AN EXTREMAL PROBLEM FOR QUASICONFORMAL MAPPINGS AND A THEOREM BY THURSTON

BY

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To Lars V. Ahlfors, on his 70th birthday

§ 1. Statement of the problem

The new extremal problem considered in this paper, and the form of the solution are suggested by Thurston's beautiful theorem on the structure of topological self-mappings of a surface (Theorem 4 in [21]). In treating this problem, however, we use none of Thurston's results and thus obtain also a new proof of his theorem.

Unless otherwise stated all surfaces considered in this paper will be assumed to be *oriented* and of finite type (p, m), that is homeomorphic to a sphere with $p \geqslant 0$ handles from which one has removed $m \geqslant 0$ disjoint continua. To avoid uninteresting special cases we assume that

$$2p - 2 + m > 0. \tag{1.1}$$

All mappings between surfaces (or between finite disjoint unions of surfaces) will be assumed bijective, topological and orientation preserving. We recall (see Mangler [15]) that two mappings of a surface are isotopic if and only if they are homotopic.

A *conformal structure* on a surface S is a mapping σ of S onto a Riemann surface. If $f: S_1 \to S_2$ is a mapping, and σ_1, σ_2 are conformal structures on S_1 and S_2, respectively, then the deviation of $\sigma_2 \circ f \circ \sigma_1^{-1}$ from conformality is measured by the *dilatation*

$$K = K(\sigma_2 \circ f \circ \sigma_1^{-1}) = K_{\sigma_1, \sigma_2}(f).$$

We recall that $1 \leqslant K \leqslant +\infty$, with $K = 1$ signifying that $\sigma_2 \circ f \circ \sigma_1^{-1}$ is conformal, and $K = +\infty$ signifying that this mapping is not even quasiconformal. If $S_1 = S_2$ and $\sigma_1 = \sigma_2$, we write $K_\sigma(f)$ instead of $K_{\sigma, \sigma}(f)$.

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Now let f be a given self-mapping of a surface S. We pose the *extremal problem* of minimizing $K_{\sigma'}(f')$ by varying the conformal structure σ' *and* by varying f' over the isotopy class of f. A solution, if it exists, is a pair (σ_0, f_0), with f_0 isotopic to f and such that $K_{\sigma_0}(f_0) \leqslant K_{\sigma'}(f')$ for all conformal structures σ' on S and all f' isotopic to f. If so, we call σ_0 an *f-minimal conformal structure*, and we call $\sigma_0 \circ f_0 \circ \sigma_0^{-1}$ an *absolutely extremal* self-mapping of the Riemann surface $\sigma_0(S)$.

We shall give necessary and sufficient conditions for the existence of an f-minimal conformal structure, shall characterize absolutely extremal mappings, and shall construct a generalized solution of our problem for the cases in which it is unsolvable as stated. An essential tool in obtaining these results is Teichmüller's theorem which we proceed to recall.

§ 2. Teichmüller mappings

Let X be a Riemann surface, not necessarily of finite type. A *Beltrami differential* M on X is a rule which assigns to every local parameter z defined in a domain $D \subset X$ an essentially bounded complex-valued measurable function $\mu(z)$, $z \in z(D)$, such that $\mu(z)\,d\bar{z}/dz$ is invariant under parameter changes. (By abuse of language we write $M = \mu(z)\,d\bar{z}/dz$ in D.) If so, $|M|$ is a real-valued function defined on X; it is required that it be essentially bounded and that its essential supremum $k = \|M\|_\infty$ satisfy $k < 1$.

A *meromorphic quadratic differential* Φ on X is a rule which assigns to every local parameter z defined in a domain $D \subset X$ a meromorphic function $\varphi(z)$, $z \in z(D)$, such that $\varphi(z)\,dz^2$ is invariant under parameter changes. (By abuse of language we write: $\Phi = \varphi(z)\,dz^2$ in D.) If, so, $|\varphi(z)|\,dxdy$ (where $z = x + iy$, x and y real) is also invariant under parameter changes and one can compute the integral $\iint_X |\Phi|$. If it is finite, Φ is called *integrable*. We call Φ *admissible* if its poles, if any, are simple. An integrable meromorphic quadratic differential is admissible, the converse is true if X is compact.

If Φ is an admissible quadratic differential on X, $P \in X$, and r the order of Φ at P, then there exists a local parameter z defined near and vanishing at P, such that

$$\Phi = \left(\frac{r+2}{2}\right)^2 z^r dz^2 \ \text{ near } P; \tag{2.1}$$

if $r = 0$, then

$$\Phi = dz^2 \quad \text{near } P. \tag{2.2}$$

The parameter z is called a *natural parameter* belonging to Φ at P. It may be multiplied by an $(r+2)-$nd root of unity but is otherwise uniquely determined.

Now let H be a quasiconformal mapping of X onto another Riemann surface Y. The

Beltrami differential M of H is defined as follows. Let z be a local parameter defined on a domain $D \subset X$, ζ a local parameter defined on $H(D) \subset Y$, and set $h: \zeta \circ H \circ z^{-1}$ (that is, let $z \mapsto \zeta = h(z)$ represent the mapping $H \mid D$). Then $M = \mu(z) d\bar{z}/dz$ in D, where μ is the *Beltrami coefficient* of the quasiconformal mapping $\zeta = h(z)$, that is, $\mu(z) = h_{\bar{z}}(z)/h_z(z)$. (The partial derivatives h_z, $h_{\bar{z}}$ are taken in the distribution sense; they are locally square integrable measurable functions.) The dilatation K of H is given by the formula

$$K = (1 + \|M\|_\infty)/(1 - \|M\|_\infty).$$

We call H a *formal Teichmüller mapping* if its Beltrami differential M is the form

$$M = k|\Phi|/\Phi$$

where $0 < k < 1$ and $\Phi \neq 0$ is an admissible quadratic differential called the *initial differential* of H. It is determined by H uniquely, except for a multiplicative positive constant.

If H: $X \to Y$ is a formal Teichmüller mapping with initial differential Φ and dilatation $K = (1+k)/(1-k)$, then there exists a unique admissible quadratic differential Ψ on Y, called the *terminal differential* of H, having the following properties.

(i) The order r of Φ at a point $P \in X$ equals the order of Ψ at $H(P)$.

(ii) If $z = x + iy$ is a natural parameter belonging to Φ at P, there exists a natural parameter $\zeta = \xi + i\eta$ belonging to Ψ at $H(P)$ such that near P the mapping H is represented by

$$\zeta = \left(\frac{z^{r+2} + 2k|z|^{r+2} + k^2 \bar{z}^{r+2}}{1 - k^2} \right)^{1/(r+2)}, \tag{2.3}$$

with $\zeta > 0$ for $z > 0$. In particular, if $r = 0$, then H is represented by

$$\xi = K^{1/2} x, \quad \eta = K^{-1/2} y. \tag{2.4}$$

(iii) We have

$$\iint_X |\Phi| = \iint_Y |\Psi|.$$

(iv) The mapping H^{-1} is a formal Teichmüller mapping with dilation K, initial differential $(-\Psi)$ and terminal differential $(-\Phi)$.

