SIMILARITY OF OPERATOR ALGEBRAS

BY

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Dedicated to M. H. Stone on his seventy fifth birthday

1. Introduction

When viewed in a certain light, Tomita's theorem (the main result of the Tomita–Takesaki theory—see [3, 14, 15, 16, 17]) appears as the combination of a result on "unbounded" similarity between self-adjoint operator algebras and the special structure of a von Neumann algebra and its commutant relative to a joint separating vector. The main purpose of this article is to introduce and develop the theory of such similarities. (See section 3.) Our secondary purpose is to present a full proof of Tomita's theorem in the style mentioned. (See section 4.) In connection with this argument, we develop a new density result (Theorem 4.10). In section 2 we prove a bounded similarity result.

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2. Bounded similarity

If $\mathcal{H}$ is a complex Hilbert space and H is an operator on $\mathcal{H}$ such that $0 < aI \leqslant H \leqslant bI$, then H is bounded and sp (H), the spectrum of H, lies in $[a, b]$. In addition, H has an inverse with spectrum in $[b^{-1}, a^{-1}]$. If $\varphi(T) = HTH^{-1}$ for T in $\mathcal{B}(\mathcal{H})$, then φ is a bounded operator on $\mathcal{B}(\mathcal{H})$ and sp (φ) (relative to $\mathcal{B}(\mathcal{B}(\mathcal{H}))$) is contained in $[ab^{-1}, a^{-1}b]$. To see this, note that left multiplication by H on $\mathcal{B}(\mathcal{H})$ has the same spectrum as H, that right multiplication by H^{-1} has the same spectrum as H^{-1}, and that these two multiplications commute.

We employ the Banach-algebra-valued, holomorphic function calculus (see, for example, [1; Chapter VII]) to discuss holomorphic functions f of an element A of a Banach algebra $\mathcal{B}$. If f is analytic on an open set containing $\mathrm{sp}_{\mathcal{B}}(A)$, we define $f(A)$ to be $(2\pi i)^{-1}\int_C f(z)\,(z-A)^{-1}dz$, where C consists of a finite number of rectifiable Jordan curves (positively oriented) constituting the boundary of an open set containing $\mathrm{sp}_{\mathcal{B}}(A)$. The theory assures us that $f(A)$, as defined, is independent of C.

LEMMA 2.1. *If $\mathcal{V}_0$ is a closed linear subspace of a complex, normed, linear space $\mathcal{V}$ stable under the bounded operator A and f is holomorphic on a compact neighborhood $\mathcal{N}$ of $\mathrm{sp}_{\mathcal{B}}(A)$, where $\mathcal{B}$ is the Banach algebra of bounded linear transformations on $\mathcal{V}$ and $\mathcal{N}$ does not disconnect the plane $\mathbf{C}$ of complex numbers, then $\mathcal{V}_0$ is stable under $f(A)$.*

Proof. Let C be a curve, disjoint from $\mathcal{N}$, in an open set O containing $\mathcal{N}$ such that f is holomorphic on O and $f(A)=(2\pi i)^{-1}\int_C f(z)(z-A)^{-1}dz$. Since $f(A)$ is the norm limit of approximating sums to the integral and $\mathcal{V}_0$ is closed, it will suffice to show that $\mathcal{V}_0$ is stable under $(z_0-A)^{-1}$ for each z_0 on C. Since $z\to(z_0-z)^{-1}$ is holomorphic on $\mathcal{N}$ and $\mathcal{N}$ does not disconnect the plane, from Runge's theorem it is the uniform limit on $\mathcal{N}$, of polynomials p_n. Since $\mathcal{N}$ is a neighborhood of $\mathrm{sp}_{\mathcal{B}}(A)$, $p_n(A)$ tends in norm to $(z_0-A)^{-1}$ (see, for example, [1; Lemma VII.3.13, p. 571]). By assumption $\mathcal{V}_0$ is stable under $p_n(A)$. Since $\mathcal{V}_0$ is closed, it is stable under $(z_0-A)^{-1}$. ∎

With reference to the following lemma, see Gardner's result [2; Corollary 3]. With the notation (H and φ) of the first paragraph of this section, we prove:

LEMMA 2.2. *If $H\mathfrak{A}H^{-1}\subseteq\mathfrak{A}$ for some closed subspace $\mathfrak{A}$ of $\mathcal{B}(\mathcal{H})$ then φ^z is defined for each complex number z, $\varphi^z(A)=H^zAH^{-z}$ for all A in $\mathcal{B}(\mathcal{H})$, and $\varphi^z(\mathfrak{A})\subseteq\mathfrak{A}$.*

Proof. Let $\psi(z)(A)$ be H^zAH^{-z}. Then $\psi(z)$ and φ^z are entire functions of z with values in $\mathcal{B}(\mathcal{B}(\mathcal{H}))$ (where $\varphi^z=(2\pi i)^{-1}\int_C\zeta^z(\zeta-\varphi)^{-1}d\zeta$). If $\mathbf{C}_s$ is $\{z\colon z\neq -|z|\}$ (i.e. $\mathbf{C}$ "slit" along the negative real axis) and r is in $(0, 1)$, then $z\to z^r$ is a one–one, holomorphic mapping on $\mathbf{C}_s$ with range $\{z\colon -r\pi<\arg z<r\pi\}$. Thus $z\to z^r$ has a one–one, holomorphic inverse, $z\to z^{1/r}$ defined on $\{z\colon -r\pi<\arg z<r\pi\}$ and having $\mathbf{C}_s$ as its range. With n a positive integer and $1/n$ in place of r, both $\varphi^{1/n}$ and $\psi(1/n)$ have spectrum in $[a^{1/n}b^{-1/n}, b^{1/n}a^{-1/n}]$ ($\subseteq\{z\colon -r\pi<\arg z<r\pi\}$). Now $\psi(1/n)^n(A)=HAH^{-1}=\varphi(A)$, and $(\varphi^{1/n})^n=\varphi$. Since $z\to z^n$ is one–one on $\{z\colon -\pi/n<\arg z<\pi/n\}$; $\psi(1/n)=\varphi^{1/n}$. As $\{1/n\}$ accumulates at 0 and $\psi(z)$, φ^z are entire; $\psi(z)=\varphi^z$ for all z in $\mathbf{C}$.

Since $\zeta\to\zeta^z$ is holomorphic on $\mathbf{C}_s$ and $\mathrm{sp}\,\varphi\subseteq[ab^{-1}, ba^{-1}]\subseteq\mathbf{C}_s$, Lemma 2.1 applies and $\varphi^z(\mathfrak{A})\subseteq\mathfrak{A}$. ∎

The bounded similarity result referred to in the introduction appears next (in slightly extended form).

THEOREM 2.3. *If $\mathfrak{A}$ and $\mathcal{B}$ are norm-closed, self-adjoint subspaces of $\mathcal{B}(\mathcal{H})$ and T is an invertible operator in $\mathcal{B}(\mathcal{H})$ such that $T\mathfrak{A}T^{-1}=\mathcal{B}$, then $U\mathfrak{A}U^*=\mathcal{B}$, where UH is the polar decomposition of T.*

Proof. Since T is invertible, $(T^*T)^{\frac{1}{2}}$ $(=H)$ is invertible and TH^{-1} $(=U)$ is a unitary operator. By assumption $UH\mathfrak{A}H^{-1}U^*=\mathcal{B}$, so that $H\mathfrak{A}H^{-1}=U^*\mathcal{B}U$. As $U^*\mathcal{B}U$ is self-adjoint, $H\mathfrak{A}H^{-1}=H^{-1}\mathfrak{A}H$; and $H^2\mathfrak{A}H^{-2}=\mathfrak{A}$. It follows from (Gardner [2; Corollary 3]) Lemma 2.2 that $H\mathfrak{A}H^{-1}=\mathfrak{A}$. Thus $UH\mathfrak{A}H^{-1}U^*=U\mathfrak{A}U^*=\mathcal{B}$. ∎

3. Unbounded similarities

Various possibilities for the meaning of "$T\mathfrak{A}T^{-1}=\mathcal{B}$" present themselves when T is a closed densely-defined operator. A weak interpretation might be: for each A in $\mathfrak{A}$, there is a dense linear subspace $\mathcal{D}_0$ of $\mathcal{D}(T^{-1})$ such that $AT^{-1}(\mathcal{D}_0)\subseteq\mathcal{D}(T)$, $TAT^{-1}|\mathcal{D}_0$ is bounded, the (unique) bounded extension of $TAT^{-1}|\mathcal{D}_0$ is in $\mathcal{B}$, and each operator in $\mathcal{B}$ is such an extension, where $\mathcal{D}(T)$ denotes the domain of T and $TAT^{-1}|\mathcal{D}_0$ denotes the restriction of TAT^{-1} to $\mathcal{D}_0$. A slightly stronger interpretation might include the assumption that $\mathcal{D}_0$ can be found independent of A in $\mathfrak{A}$. We begin our discussion with an example that indicates the need for caution even when dealing with "potentially bounded" operators.

