QUADRATIC DIFFERENTIALS AND FOLIATIONS

BY

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Introduction

This paper concerns the interplay between the complex structure of a Riemann surface and the essentially Euclidean geometry induced by a quadratic differential.

One aspect of this geometry is the "trajectory structure" of a quadratic differential which has long played a central role in Teichmüller theory starting with Teichmüller's proof of the existence and uniqueness of extremal maps. Ahlfors and Bers later gave proofs of that result. In other contexts, Jenkins and Strebel have studied quadratic differentials with closed trajectories.

Starting from the dynamical problem of studying diffeomorphisms on a C^∞ surface M, Thurston [17] invented *measured foliations*. These are foliations with certain kinds of singularities and an invariantly defined transverse measure. A precise definition is given in Chapter I, § 1. This notion turns out to be the correct abstraction of the trajectory structure and metric induced by a quadratic differential. In this language our main statement says that *given any measured foliation F on M and any complex structure X on M, there is a unique quadratic differential on the Riemann surface X whose horizontal trajectory structure realizes F.* In particular any trajectory structure on one Riemann surface occurs uniquely on every Riemann surface of that genus.

In the special case when the foliation has closed leaves, an analogous theorem was proved by Strebel [15]. Earlier Jenkins [13] had proved that quadratic differentials with closed trajectories existed as solutions of certain extremal problems. We deduce Strebel's theorem from ours in Chapter I, § 3.

By identifying the space of measured foliations with the quadratic forms on a fixed Riemann surface, we are able to give an analytic and entirely different proof of a result of Thurston's [17]; that the space of projective classes of measured foliations is homeomorphic to a sphere. This is also done in Chapter I, § 3.

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An outline of the proof of the main theorem was published in [12] but we stated the theorem only for foliations with closed leaves. In fact, this paper grew out of an attempt to find a more geometric proof of Strebel's theorem.

Then in April 1976 we heard Thurston lecture on measured foliations and diffeomorphism of surfaces and immediately realized our proof extended to any measured foliation.

Let F be any measured foliation and let Q be the vector bundle over Teichmüller space of all quadratic differentials and let $E_F \subset Q$ be those which induce F. The main ingredients in the proof are showing that E_F is nonempty and that E_F maps by a local homeomorphism to Teichmüller space. To do the latter we use the implicit function theorem and thus we need to give equations for E_F. This is fairly easy near a quadratic differential with simple zeroes, but multiple zeroes introduce major complications. What is needed is a detailed local description of the deformations of multiple zeroes. A detailed outline appears in Chapter I, § 2.

We would like to thank the numerous people who have helped us while we wrote this paper. In particular, D. Coppersmith helped with the topological structure of E_k, D. Mumford and B. Mazur with the deformation theory and F. Laudenbach and Fati with the topology of measured foliations.

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CHAPTER I

Statement and applications of the main theorem

§ 1. Measured foliations

Every holomorphic quadratic form on a Riemann surface induces a measured foliation; in this paragraph we will define this concept. The definition closely follows Thurston's. A more detailed treatment will be given in Chapter II.

Let M be a compact C^∞ surface of genus $g>1$, without boundary. A *measured foliation* F on M with singularities of order $k_1, ..., k_n$ at $x_1, ..., x_n$ is given by an open cover U_i of $M-\{x_1, ..., x_n\}$ and a non-vanishing C^∞ real-valued closed 1-form φ_i on each U_i such that

(a) $\varphi_i = \pm\varphi_j$ on $U_i \cap U_j$.

(b) At each x_i there is a local chart (u, v): $V \to \mathbf{R}^2$ such that for $z = u+iv$, $\varphi_i = \operatorname{Im}(z^{k_i/2}dz)$ on $V \cap U_i$, for some branch of $z^{k_i/2}$ in $U_i \cap V$.

Such a pair (U_i, φ_i) is called an *atlas* for F.

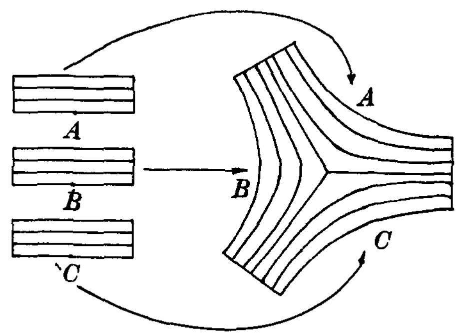


Fig. 1

Example. Let X be a Riemann surface, and q a holomorphic quadratic form on X, vanishing at $x_1, ..., x_n$ to the order $k_1, ..., k_n$. Pick an open cover of $X - \{x_1, ..., x_n\}$ by simply connected sets, and in each one set $\varphi = \operatorname{Im} \sqrt{q}$, for some branch of the square root. Near x_i a local chart z in which $q = z^{k_i} dz^2$ satisfies condition (b). Holomorphic local coordinates z in which $q = z^{k_i} dz^2$, $k_i \geqslant 0$, are called *canonical coordinates* and always exist. The foliation induced by q is denoted F_q.

It is almost but not quite true that all measured foliations are of the type above (cf. Chapter II, § 2).

Away from the singularities a measured foliation clearly induces an ordinary foliation, tangent in U_i to the vectors in the kernel of φ_i. The leaves will be leaves in the ordinary sense (i.e., maximal connected subsets of $M - \{x_1, ..., x_n\}$ for the topology which in each open set U_i has as connected subsets the fibers of the map $U_i \to \mathbf{R}$, $x \mapsto \int_{x_0}^{x} \varphi_i$). However if a leaf emanates from a singularity, then we include the singularity in the leaf.

The *measure* is the line element $|\varphi|$ induced in each U_i by $|\varphi_i|$; condition (a) guarantees the measure is well-defined; we will say that it measures a transverse length since it vanishes on vectors tangent to the leaves.

Near a singular point of order k, a model for the foliated surface can be built by taking $k+2$ rectangles $[-1, 1] \times [0, b] \subset \mathbf{R}^2$, foliated by dy, and gluing them together according to the pattern in Figure 1.

A leaf of F is called *critical* if it contains a singularity of F. The union of the compact critical leaves is called the *critical graph* denoted by Γ. An isolated multiple zero is considered part of Γ.

Let F be a measured foliation on M, defined by forms φ_i on U_i, and γ: $[a, b] \to M$ a C^1 curve. Define $l_F(\gamma) = \int_a^b |\varphi|(\gamma'(t))\,dt$.

If γ is a simple closed curve on M, define $F(\gamma)$ to be the infimum of all $l_F(\gamma_1)$ for γ_1 freely homotopic to γ.

Let S be the set of homotopy classes of simple closed curves on M. The construction above gives a map from the set of measured foliations on M to $\mathbf{R}^S$; we will call two measured foliations equivalent if their images coincide. This equivalence is clearly coarser than isotopy; we shall see that it is the finest equivalence relation coarser than isotopy with a Hausdorff set of equivalence classes.

In fact, we will show in the course of the paper that the set $\mathfrak{F}_M \subset \mathbf{R}^S$ of equivalence classes of measured foliations on M is homeomorphic to $\mathbf{R}^{6g-6} - \{0\}$. This was first proved by Thurston [17].

§ 2. The main result

Let Θ_M be the Teichmüller space of genus g. Consider the vector bundle $p: Q \to \Theta_M$ whose fiber above a point $(X, f) \in \Theta_M$ is the space $H^0(X, \Omega^{\otimes 2})$ of holomorphic quadratic forms on X. The union of these spaces can be given the structure of a vector bundle either by using the Serre duality theorem to claim that $H^0(X, \Omega^{\otimes 2})$ is the dual of the tangent space to Θ_M at (X, f) (cf. [11], Chapter IV, § 1, or [6]), and thus that Q is the cotangent bundle to Θ_M, or by invoking Grauert's direct image theorem (cf. [11] for the special case needed here).

Given any nonzero $q \in Q$ above (X, f), we can consider $f^* F_q \in \mathfrak{F}_M$. If $\mathbf{0}$ denotes the zero section of Q, the construction above defines a map $Q - \mathbf{0} \to \mathfrak{F}_M$. For any $F \in \mathfrak{F}_M$, let $E_F \subset Q - \mathbf{0}$ be the fibre above F.

MAIN THEOREM. *The restriction $E_F \to \Theta_M$ of p to E_F is a homeomorphism.*

Chapters II–IV are devoted to the proof of this theorem. We will proceed in the following steps:

(i) E_F is not empty (II, § 2)
(ii) $p|_{E_F}$ is proper (II, § 7)
(iii) $p|_{E_F}$ is injective (IV, § 7)
(iv) $p|_{E_F}$ is open (IV, § 1 and 5).

Chapter II is essentially concerned with the topology of measured foliations; many of the results are due to Thurston and are contained, explicitly or implicitly, in [17].

Chapter III is a study of the deformations of a multiple zero of a quadratic form, and is preliminary to Chapter IV.

Chapter IV is primarily concerned with finding equations for E_F in Q. This works well in a neighborhood of a quadratic form which is not the square of a 1-form, but the case of a square introduces serious difficulties that require § 2–5. Point (iii) above then follows from combining Corollary II.9, a density statement analogous to that in [5] and the Strebel Uniqueness Theorem [16].

§ 3. Applications

A holomorphic quadratic form q is called Strebel if its horizontal foliation has closed leaves. In that case the complement of the critical graph is a union of metric straight cylinders, with respect to the metric $|q|^{1/2}$, each swept out by homotopic leaves. The leaves in different cylinders are not homotopic.

Conversely, let $\mathcal{C}$ be a system of n simple closed curves on M, disjoint, not pairwise homotopic, and homotopically nontrivial. Let $E_{\mathcal{C}} \subset Q$ be the space of Strebel forms whose associated system of curves is homotopic to $\mathcal{C}$. Denote $\Pi: E_{\mathcal{C}} \to \Theta_M \times \mathbf{R}^n_+$ the map whose first factor is the canonical projection p restricted to $E_{\mathcal{C}}$ and whose second factor gives the heights of the cylinders.

THEOREM 2. *The map $\Pi: E_{\mathcal{C}} \to \Theta_M \times \mathbf{R}^n_+$ is a homeomorphism.*

This theorem was announced in [10].

THEOREM 3. (Strebel [15, 16], Jenkins [13]). *Let X be a compact Riemann surface and let $\mathcal{C}$ be a system of curves as above. Let $m_1, \ldots, m_n$ be positive real numbers. Then there exists a Strebel form q on X whose associated system of curves is homotopic to $\mathcal{C}$ and such that the ratio M_i of height to circumference (modulus) of each cylinder satisfies $M_i = Km_i$ where K is a constant independent of i. Furthermore q is unique up to a positive real multiple.*

Strebel also proved that q varies continuously with the numbers m_i, a fact which is close to the part of Theorem 2 which states that q varies continuously with the heights. Both these theorems and the next will be proved in Chapter IV.

Finally we give a new proof of a result announced by Thurston.

THEOREM 4. (Thurston [17].) *The set $\mathfrak{F}_M \subset \mathbf{R}^S$ is homeomorphic to $\mathbf{R}^{6g-6} - \{0\}$.*

Remark. The quotient $\mathbf{P}\mathfrak{F}_M$ of $\mathfrak{F}_M$ by the positive reals acting by multiplication may be identified with the unit sphere in the space of quadratic differentials on any fixed compact Riemann surface. Thurston states in [17] that $\mathbf{P}\mathfrak{F}_M$ forms a boundary for Teichmüller space in a natural way. By Teichmüller's theorem the sphere in the space of quadratic differentials also forms a boundary for Teichmüller space (depending on a choice of base-point). Kerckhoff [14] has shown that these topologies on the union of Teichmüller space and $\mathbf{P}\mathfrak{F}_M$ do not coincide.

CHAPTER II

Measured foliations and their realizations

§ 1. The orientation cover of a foliation

Let F be a measured foliation on M, with singular points $x_1, \dots, x_n$ of multiplicity $k_1, \dots, k_n$. We will construct a double cover $\tilde{M}_F$ of M ramified at the singular points of odd multiplicity, whose points above x correspond to the two orientations of F at x. In various guises, the surface $\tilde{M}_F$ will be an essential tool throughout this paper.

Let F be defined by the forms φ_i on U_i, and let $U = M - \{x_1, \dots, x_n\}$. Consider the subset of the cotangent bundle T^*U of all $\pm\varphi_i(x)$, which clearly forms an unramified double cover of U. The cover is trivial near x_i if and only if k_i is even so we may compactify it forming $\tilde{M}_F$ by adding one point above x_i if k_i is odd and two points if k_i is even. Call $\pi: \tilde{M}_F \to M$ and $\tau: \tilde{M}_F \to \tilde{M}_F$ the canonical projection and involution.

On $\tilde{M}_F$, the measured foliation π^*F is defined by the "tautological" closed form φ, with zeroes only at the $\pi^{-1}(x_i)$. It is easy to check that the index of such a zero is $k_i/2$ at both of the points in $\pi^{-1}(x_i)$ if k_i is even, and k_i+1 at the point $\pi^{-1}(x_i)$ if k_i is odd.

A parametrized curve $\gamma: [a, b] \to \tilde{M}_F$ is *increasing* if $\varphi(\gamma'(t)) > 0$ for all $t \in (a, b)$.

Remark. The surface $\tilde{M}_F$ may have two connected components. This occurs precisely if F is orientable, i.e. F is defined by a global closed one-form.

The following result is a first use of $\tilde{M}_F$.

PROPOSITION 2.1. *Every measured foliation on M has $4g-4$ singularities counting multiplicities.*

Proof. Suppose $k_1, \dots, k_m$ are odd and $k_{m+1}, \dots, k_n$ are even. The Riemann–Hurwitz formula gives

$$\chi(\tilde{M}_F) = 2(2-2g) - m.$$

On the other hand, the sum of the indices of the zeroes of φ is

$$\sum_{i=1}^{m} (k_i + 1) + 2 \sum_{j=m+1}^{n} k_j/2.$$

By the Hopf index theorem for forms,

$$\sum_{i=1}^{m} (k_i + 1) + 2 \sum_{j=m+1}^{n} k_j/2 = -\chi(\tilde{M}_F) = 2(2g-2) + m. \qquad \text{Q.E.D.}$$

This result agrees of course with the fact that a quadratic differential has $4g-4$ zeroes counting multiplicities.

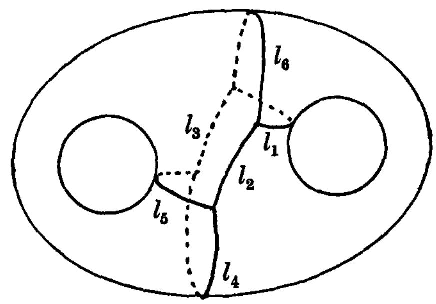


Fig. 2

§ 2. Realizable foliations

It turns out that there are measured foliations which are not given by a holomorphic quadratic form.

Example. Take two cylinders, foliated by horizontal circles, and with the measure given by the height function, and glue them together according to the pattern in Figure 2. If this measured foliation were induced by a quadratic form q, the cylinders would be straight metric cylinders for the metric $|q|^{1/2}$; in particular the top and the bottom of each would have the same length. If one writes the corresponding equations for the lengths in Figure 2, one gets

$$l_1+l_6=l_1+l_2+l_3+l_4$$
$$l_4+l_5=l_2+l_3+l_5+l_6.$$

This system has no positive solutions.

Of course, there are equivalent metric foliations which are induced by quadratic forms; for instance that obtained by collapsing l_2 and l_3 to points. The object of paragraph 2 is to show that this is always the case.

If γ is a critical segment of F (i.e. a compact critical leaf which is an interval, not a circle), we can choose a map $f\colon M\to M$ homotopic to the identity, which is a diffeomorphism on $M-\gamma$ and collapses γ to a point x. The measured foliation f_*F obtained from F by collapsing γ is defined by the open cover $f(U_i-\gamma)$ with the 1-forms $(f^{-1})^*\varphi_i$. If x_1 and x_2 are the endpoints of γ, of order k_1 and k_2 respectively, it is easy to check that the point $x=f(\gamma)$ becomes a singularity of order k_1+k_2.

Clearly f_*F is equivalent to F; we shall see in Chapter IV that the equivalence relation we have put on measured foliations is the minimal one under which isotopic foliations and those related by the collapse of a critical segment are equivalent. In the mean time we will call this minimal equivalence relation *strong equivalence*.

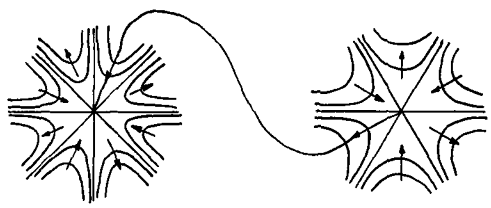


Fig. 3

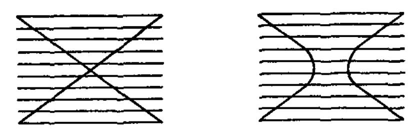


Fig. 4

Given any measured foliation F, we shall show that there is a strongly equivalent one which is induced by a holomorphic quadratic form q (for some complex structure on M): we will say that q *realizes* F.

Let F be a measured foliation on M, and φ the closed form defining π^*F on $\tilde{M}_F$. For any two points x and y in $\tilde{M}_F$ at which φ does not vanish, we say x *leads* to y if there exists an increasing curve γ: $[0, 1] \to \tilde{M}_F$ such that $\gamma(0)=x$, $\gamma(1)=y$.

PROPOSITION 2.2. *Let F be a measured foliation on M. The following conditions are equivalent:*

(a) *F can be realized by a quadratic form on M, holomorphic for some complex structure on M.*

(b) *F can be realized by a q as above, whose vertical foliation is Strebel.*

(c) *Every point x leads to every point y in the same connected component of $\tilde{M}_F$.*

Proof. (b) implies (a) is obvious. To see that (c) implies (b), suppose first that $\tilde{M}_F$ is connected. Pair up the sectors in M at all the singular points, and for each pair pick for one the sector above it in $\tilde{M}_F$ in which increasing curves leave the singularity, and for the other the sector above it in $\tilde{M}_F$ in which increasing curves go to the singularity (Figure 3). For each such pair of sectors, choose an increasing curve γ on $\tilde{M}_F$ joining their singularities, starting in one sector and ending in the other. Consider the images of these curves in M; these are transverse to F. If any of these curves intersect (even themselves) at nonsingular points, they can be changed so as to be disjoint and simple and still transverse by cutting them and reconnecting them as suggested by Figure 4. In the process we may create some simple closed curves avoiding the singularities; if so, erase them.

Let $\Gamma' \subset M$ be the graph formed by all the curves drawn. Then M cut along Γ' consists of surfaces with boundary, with a foliation transverse to the boundary, and without singularities. Then the double of each component must be a torus by Proposition 2.1, and so each component must be an annulus.

For each such annulus, pick a measured foliation tangent to the boundary, and transverse to the original foliation. The two foliations together define charts $M \to \mathbf{R}^2$ away from the singularities, and these charts are the canonical coordinates for a unique quadratic form on M, holomorphic for that structure.

In case $\tilde{M}_F$ is not connected, F is defined by a closed form φ on M which orients F. At each singularity of F, the sectors fall into two classes according to whether increasing curves leave or arrive in that sector, and there are the same number of sectors in each class. Thus they can be paired up, and the proof continues as above. This shows that (c) implies (b).

Finally, let q be holomorphic quadratic form on the Riemann surface $X = M$, and x, y two points in the same connected component of $\tilde{M}_F$ such that x does not lead to y. Consider the set N of points to which x does lead. Then it is easy to see that $N - \pi^{-1}\{x_1, ..., x_n\}$ is an open subset of $\tilde{M}_F - \pi^{-1}\{x_1, ..., x_n\}$, whose boundary in $\tilde{M}_F$ is a union of closed leaves.

