LEFSCHETZ–RIEMANN–ROCH FOR SINGULAR VARIETIES (1)

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§ 0. Introduction

0.1. The fixed point problem

Let k be an algebraically closed field, and let n be an integer prime to the characteristic of k. By an *equivariant variety* we shall mean a quasi-projective scheme X over k together with an automorphism $x: X \to X$ such that $x^n = \mathrm{id}$. The fixed point scheme will be denoted $|X|$, and its automorphism will be the identity. All morphisms $f: X \to Y$ are assumed to be equivariant, i.e., $y \circ f = f \circ x$, and the induced morphism of fixed point schemes is denoted $|f|: |X| \to |Y|$.

An *equivariant sheaf* on X is a coherent sheaf $\mathfrak{F}$ of O_X modules together with a homomorphism

$$\varphi_{\mathfrak{F}}: x^* \mathfrak{F} \to \mathfrak{F}$$

of sheaves of O_X-modules.

The Lefschetz Fixed Point Problem is to calculate, for an equivariant sheaf $\mathfrak{F}$ on a projective equivariant variety X, the alternating sum of the traces of the induced maps on the cohomology $H^i(X, \mathfrak{F})$, as a sum of contributions from the components of $|X|$. We prove a general Lefschetz–Riemann–Roch theorem which solves the fixed point problem when X is mapped to a point, just as the Hirzebruch–Riemann–Roch formula follows from a general Riemann–Roch theorem [4].

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In this situation, our theorem extends known results ([2], [7], [10]) to singular varieties, and improves the results announced in [4]. Explicit calculations of the local contributions are given for local complete intersections which generalize in a rather surprising way the Woods Hole Formula for the non-singular case. Both the statement and the proof of the theorem become particularly natural by using the formalism developed in [5].

0.2. K-groups

A *morphism* $\psi: \mathcal{F} \to \mathcal{G}$ of equivariant sheaves is a homomorphism of O_X-sheaves such that $\psi \circ \varphi_{\mathcal{F}} = \varphi_{\mathcal{G}} \circ x^*(\psi)$. The equivariant sheaves on X form an abelian category. If $f: X \to Y$ is a proper morphism of equivariant varieties, and $\mathcal{F}$ is an equivariant sheaf on X, then the higher direct image sheaves $R^i f_*(\mathcal{F})$ are equivariant sheaves on Y [7]. Define $K_0^{eq} X$ (resp. $K^0_{eq} X$) to be the Grothendieck group of all equivariant sheaves (resp. locally free sheaves) on X. Let $[\mathcal{F}]$ be the element in $K_0^{eq} X$ (resp. $K^0_{eq} X$) represented by an equivariant sheaf (resp. locally free sheaf) $\mathcal{F}$. The tensor product makes $K^0_{eq} X$ into a ring, and determines a cap product

$$K^0_{eq} X \otimes K_0^{eq} X \xrightarrow{\frown} K_0^{eq} X$$

making $K_0^{eq} X$ into a $K^0_{eq} X$-module. The structure sheaf O_X, together with its canonical endomorphism, represents the element 1 in $K^0_{eq} X$, and a *fundamental class* $[O_X]$ in $K_0^{eq} X$.

If $f: X \to Y$ is a morphism, there are induced morphisms $f^*: K^0_{eq} Y \to K^0_{eq} X$ given by $f^*[\mathcal{E}] = [f^*\mathcal{E}]$; K^0_{eq} is a contravariant functor from equivariant varieties to rings. If f is proper, define $f_*: K_0^{eq} X \to K_0^{eq} Y$ by $f_*[\mathcal{F}] = \sum (-1)^i [R^i f_* \mathcal{F}]$; K_0^{eq} is covariant for proper morphisms. There is the usual projection formula

$$f_*(f^* b \frown a) = b \frown f_* a$$

for $b \in K^0_{eq} Y$, $a \in K_0^{eq} X$. Note also that exterior powers $\Lambda^i \mathcal{E}$ of an equivariant locally free sheaf are naturally equivariant ($K^0_{eq} X$ is a λ-ring).

0.3. Trivial action

In case X is projective and the automorphism x is the identity, any equivariant sheaf $\mathcal{F}$ on X is a finite direct sum of sheaves $\mathcal{F}_a$, for $a \in k$, such that the endomorphism $\varphi_{\mathcal{F}} - aI$ is nilpotent on $\mathcal{F}_a$. This determines canonical homomorphisms

$$K^0_{eq} X \to K^0_{abs} X \otimes \mathbf{Z}[k] \tag{1}$$

$$K_0^{eq} X \to K_0^{abs} \otimes \mathbf{Z}[k] \tag{2}$$

taking $[\mathcal{F}]$ to $\sum [\mathcal{F}_a]\otimes[a]$. Here $K^0_{\mathrm{abs}}X$ (resp. $K_0^{\mathrm{abs}}X$) is the Grothendieck groups of locally free (resp. coherent) sheaves on X, without endomorphisms, and $\mathbf{Z}[k]$ is the free abelian group on the elements of k. By considering the filtration on the sheaves $\mathcal{F}_a$ given by the kernels of $(\varphi_{\mathcal{F}}-aI)^i$, one sees that (2) is always an isomorphism. If X is singular, however, (1) may fail to be an isomorphism.

0.4. The coefficient ring

It follows in particular that for a point,

$$K^0_{\mathrm{eq}}\,(\mathrm{Spec}\,(k)) = \mathbf{Z}[k];$$

the product in the ring $\mathbf{Z}[k]$ is induced by the multiplication in k. Fix a commutative $\mathbf{Z}[k]$-algebra Λ such that for each nth root of unity $a\in k$, $a\neq 1$, the element $[1]-[a]$ becomes invertible in Λ. Any finite dimensional vector space H over k with a k-linear endomorphism determines an element in $K^0_{\mathrm{eq}}\,(\mathrm{Spec}\,(k))$, and hence an element in Λ by the homomorphism from $\mathbf{Z}[k]$ to Λ. We write tr (H) for this element in Λ. Note that we may take $\Lambda = k$, and this is the usual trace. If the characteristic is positive, Λ may be taken to be the Witt ring of k, a ring of characteristic zero, and tr becomes the Brauer trace (cf. [7], [14]). The strongest results are obtained by taking Λ to be the localization of $\mathbf{Z}[k]$ at the multiplicative set generated by the above elements.