Example 3.1. With the preceding notation, we show that unitary equivalence of $\mathfrak{A}$ and $\mathcal{B}$ does not follow from the stronger interpretation noted above. In our Hilbert space $\mathcal{H}$, we choose an orthonormal basis $\{e_n\}$. Let T^{-1} be the operator that assigns $\sum_{n=1}^{\infty} n\lambda_n e_n$ to $\sum_{n=1}^{\infty}\lambda_n e_n$, with domain $\{\sum_{n=1}^{\infty}\lambda_n e_n\colon \sum_{n=1}^{\infty} n^2|\lambda_n|^2<\infty\}$. Then T^{-1} is self-adjoint. Let E_0 be the one-dimensional projection with range generated by $\sum_{n=1}^{\infty} n^{-1}e_n$ $(=x_0)$. Let $\mathcal{D}_0$ be the set of those vectors in $\mathcal{D}(T^{-1})$ such that $\sum_{n=1}^{\infty}\lambda_n=0$ (so that $\mathcal{D}_0$ is a linear space). We prove that $\mathcal{D}_0$ is dense by showing that we can approximate each e_{n_0} in norm as closely as we wish by an element of $\mathcal{D}_0$. Note, for this, that $e_{n_0}-\sum_{j=1}^{m} m^{-1}e_{n_0+j}$ $(=x_m)$ lies in $\mathcal{D}_0$ and that $\|e_{n_0}-x_m\|^2=1/m$. Since $\langle T^{-1}x, x_0\rangle=0$ for each x in $\mathcal{D}_0$; $E_0T^{-1}|\mathcal{D}_0$ is 0. It follows that $(aE_0+bI)T^{-1}|\mathcal{D}_0=bT^{-1}|\mathcal{D}_0$; so that $T(aE_0+bI)T^{-1}|\mathcal{D}_0=bI|\mathcal{D}_0$ for all scalars a and b. If $\mathfrak{A}$ is the (two-dimensional) C^*-algebra generated by E_0 and I and $\mathcal{B}$ is the algebra of scalar multiples of I, then $T\mathfrak{A}T^{-1}=\mathcal{B}$ (in the stronger sense noted above) but $\mathfrak{A}$ and $\mathcal{B}$ are not even isomorphic.

In the preceding example, $\mathcal{D}_0$ is not a core for T^{-1} (i.e. the restriction of T^{-1} to $\mathcal{D}_0$ does not have closure T^{-1}). To see this, note that the closure of the graph of the restriction of T^{-1} to a core is the graph of T^{-1}. In particular, the range of this restriction is dense in the range of T^{-1}, hence in this case, dense in $\mathcal{H}$. But x_0 is orthogonal to the range of the restriction of T^{-1} to $\mathcal{D}_0$ (this is precisely the crux of the example); so that $T^{-1}(\mathcal{D}_0)$ is not dense in $\mathcal{H}$, and $\mathcal{D}_0$ is not a core for T^{-1}. It is exactly in the failure of the lemma that follows (when $\mathcal{D}_0$ is not a core) that the pathology of the preceding example resides.

LEMMA 3.2. *If H and K are closed, densely-defined operators on the complex Hilbert space $\mathcal{H}$, $\mathcal{D}_0$ is a core for H, A is a bounded operator (with domain $\mathcal{H}$), and KAH is defined and bounded on $\mathcal{D}_0$, then KAH has domain $\mathcal{D}(H)$ and KAH is a bounded extension of $KAH \mid \mathcal{D}_0$. In addition $(KAH)^*$ is a bounded operator with domain $\mathcal{H}$ and $(KAH)^* \mid \mathcal{D}(K^*) = H^*A^*K^*$.*

Proof. Suppose $h_0 \in \mathcal{D}(H)$. Since $\mathcal{D}_0$ is a core for H, there is a sequence (h_n) in $\mathcal{D}_0$ such that $h_n \to h_0$ and $Hh_n \to Hh_0$. Now $AHh_n \to AHh_0$, since A is bounded with domain $\mathcal{H}$. By hypothesis $AHh_n \in \mathcal{D}(K)$ for each n (as $h_n \in \mathcal{D}_0$). Boundedness of $KAH \mid \mathcal{D}_0$ assures us that $(KAHh_n)$ is a Cauchy convergent sequence in $\mathcal{H}$ and, hence, has limit k in $\mathcal{H}$. But $AHh_n \to AHh_0$, $KAHh_n \to k$, and K is closed. Thus $AHh_0 \in \mathcal{D}(K)$ and $KAHh_0 = k$.

If $\|h_0\| = 1$ we can choose h_n, as above, so that $\|h_n\| = 1$. If b is the bound of the restriction of KAH to $\mathcal{D}_0$, then $\|KAHh_n\| \leqslant b$; so that $\|KAHh_0\| \leqslant b$. Thus $KAH \mid \mathcal{D}(H)$ has bound b, and KAH has domain $\mathcal{D}(H)$. With x in $\mathcal{D}(H)$ and y in $\mathcal{H}$, $|\langle KAHx, y\rangle| \leqslant b\|x\| \cdot \|y\|$; so that $y \in \mathcal{D}((KAH)^*)$, and $\langle x, (KAH)^*y\rangle = \langle KAHx, y\rangle$. Thus $\mathcal{D}((KAH)^*) = \mathcal{H}$ and $\|(KAH)^*y\| \leqslant b\|y\|$; so that $(KAH)^*$ is bounded. If we restrict y to $\mathcal{D}(K^*)$, then $\langle KAHx, y\rangle = \langle Hx, A^*K^*y\rangle$. Hence $A^*K^*y \in \mathcal{D}(H^*)$ and $\langle KAHx, y\rangle = \langle x, H^*A^*K^*y\rangle$; so that $(KAH)^*y = H^*A^*K^*y$. ∎

Remark. If H is a positive operator with inverse H^{-1} on the Hilbert space $\mathcal{H}$, E_m is the spectral projection for H corresponding to the interval $[m^{-1}, m]$, with m a positive integer, and $\mathcal{H}_m$ is $E_m(\mathcal{H})$, then $\bigcup_{m=1}^{\infty} \mathcal{H}_m$ is a core for H^k, for each integer k. To see this note that $E_m x \underset{m}{\to} x$ for each x in $\mathcal{H}$ so that $H^k E_m x = E_m H^k x \underset{m}{\to} H^k x$ for each x in $\mathcal{D}(H^k)$. We denote this particular core for H by $\mathcal{D}_0(H)$ and observe that $\mathcal{D}_0(H) = \mathcal{D}_0(H^{-1})$.

LEMMA 3.3. *If H and its inverse H^{-1} are densely-defined, positive operators on the Hilbert space $\mathcal{H}$, $\mathcal{D}_0$ is a core for H^{-1}, $\mathfrak{A}$ is a norm-closed, linear subspace of $\mathcal{B}(\mathcal{H})$ such that, for each A in $\mathfrak{A}$, HAH^{-1} is defined and bounded on $\mathcal{D}_0$, and $\varphi(A)$ is the (unique) bounded extension to $\mathcal{H}$ of $HAH^{-1} \mid \mathcal{D}_0$, then φ is a bounded linear mapping of $\mathfrak{A}$ into $\mathcal{B}(\mathcal{H})$.*

Proof. From Lemma 3.2, HAH^{-1} has domain $\mathcal{D}(H^{-1})$ and is a bounded extension of $HAH^{-1} \mid \mathcal{D}_0$. Thus HAH^{-1} is the restriction to $\mathcal{D}(H^{-1})$ of the (unique) bounded extension of $HAH^{-1} \mid \mathcal{D}_0$. We may assume, without loss of generality, that $\mathcal{D}_0$ is $\mathcal{D}(H^{-1})$.

Let E_m be the spectral projection for H corresponding to $[m^{-1}, m]$, H_m be $E_m H$, $\mathcal{H}_m$

be $E_m(\mathcal{H})$, and H'_m be the operator on $\mathcal{H}$ inverse to H_m on $\mathcal{H}_m$ and 0 on $(I-E_m)(\mathcal{H})$. If $\varphi_m(T)=H_mTH'_m$ for T in $B(\mathcal{H})$, A is in the unit ball of $\mathfrak{A}$, x and y are unit vectors in $\mathcal{H}$, and b is the bound of $HAH^{-1}|D(H^{-1})$, then $|\langle H_mAH'_mx, y\rangle| = |\langle HAH^{-1}E_mx, E_my\rangle| \leqslant b\|E_mx\|\cdot\|E_my\| \leqslant b$. Thus $\{\|\varphi_m(A)\|: m=1, 2, \ldots\}$ is bounded. As this is true for each A in $\mathfrak{A}$, $\{\|\varphi_m|\mathfrak{A}\|: m=1, 2, \ldots\}$ is bounded, say, by b_0, from the Uniform Boundedness Principle. Hence $|\langle H_mAH'_mx, y\rangle| \leqslant b_0$ for all A in the unit ball of $\mathfrak{A}$, each pair of unit vectors x and y in $\mathcal{H}$, and all m. With x and y unit vectors in $\mathcal{H}_m$, we have

$$|\langle HAH^{-1}x, y\rangle| = |\langle H_mAH'_mx, y\rangle| \leqslant b_0,$$

when A is in the unit ball of $\mathfrak{A}$. Thus $|\langle\varphi(A)x, y\rangle| \leqslant b_0$ for unit vectors x and y in $\bigcup_{m=1}^{\infty}\mathcal{H}_m$, a dense subspace of $\mathcal{H}$. As $\varphi(A)$ is bounded, $\|\varphi(A)\| \leqslant b_0$. Since this holds for all A in the unit ball of $\mathfrak{A}$; $\|\varphi\| \leqslant b_0$. ∎

PROPOSITION 3.4. *If H and its inverse H^{-1} are densely-defined, positive operators on the Hilbert space $\mathcal{H}$, $\mathcal{D}_0$ is a core for H^{-1}, and $\mathfrak{A}$ is a C^*-algebra such that HAH^{-1} is defined and bounded on $\mathcal{D}_0$ and has a (unique) bounded extension $\varphi(A)$ belonging to $\mathfrak{A}$ for each A in $\mathfrak{A}$ then φ is an automorphism of $\mathfrak{A}$ (necessarily, bounded) and there is a positive H_0 in $\mathfrak{A}''$ such that $H_0AH_0^{-1}|\mathcal{D}(H^{-1})=HAH^{-1}$ for all A in $\mathfrak{A}$. Moreover φ^z is defined for each complex z and H^zAH^{-z} has a (unique) bounded extension from $\mathcal{D}_0(H)$ to $\mathcal{H}$ equal to $\varphi^z(A)$ (in $\mathfrak{A}$) for each A in $\mathfrak{A}$.*

Proof. From Lemma 3.3, φ is bounded. From Lemma 3.2, $(HA^*H^{-1})^*$ is bounded and everywhere defined; and its restriction to $\mathcal{D}(H)$ is $H^{-1}AH$. Thus the same considerations apply, with the roles of H and H^{-1} interchanged, to yield a bounded linear mapping ψ of $\mathfrak{A}$ into $\mathfrak{A}$. Now $\psi(\varphi(A))$ restricted to $\mathcal{D}(H)$ is $H^{-1}\varphi(A)H$. Since the range of H is $\mathcal{D}(H^{-1})$ and $\varphi(A)$ restricted to $\mathcal{D}(H^{-1})$ is HAH^{-1}; $\psi(\varphi(A))|\mathcal{D}(H)=A|\mathcal{D}(H)$. As both $\psi(\varphi(A))$ and A are bounded, $A=\psi(\varphi(A))$. Symmetrically $A=\varphi(\psi(A))$. Hence φ and ψ are inverses of one another. Since the range of H is the domain of H^{-1},

$$\varphi(A)\varphi(B)|\mathcal{D}(H^{-1}) = HAH^{-1}HBH^{-1} = HABH^{-1} = \varphi(AB)|\mathcal{D}(H^{-1}).$$

Thus $\varphi(A)\varphi(B)=\varphi(AB)$; and φ is an automorphism of $\mathfrak{A}$.