Clearly $\pi^* q$ is the square of a complex valued 1-form ω_q an $\tilde{M}_F$, such that $\varphi = \operatorname{Im} \omega_q$. Define a vector field χ on $\tilde{M}_F$ by $\omega_q(\chi) = i$. This vector field has poles at the zeroes of ω_q, so it only generates a flow almost everywhere, i.e., on the complement of the critical vertical leaves. Since it points into N everywhere along the boundary, this almost everywhere defined flow sends N into its interior. This is incompatible with the fact that it preserves the measure $|\omega_q|^2$. Q.E.D.

We now come to the main result of this chapter.

THEOREM 2.3. *For any measured foliation F on M, there is a strongly equivalent F' which can be realized by a quadratic form.*

Proof. Without loss of generality, we can suppose that F has only ordinary singularities, so that $\tilde{M}_F$ is connected.

A non-empty open subset N of $\tilde{M}_F - \{x_1, ..., x_n\}$ will be called *stable* if $y \in N$ whenever there is an $x \in N$ which leads to y. A stable subset is *minimal* if it contains no smaller stable subset. It is easy to check that except at singularities the closures of stable subsets are submanifolds with boundary of $\tilde{M}_F$, and that the boundary is a subset of $\tilde{\Gamma}$. Moreover, every point of a minimal stable leads to every other point.

LEMMA 2.4. *Either there is only one minimal subset $\tilde{M}_F$, or $\pi: \tilde{M}_F \to M$ maps the union of the minimal subsets injectively onto M. The first case occurs if and only if F is realizable.*

Proof. The last statement follows immediately from Proposition 2.2. Suppose N_1 and N_2 are two minimal subsets, and that $x \in \tau(N_1) \cap N_2$. Then for any $y \in N_1$, y leads to $\tau(x)$, so x leads to $\tau(y)$, so $\tau(N_1) \subset N_2$. By symmetry, $N_1 = \tau(N_2)$. But this is clearly impossible if the boundary of N_1 is not empty, i.e. if $N_1 \neq \tilde{M}_F$. Q.E.D.

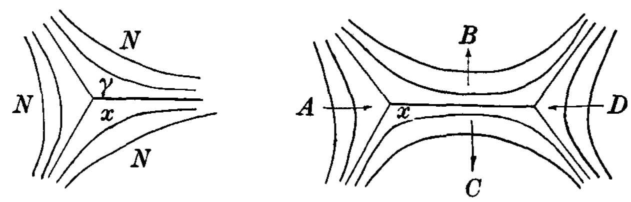


Fig. 5

This lemma suggests that we should try to simplify the set of minimal subsets for F. In order to make this precise, we shall say that one foliation F with minimal subsets $N_1, \ldots, N_n$ is *better* than another F' with minimal subsets $N'_1, \ldots, N'_m$ if $n<m$, or $n=m$ and there are more elements of $\pi_0(\tilde{M}_F-\tilde{\Gamma})$ which are subsets of $\bigcup_i N_i$ than there are elements of $\pi_0(\tilde{M}_F-\tilde{\Gamma}')$ which are subsets of $\bigcup_i N'_i$, or these numbers also are equal and there are more segments of $\tilde{\Gamma}$ contained in $\bigcup_i N_i$ than there are segments of $\tilde{\Gamma}'$ contained in $\bigcup_i N'_i$. This rather cumbersome definition does have the property that if we can make a foliation better by contracting and expanding appropriate segments of its critical graph, then iterating the process will eventually make the foliation realizable.

Suppose F is a measured foliation on M, and that N is a minimal subset of $\tilde{M}_F$, with $N \neq \tilde{M}_F$. Then there must be a segment γ of Γ with extremity x such that near x, $\pi(N) = M-\gamma$ (see Figure 5). Indeed, if all other sorts of singularities were of another type, N could be contracted into itself, contradicting minimality. Let F' be the foliation obtained by collapsing γ and expanding it the other way; we shall show that F' is better than F. Call A, B, C, D the components of $M-\Gamma$ near γ, and let $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ by those corresponding components of $\tilde{M}_F-\tilde{\Gamma}$ such that $\tilde{A}, \tilde{B}, \tilde{C} \subset N$, and $\tilde{D}$ is connected to $\tilde{B}$ and $\tilde{C}$ in $\tilde{M}_F-\pi^{-1}(\gamma)$. Label I, II, III the cases when neither $\tilde{D}$ or $\tau(\tilde{D}) \subset N$, $\tilde{D} \subset N$ and $\tau(\tilde{D}) \subset N$.

(a) In cases I and II, suppose that $\tau(D)$ leads to a minimal subset $N' \neq N$. It is then easy to see that for F' the minimal subset N disappears (it empties into N'), and that no new minimal subset is created, so in this case F' is better than F.

(b) In cases I and II, suppose $\tau(D)$ leads only to N. Let P be the set of points to which $\tau(D)$ does lead. Call P' the corresponding subset for F' and let N' be a minimal stable subset contained in P'. Then either $N' \supset N$ or $N' \supset \tau(N)$, for otherwise N' was a minimal subset for F. In particular N' is the only minimal subset of P'. If $N' \supset N$, then $N' \neq N$ since N is not stable for F', so N' is strictly larger than N and we are done. If $N' = \tau(N)$ and $\tau(D)$ leads to N, N' contains points of N and $\tau(N)$ which is impossible unless $N' = \tilde{M}_F$.

(c) Case III is easier: For F' the subset N is still a minimal subset, which contains one more segment of $\tilde{\Gamma}$ than before, and everything else is unchanged. Q.E.D.

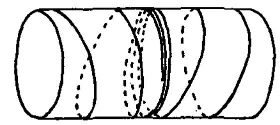


Fig. 6

§ 3. Quasi-transversal curves

On the surface M consider a measured foliation F defined by an atlas (U_i, φ_i), and a closed curve γ: $[0, 1]\to M$. Define the transverse length of γ to be

$$l_F(\gamma) = \int_0^1 |\varphi_i(\gamma'(t))|\, dt.$$

Define $F(\gamma) = \inf l_F(\gamma_1)$, where the inf is taken over all curves γ_1 homotopic to γ.

It is not quite clear that this inf is actually realized; since travel along the leaves of the foliation is free, it is conceivable that a very long path homotopic to the original one might have arbitrarily small transverse length.

Example. Consider the one form on a cylinder defined as the dot product with a unit vector field perpendicular to the vector field whose integral curves have the equator as limit cycle, as in Figure 6. Given points x and y on the two boundary components and any homotopy class of paths between them, the inf of the transverse lengths of curves from x to y in that homotopy class is zero, even though no curve realizes it.

Of course, in the example above, the form is not closed. The object of Proposition 2.5 is to show that such phenomena cannot happen for measured foliations. For this we need the concept of quasitransversal curves defined for curves that are immersions except possibly at the singularities. A closed curve γ: $S^1\to M$ is *quasitransversal to* F if at every point $t\in S^1$ either $\gamma(t)$ is a singularity of F or γ is locally near t transversal to F or an inclusion into a leaf of F. If $\gamma(t)$ is a singularity, then at least an open sector on both sides must separate the incoming and the outgoing parts of the curve. In particular, at a simple singularity, if γ comes along one critical leaf it either leaves by another, or it leaves transversally in the opposite sector.

PROPOSITION 2.5.

(a) *Every closed curve is homotopic to a quasi-transversal one.*

(b) *If γ is quasi-transversal, $l_F(\gamma) = F(\gamma)$.*

(c) *If γ_1 and γ_2 are two homotopic quasi-transversal simple closed curves, then either they are both entirely formed of leaves and are homotopic among such curves, or they include the same leaves and each transversal part of γ_1 is homotopic with endpoints fixed and through transversal curves to a transversal part of γ_2.*

Proof. For part (a) suppose first that F is induced by a quadratic differential q. Then the geodesic homotopic to γ for the metric $|q|^{1/2}$ is a quasi-transversal curve except at those singularities where it enters and leaves in adjacent open sectors. A small pertrubation near these points makes it quasi-transversal. (For details about the metric $|q|^{1/2}$ and its geodesics see [1], [3], [10].) By Theorem 2.3 all we need to show is that if γ is quasi-transverse to F and F' is obtained from F by collapsing or expanding a critical segment, then there is a γ' homotopic to γ which is quasi-transversal to F'. A few drawings will convince the reader that this is so.

(b) If $l_F(\gamma)=0$ we are done, so suppose not. Consider the covering space M_γ in which curves homotopic to γ are the only simple closed curves; this covering space is homeomorphic to an open cylinder, with γ as an equator. In this covering space any non-critical leaf intersects γ transversally at most once. Indeed, if a leaf intersects γ transversally twice, then the portion of the leaf between the intersections together with a segment of γ bound a disc. Doubling this disc along the quasi-transversal segment gives a foliated disc with the boundary a leaf. This is impossible. Thus every leaf which intersects γ transversally either is critical or goes from one end of the cylinder M_γ to the other. Let γ' be a curve homotopic to γ, so it can be lifted to a closed curve on M_γ. Then every non-critical leaf intersecting γ must intersect γ', and it is clearly possible to choose such an intersection point in a piece-wise continuous way. This defines a piece-wise continuous map of the non-critical portions of γ to γ' which is an isometry of γ onto a subset of γ'.

(c) Keeping the notations above, suppose now that γ' also is quasi-transversal. Let x be an extremity of a leaf $\alpha\subset\gamma$, at which x becomes transversal. Then one of leaves emanating from x must intersect γ'; suppose it does so at a point $x'\neq x$. Define similarly y and y' for the other end of the leaf α. Then the quadrilaterial formed of α, the leaves xx' and yy' and an appropriate segment of γ' is bounded by a leaf and a quasi-transversal segment, which is impossible, as above. So $x=x'$, $y=y'$ and α is included in both γ and γ'. Thus the leaf segments of γ and γ' coincide, and sliding along leaves provides the desired homotopy between the transversal segments.

Remark. If γ is a simple closed curve, it may be impossible to choose a quasi-transversal curve homotopic to γ *which is simple.*

§ 4. The set S and $H_1(\tilde{M}_F, \tilde{\Gamma}_F)^-$

In this paragraph all homology groups will have real coefficients. If V is a vector space with an involution τ, we will denote V^- the odd part of V, i.e. $V^-=\ker(\tau+\mathrm{Id}\colon V\to V)$.

For any element $\gamma\in S$, define $\tilde{\gamma}\in H_1(\tilde{M}_F, \tilde{\Gamma}_F)$ in the following way: replace γ by a quasi-transversal curve in its homotopy class and orient the transversal segments of $\pi^{-1}(\gamma)$ so that they are increasing. The sum of these oriented segments is a singular one-chain on $\tilde{M}_F$ whose boundary is in $C_0(\tilde{\Gamma}_F)$, so the one-chain defines a class $\tilde{\gamma} \in H_1(\tilde{M}_F, \tilde{\Gamma}_F)$ which is well-defined by Proposition 2.5. Clearly $\tau_* \tilde{\gamma} = -\tilde{\gamma}$ so $\tilde{\gamma} \in H_1(\tilde{M}_F, \tilde{\Gamma}_F)^-$.

Let $T \subset S$ be the set of homotopy classes of curves admitting a quasi-transversal representative $\gamma \in M - \Gamma$. If $F(\gamma) > 0$ the construction above gives $\tilde{\gamma} \in H_1(\tilde{M}_F - \tilde{\Gamma}_F)^-$; if $F(\gamma) = 0$ and γ is quasi-transversal then γ is an equator of a cylinder foliated by closed curves: define $\tilde{\gamma} \in H_1(\tilde{M}_F - \tilde{\Gamma}_F)^-$ to be $\pi^{-1}(\gamma)$ oriented so that increasing curves cut it from right to left.

PROPOSITION 2.6.

(i) *The classes* $\tilde{\gamma}$ *for* $\gamma \in S$ *generate* $H_1(\tilde{M}_F, \tilde{\Gamma}_F)^-$.
(ii) *The classes* $\tilde{\gamma}$ *for* $\gamma \in T$ *generate* $H_1(\tilde{M}_F - \tilde{\Gamma}_F)^-$.

Proof. First replace F by an equivalent foliation which is realizable and has simple singularities. This is possible, because, if $f: M \to M$ is a map collapsing a critical segment and $F' = f_* F$, then $f_*: H_1(\tilde{M}_F, \tilde{\Gamma}_F)^- \to H_1(\tilde{M}_{F'}, \tilde{\Gamma}_{F'})^-$ is an isomorphism, and the lifts $\tilde{\gamma}$ and $\tilde{\gamma}'$ of γ for F and F' respectively correspond under f.

Now the exact sequence

$$H_1(\tilde{\Gamma}_F) \to H_1(\tilde{M}_F) \to H_1(\tilde{M}_F, \tilde{\Gamma}_F) \to H_0(\tilde{\Gamma}_F)$$

gives a surjective map $H_1(\tilde{M}_F)^- \to H_1(\tilde{M}_F, \tilde{\Gamma}_F)^-$ since each component of Γ contains a simple singularity, so its inverse image in $\tilde{M}_F$ is connected and τ_* is the identity on $H_0(\tilde{\Gamma}_F)$.

Pick a simple closed curve α on $\tilde{M}_F$ and set $\beta = \alpha - \tau(\alpha)$. The classes of such curves β generate $H_1(\tilde{M}_F, \tilde{\Gamma}_F)^-$. If α had been put in general position with respect to $\tau(\alpha)$ avoiding the singularities, the oriented curve β may fail to be simple but will have transverse self intersections. The trick of reconnecting the segments at intersections as in Figure 4 does not change the homology class. Thus we can suppose that $\beta = \beta_1 \cup \beta_2 \cup \ldots \cup \beta_m$ where the β_i are disjoint embedded curves and each one has two connected components which are reversed by τ.

Without loss of generality, we can suppose that $\gamma = \pi(\beta)$ is connected and simple, that γ is formed of a sequence of transverse curves, and contains no singularities, i.e. $\gamma = \delta_1 * \eta_1 * \ldots * \delta_n * \eta_n$ where the δ_i and η_i are transverse segments and $*$ denotes juxtaposition. Moreover we can suppose that in β the $\pi^{-1}(\delta_i)$ are increasing and the $\pi^{-1}(\eta_i)$ are decreasing.

For each η_i pick a transverse curve η_i' with the same endpoints as η_i such that $\eta_i * \eta_i'$ is a transversal closed curve. This is possible by Proposition 2.2 (c). Now the curve $\gamma' = \delta_1 * \eta_1 * \ldots * \delta_n * \eta_n'$ is transversal and $\tilde{\gamma}'$ differs from β by $\sum_i \eta_i \tilde{*} \eta_i'$.

Another use of the disconnecting and reconnecting argument will turn γ' and the $\eta_i * \eta_i'$ into unions of simple closed curves. The simple closed curves cannot bound a disc as that would result in a foliation of the disc with transverse boundary. This proves (i).

For part (ii), it is sufficient to prove the result for each connected component of $M-\Gamma$. These are of two types: open cylinders foliated by compact leaves, for which the result is trivial, and open foliated surfaces with no closed leaves. For these the proof of part (i) works verbatim, except that the existence of the curves η_i' needs a different justification.

Let N denote one such component of $M-\Gamma$. We claim that every point $\tilde{N}_F$ leads to every other in the same component. Indeed if not, the set of points in $\tilde{N}_F$ to which a given point does lead is a submanifold of $\tilde{N}_F$ with boundary, and this boundary must consist of closed leaves, of which there are none. It is clear that the endpoints of any η_i, lifted to $\tilde{N}_F$ so that η_i leads from one to the other, are in the same component of $\tilde{N}_F$, therefore their images by τ are also in the same component, and there is a path η_i' leading from one to the other. Q.E.D.

§ 5. Poincaré duality and $H_1(\tilde{M}_F, \tilde{\Gamma}_F)^-$

In this paragraph we begin to show that if two measured foliations are equivalent they are strongly equivalent. We need to extract some information about a measured foliation F from its image in $\mathbf{R}^S$. Specifically, we will "synthesize" $H_1(\tilde{M}_F, \tilde{\Gamma}_F)^-$.

Recall that $\mathbf{R}^{(S)} \subset \mathbf{R}^S$ is the set of maps $S \to \mathbf{R}$ with finite support, i.e. finite linear combinations of elements of S.

The idea is to find the kernel of the map $\mathbf{R}^{(S)} \to H_1(\tilde{M}_F, \tilde{\Gamma}_F)^-$ defined by $\gamma \to \tilde{\gamma}$, a map which we have seen to be surjective. By Poincaré–Alexander duality [9] the algebraic intersection number gives a non-singular pairing of $H_1(\tilde{M}_F, \tilde{\Gamma}_F)$ with $H_1(\tilde{M}_F - \tilde{\Gamma}_F)$, noted $(\alpha\beta) \to \alpha \cdot \beta$, and this is still true of the odd parts, as the odd part of one is orthogonal to the even part of the other. Let $T \subset S$ be as above the set of homotopy classes α such that $\alpha \subset \tilde{M}_F - \tilde{\Gamma}_F$. Using again Proposition 2.6, the argument above can be restated as follows:

PROPOSITION 2.7. *The kernel of the canonical map* $\mathbf{R}^{(S)} \to H_1(\tilde{M}_F, \tilde{\Gamma}_F)^-$ *is the kernel of the map* $\mathbf{R}^{(S)} \to \mathbf{R}^T$ *defined by* $\gamma \to (\gamma' \to \tilde{\gamma} \cdot \tilde{\gamma}')$.

In order to use this proposition, we need to extract from the image of F in $\mathbf{R}^S$ the following information.

(a) When is an element of S in T?

(b) If $\gamma_1 \in S$ and $\gamma_2 \in T$, what is $\gamma_1 \cdot \gamma_2$?

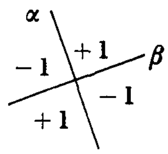


Fig. 7

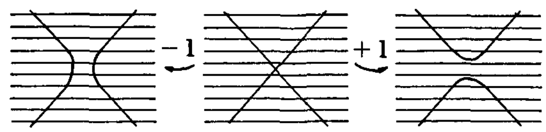


Fig. 8

Let S' be the set of homotopy classes of finite disjoint unions of simple closed curves. Recall [17] that the geometric intersection number $i(\alpha, \beta)$ of two classes $\alpha, \beta \in S$ is the minimal number of transverse intersections of curves in the homotopy classes of α and β respectively. Given $\alpha, \beta \in S$ with geometric intersection number n, we will define 2^n elements of S' indexed by $(\varepsilon_1, ..., \varepsilon_n)$ where $\varepsilon_i = \pm 1$. Suppose $\alpha \cap \beta = \{x_1, ..., x_n\}$. At each x_i label $+1$ (resp. -1) the two opposite quadrants of $M - \alpha \cup \beta$ in which α meets β on the left (resp. right); this labeling does not depend on orientations for α or β but only on the orientation of M. (See Figure 7.) Let $\gamma_{\varepsilon_1, ..., \varepsilon_n}$ be the element of S' which follows both α and β everywhere, and which at x_i turns off α onto β in both of the quadrants labeled ε_i. These elements of S' will be called the combinations of α and β. Now the answers to the questions (a) and (b) are contained in the following proposition.

PROPOSITION 2.8. *Given $\alpha \in S$, α is in T if and only if either*

(i) *$F(\alpha) > 0$ and for all $\beta \in S$, there is a unique combination $\gamma_{\varepsilon_1, ..., \varepsilon_n}$ of α and β with $F(\gamma_{\varepsilon_1, ..., \varepsilon_n})$ maximal. In that case $\tilde{\alpha} \cdot \tilde{\beta} = 2 \sum \varepsilon_i$.*

(ii) *$F(\alpha) = 0$ and there exists $\varepsilon > 0$ such that for all β with $i(\alpha, \beta) > 0$, $F(\beta) > \varepsilon$. In that case $\tilde{\alpha} \cdot \tilde{\beta} = 2i(\alpha, \beta)$.*

Proof. For both (i) and (ii) it is easy to see that if α is in T, the conclusion is true. This will be shown in step I. It is harder to show that if α is not in T, then the conclusion is not satisfied. This will be shown in step II.