0.5. A local invariant

If V is a component of $|X|$ (or a union of several connected components), and X is non-singular in a neighborhood of V, then V is also non-singular, and the conormal sheaf $\mathcal{N}$ to V in X is an equivariant locally free sheaf on V. Then $\sum(-1)^i[\Lambda^i\mathcal{N}]$ determines an element in $K^0_{\mathrm{eq}}V$, which by the homomorphism (1) of § 0.3 and base extension from $\mathbf{Z}[k]$ to Λ determines an element

$$\lambda_V X \quad \text{in } K^0_{\mathrm{abs}}V\otimes\Lambda.$$

Since the eigenvalues of the endomorphism on $\mathcal{N}$ are non-trivial nth roots of unity, our assumption on Λ makes $\lambda_V X$ invertible in $K^0_{\mathrm{abs}}V\otimes\Lambda$ (cf. [7], 4.3 and [6], VI 6.3).

0.6. The Theorem

Let X be an equivariant quasi-projective variety and assume that $|X|$ is projective. There is a canonical homomorphism

$$L^{\cdot}\colon K^0_{\mathrm{eq}}X \to K^0_{\mathrm{abs}}|X|\otimes\Lambda$$

obtained by composing the restriction (or pull-back) homomorphism from $K^0_{eq}X$ to $K^0_{eq}|X|$ with the homomorphism (1) of § 0.3 from $K^0_{eq}|X|$ to $K^0_{abs}|X|\otimes\mathbf{Z}[k]$, and then making the base extension from $\mathbf{Z}[k]$ to Λ. This $L^{\cdot}$ is a natural transformation of contravariant functors.

LEFSCHETZ–RIEMANN–ROCH THEOREM. *For each equivariant quasi-projective variety X such that $|X|$ is projective, there is a homomorphism*

$$L.: K_0^{eq}X \to K_0^{abs}|X|\otimes\Lambda$$

which is covariant for proper morphisms and compatible with cap products. For each component V of $|X|$ contained in the non-singular locus of X, the component of $L.[O_X]$ in $K_0^{abs}V\otimes\Lambda$ is $(\lambda_V X)^{-1}\frown[O_V]$, with $\lambda_V X$ as in § 0.5.

In general, write $L.[O_X]$ as a sum of terms $L_V X$ in $K_0^{abs}V\otimes\Lambda$ corresponding to the decomposition of $K_0^{abs}|X|$ into the direct sum of $K_0^{abs}V$, as V varies over the connected components of $|X|$. If $\mathcal{E}$ is an equivariant locally free sheaf on X, and X is mapped to a point, the covariance and cap product assertions in the theorem give a formula for $\sum(-1)^i\,\mathrm{tr}\,(H^i(X,\mathcal{E}))$ as a sum of terms obtained by restricting $\mathcal{E}$ to V, capping with $L_V X$, and mapping V to a point. For example, if each $V=P$ is an isolated fixed point, and $\mathcal{E}(P)$ is the fibre of $\mathcal{E}$ at P, with its induced endomorphism, we have the following corollary.

COROLLARY. $\sum_i(-1)^i\,\mathrm{tr}\,(H^i(X,\mathcal{E}))=\sum_{P\in|X|}\mathrm{tr}\,(\mathcal{E}(P))\cdot L_P X.$

In the non-singular case, this contains known formulas (cf. [2], [7], [15]) with the improvement that the equality takes place in the ring Λ. It is desirable to have an explicit computation of $L_P X$ in case P is an isolated singular point.

We give such a formula in case X is a local complete intersection at P in § 3; the expression has a denominator of the expected form, together with an interesting numerator which may well be zero (cf. [3] for further discussion of these numbers). The theorem in § 3 also gives information about $L_V X$ in case V is not a point.

The complete statement of the theorem in § 2 includes the fact that the contribution at a component V of $|X|$ depends only on a neighborhood of V in X.

0.7. The construction

We describe the construction of $L.[\mathcal{F}]$ in $K_0^{abs}|X|\otimes\Lambda$ for an equivariant sheaf $\mathcal{F}$ on X. Imbed X equivariantly in a non-singular Y, and resolve $\mathcal{F}$ by an equivariant complex $\mathcal{E}.$ of locally free sheaves on Y. The restriction of $\mathcal{E}.$ to $|Y|$ is exact off $|X|$, so the alternating sum of homology sheaves

$$\sum(-1)^i[\mathcal{H}_i(\mathcal{E}.|_{|Y|})]$$

defines an element in $K_0^{eq}|X|$, or an element η in $K_0^{abs}|X|\otimes\Lambda$, applying homomorphism (2) of § 0.3. Then

$$L.[\mathcal{F}] = |i|^*(\lambda_{|Y|}Y)^{-1}\frown\eta$$

where $|i|$ is the inclusion of $|X|$ in $|Y|$, and $\lambda_{|Y|}Y$ is the element defined in § 0.5. As in [5], the essential step is to prove that this element is independent of the choices.

0.8. Related results

The Lefschetz–Riemann–Roch maps

$$L^\cdot\colon K^0_{eq}X\to K^0_{abs}|X|\otimes\Lambda$$

and

$$L.\colon K_0^{eq}X\to K_0^{abs}|X|\otimes\Lambda$$

may be composed with non-equivariant Riemann–Roch maps

$$\tau^\cdot\colon K^0_{abs}|X|\otimes\Lambda\to H^\cdot|X|\otimes\Lambda$$

and

$$\tau.\colon K_0^{abs}|X|\otimes\Lambda\to H.|X|\otimes\Lambda$$

constructed in [4] and [5]. We may take $H^\cdot$ and $H.$ to be (1) singular cohomology and homology, if $k=\mathbf{C}$ and Λ is a $\mathbf{Q}$-algebra, or (2) the Chow rational equivalence homology–cohomology theory, for any k, if Λ is a $\mathbf{Q}$-algebra, or (3) topological K-cohomology and homology, if $k=\mathbf{C}$ and Λ is any algebra satisfying the condition in § 0.4. In (1) and (2) $\tau^\cdot$ is the Chern character. In each case, the compositions give homomorphisms

$$K^0_{eq}X\to H^\cdot|X|\otimes\Lambda$$

and

$$K_0^{eq}X\to H.|X|\otimes\Lambda$$

satisfying the same formal properties as in the main theorem in § 2. It is these versions of Lefschetz–Riemann–Roch that were referred to in [4]. They specialize to Riemann–Roch when the automorphisms are all identity maps. They were originally proved by making all the arguments of [4] equivariant, a task that is quite straightforward except perhaps in case (2). B. Moonen has also carried out part of this program in case (1). Cases (1) and (2) extend Donovan's work [7] to singular varieties in the same way that [4] extended Grothendieck–Riemann–Roch to singular varieties.