Gardner shows [2; Theorem A, p. 395] that an automorphism of a C^*-algebra is implemented by a bounded invertible operator in the reduced atomic representation of that algebra. Let $\mathfrak{A}$ acting on $\mathcal{H}_0$ be that representation and T be a bounded operator with bounded inverse such that $\varphi(A)=TAT^{-1}$ for each A in $\mathfrak{A}$. From Theorem 2.3, with UK the polar decomposition of T (i.e. $K=(T^*T)^{\frac{1}{2}}$ and $U=T(T^*T)^{-\frac{1}{2}}$), $U\mathfrak{A}U^*=\mathfrak{A}$ and $K\mathfrak{A}K^{-1}=\mathfrak{A}$. Let $\varphi_1(A)$ be UAU^* and $\varphi_2(A)$ be KAK^{-1} for A in $\mathcal{B}(\mathcal{H}_0)$. Then $\varphi=\varphi_1\varphi_2$; and φ_2 has spectrum (relative to $\mathcal{B}(\mathcal{B}(\mathcal{H}_0))$) in some closed, bounded subset of the positive real numbers. From Lemma 2.2, $K^z\mathfrak{A}K^{-z}\subseteq\mathfrak{A}$ for each complex number z, and $\varphi_2^z(A)=K^zAK^{-z}$. In particular, $t\to\varphi_2^t$ is a norm-continuous, one-parameter group of automorphisms of $\mathfrak{A}$. Hence (cf. [6; Theorem 5] or [11; 4.1.19]) there is an operator H_0 in $\mathfrak{A}''$ (recall that $\mathfrak{A}''$ acts on $\mathcal{H}$) such that $\varphi_2(A)=H_0AH_0^{-1}$ for each A in $\mathfrak{A}$. Note that $\varphi^*=\varphi^{-1}$ and $\varphi_2^*=\varphi_2^{-1}$ (for $\varphi^*(A)=\varphi(A^*)^*=(HA^*H^{-1})^*=\varphi^{-1}(A)$, and, similarly for φ_2); and $\varphi_1^*=\varphi_1$. Thus $\varphi\varphi_2^{-1}=\varphi_1=\varphi_1^*=\varphi^*\varphi_2^{-1*}=\varphi^{-1}\varphi_2$; and $\varphi^2=\varphi_2^2$. As in [7; Lemma 2], $\varphi_2^t=e^{t\delta}$ for some derivation δ of $\mathfrak{A}$. Now $(\varphi_2^t)^*=(\varphi_2^*)^t=\varphi_2^{-t}=e^{-t\delta}=(e^{t\delta})^*=e^{t\delta^*}$. Comparing series coefficients, $\delta^*=-\delta$. If A_0 in $\mathfrak{A}''$ is such that $\delta=\operatorname{ad}A_0|\mathfrak{A}$ (cf. [4, 13]), then $-\delta(A)=AA_0-A_0A=\delta^*(A)=(A_0A^*-A^*A_0)^*=AA_0^*-A_0^*A$. Hence $A_0-A_0^*\in\mathfrak{A}'$, $\delta=\operatorname{ad}\frac{1}{2}(A_0+A_0^*)|\mathfrak{A}$, and we may assume that A_0 is self-adjoint. It follows that $\varphi_2(A)=e^{\delta}(A)=e^{A_0}Ae^{-A_0}$ for each A in $\mathfrak{A}$, and H_0 can be chosen as the positive operator e^{A_0} (in $\mathfrak{A}''$).

Let E_m be the spectral projection for H corresponding to $[m^{-1}, m]$, for each positive integer m, and $\mathcal{H}_m$ be $E_m(\mathcal{H})$. We show, now, that for each A in $\mathfrak{A}$, H^zAH^{-z} has a bounded restriction to $\mathcal{D}_0(H)$ $(=\bigcup_{m=1}^{\infty}\mathcal{H}_m)$ which coincides with the restriction of $H_0^zAH_0^{-z}$ to $\mathcal{D}_0(H)$. Let H_m be E_mH, H_m^z be the operator on $\mathcal{H}$ equal to H_m^z on $\mathcal{H}_m$ and 0 on $(I-E_m)(\mathcal{H})$, and $\varphi_m(T)$ be $H_mTH_m^{-1}$ for T in $\mathcal{B}(\mathcal{H}_m)$, $m=3, 4, \ldots$ (since φ_1 and φ_2 have other meanings). Since $\varphi^2=\varphi_2^2$; $H\varphi(A)H^{-1}$ and $H_0^2AH_0^{-2}$ have the same restriction to $\mathcal{D}_0(H)$. But $H\varphi(A)H^{-1}$ restricted to $\mathcal{D}_0(H)$ is H^2AH^{-2}. Let $\eta(B)$ be $H_0^2BH_0^{-2}$ for each B in $\mathcal{B}(\mathcal{H})$. The spectrum of η relative to $\mathcal{B}(\mathcal{B}(\mathcal{H}))$ is a closed bounded subset of the positive real numbers. The same is true for the spectrum of φ_m^2 relative to $\mathcal{B}(\mathcal{B}(\mathcal{H}_m))$. Fixing m, let $\mathcal{N}$ be a closed neighborhood of both these spectra and let C be a simple, closed curve in the open, right-half plane with $\mathcal{N}$ in its interior. Note that, for each polynomial p and all x and y in $\mathcal{H}_m$, $\langle p(\eta)(A)x, y\rangle=\langle p(\varphi_m^2)(E_mAE_m)x, y\rangle$. With ζ on C, using Runge's theorem to approximate $z\to(\zeta-z)^{-1}$ uniformly on $\mathcal{N}$ by polynomials, as in Lemma 2.1, there is a sequence of polynomials p_n such that $p_n(\eta)$ tends in norm to $(\zeta-\eta)^{-1}$ and $p_n(\varphi_m^2)$ tends to $(\zeta-\varphi_m^2)^{-1}$ in norm. It follows that

$$\langle(\zeta-\eta)^{-1}(A)x, y\rangle=\langle(\zeta-\varphi_m^2)^{-1}(E_mAE_m)x, y\rangle$$

for each ζ on C. Hence

$$\begin{aligned}\langle H_0^{2z}AH_0^{-2z}x, y\rangle &= \langle\eta^z(A)x, y\rangle=\frac{1}{2\pi i}\int_C\zeta^z\langle(\zeta-\eta)^{-1}(A)x, y\rangle\,d\zeta\\ &=\frac{1}{2\pi i}\int_C\zeta^z\langle(\zeta-\varphi_m^2)^{-1}(E_mAE_m)x, y\rangle\,d\zeta=\langle\varphi_m^{2z}(E_mAE_m)x, y\rangle\\ &=\langle H_m^{2z}(E_mAE_m)H_m^{-2z}x, y\rangle=\langle AH^{-2z}x, (H^{2z})^*y\rangle.\end{aligned}$$

Thus H^zAH^{-z} has a bounded restriction to $\mathcal{D}_0(H)$, and this restriction coincides on $\mathcal{D}_0(H)$ with $H_0^zAH_0^{-z}$. ∎

THEOREM 3.5. *If T is a closed, densely-defined, linear transformation from one complex Hilbert space $\mathcal{H}$ into another $\mathcal{K}$ and T has a (closed) densely-defined inverse T^{-1} with core $\mathcal{D}_1$ such that $\mathcal{D}_1 \subseteq \mathcal{D}(TAT^{-1})$, $TAT^{-1}|\mathcal{D}_1$ has a (unique) bounded extension to $\mathcal{K}$ in the C^*-algebra $\mathcal{B}$ for each A in the C^*-algebra $\mathfrak{A}$, and each B in $\mathcal{B}$ is such an extension, then $U\mathfrak{A}U^{-1} = \mathcal{B}$, where U is the unitary transformation of $\mathcal{H}$ onto $\mathcal{K}$ appearing in the polar decomposition, UH, of T, and H^zAH^{-z} has a (unique) bounded extension to $\mathcal{H}$ in $\mathfrak{A}$ for each complex z. There is a positive H_0 in $\mathfrak{A}''$ such that $H_0AH_0^{-1}|\mathcal{D}(H^{-1}) = HAH^{-1}$ for each A in $\mathfrak{A}$.*

