptyset$, then there is a chain $G_{j,k}=\{R_j=R_1, R_2, \ldots, R_m=R_k\}$ of cubes in $\mathcal{R}$ connecting R_j to R_k, and with $m\leqslant C$.*

Proof. Let γ be an arc connecting R_j to R_k and satisfying (1.1) and (1.2). Fix a point z on γ; without loss of generality we may assume $\operatorname{dist}(z,R_j)\leqslant\operatorname{dist}(z,R_k)$. If $\operatorname{dist}(z,R_j)\leqslant 32n\varrho/\varepsilon$, then

$$d(z)\geqslant\frac{32n^3\varrho}{\varepsilon}-\frac{32n\varrho}{\varepsilon}\geqslant\frac{32n\varrho}{\varepsilon}.$$

If $\operatorname{dist}(z,R_j)>32n\varrho/\varepsilon$, then by (1.2), $d(z)\geqslant\varepsilon\cdot(32n\varrho/\varepsilon)\cdot\frac12=16n\varrho$. Thus, if $S_k\in W_1$ and $S_k\cap\gamma\neq\emptyset$, $l(S_k)\geqslant\varrho$. A suitable subset of $\{R_s\in\mathcal{R}\colon R_s\subset S_k\in W_1, S_k\cap\gamma\neq\emptyset\}$ provides us with a chain $G_{j,k}$ connecting R_j to R_k. Condition (1.1) and the estimate $\operatorname{dist}(R_j,R_k)\leqslant(2\,000n^4/\varepsilon^2)\varrho$ assure that the length of $G_{j,k}$ can be bounded by C.

For each $R_j\in\mathcal{R}'$ let P_j be the polynomial fitted to R_j. These polynomials P_j are not in general the same as those of section 3. Also construct functions $\varphi_j\in C^\infty(\mathbf{R}^n)$ supported on $\tilde{\tilde R}_j$ and satisfying $0\leqslant\varphi_j\leqslant 1$, $0\leqslant\sum_{R_j\in\mathcal{R}'}\varphi_j\leqslant 1$, $\sum_{R_j\in\mathcal{R}'}\varphi_j\equiv 1$ on $\bigcup_{R_j\in\mathcal{R}'}\tilde R_j$, and $\sum_{R_j\in\mathcal{R}'}|D^\alpha\varphi_k|\leqslant C(|\alpha|)\varrho^{-|\alpha|}$ for all α. Let $g_0=\sum_{R_j\in\mathcal{R}'}P_j\varphi_j$. The function g_0 will approximate f near $\partial\mathcal{D}$.

Lemma 4.2. *If $R_j\in\mathcal{R}'$ and $0\leqslant|\alpha|\leqslant k$, then*

$$\|D^\alpha P_j\|_{L^p(\tilde{\tilde R}_j)}\leqslant C\|D^\alpha f\|_{L^p(R_j)}+C\varrho^{k-|\alpha|}\|\nabla^k f\|_{L^p(R_j)}.$$

Proof. The lemma follows from Lemma 2.1, the triangle inequality and Lemma 2.2.

LEMMA 4.3. *If* $R_0, R_j \in \mathcal{R}'$, $\tilde{\tilde{R}}_0 \cap \tilde{\tilde{R}}_j \neq \emptyset$, *and* $0 \leq |\alpha| \leq k$, *then*

$$\|D^\alpha(P_0 - P_j)\|_{L^p(R_0)} \leq C\varrho^{k-|\alpha|}\|\nabla^k f\|_{L^p(\cup G_{0,j})}.$$

Proof. The lemma follows from Lemma 4.1 and the estimate on term II in Lemma 3.2.