Note that in the case of isolated fixed points, the use of the (trivial) Riemann–Roch theorems only weakens the result, in cases (1) and (2), by throwing away torsion.

In [11] it is proved that $L.$ induces an isomorphism

$$K_0^{\mathrm{eq}} X \otimes_{\mathbf{Z}[k]} \Lambda \to K_0^{\mathrm{abs}} |X| \otimes \Lambda.$$

The inverse is induced by the inclusion of $|X|$ in X. The Lefschetz–Riemann–Roch theorem we prove in § 2 can be deduced from this localization theorem. A similar localization theorem was first found in the non-singular case by Nielsen [10], where the result was combined with Riemann–Roch on the fixed point variety to obtain fixed point formulas.

0.9. *G*-varieties and sheaves

If a finite group G acts on a variety X, one may form the Grothendieck group $K_0^G X$ (resp. $K_G^0 X$) of coherent (resp. locally free) G-sheaves on X. As long as the order of G is prime to the characteristic, there is no difficulty in extending our results to this situation. We refer to [3] for a discussion which emphasizes this point of view, and the relation of these groups to equivariant topological K-theory. When G is cyclic, the fixed point theorem we prove in this paper is stronger, however, since we do not require the liftings of the actions on the sheaves to have finite order.

In § 4 we apply our results to calculate the homology Todd class of a quotient variety X/G in terms of data on the fixed point schemes in X of the action of the elements of G. The formula is particulary explicit for varieties arising from quasi-homogeneous polynomials.

There are other situations where questions regarding more general group actions are reduced to questions about the action of one automorphism (cf. [2]). On the other hand, it is not clear how to extend our results to endomorphisms of varieties which are not of finite order (see [8] for the case of the Frobenius, and [15] for non-singular complex manifolds, however).

0.10. Conventions

All varieties, morphisms, and sheaves will be assumed to be equivariant unless otherwise stated; adjectives such as non-singular, local complete intersection, proper, locally free, etc., refer to the underlying non-equivariant varieties, morphisms, or sheaves. As in [5], we use the word "variety" for "quasi-projective k-scheme".

0.11. Acknowledgements

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§ 1. Equivariant K-groups

1.1. Definitions

Let X be a closed (equivariant) subscheme of Y. Consider complexes $\mathcal{E}.$ of locally free sheaves on Y, exact off X, equipped with a homomorphism of complexes $\varphi_{\mathcal{E}.}: y^*\mathcal{E}. \to \mathcal{E}.$. Define $K_X^{eq} Y$ to be the free abelian group on the isomorphism classes of such complexes, modulo relations $[\mathcal{E}.] = [\mathcal{E}.'] + [\mathcal{E}.'']$ for each exact sequence

$$0 \to \mathcal{E}.' \to \mathcal{E}. \to \mathcal{E}.'' \to 0$$

and $[\mathcal{E}.] = 0$ if $\mathcal{E}.$ is exact on all of Y; $[\mathcal{E}.]$ denotes the class in $K_Y^{eq} Y$ determined by the complex $\mathcal{E}.$. The definitions and basic properties are the precise analogues of those given in [5], § 1, for the non-equivariant case, so we include only a brief summary.

If f: $Y' \to Y$ is a morphism, and $f^{-1}(X) \subset X'$, there is a *pull-back* homomorphism

$$f^*: K_X^{eq} Y \to K_{X'}^{eq} Y'$$

defined by $f^*[\mathcal{E}.] = [f^*\mathcal{E}.]$. If $X_i \subset Y_i$, there is an external *product*

$$K_{X_1}^{eq} Y_1 \otimes K_{X_2}^{eq} Y_2 \overset{\times}{\to} K_{X_1 \times X_2}^{eq}(Y_1 \times Y_2)$$

defined by $[\mathcal{E}._1] \times [\mathcal{E}._2] = [\mathcal{E}._1 \boxtimes \mathcal{E}._2]$, and the corresponding internal products. If i: $Y \to Z$ is a closed imbedding of finite Tor dimension, and X is closed in Y, there is a *Thom–Gysin* homomorphism

$$i_*: K_X^{eq} Y \to K_X^{eq} Z$$

defined by $i_*[\mathcal{E}.] = [\mathcal{F}.]$, where $\mathcal{F}. \to i_* \mathcal{E}.$ is an equivariant resolution, i.e., a resolution in the non-equivariant sense which is also a morphism of equivariant complexes of sheaves on Z. The existence of equivariant resolutions follows exactly as in the absolute case [5], App. 2 and from the fact that an equivariant sheaf on a quasi-projective scheme is the image of an equivariant locally free sheaf (cf. [7], 2.2).

There is also a *homology* map

$$h: K_X^{eq} Y \to K_0^{eq} X$$

defined by $h[\mathcal{E}.] = \sum (-1)^i [\mathcal{H}_i(\mathcal{E}.)]$, where $\mathcal{H}_i(\mathcal{E}.)$ are the homology sheaves of the complex $\mathcal{E}.$ (with the induced equivariant maps); we have used the fact that $K_0^{eq} X$ may be identified with the Grothendieck group of equivariant sheaves on Y which are supported on X. The six properties of [5] § 3.3 are equally valid in the equivariant case.

1.2. Equivariant vector bundles

An equivariant vector bundle E on an equivariant variety X has a morphism $e\colon E \to E$ so that the diagram

$$\begin{array}{ccc} E & \xrightarrow{e} & E \\ \pi \downarrow & & \downarrow \pi \\ X & \xrightarrow[x]{} & X \end{array}$$

commutes, where π is the vector bundle projection. The morphism e must respect the vector bundle structure on E; equivalently, if $\mathcal{E}$ is the locally free sheaf of sections of the *dual* bundle $E^{\vee}$, e determines a morphism φ from $x^*\mathcal{E} \to \mathcal{E}$ making $\mathcal{E}$ into an equivariant locally free sheaf in the sense of § 0.1, but with the additional condition that $\varphi^n = 1$.