The easy proof of these assertions is a slight modification of the argument in § 8 of [3]. Observe that (iii) follows at once from (2.2) and (2.4).

A formal Teichmüller mapping H: $X \to Y$ will be called a *Teichmüller mapping* if its initial differential is integrable, holomorphic (which does not prohibit simple poles at the punctures of X), and real on the ideal boundary curves of X, if any.

The Teichmüller theorem (cf. Teichmüller [19; 20], Ahlfors [1], Bers [3]) asserts that if X and Y are Riemann surfaces of finite topological type (p, m), with $2p-2+m>0$, then *every quasiconformal mapping of X onto Y is isotopic to a unique extremal mapping*, and that *a mapping $X \to Y$ is extremal if and only if it is either conformal or a Teichmüller mapping.*

§ 3. Periodic and non-periodic mappings

There is a case in which the solution of the extremal problem stated in § 1 is well known and evident.

THEOREM 1. *A self-mapping f of a surface S is isotopic to a periodic self-mapping if and only if there is a conformal structure σ on S and a mapping f' isotopic to f such that $\sigma \circ f' \circ \sigma^{-1}$ is conformal.*

Proof. The group of all conformal self-mappings of a Riemann surface X is finite except if X is of finite type $(p, m) = (0, 0)$, $(0, 1)$, $(0, 2)$ or $(1, 0)$. In view of assumption (1.1) this implies the necessity of the condition.

To prove the sufficiency, let f_0 be a mapping of a period $n+1$ isotopic to f, and let ds be a Riemannian metric on S (for some differentiable structure). Then

$$ds_0 = ds + f_0^*(ds) + \ldots + (f_0^*)^n (ds)$$

is a Riemannian metric invariant under f_0, so that f_0 is a conformal self-mapping of S equipped with the conformal structure induced by ds_0.

The argument implies the validity of the following

ADDITION TO THEOREM 1. *If the selfmapping f of S is isotopic to a periodic one, and if S has $m>0$ (ideal) boundary continua, then there is an f-minimal conformal structure σ such that $r \geq 0$ arbitrarily prescribed boundary continua of S, as well as all their images under powers of f, become punctures of $\sigma(S)$, while all other boundary continua of S become ideal boundary curves of $\sigma(S)$.*

We proceed to show that in treating the extremal problem stated in § 1 in the case when f is *not* isotopic to a periodic mapping, we may restrict ourselves to conformal structures σ on S for which $\sigma(S)$ has no ideal boundary curves. Such conformal structures will be called of the *first kind*, all others will be called of the *second kind*.

Two conformal structures, σ_1 and σ_2, on S will be called *similar* if every boundary continuum of S is either a puncture of both Riemann surfaces $\sigma_1(S)$ and $\sigma_2(S)$ or an ideal boundary curve of these two Riemann surfaces.

THEOREM 2. *Let σ be a conformal structure of the second kind on a surface S and f a self-mapping of S with $K_\sigma(f) < +\infty$. If f is not isotopic to a periodic mapping, then*

(i) *there is a conformal structure σ' on S similar to σ and a mapping f' isotopic to f, with $K_{\sigma'}(f') < K_\sigma(f)$, and*

(ii) *there is a conformal structure σ_1 of the first kind on S and a mapping f_1 isotopic to f, with $K_{\sigma_1}(f) < K_\sigma(f)$.*

Proof. We shall use Teichmüller's theorem and properties of the so-called Nielsen extension described in [7]. In view of Theorem 1 and of Teichmüller's theorem, we may assume that $\sigma\circ f\circ\sigma^{-1}$ is a Teichmüller mapping. We define the number k by the condition

$$K_\sigma(f) = (1+k)/(1-k).$$

Since $\sigma(S)$ has at least one ideal boundary curve, we can embed $\sigma(S)$ canonically into its *Schottky double* $\sigma(S)^d$. This is a Riemann surface equipped with a canonical anti-conformal involution j such that the complement of the fixed point set of j has two components: $\sigma(S)$ and another Riemann surface $j\circ\sigma(S)$, called the mirror image of $\sigma(S)$. The components of the fixed point set of j are the ideal boundary curves of $\sigma(S)$. Suppose there are n of those, and assume that $\sigma(S)$ has r punctures. Then $\sigma(S)^d$ has genus $2p+n-1$, where p is the genus of S, and is compact except for $2r$ punctures. Since $n>0$ by hypothesis, and $n+r=m$, we conclude from (1.1) that $\sigma(S)^d$ can be represented as U/G where U is the upper half-plane and G a torsion-free Fuchsian group. Since $\sigma(S)^d$ is compact, except for finitely many punctures, G is of the first kind, i.e., its limit set is $\mathbf{R}\cup\{\infty\}$.

Now let D be a component of the preimage of $\sigma(S)$ under the natural projection $\pi\colon U\to U/G=\sigma(S)^d$, and G' the stabilizer of D in G. Then $D/G'=\sigma(S)$, and $\sigma(S)$ is canonically embedded in the Riemann surface $\sigma(S)'=U/G'$, called the *Nielsen extension* of $\sigma(S)$ (cf. [7], Theorem 1 and Lemma 3). The Nielsen extension is again a Riemann surface of genus p with r punctures and n boundary curves, and the complement of the closure of $\sigma(S)$ in $\sigma(S)'$ consists of n components, each conformally equivalent to an annulus.

According to Theorem 2 of [7] every quasiconformal self-mapping of $\sigma(S)$, in particular the Teichmüller mapping $\sigma\circ f\circ\sigma^{-1}$, admits a canonical extension to $\sigma(S)'$, with the same dilatation. This extension is constructed as follows. First we extend $\sigma\circ f\circ\sigma^{-1}$ to a mapping w of $\sigma(S)^d$ onto itself which commutes with j. It is known, and easy to check, that w is a Teichmüller mapping of dilatation $K_\sigma(f)$. Since $\sigma(S)^d=U/G$, there is a quasiconformal mapping W of U onto itself inducing w, that is satisfying $w\circ\pi=\pi\circ W$. This W also induces a mapping F of $\sigma(S)'$ onto itself which is the desired extension.