Proof. From our hypothesis, $U^{-1}(\mathcal{D}_1)$ $(=\mathcal{D}_0)$ is a core for H^{-1} such that $HAH^{-1}|\mathcal{D}_0$ has a (unique) bounded extension to $\mathcal{H}$ in $U^{-1}\mathcal{B}U$, a self-adjoint family on $\mathcal{H}$. From Lemma 3.2, $(HAH^{-1}|\mathcal{D}_0)^*$ is a bounded, everywhere-defined operator on $\mathcal{H}$ in $U^{-1}\mathcal{B}U$, whose restriction to $\mathcal{D}(H)$ is $H^{-1}A^*H$. By assumption, $U(HAH^{-1}|\mathcal{D}_0)^*U^{-1}$ is the extension of $UHA_0H^{-1}U^{-1}|\mathcal{D}_1$ to $\mathcal{K}$, for some A_0 in $\mathfrak{A}$. Thus $(HAH^{-1}|\mathcal{D}_0)^*$ is the extension of $HA_0H^{-1}|\mathcal{D}_0$; and $H^{-2}A^*H^2|\mathcal{D}_0(H) = A_0|\mathcal{D}_0(H)$. From Proposition 3.4, we conclude that $H^{-2z}AH^{2z}|\mathcal{D}_0(H)$ has a (unique) bounded extension in $\mathfrak{A}$ for each A in $\mathfrak{A}$ and all complex z. In particular, $HAH^{-1}|\mathcal{D}_0(H)$ has a bounded extension $\varphi(A)$ in $\mathfrak{A}$, and φ is an automorphism of $\mathfrak{A}$. It follows that $U\varphi(A)U^{-1}|\mathcal{D}_1 = TAT^{-1}|\mathcal{D}_1$; and $U\mathfrak{A}U^{-1} = \mathcal{B}$. ∎

LEMMA 3.6. *If H is a positive, densely-defined operator with a densely-defined inverse H^{-1} on the complex Hilbert space $\mathcal{H}$, $\mathcal{D}_0$ is a core for H^{-1}, and A is a bounded, everywhere defined operator on $\mathcal{H}$ such that $\mathcal{D}_0 \subseteq \mathcal{D}(HAH^{-1})$ and $HAH^{-1}|\mathcal{D}_0$ is bounded, then, for each complex number z in the strip $\{z: 0 < \operatorname{Re} z < 1\}$ $(=S_1)$, $H^zAH^{-z}|\mathcal{D}_0$ is bounded with (unique) bounded extension $\varphi_z(A)$ to $\mathcal{H}$. If x and y are unit vectors in $\mathcal{H}$, then the function $z \to \langle \varphi_z(A)x, y\rangle$ is holomorphic on S_1, bounded by* max $\{\|A\|, \|HAH^{-1}\|\}$ *on the closure S_1^- of S_1 and continuous on S_1^-.*

Proof. Let E_m be the spectral projection for H corresponding to $[m^{-1}, m]$, with m a positive integer; and let $\mathcal{H}_m$ be $E_m(\mathcal{H})$. The operator E_mH $(=H_m)$ on $\mathcal{H}_m$ is a bounded, positive operator with a bounded inverse; so that H_m^z is defined and bounded for each complex z. From Lemma 3.2, $HAH^{-1}|\mathcal{D}_0(H)$ is bounded (with the same bound as $HAH^{-1}|\mathcal{D}_0$). If x_0 and y_0 are unit vectors in $\mathcal{H}_m$, then, with z in S_1^-, $AH^{-z}x_0 \in \mathcal{D}(H) \subseteq \mathcal{D}(H^z)$, and

$$\langle H^zAH^{-z}x_0, y_0\rangle = \langle E_mH^zAH^{-z}E_mx_0, y_0\rangle = \langle H_m^zE_mAH_m^{-z}x_0, y_0\rangle,$$

and $z \to \langle H_m^zE_mAH_m^{-z}x_0, y_0\rangle$ is entire. Now

$$|\langle H^{1+is}AH^{-1-is}x_0, y_0\rangle| \leqslant \|E_mHAH^{-1}E_m\| \leqslant \|HAH^{-1}\|$$

and $|\langle H^{is}AH^{-is}x_0, y_0\rangle| \leqslant \|A\|$. By (a variant of) the Hadamard Three Circle Theorem, $|\langle H^{z}AH^{-z}x_0, y_0\rangle| \leqslant \max\{\|A\|, \|HAH^{-1}\|\}$ for all z in S_1^- and all unit vectors x_0, y_0 in $\mathcal{D}_0(H)$. Note for this that

$$|\langle H^{z}AH^{-z}x_0, y_0\rangle| \leqslant \|H_m^z E_m A E_m H_m^{-z}\| \leqslant m^{2t}\|A\| \leqslant m^2\|A\|$$

for z $(=t+is)$ in S_1^-. Since $\mathcal{H}_m \subseteq \mathcal{H}_{m+1}$ and $\mathcal{D}_0(H)$ is dense in $\mathcal{H}$, $\|H^zAH^{-z}x_0\| \leqslant \max\{\|A\|, \|HAH^{-1}\|\}$, for each unit vector x_0 in $\mathcal{D}_0(H)$. Thus $\|\varphi_z(A)\| \leqslant \max\{\|A\|, \|HAH^{-1}\|\}$, for z in S_1^-.

Let (x_n), (y_n) be sequences of unit vectors in $\mathcal{D}_0(H)$ with limits x and y, respectively. Then

$$\begin{aligned}&|\langle\varphi_z(A)x, y\rangle - \langle H^zAH^{-z}x_n, y_n\rangle|\\ &\quad\leqslant |\langle\varphi_z(A)x, y\rangle - \langle\varphi_z(A)x_n, y\rangle| + |\langle\varphi_z(A)x_n, y\rangle - \langle H^zAH^{-z}x_n, y_n\rangle|\\ &\quad\leqslant \|\varphi_z(A)\|\cdot\|x-x_n\| + \|\varphi_z(A)\|\cdot\|y-y_n\| \to 0\end{aligned}$$

uniformly for z in S_1^-. Thus $z \to \langle\varphi_z(A)x, y\rangle$ is continuous on S_1^- and holomorphic on S_1. ■

With notation as in the preceding lemma, repeated application of it (or changes of notation in the argument) yields:

COROLLARY 3.7. *If n_1 and n_2 are positive integers, such that*

$$H^{-n_1}AH^{n_1}|\mathcal{D}_0,\ H^{-(n_1-1)}AH^{n_1-1}|\mathcal{D}_0, \ldots, H^{-1}AH|\mathcal{D}_0,\ A,\ HAH^{-1}|\mathcal{D}_0, \ldots, H^{n_2}AH^{-n_2}|\mathcal{D}_0$$

are bounded, then $z \to \langle\varphi_z(A)x, y\rangle$ is holomorphic on the strip $\{z\colon -n_1 < \operatorname{Re} z < n_2\}$ $(=S_{n_1,n_2})$, continuous on its closure, and bounded there, where $H^zAH^{-z}|\mathcal{D}_0$ is bounded for z in S_{n_1,n_2} and $\varphi_z(A)$ is its (unique) bounded extension to $\mathcal{H}$. In particular, if $H^nAH^{-n}|\mathcal{D}_0$ is bounded for all integers n, then $z \to \langle\varphi_z(A)x, y\rangle$ is entire for each pair of vectors x, y in $\mathcal{H}$; and

$$|\langle\varphi_z(A)x, y\rangle| \leqslant k_{A,n}\|x\|\cdot\|y\|,$$

where $k_{A,n} = \max\{\|A\|, \|H^nAH^{-n}|\mathcal{D}_0\|\}$ and $\operatorname{Re} z$ lies in the interval with 0 and n as end-points.

LEMMA 3.8. *If H is a positive, densely-defined operator with a densely-defined inverse H^{-1} on the complex Hilbert space $\mathcal{H}$, $\mathcal{D}_0$ is a core for H^{-1}, $\mathfrak{A}_0$ is a *-algebra of bounded operators on $\mathcal{H}$ such that, for each A in $\mathfrak{A}_0$, $\mathcal{D}_0 \subseteq \mathcal{D}(HAH^{-1})$ and $HAH^{-1}|\mathcal{D}_0$ has a (unique) bounded extension $\varphi(A)$ to $\mathcal{H}$ in $\mathfrak{A}_0$ satisfying $\|\varphi^n(A)\| \leqslant k_A^{|n|}$ for each integer n and some constant k_A (depending on A); then $H^zAH^{-z}|\mathcal{D}_0(H)$ is bounded for each complex number z and each A in $\mathfrak{A}_0$, and its (unique) bounded extension $\varphi_z(A)$ to $\mathcal{H}$ lies in $\mathfrak{A}_0''$.*

Proof. From Lemma 3.2 and our hypothesis, $H^nAH^{-n}|\mathcal{D}_0(H)$ is bounded for each integer n. Thus, from Corollary 3.7, $H^zAH^{-z}|\mathcal{D}_0(H)$ is bounded for all complex numbers z,

$z \to \langle \varphi_z(A)x, y\rangle$ is entire for each pair of unit vectors x, y in $\mathcal{H}$ and $|\langle \varphi_z(A)x, y\rangle| \leqslant k_A^n$, where $|\mathrm{Re}\, z| \leqslant n$. If $\mathfrak{A}_0'$ contains no projections other than 0 and I then $\varphi_z(A) \in \mathcal{B}(\mathcal{H}) = \mathfrak{A}_0''$.