Important examples of equivariant vector bundles are normal and tangent bundles, and their exterior powers. If X is a local complete intersection in Y, the normal bundle $N = N_{X,Y}$ is an equivariant vector bundle on X, and the conormal sheaf $\mathcal{N}_{X,Y}$ is an equivariant locally free sheaf. If Y is a non-singular variety, its tangent bundle T_Y is an equivariant bundle, and the cotangent sheaf Ω^1_Y is an equivariant locally free sheaf.

Since n is prime to the characteristic, the restriction of E to $|X|$ splits canonically into a direct sum of vector bundles $E^{(a)}$, for $a \in k$, $a^n = 1$, such that φ is multiplication by a on $E^{(a)}$. Let $E^{(\times)}$ be the direct sum of the $E^{(a)}$ for $a \neq 1$, so that $\mathcal{E}|_{|X|} = \mathcal{E}^{(1)} \oplus \mathcal{E}^{(\times)}$.

When E is regarded as an equivariant variety, the fixed point scheme $|E|$ may be identified with $E^{(1)}$, and the bundle $E^{(\times)}$ measures the extent of non-transversality in the square

$$\begin{array}{ccc} |X| & \xrightarrow{|i|} & |E| \\ \downarrow & & \downarrow \\ X & \xrightarrow{i} & E \end{array}$$

where the horizontal maps are zero-section imbeddings. The Koszul–Thom complex $\Lambda^{\cdot}\pi^*\mathcal{E}$ gives an equivariant resolution of $i_* O_X$ on E. This restricts to the complex

$$\Lambda^{\cdot}|\pi|^*\mathcal{E}^{(1)} \otimes \Lambda^{\cdot}|\pi|^*\mathcal{E}^{(\times)}$$

on $|E|$. It follows that the Thom–Gysin maps can only be compatible with the restriction to fixed point schemes if they are modified as in the following section.

1.3. Modified maps

The groups $K^{\mathrm{eq}}_X Y$ are all modules over $K^0_{\mathrm{eq}}(\mathrm{Spec}\,(k)) = \mathbf{Z}[k]$. When Λ is a $\mathbf{Z}[k]$ algebra as in § 0.4, set

$$K^{\mathrm{eq}}_X Y_{\Lambda} = K^{\mathrm{eq}}_X Y \otimes_{\mathbf{Z}[k]} \Lambda.$$

Let i: $Y \to Z$ be a closed imbedding of non-singular equivariant varieties. If X is closed in Y, define *modified Thom–Gysin maps*

$$\tilde{i}_*: K^{eq}_{|X|}|Y|_\Lambda \to K^{eq}_{|X|}|Z|_\Lambda$$

by the formula $\tilde{i}_*(\xi) = |i|_*(\lambda_{-1}\mathcal{N}^{(\times)} \cdot \xi)$ where $|i|_*$ are the Thom–Gysin maps of § 1.1, $\mathcal{N}$ is the conormal sheaf to Y in Z, and $\lambda_{-1}\mathcal{N}^{(\times)} = \sum (-1)^i \Lambda^i \mathcal{N}^{(\times)}$ in $K^0_{eq}|Y|$.

These modified Thom–Gysin maps satisfy the same properties as before. The functoriality follows from the equation

$$\lambda_{-1}(\mathcal{E}_2^{(\times)}) = \lambda_{-1}(\mathcal{E}_1^{(\times)}) \cdot \lambda_{-1}(\mathcal{E}_3^{(\times)})$$

for an exact sequence $0 \to E_1 \to E_2 \to E_3 \to 0$ of equivariant vector bundles.

The homology maps must be modified to be compatible with the modified Thom–Gysin maps. If X is a closed subscheme of a non-singular Y, j: $X \to Y$ the inclusion, and $|X|$ is projective, define the *modified homology map*

$$\tilde{h}: K^{eq}_{|X|}|Y|_\Lambda \to K_0^{abs}|X| \otimes \Lambda$$

by the formula

$$\tilde{h}(\xi) = |j|^* (\lambda_{|Y|} Y)^{-1} \frown h(\xi).$$

Here h is the unmodified map of § 1.1, and we have identified $K_0^{eq}|X|_\Lambda$ with $K_0^{abs}|X| \otimes \Lambda$ by § 0.3; $\lambda_{|Y|} Y$ is the invertible element in $K^0_{abs}|Y| \otimes \Lambda$ constructed in § 0.5.

It is easy to verify that the modified homology maps also satisfy the six properties in [5], § 3.3. The compatibility of the modified homology maps with the modified Thom–Gysin maps, for example, uses the standard exact sequence relating the tangent and normal bundles of an imbedding of non-singular varieties—which is the reason for the modification.

1.4. Deformation

Define the homomorphism

$$L: K^{eq}_X Y \to K^{eq}_{|X|}|Y|_\Lambda,$$

for X closed in Y, to be the pull-back homomorphism induced by the inclusion of $|Y|$ in Y, followed by the base extension from $\mathbf{Z}[k]$ to Λ. It is obvious that L commutes with pull-backs and products. For the Thom–Gysin maps we have the following lemma.

MAIN LEMMA. *If $i\colon Y \to Z$ is a closed imbedding of non-singular varieties, X is closed in Y, and $|X|$ is projective, then the diagram*

$$\begin{array}{ccc} K^{eq}_X Y & \xrightarrow{L} & K^{eq}_{|X|}|Y|_\Lambda \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \tilde{i}_*} \\ K^{eq}_X Z & \xrightarrow{L} & K^{eq}_{|X|}|Z|_\Lambda \end{array}$$

commutes.