The Teichmüller mapping w has (see § 2) a holomorphic integrable initial quadratic differential $\Phi \neq 0$ on $\sigma(S)^d$ which, lifted to U via π, can be written as $\varphi(z)dz^2$, with $\varphi(z)$, $z \in U$, a holomorphic function satisfying $\varphi(g(z))g'(z)^2 = g(z)$ for all $g \in G$, and

$$0 < \iint_{U/G} |\varphi(z)| \, dx\, dy < +\infty. \tag{3.1}$$

The mapping W: $U \to U$ satisfies the Beltrami equation

$$\frac{\partial W}{\partial \bar{z}} = k \frac{|\varphi(z)|}{\varphi(z)} \frac{\partial W}{\partial z}.$$

Now G' is of the second kind (its limit set is nowhere dense in $\mathbf{R} \cup \{\infty\}$), so that it is of infinite index in G, and by (3.1)

$$\iint_{U/G'} |\varphi(z)| \, dx\, dy = +\infty. \tag{3.2}$$

The mapping F: $\sigma(S)' \to \sigma(S)'$ is a formal Teichmüller mapping with dilatation $K_\sigma(f)$ and initial differential Φ^* which is induced by the function $\varphi(z)$. In view of (3.2), Φ^* is not integrable. Hence F is not a Teichmüller mapping and not extremal. Therefore there exists a self-mapping F' of $\sigma(S)'$ which is homotopic to F and whose dilatation is smaller than that of F.

It is easy to construct a mapping σ': $S \to \sigma(S)'$ and a mapping f' isotopic to f such that $F' = \sigma' \circ f' \circ (\sigma')^{-1}$. By construction σ' is similar to σ and $K_{\sigma'}(f') = K(F') < K_\sigma(f)$. This prove statement (i).

Now set $X^{(1)} = \sigma(S)'$ and let $X^{(i+1)}$ be the Nielsen extension of $X^{(i)}$, $i = 1, 2, \dots$. In view of the canonical embeddings $\sigma(S) \subset X^{(1)}$, $X^{(i)} \subset X^{(i+1)}$, we can embed $\sigma(S)$ into its infinite Nielsen extension $X = X^{(1)} \cup X^{(2)} \cup \dots$. According to Theorem 3 in [7], the complement of the closure of $\sigma(S)$ in X consists of n domains each of which is conformally equivalent to a punctured disc. The mapping F': $X^{(1)} \to X^{(1)}$ constructed above can be extended, without increasing its dilatation to a mapping F_1: $X \to X$.

It is easy to construct a mapping σ_1: $S \to X$ and a mapping f_1 isotopic to f such that $F_1 = \sigma_1 \circ f_1 \circ \sigma_1^{-1}$. By construction σ_1 is of the first kind and $K_{\sigma_1}(f_1) = K(F_1) = K(F') < K_\sigma(f)$. This proves (ii).

COROLLARY TO THEOREM 2. *Let f be a self-mapping of a surface S and σ a conformal structure on S such that $K_\sigma(f) < +\infty$ and $K_\sigma(f) \leqslant K_{\sigma'}(f')$ for all σ' similar to σ and all f' isotopic to f. Then*

(i) *either $\sigma \circ f \circ \sigma^{-1}$ is conformal or σ is of the first kind, and*

(ii) *σ is f-minimal and $\sigma \circ f \circ \sigma^{-1}$ is absolutely extremal.*

Proof. If $\sigma \circ f \circ \sigma^{-1}$ is not conformal, then it is a Teichmüller mapping, and hence not isotopic to a periodic mapping. Therefore, by Theorem 2(i), σ cannot be of the second kind.

If $\sigma \circ f \circ \sigma^{-1}$ is conformal, statement (ii) is trivial. If $\sigma \circ f \circ \sigma^{-1}$ is not conformal and hence not isotopic to a periodic mapping, σ is of the first kind, by (i), and (ii) follows from Theorem 2(ii).

The corollary permits us to restate the extremal problem of § 1 as a problem concerning the modular group acting in a Teichmüller space.

§ 4. Restatement of the problem

We first recall the definition and main properties of the *Teichmüller space* $T(p, m)$ of Riemann surfaces of finite type (p, m) with $2p-2+m>0$, cf. Ahlfors [1], Bers [4, 6], Earle [9]. We start with a surface S of type (p, m) and define two conformal structures σ_1 and σ_2 on S to be *strongly equivalent* if there is a conformal map c of $\sigma_1(S)$ onto $\sigma_2(S)$ such that $\sigma_2^{-1} \circ c \circ \sigma_1$ is isotopic to the identity. The strong equivalence classes $[\sigma]$ of structures of the first kind are the points of $T(p, m)$ and the distance (*Teichmüller distance*) between two points $[\sigma_1]$ and $[\sigma_2]$ is defined as

$$\langle [\sigma_1], [\sigma_2] \rangle = \tfrac{1}{2} \log \inf K(g), \quad g \text{ isotopic to } \sigma_2 \circ \sigma_1^{-1}, \tag{4.1}$$

or

$$\langle [\sigma_1], [\sigma_2] \rangle = \tfrac{1}{2} \log K(h), \quad h \text{ extremal and isotopic to } \sigma_2 \circ \sigma_1^{-1}. \tag{4.1'}$$

With this metric $T(p, n)$ is a complete metric space homeomorphic to $\mathbf{R}^{6p-6+2m}$. (That $T(p, m)$ also has a natural complex structure which can be realized by embedding $T(p, m)$ as a bounded domain of holomorphy in $\mathbf{C}^{3p-3+m}$, and that, according to a theorem by Royden, the Teichmüller metric is the Kobayashi metric need not concern us here.)

Kravetz [13] showed that the Teichmüller space $T(p, m)$ is a *straight line space* in the sense of Busemann [8]; see Linch [14] concerning this proof. This means, in particular, that any two distinct points $\tau_1, \tau_2 \in T(p, n)$ lie on a unique *line*, i.e. an isometric image of $\mathbf{R}$, and that a line l through two points τ_1 and τ_2 also contains all points τ with $\langle \tau_1, \tau \rangle + \langle \tau, \tau_2 \rangle = \langle \tau_1, \tau_2 \rangle$.

The *modular group* Mod (p, m) is the group of self-mappings of $T(p, m)$ of the form

$$[\sigma] \mapsto f^*([\sigma]) = [\sigma \circ f]$$

where f is a self-mapping of S. It is clear that the *modular transformation* f^* depends only on the isotopy class of f, and that every f^* is an *isometry* of $T(p, m)$ and hence maps lines into lines. (Moreover, Mod (p, m) is a group of holomorphic automorphisms of $T(p, m)$, indeed, according to a theorem by Royden, the full group of holomorphic self-mappings of this space, provided that $3p-3+m>1$.)

The definition implies that two Riemann surfaces $\sigma_1(S)$ and $\sigma_2(S)$ are conformally equivalent if and only if $f^*([\sigma_1])=[\sigma_2]$ for some $f^*\in \text{Mod}\,(p, m)$.

We introduce a classification of elements of the modular group, similar to the classification of elements of the elliptic modular group $SL(2, \mathbf{Z})/\pm I$. (The elliptic modular group may be identified with Mod (1, 1), and the principal congruence subgroup modulo 2 of the elliptic modular group may be identified with Mod (0, 4).)