Step I. (i) If $\alpha \in T$ and $F(\alpha) > 0$, we may represent α by a curve transversal to F, β by a quasi transversal curve such that the intersections of α and β are transversal. Then Figure 8 makes it clear that for exactly one $\varepsilon_1, ..., \varepsilon_n$ is $\gamma_{\varepsilon_1, ..., \varepsilon_n}$ quasi-transversal, and for all others the transverse length is less.

Moreover, both inverse images of $x_i \in \alpha \cap \beta$ contribute ε_i to $\tilde{\alpha} \cdot \tilde{\beta}$.

(ii) If $\alpha \in T$ and $F(\alpha) = 0$, α can be realized as the equator of a cylinder foliated under F by equators, and of transverse height $h > 0$. Then if $i(\alpha, \beta) = n$, β must cross the cylinder from top to bottom n times, and $F(\beta) \geq nh > 0$.

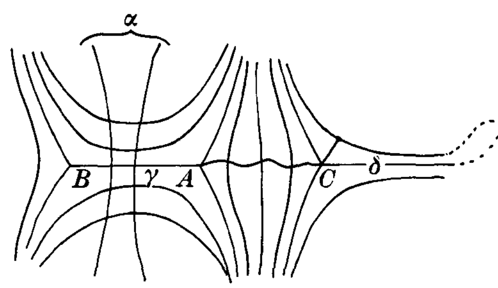


Fig. 9

Moreover, if β is realized so as to intersect α transversely in n points, then each point of $\pi^{-1}(\alpha \cap \beta)$ contributes 1 to $\tilde{\alpha} \cdot \tilde{\beta}$.

Step II. We will need two facts whose proofs are left to the reader; the techniques of [7] can be used to prove (b).

(a) Let F be a realizable foliation, and γ a non-critical curve. Then either γ is closed, or $\bar{\gamma} \subset M$ is a region whose boundary is contained in Γ.

(b) If α and β are two transversal simple closed curves on M, which intersect in more than $i(\alpha, \beta)$ points, then there is an embedded disc in M bounded by a segment of α and a segment of β.

(i) Let α be a quasi-transversal curve homotopic to a simple curve, with $\alpha \cap \Gamma \neq \emptyset$, and $F(\alpha) > 0$. Then α must cross a 1-cell of Γ, or follow one, or do both of these things.

Suppose there is a quasi-transversal curve β homotopic to a simple curve, transversal to α, which follows some 1-cells of Γ which α crosses, or crosses some 1-cells of Γ which α follows, or both, and that these are only points of $\alpha \cap \beta \cap \Gamma$.

Consider the two combinations of $\alpha \cap \beta$ obtained by choosing the following quadrants:

The unique choice which makes the combination quasi-transversal, as in step I, at points not in Γ;

The same choice $+1$ or -1 at all points of $\alpha \cap \beta \cap \Gamma$.

It is not hard to show that both of these combinations are arbitrarily close to quasi-transversal curves of maximal length $F(\alpha) + F(\beta)$.

It remains to show that such a β exists. Suppose F realizable.

If α follows a 1-cell γ, such a β exists by Proposition II.2. Indeed a transversal curve exists which leaves γ on one side and returns on the other; such a curve can be made simple by disconnecting and reconnecting at the intersections.

If α is transversal to Γ, the argument is more delicate.

Let γ be a critical segment of Γ which α crosses; we need to find a quasi-transversal

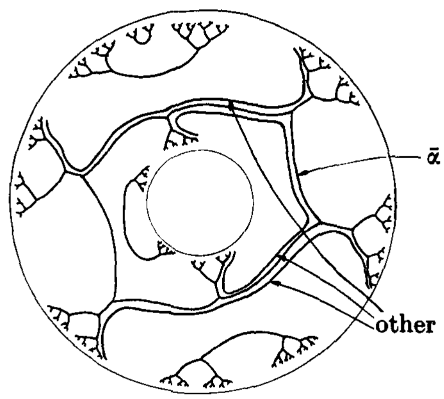


Fig. 10

simple closed curve which follows γ. Let A and B be the end points of γ, which we may assume distinct. If the leaves in the sector opposite to γ at A are closed, the argument is easy, so suppose not. Then the critical ones among them are dense, so we may assume a neighborhood of γ looks like Figure 9, where C is different from A or B. Follow the leaf δ; it is possible to return to C either along a critical leaf, or by following δ till it returns near C and making a short transverse leap. In either case there is a quasi-transverse path that leaves A, goes to C transversely, follows δ, returns to C as above, and returns to A transversely.

Repeat the argument for B and patch the paths together.

(ii) Now suppose α is in Γ, and not homotopic to the equator of a cylinder. Consider a covering space $\tilde{M}$ of M in which a lift $\tilde{\alpha}$ of α is the only simple closed curve (remark that $\tilde{\alpha}$ actually is simple).

If we draw the lifts of α "infinitesimally separated", $\tilde{M}$ might look like Figure 10. Statements about intersections are to be understood in the sense of this "infinitesimal separation", i.e. intersections are considered to exist only if they cannot be avoided by an arbitrarily small isotopy. In particular, there are distinct points $\tilde{A}$ and $\tilde{B}$ on $\tilde{\alpha}$ from which critical leaves leave $\tilde{\alpha}$ on opposite sides, which do not intersect other lifts of α near $\tilde{\alpha}$. Let $\tilde{\delta}_1$ be the leaf leaving $\tilde{A}$. Then $\tilde{\delta}_1$ can be joined to some other lift $\tilde{A}'$ of A without intersecting any other lifts of α by a path with either transversal length, or arbitrarily short transversal length, by fact (a). Pick a similar curve $\tilde{\delta}_2$ leaving $\tilde{\alpha}$ from the point $\tilde{B}$. There are several cases to consider depending on whether the images δ_1 and δ_2 of $\tilde{\delta}_1$ and $\tilde{\delta}_2$ in M are simple or not, and intersect or not, and whether they return to the same side of α that they left α or not. If they are not simple, they can be made simple by disconnecting and reconnecting.

If they intersect, it is possible to follow δ_1 from A to the first point of intersection, then δ_2 back to B, and a segment of α from B to A, to produce a simple closed curve β on M with algebraic intersection $\alpha\cdot\beta=1$. If δ_1 (or δ_2) returns to α on the opposite side from which it left, the same construction is possible.

Finally if both δ_1 and δ_2 are disjoint and simple, it is possible to follow δ_1, a segment of α, δ_2 and the other segment of α to produce a simple closed curve β, with $i(\alpha,\beta)=2$ by fact (b). Indeed, the intersection points exist, and any disc bounded by a segment of α and a segment of β would be visible in $\bar{M}$. Q.E.D.

COROLLARY 2.9. *If F and F' are two measured foliations on M with the same image in $\mathbf{R}^S$, then there is a unique isomorphism*

$$H_1(\tilde{M}_F, \tilde{\Gamma}_F)^- \to H_1(\tilde{M}_{F'}, \tilde{\Gamma}_F)^-$$

such that the diagram

$$\begin{array}{ccc} & \nearrow & H_1(\tilde{M}_F, \tilde{\Gamma}_F)^- \\ \mathbf{R}^{(S)} & & \downarrow \\ & \searrow & H_1(\tilde{M}_{F'}, \tilde{\Gamma}_{F'})^- \end{array}$$

commutes.

Proof. This is precisely the content of Proposition 2.7 and Proposition 2.8.

§ 6. Foliations with closed leaves

Although it seems quite difficult to get any precise geometric information about a measured foliation from its image in $\mathbf{R}^S$, this is not the case for foliations with closed leaves. In this paragraph we will see that the image in $\mathbf{R}^S$ determines the cylinders and their heights; this will be useful in Chapter IV, § 3.

LEMMA 2.10. *Let F be measured foliation with closed leaves, and F' another measured foliation with same image in $\mathbf{R}^S$. Then*

(i) *F' also has closed leaves,*

(ii) *For each cylinder for F there is a corresponding one for F' with homotopic equator,*

(iii) *Corresponding cylinders have the same height.*

Proof. (i) Foliations with closed leaves are distinguished by the fact that the image of S in $\mathbf{R}$ for such foliations is discrete.

(ii) The equators of cylinders for F are homotopic to those simple closed curves γ such that $F(\gamma)=0$, and for any γ such that $i(\gamma,\gamma')\neq 0$, $F(\gamma')\neq 0$.

(iii) We have seen that the image of F in $\mathbf{R}^S$ determines the homotopy classes $\gamma_1, \ldots, \gamma_n$ of the equators of the cylinders. It is not hard to see that for any $j=1, \ldots, n$ there is a simple closed curve γ_j' such that $i(\gamma_j, \gamma_j')=1$ or 2, and $i(\gamma_k, \gamma_j')=0$ for $k \neq j$. Then the height of the cylinder with equator γ_j is either $F(\gamma_j')$ or $1/2F(\gamma_j')$ depending on the intersection number. Q.E.D.

Remark. It is fairly easy to prove that in this case F and F' are equivalent.

§ 7. The map $E_F \to \Theta_M$ is proper

LEMMA 2.11. *The map $Q-\{0\} \to \mathbf{R}^S$ defined by $q \to (\gamma \to F_q(\gamma))$ is continuous.*

Proof. In each homotopy class there is either a unique geodesic in the metric $|q|^{1/2}$ or an annulus swept out by geodesics. An application of Ascoli's theorem shows the transverse length varies continuously in $Q-\{0\}$.

LEMMA 2.12. *The map $p: E_F \to \Theta_M$ is proper.*

Proof. Suppose K is compact in Θ_M and $q_n \in E_F \cap p^{-1}(K)$ is a quadratic form on X_n. If $\|q_n\| = \int_{X_n} |q_n|$ is not bounded above then since the images of q_n in $\mathbf{R}^S$ coincide, the images of $q_n' = q_n/\|q_n\|$ in $\mathbf{R}^S$ converge to zero. However q_n' is in the unit sphere in Q which is proper over Θ_M so some subsequence converges to $q_0 \neq 0$. By the continuity of the map to $\mathbf{R}^S$, the image of q_0 is zero. This is clearly impossible. A similar argument shows that $\|q_n\|$ bounded away from zero. Therefore a subsequence converges to q_0 and since E_F is closed, as it is the inverse image of a point, $q_0 \in E_F$.

CHAPTER III

The space E_k

§ 1. A versal deformation of $z^k dz^2$

Let P_k be the space of quadratic differentials on $\mathbf{C}$ of the form $(z^k + p(z))dz^2$, with p a polynomial of degree at most $k-2$. We wish to show that P_k is a universal deformation of $z^k dz^2$; this is ordinarily stated in terms of germs, but we will prove a slightly stronger statement which pays attention to domains of definition. The germified statement follows from Proposition 3.1 by a straightforward inductive limit argument.

Our proof rests on the inverse function theorem for Banach spaces. Let U be a simply connected neighborhood of 0 in $\mathbf{C}$, and let $B(U)$ (resp. $B^1(U)$) be the Banach space of functions analytic and bounded in U, with the uniform norm (resp. analytic functions on U with bounded derivatives, with $\|f\| = \sup_{z\in U}(|f(z)| + |f'(z)|)$). Similarly, let $B(U, \Omega^{\otimes 2})$ (resp. $B^1(U, \Omega^{\otimes 2})$) be the Banach space of quadratic differentials of the form $f(z)dz^2$ with $f \in B(U)$ (resp. $f \in B^1(U)$).

PROPOSITION 3.1. *There is a unique analytic map* $\alpha = (\alpha_1, \alpha_2)$ *from any sufficiently small neighborhood of* $z^k dz^2$ *in* $B(U, \Omega^{\otimes 2})$ *to a neighborhood of* $(id, z^k dz^2)$ *in* $B^1(U) \times P_k$ *such that*

$$\alpha_1(\varphi)^*(\alpha_2(\varphi)) = \varphi.$$

Proof. Consider the map $F: B^1(U) \times P_k \to B(U, \Omega^{\otimes 2})$ defined by $(f, q) \mapsto f^* q$. The map F is well defined as we can find a bound for f^*q in terms of a bound for q and a bound on both f and f'.

We wish to compute the derivative of F. The tangent space to $B^1(U)$ at the identity should be thought of as vector fields $\chi(z)d/dz$ with $\chi \in B^1(U)$, whereas the tangent space to P_k is the space $\bar{P}_k$ of polynomial quadratic differentials of degree at most $k-2$. An easy calculation shows that the derivative of F at $(id, z^k dz^2)$ is

$$(\chi, p(z)dz^2) \mapsto L_\chi(z^k dz^2) + p(z)dz^2,$$

where L_χ is the Lie derivative. If we can show that the linear map above is an isomorphism, the proposition will follow from the inverse function theorem.

A calculation to first order shows that $L_\chi(z^k dz^2) = kz^{k-1}\chi(z) + 2z^k \chi'(z)$. Thus we must show that given any $\varphi \in B(U)$ there exist a unique $\chi \in B^1(U)$ and p polynomial of degree at most $k-2$ such that

$$kz^{k-1}\chi(z) + 2z^k\chi'(z) + p(z) = \varphi(z).$$

Clearly p must be the $k-2$ jet of φ at 0; set $\psi(z) = (\varphi(z) - p(z))/z^{k-1}$, we must show that there is a unique solution $\chi \in B^1(U)$ to the differential equation $k\chi + 2z\chi' = \psi$. Using the integrating factor $1/2z^{k/2-1}$, we find that the unique solution analytic at zero is

$$\chi(z) = z^{-k/2} \int_0^z \tfrac{1}{2}\psi(\zeta)\, \zeta^{k/2-1}\, d\zeta.$$

It is clear from the formula that χ is bounded if ψ is bounded, and $\chi' = (2z)^{-1}(\psi - k\chi)$ gives a bound for χ'. Q.E.D.

§ 2. Statement of the main result

Let E_k be the set of $q \in P_k$ such that any two zeroes are connected by the critical graph Γ_q. Pick $A > 0$ on the real axis and let $U \subset P_k$ be the set of q having all their roots in

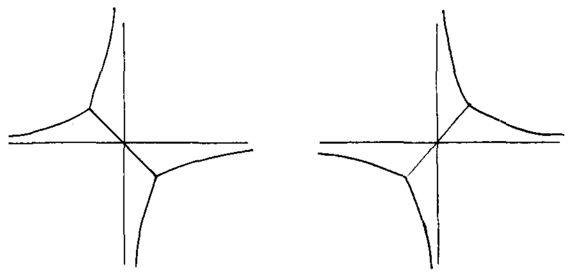


Fig. 11

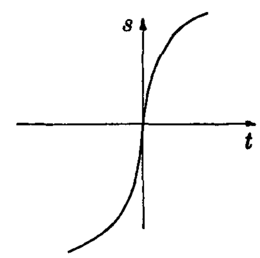


Fig. 12

the disc of radius A. We can define a continuous function s: $E_k \cap U \to R$ by $s(q) = \mathrm{Im} \int_A^{\Gamma_q} \sqrt{q}$ since a branch of $\sqrt{q}$ can be chosen continuously at A, and any two paths from A to Γ_q differ up to homotopy by a path in Γ_q which contributes only to the real part of the integral. We shall consider $E_k \subset P_k \times \mathbf{R}$ embedded by $q \mapsto (q, s(q))$. (Actually, it is U which is embedded in $P_k \times \mathbf{R}$, but as the result we are after is local, we will frequently speak of E_k when we only mean a neighborhood of $z^k dz^2$.)

The object of the rest of this chapter is to show that *E_k is a C^1 submanifold of $P_k \times \mathbf{R}$ and to compute its tangent space at $z^k dz^2$.*

Example. If $k = 2$, it is easy to show that $q = (z^2 + a)dz^2$ is in E_2 if and only if a is purely imaginary. In that case the critical graph looks like Figure 11, depending on whether a/i is positive or negative. The function $s(t) = \mathrm{Im} \int_A^{\Gamma_t} \sqrt{z^2 + it}\, dz$ has an asymptotic development $s(t) = -\frac{1}{4} t \log |t| + O(t)$ and is not differentiable at $t = 0$; its graph looks like Figure 12.

Thus although E_2 is a submanifold of both P_2 and $P_2 \times \mathbf{R}$, the induced C^1 structures *do not coincide*.

We do not know whether E_k is in general a differentiable submanifold of P_k, but if it is, the induced differentiable structure does not coincide with the one we shall describe here. The extra differentiable function s will be crucial for our purposes.

Remark. It appears likely that for $k \geq 4$, k even, E_k is a topological submanifold of P_k with a tangent space at each point which does not depend continuously on the point.

THEOREM 3.2. (a) *The space E_k is near $z^k dz^2$ a C^1 submanifold of $P_k \times \mathbf{R}$, of real dimension $k-1$.*

(b) *The tangent space to E_k at $z^k dz^2$ is the space of pairs (p, s) with $p = (a_{k-2} z^{k-2} + \dots + a_0) dz^2$, such that $a_0 = \dots = a_{\frac{1}{2}k-1} = 0$ and s arbitrary, if k is even; $a_0 = \dots = a_{\frac{1}{2}(k-3)} = 0$ and $s = \mathrm{Im} \int_A^0 (p/\sqrt{z^k}) dz$, if k is odd.*

Remark. If k is odd then E_k is in fact a submanifold of P_k.

The proof of Theorem 3.2 will require the remainder of this chapter. The organizing principle is the following criterion.

PROPOSITION 3.3. *Let $U \subset \mathbf{R}^n \times \mathbf{R}^m$ be closed in a neighborhood of* 0 *with $U \cap (\{0\} \times \mathbf{R}^m) = \{0\}$ and satisfy* 0 *has a basis of neighborhoods V in U such that*

(i) *$V - \{0\}$ is connected and $\neq \emptyset$.*

(ii) *For all $u \in U$, $u \neq 0$, U is near u the graph of a C^1 map $\mathbf{R}^m \to \mathbf{R}^m$;*

(iii) *$\lim_{u\to 0} T_u U$ exists and is $\mathbf{R}^n \times \{0\}$;*

(iv) *Either $n > 2$ or the projection $U \to \mathbf{R}^n$ is injective.*

Then U is a C^1 submanifold of $\mathbf{R}^n \times \mathbf{R}^m$ near 0, *and $T_0 U = \mathbf{R}^n \times \{0\}$.*

Proof. The projection map $T: U - \{0\} \to \mathbf{R}^n$ is a local homeomorphism near 0 by (ii). Since U is closed and $U \cap (\{0\} \times \mathbf{R}^m) = \{0\}$, it is onto a neighborhood W of 0. Taking W small enough and V' the component of $T^{-1}(W)$ containing 0, the map $V' - \{0\} \to W - \{0\}$ is a covering map. If $n > 2$, $\mathbf{R}^n - \{0\}$ is simply connected so the covering space is trivial and single sheeted by (i). Condition (iv) guarantees that the same is true if $n = 2$. Thus U is in a neighborhood of $\{0\}$ the graph of a map $f: \mathbf{R}^n \to \mathbf{R}^m$ which is C^1 in $\mathbf{R}^n - \{0\}$.

One form of L'Hospital's rule says that if $f: \mathbf{R}^n \to \mathbf{R}^m$ is continuous, differentiable except at 0, and $\lim_{x\to 0} d_x f$ exists, then f is differentiable at zero and $d_0 f = \lim_{x\to 0} d_x f$. By (iii), this result can be applied to the map f above. Q.E.D.

Remark. The curve $y^2 = x^3$ in $\mathbf{C}^2$, with the projection $(x, y) \mapsto x$, shows that (iv) is necessary if $n = 2$.