Proof. Construct the deformation diagram of [5], § 2, i.e.,

$$\begin{array}{ccccc} Y & \xrightarrow{j_1} & Y \times \mathbf{A}^1 & \xleftarrow{j_0} & Y \\ \downarrow{\scriptstyle i} & & \downarrow & & \downarrow{\scriptstyle \tilde{i}} \\ Z & \xrightarrow[k_1]{} & W & \xleftarrow[k_0]{} & N \end{array}$$

and note by its construction that all the varieties and maps are equivariant; here $\mathbf{A}^1$ is the affine line over k, with the identity automorphism. The induced diagram of fixed point schemes is

$$\begin{array}{ccccc} |Y| & \xrightarrow{|j_1|} & |Y| \times \mathbf{A}^1 & \xleftarrow{|j_0|} & |Y| \\ \downarrow{\scriptstyle |i|} & & \downarrow & & \downarrow{\scriptstyle |\tilde{i}|} \\ |Z| & \xrightarrow[|k_1|]{} & |W| & \xleftarrow[|k_0|]{} & |N| \end{array}$$

which is the deformation diagram for the inclusion of $|Y|$ in $|Z|$. All the squares of the above diagrams are transversal (Tor independent). The proof now concludes precisely as in [5], § 2. (Note that we have already checked the compatibility of L with the modified Thom–Gysin maps for the normal bundle situation.)

§ 2. The Lefschetz–Riemann–Roch theorem

Definition 2.1. Let X be a quasi-projective, equivariant scheme such that $|X|$ is projective. Define a homomorphism

$$L_\cdot\colon K_0^{eq} X \to K_0^{abs}|X| \otimes \Lambda$$

as follows. Choose an equivariant imbedding of X in a non-singular quasi-projective Y (cf. Lemma (i) below), and then $L_\cdot$ is the composite

$$K_0^{eq} X \xleftarrow[\cong]{h} K^{eq}_X Y \xrightarrow{L} K^{eq}_{|X|}|Y|_\Lambda \xrightarrow[\cong]{\bar{h}} K_0^{abs}|X| \otimes \Lambda$$

where L, h and $\bar{h}$ are defined in § 1.

THEOREM 2.2. *The homomorphism $L.$ is independent of the imbedding, and satisfies:*

1 (covariance). *For every proper morphism $f: X \to X'$, the diagram*

$$\begin{array}{ccc} K_0^{eq} X & \xrightarrow{L.} & K_0^{abs}|X| \otimes \Lambda \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle |f|_*} \\ K_0^{eq} X' & \xrightarrow{L.} & K_0^{abs}|X'| \otimes \Lambda \end{array}$$

commutes.

2 (module). *For every X, the diagram*

$$\begin{array}{ccc} K^0_{eq} X \otimes K_0^{eq} X & \xrightarrow{L^{\cdot} \otimes L.} & (K^0_{abs}|X| \otimes \Lambda) \otimes_\Lambda (K_0^{abs}|X| \otimes \Lambda) \\ {\scriptstyle \frown}\downarrow & & \downarrow{\scriptstyle \frown} \\ K_0^{eq} X & \xrightarrow{L.} & K_0^{abs}|X| \otimes \Lambda \end{array}$$

commutes.

3 (product). *For every X_1, X_2, the diagram*

$$\begin{array}{ccc} K_0^{eq} X_1 \otimes K_0^{eq} X_2 & \xrightarrow{L. \otimes L.} & (K_0^{abs}|X_1| \otimes \Lambda) \otimes_\Lambda (K_0^{abs}|X_2| \otimes \Lambda) \\ {\scriptstyle \times}\downarrow & & \downarrow{\scriptstyle \times} \\ K_0(X_1 \times X_2) & \xrightarrow{L.} & K_0^{abs}(|X_1 \times X_2|) \otimes \Lambda \end{array}$$

commutes.

4 (restriction). *If $j: U \to X$ is the inclusion of an open equivariant subscheme in X, then the diagram*

$$\begin{array}{ccc} K_0 X & \xrightarrow{L.} & K_0^{abs}|X| \otimes \Lambda \\ {\scriptstyle j^*}\downarrow & & \downarrow{\scriptstyle |j|^*} \\ K_0 U & \xrightarrow{L.} & K_0^{abs}|U| \otimes \Lambda \end{array}$$

commutes.

5. *If X is non-singular, then*

$$L.[O_X] = (\lambda_{|X|} X)^{-1} \frown [O_{|X|}]$$

where $\lambda_{|X|} X$ is the class defined in § 0.5.

6. *If the automorphism x of X is the identity, then $L.: K_0^{eq} X \to K_0^{abs}|X| \otimes \Lambda$ is the homomorphism* (2) *defined in* § 0.3.

The proof is entirely analogous to the proof of the Riemann–Roch theorem in [5], § 4. Only in steps (5) and (7), and the proof of the module property in that proof were facts required that did not follow formally from the general properties. The analogous facts needed for Lefschetz–Riemann–Roch are given in the following lemma, parts (iii), (ii), (iv) respectively.

LEMMA 2.3. (i) *If X is an equivariant quasi-projective variety, then there is a projective space $P = \mathbf{P}^N$, whose automorphism is given by a diagonal matrix, an open equivariant subvariety U of P, and a closed equivariant imbedding of X in U.*

(ii) *With P as in* (i), *and any equivariant variety X, the product mapping*

$$K_0^{eq} X \otimes K_0^{eq} P \xrightarrow{\times} K_0^{eq}(X \times P)$$

is an isomorphism.

(iii) *With P as in* (i), *and f the mapping of P to a point, the diagram*

$$\begin{array}{ccc} K_0^{eq} P & \xrightarrow{L.} & K_0^{abs}|P| \otimes \Lambda \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle |f|_*} \\ K_0^{eq}(\text{pt.}) & \xrightarrow{L.} & K_0^{abs}(\text{pt.}) \otimes \Lambda \end{array}$$

commutes; the horizontal maps $L.$ are defined by imbedding the varieties in themselves.

(iv) *If $\mathcal{E}$ is an equivariant locally free sheaf on an equivariant variety X, there is a nonsingular equivariant variety Y, with an equivariant locally free sheaf $\tilde{\mathcal{E}}$ on Y, and an equivariant morphism $f: X \to Y$ such that $f^*\tilde{\mathcal{E}}$ and $\mathcal{E}$ are isomorphic* (*as equivariant locally free sheaves*).

Proof. Parts (i)–(iii) are variations of rather standard facts. We sketch the proofs in geometric language, and refer to the literature for alternative descriptions.