For $s>0$ let $\mathcal{D}_s = \{x \in \mathcal{D}: d(x) \leq s\}$. Fix a value of $s \in (0,1)$ so that $\|f\|_{L^p_k(\mathcal{D}\setminus\mathcal{D}_{2s})} \leq \eta$. Let $\psi \in C^\infty(\mathbf{R}^n)$ satisfy $0 \leq \psi \leq 1$, $\psi \equiv 1$ on $\mathcal{D}_s$, $\psi \equiv 0$ on $\mathbf{R}^n \setminus \mathcal{D}_{s/2}$, and $|D^\alpha \psi| \leq C(|\alpha|)s^{-|\alpha|}$ for all α. Let $\zeta \in C^\infty(\mathbf{R}^n)$ be supported on $\{\|x\|<1\}$ and satisfy $\int_{\mathbf{R}^n} \zeta\, dx = 1$. For $t>0$, set $\zeta_t(x) = t^{-n}\zeta(x/t)$, and let $f * \zeta_t$ denote the convolution of f with ζ_t. Now fix a value of $t \in (0, s/2)$ so that

$$\|f - f*\zeta_t\|_{L^p_k(\mathcal{D}^0_{s/2})} \leq \eta s^k.$$

Let $g_1 = g_0(1-\psi) = (\sum_{R_j\in\mathcal{R}'} P_j\varphi_j)(1-\psi)$ and let $g_2 = (f*\zeta_t)\psi$. Then $g_1, g_2 \in C^\infty(\mathbf{R}^n)$ and by Lemma 4.2 there is a number $M<\infty$ such that $|D^\alpha g_j| \leq M$, $0 \leq |\alpha| \leq k$, $j=1,2$. To show $\|f-(g_1+g_2)\|_{L^p_k(\mathcal{D})} \leq C\eta$, we need only show that for every α, $0\leq|\alpha|\leq k$, $\|D^\alpha(f-(g_1+g_2))\|_{L^p(\mathcal{D}\setminus\mathcal{D}_s)} \leq C\eta$, because

$$\|D^\alpha(f-(g_1+g_2))\|_{L^p(\mathcal{D}_s)} = \|D^\alpha(f-g_2)\|_{L^p(\mathcal{D}_s)} \leq \eta.$$

Fix α, $0 \leq |\alpha| \leq k$, and write

$$D^\alpha(f-(g_1+g_2)) = \sum_{\beta\leq\alpha} C_{\alpha,\beta}(D^{\alpha-\beta}\psi)(D^\beta(f-f*\zeta_t)) + \sum_{\beta\leq\alpha} C_{\alpha,\beta}(D^{\alpha-\beta}(1-\psi))(D^\beta(f-g_1)).$$

It is only necessary to check that all elements on the right-hand side of the above equality have small L^p norm on $\mathcal{D}\setminus\mathcal{D}_s$. Since $|D^{\alpha-\beta}\psi| \leq Cs^{-|\alpha-\beta|}$, the manner in which we have picked t yields

$$\Big\|\sum_{\beta\leq\alpha} |D^{\alpha-\beta}\psi|\,|D^\beta(f-f*\zeta_t)|\Big\|_{L^p(\mathcal{D}/\mathcal{D}_s)} \leq C\eta. \tag{4.2}$$

We now handle the other terms in (4.1). Notice that $(1-\psi)\chi_{\mathcal{D}}$ is supported in $\mathcal{D}\setminus\mathcal{D}_s$ and $D^\alpha(1-\psi)$ is supported in $\mathcal{D}_{s/2}\setminus\mathcal{D}_s$ whenever $\alpha \neq 0$. The triangle inequality and Lemmas 4.1–4.3 applied to the function $(1-\psi)(D^\alpha(f-g_1))$ yield

$$\begin{aligned}\|(1-\psi)(D^\alpha(f-g_1))\|_{L^p(\mathcal{D}\setminus\mathcal{D}_s)} &\leq C\|D^\alpha f\|_{L^p(\mathcal{D}\setminus\mathcal{D}_s)} + C\|D^\alpha f\|_{L^p(\mathcal{D}\setminus\mathcal{D}_{2s})}\\ &\quad + C\varrho^{k-|\alpha|}\|\nabla^k f\|_{L^p(\mathcal{D}_{2s}\setminus\mathcal{D})} \leq C\eta,\end{aligned} \tag{4.3}$$