For an appropriate decomposition of $P_k \times \mathbf{R}$, E_k will satisfy near $(z^k dz^2, 0)$ the conditions of Proposition 3.5.

The justification of (ii) will be given in § 4, with preliminaries in § 3. The justification of (iii) will be given in § 5, and will follow easily from (ii) and a homogeneity property of E_k. The justification of (i) will be given in § 6, and will require an entirely different approach to E_k. We shall show that E_k has a natural simplicial structure, and that with this structure it is a piecewise linear manifold. The study of E_3 needed to justify (iv) will be given in § 7, using elementary but delicate analysis of the differential equation defined by a quadratic form.

As the entire proof is by induction on k, the following assumption will be in force till the end of the chapter.

Inductive assumption: k is an integer > 1, and the statement of Theorem 3.2 is true for all $k' < k$.

Remark. The space $E_1 \subset P_1 \times \mathbf{R} = \mathbf{R}$ is just the point 0, and Theorem 2.2 is true in that case. We have already constructed E_2 in the example above, but the construction will in principle be repeated in the general proof.

§ 3. Preliminaries on the topology of Riemann surfaces

In this paragraph we will collect three results that will be useful in the proof of Proposition 3.6 (which will be quite elaborate enough without interruptions).

We will identify elements of P_k with the associated polynomials. All homology groups will be with coefficients $\mathbf{Z}$, but cohomology groups will have coefficients in whatever sheaf is indicated.

Let $q_0 \in P_k$ be a polynomial. We will denote $\hat{X}_{q_0}$ the curve in $\mathbf{C}^2$ of equation $y^2 = q_0(z)$. This curve is non-singular if and only if q_0 has only simple zeroes. Denote $\tilde{X}_{q_0}$ the normalization of $\hat{X}_{q_0}$, and $\bar{X}_{q_0}$ the non-singular compactification of $\tilde{X}_{q_0}$. In all cases, the projection on $\mathbf{C}$ will be denoted by π.

Remarks. (a) The Riemann surface $\tilde{X}_{q_0}$ is "the Riemann surface of $\sqrt{q_0}$", in particular it carries a canonical differential ω_{q_0}.

(b) $\bar{X}_{q_0}$ is obtained by adding one or two points at ∞ to $\tilde{X}_{q_0}$ depending on whether k is odd or even; these will be denoted ∞ or ∞_1 and ∞_2 respectively.

(i) *Period matrices.*

The following fact is just one way of saying that the imaginary part of the period matrix of a compact Riemann surface is non-degenerate.

PROPOSITION 3.4. *If X is any compact Riemann surface, the map $H^0(X, \Omega_X) \to$ Hom $(H_1(X); \mathbf{R})$ defined by*

$$\varphi \mapsto \left(\gamma \mapsto \operatorname{Im} \int_\gamma \varphi\right)$$

is an isomorphism of real vector spaces.

Proof. Recall from [10, p. 71] that $H^1(X, \mathbf{C}) = H^0(X, \Omega_X) \oplus \overline{H^0(X, \Omega_X)}$ under the de Rham map. The map described in the proposition sends φ to its imaginary part, i.e. $\varphi \mapsto \frac{1}{2}i(\bar{\varphi} - \varphi)$, and is injective since the sum above is direct. Both spaces have real dimension $2g$, thus this map is an isomorphism. Q.E.D.

COROLLARY 3.5. (a) *If k is odd, the map*

$$H^0(\bar{X}_{q_0}, \Omega) \to \operatorname{Hom}(H_1(\tilde{X}_{q_0}); \mathbf{R})$$

defined in Proposition 3.4 is an isomorphism.

(b) *If k is even and $\Omega(\mathbf{R}\infty)$ is the sheaf of meromorphic differentials on $\bar{X}_{q_0}$, holomorphic except at ∞_1 and ∞_2 and having there at most simple poles with real residues, the map*

$$H^0(\bar{X}_{q_0}, \Omega(\mathbf{R}\infty)) \to \operatorname{Hom}(H_1(\tilde{X}_{q_0}); \mathbf{R})$$

defined in Proposition 3.4 is an isomorphism.

Proof. Part (a) is clear, since removing one point from a compact surface does not change its first homology group.

For part (b) suppose first that q_0 is a square. Then the result is trivial as both sides are 0. The case when q_0 is not a square follows from the five lemma and the following commutative diagram (the subscript q_0 is dropped for convenience)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(\bar{X}, \Omega) & \longrightarrow & H^0(\bar{X}, \Omega(\mathbf{R}\infty)) & \xrightarrow{\text{res}} & \mathbf{R} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Hom}(H_1(\bar{X}); \mathbf{R}) & \longrightarrow & \operatorname{Hom}(H_1(\tilde{X}); \mathbf{R}) & \longrightarrow & \mathbf{R}\times\mathbf{R} & \longrightarrow \mathbf{R} \longrightarrow & 0
\end{array}$$

where the map res is $\omega \to \operatorname{res}_{\infty_1}(\omega) = -\operatorname{res}_{\infty_2}(\omega)$; the bottom exact sequence is extracted from the transpose of the homology exact sequence of the pair $(\bar{X}, \tilde{X})$. The last two terms are computed by excision; the last map is addition and the map from $\operatorname{Hom}(H_1(\tilde{X}); \mathbf{R})$ is onto the line $x+y=0$. The vertical maps are given by the imaginary parts of integrals as in Proposition 3.4. The left-hand map is an isomorphism by Proposition 3.4, the right-hand map is an isomorphism onto $x+y=0$ because the integral around a loop is $2\pi i$ times the residue, so the map in the center is an isomorphism. Q.E.D.

(ii) *The pair* $(\hat{X}_{q_0}, \tilde{X}_{q_0})$

Topologically, $\hat{X}_{q_0}$ can be obtained from $\tilde{X}_{q_0}$ by identifying the pairs of points above the even zeroes of q_0. It is more convenient (and équivalent up to homotopy) to think of $\hat{X}_{q_0}$ as $\tilde{X}_{q_0}$ to which line segments joining the above pairs of points have been added. Thus we can think of $\tilde{X}_{q_0}$ as a subset of $\hat{X}_{q_0}$.

The long homology sequence of the pair looks rather different depending on whether q_0 is a square or not; in both cases $\hat{X}_{q_0}$ is connected, but $\tilde{X}_{q_0}$ has two connected (contractible) components if q_0 is a square, and one otherwise.

If q_0 is not a square, and has m even zeroes, the exact sequence

$$0 \to H_1(\tilde{X}_{q_0}) \to H_1(\hat{X}_{q_0}) \to \mathbf{Z}^m \to 0$$

can be extracted from the long exact sequence, where $H_1(\hat{X}_{q_0}, \tilde{X}_{q_0}) \cong \mathbf{Z}^m$ is computed by excision; the inclusion of the line segments described above can be taken as generators of $\mathbf{Z}^m$.

If q_0 is a square and has m even zeroes (and no odd zeroes of course) the exact sequence

$$0 \to H_1(\hat{X}_{q_0}) \to \mathbf{Z}^m \to \mathbf{Z} \to 0$$

can be extracted from the long exact sequence, where again the generators of $\mathbf{Z}^m$ can be represented by the line segments, and, if they are all oriented going from one component of $\tilde{X}_{q_0}$ to the other, the last map may be taken to be addition.

(iii) *Vanishing homology and the local system* $H_1(\hat{X}_{q_0})$.

Pick small disjoint discs D_i around the zeroes of q_0 and consider the space $U \subset P_k$ of q which vanish in each disc as many times as q_0 (counting multiplicities). The $H_1(\hat{X}_q)$ form the fibres of a local system over U only if q_0 has only simple zeroes. However, each $\hat{X}_q$ comes with a canonical homotopy class of maps to $\hat{X}_{q_0}$ given by collapsing the inverse images of the discs D_i to points. The kernel in $H_1(\hat{X}_q)$ of the induced map to $H_1(\hat{X}_{q_0})$ is called the vanishing homology, and the quotients of the $H_1(\hat{X}_q)$ by the vanishing homology do fit together to form a trivial local system over U, which we shall denote by abuse of notation $H_1(\hat{X}_{q_0})$.

§ 4. Local equations for E_k

The object of this paragraph is to prove that E_k satisfies condition (ii) of Proposition 3.3, for an appropriate decomposition of $P_k \times \mathbf{R}$. This will use the inductive hypothesis for k' the order of the zeroes of q_0. The main tool in the proof is the non-degeneracy of the imaginary part of the period matrix for a Riemann surface; (i.e. Corollary 3.5) this is used in Lemma 3.8 and is the crucial computation to show that the implicit function theorem can be applied.

Because the statements for k even and odd are different, we shall frequently have to go through arguments twice; this seems to be inherent in the problem, as the arguments are sometimes different in essential ways. The case when q_0 is a square will also require separate treatment.

Notation. If k is even, denote

$$H_k = \{z^k + a_{k-2}z^{k-2} + \dots + a_{\frac{1}{2}k}z^{\frac{1}{2}k}\} \times \mathbf{R}$$

$$L_k = \{a_{\frac{1}{2}k-1}z^{\frac{1}{2}k-1} + \dots + a_0\}.$$

If k is odd, denote

$$H_k = \{z^k + a_{k-2}z^{k-2} + \dots + a_{(k-1)/2}z^{(k-1)/2}\}$$

$$L_k = \{a_{(k-3)/2}z^{(k-3)/2} + \dots + a_0\} \times \mathbf{R}.$$

In both cases, $P_k \times \mathbf{R} = H_k \times L_k$. In either case H_k is an affine space. Denote by $\bar{H}_k$ the linear part. The terms "high coefficients" and "low coefficients" will be used accordingly; the term "middle coefficient" will be used only if k is even, and refers to $a_{\frac{1}{2}k-1}$.

PROPOSITION 3.6. *For any $q_0 \in P_k$ different from $z^k dz^2$ but sufficiently close, $E_k \subset P_k \times \mathbf{R}$ is locally near q_0 the graph of a C^1 map $H_k \to L_k$.*

The proof is divided into two steps and will take the remainder of this paragraph. The first describes an intermediate space F_k consisting of $q \in P_k$ with a critical graph which is locally connected near the zeroes of q_0 (see Figure 13). The second step deals with connecting up the critical graph.

Step 1. It is convenient to reformulate the inductive hypothesis to state: for all $k' < k$,

$$T_{z^{k'}dz^2} E_{k'} = \left\{ (p, s) \,\middle|\, \frac{p}{\sqrt{z^{k'}}dz} \text{ is holomorphic on } \tilde{X}_{z^{k'}dz^2} \text{ and } s \text{ is arbitrary} \right\} \quad \text{if } k' \text{ is even;}$$

$$= \left\{ (p, s) \,\middle|\, \frac{p}{\sqrt{z^{k'}}dz} \text{ is holomorphic on } \tilde{X}_{z^{k'}dz^2} \text{ and } s = \operatorname{Im} \int_A^0 \frac{p}{\sqrt{z^{k'}}dz} \right\} \quad \text{if } k' \text{ is odd.}$$

Indeed it is clear that $p/(\sqrt{z^{k'}}dz)$ is holomorphic if and only if p vanishes at least to the order $k'/2$ (k' even) or $(k'-1)/2$ (k' odd).

Let $q_0 \in E_k$ be sufficiently close to $z^k dz^2$, $q_0 \neq z^k dz^2$, and let $x_1, \ldots, x_n$ be the zeroes of q_0, of order $k_1, \ldots, k_n$; suppose $k_1, \ldots, k_m$ even and $k_{m+1}, \ldots, k_n$ odd. Pick disjoint discs D_i centered at x_i and points $A_i \in \pi^{-1}(\partial D_i \cap \Gamma_{q_0})$. Let $U \subset P_k$ be a simply connected neighborhood of q_0 consisting of forms q with k_i zeroes in D_i, $i = 1, \ldots, n$.

By Proposition 3.1, there is an analytic map

$$f\colon U \to \prod_{i=1}^{n} P_k$$

classifying the deformations of the zeroes of q_0. Denote still $f\colon U \times \mathbf{R}^n \to \prod_{i=1}^n (P_{k_i} \times \mathbf{R})$ the map above extended by the identity on the second factor and consider

$$F_k = f^{-1}\left(\prod_{i=1}^{n} E_{k_i} \right) \subset P_k \times \mathbf{R}^n.$$

LEMMA 3.7. (a) *F_k is a C^1 submanifold of $P_k \times \mathbf{R}^n$.*

(b)
$$T_{q_0} F_k = \left\{ (p, s_1, \ldots, s_n) \,\middle|\, \frac{p}{\sqrt{q_0}} \in H^0\left(\bar{X}_{q_0}, \Omega\left(\frac{k}{2}\infty_1 + \frac{k}{2}\infty_2 \right) \right),\ s_i = \operatorname{Im} \int_{A_i}^{x_i} p/\sqrt{q_0},\quad i = m+1, \ldots, n \right\}$$

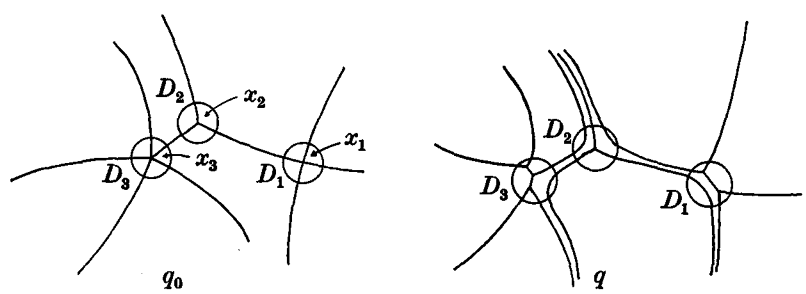


Fig. 13

if k is even

$$T_{q_0}F_k=\left\{(p, s_1, \ldots, s_n)\left|\frac{p}{\sqrt{q_0}}\in H^0(\bar{X}_{q_0}, \Omega((k-1)\infty)), s_i=\operatorname{Im}\int_{A_i}^{x_i}=p/\sqrt{q_0}, \quad i=m+1, \ldots, n\right.\right\}$$

if k is odd.

Proof. Statement (a) follows from the fact that f is a submersion, which in turn comes from the fact that there is no restriction on the zeroes of a polynomial. Part (b) follows from the (restatement of) inductive hypothesis, which says precisely that $p/\sqrt{q_0}$ is holomorphic on $\hat{X}_{q_0}$. The computation of the order of the pole at ∞ is left to the reader. Q.E.D.

Remark. Elements of F_k have critical graphs that are locally connected near the zeroes of q_0, cf. Figure 13.

We shall denote $\Gamma_i(q)$ the part of the critical graph of $q\in F_k$ which contains the zeroes of q in D_i.

Step 2. Recall the local system $\hat{X}_{q_0}$ defined in § 3, (iii). Notice that for any q in F_k, the integral of ω_q over a vanishing homology class is real. Thus the map

$$g: F_k \to \operatorname{Hom}(H_1(\hat{X}_{q_0}); \mathbf{R}) \quad \text{given by} \quad g(q): \gamma \mapsto \operatorname{Im}\int_\gamma \omega_q$$

is well defined. The map g is C^1 because of the differentiable structure on F_k (any time a loop goes through x_i to get from one sheet of $\hat{X}_{q_0}$ to the other, the function s_i is called in).

An essential remark is that $E_k=g^{-1}(0)$. This is clear pointwise: for any two zeroes, x_i, x_j of q there is a cycle γ going through both of them which covers a line in $\mathbf{C}$ once forwards and once backwards. The integral $\int_\gamma \omega_q$ is real, so $\Gamma_i(q)=\Gamma_j(q)$, and the critical graph of q is connected. However, this remark demands a bit of amplification. The space E_k of Theorem 3.2 lies in $P_k\times\mathbf{R}$, whereas $g^{-1}(0)\subset P_k\times\mathbf{R}^n$. Pick curves γ_i on $\hat{X}_{q_0}$ joining A to A_i; if they are chosen outside of $\pi^{-1}(D_i)$ they define unique homotopy classes of curves on all $\hat{X}_q$. Define differentiable functions $h_i(q) = \operatorname{Im} \int_{\gamma_i} \omega_q$ on P_k. On $g^{-1}(0)$, $s_i + h_i = s_j + h_j$ for all i, j. Therefore if $P_k \times \mathbf{R}$ is embedded in $P_k \times \mathbf{R}^n$ by $(q, s) \mapsto (q, s - h_1(q), \ldots, s - h_n(q))$, the space $g^{-1}(0)$ lies in the image. So in order to prove Proposition 3.6, it is enough to show that $g^{-1}(0)$ is a C^1 submanifold of $P^k \times \mathbf{R}^n$, and that, for $P_k \times \mathbf{R}$ embedded in $P_k \times \mathbf{R}^n$ as above, $\bar{H}_k$ gives local coordinates on $g^{-1}(0)$.

Let $V = H^0(X_{q_0}, \Omega(\mathbf{R}\infty)) \times \mathbf{R}^m \subset T_{q_0} F_k$. We remark that if k is odd $H^0(\bar{X}_{q_0}, \Omega(\mathbf{R}\infty)) = H^0(\bar{X}_{q_0}, \Omega)$ as there is a single point at ∞. Suppose first k is even and q_0 is a square.

Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & & & \mathbf{R}^m & \longrightarrow & V & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow{\scriptstyle d_{q_0} g|_V} & & \\ 0 & \longrightarrow & \mathbf{R} & \longrightarrow & \mathbf{R}^m & \longrightarrow & \operatorname{Hom}(H_1(\hat{X}_{q_0}); \mathbf{R}) & \longrightarrow & 0 \end{array}$$

where the bottom exact sequence is described in § 3, (ii), and the bottom inclusion $\mathbf{R} \to \mathbf{R}^m$ is the diagonal map. The diagram commutes so $d_{q_0} g|_V$ is an isomorphism restricted to any subspace complementary to the diagonal. By the implicit function theorem E_k is a C^1 submanifold of P_k and the high coefficients and s are coordinates on E_k near q_0 since $d_{q_0} s$ does not vanish on the diagonal. Comparing with the remark made in the last paragraph, the vector in $P_k \times \mathbf{R}$ with non-zero component only in the $\mathbf{R}$ direction is tangent to E_k to q_0. This proves Proposition 3.6 if q_0 is a square.

Now suppose q_0 is not a square.

LEMMA 3.8. *The map* $d_{q_0} g|_V \colon V \to \operatorname{Hom}(H_1(\hat{X}_{q_0}; \mathbf{R})$ *is an isomorphism.*

Proof. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{R}^m & \longrightarrow & V & \longrightarrow & H^0(\bar{X}_{q_0}, \Omega(\mathbf{R}\infty)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle d_{q_0} g|_V} & & \downarrow{\wr} & & \\ 0 & \longrightarrow & \mathbf{R}^m & \longrightarrow & \operatorname{Hom}(H_1(\hat{X}_{q_0}); \mathbf{R}) & \longrightarrow & \operatorname{Hom}(H_1(\tilde{X}_{q_0}; \mathbf{R}) & \longrightarrow & 0 \end{array}$$

where the bottom exact sequence is the transpose of the one described in § 3, (ii) and the right-hand vertical map is the isomorphism described in Corollary 3.5. The diagram commutes by differentiation under the integral sign, and the lemma follows from the five lemma. Q.E.D.

By the implicit function theorem, $E_k = g^{-1}(0)$ is a C^1 submanifold of F_k embedded in $P_k \times \mathbf{R}^n$ and any set of coordinates complementary to V will be local coordinates on E_k, in particular the high coefficients and the imaginary part of the middle coefficient if k is even. This proves Proposition 3.6 if k is odd. If k is even, we need to show that the imaginary part of the middle coefficient can be traded for s as a coordinate. Lemma 3.9 and the implicit function theorem allow precisely this. Let $h(q) = \operatorname{Im} a_{\frac{1}{2}k-1}$.