For (i), choose a (non-equivariant) closed imbedding of X in an open subvariety V of a projective space Q. Let $Q_i = Q$ and $V_i = V$ for $i = 1, ..., n$, and define an automorphism on the product $Q_1 \times ... \times Q_n$ by sending $(p_1, ..., p_n)$ to $(p_2, p_3, ..., p_n, p_1)$. Imbed X equivariantly in $V_1 \times ... \times V_n$ by sending $p \in X$ to $(x(p), x^2(p), ..., x^n(p))$. Use the Segre imbedding to imbed $Q_1 \times ... \times Q_n$ in a projective space $\mathbf{P}^N$. The above automorphism extends canonically to a linear automorphism of order n on $\mathbf{P}^N$, and there is a canonical open set U of $\mathbf{P}^N$ that intersects $Q_1 \times ... \times Q_n$ in $V_1 \times ... \times V_n$. The automorphism is diagonizable since n is prime to the characteristic.

A simple proof of (ii) can be obtained by following Quillen ([12], § 8). A matrix for the automorphism p of P gives a homomorphism from $p^*\mathcal{O}(1)$ to $\mathcal{O}(1)$, so all the sheaves

$\mathcal{O}(k)$ become equivariant. As in [12], we can confine our attention to those equivariant sheaves $\mathcal{F}$ on $X \times P$ such that $H^i(\mathcal{F}(-i))=0$ $i>0$; such sheaves have canonical resolutions

$$0 \to T_N(\mathcal{F}) \otimes \mathcal{O}(-N) \to \dots \to T_0(\mathcal{F}) \otimes \mathcal{O} \to \mathcal{F} \to 0$$

for coherent sheaves $T_i(\mathcal{F})$ on X. The $T_i(\mathcal{F})$ are exact functors of $\mathcal{F}$, from which it follows that the above is an equivariant resolution of $\mathcal{F}$ (by coherent sheaves). If $\mathcal{F}$ is locally free, then the $T_i(\mathcal{F})$ are locally free on X. This proves (ii). This same argument gives the expected relation between $K_0^{\mathrm{eq}}(P(E))$ (resp. $K^0_{\mathrm{eq}}P(E)$) and $K_0^{\mathrm{eq}}X$ (resp. $K^0_{\mathrm{eq}}X$) for any equivariant vector bundle E on X.

When X is a point, this shows that $K_0^{\mathrm{eq}}P$ is generated over $\mathbf{Z}[k]$ by the elements l^i, $l=[\mathcal{O}(1)]$, $i=0, 1, \dots, N$.

Let ι be the inclusion of $|P|$ in P, and consider the homomorphisms

$$K_0^{\mathrm{eq}}|P|_\Lambda \xrightarrow{\iota_*} K_0^{\mathrm{eq}}P_\Lambda \cong K^0_{\mathrm{eq}}P_\Lambda \xrightarrow{\iota^*} K^0_{\mathrm{eq}}|P|_\Lambda \cong K_0^{\mathrm{eq}}|P|_\Lambda$$

of free Λ-modules of rank $=N+1$; the unnamed isomorphisms are Poincaré duality isomorphisms. We have seen in § 1.4 that the composite is multiplication by the unit $\lambda_{|P|}P$. Therefore ι_* and ι^* are isomorphisms. To prove (iii), consider the diagram

$$\begin{array}{ccccc} K_0^{\mathrm{eq}}|P| & \xrightarrow{\iota_*} & K_0^{\mathrm{eq}}P & \xrightarrow{f_*} & K_0^{\mathrm{eq}}(\mathrm{pt.}) \\ L.\downarrow & & L.\downarrow & & L.\downarrow \\ K_0^{\mathrm{abs}}|P| \otimes \Lambda & \xrightarrow{\mathrm{id}} & K_0^{\mathrm{eq}}|P| \otimes \Lambda & \xrightarrow{|f|_*} & K_0^{\mathrm{abs}}(\mathrm{pt.}) \otimes \Lambda, \end{array}$$

where the vertical maps are defined by using the imbeddings of the varieties in themselves. The left square commutes by the analogue of Step (1) of [5], § 4, and the outside rectangle clearly commutes. The desired commutativity of the right square follows from the surjectivity of ι_*, after base extension from $\mathbf{Z}[k]$ to Λ. (One can also prove (iii) by the deformation argument of [5], Appendix 3.)

To prove (iv), choose a (non-equivariant) non-singular variety Q with a locally free sheaf $\mathcal{F}$, and a morphism g: $X \to Q$ such that $g^*\mathcal{F} = \mathcal{E}$ ([4] Appendix § 3.2). Replace Q by the equivariant variety $Q_1 \times \dots \times Q_n$ just as in the proof of (i), and $\mathcal{F}$ by $\mathrm{pr}_n^*(\mathcal{F})$, where pr_n is the projection to the last factor. Thus we may assume g: $X \to Q$ is an equivariant morphism $g^*\mathcal{F} = \mathcal{E}$, but $\mathcal{F}$ is not yet equivariant.

Now let H_i be the vector bundle on Q whose sheaf of sections is

$$\mathcal{H}_i = \mathcal{H}om((q^i)^*\mathcal{F}. (q^{i-1})^*\mathcal{F})$$

for $i=1, 2, ..., n$, and let

$$Y = H_1 \times_Q \dots \times_Q H_n$$

be the fibre product, with projection π: $Y \to Q$. The canonical isomorphisms $\mathcal{H}_i \cong q^* \mathcal{H}_{i-1}$ determine an automorphism of Y permuting the factors H_i, so that π is an equivariant morphism. There is a canonical homomorphism from $q^*\mathcal{J}$ to $\mathcal{J}$ when these sheaves are pulled back to H_1; since π factors through the projection from Y to H_1, this makes $\tilde{\mathcal{E}} = \pi^*\mathcal{J}$ into an equivariant sheaf on Y. Since the homomorphism $\varphi_{\mathcal{E}}$: $x^*\mathcal{E} \to \mathcal{E}$ determines homomorphisms $(x^{i-1})^*(\varphi_{\mathcal{E}})$: $(x^i)^*\mathcal{E} \to (x^{i-1})^*\mathcal{E}$, the mapping g: $X \to Q$ factors through Y, $g = \pi \circ f$, in such a way that $f^*\tilde{\mathcal{E}}$ is equivariantly isomorphic to $\mathcal{E}$.

