# PERIODIC MINIMAL SURFACES AND WEYL GROUPS

BY

T. NAGANO and B. SMYTH

*University of Notre Dame*
*Notre Dame, Indiana 46556, U.S.A.*

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## § 1. Introduction

The observation that the Jacobi map of a compact Riemann surface $X$ is universal among all harmonic maps of $X$ into real tori is the basis for our investigation of periodic minimal surfaces in Euclidean space [11], [12] and [14]. This paper continues this work; some of the results were announced at the U.S.-Japan Seminar on Minimal Surfaces in 1977 [13].

Roughly, the first half extends our work [14] to minimal surfaces with symmetry in arbitrary codimension. The main result is that to any such conformal minimal immersion of a fixed compact Riemann surface in flat $n$-tori there corresponds a certain complex subvariety of its Jacobi variety and this correspondence is essentially unique (Theorem 3). An essential step in the proof, but also of intrinsic interest, is the observation that the image of any such immersion is homologous to zero (Theorem 2).

In the second half we develop an idea going back to the H. A. Schwarz Preisschrift [18] of 1867 to construct a remarkable family of such surfaces. We begin by solving a geometric problem of Schoenflies [17] in $n$ dimensions; the solution shows how the root

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system of any simple Lie group of rank $n$ can be used to construct minimal surfaces in $\mathbf{R}^n$ with the periodicity of the root lattice and the symmetry of the corresponding Weyl group. Their quotients by the root lattice are called *Schwarz surfaces*. Applied to these surfaces the general results of § 1–§ 4 reveal a number of interesting analytic and geometric features. The Riemann-Roch formula of Chevalley and Weil [3, 19] is the other main tool for this part of the work; the Atiyah-Singer theorem for orbit spaces [1, Th. 4.7] could also have been used. For example the Chevalley-Weil formula and Theorem 3 imply a rigidity result for the Schwarz surfaces (Theorem 8).

Before describing the results we first explain the natural equivalence among minimal immersions.

A periodic minimal surface in $n$-space can be replaced by a compact minimal surface $X$ in a flat $n$-torus $T^n$ and the conformal structure induced by the immersion $f: X \to T$ makes $X$ a Riemann surface. The original minimal surface is studied via the Jacobi variety of this Riemann surface. The *associate surfaces* or *associate immersions of $f$* arise naturally. The coordinate functions of a minimal surface in $\mathbf{R}^n$ are harmonic so that, when the surface is simply-connected, their conjugates may be used to give a new minimal immersion (called the conjugate immersion), or indeed a 1-parameter family of such immersions with the additional property that the induced metric is the same for all immersions; for example the helicoid is deformed to the catenoid in this way (cf. [5]). When we begin with a periodic minimal surface it is natural to investigate the associate immersions for periodicity. In principle, these associates can be determined from the Jacobi variety of $X$; the definitions are given anew in § 2, the equivalence with the classical definition having been checked in [14].

Among the results for the 3-dimensional case were:

(i) Each associate of the lifted minimal immersion $\tilde{f}: \tilde{X} \to \mathbf{R}^3$ is either dense in $\mathbf{R}^3$ or else projects to another minimal immersion of $X$ into another flat torus [14].

(ii) *The boundary theorem.* $f(X)$ is a boundary in $T^3$ if either $f$ is an imbedding [9] or $f$ has irreducible symmetry [14].

(iii) *The uniqueness theorem.* Two conformal minimal immersions

$$f_\alpha: X \to T^3_\alpha \quad (\alpha = 1, 2)$$

which are homologous to zero(1) and have the same complex kernel (see § 2) are associates [14].

(1) This condition should have appeared in the hypothesis of Theorem 3 [14].

Thus for a given compact Riemann surface $X$ these last two results open the way for a classification of all conformal minimal imbeddings (resp. immersions with irreducible symmetry) of $X$ in flat 3-tori in terms of certain complex codimension 3 linear subvarieties of the Jacobi variety of $X$.

Theorems 1, 2 and 3 extend these last three results to arbitrary codimension. In extending our earlier work to higher codimension we have had in mind the assumption of a high degree of symmetry both for aesthetic reasons and because of the wealth of examples with the symmetry of an irreducible Weyl group indicated in our work [12] and dealt with in more detail here. The correct condition is *absolute irreducible symmetry*, as defined in § 2, and it automatically holds for surfaces with the symmetry of an irreducible Weyl group—in particular for all surfaces with irreducible symmetry in the classical case $n=3$.

Any conformal minimal immersion $f: X \to T$ determines a complex linear subvariety of the Jacobi variety $A(X)$ of $X$ called the *complex kernel* (see § 2). The question of whether this subvariety is closed is the decisive one. For surfaces with absolute irreducible symmetry, closedness is equivalent to the existence of an associate (Theorem 4). For surfaces with irreducible Weyl symmetry closedness implies that the associate $f_\theta$ exists for a dense set of angles $\theta$ in $[0, \pi)$ and that the Jacobi variety $A(X)$ splits off an elliptic curve (Theorem 5). It follows that the coordinate functions of such surfaces are expressible in terms of the real parts of elliptic integrals, a fact first observed by Schwarz in the analysis of one of his surfaces.

The treatment of the Schwarz surfaces begins in § 5. The original idea of Schwarz was to take a skew-quadrilateral in $\mathbf{R}^3$, the reflexions in the edges of which generate a discrete uniform subgroup of the group of motions of $\mathbf{R}^3$; the problem of determining all such quadrilaterals was settled by Schoenflies [17]. The general solution of the Plateau problem was not then available but a minimal surface of disk type spanning the quadrilateral was found using the Schwarz-Christoffel transformation and the Weierstrass representation for simply-connected minimal surfaces in $\mathbf{R}^3$. The discrete group generated by the quadrilateral was then applied to continue the surface throughout $\mathbf{R}^3$, the total surface being analytic on account of the Schwarz reflexion principle. The result was a periodic minimal surface.

The complete solution of this problem of Schoenflies for $\mathbf{R}^n$ is Theorem 6: The reflexions in the edges of a polygon $P$ in $\mathbf{R}^n$ generate a discrete uniform subgroup of the group of motions of $\mathbf{R}^n$ if and only if $P$ is a *root polygon*, i.e. its edges are integer multiples of the roots of some root system $R$.

A solution of the Plateau problem for a root polygon $P$ may then be continued throughout $\mathbf{R}^n$ to obtain a periodic minimal surface in $\mathbf{R}^n$. Dividing out by the period lattice we obtain a compact minimal surface $f\colon X \to T^n$ in a flat $n$-torus, and this we call a *Schwarz surface*.(1) Writing $W_0 = \mathrm{SO}(n) \cap W$, where $W$ is the Weyl group of the root system in question, we note a number of features that the Schwarz surfaces have in common:

(i) each has symmetry $W_0$.

(ii) the quotient Riemann surface $X/W_0$ is the sphere.

(iii) the dimension of the space of $W_0$-invariant differentials of every even degree is known in terms of the polygon $P$

and, furthermore, when $W_0$ is irreducible

(iv) the multiplicity of the representation of $W_0$ induced by $f$ on the space of abelian differentials is known in terms of $P$.

(v) the conditions of

(a) closed complex kernel

(b) the existence of an associate, and

(c) the existence of the conjugate

are all equivalent.

The classical formula of Chevalley-Weil [3, 19] or, alternately, the Atiyah-Singer theorem for orbit spaces [1, Th. 4.7], is the essential tool for (iii) and (iv). The Chevalley-Weil formula also leads to

(vi) the existence of the conjugate for any primitive Schwarz surfaces (Theorem 7)

but we give a more geometric proof.

It follows that the work of § 2–§ 4 applies to all of these Schwarz surfaces. Most of the results are new even for the classical ones studied by Schwarz. So much information on the conformal invariants of the Schwarz surfaces is available or within reach, via root systems, that they might prove a useful fund of examples in the theory of Riemann surfaces itself.

The Schwarz surfaces will, in general, have singularities, so it is worth mentioning that the general results of § 2–§ 4 hold also for surfaces with singularities, the proofs needing little or no change. The substance of the final section is that every irreducible root system gives rise to many *nonsingular* Schwarz surfaces (Theorem 9).

## § 2. Preliminaries

Let $X$ be a compact Riemann surface of genus $p > 1$ and $G$ a subgroup of the automorphism group Aut $(X)$ of $X$. A minimal immersion $f$ of $X$ into a flat $n$-torus $T^n$ for which

(1) The Schwarz surfaces treated here are understood to be without singularities.

the conformal structure induced on $X$ coincides with the given conformal structure on $X$, will be called a *conformal minimal immersion*; say *f has symmetry G* if $G$ extends under $f$ to a group of affine transformations of $T$; when the corresponding linear representation of $G$ is irreducible we say *f has irreducible symmetry G*. If the complexification of this representation is also irreducible we say $f$ has *absolute irreducible symmetry G*, as happens when $G$ is the Weyl group of an irreducible root system [2]. Because irreducibility and absolute irreducibility coincide in odd dimensions, the latter notion does not appear in our previous work on the case $n=3$.