We note first that E_k has the following homogeneity property: if

$$q = (z-r_1) \dots (z-r_k)dz^2 \in E_k \quad \text{then for } t \geq 0, \quad q^t = (z-tr_1)(z-tr_2)\dots(z-tr_k)dz^2 \in E_k.$$

We will speak of the line through q.

LEMMA 3.9. *Suppose k is even and q is not a square. If q is sufficiently close to $z^k dz^2$, then $\partial s/\partial h \neq 0$ at q.*

Comment. The partial derivative is taken in the local system of coordinates above.

Proof. Since q is not a square it has an odd zero. If $\gamma \in H_1(\hat{X}_q)$ covers a circle of radius A, $\int_\gamma \omega_q = 2\pi i \operatorname{res}_{\infty_1} \omega_q$, and the change of variables $\zeta = z^{-1}$ and a power series development of the square root show that $\operatorname{res}_\infty \omega_q = \frac{1}{2} a_{\frac{1}{2}k-1} +$ higher order terms in the high coefficients. But a drawing shows that $\int_\gamma \omega_q$ is real so $\pi \operatorname{Re} a_{\frac{1}{2}k-1} + \operatorname{Re}$ (higher order terms) $= 0$. We conclude that $\partial (\operatorname{Re} a_{\frac{1}{2}k-1})/\partial h = 0$.

Now suppose $q_n \to z^k dz^2$, $\partial s/\partial h(q_n) = 0$. Let z_n be an odd zero of q_n. Find $\hat{q}_n$ on the same line as q_n so that $\hat{q}_n \to \hat{q}_0 \not\equiv z^k dz^2$. Also $q_n = \hat{q}_n^{t_n}$, $t_n \to 0$.

Case I. $\hat{q}_0$ not a square. Let $(\hat{p}_n(z), s) = (iz^{\frac{1}{2}k-1} + \dots + a_0, s)$ tangent to E_k at $\hat{q}_n$. Then $\hat{p}_n$ converges. The vector $p_n(z) = (iz^{\frac{1}{2}k-1} + t_n a_{\frac{1}{2}k-2} + \dots + t_n^{\frac{1}{2}k-1} a_0, s_n)$ is tangent at q_n by homogeneity. The condition $\partial s/\partial h = 0$ means $s_n = \operatorname{Im} \int_A^{z_n} (p_n(z)/q_n^{1/2}(z))dz = 0$. Now $p_n(z)/q_n^{1/2}(z)$ converges to i/z. This contradicts Fatou's lemma.

Case II. $\hat{q}_0$ is a square. Let $(p_n(z), 1) = (ia_{\frac{1}{2}k-1} z^{\frac{1}{2}k-1}, \dots, a_0, 1)$ tangent to E_k at $\hat{q}_n$. Now

$$1 = \operatorname{Im} \int_A^{z_n/t_n} \frac{\hat{p}_n(z)}{\hat{q}_n^{1/2}(z)}\, dz.$$

Since $(0, \dots, 0, 1)$ is tangent at $\hat{q}_0$, the coefficients a_j all converge to zero. Again $(p_n, 0) = (ia_{\frac{1}{2}k-1} z^{\frac{1}{2}k-1} + t_n a_{\frac{1}{2}k-2} z^{\frac{1}{2}k-2} + \dots + t_n a_0, 0)$ is tangent to E_k at q_n. Let $\omega_1, \dots, \omega_k$ be the set of zeroes of q_n. With the substitution $u = t_n/z$,

$$0 = \operatorname{Im} \int_A^{z_n} \frac{p_n(z)}{q_n^{1/2}(z)}\, dz = -\operatorname{Im} \int_{t_n/A}^{1/\hat{z}_n} \frac{u^{\frac{1}{2}k} t_n}{t_n^{\frac{1}{2}k} u^2} \frac{\left(ia_{\frac{1}{2}k-1} \left(\frac{t_n}{u}\right)^{\frac{1}{2}k-1} + \dots \right) du}{(1-u\omega_1)^{1/2} \dots (1-u\omega_k)^{1/2}}$$

$$= -\operatorname{Im} \int_{t_n/A}^{1/\hat{z}_n} \left(\frac{a_{\frac{1}{2}k-1}}{u} + a_{\frac{1}{2}k-2} + \dots + a_0 u^{\frac{1}{2}k-2} \right) \frac{du}{(1-u\omega_1)^{1/2} \dots (1-u\omega_k)^{1/2}}.$$

We break this up into two integrals, the first from $1/A$ to $1/\hat{z}_n$, the second from t_n/A to $1/A$. The first integral is 1 by the change of variables $z = 1/u$. Therefore

$$\operatorname{Im} \int_{t_n/A}^{1/A} \left(\frac{a_{\frac{1}{2}k-1}}{u} + \dots + a_0 u^{\frac{1}{2}k-2} \right) \frac{du}{(1-u\omega_1)^{1/2} \dots (1-u\omega_k)^{1/2}} = 1.$$

The denominator has limit 1 as $u \to 0$ and $n \to \infty$.

Since a_i goes to zero, $1 = \lim_{n\to\infty} \int_{t_n/A}^{1/A} [ia_{\frac{1}{2}k-1}/u]\,du = -\lim_{n\to\infty} a_{\frac{1}{2}k-1} \log t_n$. We have

$$0 = \int_A^{z_n} \frac{-\log t_n\, p_n(z)}{q_n^{1/2}(z)}\, dz.$$

But $-(\log t_n) p_n(z)/q_n^{1/2}(z)$ has limit i/z again contradicting Fatou's lemma.

§ 5. The tangent space to E_k at $z^k dz^2$

The object of this paragraph is to prove that E_k satisfies condition (iii) of Proposition 3.3. This will follow from Proposition 3.6, homogeneity, and the computations in the following two lemmas.

LEMMA 3.10. *For a sufficiently small neighborhood V of $z^k dz^2$, if $(p, 1) = (ia_{\frac{1}{2}k-1} z^{\frac{1}{2}k-1} + \dots + a_0, 1)$ is tangent to E_k at q a nonsquare, then $a_{\frac{1}{2}k-1} < 0$. In particular $\partial s/\partial h < 0$.*

Proof. Let $\omega_1, \dots, \omega_k$ be the roots of q, q^t is the line joining q to $z^k dz^2$. By Lemma 3.9 the vector $(p, 1) = (ia_{\frac{1}{2}k-1} z^{\frac{1}{2}k-1} + a_{k-2} z^{\frac{1}{2}k-2} + \dots + a_0, 1)$ is tangent to E_k at q for q near $z^k dz^2$. By homogeniety there exists $f(t) > 0$ such that $1 = \operatorname{Im} \int_A^{t\omega_1} f(t) p_t/(q^t)^{1/2}$, where $f(1) = 1$ and $p_t(z) = ia_{\frac{1}{2}k-1} z^{\frac{1}{2}k-1} + ta_{\frac{1}{2}k-1} z^{\frac{1}{2}k-2} + \dots + t^{\frac{1}{2}k-1} a_0$. The change of variables $u = t/z$ gives

$$1 = f(t) - \operatorname{Im} \int_{t/A}^{1/A} \frac{f(t) \left(\dfrac{ia_{\frac{1}{2}k-1}}{u} + a_{\frac{1}{2}k-2} + \dots + a_0 u^{\frac{1}{2}k-2} \right)}{(1-u\omega_1)^{1/2} \dots (1-u\varphi_k)^{1/2}}\, du.$$

The second term is of order $a_{\frac{1}{2}k-1} f(t) \log t$. Since $f(t) > 0$, $a_{\frac{1}{2}k-1} < 0$. Q.E.D.

LEMMA 3.11. *Suppose $q_n \to z^k dz^2$, q_n not a square and $(p_n, 1)$ is tangent to E_k at q_n where $p_n(z) = (ia_{\frac{1}{2}k-1} z^{\frac{1}{2}k-1} + \dots + a_0)$. Then $\lim_{n\to\infty} a_i = 0$; $i = 0, \dots, \frac{1}{2}k - 1$.*

Proof. Find $\hat{q}_n$ so that $q_n = \hat{q}_n^{t_n}$ and $\hat{q}_n \to q_0 \neq z^k dz^2$. Suppose ω is an odd root of $\hat{q}_n$ and $(\hat{p}_n, 1) = (i\hat{a}_{\frac{1}{2}k-1} z^{\frac{1}{2}k-1} + \dots + \hat{a}_0, 1)$ tangent at $\hat{q}_n$. Then again by homogeneity $a_{\frac{1}{2}k-j} = f(n) \hat{a}_{\frac{1}{2}k-j} t_n^{j-1}$, $j = 1, \dots, \frac{1}{2}k$ for some positive function $f(n)$.

Assume first q_0 is a square. Then the coefficients $\hat{a}_{\frac{1}{2}k-j}$ go to zero and computing as before,

$$1 = f(n) + f(n) \log t_n \hat{a}_{\frac{1}{2}k-1} + f(n)\, C(t_n)$$

where $C(t_n) \to 0$. By Lemma 3.10, $\hat{a}_{\frac{1}{2}k-1} < 0$. Therefore $f(n)$ is bounded and $\lim_{n\to\infty} p_n(z) = 0$.

If q_0 is not a square, the coefficients converge; $\hat{a}_{\frac{1}{2}k-1}$ to a nonzero limit. Again $1 = f(n) + f(n) \log t_n \hat{a}_{\frac{1}{2}k-1} - f(n)\hat{a}_{\frac{1}{2}k-1} \log 1/A + f(n)\, C(t_n)$ where once again $C(t_n) \to 0$. This time $\lim_{n\to\infty} f(n) = 0$ and the lemma is proved.

PROPOSITION 3.12. *The tangent space* $T_q E_k \to \bar{H}_k$ *as* $q \to z^k dz^2$.

Proof. We start with k even. Suppose q is not a square and v, tangent at q, has 0 in all components of $\bar{H}_k$ except a 1 in the $\mathbf{R}$ direction. Then by Lemma 3.11, the L_k components go to zero as $q \to z^k dz^2$. This automatically holds if q is a square as the L_k directions are already zero. Now we let $(p, 0) = (cz^l + a_{\frac{1}{2}k-1} z^{\frac{1}{2}k-1} + \ldots + a_0, 0)$ be tangent to E_k at q where $k - 2 \geq l \geq k/2$. If $q \to 0$ along the square locus the homogeneity shows that the coefficients $a_{\frac{1}{2}k-1}, \ldots, a_0$ go to zero. Otherwise we pull back q as in the previous two lemmas by a factor $1/t$ to $\hat{q}$ with odd root ω and we suppose $\hat{q} \to q_0 \neq z^k dz^2$. Let $(cz^l + \hat{a}_{\frac{1}{2}k-1} z^{\frac{1}{2}k-1} + \ldots + \hat{a}_0, 0)$ tangent at $\hat{q}$. The coefficients $\hat{a}_{\frac{1}{2}k-1}, \ldots, \hat{a}_0$ are bounded. We compute

$$s = \operatorname{Im} \int_A^{t\omega} \frac{(cz^l + t^{l+1-\frac{1}{2}k} \hat{a}_{\frac{1}{2}k-1} z^{\frac{1}{2}k-1} + \ldots + t^l \hat{a}_0)}{q^{1/2}}\, dz.$$

By the change of variable $u = t/z$ we find the above integral has limit

$$\operatorname{Im} \frac{cA^{l-\frac{1}{2}k+1}}{l - \frac{1}{2}k + 1},$$

since

$$\operatorname{Im} \int_A^{\omega} \frac{(cz^l + \ldots + \hat{a}_0)}{(\hat{q})^{1/2}}\, dz = 0.$$

Now

$$(p, 0) = (cz^l + t^{l+1-\frac{1}{2}k} \hat{a}_{\frac{1}{2}k-1} z^{\frac{1}{2}k-1} + \ldots + t^l a_0, s) - (b_{\frac{1}{2}k-1} z^{\frac{1}{2}k-1} + \ldots + b_0, s)$$

for some tangent vector $(b_{\frac{1}{2}k-1} z^{\frac{1}{2}k-1} + \ldots + b_0, s)$. By Lemma 3.11 the coefficients b_j go to zero since s is bounded. Therefore $(p, 0)$ converges to $(cz^l, 0)$. This proves the proposition if k is even. The proof is trivial for k odd as there is no middle coefficient. The tangent vector $(a_l z^l + a_{(k-3)/2} z^{(k-3)/2} + \ldots + a_0, s)$ converges by the change of variables to

$$\left(a_l z^l, \; -\operatorname{Im} \frac{a_l A^{l-\frac{1}{2}k-1}}{l - \frac{1}{2}k + 1}\right). \qquad \text{Q.E.D.}$$

We have justified (iii) of Proposition 3.3.

§ 6. The simplicial structure of E_k

In this paragraph we shall exhibit a simplicial complex S_n such that E_k is the cone over S_{k+2}. Roughly speaking, $q \in E_k$ lies in a low dimensional simplex if its zeroes are very degenerate; any attempt to draw deformations of critical graphs will make it clear that some such structure must exist. Then we will show that S_n is homeomorphic to the sphere of dimension $n-4$, which justifies (i) of Proposition 3.3.

The following elementary lemma from linear algebra will be necessary in Proposition 3.14. The indices should be interpreted circularly, i.e. $n+1=1$.

LEMMA 3.13. *Let $a_1, \ldots, a_n$ be reals, such that $\sum_{i=1}^n (-1)^i a_i = 0$ if n is even. Then there exist $t_1, \ldots, t_n \in \mathbf{R}$ with $t_i \geq 0$ such that if $a_i = t_{i-1} + t_i$, the system of linear equations $x_i + x_{i+1} = a_{i+1}$ can be solved for real $x_1, \ldots, x_n$ with $x_i \geq 0$, all $i = 1, \ldots, n$.*

Remark. Of course, all the interest of the lemma is the positivity of the x_i.

Proof. Set (the upper indices refer to n odd, the lower even)

$$\begin{aligned}
2x_1 &= a_1 + a_2 - a_3 + \ldots \mp a_n \\
2x_2 &= -a_1 + a_2 + a_3 - \ldots \pm a_n \\
&\vdots \\
2x_{n-1} &= \mp a_1 \pm a_2 \ldots + a_{n-1} + a_n \\
2x_n &= \pm a_1 \mp a_2 \ldots - a_{n-1} + a_n.
\end{aligned}$$

Adding successive equations shows that we do have a solution to the system of equations; the last one says $x_n = 0$ if n is even, so although the difference of the last and the first gives $2x_1 - 2x_n = 2a_1$, it still works. Now increasing t_i changes x_i by t_i without changing the x_j, $j \neq i$. Q.E.D.

Let $n \geq 2$ be an integer and let D be the closed unit disc in $\mathbf{C}$; we shall be interested in closed graphs $\Gamma \subset D$ satisfying the following conditions:

(i) Γ is contractible.

(ii) The nth roots of 1 are nodes of Γ, are the only points in ∂D, and bound only one edge.

(iii) Every node in the interior of D is the boundary of at least 3 edges.

Remark. Let $\bar{\Gamma}$ be the graph obtained by adding the unit circle to Γ. Parts (i) and (ii) show that $\chi(\bar{\Gamma}) = 1 - n$; if c_0 and c_1 are the number of nodes and edges of $\bar{\Gamma}$, (ii) and (iii) show that $c_1 \geq \frac{3}{2} c_0$, $c_0 \leq 2(n-1)$, and $c_1 \leq 3(n-1)$. There are therefore *in the interior* at

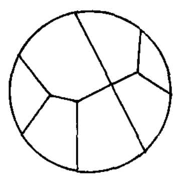


Fig. 14

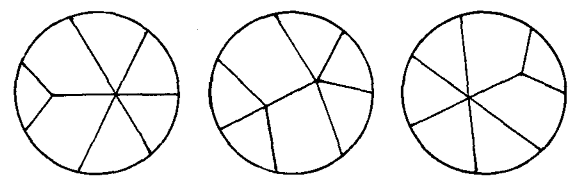


Fig. 15

most $n-2$ vertices, and $n-3$ edges which do not intersect the boundary. A few drawings will convince the reader that these inequalities are true, and strict if and only if there is a vertex which bounds at least four edges.

Two of these graphs Γ_1 and Γ_2 will be called equivalent if there is a homeomorphism of D onto itself which is the identity on ∂D and sends Γ_1 onto Γ_2. The remark above shows that there are only finitely many equivalence classes; in the future when speaking of graphs, we will mean "equivalence classes of".

Recall that a finite simplicial complex $X=(V, \zeta)$ is a finite set V of vertices, and a set ζ of subsets of V called simplices which contains the singletons, and contains every subset of a set in ζ. The topological realization of X is the set $|X|=\{f\in\mathbf{R}^V\}$ such that (i) $f(v)\geqslant 0$, (ii) $\sum_{v\in V}f(v)=1$, (iii) support $f\in\zeta$; with the induced topology from $\mathbf{R}^V$ (which is a finite dimensional vector space).

Define the simplicial complex S_n: The vertices are the n-graphs with exactly one interior edge, and the m-dimensional simplices correspond one-to-one with the graphs with $m+1$ interior edges, its vertices being the graphs obtained by collapsing all but one edge.

Example. The simplex of dimension 2 corresponding to Figure 14 has as its vertices the three graphs in Figure 15.

Remark. A point of $|S_n|$ is a graph Γ with interior edges $\gamma_1, \dots, \gamma_m$ and homogeneous coordinates $a_1, \dots, a_m$ with $a_i>0$ and $\sum a_i=1$.

Recall that if X is a topological space, the *cone* CX over X is the quotient of $X\times[0,\infty)$ by the equivalence relation collapsing $X\times\{0\}$ to a point.

We shall now construct a map $f\colon E_k\to C|S_{k+2}|$. Pick $q\neq z^k dz^2$ in E_k and let $\gamma_1, \dots, \gamma_m$ be the bounded segments of Γ_q.

There is exactly one unbounded critical leaf asymptotic to each ray $\theta=2\pi m/(k+2)$, $m=0, \dots, k-1$. Therefore the homomorphism $z\mapsto z/(|z|^2+1)^{\frac{1}{2}}$ of $\mathbf{C}$ onto the open unit disc maps the critical graph of q to a $(k+2)$-graph of the sort considered above.

The cone factor of $f(q)$ will be $t=\sum_i\int_{\gamma_i}|q|^{1/2}$, and the $|S_{k+2}|$ factor will be the point

Fig. 16

of the simplex corresponding to the graph of q, with homogeneous coordinate $(\int_{\gamma_i}|q|^{1/2})/t$ for the vertex corresponding to γ_i.

We set $f(z^k dz^2)$ to be the summit of the cone.

Proposition 3.14. *The map $f: E_k \to C|S_{k+2}|$ is a homeomorphism.*

Proof. The map f is clearly continuous: we shall show that it is bijective. The continuity of the inverse follows from the homogeneity of both spaces.

Pick a point $(\Gamma, l) \in C|S_{k+2}|$ where Γ is a $(k+2)$-graph and l associates a homogeneous coordinate to each interior edge. We shall construct a Riemann surface X_Γ with a quadratic form q by gluing $k+2$ copies H_m of the upper half plane according to the pattern (Γ, l). To be more explicit, for each $m = 1, \ldots, k+2$ consider the injective path in the graph from $e^{2\pi i m/(k+2)}$ to $e^{2\pi i(m+1)/(k+2)}$; this will go through interior edges $\gamma_{m_1}, \ldots, \gamma_{m_j}$ with homogeneous coordinates $l_{m_1}, \ldots, l_{m_j}$. Mark off on the real axis in H_m contiguous line segments of length $l_{m_j}, \ldots, l_{m_1}$ as indicated in Figure 16:

In the union of the H_i identify isometrically the line segments that correspond to the same edges of the graph Γ (including the unbounded ones).

The resulting space X_Γ carries a unique structure of a Riemann surface restricting to the standard one in the H_i, and a unique holomorphic quadratic form q restricting in each H_m to dz^2.