2.4. Uniqueness

Property 6, which has no analogue in the non-equivariant case, follows immediately from the construction of $L.$. For a general projective X, if ι is the inclusion of $|X|$ in X, it follows from Properties 1 and 6 that $L. \circ \iota_*$ is the homomorphism $K_0^{eq}|X| \to K_0^{abs}|X| \otimes \Lambda$ of § 0.3. Since ι_* becomes an isomorphism after base extension to Λ [11], it follows that $L.$ is the only homomorphism satisfying Properties 1 and 6 of the theorem.

§ 3. Local invariants

3.1. Local complete intersections

Let V be a connected component of $|X|$, and let $L_V X$ be the component of $L.[O_X]$ in $K_0^{abs} V \otimes \Lambda$ (cf. § 0.5). We will describe $L_V X$ in case X is a local complete intersection in a neighborhood of V. By Property 4 of the theorem, $L_V X$ depends only on an equivariant neighborhood of V in X, so we may assume X is itself a local complete intersection, and that $V = |X|$. For any equivariant imbedding of X in a non-singular Y, there is an equivariant homomorphism

$$\mathcal{N} \xrightarrow{d} i^*\Omega_Y^1$$

of locally free sheaves on X; $\mathcal{N}$ is the conormal sheaf to the imbedding i of X in Y. Define

$$\lambda_V X = \lambda_{-1}(i^*\Omega_Y^{1(\times)})/\lambda_{-1}(\mathcal{N}^{(\times)})$$

in $K^0_{abs} V \otimes \Lambda$, and define an integer

$$e_V X = \text{rank } \mathcal{N}^{(1)} - \text{rank } (\iota^*\Omega_Y^{1(1)}) + \dim V,$$

where the superscripts (1) and $(\times)$ denote the trivial and non-trivial eigenspaces of the restrictions of the bundles to V, as in § 1.2. The usual proof of the uniqueness of the cotangent complex ([6], VIII 2.2) extends immediately to the equivariant case to show that these invariants are independent of the imbedding in Y.

PROPOSITION 3.2. (i) $e_V X \geqslant 0$.

(ii) *If* $e_V X = 0$, *then* V *is a local complete intersection in* $|Y|$ *of codimension equal to the rank of* $\mathcal{N}^{(1)}$, *and*

$$L_V X = (\lambda_V X)^{-1} \frown [O_V].$$

(iii) *If* $e_V X > 0$, *then* $L_V X$ *belongs to the submodule of* $K_0^{\text{abs}} V \otimes \Lambda$ *generated by sheaves whose support has dimension* $<$ dim V.

Proof. We first assume there is an equivariant bundle E on Y of rank equal to the codimension of X in Y, and an equivariant section s of E which vanishes precisely (scheme-theoretically) on X. Since this is always the case locally on Y, this case suffices to prove (i), (iii), and the first assertion in (ii), as well as the entire proposition if V is a point.

Let $\mathcal{E}$ be the sheaf of sections of $E^{\vee}$. The section s determines a Koszul complex $\Lambda^{\cdot}\mathcal{E}$ which is an equivariant resolution of O_X on Y. Corresponding to the decomposition $E|_{|Y|} = E^{(1)} \oplus E^{(\times)}$, the restrictions of s to Y decomposes into $s^{(1)} \oplus s^{(\times)}$. Then $s^{(\times)} = 0$ since s is equivariant and $s^{(1)}$ vanishes precisely on $X \frown |Y| = V$. It follows that $\operatorname{codim}(V, |Y|) \leqslant \operatorname{rank} E^{(1)}$. Since $\mathcal{E}|_X = \mathcal{N}$, inequality (i) follows. The Koszul complex $\Lambda^{\cdot}\mathcal{E}$ restricts to the tensor product of $\Lambda^{\cdot}\mathcal{E}^{(\times)}$ and $\Lambda^{\cdot}\mathcal{E}^{(1)}$ on $|Y|$. The complex $\Lambda^{\cdot}\mathcal{E}^{(1)}$ is the Koszul complex determined by the section $s^{(1)}$, while $\Lambda^{\cdot}\mathcal{E}^{(\times)}$ is a complex of locally free sheaves with zero boundary homomorphism.

If $e_V X = 0$, the complex $\Lambda^{\cdot}\mathcal{E}^{(1)}$ is a resolution of O_V on $|Y|$, and (ii) follows from the definition of $L_{\cdot} O_X$. If $e_V X > 0$, however, the alternating sum of the homology of the complex $\Lambda^{\cdot}\mathcal{E}^{(1)}$ is zero when localized at a generic point of a top-dimensional component of V ([13]), from which (iii) follows.

We sketch the proof of the formula in (ii) in the general case. Consider the fibre square

$$\begin{array}{ccc} |X| & \longrightarrow & X \\ \downarrow & & \downarrow \\ |Y| & \longrightarrow & Y \end{array}$$

and deform both inclusions $X \subset Y$ and $|X| \subset |Y|$ to the normal bundles N and N' respectively, by the process of § 1.4. In general we have an inclusion of N' in $N^{(1)}$, but under the assumption $e_V X = 0$, the previous local description shows that $N' = N^{(1)}$. We must therefore show that if we resolve O_X on Y, and restrict to $|Y|$, we have the same alternating sum of homology as when we resolve O_X on N and restrict to N'; in symbols,

$$\sum(-1)^i[\mathrm{Tor}_i^{O_Y}(O_{|Y|}, O_X)] = \sum(-1)^i[\mathrm{Tor}_i^{O_N}(O_{N'}, O_X)]$$

in $K_0^{\mathrm{eq}}|X|$. This is the equivariant analogue of the formula proved in [9] § 6, and it is proved in the same way.

3.3. Isolated fixed points

In case P is an isolated fixed point in X, and X is defined in a non-singular Y by a regular sequence of functions $f_1, \ldots, f_r$, we may assume $y^*(f_i) = b_i f_i$ for some nth roots of unity $b_1, \ldots, b_r$. Let $a_1, \ldots, a_m$ be the eigenvalues, counted with multiplicity, of the induced action on the tangent space to Y at P. Then

$$e_P X = \#\{i \mid b_i = 1\} - \#\{j \mid a_j = 1\}$$

and the local invariant is given by the formula

$$L_P X = \begin{cases} 0 & \text{if } e_P X > 0 \\ \dfrac{\prod_i'([1]-[b_i])}{\prod_j'([1]-[a_j])} \cdot \mathrm{length}\,(O_P|X|) & \text{if } e_P X = 0 \end{cases}$$

where the products are over the indices corresponding to eigenvalues which are not one; we have identified $K_0^{\mathrm{abs}}(P) \otimes \Lambda$ with Λ and used the same notation $[c]$ for an element in $\mathbf{Z}[k]$ as for its image in Λ.