Let $\mathfrak{h}$ denote the space of holomorphic 1-forms on $X$. Each 1-cycle $\sigma$ on $X$ determines an element $\int_\sigma \in \mathfrak{h}^*$. These elements form a lattice in $\mathfrak{h}^*$ denoted $\Delta$ and called the period lattice. Fixing $x_0 \in X$, the surface is mapped into the complex torus $A(X)=\mathfrak{h}^*/\Delta$ by $a(x)=\int_{x_0}^{x} \bmod \Delta$. Then

$$a\colon X \to A(X)$$

is called the Jacobi map of $X$ into the complex torus $A(X)$ with base point $x_0$. This map is easily seen to be holomorphic and is well known to be an imbedding and universal among all holomorphic maps of $X$ into complex tori. But it is also easy to see that $a$ is universal in the class of all *harmonic* maps of $X$ into real flat tori [10]. It is entirely this circumstance which allows us to treat compact minimal surfaces in real flat tori.

Given $f\colon X \to T^n$ a conformal minimal immersion of the compact Riemann surface $X$ into a flat torus $T$, we may assume $f(x_0)=\mathrm{id}_T$. Then universality says that $f=h\circ a$, where $h\colon A \to T$ is a real homomorphism of tori. The kernel of $h$ determines a real subspace $U$ of the tangent space to $A(X)$ at the identity called the *real kernel of f* and its maximal complex subspace $V$ is called the *complex kernel of f*. On occasion the linear subvariety of $A(X)$ passing through the identity and tangent to $V$ will also be called the complex kernel and also denoted $V$. We may, and will, assume $f(X)$ lies in no subtorus of $T^n$ so it follows that $\dim_{\mathbf{R}} U = 2p-n$, where $p$ is the genus of $X$.

The notion of an associate of a minimal immersion of a simply-connected domain into Euclidean space is a familiar one going back to Bonnet (cf. [5]). In our context the appropriate definition is arrived at as follows: take a lift $\tilde{f}\colon \tilde{X} \to \tilde{T}$ to universal covers and let $\tilde{f}_\theta$ $(0 \leq \theta < \pi)$ denote an associate of $\tilde{f}$ in the classical sense; when this projects to a map $f_\theta$ of $X$ into some torus $T_\theta$ we call $f_\theta$ *an associate* of the minimal immersion $f$. It will of course be a minimal immersion with respect to the projected flat metric on $T_\theta$ and will even induce the same metric on $X$ as $f$. This definition of associate is equivalent to the following simpler description: $e^{i\theta}$ can be considered as a complex linear transformation of $T_e(A)$ and if $e^{i\theta}U$ determines a real subtorus of $A$, consider

$$X \xrightarrow{a} A \xrightarrow{h_\theta} T_\theta$$

where $h_\theta$: $A \to T_\theta$ is the homomorphism determined by dividing out by this subtorus. Then $T_\theta$ has a natural flat metric with respect to which $f_\theta = h_\theta \circ a$ is a minimal immersion inducing the same Riemannian metric on $X$ as $f$. The details and the proof of equivalence we gave in [14]. Whether $U_\theta$ determines a torus or not, if we denote by $\tilde{a}: \tilde{X} \to \widetilde{A(X)}$ a lift of $a$ to universal covers and by $\tilde{h}_\theta$ the projection $\widetilde{A(X)} \to \widetilde{A(X)}/U_\theta$ then $\tilde{f}_\theta = \tilde{h}_\theta \circ \tilde{a}$ defines the associate of $\tilde{f}$ in the classical sense (after the natural identification of $\widetilde{A(X)}/U_\theta$ with $\widetilde{A(X)}/U = \mathbf{R}^n$) [13]. Clearly $\tilde{f}_\theta(\tilde{X})$ is dense in $\mathbf{R}^n$ if the linear subvariety of $A(X)$ determined by $U_\theta$ is dense in $A(X)$.

Note that all associates of $f$ have the same complex kernel as $f$ and if $f$ has irreducible (resp. absolutely irreducible) symmetry $G$, the same will be true of its associates. When an associate $f_\theta$ of $f$ does exist then $V = U \cap U_\theta$ and so determines a complex subtorus of $A$.

## § 3. Surfaces with absolutely irreducible symmetry

Given a compact Riemann surface $X$, if we look to classify all conformal minimal immersions of $X$ in flat tori with absolute irreducible symmetry $G$ (some subgroup of the automorphism group of $X$) the problem splits into two parts:

1. Determine all complex subspaces $V$ of $T_e(A)$ that can occur as complex kernels of some such immersion.

2. Determine the relation between all such immersions having a given $V$ as complex kernel.

Nothing much is yet known for 1., but Theorem 3 answers 2. completely; all such immersions are associates.

The result that any such immersion is homologous to zero extends the Boundary Theorem of Meeks [9]. This is proved in Theorem 2 and is essential for our proof of Theorem 3.

Let $X$ be a compact Riemann surface and $f: X \to T^n$ a conformal minimal immersion of $X$ into a flat torus $T$. We may normalise $f$ by assuming $f(x_0) = e$ for some $x_0 \in X$ and furthermore that $f(X)$ lies in no subtorus of $T$. We can factor $f = h \circ a$, where $h$: $A(X) \to T$ is a homomorphism and $a$: $X \to A(X)$ the Jacobi map. The group Aut $(X)$ extends under $a$ to a group of complex affine transformations of $A(X)$, and if a subgroup $G$ of Aut $(X)$ extends under $f$ to a group of affine transformations of $T$ then $h$ is equivariant with respect to the actions of $G$ on $A(X)$ and $T$; to see this we use the fact that the curve $a(X)$ generates

$A(X)$. If $f$ has irreducible symmetry $G$ then the linear part of the action of $G$ on $T$ is irreducible. If $f$ has absolute irreducible symmetry $G$ then the complexification of this latter representation is also irreducible. The linear part of the action of $G$ on $A$ leaves the kernels $U$ and $V$ invariant, and absolute irreducibility of $f$ is equivalent to saying that the induced action of $G$ on the complex space $T_e(A)/V$ is irreducible.

First we collect a few simple properties of these immersions.

LEMMA 1. *Let $f\colon X \to T^n$ be a conformal minimal immersion with irreducible symmetry $G$. Then*

(i) $\dim_{\mathbf{R}} U = 2p - n$, *where $p$ is the genus of $X$,*

(ii) $\dim_{\mathbf{C}} V = p - n$, *if $f$ is not holomorphic,*

(iii) *the linear subvariety of $A(X)$ determined by $V$ is either a torus or else is dense in that determined by $U$.*

*Proof.* Since $f(X)$ does not lie in a subtorus of $T^n$ it follows that the kernel of $h\colon A(X) \to T$ has real dimension $2p-n$, and this proves (i).

Let $J$ denote the complex structure on $A(X)$. If $f$ is not holomorphic then $U \supsetneqq V$ and so $JU/V$ is carried isomorphically by $h$ to a nonzero subspace of $T_e(T)$. Now irreducibility implies that $\dim_{\mathbf{R}} (JU/V) = \dim_{\mathbf{R}} T = n$. Hence $\dim_{\mathbf{C}} V = p - n$, proving (ii).

Note that if the linear subvariety determined by $V$ is not a complex subtorus of $A(X)$ then its closure lies in the kernel of $h$ and so has tangent space $\bar{V}$ at the identity satisfying $V \subsetneqq \bar{V} \subset U$. Clearly $J\bar{V}/V$ is isomorphic under $h$ to a nonzero invariant subspace of $T_e(T)$. By irreducibility this subspace must be $T_e(T)$ itself and it follows easily enough that $\bar{V} = U$. This proves (iii).

THEOREM 1. *Let $f\colon X \to T^n$ be a conformal minimal immersion of a compact Riemann surface $X$ into a flat $n$-torus $T^n = \mathbf{R}^n/L$ with irreducible symmetry $G$. Let $\tilde{f}\colon \tilde{X} \to \mathbf{R}^n$ denote a lift of $f$ to universal covers and $\tilde{f}_\theta$ the associate of $\tilde{f}$ corresponding to the angle $\theta$. Then either*

(i) *$\tilde{f}_\theta$ projects to a conformal minimal immersion of $X$ into some flat $n$-torus $T^n_\theta$ or*

(ii) *$\tilde{f}_\theta(\tilde{X})$ is dense in $\mathbf{R}^n$.*

*Proof.* The proof is along the same lines as Theorem 1 in our previous paper [14]. Assuming that $\tilde{f}_\theta$ does not project to a conformal minimal immersion of $X$ into some flat torus is equivalent to assuming that $U_\theta = e^{i\theta}U$ does not determine a subtorus of $A(X)$. Supposing this to be the case, consider the closure in $A(X)$ of the linear subvariety determined by $U_\theta$ and denote its tangent space at $e$ by $\bar{U}_\theta \supsetneqq U_\theta$. Now $G$ acts as a group of complex transformations on $A(X)$ and, in its linear action, will leave $U_\theta$—and therefore $\bar{U}_\theta$ also—invariant. The induced action on $T_e(A)/U_\theta$ is irreducible and $\bar{U}_\theta/U_\theta$ is an invariant subspace of it. Hence $\bar{U}_\theta = T_e(A)$ and this means that the linear subvariety determined by $U_\theta$ is dense in $A(X)$. By the remarks at the end of § 2, $\tilde{f}_\theta(\tilde{X})$ is dense in $\mathbf{R}^n$.