Suppose E' is a projection in $\mathfrak{A}_0'$ distinct from 0 and I; and let x_0, y_0 be unit vectors in $E'(\mathcal{H})$, $(I-E')(\mathcal{H})$, respectively. Then

$$\langle \varphi_n(A)x_0, y_0\rangle = \langle \varphi^n(A)E'x_0, (I-E')y_0\rangle = 0,$$

for each positive integer n, since $\varphi^n(A)$ is in $\mathfrak{A}_0$. Let $f(z)$ be $k_A^{-(z+1)}\langle \varphi_z(A)x_0, y_0\rangle$, for z in $\mathbf{C}_r$, the (open) right half-plane. Then $|f(z)| \leqslant 1$ for z in $\mathbf{C}_r$ and $f(n) = 0$ for each positive integer n. Thus $f(z) = (z-1)^k f_1(z)$, where f_1 is bounded and holomorphic on $\mathbf{C}_r$. Multiplying by a positive scalar, we may assume that $|f_1(z)| \leqslant 1$ for z in $\mathbf{C}_r$. Let $F_n(z)$ be $(2-z)(3-z)\ldots(n-z)/n!$. With ε positive, $1-\varepsilon \leqslant |F_n(z)|$ for all z near the imaginary axis. Thus f_1/F_n is bounded and holomorphic on $\mathbf{C}_r$ and $|f_1(z)/F_n(z)| \leqslant (1-\varepsilon)^{-1}$ for z near the imaginary axis. From the Phragmen–Lindelöf theorem, $|f_1(z)/F_n(z)| \leqslant 1$ for z in $\mathbf{C}_r$. In particular $|f_1(1)| \leqslant |F_n(1)| = 1/n$. It follows that $f_1(1) = 0$ and that 1 is a zero of infinite order for f. Hence f is identically 0 on $\mathbf{C}_r$; and $(I-E')\varphi_z(A)E' = 0$ for each projection E' in $\mathfrak{A}_0'$, each A in $\mathfrak{A}_0$ and each complex z. From this

$$(I-E')\varphi_z(A)E' = 0 = E'\varphi_z(A)(I-E');$$

and $E'\varphi_z(A) = \varphi_z(A)E'$. Thus $\varphi_z(A) \in \mathfrak{A}_0''$. ∎

Theorem 3.9. *If T is a closed, densely-defined transformation from one complex Hilbert space $\mathcal{H}$ into another $\mathcal{K}$, T has densely-defined inverse T^{-1} with core $\mathcal{D}_1$ such that $TAT^{-1}|\mathcal{D}_1$ has a (unique) bounded extension in a *-algebra of operators $\mathcal{B}_0$ acting on $\mathcal{K}$ for each A in a *-algebra of operators $\mathfrak{A}_0$ acting on $\mathcal{H}$, each B in $\mathcal{B}_0$ is such an extension, and $\|H^nAH^{-n}|\mathcal{D}_0(H)\| \leqslant k_A{}^{|n|}$ for each integer n and some constant k_A (depending on A), where UH is the polar decomposition of T and $\mathcal{D}_0 = U^{-1}(\mathcal{D}_1)$; then $U\mathfrak{A}_0''U^{-1} = \mathcal{B}_0''$, $H^zAH^{-z}|\mathcal{D}_0(H)$ is bounded for each complex number z and each A in $\mathfrak{A}_0$, and the (unique) bounded extension of $H^zAH^{-z}|\mathcal{D}_0(H)$ to $\mathcal{H}$ lies in $\mathfrak{A}_0''$. In particular, $t \to H^{it}$ is a strong-operator-continuous, one-parameter unitary group which gives rise to a one-parameter group of *-automorphisms of $\mathfrak{A}_0''$.*

Proof. Arguing precisely as in the proof of Theorem 3.5, we conclude that, with A in $\mathfrak{A}_0$, $H^{-2}AH^2|\mathcal{D}_0(H) = A_0|\mathcal{D}_0(H)$ for some A_0 in $\mathfrak{A}_0$. By hypothesis $H^{-2n}AH^{2n}|\mathcal{D}_0(H)$ is bounded and $\|H^{-2n}AH^{2n}|\mathcal{D}_0(H)\| \leqslant k_A^{|2n|}$. From Lemma 3.8, $H^{-2z}AH^{2z}|\mathcal{D}_0(H)$ is bounded for each complex z and each A in $\mathfrak{A}_0$ and its (unique) bounded extension to $\mathcal{H}$ lies in $\mathfrak{A}_0''$. In particular, $H^{it}AH^{-it} \in \mathfrak{A}_0''$ for each A in $\mathfrak{A}_0$ – hence, for each A in $\mathfrak{A}_0''$. At the same time, the (unique) bounded extension $\varphi(A)$ of $HAH^{-1}|\mathcal{D}_0$ is in $\mathfrak{A}_0''$. Since $U\varphi(A)U^{-1}|\mathcal{D}_1 = TAT^{-1}|\mathcal{D}_1$ and, by assumption, $TAT^{-1}|\mathcal{D}_1$ has a (unique) bounded extension to $\mathcal{K}$ in $\mathcal{B}_0$; $U\varphi(A)U^{-1} \in \mathcal{B}_0$.

On the other hand, given B in $\mathcal{B}_0$, by hypothesis, there is an A in $\mathfrak{A}_0$ such that B is the unique extension of $TAT^{-1} | \mathcal{D}_1 (= U\varphi(A)U^{-1} | \mathcal{D}_1)$. Hence $B = U\varphi(A)U^{-1}$; and $U^{-1}BU = \varphi(A) \in \mathfrak{A}_0''$. Thus $U^{-1}\mathcal{B}_0''U \subseteq \mathfrak{A}_0''$.

We note, next, that the hypotheses apply with the rôles of T and $\mathfrak{A}_0$ interchanged with those of T^{-1} and $\mathcal{B}_0$, from which we can conclude, as above, that $U\mathfrak{A}_0''U^{-1} \subseteq \mathcal{B}_0'' \subseteq U\mathfrak{A}_0''U^{-1}$, and, hence, that $U\mathfrak{A}_0''U^{-1} = \mathcal{B}_0''$. To see this note that

$$T^{-1}BT | \mathcal{D}_0(H) = H^{-1}U^{-1}BUH | \mathcal{D}_0(H) = H^{-1}\varphi(A)H | \mathcal{D}_0(H) = A | \mathcal{D}_0(H);$$

that is, $T^{-1}BT | \mathcal{D}_0(H)$ has a bounded extension A in $\mathfrak{A}_0$ and each A in $\mathfrak{A}_0$ is such an extension. For the growth condition on the bound, let WK^{-1} be the polar decomposition of T^{-1}, where $K^{-1} = (T^{-1*}T^{-1})^{\frac{1}{2}} = (TT^*)^{-\frac{1}{2}}$. Then $K = (TT^*)^{\frac{1}{2}}$, and KU is a polar decomposition for T. Since $T = KU = KW^{-1}$, we have $W^{-1} = U$ and $K = UHU^{-1}$. Thus

$$K^nBK^{-n} = UH^nU^{-1}(U\varphi(A)U^{-1})UH^{-n}U^{-1} = UH^n\varphi(A)H^{-n}U^{-1};$$

so that $K^nBK^{-n} | \mathcal{D}_0(K)$ is bounded and

$$\|K^nBK^{-n} | \mathcal{D}_0(K)\| = \|H^{n+1}AH^{-(n+1)} | \mathcal{D}_0(H)\| \leqslant k_A^{|n+1|}$$

for all integers n, which establishes the symmetry between the rôles of T and $\mathfrak{A}_0$ and those of T^{-1} and $\mathcal{B}_0$. ∎

4. The Tomita–Takesaki theory

Throughout this section $\mathcal{R}$ denotes a von Neumann algebra acting on the Hilbert space $\mathcal{H}$ and x_0 is a separating and generating unit vector for $\mathcal{R}$. Let $\bar{\mathcal{H}}$ denote the Hilbert space conjugate to $\mathcal{H}$ (so that $\overline{ax+y} = \bar{a}\bar{x} + \bar{y}$ and $\langle \bar{x}, \bar{y} \rangle = \langle y, x \rangle$). With z in $\bar{\mathcal{H}}$, we denote by $\bar{z}$ the element of $\mathcal{H}$ corresponding to z. With T an operator on $\mathcal{H}$, let $\bar{T}\bar{x}$ be $\overline{Tx}$. Then $T \to \bar{T}$ is a conjugate-linear, *-isomorphism of $\mathcal{B}(\mathcal{H})$ onto $\mathcal{B}(\bar{\mathcal{H}})$. Let S_0Ax_0 be $\overline{A^*x_0}$ and $F_0\overline{A'x_0}$ be A'^*x_0, where $A \in \mathcal{R}$ and $A' \in \mathcal{R}'$. We shall note (Lemma 4.3) that S_0 and F_0 are preclosed. Let $J\Delta^{\frac{1}{2}}$ be a polar decomposition of the closure S of S_0. In this notation, Tomita's theorem asserts that:

$$J\mathcal{R}J^* = \bar{\mathcal{R}}' \quad \text{and } A \to \Delta^{it}A\Delta^{-it} \quad \text{is a *-automorphism of } \mathcal{R} \text{ for each real } t.$$

The relation of this theory to unbounded similarity theory lies in the identity

$$SAS^{-1}\bar{B}\bar{C}\bar{x}_0 = \bar{B}\bar{C}\bar{A}^*\bar{x}_0 = \bar{B}SAS^{-1}\bar{C}\bar{x}_0;$$

so that, if SAS^{-1} is bounded, its extension to $\bar{\mathcal{H}}$ is in $\bar{\mathcal{R}}'$. In the results that follow, we locate strong-operator-dense, self-adjoint subalgebras of $\mathcal{R}$ and $\bar{\mathcal{R}}'$ between which S effects an unbounded similarity satisfying the growth condition of Theorem 3.9.