This generalizes the Woods Hole Formula $\det(1 - df_P)^{-1}$ to the singular case. See [11] for examples where $L_P X = 0$.

§ 4. Group actions

4.1. Quotient varieties

In this section G will be a finite group of order n prime to the characteristic of the ground field k; the coefficient ring Λ will be k if $\mathrm{char}(k) = 0$, or the quotient field of the Witt ring of k if $\mathrm{char}(k) \neq 0$.

If G acts trivially on a projective scheme $\mathfrak{X}$, and $\mathfrak{F}$ is a coherent G-sheaf on $\mathfrak{X}$, then

$$n[\mathfrak{F}^G] = \sum [\mathfrak{F}(g, a)] \otimes [a] \tag{1}$$

in $K_0 \mathfrak{X} \otimes \Lambda$. Here $\mathfrak{F}^G$ denotes the G-invariant subsheaf of $\mathfrak{F}$, $\mathfrak{F}(g, a)$ is the subsheaf of $\mathfrak{F}$ on which g acts by multiplication by a; the sum is over all g in G and all nth roots of unity a in k. When $\bar{X}$ is a point, this amounts to a well-known formula for (Brauer) characters of G (cf. [14], Chap. 18.1 (ix)); the general case follows formally from this as in § 0.3.

Now let G act on a projective scheme X, with quotient $\bar{X}=X/G$, and quotient map $\pi\colon X\to\bar{X}$. For each g in G, let X^g be the fixed point subscheme of the action of g on X, and let $\pi^g\colon X^g\to\bar{X}$ be the induced map. Let $L_\cdot^{(g)}$ denote the transformation constructed in § 2 for the endomorphism g on X (or $\bar{X}$). If (1) is applied to the sheaf $\pi_* O_X$, it gives

$$n[O_{\bar{X}}]=\sum_{g\in G} L_\cdot^{(g)}(\pi_* O_X). \tag{2}$$

By the covariance of $L_\cdot^{(g)}$, this implies the formula

$$n[O_{\bar{X}}]=\sum_{g\in G} \pi_*^g L^{(g)}X \tag{3}$$

in $K_0\bar{X}\otimes\Lambda$, where $L^{(g)}X=L_\cdot^{(g)}O_X$ in $K_0(X^g)\otimes\Lambda$. This is similar to the procedure followed by Zagier [16].

If $\tau\colon K_0\bar{X}\to H.\bar{X}$ is one of the Riemann–Roch transformations constructed in [4], [5], with $H.$ either ordinary homology or Chow theory (or topological K-theory), τ may be applied to (3) to give a formula for the homology Todd class (or K-theory orientation class) of $\bar{X}$:

$$\tau(\bar{X})=\frac{1}{n}\sum_{g\in G}\pi_*^g\,\tau(L^{(g)}X). \tag{4}$$

This gives a Riemann–Roch formula for the Euler characteristic of locally free sheaves E on $\bar{X}$ in terms of invariants of the actions of G on X. For example, if the hypotheses of Proposition 3.2 (ii) are satisfied for each g in G, the formula

$$\sum_i(-1)^i\dim H^i(\bar{X},E)=\frac{1}{n}\sum_{g\in G}\int_{X^g} ch(\pi^{g*}E)\smile ch(\lambda_{X^g}X)^{-1}\smile \mathrm{td}\,(T_{X^g}) \tag{5}$$

results. Here X^g is a local complete intersection with virtual tangent bundle T_{X^g}, and the integral takes the degree of the cohomology class of highest codimension.

4.2. Weighted homogeneous varieties

Fix positive integers $m_0, \ldots, m_r$. Grade the polynomial ring $k[z_0, \ldots, z_r]$ by giving z_i the degree m_i. A homogeneous ideal I in this ring has polynomial generators f such that

$$f(t^{m_0}z_0, \ldots, t^{m_r}z_r)=t^d f(z_0, \ldots, z_r)$$

for some d. The multiplicative group $\mathbf{G}_m$ acts on k^{r+1} by $t\cdot(z_0, ..., z_r)=(t^{m_0}z_0, ..., t^{m_r}z_r)$, preserving $V=V(I)$, and the quotient $\bar{X}=V-\{0\}/\mathbf{G}_m$ is a projective scheme:

$$\bar{X} = \text{Proj}\,(k[z_0, ..., z_r]/I).$$

Let φ: $k^{r+1}\to k^{r+1}$ be the map $\varphi(x_0, ..., x_r)=(x_0^{m_0}, ..., x_r^{m_r})$. Then φ^*I is a homogeneous ideal in $k[x_0, ..., x_r]$ with its usual grading, and hence defines a projective scheme X; φ induces a morphism π: $X\to\bar{X}$.

Let $G=\mu_{m_0}\times ... \times\mu_{m_r}$ be the product of the groups of m_ith roots of unity, acting as usual on k^{r+1}; φ may be identified with the quotient map from k^{r+1} to k^{r+1}/G. Then G acts on X, and $\bar{X}=X/G$, with quotient map π.

If I is generated by a regular sequence of weighted homogeneous polynomials, then X will be a complete intersection in $\mathbf{P}^r$. The fixed point subschemes of the various g in G are the intersections of X with projective subspaces of $\mathbf{P}^r$ obtained by setting some of the coordinate functions equal to zero. The calculation of the $L^{(g)}X$ may be made as explicitly as desired.

For example, if I is generated by a single weighted homogeneous polynomial f, and $f(P_i)\neq 0$, where P_i is the point with a 1 in the ith place and 0 elsewhere, then each of the coordinate subspaces meets X properly, and the last formula of § 4.1 holds. In particular f may be the "Brieskorn polynomial"

$$f=z_0^{a_0}+z_1^{a_1}- \dots +z_r^{a_r}.$$

In this case, when $k=\mathbf{C}$, Atiyah [1] has an interesting Riemann–Roch formula identifying the Euler characteristics of the sheaves $O_{\bar{X}}(m)$ with the index of an elliptic operator on the Brieskorn manifold obtained by intersecting V with a sphere S^{2r+1}.

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