THEOREM 2. *Let $f: X \to T^n$ be a conformal minimal immersion of a compact Riemann surface $X$ into a flat torus $T^n$. If $f$ has absolutely irreducible symmetry then $f$ is homologous to zero.*

*Proof.* Let $H$ denote the space of real harmonic 1-forms on $X$ obtained by pulling back all of the linear 1-forms from $T$ by $f$. Then

$$I(\eta_1, \eta_2) = \int_X \eta_1 \wedge \eta_2, \quad \eta_1, \eta_2 \in H$$

defines an alternating real bilinear form on $H$. $G$ acts on the space of all harmonic 1-forms leaving the subspace $H$ invariant because $G$ extends under $f$ to a group of affine transformations of $T$. For the same reason the action of $G$ on $H$ is absolutely irreducible. Since $G$ consists of holomorphic transformations of $X$ it follows from the degree formula that it preserves the form $I$. By irreducibility we have either $I=0$ (so $f$ is homologous to zero) or $I$ is nondegenerate. But in the latter case $I$ determines a nonsingular skew symmetric (with respect to the $G$-invariant inner product on $H$) linear transformation commuting with the representation. By Schur's lemma this contradicts the absolute irreducibility of $G$ acting on $H$.

Next we come to the second classification problem mentioned in the beginning of this section.

THEOREM 3. *Let $f_\alpha: X \to T_\alpha$ $(\alpha = 1, 2)$ be conformal minimal immersions of a compact Riemann surface $X$ in flat tori with absolutely irreducible symmetry $G$.*

*If $f_1$ and $f_2$ have the same complex kernel in the Jacobi variety of $X$ then they must be associates.*

*Proof.* Since the immersion $f_\alpha$ has absolute irreducible symmetry it cannot be holomorphic, so certainly $U_\alpha \supsetneqq V_\alpha$. By assumption $V_1 = V_2 = V$ (say), and if $V$ does not determine a complex subtorus of $A(X)$ then, by Lemma 1, (iii), the linear subvariety of $A(X)$ it determines is dense in that determined by $U_\alpha$. Hence $U_1 = U_2$ and $f_1$ and $f_2$ are (trivial) associates.

For the rest of the proof we can therefore assume $V$ determines a complex subtorus of $A(X)$. Denoting the quotient complex torus by $A'$, we have the commutative diagram

$$\begin{tikzcd}
 & T_1 \\
X \arrow[ur, "f_1"] \arrow[r, "a'"] \arrow[dr, "f_2"'] & A' \arrow[u, "h_1"'] \arrow[d, "h_2"] \\
 & T_2
\end{tikzcd}$$

$J$ will also denote the complex structure on $A'$ and $U'_\alpha$ the tangent space to the kernel of $h_\alpha$ at the identity in $A'$. Identifying the universal covers of $T_1$ and $T_2$ with a fixed Euclidean space $\mathbf{R}^n$ this diagram lifts to

$$\begin{tikzcd}
 & \mathbf{R}^n \\
\tilde{X} \arrow[ur, "\tilde{f}_1"] \arrow[r, "\tilde{a}'"] \arrow[dr, "\tilde{f}_2"'] & \tilde{A}' \arrow[u, "h_1"'] \arrow[d, "h_2"] \\
 & \mathbf{R}^n
\end{tikzcd}$$

where $\tilde{X}$ (resp. $\tilde{A}'$) is the universal cover of $X$ (resp. $A'$). Writing $\mathbf{C}^n = \mathbf{R}^n + i\mathbf{R}^n$ and $\pi$ for the projection onto the real part, we have the diagram

$$\begin{tikzcd}
\mathbf{C}^n \arrow[r, "\pi"] \arrow[dr, "\varrho_1"] & \mathbf{R}^n \\
\tilde{X} \arrow[u, "F_1"] \arrow[r, "\tilde{a}'"] \arrow[d, "F_2"'] & \tilde{A}' \arrow[u, "h_1"'] \arrow[d, "h_2"] \\
\mathbf{C}^n \arrow[ur, "\varrho_2"'] \arrow[r, "\pi"'] & \mathbf{R}^n
\end{tikzcd}$$

where $F_\alpha$ denotes the complexification of the harmonic vector-valued function $\tilde{f}_\alpha = h_\alpha \circ \tilde{a}'$ on the simply-connected Riemann surface $\tilde{X}$ and the complex isomorphisms $\varrho_\alpha$ are as defined below. Now $h_\alpha \colon JU'_\alpha \to \mathbf{R}^n$ is a real isomorphism, so its inverse $\varrho_\alpha \colon \mathbf{R}^n \to JU'_\alpha$ can be complexified to give a complex isomorphism from $\mathbf{C}^n$ to $\tilde{A}'$ and there is no difficulty in verifying that $\varrho_\alpha \circ F_\alpha = \tilde{\alpha}'$. Thus $F_2 = k \circ F_1$ where $k = \varrho_2^{-1} \circ \varrho_1$. Let $\Gamma_\alpha : \tilde{X} \to P^{n-1}(\mathbf{C})$ be the Gauss map of $F_\alpha$; then $\Gamma_2 = k \circ \Gamma_1$ ($k$ being considered as a projective transformation of $P^{n-1}(\mathbf{C})$). Now $G$ has two representations in the orthogonal group of $\mathbf{R}^n$—one for each $f_\alpha$. So for any $\sigma \in G$ we have $\Gamma_\alpha \circ \sigma = \sigma_\alpha \circ \Gamma_\alpha$ where $\sigma_\alpha$ is an orthogonal transformation of $\mathbf{R}^n$. We have

$$\sigma_2 \circ k \circ \Gamma_1 = \sigma_2 \circ \Gamma_2 = \Gamma_2 \circ \sigma = k \circ \Gamma_1 \circ \sigma = k \circ \sigma_1 \circ \Gamma_1.$$

On the other hand $\Gamma_1(X)$ lies in no hyperplane in $P^{n-1}(\mathbf{C})$, by the irreducible symmetry of $f_1$. Therefore

$$k \circ \sigma_1 = \sigma_2 \circ k$$

or

$${}^t\sigma_1 {}^t k k \sigma_1 = {}^t k {}^t\sigma_2 \sigma_2 k = {}^t k k.$$

By absolute irreducibility and Schur's lemma, ${}^t k k = \lambda I$ for some $\lambda \in \mathbf{C} - \{0\}$.

Of itself, this last identity is not enough to prove the theorem; if $k$ can be proved to be a complex multiple of a real orthogonal matrix it would follow that $U_2' = e^{i\theta} U_1'$ for some real number $\theta$ and so $f_1$ and $f_2$ would be associates. As in our proof in dimension three [14], Theorem 2 plays a key role at this point.

Write

$$k = \mu e^{i\theta}(C + iD)$$

where $\mu > 0$, $\theta$ is real and $C + iD$ is complex orthogonal; there is no harm in taking $\mu = 1$ in the rest of the proof. Let $\varphi$ and $\psi$ be linear holomorphic 1-forms on $\mathbf{C}^n$ and let $\varphi_\alpha$ and $\psi_\alpha$ denote the corresponding forms on $\tilde{X}$ induced by $F_\alpha$. These forms project to $X$ and it is there we will be considering them. First

$$\begin{aligned} 0 &= \operatorname{Re} \int_X \varphi_2 \wedge \bar{\psi}_2 \\ &= \operatorname{Re} \int_X e^{i\theta}(C + iD)\varphi_1 \wedge e^{-i\theta}(C - iD)\bar{\psi}_1 \\ &= \operatorname{Re} \int_X \{(C\varphi_1 \wedge \overline{C\psi_1} + D\varphi_1 \wedge \overline{D\psi_1}) + i(D\varphi_1 \wedge \overline{C\psi_1} - C\varphi_1 \wedge \overline{D\psi_1})\} \end{aligned}$$

the left-hand side being zero since $f_2$ is homologous to zero by Theorem 2. Since the same is true of $f_1$ and $C$ and $D$ are real, the real parts of the first two integrals on the right vanish and what remains is

$$\operatorname{Im} \int_X (D\varphi_1 \wedge \overline{C\psi_1} - C\varphi_1 \wedge \overline{D\psi_1}) = 0.$$

Choosing $\varphi_1 = {}^t C \Theta$ and $\psi_1 = {}^t D \Theta$ this becomes

$$\operatorname{Im} \int_X (D^t C\Theta \wedge \overline{C^t D\Theta} - C^t C\Theta \wedge \overline{D^t D\Theta}) = 0.$$

Recalling that $C+iD$ is complex orthogonal this can be written

$$\operatorname{Im} \int_X \{C^t D\Theta \wedge \overline{C^t D\Theta} + (I + D^t D)\,\Theta \wedge \overline{D^t D\Theta}\} = 0.$$

Now $D^t D$ is a positive semidefinite real symmetric matrix and if $\Theta$ corresponds under $F_1$ to a positive eigenvalue of ${}^tDD$ then the above integral would have positive imaginary part. This contradiction shows that $D=0$, so that $k=\mu e^{i\theta}C$ where $C$ is real orthogonal. And, by the remarks above, this ends the proof.