as soon as ϱ is small enough with respect to s. Now fix a multi-index β, $0\leq\beta\leq\alpha$, $\beta\neq\alpha$. For $R_0 \in \mathcal{R}'$, $R_0 \cap \{\mathcal{D}_{s/2}\setminus\mathcal{D}_s\} \neq \emptyset$, write

$$|D^\beta(f-g_1)| \leq |D^\beta(f-P_0)| + \Big|D^\beta \sum_{R_j\in\mathcal{R}'} (P_0-P_j)\varphi_j\Big|.$$

Combining Lemmas 2.2, 4.1, and 4.3 with the estimate (3.3), we obtain

$$\begin{aligned}\|(D^{\alpha-\beta}(1-\psi))\,(D^{\beta}(f-g_1))\|_{L^p(\mathcal{D}\setminus\mathcal{D}_s)} &= \|(D^{\alpha-\beta}(1-\psi))\,(D^{\beta}(f-g_1))\|_{L^p(\mathcal{D}_{s/2}\setminus\mathcal{D}_s)}\\ &\leqslant Cs^{-|\alpha-\beta|}\|D^{\beta}(f-g_1)\|_{L^p(\mathcal{D}_{s/2}\setminus\mathcal{D}_s)}\\ &\leqslant Cs^{-|\alpha-\beta|}C\varrho^{k-|\beta|}\|\nabla^k f\|_{L^p(\mathcal{D}\setminus\mathcal{D}_{2s})}\leqslant\eta, \qquad (4.4)\end{aligned}$$

as soon as ϱ is small enough with respect to s. To obtain the inequalities (4.3) and (4.4) we have used the fact that when $R_0,\ R_j\in\mathcal{R}',\ R_0\cap\{\mathcal{D}\setminus\mathcal{D}_s\}\neq\emptyset,\ \bar{\bar{R}}_0\cap\bar{\bar{R}}_j\neq\emptyset$, we then have $\bigcup G_{0,j}\subset\mathcal{D}\setminus\mathcal{D}_{2s}$, if ϱ is small enough. Fix a value of $\varrho>0$ so that estimates (4.3) and (4.4) hold. By (4.2)–(4.4) we then obtain

PROPOSITION 4.4. $\|f-(g_1+g_2)\|_{L^p_k(\mathcal{D})}\leqslant C\eta.$

The above proposition completes the proof of Theorem 1 for the case where $1\leqslant p<\infty$. For the case where $p=\infty$ we need the usual weak approximation of $f\in L^{\infty}_k(\mathcal{D})$ (see [29], page 188). The argument of this section produces for each $\eta>0$ a function $g\in C^{\infty}(\mathbf{R}^n)$ satisfying $\|f-g\|_{L^{\infty}_{k-1}(\mathcal{D})}\leqslant\eta$ and $\|g\|_{L^{\infty}_k(\mathcal{D})}\leqslant C\|f\|_{L^{\infty}_k(\mathcal{D})}$. This is sufficient for our purposes.

§ 5. Proof of Theorem 2

Suppose that for all pairs of points $z_1, z_2\in\mathcal{D}$ there is an arc γ joining z_1 to z_2 and such that

$$\frac{|x-y|\,|z_1-z_2|}{|x-z_i|\,|y-z_j|}\geqslant\varepsilon, \quad i,j=2,2, \quad i\neq j,$$