Lemma 4.1. *If $x \in \mathcal{D}(F_0^*)$ and $\bar{y} \in \mathcal{D}(S_0^*)$ then there are closed operators L_x and R_y affiliated with $\mathcal{R}$ and $\mathcal{R}'$, respectively, such that $L_x A' x_0 = A'x$ and $R_y A x_0 = Ay$, for each A in $\mathcal{R}$ and A' in $\mathcal{R}'$. In addition $\mathcal{R}'x_0 \subseteq \mathcal{D}(L_x^*)$, $\mathcal{R}x_0 \subseteq \mathcal{D}(R_y^*)$; $L_x^* B' x_0 = B'\overline{F_0^* x}$, and $R_y^* B x_0 = B\overline{S_0^* \bar{y}}$.*

Proof. With A', B' in $\mathcal{R}'$,

$$\langle L_x A' x_0, B' x_0 \rangle = \langle x, F_0 \bar{B}'^* \bar{A}' \bar{x}_0 \rangle = \langle A' x_0, B' \overline{F_0^* x} \rangle.$$

Hence $B'x_0 \in \mathcal{D}(L_x^*)$ and $L_x^* B' x_0 = B'\overline{F_0^* x}$. Since L_x^* is densely defined, there is a closed operator L_x (mapping $\mathcal{R}'x_0$ as defined). Now $U'^* L_x U' A' x_0 = L_x A' x_0$ for each unitary operator U' in $\mathcal{R}'$. Since $\mathcal{R}'x_0$ is a core for L_x, $L_x \eta \mathcal{R}$. (See Remark 4.2.) Similarly for $\mathcal{R}y$. ■

Remark 4.2. If A is a closed, densely-defined operator with core $\mathcal{D}_0$, and $U'^* A U' x = Ax$ for each x in $\mathcal{D}_0$ and each unitary operator U' in $\mathcal{R}'$, then $A \eta \mathcal{R}$ (that is, $\mathcal{D}(U'^* A U') = \mathcal{D}(A)$ and $U'^* A U' y = Ay$ for all y in $\mathcal{D}(A)$). To see this, note that, with y in $\mathcal{D}(A)$, there is a sequence (y_n) in $\mathcal{D}_0$ such that $y_n \to y$ and $Ay_n \to Ay$ (since $\mathcal{D}_0$ is a core for A). Now $U'y_n \to U'y$ and $AU'y_n = U'Ay_n \to U'Ay$. Since A is closed, $U'y \in \mathcal{D}(A)$ and $AU'y = U'Ay$. Thus $\mathcal{D}(A) \subseteq U'^*(\mathcal{D}(A))$. Applied to U'^*, we have $\mathcal{D}(A) \subseteq U'(\mathcal{D}(A))$; so that $U'(\mathcal{D}(A)) = \mathcal{D}(A)$. Hence $\mathcal{D}(U'^* A U') = \mathcal{D}(A)$ and $U'^* A U' y = Ay$ for each y in $\mathcal{D}(A)$.

Lemma 4.3. *The operators S_0 and F_0 are preclosed linear operators and their closures S and F satisfy: $S \subseteq F_0^*$, $F \subseteq S_0^*$.*

Proof. With A in $\mathcal{R}$ and A' in $\mathcal{R}'$,

$$\langle S_0 A x_0, \bar{A}' \bar{x}_0 \rangle = \langle A x_0, A'^* x_0 \rangle,$$

so that $\bar{A}'\bar{x}_0 \in \mathcal{D}(S_0^*)$ and $S_0^* \bar{A}' \bar{x}_0 = F_0 \bar{A}' \bar{x}_0$. Thus S_0 is preclosed and $F_0 \subseteq S_0^*$. ■

Lemma 4.4. *If $T \eta \mathcal{R}$ and $x_0 \in \mathcal{D}(T) \cap \mathcal{D}(T^*)$ then $\overline{Tx_0} \in \mathcal{D}(S)$. If $T' \eta \mathcal{R}'$ and $x_0 \in \mathcal{D}(T') \cap \mathcal{D}(T'^*)$ then $\overline{T'x_0} \in \mathcal{D}(F)$. Moreover $S\overline{Tx_0} = \overline{T^* x_0}$ and $F\overline{T'x_0} = \overline{T'^* x_0}$.*

Proof. Let VH be the polar decomposition of T. Let E_n be the spectral projection for H corresponding to $[-n, n]$ and H_n be HE_n ($\supseteq E_n H$). Then $\overline{VH_n x_0} \to \overline{Tx_0}$, and $S_0 \overline{VH_n x_0} = \overline{H_n V^* x_0} \to \overline{T^* x_0}$. Thus $\overline{Tx_0} \in \mathcal{D}(S)$, and $S\overline{Tx_0} = \overline{T^* x_0}$. Similarly $\overline{T'x_0} \in \mathcal{D}(F)$ and $F\overline{T'x_0} = \overline{T'^* x_0}$. ■

Corollary 4.5. *The operators S and F are each others adjoints.*

Proof. From Lemma 4.3, $S \subseteq F_0^*$. If $x \in \mathcal{D}(F_0^*)$, from Lemma 4.1, there is a closed operator L_x affiliated with $\mathcal{R}$ such that $x_0 \in \mathcal{D}(L_x) \cap \mathcal{D}(L_x^*)$. From Lemma 4.4, $x = L_x x_0 \in \mathcal{D}(S)$. Thus $S = F_0^*$. Similarly, $F = S_0^*$; so that $F^* = S_0^{**} = S$ and $S^* = F_0^{**} = F$. ■

Since S is a closed operator, it has polar decompositions $J\Delta^{\frac{1}{2}}$ and $\bar{\Delta}_1^{\frac{1}{2}}J$, where J is an isometric linear transformation from $\mathcal{H}$, the closure of the range of $S^*(=F)$, onto the closure of the range of the range of S (viz. $\bar{\mathcal{H}}$), $\Delta = FS$, and $\bar{\Delta}_1 = SF$. Let $\tilde{J}x$ be $\overline{J^*\bar{x}}$. Then $\tilde{J}$ is a unitary transformation of $\mathcal{H}$ onto $\bar{\mathcal{H}}$. Since S^{-1} is a closed operator (obtained by interchanging the rôles of $\mathcal{R}$ and $\bar{\mathcal{R}}$, x_0 and $\bar{x}_0$, and $\mathcal{H}$ and $\bar{\mathcal{H}}$) with polar decomposition $\Delta^{-\frac{1}{2}}J^*$, we have

$$\langle \bar{\Delta}^{-\frac{1}{2}}\tilde{J}Ax_0, \bar{y}\rangle = \langle y, \Delta^{-\frac{1}{2}}J^*\bar{A}\bar{x}_0\rangle = \langle y, A^*x_0\rangle = \langle \bar{A}^*\bar{x}_0, \bar{y}\rangle = \langle SAx_0, \bar{y}\rangle,$$

for each A in $\mathcal{R}$. Thus $\bar{\Delta}^{-\frac{1}{2}}\tilde{J}$ is a polar decomposition for S. From uniqueness of the polar decomposition for S, $\bar{\Delta}^{-\frac{1}{2}} = \bar{\Delta}_1^{\frac{1}{2}}$ and $\tilde{J} = J$. It follows that $J\Delta^{\frac{1}{2}} = \bar{\Delta}^{-\frac{1}{2}}J$, from which we have:

Lemma 4.6. *For each real* t,

$$J\Delta^t J^* = \bar{\Delta}^{-t}, \quad (SF)^t = \bar{\Delta}_1^t = (\overline{FS})^{-t} = \bar{\Delta}^{-t}.$$

Among other things, Lemma 4.6 tells us that if we interchange $\mathcal{R}$ and $\mathcal{R}'$ and let $\tilde{S}A'x_0$ be $\bar{A}'^*\bar{x}_0$, $\tilde{F}\bar{A}\bar{x}_0$ be A^*x_0, and $\tilde{\Delta}$ be $\tilde{F}\tilde{S}$, then $\tilde{\Delta} = \Delta^{-1}$. Thus statements proved for $\mathcal{R}$ and Δ apply to $\mathcal{R}'$ and Δ^{-1}. In view of this symmetry, we need prove only the first assertion of the crucial "bridging lemma" that follows.

Lemma 4.7. *If* $x = (\Delta - aI)^{-1}A_0'x_0$, *where* $a \neq |a|$ *and* $A_0' \in \mathcal{R}'$ *then* $L_x \in \mathcal{R}$ *and* $\|L_x\| \leqslant a_0\|A_0'\|$, *where* $a_0 = (2|a| - 2\operatorname{Re} a)^{-\frac{1}{2}}$. *If* $y = (\Delta^{-1} - aI)A_0x_0$, *where* $A_0 \in \mathcal{R}$, *then* $R_y \in \mathcal{R}'$ *and* $\|R_y\| \leqslant a_0\|A_0\|$.

Proof. Since Δ is positive, $\Delta(\Delta - aI)^{-1}$ is bounded. Thus $x \in \mathcal{D}(\Delta) \subseteq D(\Delta^{\frac{1}{2}}) = \mathcal{D}(S) = \mathcal{D}(F_0^*)$. From Lemma 4.1, $L_x \eta \mathcal{R}$. Let UH and KU be the polar decompositions of L_x. Let M and N be the spectral projections for H and K corresponding to the same closed, finite subinterval of $(a_0\|A_0'\|, \infty)$. Then U, M, and N are in $\mathcal{R}$, $UMH = KNU$, and

$$SNx = SNL_xx_0 = SNKUx_0 = \overline{U^*KNx_0} = \overline{MHU^*x_0} = \overline{ML_x^*x_0} = \bar{M}Sx.$$

If $N \neq 0$ then $Nx_0 \neq 0$. By choice of N,

$$\begin{aligned}
\|A_0'\|^2\|Nx_0\|^2 &< a_0^{-2}\|KNx_0\|^2 = a_0^{-2}\|U^*KNx_0\|^2 \\
&= a_0^{-2}\|MHU^*x_0\|^2 = a_0^{-2}\|ML_x^*x_0\|^2 = a_0^{-2}\|M\overline{Sx}\|^2 \\
&= a_0^{-2}\langle \bar{M}Sx, Sx\rangle = a_0^{-2}\langle SNx, Sx\rangle = a_0^{-2}\langle Nx, \Delta x\rangle \\
&= 2|a|\langle Nx, \Delta x\rangle - 2\operatorname{Re}\langle aNx, \Delta x\rangle \leqslant \|N\Delta x\|^2 \\
&\qquad + |a|^2\|Nx\|^2 - 2\operatorname{Re}\langle aNx, N\Delta x\rangle = \|N(\Delta - aI)x\|^2 \\
&= \|NA_0'x_0\|^2 \leqslant \|A_0'\|^2\|Nx_0\|^2.
\end{aligned}$$

Thus $N = 0$, L_x is bounded, and $\|L_x\| \leqslant a_0\|A_0'\|$. ∎

When $Ax_0 = A'x_0$ with A in $\mathcal{R}$ and A' in $\mathcal{R}'$, we shall say that A' is the reflection of A (about x_0) and that A is the reflection of A'.

Definition 4.8. A reflection sequence (of operators for $\mathcal{R}$ and $\mathcal{R}'$ relative to x_0) is a sequence $(\ldots, A'_{-3}, A_{-2}, A'_{-1}, A_0, A'_1, A_2, \ldots)$ such that each operator is the reflection of the adjoint of the operator following it, and there is a constant k such that $\|A_n\| \leqslant k^{|n|}$, $\|A'_m\| \leqslant k^{|m|}$.