The next result gives a geometric interpretation of closedness of the complex kernel in the Jacobi variety.

THEOREM 4. *Let $f\colon X \to T$ be a conformal minimal immersion of a compact Riemann surface into a flat torus $T$ with absolutely irreducible symmetry $G$. The complex kernel of $f$ is a complex subtorus of the Jacobi variety $A(X)$ if and only if $f$ has an associate.*

*Proof.* By the final remark of § 2 we need only prove that closedness of the complex kernel $V$ implies the existence of an associate. Denote the quotient of $A$ by this complex subtorus by $A'$ as in Theorem 3. We have the diagram

$$\begin{tikzcd} & T \\ X \arrow[ur, "f"] \arrow[r, "\alpha'"'] & A' \arrow[u, "h"'] \end{tikzcd}$$

and we denote by $U'$ the tangent space to the kernel of $h$ at the identity. $G$ acts linearly on the $\mathbf{Z}$-module $\Delta$, which is the lattice of $A'$, leaving invariant the submodule determined by $U'$, which for convenience we also denote $U'$. By Maschke's theorem [4] there exists a complementary lattice $U''$ in $\Delta$ which is invariant by $G$; the corresponding real subspace of $T_e(A')$ will also be denoted $U''$. When $U''=JU'$ as vector spaces, the conjugate of $f$ exists, being obtained by dividing out by the real *subtorus* determined by $U''$. We may therefore assume $U'' \neq JU'$ and it then follows further, by irreducibility, that $U'' \cap JU' = \{0\}$ as vector spaces. Each $u'' \in U''$ may be written uniquely as $u'' = u_1 + Ju_2$ where $u_\alpha \in U'$; by the previous remark $u_1 \neq 0$. Similarly $u_2 \neq 0$ since $U'' \cap U' = \{0\}$. It is easy to see that this determines a real linear isomorphism

$$\varrho\colon U' \to U'$$

defined by $\varrho(u_1)=u_2$ which commutes with the action of $G$. By absolute irreducibility of this action and Schur's lemma we have $\varrho=\lambda I$ for some $\lambda\in\mathbf{R}$. Therefore $u''=u_1+\lambda Ju_1$, i.e. $u''=\cos\theta\ \ u+\sin\theta\ \ Ju$ where $\theta=\tan^{-1}\lambda$ for some $u\in U'$. This shows that $U''=e^{i\theta}U'$ and since, by our choice above, $U''$ contains a lattice it follows that the associate $f_\theta$ exists.

## § 4. Surfaces with irreducible Weyl symmetry

The surfaces of greatest interest are those with the symmetry of an irreducible Weyl group and the results of § 3 can be further refined for these. Let $W$ be an irreducible Weyl group acting on $\mathbf{R}^n$ and $W_0=\mathrm{SO}(n)\cap W$. The existence of a compact Riemann surface $X$, a flat torus $T^n$ and a conformal minimal immersion $f\colon X\to T^n$ with irreducible symmetry $W_0$ is shown in § 5. We first see that in such a case the lattice of $T$ is a finite extension of the root lattice; this is then used to prove Theorem 5, which extends our earlier work on the 3-dimensional case [14].

LEMMA 2. *Let $W$ be the Weyl group of any root system $R$ in $\mathbf{R}^n$, and $W_0=\mathrm{SO}(n)\cap W$. If $L_W$ is the lattice spanned by a root system $R$, then for any other $W_0$-invariant lattice $L$ there is an epimorphism*

$$\mathbf{R}^n/L_W\to\mathbf{R}^n/L$$

*which is $W$-equivariant.*

*Proof.* Taking two roots $\alpha,\beta\in R$, it is easily seen that for any $x\in L$,

$$x-s_\beta s_\alpha(x)=2\frac{\langle x,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha+2\frac{\langle x,\beta\rangle}{\langle\beta,\beta\rangle}\beta-4\frac{\langle x,\alpha\rangle\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle\langle\beta,\beta\rangle}\beta$$

is also in $L$ since $s_\beta s_\alpha\in W_0$; here $s_\alpha$ stands for reflexion in the hyperplane through $O$ orthogonal to the root $\alpha$. Assuming $\langle\alpha,\beta\rangle=0$, we obtain

$$2\frac{\langle x,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha+2\frac{\langle x,\beta\rangle}{\langle\beta,\beta\rangle}\beta\in L$$

for all $x\in L$. Since $L$ spans $\mathbf{R}^n$, it follows that some multiple of $\alpha$ is in $L$. Denote by $c(\alpha)\in\mathbf{R}^+$ the smallest number such that $c(\alpha)\alpha\in L$. Substituting $x=c(\alpha)\alpha$ in the above equation, we obtain

$$2c(\alpha)\alpha-2c(\alpha)\frac{\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle}\beta\in L,$$

from which

$$2c(\alpha)\frac{\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle}\beta\in L,$$

i.e.

$$2c(\alpha)\frac{\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle}=m_{\alpha\beta}c(\beta)$$

for some $m_{\alpha\beta}\in\mathbf{Z}$. Therefore

$$2\frac{\langle c(\alpha)\alpha,\, c(\beta)\beta\rangle}{\langle c(\beta)\beta,\, c(\beta)\beta\rangle}=m_{\alpha\beta}.$$

Hence $R'=\{c(\alpha)\alpha\,|\,\alpha\in R\}$ is a root system for $W$. If $W$ is irreducible it follows that $R'$ is similar to $R$ or to its inverse root system $R^{\vee}$ (cf. [2], p. 144, for the definition of $R^{\vee}$). Therefore $cR=R'\subseteq L$ for some positive real number $c$. The linear automorphism $cI$ on $\mathbf{R}^n$ induces the desired epimorphism. When $W$ is reducible this argument applies to the individual irreducible root systems in $R$ to give the result.

THEOREM 5. *Let $W$ be an irreducible Weyl group acting on $\mathbf{R}^n$ as usual and $W_0$ the subgroup of proper motions.*

*Let $X$ be a compact Riemann surface with $W_0\subset\operatorname{Aut}(X)$ and $f\colon X\to T^n$ a conformal minimal immersion of $X$ into a flat $n$-torus with $W_0$ extending under $f$ to a group of affine transformations of $T^n$ so that the corresponding linear representation of $W_0$ is the standard one.*

*If the kernel $V$ of $f$ is closed then*

(i) *the associates $f_\theta\colon X\to T^n_\theta$ exist for a dense set of angles $(0<\theta<\pi)$.*

(ii) *The abelian variety $A(X)/V$ is isogenous to the $n$-fold product of some elliptic curve.*

(iii) *The conjugate of $f$ exists when $X$ admits an antiholomorphic involution extending under $f$ to $T$.*

*Proof.* (i) As in the proof of Theorem 3 we consider the diagram

$$\begin{tikzcd} & T \\ X \arrow[ur, "f"] \arrow[r, "a'"'] & A'=A/V. \arrow[u, "h"'] \end{tikzcd}$$

The group $W_0$ extends to a group of complex transformations of $A(X)$ preserving the foliation determined by $U$ and therefore that determined by $V$ also. Hence $W_0$ acts on $A'$. The induced linear representation of $W_0$ leaves invariant the lattice $\Delta$ of $A'$. Via $J$ we see that the action of $W_0$ on $U'$ is isomorphic to the standard linear action of $W_0$ on $\mathbf{R}^n$. The existence of $f$ means that $\Delta\cap U'$ is a lattice in $U'$. Theorem 4 tells us that $\Delta\cap U'_\theta$ is a lattice in $U'_\theta$ for some $\theta\neq 0 \bmod \pi$; because $W_0$ consists of complex (and not conjugate complex) transformations it leaves $U'_\theta$—and therefore $\Delta\cap U'_\theta$—invariant. Thus $\Delta_0=\Delta\cap U'$ and

$\Delta_\theta = e^{-i\theta}(\Delta \cap U'_\theta)$ are lattices in $U'$ invariant by the action of $W_0$. By the lemma above, $\Delta_0$ and $\Delta_\theta$ contain multiples $c(0)L_W$ and $c(\theta)L_W$, respectively, of the standard root lattice $L_W$ in $U'$. We can assume $c(0)=1$ and write $c(\theta)=r$. Now for any pair of integers $m, n \in \mathbf{Z}$ it can be checked that $r'e^{i\theta'}L_W \subset \Delta$ where

$$\tan\theta' = \frac{mr\sin\theta}{mr\cos\theta + n} \quad (0<\theta'<\pi)$$

and $r'$ is a certain positive real number. Hence $\Delta \cap U'_{\theta'}$ is a lattice in $U'_{\theta'}$ and so the associate $f_{\theta'}$ exists. This set of angles is clearly dense in $[0, \pi]$.