for all pairs of points x, y, $x\in\gamma$, $y\in\mathcal{D}^c$. With a little bit of work one can see that then $\mathcal{D}$ is an (η,∞) domain for some $\eta=\eta(\varepsilon)>0$. Conversely, if $\mathcal{D}$ is an (ε,∞) domain, then (5.1) holds for some $\varepsilon=\varepsilon(\eta)>0$. This observation is due to Olli Martio. The advantage of Martio's definition is that the estimate in (5.1) is invariant under Möbius transformations. In proving Theorem 2 we may therefore assume that $\mathcal{D}$ is unbounded. A look at the estimates of section 4 shows that $C^{\infty}(\mathbf{R}^n)$ functions are dense in $E(\mathcal{D})$. For each cube $Q_j\in W_2$ select a reflected cube Q_j^* as in section 2. Since $\mathcal{D}$ is unbounded, Lemmas 2.4–2.8 remain valid if we replace W_3 by W_2 in their statements. For $f\in E(\mathcal{D})$ and $Q_j\in W_2$, let P_j be the constant given by

$$\int_{Q_j^*}(f-P_j)\,dx=0.$$

Let $\{\varphi_j\}$ be the usual partition of unity on $(\mathcal{D}^c)^{\circ}$ and put

$$\Lambda f=\sum P_j\varphi_j$$

on $(\mathcal{D}^c)_0^\circ$. Then if $1 \leqslant r \leqslant n$, Lemmas 2.8 and 3.1 yield

$$\begin{aligned}\left\|\frac{\partial}{\partial x_r}\Lambda f\right\|_{L^n(Q_0)} &= \left\|\sum (P_0-P_j)\frac{\partial}{\partial x_r}\varphi_j\right\|_{L^n(Q_0)} \\ &\leqslant Cl(Q_0)^{-1}\sum_{\substack{Q_j\in W_2\\ Q_0\cap Q_j\neq\varnothing}}\|P_0-P_j\|_{L^n(Q_0)} \\ &\leqslant Cl(Q_0)^{-1}l(Q_0)\|\nabla f\|_{L^n(\cup F(Q_0))}.\end{aligned}$$

Consequently, $\|\Lambda f\|_{E((\mathcal{D}^c)^\circ)} \leqslant C\|f\|_{E(\mathcal{D})}$. The argument of section 3 shows that

$$\left(\frac{\partial}{\partial x_r}f\right)\chi_{\mathcal{D}} + \left(\frac{\partial}{\partial x_r}\Lambda f\right)\chi_{(\mathcal{D}^c)^\circ}$$

is a weak derivative of f. Theorem 2 is proved.

§ 6. Quasicircles

In this section we prove Theorems 3 and 4. To do this we first give an alternative proof of Theorem B. To this end, fix a bounded Jordan curve Γ which is not a quasicircle, and let $\mathcal{D}$ be the domain interior to Γ. Let M be a large positive integer. Since Γ is not a quasicircle, we can find points z_1, z_2, z_3, z_4 on Γ such that z_3 and z_4 lie on different components of $\Gamma\setminus\{z_1, z_2\}$ and such that $|z_1-z_4| \geqslant |z_1-z_3| = e^M|z_1-z_2|$. Then $\Gamma\setminus\{z_1, z_2, z_3, z_4\}$ is divided into four disjoint open arcs $\widehat{z_1z_3}$, $\widehat{z_3z_2}$, $\widehat{z_2z_4}$, $\widehat{z_4z_1}$, and we may assume without loss of generality that these arcs are given by the counter-clockwise orientation on Γ. Let φ be a conformal mapping from $\mathcal{D}$ to the unit disk, Δ. The map φ indices a homeomorphism from Γ onto T. Let $\varphi(z_j) = w_j$, $1 \leqslant j \leqslant 4$, and let $I_1 = \widehat{w_1w_3}$, $I_2 = \widehat{w_3w_2}$, $I_3 = \widehat{w_2w_4}$, $I_4 = \widehat{w_4w_1}$ be the four disjoint open arcs of $\mathbf{T}\setminus\{w_1, w_2, w_3, w_4\}$ thus obtained. Let I_j be an arc of smallest Euclidean arclength among the collection $\{I_1, I_2, I_3, I_4\}$. We may assume $I_j = I_1$; the other three cases are handled in exactly the same fashion. Let $\tilde{I}_1$ denote the open arc of T having the same center as I_1 and length $|\tilde{I}_1| = 3|I_1|$. Then by assumption, $\tilde{I}_1 \cap I_3 = \varnothing$. Therefore there is a function $\tau \in C^\infty(\mathbf{R}^2)$ such that $0 \leqslant \tau \leqslant 1$, $\tau \equiv 1$ on I_1, $\tau \equiv 0$ on I_3, and