For $\chi\in \text{Mod}\,(p, m)$, let $a(\chi)$ denote the infimum of $\langle\tau, \chi(\tau)\rangle$ for $\tau\in T(p, m)$. We shall say that χ is *elliptic* if it has a fixed point in $T(p, m)$, *parabolic* if there is no fixed point but $a(\chi)=0$, *hyperbolic* if $a(\chi)>0$ and there is a $\tau\in T(p, m)$ with $\langle\tau, \chi(\tau)\rangle=a(\chi)$, *pseudo-hyperbolic* if $a(\chi)>0$ and $\langle\chi, \tau(\chi)\rangle>a(\chi)$ for all $\tau\in T(p, m)$. The property of being elliptic, parabolic, hyperbolic or pseudohyperbolic is preserved by inner automorphisms of the modular group.

A point $\tau\in T(p, m)$ will be called *χ-minimal* if $\langle\tau, \chi(\tau)\rangle=a(\tau)$.

PROPOSITION 1. *A conformal structure σ of the first kind, on a surface S, is f-minimal if and only if $[\sigma]$ is f^*-minimal.*

Proof. Let h denote the unique extremal mapping in the isotopy class of $\sigma\circ f\circ\sigma^{-1}$. If σ is f-minimal and f_1 is isotopic to f, then, for every conformal structure σ_1 of the first kind

$$K(\sigma_1\circ f_1\circ\sigma_1^{-1})\geqslant K(\sigma\circ h\circ\sigma^{-1})$$

so that, by definition (4.1′),

$$\langle[\sigma_1], f^*([\sigma_1])\rangle\geqslant\langle[\sigma], f^*([\sigma])\rangle;$$

hence $[\sigma]$ is f^*-minimal. The converse statement is verified in the same way, taking into account the Corollary to Theorem 2.

THEOREM 3. *An element of the modular group is elliptic if and only if it is periodic.*

This is a well-known result. The necessity is, of course, trivial. If f^* has a fixed point, there is a conformal structure σ on the surface considered and an f_1 isotopic to f such that $\sigma\circ f_1\circ\sigma^{-1}$ is conformal. Hence there is an integer $n>0$ with $(\sigma\circ f_1\circ\sigma^{-1})^n=\text{id}$, and $(f^*)^n=(f_1^*)^n=(\text{id})^*=\text{id}$.

The sufficiency statement is a rather deep result going back to Nielsen. A brief proof, based on the Paul Smith periodicity theorem and on the fact that the Teichmüller space is a cell, is due to Fenchel [12], see also Kravetz [13]. A new proof is contained in Thurston's theory.

Theorem 3 deepens Theorem 1. The two theorems, together with the Corollary to Theorem 2 and with Proposition 1 imply

PROPOSITION 2. *Let f be a self-mapping of a surface. An f-minimal conformal structure exists if and only if the modular transformation f^* is elliptic or hyperbolic.*

Propositions 1 and 2 constitute the desired restatement of the extremal problem of § 1.

§ 5. Irreducible mappings

A finite non-empty set of disjoint Jordan curves $\{C_1, ..., C_r\}$ on a surface S will be called *admissible* if no C_i can be continuously deformed into a point, a boundary continuum of S, or into a C_j with $j \neq i$. Following Thurston we say that a mapping f': $S \to S$ is *reduced* by $\{C_1, ..., C_r\}$ if this set is admissible and

$$f'(C_1 \cup ... \cup C_r) = C_1 \cup ... \cup C_r.$$

A self-mapping f of S will be called *reducible* if it is isotopic to a reduced mapping, *irreducible* if it is not.

THEOREM 4. *If f is an irreducible self-mapping of a surface S, the modular transformation f^* induced by f is either elliptic or hyperbolic.*

The proof will be preceded by several lemmas. In these all Riemann surfaces X are assumed to have U as the universal covering surface, and all geometric concepts refer to the Poincaré metric. We recall, that every free homotopy class of closed curves on X contains a unique geodesic, that a geodesic freely homotopic to a Jordan curve is itself a Jordan curve, and that no closed geodesic on X can be deformed into a point oı a puncture on X.

LEMMA 1 (Wolpert [22]). *Let f be a quasiconformal mapping of a Riemann surface X onto a Riemann surface $f(X)$ and C a closed geodesic on X of length l. Then $f(C)$ is freely homotopic to a closed geodesic on $f(X)$ of length l' with*

$$l' < K(f)l. \tag{5.1}$$

This sharp inequality replaces a cruder one, based on Mori's theorem [17], which I used originally. Wolpert's elegant proof is reproduced for the convenience of the reader.

Represent X and $f(X)$ as $X = U/G$, $f(X) = U/G'$, G and G' torsion free Fuchsian groups, and lift the mapping f: $X \to f(X)$ to a mapping F: $U \to U$. Then $FGF^{-1} = G'$ and $K(F) = K(f)$.

Let $g \in G$ be a hyperbolic element whose axis projects onto C under the natural mapping $U \to U/G = X$. Then the axis of $g' = F \circ g \circ F^{-1}$ projects onto the geodesic freely homotopic to $f(C)$. Let G_0 and G_0' be the cyclic groups generated by g and g', respectively. Then $FG_0F^{-1} = G_0'$ and F induces a mapping f_0 of the annulus U/G_0 onto the annulus U/G_0', with $K(f_0) = K(F) = K(f)$. Hence the module of the annulus U/G_0 is at most $K(f)$ times the module of U/G_0'. But the module of U/G_0 is $2\pi^2/l$, that of U/G_0' is $2\pi^2/l'$. Inequality (5.1) follows.

LEMMA 2. *There is a number $\delta_0 > 0$ such that any two distinct geodesic Jordan curves of length not exceeding δ_0, on some Riemann surface X, are disjoint.*

This is a known result; we include a proof for the sake of completeness.

Let C be a geodesic Jordan curve on X of length l. A *collar* D about C of width $2b$ is a domain homeomorphic to an annulus, with $C \subset D \subset X$, bounded by two Jordan curves C' and C'', freely homotopic to C, such that every point on C' or on C'' has distance b from C. If C_1 is a geodesic Jordan curve intersecting C, then C_1 must contain an arc in D joining a point on C' to a point on C''. Hence the length l_1 of C_1 satisfies $l_1 > 2b$.

According to the Keen-Halpern collar lemma, in the sharp form due to Matelski [16], there always is a collar about C of width $2b$, with

$$\sinh b = [2 \sinh (l/2)]^{-1}.$$

Let $\delta_0 > 0$ be so small that $2b > \delta_0$ for $l < \delta_0$. This δ_0 has the required property.

LEMMA 3. *Let X be a Riemann surface of type (p, m), and assume that there is a geodesic Jordan curve C on X of length l. Then every irreducible self-mapping f of X satisfies*

$$K(f) \geqslant (\delta_0/l)^{1/(3p-3+m)} \tag{5.2}$$

where δ_0 is the number from Lemma 2.