(ii) Following the proof of (i) we see that $L_W \oplus re^{i\theta}L_W$ is a lattice in the universal cover of $A'$ and is contained in the lattice $\Delta$ of $A'$. Take any generating set $\{e_1, \ldots, e_n\}$ for $L_W$, and consider the lattice $Z$ generated by the $\{e_\alpha, re^{i\theta}e_\alpha\}$ in the complexification of $U'$. We have an obvious linear isomorphism

$$\varphi\colon (U')^{\mathbf{C}} \to T_e(A')$$

with $\varphi(Z) \subset L_W \oplus re^{i\theta}L_W \subset \Delta$. Hence $\varphi\colon (U')^{\mathbf{C}}/Z \to A'$ is an isogeny. On the other hand the torus $(U')^{\mathbf{C}}/Z$ is, as a complex torus, isomorphic to $C \times \overset{(n)}{\cdots} \times C$ where $C$ is the elliptic curve $\mathbf{C}/\{1, re^{i\theta}\}$.

(iii) Denote this antiholomorphic involution $s$. With the notation of our proof of (i), we consider the induced linear action on $T_e(A')$. For $\alpha \in L_W$, either $\alpha^+ = s\alpha + \alpha$ or $\alpha^- = s\alpha - \alpha$ is nonzero; say $\alpha^+$. We have $\alpha^+ \in \Delta_0 = \Delta \cap U'$, and since $\Delta_0$ contains the root lattice $L_W$ it follows that $q\alpha^+ \in L_W$ for some integer $q$. Because the immersion $f$ has absolute irreducible symmetry and the complex kernel is closed, an associate $f_\theta$ exists by Theorem 4, i.e. $U'_\theta$ determines a subtorus in $A'$. Recalling the proof of (i), there is a real number $r$ such that $re^{i\theta}L_W$ is contained in $\Delta \cap U'_\theta$. In particular $z = re^{i\theta}q\alpha^+ \in \Delta$. Using the fact that $s$ is an antiholomorphic involution we have

$$z - sz = 2rq\sin\theta\, J\alpha^+ \in \Delta \cap JU'$$

and letting $W$ act on this element we see $\Delta \cap JU'$ is a lattice in $JU'$, i.e. $JU'$ determines a subtorus of $A'$, and so the conjugate exists. If $\alpha^+ = 0$, a similar argument is applied to $\alpha^-$.

## § 5. Weyl groups, root polygons and Schwarz surfaces

We construct here compact minimal surfaces in flat tori with the symmetry of an irreducible Weyl group—the construction is good for any Weyl group—and since an irreducible Weyl group is absolutely irreducible (cf. Bourbaki [2], p. 66) the work of the preceding sections applies. Because the construction gives the classical examples of Schwarz [18] when $n=3$, we call the surfaces generated by this construction *Schwarz surfaces.*

We begin by solving a problem which first interested Schoenflies [17] and which he solved in the special case of quadrilaterals in 3-space:

*Let $P$ be a polygon in Euclidean $n$-space and $K(P)$ the group generated by the reflexions $s_1, ..., s_m$ in the edges of $P$. For what $P$ is $K(P)$ a discrete uniform subgroup (i.e. contains a lattice) of the Euclidean group?*

THEOREM 6. *$K(P)$ is a discrete uniform subgroup of the Euclidean group if and only if $P$ is a root polygon (i.e. there is a root system $R$ and every edge of $P$ is an integral multiple of some root vector in $R$).*

*Proof.* If $P$ is a root polygon then after a translation of one vertex to the origin, the vertices lie on the root lattice. The group $K(P)$ is a subgroup of the affine Weyl group with reflexion in the origin adjoined. Since the latter group is a discrete uniform subgroup of the Euclidean group so also is $K(P)$.

Conversely assume $K=K(P)$ is a discrete uniform subgroup of the Euclidean group; then the maximum translation subgroup $L$ of $K$ is a lattice group. Fixing any point $O$ as origin and taking the linear parts of the elements of $K$, we have a homomorphism of $K$ into the orthogonal group (with respect to $O$) with kernel $L$. Thus $L$ is a normal subgroup of $K$ and the image of the homomorphism, which is isomorphic to $K/L$, leaves the lattice $L(O)$ invariant and so is finite. Let $W$ denote the group generated by the reflexions $r_1, ..., r_m$ in the hyperplanes through $O$ orthogonal to the edges of $P$. The correspondence

$$W \to K/L$$

defined by $r_i \to (Ls_i)$ determines an isomorphism from $W$ onto $K/L$ except when $n$ is odd and $-I \in W$, in which case the kernel is $\{\pm I\}$. Since $W$ is generated by hyperplane reflexions and leaves the lattice $L(O)$ invariant, it must be a Weyl group [2]. If $R$ is a root system corresponding to $W$ then it is already clear that the edges of $P$ are parallel to roots in $R$.

The rationality part of the proof is more delicate. Fixing representatives $\{k_1, ..., k_N\}$ in $K$ for the elements of $K/L$ and fixing a point $x$ in Euclidean space, we denote by $y$ the barycentre of $\{k_1 x, ..., k_N x\}$. The point $y$ is independent of these representatives and likewise of the point $x$ (to within a translation in the lattice $L'=N^{-1}L$). Furthermore for each $k \in K$ we have $k(y)=t_k(y)$ for some translation $t_k \in L'$. Choosing $y$ as origin $O$ and writing $\Lambda = L'(O)$, we see that $K$ can be considered as a subgroup of the symmetry group of the lattice $\Lambda$. The Weyl group $W$ considered as fixing $O$ has proper part $W_0$ contained in $K$ and so leaves $\Lambda$ invariant. Applying Lemma 2 to $W_0$ (or, if necessary, to its irreducible components) we obtain a root system $R$ for $W$ whose root lattice $L_W$ lies in $\Lambda$. If we continue the proof with $W$ irreducible, the modifications to be made in the general case suggest themselves.

Labelling the vertices of $P$ in succession $\{v_1, \ldots, v_m\}$, each edge $e_i = v_{i+1} - v_i$ is parallel to some root $\alpha_i = \varrho_i e_i$. It is required to show that the numbers $\varrho_i$ are rationally related. It is well-known that among the roots of an irreducible root system at most two lengths occur and their squares are rationally related, i.e. all of the $\|\varrho_i e_i\|^2$ are rational multiples of some number $\varrho$. Reflexion in the $i$th edge of $P$ being denoted $s_i$, we have the identities of the kind

$$\begin{aligned} s_i(O) &= 2\left(v_i - \frac{\langle v_i, e_i\rangle}{\langle e_i, e_i\rangle} e_i\right) \\ &= 2\left(v_{i+1} - \frac{\langle v_{i+1}, e_i\rangle}{\langle e_i, e_i\rangle} e_i\right) \end{aligned}$$

and these lead to

$$s_{i+1}(O) - s_i(O) = -2\frac{\langle v_{i+1}, e_{i+1}\rangle e_{i+1}}{\langle e_{i+1}, e_{i+1}\rangle} + 2\frac{\langle v_{i+1}, e_i\rangle e_i}{\langle e_i, e_i\rangle}$$

and the right-hand side is in $\Lambda$. Since $\varrho_i e_i, \varrho_{i+1} e_{i+1} \in \Lambda$, we see that the numbers $\langle v_{i+1}, z\rangle / \langle z, z\rangle \in \mathbf{Q}$, when $z = \varrho_i e_i$ or $\varrho_{i+1} e_{i+1}$. But as the $\langle z, z\rangle$ are then rational multiples of the number $\varrho$, the same is true of $\langle v_{i+1}, z\rangle = \langle v_{i+1}, \varrho_i e_i\rangle$ or $\langle v_{i+1}, \varrho_{i+1} e_{i+1}\rangle$. Likewise the same must also be true of $\langle v_i, \varrho_i e_i\rangle$ and $\langle v_i, \varrho_{i-1} e_{i-1}\rangle$. In particular it is true of $\langle v_{i+1} - v_i, \varrho_i e_i\rangle = \langle e_i, \varrho_i e_i\rangle$. But $\langle \varrho_i e_i, \varrho_i e_i\rangle$ is a rational multiple of $\varrho$. Hence $\varrho_i \in \mathbf{Q}$, completing the proof.

Proceeding toward the construction of the Schwarz surfaces, we take a root polygon $P$ of any root system in $\mathbf{R}^n$. As a further condition on $P$ *we assume that among all solutions of the Plateau problem for the boundary $P$ there is one, say $\psi: \Delta \to \mathbf{R}^n$ (here $\Delta$ denotes the closed unit disk in $\mathbf{R}^2$) with no singularities in the interior or along the edges or at the vertices of $P$.* We denote one such solution surface $\Sigma$.

*Remark.* As we will see in § 7, there are many root polygons meeting this requirement. But even for polygons where this is not so, one can still proceed with the rest of the construction to obtain a compact Riemann surface conformally immersed as a minimal surface with singularities into a flat torus. To be sure the construction is then more complicated but as we pointed out in the introduction, all of the arguments apply equally well to such generalised minimal surfaces.

Writing $K = K(P)$, consider the surface $\bigcup_{k\in K} k(\Sigma)$ invariant by the lattice group $L$. By the Schwarz reflexion principle, the Plateau solutions $k(\Sigma)$ fit together analytically to make up a complete nonsingular minimal surface in $\mathbf{R}^n$. On the product $K\times\Delta$ define an equivalence relation by $(k, p)\sim(h, q)$ if $p=q$ and $h^{-1}k\psi(p)=\psi(q)$. With the quotient topology, $M=K\times\Delta/\sim$ is a differentiable surface and $\Psi(k, p)=k\psi(p)$ extends the map $\psi$ to a map $\Psi$ of $M$ into $\mathbf{R}^n$. The obvious actions of $K$ on $M$ and $\mathbf{R}^n$ are equivariant for $\Psi$; moreover $L$ acts freely on $M$.