Lemma 3.15. *The one point compactification $\overline{X}_\Gamma$ of X_Γ is conformally equivalent to the Riemann sphere. The quadratic form q can be extended as a meromorphic form to $\overline{X}_\Gamma$ with a pole of order $k+4$ at infinity.*

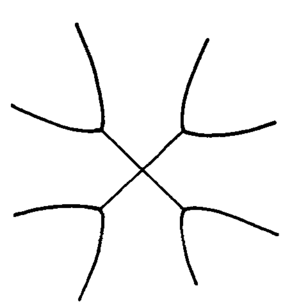


Fig. 17

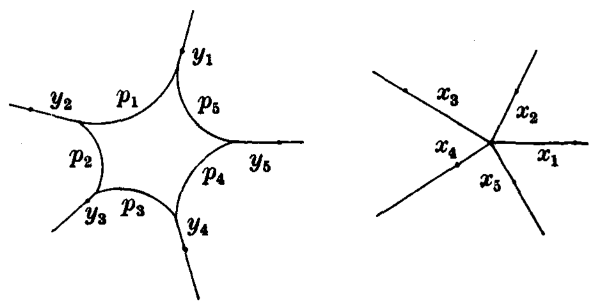


Fig. 18

Proof. We shall need the following fact: if k is even, $(z^k + iaz^{\frac{1}{2}k-1})dz^2$ is in E_k; its critical graph for $k=6$ is drawn in Figure 17. For any γ bounded segment of the critical graph,

$$\int_\gamma |q|^{1/2} = \frac{\pi}{k+2}|a|.$$

The idea of the proof now is to show that the complement of a critical part of Γ_q in X_Γ is isometrically isomorphic to the complement of a compact part of the critical graph of $z^k dz^2$ if k is odd, and of $(z^k + iaz^{\frac{1}{2}k-1})dz^2$ for an appropriate value of a if k is even.

If X is cut along the compact segments of Γ_q the result is a union of H_m each connected to the next along some ray on the right of the real axis, the unattached part is a segment of length p_m ($l_{m_1} + \ldots + l_{m_j}$ in the above notation); clearly the lengths p_m completely classify the complement of the critical graph.

If k is odd, we wish to cut out further points y_m on the mth ray of Γ_q, and out to points x_m on the mth ray of $z^k dz^2$ so that the resulting surfaces with quadratic differentials will be isomorphic. Consulting Figure 19 we see that this means $y_m + p_m + y_{m+1} = x_m + x_{m+1}$. Lemma 3.13 guarantees this can be done.

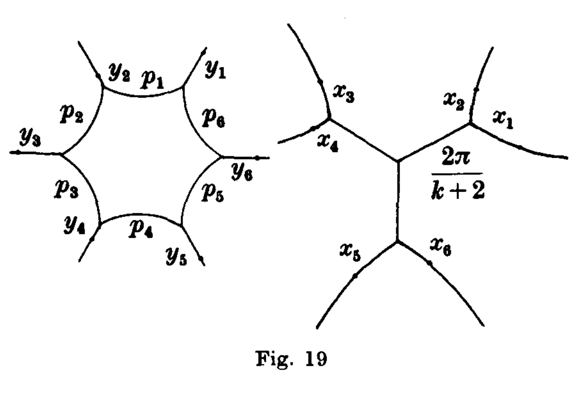


Fig. 19

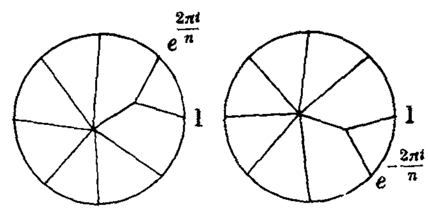


Fig. 20

If k is even, a similar proof works, but the fact that one is now comparing to $(z^k + iaz^{\frac{1}{2}k-1})dz^2$ allows one to subtract

$$\frac{2}{k+2}\sum(-1)^i p_i$$

from p_m for even m which makes the alternating sum vanish. Lemma 3.13 can then be applied by choosing $a = (1/\pi)\sum(-1)^i p_i$.

Thus q can be written in the coordinate ζ: $X_\Gamma \xrightarrow{\sim} \mathbf{C}$ as a polynomial quadratic differential of degree k. A unique translation will make the coefficient of the linear team vanish, and a multiplication unique up to a $k+2$ root of 1 will make the leading coefficient 1. The root of one is uniquely specified by requiring that the ray of Γ_q previously going to 1 be asymptotic to the positive real axis. Q.E.D.

The next proposition is purely topological; it requires only the definition of S_n. In order to prove (i) of Proposition 3.3, all we need is that S_n is connected, so we will be a bit sketchy on the full proof, as we get it also from the general inductive argument.

PROPOSITION 3.16. *The simplicial complex S_n is homeomorphic to the sphere of dimension $n-4$.*

Proof. We shall construct a homeomorphism by induction. Suppose that Proposition 3.16 is true for all $n' < n$. We shall give a decomposition of S_n into two "polar" zones homeomorphic to balls of dimension $n-4$ and a "temperate" zone which is homotopy equivalent to S_{n-1}. The polar zones U_1 and U_2 will be the closure of the stars of the two vertices v_1 and v_2 drawn below:

Each of these is simplicially equivalent to the cone over S_{n-1}; recall that the cone of summit v over a simplicial complex (V, ζ) is the simplicial complex (V', ζ') where $V' = V \cup \{v\}$, and $\sigma \in \zeta'$ if and only if either $\sigma \in \zeta$, or $\sigma = \tau \cup \{v\}$, with $\tau \in \zeta$. Indeed, each vertex of S_{n-1} can be identified with a vertex of U_1 (resp. U_2) other than v_1 (resp. other than v_2) by joining the node where the edge coming from 1 meets the interior of the graph to $e^{\pi i/(n-1)}$

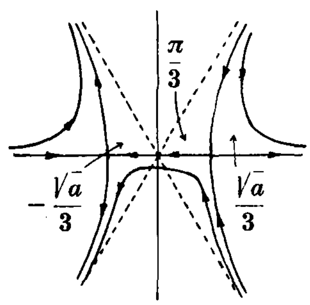


Fig. 21

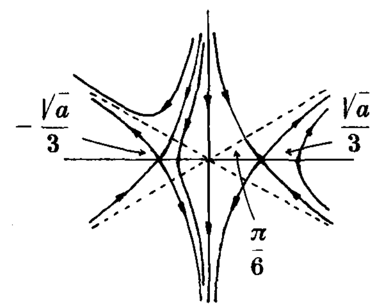


Fig. 22

(resp. $e^{-\pi i/(n-1)}$) and adjusting the positions of the points on the unit circle so that they form the nth roots of 1 (keeping 1 fixed). Then the vertex v_1 (resp. v_2) and the vertices of S_n constructed above clearly span a simplicial complex equivalent to the cone over S_{n-1}. So by the inductive assumption U_1 and U_2 are balls.

Consider the simplicial complex S'_n found by removing the stars of the vertices v_1 and v_2 from S_n.

Map $S'_n \to S_{n-1}$ by simply erasing the segment joining 1 to the interior graph, and re-adjusting the roots rotating counterclockwise. This is well defined since there will still be at least one interior edge. In the topological realization, the fibres are easily seen to be intervals, sometimes degenerate (i.e. points). Collapsing these intervals gives a homotopy equivalence of S_n with the $(n-4)$-sphere, and a little extra work shows that S_n is homeomorphic with the sphere. (Topologically $|S'_n| = |S_n| - (|\mathring{U}_1| \cup |\mathring{U}_2|)$ is the complement of the open stars.)

§ 7. The space E_3

The object of this paragraph is to prove (iv) of Proposition 3.3, i.e. that the map which sends $(z^3+az+b)dz^2 \in E_3$ to the linear coefficient a is single sheeted. The proof is nothing but a detailed look at the differential equation $(z^3+az+b)(z')^2=1$, using elementary techniques, mainly drawing the field of slopes.

Essentially, everything can be deduced from the following two drawings: let $a>0$ be real, and consider the orthogonal families of curves

$$\operatorname{Im}(z^3-az)=u$$
$$\operatorname{Re}(z^3-az)=v;$$

they are represented in Figures 21 and 22, and the arrows on the first (second) correspond to increasing v (resp. u).

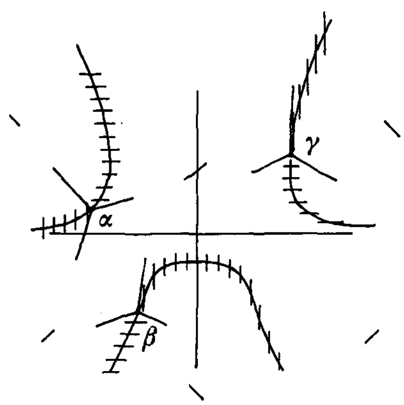


Fig. 23

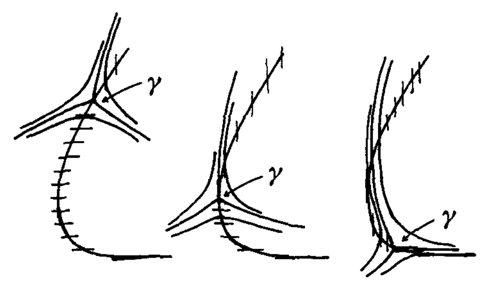


Fig. 24

Remark. For any given value of u (resp. v) a curve of the first family generally has three branches, which are usually disjoint, but sometimes degenerate. In particular, for $u=0$, the curve consists of the real axis and the hyperbola $3x^2-y^2=a$.

PROPOSITION 3.17. *Let $a>0$ be real. Then $(z^3-az-b)dz^2$ is in E_3 if and only if*

$$b=-\frac{a}{3}\sqrt{\frac{4a}{3}}.$$

Proof. By symmetry, it is enough to prove the result if $\operatorname{Im} b \geqslant 0$. The case when b is real will be left to the reader, and is in any case a straightforward limiting case of the case $\operatorname{Im} b>0$.

If $\operatorname{Im} b>0$, there is exactly one root of the polynomial z^3-az-b on each of three branches of the cubic $\operatorname{Im}(z^3-az)=\operatorname{Im} b$ which in this case looks like Figure 23. Call the one in the second quadrant α the one with negative imaginary part β and the one in the first quadrant γ.

LEMMA 3.18. $\operatorname{Re}\alpha<\operatorname{Re}\beta<\operatorname{Re}\gamma$.

Proof. Start with b real, and follow the roots as the imaginary part increases. A look at the arrows in Figure 22 shows that for each of the possible initial positions of the roots, they evolve in such a way as to satisfy the lemma. Q.E.D.

Now consider the differential equation $(z^3-az-b)(z')^2=1$. The slope field is drawn in Figure 23. It is clear that the only way in which the quadratic form could have a connected graph is if both α and β were connected to γ by the critical graph. Thus two of the critical rays emanating from γ should leave the first quadrant. But this does happen, as Figure 24, which gives all the possibilities for γ, should make clear.

CHAPTER IV

Proof of the theorems

In this chapter we will pull together the results of Chapters II and III to prove the main result.

§ 1. The local homeomorphism

Let $q \in Q$ be a holomorphic quadratic form on X, and F the underlying measured foliation.

PROPOSITION 4.1. *The projection $p: Q \to \Theta$ induces an open mapping $E_F \to \Theta$ at q.*

Proof. In this paragraph we will prove the result only if q is not a square. The case where q is a square requires different techniques, and will be treated in § 5 with preliminaries in § 2, 3, 4. We invite the reader to compare the present proof with that of Proposition 3.6.

Step 1. Suppose that q vanishes at $x_1, ..., x_n$ to the orders $k_1, ..., k_n$. Call $\Xi \to \Theta$ the universal curve over Teichmüller space, and consider the curve $p^*\Xi \to Q$. This curve is smooth over Q, so for each x_i there is a neighborhood U_i of q in Q, an open subset $W_i \subset X$ with $x_i \in W_i$ and an embedding α_i: $U_i \times W_i \to p^*\Xi$ commuting with the projections to Q and which restricts to the inclusion $W_i \subset X$ on $\{q\} \times W_i$.

The curve $p^*\Xi$ carries the tautological relative quadratic form (the one which restricts to q on the fibre above q), and using the embedding $U_i \times W_i \hookrightarrow p^*\Xi$, above this yields a family of quadratic forms on W_i parametrized by U_i. Choose a local coordinate z on W_i such that $q = z^{k_i} dz^2$ in W_i. Proposition 3.1 gives a map f_i: $U_i' \to P_{k_i}$ for some U_i' neighborhood of q in U_i classifying the deformation of $z^{k_i} dz^2$ given by the above family.

Set $U = \bigcap U_i'$ and consider the map f: $U \to \prod_i P_{k_i}$ whose ith entry is f_i; as a first step in constructing E_F we wish to consider $f^{-1}(\prod_i E_{k_i})$. Since E_k is not quite a submanifold of P_k if k is even this is not the right thing to do; call still f: $U \times \mathbf{R}^n \to \prod_i P_{k_i} \times \mathbf{R}^n$ the map f above extended by the identity on the second factor. Now since E_k is a submanifold of $P_k \times \mathbf{R}$, we consider $V \subset U \times \mathbf{R}^n$ defined by $V = f^{-1}(\prod_i E_{k_i})$.

Remark. Points of V correspond to quadratic forms q' near q whose critical graph is connected "near the zeroes of q". The drawing in Figure 25 illustrates what such a q and q' might look like.

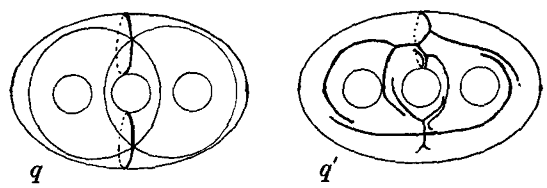


Fig. 25

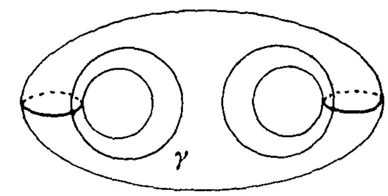


Fig. 26

Lemma 4.2. (i) *The set V is a C^1 submanifold of $U \times \mathbf{R}^n$, and the projection p: $U \to \Theta$ induces a submersion $V \to \Theta$.*

(ii) *The vertical tangent space $T_{V/\Theta, q}$ is the set of $(q', s) \in \Omega^{\otimes 2}(X) \times \mathbf{R}^n$ such that q'/ω_q is holomorphic on $\tilde{X}_q$, s_i arbitrary.*

Proof. This is an application of the implicit function theorem. Recall from Chapter 3, § 4, the spaces H_k, $L_k \subset P_k$ (resp. $P_k \times \mathbf{R}$ if k even). Define $E_k \subset P_k$ (resp. $P_k \times \mathbf{R}$) as $h^{-1}(0)$ for an appropriate mapping $h: P_k \to L_k$ (resp. $P_k \times \mathbf{R} \to L_k$). This is possible by Proposition 3.6, and $d_0 h$ is the projection $\tilde{H}_k \times L_k \to L_k$.

Now V is defined in $U \times \mathbf{R}^n$ as $(h \circ f)^{-1}(0)$. The lemma will be proved if we show that $d_q(h \circ f)|_{\Omega^{\otimes 2}(X) \times \mathbf{R}^n}$ is surjective, and that its kernel is our candidate for the vertical tangent space. By Proposition 3.1 $d_q f$ sends any q' to its $(k_i - 2)$-jet at each x_i and is the identity on the second factor. The derivative of h further truncates q' to its $[k_i/2] - 1$ jet at each x_i and vanishes on the s_i.

Let $J_k(x)$ be the space of k-jets at x of quadratic forms on X. We see that $d_q(h \circ f)|_{\Omega^{\otimes 2}(X) \times \mathbf{R}^n}$ is the map $\Omega^{\otimes 2}(X) \times \mathbf{R}^n \to \oplus J_{[k_i/2]-1}(x_i)$ which vanishes on the second factor, and sends $q' \in \Omega^{\otimes 2}(X)$ to its $[k_i/2] - 1$-jets at the x_i. This part fits into the exact sequence

$$H^0(X, \Omega^{\otimes 2}) \to \oplus J_{[k_i/2]-1}(x_i) \to H^1(X, \Omega^{\otimes 2}(-\textstyle\sum [k_i/2] x_i))$$

coming from the exact sequence of sheaves

$$0 \to \Omega^{\otimes 2}(-\textstyle\sum [k_i/2] x_i) \to \Omega^{\otimes 2} \to \oplus J_{[k_i/2]-1}(x_i) \to 0.$$

The above H^1 is dual to $H^0(X, T_X(\sum [k_i/2] x_i)$ by Serre duality. If the sheaf $T_X(\sum [k_i/2] x_i)$ has a non-zero section χ it is easy to check $q = c\chi^{-2} = (c^{1/2}\chi^{-1})^2$, and thus q is a square. This proves Lemma 4.2.

Remark. (a) The one-form χ^{-1} is the unique one such that $\chi^{-1}(\chi) = 1$.

(b) Although it is of course necessary that the zeroes of q be of even order for q to be a square, it is not sufficient. The Strebel form in Figure 26 is a counter-example. Indeed the foliation is not orientable around the loop γ.

Step 2. Recall that there is a local system over U with fibre $H_1(\hat{X}_q)$, and it can of course be restricted to V. Define a map

$$g\colon V \to \operatorname{Hom}(H_1(\hat{X}_q); \mathbf{R})$$

by the formula $q' \mapsto (\gamma \mapsto \operatorname{Im} \int_\gamma \omega_{q'})$. This map is well defined although γ, as a cohomology class on the Riemann surface above q', is only defined up to elements of the vanishing homology. This is true because in V the integrals of $\omega_{q'}$ over such vanishing classes are real.

Moreover g is of class C^1, for this is precisely the effect of adding the coordinates s_i.

LEMMA 4.3. *Let $W = g^{-1}(g(q))$; then $W \subset U \cap E_F$.*

Proof. Given any $\gamma \in S$ and any $q' \in U$, denote by $\tilde{\gamma} \in H_1(\tilde{X}'_{q'}, \tilde{\Gamma})$ the lift defined in Chapter 2, § 4. Then the transverse length $F_{q'}(\gamma)$ is given by $\frac{1}{2} \operatorname{Im} \int_{\tilde{\gamma}} \omega_{q'}$. Clearly the values of such integrals are constant in W.

Now clearly Proposition 4.1 follows from the following lemma.

LEMMA 4.4. *The map $p\colon W \to \Theta$ is a local homeomorphism at q.*

Proof. First observe that under the map $q' \mapsto q'/\omega_q$ the space of q' whose images are holomorphic on $\tilde{X}_q$ is identified with $H^0(\tilde{X}_q, \Omega)^-$, there is therefore a canonical isomorphism of the vertical tangent space to V with $\mathbf{R}^n \times H^0(\tilde{X}_q, \Omega)^-$. Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{R}^n & \longrightarrow & T_{V/\theta, q} & \longrightarrow & H^0(\tilde{X}_q, \Omega)^- & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle d_q g} & & \downarrow{\scriptstyle \wr} & & \\
0 & \longrightarrow & \mathbf{R}^n & \longrightarrow & \operatorname{Hom}(H_1(\hat{X}_q)^-, \mathbf{R}) & \longrightarrow & \operatorname{Hom}(H_1(\tilde{X}_q)^-, \mathbf{R}) & \longrightarrow & 0
\end{array}$$

where the top line comes from the argument above and the bottom one from the homology exact sequence of the pair $(\hat{X}_q, \tilde{X}_q)$ as in III, § 3, (iii).

Exactly as in Proposition 3.6 the diagram commutes and the maps at the right and left are isomorphisms, so $d_q g$ is an isomorphism. Q.E.D.