Proof. Assume that (5.2) does not hold, so that

$$K(f)^{3p-3+m}\, l < \delta_0. \tag{5.3}$$

Set $C_1 = C$ and let C_{j+1} be the geodesic Jordan curve freely homotopic to $f^j(C_1)$. By (5.3) and Lemma 1, the lengths of the curves $C_1, \ldots, C_{3p-2+m}$ do not exceed δ_0. Not all of them can be disjoint, and we conclude, by Lemma 2, that there is an $r > 0$ such that the curves $C_1, C_2, \ldots, C_r$ are disjoint and $C_{r+1} = C_1$.

By construction $f(C_j)$ is freely homotopic to C_{j+1}, $j = 1, \ldots, r-1$, and $f(C_r)$ is freely homotopic to C_1. Therefore (cf. Epstein [11]) there is a mapping f' isotopic to f such that

$f'(C_j) = C_{j+1}$, $j = 1, ..., r-1$, $f'(C_r) = C_1$. This f' is reduced by $\{C_1, ..., C_r\}$, so that f is reducible.

LEMMA 4. *Let $\{[\sigma_j]\} \subset T(m, p)$ be a sequence such that the lengths of all geodesic Jordan curves on the Riemann surfaces $\sigma_j(S)$ have a positive lower bound. Then there is a subsequence $\{[\sigma_{j_n}]\}$ and a sequence $\{\chi_n\} \subset \operatorname{Mod}(p, m)$ such that the sequence $\{\chi_n([\sigma_{j_n}])\}$ converges.*

Proof. A theorem, going back to Fricke, asserts that Mod (p, m) acts properly discontinuously on $T(p, m)$, so that the quotient $R(p, m) = T(p, m)/\operatorname{Mod}(p, m)$ is a Hausdorff space (see, for instance, [4], p. 100, for a proof). The *Riemann space* $R(p, m)$ is the space of conformal equivalence classes of Riemann surfaces X of genus p with m punctures, it can be identified with the space of conjugacy classes of torsion free Fuchsian groups Γ, acting on U, which represent such surfaces (such Γ are all of the first kind). This latter space has a natural topology, equivalent to the quotient topology in $R(p, m)$, and the set of groups Γ for which the absolute values of the traces of hyperbolic elements are bounded away from 2 is compact. This is a theorem by Mumford [18] as extended in [5] (see Matelski [16] for an extension to groups of the second kind). It follows that the set of points in $R(p, m)$ corresponding to all Riemann surfaces on which the lengths of all geodesic Jordan curves are not less than an $\varepsilon > 0$ is compact. This implies the desired statement.

Proof of Theorem 4. Let (p, m) be the type of S. We recall the notation, introduced in § 4,

$$a(f^*) = \inf \langle [\sigma], f^*([\sigma]) \rangle, \quad [\sigma] \in T(p, m),$$

and consider a sequence $\{[\sigma_j]\} \subset T(p, m)$ with

$$\lim_{j \to \infty} \langle [\sigma_j], f^*[\sigma_j] \rangle = a(f^*). \tag{5.4}$$

Let h_j be the extremal mapping in the isotopy class of $\sigma_j \circ f \circ \sigma_j^{-1}$. Then

$$\lim_{j \to \infty} \tfrac{1}{2} \log K(h_j) = a(f^*) \tag{5.5}$$

and, on the other hand,

$$h_j = \sigma_j \circ f_j \circ \sigma_j^{-1}$$

where f_j is isotopic to f. By (5.4), there is a number $A > 1$ such that

$$K(h_j) = K_{\sigma_j}(f_j) \leqslant A, \quad j = 1, 2, \ldots. \tag{5.6}$$

Since f is irreducible, by hypothesis, so are f_j and h_j. By (5.6) and Lemma 3, we conclude that no geodesic Jordan curve on any Riemann surface $\sigma_j(S)$ has a length less than $l_0 = \delta_0 A^{3-3p-m}$.

By Lemma 4 we may assume, selecting if need be a subsequence, that there is a sequence $\{\chi_j\}$ of elements of Mod (p, m) such that the sequence $\{\tau_j\}$, where $\tau_j = \chi_j([\sigma_j])$, converges. We set

$$\tau = \lim_{j\to\infty} \tau_j. \tag{5.7}$$

Since each χ_j is an isometry,

$$\langle \tau_j, \chi_j \circ f^* \circ \chi_j^{-1}(\tau_j) \rangle = \langle [\sigma_j], f^*([\sigma_j]) \rangle$$

so that by (5.1)

$$\lim_{j\to\infty} \langle \tau_j, \chi_j \circ f^* \circ \chi_j^{-1}(\tau_j) \rangle = a(f^*). \tag{5.8}$$

Together with (5.7) this implies that we may assume, selecting if need be a subsequence, that the sequence $\{\chi_j \circ f^* \circ \chi_j^{-1}(\tau_j)\}$ converges to some $t \in T(p, m)$. Since each $\chi_j \circ f^* \circ \chi_j^{-1}$ is an isometry, we conclude that

$$\lim_{j\to\infty} \chi_j \circ f^* \circ \chi_j^{-1}(\tau) = t.$$

This implies that, for any $\varepsilon > 0$,

$$\langle (\chi_j \circ f^* \circ \chi_j^{-1})^{-1} \circ \chi_i \circ f^* \circ \chi_i^{-1}(\tau), \tau \rangle \leqslant \varepsilon$$

if j and i are large enough. Since Mod (p, m) acts properly discontinuously we may assume, selecting if need be a subsequence, that $\chi_j \circ f^* \circ \chi_j^{-1}$ is constant for large enough j, say for $j \geqslant j_0$. Setting $\chi_{j_0} = \chi$ we now conclude from (5.8) that $\langle \tau, \chi \circ f^* \circ \chi^{-1}(\tau) \rangle = a(f^*)$ and therefore

$$\langle [\sigma], f^*([\sigma]) \rangle = a(f^*)$$

where $[\sigma] = \chi^{-1}(\tau)$. Hence f^* is hyperbolic or elliptic.

§ 6. Absolutely extremal mappings

In this section we characterize absolutely extremal self-mappings of a Riemann surface. We recall (cf. § 1) that these are quasiconformal mappings whose dilatation cannot be decreased by varying the mapping within its homotopy class and by varying the conformal structure of the underlying surface.

We begin with an elementary result.