In constructing a compact Riemann surface from $M$ we must take account of the cases where $M$ is itself not orientable. With the earlier notation, we let

$$K_0 = \{k\in K \mid k = s_{\alpha_1} \dots s_{\alpha_r},\ r \text{ even}\}.$$

There are two cases to consider, (1) where the Weyl group $W\ni -I$ and $n$ is odd and (2) otherwise. In the latter case no odd product of our generators of $K$ could be a translation. In particular $L_0=L$ and the generators are not in $K_0$. Thus $K_0 \subsetneqq K$. It follows that $M$ is orientable and we take $X=M/L$, obtaining a compact Riemann surface on which $W=K/L$ acts as a group of transformations with $W_0\approx K_0/L$ contained in Aut $(X)$. In case (1), some odd product of generators of $K$ is a translation; if the identity (trivial translation) of $K$ does not occur as such a product then $K_0$ and $L_0$ are index 2 subgroups of $K$ and $L$ respectively. Again $M$ is orientable and we take $X=M/L_0$, and $K/L_0$ acts as a group of transformations with $W_0\approx K_0/L_0$ contained in Aut $(X)$. Finally if in case (1) some odd product of generators of $K$ is the identity, then $K_0=K$ and $L_0=L$, and $M$ is certainly not orientable. The action of $K$ lifts to an orientation-preserving action on the two-fold cover $\tilde{M}$ of $M$ and we take $X=\tilde{M}/L$ on which $W_0\approx K/L$ acts as a group of holomorphic transformations.

In summary each root polygon $P$ with the above properties determines a compact Riemann surface $X$ and a conformal minimal immersion $f$ of $X$ into the flat torus $T=\mathbf{R}^n/L_0$; this we call a *Schwarz surface*. The rotational part $W_0$ of the corresponding Weyl group $W$ acts on $X$ as a group of automorphisms and on $T$ as a group of orientation-preserving motions equivariantly with respect to $f$. Moreover $X$ admits an antiholomorphic involution extending under $f$ to a motion of $T$; such an involution arises from reflexion in any edge of $P$ except in the last case of the above construction where $M$ is non-orientable, but in that case it arises from the natural involution on the 2-fold cover $\tilde{M}$ of $M$. As an immediate consequence of Theorem 5, we have

PROPOSITION 1. *If $f\colon X\to T$ is a Schwarz surface of an irreducible Weyl group then the following are equivalent:*

(a) *the complex kernel of $f$ is closed*

(b) *an associate of $f$ exists*

(c) *the conjugate of $f$ exists.*

*Moreover, when these conditions hold, the abelian variety $A(X)/V$ is isogenous to the $n$-fold product of some elliptic curve, so that $f$ is given by the real parts of elliptic integrals on $X$.*

*Remark.* These conditions are verified for a special class of Schwarz surfaces in Theorem 7, but we consider it likely that they hold for all Schwarz surfaces of an irreducible Weyl group.

Almost all of the remaining discussion of Schwarz surfaces in this section is quite general so we will explicitly mention the one occasion where irreducibility of the corresponding Weyl group is called for.

PROPOSITION 2. *Let $X$ be the Schwarz surface determined by a root polygon $P$, then*

(i) *the genus $p$ of $X$ is given by*

$$2(p-1)=\#W_0\left\{m-2-\sum\frac{\nu_k}{k}\right\}$$

*where $W_0$ denotes the orientation-preserving part of the Weyl group determined by $P$, $m$ is the number of vertices of $P$, $\nu_k$ is the number of vertices of $P$ with angle $\pi/k$ and the summation is over $k=2, 3, 4,$ or $6$.*

(ii) *The projection $\pi: X\to X/W_0=X_0$ branches over exactly $m$ points in $X_0$, $X_0$ is the Riemann sphere and the branch points in $X$ are precisely those points arising from the vertices of $P$; moreover $X_0$ is the Riemann sphere.*

(iii) *$X$ has no $W_0$-invariant abelian differentials and the number of such invariant holomorphic differentials of even order $l$ is given by*

$$N_l=m(l-1)-(2l-1)-\sum\nu_k\left[\frac{l-1}{k}\right]$$

*where $[x]$ stands for the integral part of a real number $x$.*

*Proof.* The product $\sigma$ of the reflexions in the two edges leading into a vertex $v$ of $P$ fixes the normal space at that vertex and is a rotation through an angle $2\alpha$ in the plane of that vertex, where $\alpha$ denotes the angle between these edges. Therefore $\sigma_*$ has trace $(n-2)+2\cos 2\alpha$; bearing in mind that $\sigma_*$ preserves a lattice in $\mathbf{R}^n$, this trace must be integral. The fact that $\alpha$ is acute now gives $\alpha=\pi/2, \pi/3, \pi/4$ or $\pi/6$. This also follows from the fact, mentioned above, that $W$ is a Weyl group and the edges of $P$ are parallel to roots of $W$.

(i) The construction of $X$ carries with it a simplicial subdivision of $X$ in which $F = 2\#W_0$ faces, $E$ edges and $V$ vertices appear. The edges and vertices arise from edges and vertices in $P$ and the vertices arising from a vertex with angle $\pi/k$ in $P$ will be called $k$-vertices; the number of $k$-vertices in $X$ will be denoted $\nu_{(k)}$. As $2k$ faces abut each $k$-vertex and each face touches $\nu_k$ such vertices, we have $2k\nu_{(k)} = \nu_k F$. Trivially $2E = mF$, so the Euler number of $X$ is

$$\begin{aligned}2-2p &= V-E+F\\ &= \sum \nu_{(k)} - \frac{m}{2}F + F\\ &= F\left(\sum \frac{\nu_k}{2k} - \frac{m}{2} + 1\right)\\ &= \#W_0\left(\sum \frac{\nu_k}{k} - m + 2\right)\end{aligned}$$

and this proves (i).

(ii) The Riemann-Hurwitz formula for the projection

$$\pi\colon X \to X/W_0 = X_0$$

says

$$2(1-p) = \#W_0 \cdot 2(1-p_0) - B,$$

where $p$ and $p_0$ are the respective genera and $B$ is the sum of the branching orders of $\pi$. As $\pi$ certainly branches at the vertex points in $X$ with branching order $k-1$ at each $k$-vertex, it follows that

$$\begin{aligned}B \geq \sum (k-1)\nu_{(k)} &= \sum (k-1)\frac{\nu_k}{2k}F\\ &= \#W_0 \sum \left(\nu_k - \frac{\nu_k}{k}\right)\\ &= \#W_0\left(m - \sum \frac{\nu_k}{k}\right)\\ &= 2(p-1) + 2\#W_0\end{aligned}$$

from the computation of the previous paragraph. Returning to the Riemann-Hurwitz formula we see $p_0 = 0$ and the inequality for $B$ becomes an equality; in other words $\pi$ branches exactly on the set of vertices in $X$. The $m$ vertices of $P$ may be considered as vertices in $X$ and no two such vertices are $W_0$-related; further each vertex in $X$ is $W_0$-related to one of these $m$; in short, branching occurs precisely over the projection of these

$m$ points in $X_0$. This fact is essential for the application of the Chevalley-Weil formula in (iii).

(iii) An abelian differential on $X$ invariant by $W_0$ determines an abelian differential on $X_0 = X/W_0$ which, by (ii), is the Riemann sphere. The differential is therefore trivial. Let $D_l$ denote the space of holomorphic differentials of order $l$ on $X$. In the natural representation of $W_0$ on $D_l$, the multiplicity of any irreducible component representation is given by a classical formula of Chevalley and Weil [19], which we now recall.

Let $M$ be an irreducible factor of degree $r$ (i.e. its dimension as a complex subspace of $D_l$). These points over which

$$\pi: X \to X/W_0 = X_0$$

branches are labelled $\{C_\mu\}_{\mu=1}^m$. The isotropy groups of $W_0$ at all points over a given $C_\mu$ are cyclic, of order $k_\mu$ say, and conjugate in $W_0$; considering any generator of such a group as an $r \times r$ matrix via the representation of $W_0$ on $M$, we denote by $N_{\mu\alpha}$ the multiplicity of $e^{(2\pi i\alpha)/k_\mu}$, $0 \leqslant \alpha \leqslant k_\mu$, as a characteristic root of this matrix. Then the Chevalley-Weil formula states that the multiplicity of this representation in the representation of $W_0$ on $D_l$ is

$$N_l = r(2l-1)\,(p(X_0)-1) + \sum_{\mu=1}^{m} \sum_{\alpha=0}^{k_\mu - 1} N_{\mu\alpha} \left\{ (l-1)\left(1 - \frac{1}{k_\mu}\right) + \left\langle \frac{l-1-\alpha}{k_\mu} \right\rangle \right\} + \sigma$$

where $p(X_0)$ is the genus of $X_0$, $\langle x \rangle$ stands for the nonintegral part of $x$, and $\sigma$ is 1 when both $l=1$ and the representation is the identity representation, and is otherwise zero.