§ 2. The tangent space to Q

The proof given in the last section fails if q is a square as the map $H^0(X, \Omega^{\otimes 2}) \to \oplus P_{[k_i/2]-1}$ is not onto. In fact we need to investigate the classifying map $f\colon U \to \oplus P_{k_i}$ more closely. In order to do this we need to identify the tangent space to Q in terms analogous to the Kodaira–Spencer identification $H^1(X, T_X) = T_X \Theta_M$ (T_X is the sheaf of germs of holomorphic vector fields on X).

There are several ways of obtaining this isomorphism; one in terms of the Dolbeault resolution was found by Earle and Eells and is described in [6]. The one in terms of Cech cohomology will be more convenient for our purposes; it is less well adapted to actually constructing the various spaces, but as we already know that Q exists, this will not matter.

The idea is to differentiate the change of coordinates with respect to the parameters. Let $\Pi: \Xi_M \to \Theta_M$ be the universal curve over Teichmüller space and $\theta \in \Theta_M$ correspond to the Riemann surface X (in this discussion the Teichmüller marking is irrelevant). Since Π is a submersion, every point $x \in X$ has a neighborhood U in X such that there is an open neighborhood S of θ in Θ_M and an isomorphism $\alpha: U \times S \to \Xi_M$ onto an open subset which commutes with the projections on Θ_M. Such a pair (U, α) is called a relative coordinate chart.

Pick relative coordinate charts (U_i, α_i) such that the U_i form a cover of X. Then for any V_{ij} relatively compact in $U_i \cap U_j = U_{ij}$ the map $\alpha_{ij}(s) = \alpha_j^{-1}(s) \circ \alpha_i(s)$: $V_{ij} \to U_{ij}$ can be defined for s sufficiently near θ. These $\alpha_{ij}(s)$ are called relative change of coordinate maps.

Since $\alpha_{ij}(\theta)$ is the identity, the derivative of α_{ij} with respect to s at $s=\theta$ is a vector field on V_{ij}, and since V_{ij} was arbitrary in U_{ij}, we actually get a vector field χ_{ij} on U_{ij}. These define a map $T_\theta \Theta_M \to C^1(U, T_X)$.

One must check that the image falls in the cocycles and that it does not depend on the relative coordinate charts chosen. The first point is settled by differentiating the relation $\alpha_{jk} \circ \alpha_{ij} = \alpha_{ik}$. For the second suppose (U_i, α_i') is another relative atlas (with the same sets U_i), and denote with a prime everything coming from the new atlas. Then if $\beta_i(s) = \alpha_i^{-1}(s) \circ \alpha_i'(s)$ (defined on any relatively compact subset $V_i \subset U_i$), the derivative at θ of the relation $\beta_j^{-1}(s) \circ \alpha_{ij}(s) \circ \beta_i(s) = \alpha_{ij}'(s)$ gives $\chi_{ij}' - \chi_{ij} = \chi_i - \chi_j$, i.e. the cocycles χ and χ' are cohomologous. The general case is deduced from this one by refining the covering.

All of this gives a map $T_\theta \Theta_M \to H^1(X, T_X)$. It is not quite obvious that it is either injective or surjective, and both must be proved either from a construction of Teichmüller space or from functorial properties of the deformation functor. We will not do this here, but refer to [6] or [11] for detailed proofs.

The analogous description of the tangent space to Q is slightly more difficult. We could describe a Riemann surface near a given one using the same charts, and slightly deformed change of coordinate maps, but we cannot describe an arbitrary Riemann surface *with a quadratic differential* near a given one by gluing together coordinate patches of the original Riemann surface by perturbed coordinate transformations which preserve the *original* quadratic form. The difficulty is that we do not get enough points of Q this way, because the multiplicities of the zeroes of the original quadratic differential will be preserved by this operation, and we will not have allowed the deformations which break up a multiple zero into several less degenerate ones.

Thus to describe deformations of a Riemann surface with a quadratic differential we need to do the following operation: cover the original surface with coordinate patches, and vary both the quadratic differential in each patch, and the change of coordinate maps, subject to the obvious compatibility condition that the deformed change of coordinate maps send the deformed quadratic forms into each other.

The appropriate language for describing this construction is hypercohomology. The reader can find excellent treatments of the subject in [2] and [8]. We shall only develop what is strictly necessary for our purposes.

Let X be a Riemann surface (as always, compact of genus $g \geqslant 2$) and q_0 a quadratic form on X.

Consider the complex of sheaves $L^\cdot$:

$$0 \longrightarrow T_X \xrightarrow{L_\chi} \Omega_X^{\otimes 2} \longrightarrow 0$$

where L_χ is the Lie derivative. The first hypercohomology group of this complex is our candidate for $T_{q_0}Q$.

In order to construct a map $T_{q_0}Q \to \mathbf{H}^1(L^\cdot)$, consider the family $p^*\Xi \to Q$; the fibre over (X, q) is the Riemann surface X, and carries the quadratic form q. Repeat the construction of the previous paragraph: above a small neighborhood S of q_0 in Q pick relative coordinate charts (α_i, U_i) and consider the pairs

$$\alpha_{ij}(s) = \alpha_j^{-1}(s) \circ \alpha_i(s), \quad \varphi_i(s) = \alpha_i(s)^* q.$$

These satisfy the relations

$$\alpha_{jk}(s) \circ \alpha_{ij}(s) = \alpha_{ik}(s) \quad \text{and} \quad \alpha_{ij}(s)^* \varphi_j(s) = \varphi_i(s).$$

If we let $\chi_{ij} = d_{q_0}\alpha_{ij}$, $\psi_i = d_{q_0}\varphi_i$, the derivatives of the relations above give

$$\chi_{ij} + \chi_{jk} = \chi_{ik} \quad \text{and} \quad L_{\chi_{ij}} q_0 = \psi_i - \psi_j.$$

Of course, all the computations above should be understood restricted to the appropriate domains.

Thus we have an element $(\chi, \psi) \in C^1(U, T_X) \oplus C^0(U, \Omega^{\otimes 2}) = C^1(L^\cdot)$, and the above relations say exactly that the image is a cocycle.

Proposition 4.5. *The induced map $T_q Q \to \mathbf{H}^1(L^\cdot)$ does not depend on the relative atlas (α_i, U_i), and is an isomorphism.*

Proof. The first part is straightforward: pick another atlas (α_i', U_i') and denote with a prime everything coming from the new atlas. Define maps $\beta_i(s) = \alpha_i^{-1}(s) \circ \alpha_i'(s)$, then we have the relations

$$\beta_j^{-1}(s) \circ \alpha_{ij}(s) \circ \beta_i(s) = \alpha_{ij}'(s) \quad \text{and} \quad \beta_i(s)^* \varphi_i(s) = \varphi_i'(s).$$

Differentiating these relations with respect to s and setting $\chi_i = d_{q_0}\beta_i$, we get $\chi_{ij} - \chi_{ij}' = \chi_j - \chi_i$ and $L_{\chi_i} q_0 + \psi_i = \psi_i'$. This is just what is needed to claim that (χ, ψ) and (χ', ψ') are cobordant.

The second part comes from the exact sequence of complexes

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & \Omega_X^{\otimes 2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_X & \longrightarrow & \Omega_X^{\otimes 2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_X & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

This leads to the long exact sequence

$$H^0(X, T_X) \to H^0(X, \Omega_X^{\otimes 2}) \to H^1(L^\cdot) \to H^1(X, T_X) \to H^1(X, \Omega_X^{\otimes 2}).$$

The two end terms are zero. It is clear from the construction that the map $H^0(X, \Omega_X^{\otimes 2}) \to H^1(L^\cdot)$ is induced by the inclusion of the fibre $H^0(X, \Omega^{\otimes 2}) \to Q$, and similarly that the map $H^1(L^\cdot) \to H^1(X, T_X)$ commutes with the derivative $T_q Q \to T_\theta \Theta_M$ of the natural projection. Since $T_\theta \Theta_M = H^1(X, T_X)$, the result is proved. Q.E.D.

Remark. It is clear that a similar description is possible for the tangent space to any universal space of compact manifolds with tensors. In general, the long exact sequence above does not have vanishing end-terms; in the general case the map $H^0(X, T_X) \to H^0(X, \Omega_X^{\otimes 2})$ expresses the fact that the universal space is not the total space of the vector bundle but its quotient by Aut (X). The map $H^1(X, T_X) \to H^1(X, \Omega_X^{\otimes 2})$ measures the obstruction to extending a tensor when the underlying manifold is deformed.

There is a spectral sequence which relates hypercohomology groups of a complex of sheaves and the cohomology groups of the associated cohomology sheaves.

The kernel Λ_{q_0} of $L_\chi q_0$: $T_X \to \Omega_X^{\otimes 2}$ is called the sheaf of locally constant vector fields. This terminology is justified by observing that if $q_0 = dz^2$ in the local coordinate z and if $\chi = \chi(z)d/dz$ then $L_\chi q_0 = 2\chi'(z)dz^2$, so χ is in the kernel if and only if $\chi(z)$ is constant. It is easy to check that there are no non-vanishing locally constant vector fields on a connected open set in which q_0 has a zero.

The cokernel of $L_\chi q_0$: $T_X \to \Omega_X^{\otimes 2}$ is a skyscraper sheaf supported by the multiple zeroes of q_0; this is the content of Proposition 3.1. In fact the proposition tells us that the stalk of the cokernel at a point x where q_0 has a zero of order k is the quotient $\overline{P_k(x)}$ of the germs of quadratic forms at x by those that vanish at least to the order $k-1$.

To be more precise consider the map f: $U \to \oplus P_{k_i}$ classifying the deformations of the zeroes of q_0.

Proposition 4.6. *The map* $\mathbf{H}^1(L^\cdot) \to P_k(x)$ *induced by the sheaf map* $\Omega_X^{\otimes 2} \to P_k(x)$ *is the derivative of* f *at* q_0.

Proof. This is just a restatement of Proposition 3.1.

If q_0 has zeroes $x_1, \ldots, x_n$ of order $k_1, \ldots, k_n$, we wish to see how $\mathbf{H}^1(L^\cdot)$ is made up of $H^1(X, \Lambda_{q_0})$ and of the $P_{k_i}(x_i)$.

Proposition 4.7. (a) *If* q_0 *is not a square, the inclusion* $\Lambda_{q_0} \to T_X$ *and the projection* $\Omega_X^{\otimes 2} \to P_{k_i}(x_i)$ *induce an exact sequence*

$$0 \to H^1(X, \Lambda_{q_0}) \to \mathbf{H}^1(L^\cdot) \to \bigoplus_i P_{k_i}(x_i) \to 0$$

(b) *If* $q_0 = \varphi^2$ *is the square of a one-form, the maps above induce an exact sequence*

$$0 \to H^1(X, \Lambda_{q_0}) \to \mathbf{H}^1(L^\cdot) \to \bigoplus_i P_{k_i}(x_i) \to \mathbf{C} \to 0$$

Remark. Part (b) is what is needed in the main theorem.

Proof. There is a spectral sequence [8] with $E_2^{p,q} = H^1(X, h^q(L^\cdot))$ which converges to $\mathbf{H}^{p+q}(L^\cdot)$. In our case, the E_2 term looks like

$$\begin{array}{l|llll}
q\uparrow & \bigoplus_i P_{k_i}(x_i) & 0 & 0 & 0 \\
 & & \quad\searrow^{d_2} & & \\
 & H^0(X, \Lambda_{q_0}) & H^1(X, \Lambda_{q_0}) & H^2(X, \Lambda_{q_0}) & 0 \\
\end{array} \qquad \longrightarrow p$$

If q_0 is not a square, Λ_{q_0} is not orientable, so $H^2(X, \Lambda_{q_0})=0$ and the terms along the anti-diagonal $p+q=1$ are the graded group for the appropriate filtration of $\mathbf{H}^1(L^{\cdot})$, giving the exact sequence and proving (a).

If q_0 is a square, $H^2(X, \Lambda_{q_0}) \cong \mathbf{C}$, so the derivative is onto a hyperplane. We compute the image of f by Cauchy's Theorem.

Consider the map α_i: $W \to \mathbf{C}$ defined in a neighborhood of $z^{k_i}dz^2$ in P_{k_i} by $q \to \int_{\gamma_i} q^{1/2}$, where γ_i is a circle of large radius and the branch of the square root along γ which passes through $z^{\frac{1}{2}k_i}dz$ is chosen continuously in W; this is possible because k_i is even.

Suppose $q=\varphi^2$; the map f_i: $U \to P_{k_i}$ depends on the choice of a local coordinate near x_i for which $q=z^{k_i}dz^2$; pick one such that also $\varphi = z^{\frac{1}{2}k_i}dz$ (half of the possible local coordinates will work, for the other half $\varphi = -z^{\frac{1}{2}k_i}dz$).

LEMMA 4.8. *The map f: $U \to \prod P_{k_i}$ is a submersion onto the submanifold of $\prod P_{k_i}$ defined by the equation $\sum \alpha_i(q_i)=0$.*

Proof. Pick small discs D_i around the x_i, with boundary circles γ_i'. If U is a sufficiently small neighborhood of q, then every $q' \in U$ has a square root in $X - UD_i$ and we can pick the one which can be continued from φ. By Cauchy's theorem $\sum \int_{\gamma_i} \sqrt{q'}=0$ which translates to $\sum \alpha_i(q_i)=0$ when looked at from the inside rather than the outside of the discs. By Proposition 4.7 (b), f must be a submersion onto this submanifold.

We see that we must understand the set $Z \subset \prod E_k$ defined by $\sum \alpha_i(q_i)=0$.

§ 3. Perverse manifolds

The space Z defined in the previous paragraph turns out to be a most peculiar object: a differentiable manifold which is not of class C^1. Such manifolds are badly behaved: the implicit theorem cannot be applied to them. In fact, it is not quite clear what the right definition is, since the equivalence of the definitions of C^1 manifolds by parametrizations and by equations uses the implicit function theorem.

Definition. A subset X of $\mathbf{R}^n$ is a differentiable manifold of dimension d if for every $x \in X$ there is a neighborhood U of x in $\mathbf{R}^n$, an open subset V of $\mathbf{R}^d$ and an injective map α: $V \to \mathbf{R}^n$, $\alpha(v)=x$, α has an injective derivative at v, and $\alpha(V) \cap U = X \cap U$. The tangent space at x is the image of $d_v\alpha$. A function on X is differentiable if it is the restriction to X of a differentiable function on a neighborhood of X.

The manifold is called *perverse* if α is not of class C^1.

Example. Consider in R^3 the set defined by the equation $z=e^{-(1/x^2+y^2)}\sqrt[3]{xy}$. The following drawing represents this surface; notice that arbitrarily near 0 there are points

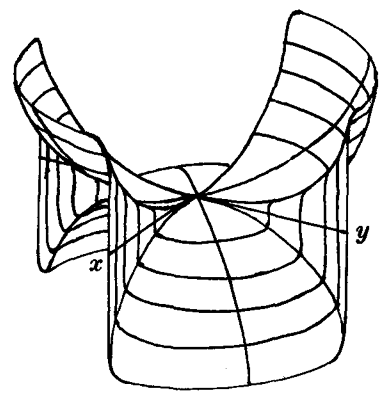


Fig. 27

with vertical tangent space. Of course (x, y) parametrizes the surface, and z is a differentiable (but not C^1) function of (x, y) at 0 so we do have a perverse manifold. No oblique linear projection onto the $x-y$ plane is injective near 0, even though such a projection is the identity on the tangent space at 0, so the inverse function theorem is false for this perverse manifold.

Even for perverse manifolds, there is a weak sort of implicit function theorem.

LEMMA 4.9. *Let X be a differentiable manifold and $x \in X$, and $f: X \to \mathbf{R}^m$ a differentiable map whose derivative at x is surjective. Then f is open at x.*

Proof. Using a local parameter we can suppose that f is a map defined on an open subset $V \subset \mathbf{R}^d$ with surjective derivative at 0. Pick a subspace $E \subset \mathbf{R}^d$ on which $d_0 f$ is an isomorphism. It is enough to prove $f|_{V \cap E}$ is open at 0. Now by the definition of the derivative

$$f(x) = f(0) + d_0 f(x) + \varepsilon(x)$$

and for δ sufficiently small, $\|\varepsilon(x)\| \leq \|d_0 f(x)\|$ if $\|x\| < \delta$.

Therefore restricted to the sphere $S_{\delta'}$, of radius $\delta' < \delta$, the map $x \to f(x) - f(0)$ is homotopic to the map $x \to d_0 f$, as maps $S_{\delta'} \cap E \to \mathbf{R}^m - \{0\}$. But this last map is of degree ± 1.

Now it is a standard result in algebraic topology that if $f: B^m \to \mathbf{R}^m$, $f(0) = 0$, is a continuous map such that $f|_{S_\delta^{m-1}}: S_\delta^{m-1} \to \mathbf{R}^m - \{0\}$ is of degree $\neq 0$ for every sphere about the origin, then the image of B_δ^m is a neighborhood of 0 for every ball B_δ^m. See also [4, p. 269].

§ 4. The space Z

PROPOSITION 4.10. *The set $Z \subset \prod P_{k_i} \times \mathbf{R}^n$ is a perverse submanifold at $z^{k_i} dz^2$ $i = 1, \ldots, n$ and the tangent space at that point is the hyperplane defined in $\prod \bar{H}_{k_i}$ by the equation $\sum (k_i + 2) = 0$.*

Proof. As in Lemma 3.9 we let $h(q_n)$ be the imaginary part of the middle coefficient of q_n. The high coefficients on E_{k_n} are $a_{k-2}, \ldots, a_{\frac{1}{2}k}$. We will solve for s_n on Z as a function of the other variables. By Lemma 3.10 on the complement of the square locus of E_{k_n}, $\partial s_n/\partial h<0$ near $z^{k_n}dz^2$. Also the line $(a_{k-2}, \ldots, a_{\frac{1}{2}k})$ constant in E_{k_n} intersects the square locus in at most one point since the high coefficients of any polynomial determine the polynomial if it is a square.

Now for $q_n=(z^{k_n}+a_{k-2}z^{k_n-2}+\ldots+a_0)dz^2$, by Lemma 3.9, $\alpha_n(q_n)=\pi h+\pi i \operatorname{Re} a_{\frac{1}{2}k-1}+$ higher order terms in the high coefficients. Therefore α_n is differentiable and $\partial\alpha_n/\partial h=\pi$. Then

$$\frac{\partial\alpha_n}{\partial s_n}=\frac{\partial\alpha_n}{\partial h}\frac{\partial h}{\partial s_n}<0$$

on each line except possibly at a square and α_n is a strictly decreasing function of s_n.

To make use of this information we make use of the fact that for $t\in\mathbf{R}$, $p_t(z)=(z^k+itz^{\frac{1}{2}k-1})dz^2\in E_k$. By the change of variables $\zeta=1/z$ and a Taylor series expansion of the square root near $\zeta=0$, we find $s=(1/(k+2))t\log(t)+O(t)$ and $\alpha(p_t)=\pi|t|$.

From the continuity of α_n and the monotonicity on each line we conclude there is a neighborhood $U\times V$ of $(0,0)\in\mathbf{R}\times\mathbf{R}^{\frac{1}{2}k-1}$ such that for $\alpha=\alpha_1+\ldots+\alpha_{n-1}\in U$ and $(a_{k-2}, \ldots, a_{\frac{1}{2}k})\in V$ there is a unique s_n and $q_n\in E_{k_n}$ with local parameters $(a_{k-2}, \ldots, a_{\frac{1}{2}k}, s_n)$ such that $\alpha_n(q_n)=-\alpha$.