THEOREM 5. *If $\chi \in$ Mod (p, m) is of infinite order, then a $\tau \in T(p, m)$ is χ-minimal if and only if χ leaves a line through τ invariant.*

Proof. Assume that τ is χ-minimal. Since χ is of infinite order, neither χ nor χ^2 are elliptic, so that τ, $\chi(\tau)$ and $\chi^2(\tau)$ are distinct points. Let τ_1 be the mid-point of the segment $(\tau, \chi(\tau))$, τ_2 that of the segment $(\chi(\tau), \chi^2(\tau))$. Then

$$\langle \tau, \tau_1 \rangle = \langle \tau_1, \chi(\tau) \rangle = \tfrac{1}{2}\langle \tau, \chi(\tau) \rangle = \tfrac{1}{2}a(\chi), \tag{6.1}$$

where $a(\chi)$ is the infimum of all distances $\langle \tau', \chi(\tau') \rangle$. Similarly

$$\langle \chi(\tau), \tau_2 \rangle = \langle \tau_2, \chi^2(\tau) \rangle = \tfrac{1}{2}\langle \chi(\tau), \chi^2(\tau) \rangle$$

and, since χ is an isometry,

$$\langle \chi(\tau), \tau_2 \rangle = \langle \tau_2, \chi^2(\tau) \rangle = \tfrac{1}{2}a(\chi)$$

and

$$\chi(\tau_1) = \tau_2.$$

By the triangle inequality,

$$\langle \tau_1, \tau_2 \rangle \leqslant a(\chi),$$

and by the definition of $a(\chi)$

$$\langle \tau_1, \tau_2 \rangle \geqslant a(\chi).$$

Hence $\chi(\tau)$ is the midpoint of the segment (τ_1, τ_2). Therefore τ_1, $\chi(\tau)$ and τ_2 lie on a line l. This line also contains τ and $\chi^2(\tau)$ and is therefore invariant under χ.

Suppose next that a line l through τ is invariant under χ. This line contains the points $\chi(\tau), \chi^2(\tau), \ldots, \chi^n(\tau)$. Now let τ' be any point in $T(p, m)$. Using the fact that χ is an isometry, and the triangle inequality, we have

$$\begin{aligned} n\langle \tau, \chi(\tau) \rangle &= \langle \tau, \chi(\tau) \rangle + \langle \chi(\tau), \chi^2(\tau) \rangle + \ldots + \langle \chi^{n-1}(\tau), \chi^n(\tau) \rangle \\ &= \langle \tau, \chi^n(\tau) \rangle \leqslant \langle \tau, \tau' \rangle + \langle \tau', \chi^n(\tau') \rangle + \langle \chi^n(\tau'), \chi^n(\tau) \rangle \\ &\leqslant \langle \tau, \tau' \rangle + \langle \tau', \chi(\tau') \rangle + \langle \chi(\tau'), \chi^2(\tau') \rangle + \ldots + \langle \chi^{n-1}(\tau'), \chi^n(\tau') \rangle + \langle \chi^n(\tau'), \chi^n(\tau) \rangle \\ &= 2\langle \tau, \tau' \rangle + n\langle \tau', \chi(\tau') \rangle. \end{aligned}$$

Since n is arbitrary, $\langle \tau, \chi(\tau) \rangle \leqslant \langle \tau', \chi(\tau') \rangle$ and since τ' is arbitrary, τ is χ-minimal.

COROLLARY 1 TO THEOREM 5. *A non-periodic element of* Mod (p, m) *is hyperbolic if and only if it leaves a line invariant.*

COROLLARY 2 TO THEOREM 5. *If f is a self-mapping of a surface S and f^* is hyperbolic, then f has infinitely many essentially distinct f-minimal conformal structures.*

Proof. If σ is f-minimal, $[\sigma]$ lies on a line l invariant under f^*. Every σ_1 such that $[\sigma_1] \in l$ is f-minimal, and a conformal mapping $\sigma(S) \to \sigma_1(S)$ exists only if there is a modular transformation χ with $\chi([\sigma]) = [\sigma_1]$.

Remarks. It follows from Thurston's theory that a modular transformation has at most one invariant line. It would be desirable to prove this using quasiconformal mappings. Earle [10] showed that the function $\langle \tau, f^*(\tau) \rangle$ is differentiable at all points of $T(p, m)$ at which it does not vanish, and that every critical point of this function is an absolute maximum point.

THEOREM 6. *Let X be a Riemann surface of finite type (p, m), with $2p-2+m>0$, compact except for m punctures. A mapping w: $X \to X$ is absolutely extremal if and only if it is either conformal or a Teichmüller mapping satisfying the following two equivalent conditions:*

(a) *the mapping $w^2 = w \circ w$ is also a Teichmüller mapping and*

$$K(w^2) = K(w)^2, \tag{6.1}$$

(b) *the initial and terminal quadratic differential of w coincide.*

Proof. We may assume that w is not conformal and is a Teichmüller mapping. To conform to our previous notations we set $X = \sigma(S)$, $w = \sigma \circ f \circ \sigma$. Since w is a Teichmüller mapping, f^* is of infinite order, and w is absolutely extremal if and only if σ is f-minimal, that is, if and only if $[\sigma]$ is f^*-minimal, that is, if and only if the line through $[\sigma]$ and $f^*([\sigma]) = [\sigma \circ f]$ coincides with the line through $f^*([\sigma])$ and $(f^*)^2([\sigma]) = [\sigma \circ f^2]$, that is, if and only if

$$\langle [\sigma], [\sigma \circ f] \rangle + \langle [\sigma \circ f], [\sigma \circ f^2] \rangle = \langle [\sigma], [\sigma \circ f^2] \rangle. \tag{6.2}$$

Since w is a Teichmüller mapping, hence extremal, the extremal mapping in the isotopy classes of $\sigma \circ f \circ \sigma^{-1}$ and of $(\sigma \circ f^2) \circ (\sigma \circ f)^{-1}$ is w, and condition (6.2) becomes $2\langle [\sigma], [\sigma \circ f] \rangle = \langle [\sigma], [\sigma \circ f^2] \rangle$ or

$$K(w)^2 = K(h), \quad h \text{ the Teichmüller mapping isotopic to } w^2. \tag{6.3}$$

Assume now that w is absolutely extremal, so that (6.3) holds. Since $K(w^2) \leqslant K(w)^2$ and $K(h) \leqslant K(w^2)$ for all mappings w: $X \to X$, (6.3) implies that $K(h) = K(w^2)$ so that, by Teichmüller's theorem, $w^2 = h$ and (a) holds.

On the other hand, if (a) holds, $h = w^2$ and (6.1) implies (6.3), so that w is absolutely extremal.

Next we represent X as U/G, G Fuchsian and torsion free, and, using the canonical mapping π: $U \to U/G = X$, we lift w to a mapping W: $U \to U$. Then $w^2 = w \circ w$ lifts to $W^2 = W \circ W$. Since w is a Teichmüller mapping, W is a formal Teichmüller mapping; it satisfies the Beltrami equation $W_{\bar z}(z) = \mu(z) W_z(z)$ with the Beltrami coefficient

$$\mu(z) = k \frac{|\varphi(z)|}{\varphi(z)} \tag{6.4}$$

where

$$\frac{1+k}{1-k} = K(W) = K(w)$$

and $\varphi(z)$, $z \in U$, is holomorphic, and satisfies

$$\varphi(g(z))\, g'(z)^2 = \varphi(z), \quad g \in G,$$

$$\iint_{U/G} |\varphi(z)|\, dx\, dy = 1. \tag{6.5}$$

Under π the quadratic differential $\varphi(z)dz^2$ projects onto the initial quadratic differential Φ of w.