For our application here $p(X_0)=0$ by (ii), $\sigma=0$ since $l>1$, and $r=1$, $N_{\mu 0}=1$ and $N_{\mu\alpha}=0$ for $\alpha>0$, since we are interested in counting the multiplicity of the trivial representation in that of $W_0$ on $D_l$. The Chevalley-Weil formula now gives (iii).

COROLLARY. *The space of $W_0$-invariant holomorphic quadratic differentials on $X$ has dimension $m-3$, where $m$ is the number of vertices of $P$.*

It is a classical fact found in the works of Riemann, Schwarz and Weierstrass that a conformal minimal immersion of a Riemann surface in $\mathbf{R}^3$ induces on the surface a holomorphic quadratic differential (cf. H. Hopf [6]): the same will be true for a minimal surface in a flat 3-torus. Moreover any automorphism extending to an isometry of the ambient space will leave this differential invariant. For the classical surfaces of Schwarz we have $n=3$ and $m=4$, so by the corollary above we have a unique $W_0$-invariant quadratic differential. This suggests that there essentially is no other way to realise these Riemann surfaces as $W_0$-symmetric minimal surfaces in flat 3-tori. The precise statement is the rigidity theorem of the next section for which the next result is a key step.

PROPOSITION 3. *In the natural representation of $W_0$ on the space of abelian differentials the representation determined by*

$$f\colon X \to T^n$$

*occurs with multiplicity $m-n$, if $W_0$ is irreducible.*

*Proof.* The space of real harmonic 1-forms on $T^n$ pulls back, under $f$, to an $n$-dimensional space of harmonic 1-forms on $X$ which is invariant under the natural action of $W_0$. Moreover this action—being assumed irreducible—is absolutely irreducible since the action is determined by that of a Weyl group. Hence $W_0$ acts irreducibly on the corresponding complex $n$-dimensional subspace $K$ of abelian differentials in $D_1$. In the decomposition of $D_1$ into irreducible $W_0$-submodules, the multiplicity $N$ of any factor can, in principle, be counted by the formula of Chevalley-Weil cited above. Fortunately this calculation is within reach for the submodule $K$. In this instance $r=n$, $l=1$, $p(X_0)=1$ and $\sigma=0$ in the formula, so the multiplicity is given by

$$N = -n + \sum_{\mu=1}^{m} \sum_{\alpha=0}^{k_\mu-1} N_{\mu\alpha} \left\langle \frac{-\alpha}{k_\mu} \right\rangle$$

where $\langle x \rangle$ stands for the nonintegral part of a number $x$. By Proposition 2 (iii), all branch points over $C_\mu$ are $k_\mu$-vertices; a generator of the isotropy group of $W_0$ at such a point is given by the product of the reflexions in the edges of $P$ emanating from it; since this fixes the normal $(n-2)$-dimensional subspace and rotates the tangent plane through an angle $2\pi/k_\mu$, the characteristic values have multiplicities given by

$$(k_\mu > 2) \quad N_{\mu 0} = n-2, \quad N_{\mu 1} = 1, \quad N_{\mu(k_\mu-1)} = 1, \quad N_{\mu\alpha} = 0 \text{ otherwise}$$

$$(k_\mu = 2) \quad N_{\mu 0} = n-2, \quad N_{\mu 1} = 2.$$

In either case

$$\sum_{\alpha=0}^{k_\mu-1} N_{\mu\alpha}\left(1 - \frac{\alpha}{k_\mu}\right) = 1.$$

Thus the Chevalley-Weil formula gives

$$N = m - n$$

as the multiplicity of this representation.

## § 6. The primitive Schwarz surfaces

Those Schwarz surfaces determined by root polygons with $(n+1)$ edges in $\mathbf{R}^n$ we will call primitive here. The main result of this section is the existence of the conjugate for the primitive Schwarz surfaces. Even for the classical surfaces of Schwarz (which are primitive) this result was known only in a couple of cases which were carefully studied by Schwarz [18] and Neovius [15]. In particular the complex kernel is closed for the primitive surfaces. Thus the assumption of the existence of an associate in our earlier work [14], as well as in the parallel work of Meeks [9], is of wide occurrence.

THEOREM 7. *If $f: X \to T^n$ is a primitive Schwarz surface then the conjugate immersion*

$$f_\pi: X \to T^n_\pi$$

*exists.*

*Remark.* It can easily be verified that a root polygon with $n+1$ edges in $\mathbf{R}^n$ can only come from an irreducible root system. Thus all primitive surfaces have irreducible Weyl symmetry. Theorem 5 now gives the existence of infinitely many associates for the primitive surfaces.

*Proof.* $P$ denotes the $(n+1)$-gon and $\psi: \Delta \to \mathbf{R}^n$ a solution of the Plateau problem for $P$. Writing $\psi = (\psi^1, ..., \psi^n)$, the functions $\psi^\alpha$ are real harmonic functions on the interior of $\Delta$. The harmonic conjugate $\varphi$ of $\psi$ is therefore defined on the interior and extends continuously to the boundary because $\psi$ can be analytically continued across the boundary by the Schwarz reflexion principle. $\varphi$ gives a minimal surface in $\mathbf{R}^n$ (unique to within translation) called the conjugate surface to $\psi$. A well-known and easily proved classical fact (cf. Darboux [5], p. 379) says that a segment of $\partial\Delta$ mapped by $\psi$ to an edge $e$ of $P$ is mapped by $\varphi$ to a curve lying in a hyperplane $H$ perpendicular to $e$ and the surface $\varphi$ may be analytically continued by reflexion in this hyperplane. Applying this to each edge, the surface $\varphi(\Delta)$ is seen to be bounded by a convex polyhedron formed by $n+1$ hyperplanes $\{H_1, ..., H_{n+1}\}$ respectively perpendicular to the edges $\{e_1, ..., e_{n+1}\}$ of $P$ and to meet each of these hyperplanes orthogonally. The edges being parallel to roots by Theorem 6, the hyperplanes are orthogonal to roots; they will be called *root hyperplanes.*

LEMMA 3. *The group generated by reflexions in the $n+1$ root hyperplanes $\{H_1, ..., H_{n+1}\}$ is a discrete uniform subgroup of the group of motions of $\mathbf{R}^n$.*

Granting this for the moment, it follows that this group applied to $\varphi(\Delta)$ gives a periodic minimal surface in $\mathbf{R}^n$ and dividing out by the appropriate lattice, as in § 5, we obtain the conjugate immersion of the Schwarz surface determined by $P$, completing the proof of Theorem 7.

*Proof of Lemma* 3. Let $\alpha_1, ..., \alpha_{n+1}$ be roots of $W$ corresponding to the hyperplanes $H_1, ..., H_{n+1}$. We can assume the first $n$ of these pass through the origin and then $H_{n+1}$

has equation $x \cdot \alpha_{n+1} = \mu$. After a change of metric by a scale factor, this becomes $x \cdot \alpha_{n+1} = 1$; all of the hyperplanes are now of the form $x \cdot \alpha_i = \beta_i \in \mathbf{Z}$, so that the group whose generators are the reflexions in these hyperplanes is a subgroup of the affine Weyl group [2], and the lemma follows.

In general, the primitive Schwarz surfaces are rigid in the appropriate sense.

THEOREM 8. *Let $f: X \to T^n$ be a primitive Schwarz surface, $W$ the corresponding Weyl group and $W_0 = \mathrm{SO}(n) \cap W$. Any other conformal minimal immersion*

$$f_1: X \to T_1^n$$

*of $X$ into a flat torus $T_1^n$ with irreducible symmetry $W_0$ is an associate of $f$ provided $W_0$ has but one irreducible complex representation of degree $n$.*

*Proof.* The group $W_0$ acts on the space of abelian differentials on $X$. Each of the immersions $f$ and $f_1$ determines an irreducible $W_0$-submodule of this space of complex dimension $n$. By assumption these submodules are equivalent. But the Chevalley-Weil formula was used in Proposition 3 to count the multiplicity of the $W_0$-submodule determined by $f$ and it is 1 for the primitive surfaces. Hence these two $W_0$-submodules coincide. It follows easily that $f$ and $f_1$ have the same complex kernel. By Theorem 3, $f$ and $f_1$ are associates.

## § 7. Regularity for the Schwarz surfaces

The construction of the Schwarz surfaces in § 5 can lead to minimal surfaces with singularities (i.e. the maps may fail to be immersions at finitely many points). While the work of § 2–§ 4 applies even in the presence of singularities, it is of interest to know the existence of nonsingular Schwarz surfaces with the symmetry of each of the irreducible Weyl groups. Once existence is shown it will be clear from our proof that such surfaces are abundant.

THEOREM 9. *For every irreducible root system $R$ in $\mathbf{R}^n$ there exists a nonsingular primitive Schwarz surface $f: X \to T^n$ with symmetry $W_0 = \mathrm{SO}(n) \cap W$, where $W$ is the Weyl group of $R$.*

The proof of this result requires that we produce a root polygon $P$ (corresponding to $R$), for which one solution of the Plateau problem will be regular in the interior, on the edges and at the vertices. The construction of § 5 will then give a nonsingular Schwarz surface. The main step in the proof is the observation that for a certain class of polygons in $\mathbf{R}^n$ all solutions of the Plateau problem have this kind of regularity. Finally we must show that each of the irreducible root systems gives rise to a *root polygon* in the aforementioned class. This is done for the root systems $A_n$, $B_n$, $C_n$ and $D_n$, the exceptional ones being left to the reader.