It remains to compute the derivatives of s_n with respect to the high coefficients of $\prod_{i=1}^n E_{k_i}$ and $s_1, \ldots, s_{n-1}$. If a_j is a high coefficient of $\prod_{i=1}^{n-1} E_{k_i}$ then for $p=(z^k+ta_jz^j)dz^2$, $\alpha(p)=O(t^2)$. The line $(z^{k_n}+it^2z^{\frac{1}{2}k-1})$ in E_{k_n} gives $\alpha_n=-\pi t^2$ and $s_n=o(t^2\log(t^2))$. Thus $\partial s_n/\partial a_j=s_n'(0)=0$.

The case where a_j is a high coefficient of E_{k_n} is somewhat different. Since there is no variation in $\prod_{i=1}^{n-1} E_{k_i}$, $\alpha=\alpha_1+\ldots+\alpha_{n-1}=0$, and $\alpha_n=O(t^2)$. The compensating coefficient h is then also $O(t^2)$. But now the Taylor series for

$$\sqrt{z^{k_n}+ta_jz^j+ha_{\frac{1}{2}k_n-1}z^{\frac{1}{2}k_n-1}}$$

near $z=\infty$ gives $s_n=O(h\log|h|)=O(t^2\log|t|^2)$ and again $\partial s_n/\partial a_j=0$.

Finally the expansion $s=(1/(k+2))t\log|t|+O(t)$ and $\alpha(t)=\pi|t|$ for $p(z)=(z^k+itz^{\frac{1}{2}k-1})dz^2$ shows that $\partial s_n/\partial s_i=(k_i+2)/(k_n+2)$. The tangent space to Z is the image of the derivative, the hyperplane $\prod_{i=1}^n (k_i+2)s_i=0$.

§ 5. The map $E_F\to\Theta_M$ is open

In this paragraph we pull together the results of the previous three paragraphs to prove that the map $E_F\to\Theta_M$ is open near a square.

Recall from § 1 the space $V\subset U\times\mathbf{R}^n$.

PROPOSITION 4.11. (i) *If q_0 is a square, V is at q_0 a perverse submanifold of $U\times\mathbf{R}^n$ and the derivative of the projection $p\colon U\to\Theta_M$ is surjective on the tangent space to V.*

(ii) *The vertical tangent space $T_{V/\Theta},\, q_0$ is the set of $(q', s_1, \ldots, s_n)\in H^0(X, \Omega^{\otimes 2})\times\mathbf{R}^n$ such that q'/ω_{q_0} is holomorphic on $\tilde{X}_{q_0}$ and $\sum_{i=1}^n (k_i+2)s_i=0$.*

Proof. By Lemma 4.8 the map $f\colon U\times\mathbf{R}^n\to\prod P_{k_i}\times\mathbf{R}^n$ is a submersion onto the hyperplane $\sum\alpha_i(q)=0$ whose tangent space at $z^{k_i}dz^2$, $i=1,\ldots,n$ is the hyperplane $\sum a_{\frac{1}{2}k_i-1}=0$. It is easy to see that the inverse image of a perverse manifold by a submersion is again a perverse manifold so $V=f^{-1}(Z)$ is a perverse submanifold of $U\times\mathbf{R}^n$ at $(q_0, 0)$ and $T_{q_0,0}V$ is the inverse image of T_0Z under $d_{(q_0,0)}f$. By Lemma 4.2 the truncation map sends $H^0(X,\Omega^{\otimes 2})$ onto a hyperplane in $\oplus J_{\frac{1}{2}k_i-1}(x_i)$ and that hyperplane must be $\sum_{i=1}^n a_{\frac{1}{2}k_i-1}=0$. Therefore $T_{(q_0,0)}V$ must map surjectively onto the tangent space to Θ_M. Part (ii) follows immediately.

We again define a map $g\colon V\to\operatorname{Hom}(H_1(\hat{X})^-, \mathbf{R})$ by $q'\mapsto(\gamma, \operatorname{Im}\int_\gamma\omega_{q'})$ and let $W=g^{-1}g(q_0)$. Then $W=U\cap E_F$ as before. Now g is merely differential at $(q_0, 0)$.

LEMMA 4.12. *The map $p\colon W\to\Theta$ is open at $(q_0, 0)$.*

Proof. We observe as in Lemma 4.4 that there is a canonical isomorphism of the vertical tangent space to V with the hyperplane $\sum_{i=1}^n (k_i+2)s_i=0$ in $\mathbf{R}^n\times H^0(\tilde{X}_{q_0}, \Omega)^-$. Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{R}^{n-1} & \hookrightarrow & T_{V/\theta, q_0} & \longrightarrow & H^0(\tilde{X}_{q_0}, \Omega^-) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow{\scriptstyle d_{q_0}g} & & \downarrow\wr\!\| & & \\
\mathbf{R} & \xrightarrow{\Delta} & \mathbf{R}^n & \hookrightarrow & \operatorname{Hom}(H_1(\hat{X}_{q_0})^-, \mathbf{R}) & \longrightarrow & \operatorname{Hom}(H_1(\tilde{X}_{q_0})^-, \mathbf{R}) & \longrightarrow & 0
\end{array}$$

Here the left hand vertical map is the inclusion map onto the hyperplane. The bottom sequence is the transpose of the homology exact sequence of $(\hat{X}_{q_0}, \tilde{X}_{q_0})$ as in III, § 3 (ii) and Δ is the diagonal map. The diagram commutes as in Proposition 3.6 and since the hyperplane $\sum_{i=1}^n (k_i+2)s_i=0$ is complementary to the diagonal, $d_{(q_0,0)}g$ is an isomorphism.

Now the map $V\xrightarrow{(p,g)}\Theta_M\times\operatorname{Hom}(H_1(\hat{X}_{q_0})^-; \mathbf{R})$ has surjective derivative so by Lemma 4.9 it is open at $(q_0, 0)$. Therefore p restricted to W is open.

§ 6. Strebel's uniqueness theorem

We present here a proof that if F is a measured foliation on X with compact leaves, there is a unique holomorphic quadratic form on X inducing F. Strebel's theorem gives uniqueness, up to a real multiple, of a form with given moduli. Both results follow from

Lemma 4.12. In § 7, we will combine the first result with a density argument to prove uniqueness in the general case.

Let q be a holomorphic quadratic form on X which is Strebel, and C a cylinder for q with height h and circumference c. Let C' an abstract straight cylinder of height h' and circumference c'. Let $f: C' \to X$ be an injective holomorphic map.

LEMMA 4.13. (Strebel [15, 16]). *If the equators of $f(C')$ are homotopic to those of C, then*

$$\int_{f(C')} |q| \geqslant \frac{h'c^2}{c'}.$$

Equality is realized only if f is an inclusion of a subcylinder of C.

Proof. The proof is an application of the length area method. By a change of scale on C' we can suppose $c = c'$. For any γ equator of C', the length of $f(\gamma)$ is $\geqslant c$, since c is the length of the geodesic in its homotopy class. Picking coordinates $x \in R/cZ$, $y \in [0, h']$ on C', this can be written

$$\int_\gamma \left| q\left(\frac{\partial f}{\partial x} \otimes \frac{\partial f}{\partial x}\right)\right|^{1/2} dx \geqslant c.$$

Introducing a factor of 1 in the integrand, we get from the Schwarz inequality

$$c \int_\gamma \left| q\left(\frac{\partial f}{\partial x} \otimes \frac{\partial f}{\partial x}\right)\right| dx \geqslant c^2.$$

Since f is holomorphic, $|q(\partial f/\partial x \otimes \partial f/\partial x)| = |q(\partial f/\partial x \otimes \partial f/\partial y)|$ by Cauchy–Riemann. Therefore,

$$\int_{f(C')} |q| = \int_{C'} f^*|q| = \int_0^{h'} \left(\int_0^c \left| q\left(\frac{\partial f}{\partial x} \otimes \frac{\partial f}{\partial y}\right)\right| dx\right) dy \geqslant ch'.$$

This proves the first part of the lemma. In case equality is realized, there must be equal signs throughout the proof, and in particular $f(\gamma)$ must be a geodesic, so f must send equators to equators, and since f is holomorphic it must be an isometric inclusion. Q.E.D.

For a cylinder with circumference c and height h define the modulus M to be h/c. The following uniqueness theorem is an easy consequence of Lemma 4.13.

PROPOSITION 4.14. *Let q be a holomorphic form on X with underlying foliation F with closed leaves and q another holomorphic quadratic form with underlying foliation F'.*

(a) *If the images of F and F' in $\mathbf{R}^S$ coincide, then $q = q'$.*
(b) (Strebel). *If F' also has closed leaves with cylinders homotopic to the cylinders of F and if M_i and N_i are the moduli of q and q' then* $\max N_i/M_i \geqslant 1$, $\max M_i/N_i \geqslant 1$, *equality holding in either case if and only if $q' = rq$ for $r \in \mathbf{R}$.*

Proof. For the proof of (a) we recall from Lemma 2.9 that F' also has closed leaves, and that if $C_1, \ldots, C_n$ are the cylinders for F, with heights h_j and circumferences c_j, then the cylinders for F' can be indexed $C_1', \ldots, C_n'$ so that equators of C_j are homotopic to equators of C_j'. Moreover the height of C_j' is h_j; call c_j the circumference of C_j'.

Applying Lemma 4.13 to the inclusion $f_j'\colon C_j' \to X$ and the cylinder C_j, we get

$$\int_{f(C_j')} |q| \geqslant \frac{h_j c_j^2}{c_j'},$$

and summing over j this gives

$$\sum h_j c_j \geqslant \sum \frac{h_j c_j^2}{c_j'}.$$

Similarly, applying Lemma 4.13 to the inclusion $f_j\colon C_j \to X$ and C_j' and summing, we get

$$\sum h_j c_j' \geqslant \sum \frac{h_j c_j'^2}{c_j}.$$

Set $u_j = h_j c_j$ and $v_j = h_j c_j'$; by adding the inequalities above we get

$$\sum (u_j + v_j) \geqslant \sum \left(\frac{u_j^2}{v_j} + \frac{v_j^2}{u_j}\right).$$

But for any positive reals u and v, we have $u + v \leqslant u^2/v + v^2/u$ with equality only if $u = v$, since $u^2/v + v^2/u - u - v = (1/u + 1/v)(u-v)^2$.

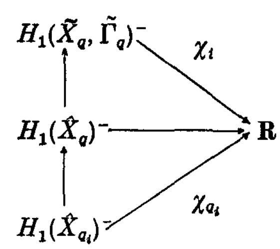
Therefore $u_j = v_j$ and $c_j = c_j'$ for all j. Moreover applying the second part of Lemma 4.13, we see that $C_j = C_j'$ and thus $q = q'$. Q.E.D.

Part (b) is easier and follows from the inequalities $\sum h_j c_j \geqslant \sum (h_j' c_j^2)/c_j'$ and $\sum h_j' c_j' \geqslant \sum (h_j c_j'^2)/c_j$.

§ 7. The uniqueness theorem

In order to apply Strebel's uniqueness theorem in the more general situation, we need a density statement analogous to that in [5]. If q is a quadratic form on X, we denote $\chi_q\colon H_1(\hat{X}_q)^- \to \mathbf{R}$ the canonical element $\gamma \mapsto \operatorname{Im} \int_\gamma \omega_q$. This also gives $\chi_q\colon H_1(\overline{X}_q, \tilde{\Gamma}_q)^- \to \mathbf{R}$. Furthermore there is a canonical map $H_1(\hat{X}_q)^- \to H_1(\tilde{X}_q, \tilde{\Gamma}_q)^-$ given by erasing any segment joining even zeroes (cf. III § 3 (ii)).

LEMMA 4.15. *Let q be a holomorphic quadratic form on X. For any sequence χ_i of elements of* Hom $(H_1(\tilde{X}_q, \tilde{\Gamma}_q)^-; \mathbf{R})$ *converging to χ_q, there is a sequence $q_i \in H^0(X, \Omega^{\otimes 2})$, for i large, converging to q, such that the diagram*



commutes where the map $H_1(\hat{X}_{q_i})^- \to H_1(\hat{X}_q)^-$ *is induced by a map* $\hat{X}_{q_i} \to \hat{X}_q$ *collapsing the vanishing homology.*

Proof. This is an immediate consequence of the fact that the map $V \cap H^0(X, \Omega^{\otimes 2}) \to \mathrm{Hom}\,(H_1(\hat{X}_q)^-; \mathbf{R})$ which is the restriction of g (cf. IV, § 1) is open. But this follows from the implicit function theorem and Lemma 4.4 if q is not a square, and from Lemma 4.12 if q is a square.

PROPOSITION 4.16. *If q and q' are holomorphic quadratic forms on X with underlying measured foliations F and F', and if F and F' have the same image in $\mathbf{R}^S$, then $q = q'$.*

Proof. By Corollary 2.9 there is a canonical isomorphism $H_1(\tilde{X}_q, \tilde{\Gamma}_q)^- = H_1(\tilde{X}_{q'}, \tilde{\Gamma}_{q'})^-$. Pick sequences χ_i and χ_i' in $\mathrm{Hom}\,(H_1(\tilde{X}_q, \tilde{\Gamma}_q)^-; \mathbf{R})$ and $\mathrm{Hom}\,(H_1(\tilde{X}_{q'}, \tilde{\Gamma}_{q_i})^-; \mathbf{R})$ converging to χ_q and $\chi_{q'}$ resp., commuting with the isomorphism above and formed entirely of homomorphisms of rank one. Then by Proposition 2 of [5] the sequences q_i and q_i' determined by Lemma 4.15 are entirely formed of quadratic forms whose horizontal foliations have closed leaves. Moreover the images of q_i and q_i' in $\mathbf{R}^S$ coincide by Corollary 2.9. Therefore $q_i = q_i'$ by Proposition 4.14 and $q = q'$. Q.E.D.

§ 8. Proof of the Theorems

In this paragraph we pull our results together.

Proof of the Main Theorem. By Lemmas 4.4 and 4.11 the map $E_F \to \Theta_M$ is open. Moreover, if $q, q' \in E_F$ lie on the same Riemann surface, by Proposition 4.16 they must be equal. Therefore the map is one-to-one, open, and therefore a homeomorphism onto its image. However E_F is proper over Θ_M by Lemma 2.15 so the image must be closed.

Proof of Theorem 2. It is not hard to construct a measured foliation F with closed leaves homotopic to the curves C and with the given heights. This can be done by a downward induction on the number of curves starting with $n = 3g - 3$. In that case the complement of the curves is union of spheres with three holes. The critical graph is drawn according

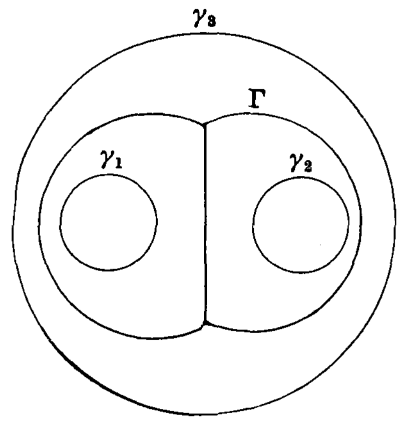


Fig. 28

to Figure 28. Once a measured foliation with k cylinders has been constructued, collapsing an appropriate cylinder gives a measured foliation with $k-1$ cylinders.

Now the main theorem shows that E_F maps homeomorphically to Θ_M. Then E_C maps bijectively onto $\Theta_M \times \mathbf{R}_+^n$ and it is clear that it is continuous in both directions.

Proof of Theorem 3. The uniqueness is of course Proposition 4.14. For the existence we proceed by induction on the number of cylinders p. The case $p=1$ is true by Theorem 2. Assume the theorem true for $k-1$ cylinders. Let Y_0^k be the open first quadrant in $\mathbf{R}^k$, Y^k its closure minus the origin. Each vector $h=(h_1, ..., h_k)$ in Y_0^k determines a unique quadratic form q with height vector h and cylinders homotopic to the curves γ_i. Let $(l_1, ..., l_k)$ be the corresponding circumference vector and consider the map

$$(h_1, \ldots, h_k) \to \left(\frac{h_1}{l_1}, \ldots, \frac{h_k}{l_k}\right).$$

The length of a geodesic in a fixed homotopy class is continuous on $H^0(X, \Omega^{\otimes 2})$ so the map extends to Y^k. We restrict the extension to the intersection of Y^k with the sphere S^{k-1} and follow with the map to S^{k-1} which is a retraction along lines. By Proposition 4.14 (b) the composition is injective. The restriction to $(Y^k - Y_0^k) \cap S^{k-1}$ is a homeomorphism onto itself by the induction hypothesis as we are reduced to considering $k-1$ cylinders. But $Y^k \cap S^{k-1}$ is a disc and $(Y^k - Y_0^k) \cap S^{k-1}$ is the boundary. An injective mapping of a disc which is a homeomorphism on the boundary is also a homeomorphism. The theorem follows.

Proof of Theorem 4. By Proposition 4.16, for each Riemann surface X, $H^0(X, \Omega^{\otimes 2}) - \{0\}$ maps injectively into $\mathbf{R}^S$. The main theorem associates equivalence classes of measured foliations with $H^0(X, \Omega^{\otimes 2})$. Therefore the map of measured foliations to $\mathbf{R}^S$ is an injection and the image is homeomorphic to $\mathbf{R}^{6g-6} - \{0\}$.

We also obtain the following purely topological result, originally due to Thurston.

PROPOSITION 4.17. *Equivalent measured foliations are strongly equivalent.*

Proof. We actually showed (in Prop. 4.1 and Lemma 4.12) that the map $p: E_F \to \Theta_M$ is open under the hypothesis of strong equivalence; the injectivity of p (Prop. 4.16) was shown for (weak) equivalence. The result follows immediately.

References

[1]. AHLFORS, L., On quasiconformal mappings. *Journal d'Analyse Math.*, (1954), 1–54.
[2]. ATIYAH, M. F., BOTT, R. & GÅRDING, L., Lacuna for hyperbolic differential operators II. *Acta Math.*, 131 (1973), 145–205.
[3]. BERS, L., Quasiconformal Mappings and Teichmüller's Theorem in *Analytic functions*. R. Nevanlinna et al., eds., Princeton University Press, 1960.
[4]. DIEUDONNÉ, J., *Foundations of Modern Analysis*. Academic Press. New York, 1960.
[5]. DOUADY, A. & HUBBARD, J., On the density of Strebel differentials. *Inventiones Math.*, 30 (1975), 175–179.
[6]. EARLE, C. & EELLS, J., A fibre bundle description of Teichmüller theory. *J. Differential Geometry*, 3 (1969), 19–45.
[7]. EPSTEIN, D., Curves on 2-manifolds and isotopies. *Acta Math.*, 115 (1966), 83–107.
[8]. GODEMENT, R., *Topologie Algebrique et Theorie des Faisceaux*, Paris, 1958.
[9]. GREENBERG, M., *Lectures on Algebraic Topology*. W. A. Benjamin, Reading, Mass. (1966).
[10]. GUNNING, R., *Lectures on Riemann Surfaces*. Princeton University Press, (1966).
[11]. HUBBARD, J., Sur les sections analytiques de la courbe universelle de Teichmüller. *Mem. Amer. Math. Soc.*, 166 (1976).
[12]. HUBBARD, J. & MASUR, H., On the existence and uniqueness of Strebel differentials. *Bull. Amer. Math. Soc.*, 82 (1976), 77–79.
[13]. JENKINS, J. A., On the existence of certain general extremal metrics. *Ann. of Math.*, 66 (1957), 440–453.
[14]. KERCKHOFF, S., The Asymptotic Geometry of Teichmüller Space, Thesis, Princeton University, 1978.
[15]. STREBEL, K., Über quadratische differentiale mit geschlossen trajectorien and extremale quasikonforme abbildungen. In *Festband zum 70. Geburstag von Rolf Nevanlinna*, Springer-Verlag, Berlin, 1966.
[16]. —— On quadratic differentials and extremal quasiconformal mappings. Lecture Notes, University of Minnesota, 1967.
[17]. THURSTON, W., On the geometry and dynamics of diffeomorphisms of surfaces. Preprint.

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