The inverse mapping W^{-1} is, according to § 2, also a formal Teichmüller mapping, with Beltrami coefficient

$$\varrho(z) = -k \frac{|\psi(z)|}{\psi(z)} \tag{6.6}$$

where $\psi(z)$, $z \in U$, is a holomorphic function. Since W^{-1} is a lift of w^{-1}, $\psi(z)dz^2$ projects onto the terminal differential Ψ of w, so that

$$\psi(g(z))\, g'(z)^2 = \psi(z), \quad g \in G,$$

and

$$\iint_{U/G} |\psi(z)|\, dx\, dy = 1. \tag{6.7}$$

Computing the Beltrami coefficients of both sides of the identity $W^{-1}(W(z)) = z$ we obtain

$$\hat{\varrho}(z) + \mu(z) = 0 \tag{6.8}$$

where

$$\hat{\varrho}(z) = \varrho(W(z))\, \overline{W_z(z)} / W_z(z).$$

On the other hand, the Beltrami coefficient $\nu(z)$ of $W^2 = W(W(z))$ is easily computed to be

$$\nu(z) = \frac{\hat{\mu}(z) + \mu(z)}{1 + \hat{\mu}(z)\overline{\mu(z)}} \tag{6.9}$$

where

$$\hat{\mu}(z) = \mu(W(z))\, \overline{W_z(z)} / W_z(z).$$

Since $|\hat{\mu}(z)| = |\mu(z)| = k$ a.e.,

$$|\nu(z)|^2 = \frac{2k^2 + 2 \operatorname{Re} \hat{\mu}(z)\overline{\mu(z)}}{1 + k^4 + 2 \operatorname{Re} \hat{\mu}(z)\overline{\mu(z)}}.$$

Assume that (a) holds. Then w^2 is a Teichmüller mapping, W^2 a formal Teichmüller mapping, and $|\nu(z)|$ is constant a.e. Condition (6.1) becomes

$$\frac{1+|\nu(z)|}{1-|\nu(z)|}=\left(\frac{1+k}{1-k}\right)^2$$

or $\operatorname{Re}\hat{\mu}\bar{\mu}=k^2$. Since $|\hat{\mu}|=|\mu|=k$, this implies that $\hat{\mu}(z)\overline{\mu(z)}=k^2$ or

$$\hat{\mu}(z)=\mu(z). \tag{6.10}$$

Together with (6.8) this yields $\hat{\varrho}(z)+\hat{\mu}(z)=0$, that is, $\varrho(z)+\mu(z)=0$, and, by (6.4) and (6.6),

$$\frac{|\varphi(z)|}{\varphi(z)}=\frac{|\psi(z)|}{\psi(z)},$$

so that the meromorphic function $\varphi(z)/\psi(z)$ is positive a.e. and thus a positive constant. By (6.5) and (6.7) this constant is 1, hence $\varphi=\psi$ and $\Phi=\Psi$. Condition (b) holds.

Now we assume (b). Then $\varphi=\psi$, hence $\varrho+\mu=0$ and $\hat{\varrho}+\hat{\mu}=0$, and by (6.8) we see that $\hat{\mu}=\mu$. Substitution into (6.9) yields

$$\nu(z)=\frac{2k}{1+k^2}\frac{|\varphi(z)|}{\varphi(z)}.$$

Hence W^2 is a formal Teichmüller mapping with dilatation $(1+k)^2/(1-k)^2=K(w)^2$ and initial quadratic differential $\varphi(z)dz^2$, and w^2 a Teichmüller mapping with $K(w^2)=K(w)^2$. Condition (a) holds.

§ 7. Reducible mappings

In this section we shall establish

THEOREM 7. *If f is a reducible self-mapping of a surface S, not isotopic to a periodic mapping, then the modular transformation f^* induced by f is either parabolic or pseudohyperbolic.*

We note the

COROLLARY TO THEOREM 7. *Let $f: S\to S$ be given. An f-minimal conformal structure exists on S if and only if f is either isotopic to a periodic mapping or irreducible.*

Proof of the Corollary. Compare Proposition 2, Theorem 1, Theorem 4 and Theorem 7. We shall need some technical lemmas.

Let $f: S \to S$ be reduced by the r curves $\{C_1, ..., C_r\}$. We shall say that f is *completely reduced* by $\{C_1, ..., C_r\}$ if, for every component S_1 of $S - \{C_1 \cup ... \cup C_r\}$, and for the smallest positive integer n with $f^n(S_1) = S_1$, the map $f^n | S_1$ is irreducible.

LEMMA 5. *A reducible mapping $f: S \to S$ is isotopic to a completely reduced mapping.*

Proof. Let S be of type (p, m). A set of Jordan curves $\{C_1, ..., C_r\}$ can be admissible only if $r \leqslant 3p - 3 + m$. Hence, if f is reducible, there are r, $0 < r < 3p - 2 + m$, curves $\{C_1, ..., C_r\}$ such that there is a mapping f' isotopic to f which is reduced by $\{C_1, ..., C_r\}$, and there is no mapping isotopic to f which is reduced by $r' > r$ curves. We claim that f' is completely reduced by $\{C_1, ..., C_r\}$.

Indeed, assume that S_1 is a component of $S - \{C_1 \cup ... \cup C_r\}$, that the components S_1, $S_2 = f'(S_1), ..., S_n = f'(S_{n-1})$ are all distinct, that $f'(S_n) = S_1$, and that $(f')^n | S_1$ is reducible. Then there are $q > 0$ curves $\Gamma_1, ..., \Gamma_q$ on S_1 and a mapping $h: S_1 \to S_1$, isotopic to the identity, such that $h \circ (f')^n$ is reduced by $\{\Gamma_1, ..., \Gamma_q\}$. We may choose h so that it is the restriction of a mapping $H: S \to S$ with $H | S - S_1 = \text{id}$. Set $f'' = H \circ f'$. Then f'' is isotopic to f and is reduced by the $r + nq$ curves $C_1, ..., C_r, \Gamma_1, ..., \Gamma_q, f'(\Gamma_1), ..., f'(\Gamma_q), ..., (f')^{n-1}(\Gamma_q)$. This contradicts the definition of r.

LEMMA 6. *Let X be a Riemann surface of finite type, $C_1, ..., C_r$ disjoint geodesic Jordan curves on X, Y a component of $X - \{C_1 \cup ... \cup C_r\}$, G a torsion free Fuchsian group such that $U/G = X$. Let Δ be a component of the preimage of Y under the canonical mapping $U \to U/G = X$, and let G_1 be the stabilizer of Δ in G. Then U/G_1 is the Nielsen extension of Y.*

The proof is an easy modification of the proof of Lemma 3 in [7].