Let $P$ be any polygon in $\mathbf{R}^n$ with

(i) $n+1$ edges and not lying in any hyperplane in $\mathbf{R}^n$; the edge-vectors are labelled $e_1, \ldots, e_{n+1}$ (in order) with $e_1$ emanating from the origin $O$, the remaining vertices being then denoted $v_i = \sum_{k=1}^{i} e_k$ for $i = 1, 2, \ldots, n$.

Any solution $\psi: \Delta \to \mathbf{R}^n$ of the Plateau problem for $P$ is then regular in the interior and at the boundary (Lemma 4). This part of the proof follows the lines of Lawson [7]. If further

(ii) $\langle e_i, e_j \rangle \leqslant 0$ for $1 \leqslant i < j \leqslant n+1$ and

(iii) the acute angle at each vertex is an integral divisor of $\pi$,

regularity at the vertices follows (Lemma 5).

LEMMA 4. *If $P$ satisfies* (i) *then $\psi$ is regular in the interior and at the boundary.*

*Proof.* Any 4-gon in $\mathbf{R}^3$ has one-one convex-parallel projection into some 2-plane. Dropping any one vertex $v_i$ from $P$, the remaining ones $\{v_0, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n\}$ determine an $n$-gon $P_n$ in some $\mathbf{R}^{n-1} \subset \mathbf{R}^n$. By the similar assumption on $P$, the $n$-gon $P_n$ will not lie in any hyperplane of $\mathbf{R}^{n-1}$; hence $P_n$ has no self-intersection. Denote the mid-point of $v_{i-1}v_{i+1}$ by $m_i$. Parallel projection of $\mathbf{R}^n$ along the direction $v_i m_i$ into the hyperplane $\mathbf{R}^{n-1}$ carries $P$ onto the $n$-gon $P_n$ monotonically. By induction *there is a parallel projection of $\mathbf{R}^n$ onto some 2-plane $\mathbf{R}^2 \subset \mathbf{R}^n$ carrying $P$ one-one onto the boundary of a convex region in $\mathbf{R}^2$.*

Under these circumstances a theorem of Rado (cf. [16], or [8]) ensures the interior regularity of any solution $\psi: \Delta \to \mathbf{R}^n$ of the Plateau problem for $P$. Furthermore Rado's result says that this solution can be realised as the graph of some continuous $\mathbf{R}^{n-2}$-valued function defined on the above region in $\mathbf{R}^2$, analytic in the interior. In particular $\psi$ is one-one on $\Delta$, a fact which will be used below.

Next we treat boundary regularity. Reflexion of our Plateau solution $\psi$ in any one edge $e_i$ of $P$ analytically continues this solution and in the composite surface the only interior points where a singularity could occur are interior to $e_i$. By the Riemann mapping theorem, the composite solution can be considered as another map $\psi': \Delta \to \mathbf{R}^n$ with $\psi'[-1, 1] = e_i$ and we assume $\psi'$ singular at some point $p \in (-1, 1)$. A simple induction argument, using condition (i), shows $P$ lies on the boundary of the polyhedron which is its convex hull; let $H$ be the support hyperplane of any one of its faces which contain $e_i$. Since a minimal surface lies in the convex hull of its boundary, $\psi'$ maps the upper and lower halves of $\Delta$ to either side of $H$. By the remarks at the end of the preceding paragraph, $\psi'$ is one-one on both these half-disks. Now for any small disk $D$ centred at $p$, the minimal surface $\psi'\colon D\to\mathbf{R}^n$ is analytic and $\psi'|_{\partial D}$ is a Jordan curve. The proof of Proposition 5 on page 97 in [8] implies that any hyperplane through $\psi'(p)$ meets $\psi'(\partial D)$ in at least four components if $\psi'$ is singular at $p$. As $H$ meets $\psi'(\partial D)$ in just two points, $\psi'$ must be regular at $p$.

LEMMA 5. *Assume* (ii) *for* $P$. *For each* $i$, $\psi(\Delta)$ *lies in the wedge-region in* $\mathbf{R}^n$ *which is the product of the normal space at* $v_i$ *and the sector determined by* $\{e_{i+1}, -e_i\}$ *in the plane of that vertex.*

*Proof.* In the plane of this vertex we choose $v_i$ as origin and $\{e_{i+1}, -e_i\}$ as basis and denote the resulting coordinates $(x, y)$. By (ii) the orthogonal projection of $e_j$ into this plane is either a point or else a vector making an obtuse angle (i.e. $\geqslant\pi/2$) with $e_i$. Hence, as $e_j$ $(j\neq 1, i+1)$ is traversed positively, $y$ is non-decreasing and similarly $x$ is non-increasing. In particular the projection of $P$ lies in the sector $\{(x, y)\,|\,x\geqslant 0, y\geqslant 0\}$, so that the convex hull of $P$ lies in the region mentioned in the lemma. But $\psi(\Delta)$ lies in the convex hull of $P$ by the maximum principle, so the lemma is proved.

*Remark.* It follows from the proof that each of the hyperplanes $x=0$ and $y=0$ meets $P$ in at most two connected components.

LEMMA 6. *Assume* (i), (ii) *and* (iii) *for* $P$. *Then* $\psi$ *is regular at the vertices.*

*Proof.* If the angle at $v_i$ is $\pi/k$, we may reflect our solution $\psi$ for $P$ around the vertex $v_i$, in total $2k$ times, obtaining $\psi''\colon\Delta\to\mathbf{R}^n$ with $\psi''(0)=v_i$. Certainly $\psi''$ is analytic (and regular) on $\Delta-\{0\}$ but it is also analytic on $\Delta$ as we now show. If $\varphi$ denotes the harmonic conjugate of $\psi$ then, after normalisation by a suitable translation, $\varphi(\Delta)$ meets orthogonally the hyperplanes through $v_i$ orthogonal to $e_i$ and $e_{i+1}$ (cf. Darboux [5]). Reflexion in these hyperplanes continues $\varphi$ analytically and repeated reflexion defines a single-valued continuous $\varphi''\colon\Delta\to\mathbf{R}^n$ analytic on $\Delta-\{0\}$ and conjugate to $\psi''$ on $\Delta-\{0\}$. Riemann's theorem on removable singularities guarantees $\psi''+i\varphi''$ is holomorphic on $\Delta$ and in particular $\psi''$ is analytic on $\Delta$.

Next we show $\psi''$ is regular at $O$. By Lemma 5 each reflexion of $\psi(\Delta)$ around $v_i$ lies in a wedge-region and these wedge-regions have mutually disjoint interiors. Since $\psi$ is also one-one, from the proof of Lemma 1, it follows at once that $\psi''(\partial D)$ is a Jordan curve for any small disk $D$ centred at $O$; furthermore $\psi''\colon D\to\mathbf{R}^n$ is an analytic minimal surface by the above. If $\psi''$ were singular at $O$ then by the proof of Proposition 5 on page 97 in [8], any hyperplane passing through $\psi''(O)$ meets $\psi''(\partial D)$ in at least four components. By Lemma 5 and the remark thereafter, the hyperplane $x=0$ meets $\psi''(\partial D)$ in just two components, so $\psi''$ must be regular at $O \in \Delta$.

It remains now to show that for any irreducible root system $R$ in $\mathbf{R}^n$ a root polygon can be found satisfying (i), (ii) and (iii) above. Of course (iii) is redundant for root polygons so we simply look for root $(n+1)$-gons satisfying (ii), a condition that is simply checked. If $\{\alpha_1, ..., \alpha_n\}$ is a basis of simple roots for $R$, consider the $(n+1)$-gon defined by

$$e_i = \alpha_i, \quad 1 \leqslant i \leqslant n,$$
$$e_{n+1} = -\sum_{i=1}^{n} \alpha_i.$$

For the root systems $A_n$, $B_n$ and $D_n$ this defines a root $(n+1)$-gon satisfying (ii) (cf. Bourbaki [2], Planche I, II and IV). For $C_n$ and the remaining exceptional root systems slight modification of this polygon can be found which satisfy (ii). For $C_n$ (with the notation of Bourbaki [2], Planche III) we take

$$\begin{aligned} e_i &= 2\alpha_i, \quad 1 \leqslant i \leqslant n-3, \\ e_{n-2} &= \alpha_{n-2}, \\ e_{n-1} &= \alpha_{n-2} + 2\alpha_{n-1} + \alpha_n, \\ e_n &= \alpha_n, \\ e_{n+1} &= -2\sum_{i=1}^{n} \alpha_i. \end{aligned}$$

The details for the exceptional systems are left to the reader.

*Remarks.* (a) Since the properties (i), (ii) and (iii) are invariant under permutations of the edges of $P$, it follows that there are many noncongruent root polygons satisfying (i), (ii) and (iii). Consequently there are as many nonsingular Schwarz surfaces.

(b) In showing the existence of *nonsingular* surfaces for each Weyl group, Theorem 9 sharpens our earlier existence result [12].

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