LEMMA 4.9. *The elements in $\mathcal{R}$ that belong to a reflection sequence form a $*$-subalgebra $\mathcal{R}_0$ of $\mathcal{R}$.*

Proof. If A and B are in the reflection sequences $(\ldots, A'_{-1}, A_0, A'_1, \ldots)$ and $(\ldots, B'_{-1}, B_0, B'_1, \ldots)$, renumbering, we may assume that $A = A_0$ and $B = B_0$. Then $aA + B$ belongs to the reflection sequence

$$(\ldots, \bar{a}A'_{-1} + B'_{-1}, aA_0 + B_0, \bar{a}A'_1 + B'_1, aA_2 + B_2, \ldots);$$

while AB belongs to the reflection sequence,

$$(\ldots, A_{-2}B_{-2}, A'_{-1}B'_{-1}, A_0B_0, A'_1B'_1, \ldots).$$

Moreover A^* belongs to the "adjoint" reflection sequence

$$(\ldots, A_2^*, A_1'^*, A_0^*, A'^*_{-1}, A^*_{-2}, \ldots). \qquad \blacksquare$$

We will speak, too, of a reflection sequence of vectors, $(\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots)$, when $y_{-2} = A_{-2}x_0$, $y_{-1} = A'_{-1}x_0$, $y_0 = A_0x_0$, $y_1 = A'_1x_0$, $y_2 = A_2x_2$ and $(\ldots, A_{-2}, A'_{-1}, A_0, A'_1, A_2, \ldots)$ is a reflection sequence of operators. Note that a vector y_0 lies in a reflection sequence of vectors if and only if $y_0 \in \mathcal{D}(\Delta^n)$ and $\Delta^n y_0 \in \mathcal{R}x_0 \cap \mathcal{R}'x_0$ for each integer n, and provided the norm-growth condition holds for the associated reflection sequence of operators. To see this, if $y_0 = A_0x_0 = A_1'^*x_0$, let y_1 be A'_1x_0 and let y_{2n} be $\Delta^{-n}y_0$ $(= A_{2n}x_0)$ and y_{2n+1} be $\Delta^n y_1$. Then $\bar{A}_2\bar{x}_0 = \bar{y}_2 = \bar{\Delta}^{-1}\bar{y}_0 = SF\bar{A}_1'^*\bar{x}_0 = SA'_1x_0$; so that $S^{-1}\bar{A}_2\bar{x}_0 = A_2^*x_0 = A'_1x_0$. Since $y_1 = F\bar{y}_0$; we have

$$\Delta^n y_1 = \Delta^{n+\frac{1}{2}}J^*\bar{y}_0 = J^*J\Delta^{n+\frac{1}{2}}J^*\bar{y}_0 = J^*\bar{\Delta}^{-n-\frac{1}{2}}\bar{y}_0 = F\bar{\Delta}^{-n}\bar{y}_0 = F\bar{A}'^*_{2n+1}\bar{x}_0 = A'_{2n+1}x_0$$

for some A'_{2n+1} in $\mathcal{R}'$. Thus

$$\bar{A}'_{-1}\bar{x}_0 = \bar{y}_{-1} = \bar{\Delta}^{-1}\bar{y}_1 = SF\bar{A}'_1\bar{x}_0 = SA_1'^*x_0 = SA_0x_0 = \overline{A_0^*x_0}.$$

Continuing in this way, and assuming that $\|A_{2n}\| \leqslant k^{|2n|}$, $\|A'_{2n+1}\| \leqslant k^{|2n+1|}$ for some constant k, we construct the reflection sequence of vectors $(\ldots, y_{-1}, y_0, y_1, \ldots)$.

If $A^*x_0 = A'x_0$ with A in $\mathcal{R}$ and A' in $\mathcal{R}'$, then, with B in $\mathcal{R}$,

$$SAS^{-1}\bar{B}\bar{x}_0 = SAB^*x_0 = \bar{B}\bar{A}^*\bar{x}_0 = \bar{B}\bar{A}'\bar{x}_0 = \bar{A}'\bar{B}\bar{x}_0.$$

Thus $SAS^{-1}|\bar{\mathcal{R}}\bar{x}_0$ has a (unique) bounded extension $\bar{A}'$ to $\mathcal{H}$ and $\bar{A}'\in\bar{\mathcal{R}}'$. If A_0 is in a reflection sequence then $A_0^*x_0=A'_{-1}x_0$; so that $SA_0S^{-1}|\bar{\mathcal{R}}\bar{x}_0$ has a (unique) bounded extension to $\bar{\mathcal{H}}$ and this extension, $\bar{A}'_{-1}$ lies in a reflection sequence of operators for $\bar{\mathcal{R}}$ and $\bar{\mathcal{R}}'$ relative to $\bar{x}_0$. It follows that S induces a similarity (unbounded) of $\mathcal{R}_0$ onto the *-subalgebra of elements in $\bar{\mathcal{R}}'$ that lie in a reflection sequence. The conditions of Theorem 3.9 apply and yield the main theorem of the Tomita–Takesaki theory once we note that $\mathcal{R}_0''=\mathcal{R}$. For this last, we must produce an abundance of vectors and operators in reflection sequences. Having done this, we employ the density theorem (of independent interest) whose proof follows. In [5] we gave an example of a type I_∞ factor and a proper type I_∞ subfactor and a unit generating and separating vector for both. This cannot occur in the finite-dimensional case (nor even for finite von Neumann algebras—and that forms the basis for the results of [5]). In Theorem 4.10 we supply the condition on the generating vector that is needed to return the conclusion to the classical framework.

THEOREM 4.10. *If $\mathcal{R}$ is a von Neumann algebra acting on the Hilbert space $\mathcal{H}$, $\mathcal{R}_0$ is a self-adjoint subalgebra of $\mathcal{R}$ and x_0 is a unit vector in $\mathcal{H}$ that is separating and generating for $\mathcal{R}$, then the following three statements are equivalent:*

(i) *$\mathcal{R}_0$ is strong-operator dense in $\mathcal{R}$;*

(ii) *$(\mathcal{R}_0)_{sa}x_0$ is dense in $(\mathcal{R})_{sa}x_0$;*

(iii) *$\mathcal{R}_0x_0$ is a core for $\Delta^{\frac{1}{2}}$.*

Proof. (i)→(ii). Since $\mathcal{R}_0$ is weak-operator dense in $\mathcal{R}$ and the adjoint operation is weak-operator continuous, $(\mathcal{R}_0)_{sa}$ is weak-operator dense in $(\mathcal{R})_{sa}$. As $(\mathcal{R}_0)_{sa}$ and $(\mathcal{R})_{sa}$ are convex, $(\mathcal{R}_0)_{sa}$ is strong-operator dense in $(\mathcal{R})_{sa}$.

(ii)→(iii). Since $\mathcal{R}x_0$ is a core for $\Delta^{\frac{1}{2}}$, given A in $\mathcal{R}$, it will suffice to find operators A_n in $\mathcal{R}_0$ such that $A_nx_0\to Ax_0$ and $\Delta^{\frac{1}{2}}A_nx_0$ $(=J^*SA_nx_0=J^*\bar{A}_n^*\bar{x}_0)\to\Delta^{\frac{1}{2}}Ax_0$ $(=J^*\bar{A}^*\bar{x}_0)$, or, equivalently, such that $A_n^*x_0\to A^*x_0$ (since J^* and $x\to\bar{x}$ are isometries). Now $A=H_1+iH_2$, with H_1 and H_2 self-adjoint operators in $\mathcal{R}$. By assumption, there are self-adjoint operators K_{1n} and K_{2n} in $\mathcal{R}_0$ such that $K_{1n}x_0\to H_1x_0$ and $K_{2n}x_0\to H_2x_0$. If $A_n=K_{1n}+iK_{2n}$, then $A_n\in\mathcal{R}_0$, $A_nx_0\to Ax_0$, and $A_n^*x_0\to A^*x_0$.

(iii)→(i). We show that $\mathcal{R}_0'\subseteq\mathcal{R}'$ by showing that each self-adjoint H' in $\mathcal{R}_0'$ lies in $\mathcal{R}'$. Since $\mathcal{R}_0\subseteq\mathcal{R}$, we have $\mathcal{R}'\subseteq\mathcal{R}_0'$; so that $\mathcal{R}_0'=\mathcal{R}'$ and $\mathcal{R}_0''=\mathcal{R}''=\mathcal{R}$. With A_n in $\mathcal{R}_0$,

$$\langle SA_nx_0,\bar{H}'\bar{x}_0\rangle=\langle\bar{A}_n^*\bar{x}_0,\bar{H}'\bar{x}_0\rangle=\langle\bar{H}'\bar{x}_0,\bar{A}_n\bar{x}_0\rangle$$

If $x\in\mathcal{D}(\Delta^{\frac{1}{2}})$, by assumption, there are operators A_n in $\mathcal{R}_0$ such that $A_nx_0\to x$ and $\Delta^{\frac{1}{2}}A_nx_0$ $(=J^*\bar{A}_n^*\bar{x}_0)\to\Delta^{\frac{1}{2}}x$. In this case $\langle SA_nx_0,\bar{H}'\bar{x}_0\rangle=\langle J\Delta^{\frac{1}{2}}A_nx_0,\bar{H}'\bar{x}_0\rangle\to\langle J\Delta^{\frac{1}{2}}x,\bar{H}'\bar{x}_0\rangle=$

$\langle Sx, \bar{H}'\bar{x}_0\rangle$; and $\langle \bar{H}'\bar{x}_0, \bar{A}_n\bar{x}_0\rangle = \langle A_n x_0, H'x_0\rangle \to \langle x, H'x_0\rangle$. Thus $\langle Sx, \bar{H}'\bar{x}_0\rangle = \langle x, H'x_0\rangle$. It follows that $\bar{H}'\bar{x}_0 \in \mathcal{D}(S^*)$ $(=\mathcal{D}(F))$ and $F\bar{H}'\bar{x}_0 = H'x_0$. Hence the mapping $Ax_0 \to AH'x_0$ has closure H_0' affiliated with $\mathcal{R}'$, where A takes on values in $\mathcal{R}$, from Lemma 4.1. If $A \in \mathcal{R}_0$ then $H_0'Ax_0 = AH'x_0 = H'Ax_0$, since $H' \in \mathcal{R}_0'$. With x in $\mathcal{H}$ and A_n in $\mathcal{R}_0$ such that $A_n x_0 \to x$, we have $H_0'A_n x_0 = H'A_n x_0 \to H'x$. Since H_0' is closed, $x \in \mathcal{D}(H_0')$ and $H_0'x = H'x$. Thus $H_0' = H' \in \mathcal{R}'$. ∎

In the discussion that follows, we complete the proof by showing that vectors in $(\mathcal{R}x_0) \cap E(k^{-1}, k)(\mathcal{H})$, where $E(k^{-1}, k)$ is the spectral projection for Δ (and also Δ^{-1}) corresponding to the interval (k^{-1}, k), lie in a reflection sequence; and that the set of these vectors, with k taking values in $(1, \infty)$, is a core for $\Delta^{\frac{1}{2}}$. Thus $\mathcal{R}_0 x_0$ is a core for $\Delta^{\frac{1}{2}}$; and the density theorem (4.10) just proved establishes that $\mathcal{R}_0'' = \mathcal{R}$.