$$\|\tau\|_{E(\Delta)} \leqslant 100.$$

Let $f = \tau\circ\varphi$ on $\mathcal{D}$. Then $\|f\|_{E(\mathcal{D})} \leqslant 100$ and

$$\|f\|_{L_1^2(\mathcal{D})} \leqslant \pi^{1/2}\,\text{radius}\,(\mathcal{D}) + 100.$$

Suppose now that F is an extension of f to $\mathbf{R}^2$ and suppose $F \in E$. By its construction, $F \equiv 1$ on $\widehat{z_1 z_3}$ and $F \equiv 0$ on $\widehat{z_2 z_4}$. If $|z_1 - z_2| < r < |z_1 - z_3|$, the circle $\{|z - z_1| = r\}$ intersects both the arcs $\widehat{z_1 z_3}$ and $\widehat{z_2 z_4}$. Consequently,

$$\int_0^{2\pi} |\nabla F(z_1 + re^{i\theta})|^2 r\, d\theta \geqslant \frac{1}{r},$$

for almost every such r. Since $|z_1 - z_3| = e^M |z_1 - z_2|$, we obtain

$$\|F\|_E^2 \geqslant \int_{|z_1 - z_2|}^{e^M |z_1 - z_2|} \int_0^{2\pi} |\nabla F(z_1 + re^{i\theta})|^2 r\, d\theta\, dr \geqslant M.$$

By standard patching arguments, there is $f \in L_1^2(\mathcal{D})$ such that no extension of f to $\mathbf{R}^2$ lies in E. An application of the Riemann mapping theorem now completes our proof of Theorem B.

To complete the proof of Theorem 4, notice that the implications (iii) ⇒ (iv) ⇒ (iii) ⇒ (i), (ii) follow from Theorems C and 1. The counterexample of this section can be easily modified to show that if condition (iv) fails, conditions (i) and (ii) also fail simultaneously.

The proof of Theorem 3 is similar. Suppose $\mathcal{D}$ is finitely connected and suppose further that $\mathcal{D}$ is not an (ε, δ) domain for any values of $\varepsilon, \delta > 0$. By Theorem 4 we may assume $\mathcal{D}$ is unbounded. Since $\mathcal{D}$ is conformally equivalent to the unit disk minus a finite number of points and disks; our method of proof shows we may assume that $\partial\mathcal{D}$ consists of a finite number of bounded Jordan curves plus a (possibly infinite) number of unbounded Jordan curves. Call the collection of all boundary curves $\{\Gamma_j\}$. Fix a value of $\delta > 0$. By Theorem C one of the following conditions must fail:

(A) Every bounded Γ_j satisfies condition (1.3) for $M = 1/\delta$.

(B) If Γ_j and Γ_k are distinct unbounded curves, then dist $(\Gamma_j, \Gamma_k) \geqslant \delta$.

(C) If $z_1, z_2 \in \Gamma_j$ (Γ_j unbounded) and $|z_1 - z_2| \leqslant \delta$, then diam $(\gamma_j) \leqslant (1/\delta)|z_1 - z_2|$, where γ_j is the smaller arc between z_1 and z_2.

In each of the above cases, the counterexample of this section can be localized by using smooth cut-off functions to show that $\mathcal{D}$ is not an E.D.S.

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