The essential steps in the argument that follows are drawn from part (Lemmas 3–7) of Haagerup's argument [3]. Using the Bridging Lemma (4.7) and some preliminary analysis of the special functions involved, we shall prove:

Lemma 4.11. *If $f_a(t) = \exp(-|t-a|)$ with a real, and $A \in \mathcal{R}$, then $f_a(\log \Delta)Ax_0 = Bx_0$, where $B \in \mathcal{R}$ and $\|B\| \leq \|A\|$.*

Assuming this result, for the time, we prove:

Lemma 4.12. *If $A_0 x_0 \in E(k^{-1}, k)(\mathcal{H})$ for some k greater than 1 and $A_0 \in \mathcal{R}$, then $\Delta^n A_0 x_0 = A_n x_0$, where $A_n \in \mathcal{R}$ and $\|A_n\| \leq k^{|n|}\|A_0\|$. In addition $A_0 x_0 = A'x_0$, where $A' \in \mathcal{R}'$ and $\|A'\| \leq k^{\frac{1}{2}}\|A_0\|$. The statement obtained by interchanging $\mathcal{R}$ and $\mathcal{R}'$ in the preceding is also valid.*

Proof. Since $k \exp(-|t - \log k|)$ and $\exp t$ coincide on $[-\log k, \log k]$; we have

$$\Delta A_0 x_0 = k f_{\log k}(\log \Delta) A_0 x_0 = A_1 x_0,$$

where $A_1 \in \mathcal{R}$ and $\|A_1\| \leq k\|A_0\|$. (The last equality uses Lemma 4.11.) Replacing t by $-t$, we also have

$$\Delta^{-1} A_0 x_0 = k f_{-\log k}(\log \Delta) A_0 x_0 = A_{-1} x_0,$$

with A_{-1} in $\mathcal{R}$ and $\|A_{-1}\| \leq k\|A_0\|$. Since $A_1 x_0 \in (\mathcal{R}x_0) \cap E(k^{-1}, k)(\mathcal{H})$, it follows from what we have proved that $\Delta A_1 x_0 = A_2 x_0$, where $A_2 \in \mathcal{R}$ and $\|A_2\| \leq k^2\|A_0\|$. In addition $A_2 x_0 \in (\mathcal{R}x_0) \cap E(k^{-1}, k)(\mathcal{H})$. Continuing, we construct A_n with the desired properties.

As Δ^{-1} plays the rôle of Δ when $\mathcal{R}$ and $\mathcal{R}'$ are interchanged (with the same x_0) and $E(k^{-1}, k)$ is the spectral projection corresponding to (k^{-1}, k) for both Δ and Δ^{-1}, we can apply the result just established to $\mathcal{R}'$ and Δ^{-1} with the only modification of the conclusion being the replacement of $\mathcal{R}$ by $\mathcal{R}'$.

From the Bridging Lemma (4.7), $(kI+\Delta^{-1})^{-1}A_0x_0=A_0'x_0$, where $A_0'\in\mathcal{R}'$ and $\|A_0'\|\leqslant(4k)^{-\frac{1}{2}}\|A_0\|$. Thus $A_0x_0=(kI+\Delta^{-1})A_0'x_0=kA_0'x_0+A_1'x_0$, where $A_1'\in\mathcal{R}'$ and $\|A_1'\|\leqslant k\|A_0'\|$. (Note for this that $A_0'x_0=(kI+\Delta^{-1})^{-1}A_0x_0\in E(k^{-1},k)(\mathcal{H})$ and apply the result of the preceding paragraph.) Letting A' be $kA_0'+A_1'$, the last assertion of this lemma follows. ■

We conclude from Lemma 4.12 that each y in $(\mathcal{R}x_0)\cap E(k^{-1},k)(\mathcal{H})$ (or in $(\mathcal{R}'x_0)\cap E(k^{-1},k)(\mathcal{H})$) lies in a reflection sequence. We want, next, to show that the set of such vectors (as k takes values in $(1,\infty)$) forms a core for $\Delta^{\frac{1}{2}}$. We prove this in the lemma that follows.

LEMMA 4.13. *The linear manifold* $\bigcup_{n=2}^{\infty}(\mathcal{R}x_0)\cap E(n^{-1},n)(\mathcal{H})$ $(=\mathcal{D})$ *is a core for* $\Delta^{\frac{1}{2}}$.

Proof. If $A\in\mathcal{R}$ and

$$g_n(t)=e^{-|t|}-(e^n+e^{-n})^{-1}(e^{-|t-n|}+e^{-|t+n|})$$

with n an integer greater that 1, then (g_n) is an increasing sequence of positive functions vanishing outside (but not on) $(-n,n)$ and converging at each t to $\exp(-|t|)$. (Note, for this, that $g_n(t)=g_n(-t)$; so that we may assume $0\leqslant t$; and write $g_n(t)$ as $\exp(-t)[1-(\exp(2n)+1)^{-1}(\exp(2t)+1)]$ when $0\leqslant t\leqslant n$.) From Lemma 4.11, $g_n(\log\Delta)Ax_0=Bx_0$, where $B\in\mathcal{R}$. Moreover $g_n(\log\Delta)E_n=g_n(\log\Delta)$, where $E_n=E(\exp(-n),\exp n)$, since g_n vanishes outside $(-n,n)$; and $g_n(\log\Delta)E_n(\mathcal{H})$ is dense in $E_n(\mathcal{H})$ since g_n does not vanish on $(-n,n)$. Thus $g_n(\log\Delta)Ax_0=g_n(\log\Delta)E_nAx_0\in\mathcal{D}$ for each A in $\mathcal{R}$ and all n $(=2,3,\ldots)$. Since $\{E_nAx_0\colon A\in\mathcal{R}\}$ is dense in $E_n(\mathcal{H})$; $\{g_n(\log\Delta)E_nAx_0\colon A\in\mathcal{R}\}$ is dense in $E_n(\mathcal{H})$. If $y\in E_n(\mathcal{H})$, we can, therefore, choose y_m in $\mathcal{D}\cap E_n(\mathcal{H})$ such that (y_m) tends to y. As $\Delta^{\frac{1}{2}}$ is bounded on $E_n(\mathcal{H})$, $\Delta^{\frac{1}{2}}y_m\to\Delta^{\frac{1}{2}}y$. Hence $(y,\Delta^{\frac{1}{2}}y)$ is in the closure of the graph of $\Delta^{\frac{1}{2}}\mid\mathcal{D}$. Since $\bigcup_{n=2}^{\infty}E_n(\mathcal{H})$ is a core for $\Delta^{\frac{1}{2}}$, $\mathcal{D}$ is a core for $\Delta^{\frac{1}{2}}$. ■

It remains to prove Lemma 4.11.

Proof of Lemma 4.11. If

$$h_a(t)=[\cosh(t-a)]^{-1}(=2[e^{t-a}+e^{a-t}]^{-1})$$

then

$$h_a(\log\Delta)=2(e^{-a}\Delta+e^{a}\Delta^{-1})^{-1}=2i(\Delta+ie^{a}I)^{-1}(\Delta^{-1}+ie^{-a}I)^{-1}.$$

From the Bridging Lemma, with A in $\mathcal{R}$, we have $h_a(\log\Delta)Ax_0=B_0x_0$, where $B_0\in\mathcal{R}$ and $\|B_0\|\leqslant\|A\|$. We use the fact that, for all real t,

$$e^{-|t|}=\sum_{n=1}^{\infty}a_n[\cosh t]^{-(2n-1)}$$

and convergence is uniform on the reals, where $0 < a_n$ and $\sum_{n=1}^{\infty} a_n = 1$. (This can be proved by studying the inverse to $s \to 2s(s^2+1)^{-1}$ on $[-1, 1]$ and letting s be $\exp(-t)$.) From this, we have

$$f_a(\log \Delta) = \sum_{n=1}^{\infty} a_n [h_a(\log \Delta)]^{2n-1},$$

where convergence is in the operator-norm topology. Thus, for each A in $\mathcal{R}$,

$$f_a(\log \Delta) A x_0 = \sum_{n=1}^{\infty} a_n [h_a(\log \Delta)]^{2n-1} A x_0 = \sum_{n=1}^{\infty} a_n B_n x_0,$$

where $B_n \in \mathcal{R}$ and $\|B_n\| \leq \|A\|$. Since $0 \leq a_n$ and $\sum a_n = 1$; we have that $\sum_{n=1}^{\infty} a_n B_n$ converges (in norm) to an operator B in $\mathcal{R}$ and $\|B\| \leq \|A\|$. ∎

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