LEMMA 7. *Let $X_1, ..., X_N$ be Riemann surfaces, $g_j: X_j \to X_{j+1}$ (where $X_{N+1} = X_1$) mappings, $P_1^1, ..., P_k^1$ distinct points on X_1, and set $P_i^{j+1} = g_j(P_i^j)$, $i = 1, ..., k$, $j = 1, ..., N-1$. Assume that $g_N(P_i^N) = P_{\pi(i)}^1$ where π is a permutation of $(1, ..., k)$.*

Assume also that either

(I) *all g_j are conformal or,*

(II) *all g_j are formal Teichmüller mappings with the same dilatation, and the initial quadratic differential of g_j is the terminal quadratic differential of g_{j-1} (of g_N if $j = 1$).*

Let $\varepsilon > 0$ be a given number.

Then there are (for $j = 1, ..., N$, $i = 1, ..., k$) quasiconformal mappings g_j', isotopic to g_j and satisfying

$$K(g_j') < K(g_1) + \varepsilon, \tag{7.1}$$

$$g_j'(P_i^j) = g_j(P_i^j), \tag{7.2}$$

and, on each X_j, k *disjoint Jordan arcs* α_i^j *emanating from the points* P_i^j, *with* $g_j'(\alpha_i^j)=\alpha_i^{j+1}$, $g_N'(\alpha_i^N)=\alpha_{\pi(i)}^1$, *and local parameters* z_i^j *defined near and vanishing at* P_i^j, *such that, in terms of* z_i^j *the arc* α_i^j *is given by* $0 \leqslant \operatorname{Re} z_i^j \leqslant 1$, $\operatorname{Im} z_i^j = 0$, *and the mapping* $g_j' \mid \alpha_i^j$ *is given by* $\operatorname{Re} z_i^{j+1} = \operatorname{Re} z_i^j$, *or* $\operatorname{Re} z_i^1 = \operatorname{Re} z_i^N$ *if* $j=N$.

Proof. Without loss of generality we assume that π is a cyclic permutation, for otherwise we could have considered separately each cycle entering into π. Let $\gamma = g_N \circ g_{N-1} \circ \ldots \circ g_1$. We may assume that

$$\gamma(P_i^1)=P_{i+1}^1, \quad i=1, \ldots, k-1; \quad \gamma(P_k^1)=P_1^1 \tag{7.3}$$

so that $\gamma^k(P_i^1)=P_i^1$, $i=1, \ldots, k$.

Suppose first that (I) holds. Then $\gamma\colon X_1 \to X_1$ is conformal, so is γ^k, and hence there is an integer $a>0$ with $\gamma^{ak}=\mathrm{id}$. It follows that there is a local parameter ζ defined in a domain $\Delta \subset X_1$ containing the point P_1^1, such that $\gamma^k(\Delta)=\Delta$, the k domains $\Delta, \gamma(\Delta), \ldots, \gamma^{k-1}(\Delta)$ are disjoint, $\zeta(P_1^1)=0$, $\zeta(\Delta)$ contains the disc $|\zeta|<2$, and in Δ the mapping γ^k is represented by the rotation $\zeta \mapsto \theta\zeta$ where $\theta^a=1$.

Let $0<\delta<1$ and define a quasiconformal mapping $h\colon X_1 \to X_1$ such that h is the identity outside the disc $|\zeta|<1$, coincides with the rotation $\zeta \mapsto \theta^{-1}\zeta$ in the disc $|\zeta|<\delta$, and is affine in the variables $\log|\zeta|$ and $\arg \zeta$ in the annulus $\delta<|\zeta|<1$. If δ is chosen small enough,

$$K(h)<1+\varepsilon.$$

Also, $h \circ \gamma^k$ coincides with the identity along the arc $0<\operatorname{Re}\zeta<\delta$, $\operatorname{Im}\zeta=0$.

Now we define the mappings $g_1', \ldots, g_N'$ by

$$g_j'=g_j, \quad j=1, \ldots, N-1; \quad g_N'=h \circ g_N, \tag{7.4}$$

the parameter z_1^1 by

$$z_1^1=\zeta/\delta \tag{7.5}$$

and determine the arcs α_i^j and the parameters z_i^j, $j=1, \ldots, N$, $i=1, \ldots, k$, $(i,j) \neq (1,1)$, by the requirements of the lemma.

Assume next that (II) holds. We denote the common dilatation of the mappings g_j by A. Let Φ_j denote the initial quadratic differential of the mapping g_j. By hypothesis (II) and by (7.3) all Φ_j have at all points P_i^j the same order $r \geqslant -1$. Let Z_i^j denote a natural parameter belonging to Φ_j at P_i^j. We choose a domain $\Delta \subset X_1$ containing P_1^1 and so small that $\zeta=Z_1^1$ is defined in $\Delta \cup \gamma^k(\Delta)$, and all k domains $\gamma(\Delta), \gamma^2(\Delta), \ldots, \gamma^{k-1}(\Delta)$ and $\gamma^k(\Delta) \cup \Delta$ are disjoint. The mapping γ^k is easily seen to be a formal Teichmüller mapping with dilatation A^{Nk} and having Φ_1 as its initial and terminal quadratic differential.

Set $\zeta = Z_1^1$ and let the numbers δ and δ' be such that $0 < \delta < \delta'$ and the disc $|\zeta| \leqslant A^{Nk}\delta'$ is contained in Δ. Along the segment $0 \leqslant \operatorname{Re}\zeta \leqslant \delta'$, $\operatorname{Im}\zeta = 0$, the mapping γ^k coincides with the similarity $\zeta \mapsto \theta A^{Nk}\zeta$ where $\theta^{r+2} = 1$. We define a quasiconformal mapping $h: X_1 \to X_1$ such that h coincides with the identity outside the disc $|\zeta| < \delta'$, coincides with the similarity $\zeta \mapsto \theta^{-1}A^{-Nk}\zeta$ in the disc $|\zeta| < \delta$, and is affine in the variables $\log|\zeta|$ and $\arg\zeta$ in the annulus $\delta < |\zeta| < \delta'$. If the ratio δ/δ' is small enough, $K(h) < 1 + (\varepsilon/A)$, so that

$$K(h \circ g_N) < A + \varepsilon.$$

Also, $h \circ \gamma^k$ coincides with the identity along the arc $0 \leqslant \operatorname{Re}\zeta \leqslant \delta$, $\operatorname{Im}\zeta = 0$.

Next we define the mappings $g_1', \ldots, g_N'$ by (7.4), the parameter z_1^1 by (7.5) and proceed as before. Of course, we will have $z_i^j = c_i^j Z_i^j$ where the c_i^j are constants with $(c_i^j)^r > 0$.

Proof of Theorem 7. In view of Lemma 5 we may assume that f is completely reduced by $r \geqslant 1$ Jordan curves $C_1, \ldots, C_r$. The components of $S - \{C_1 \cup \ldots \cup C_r\}$ can be denoted by S_{kj}, $k = 1, \ldots, q$, $j = 1, \ldots, N_k$ in such a way that

$$f(S_{kj}) = S_{k,j+1}$$

where we agree, once and for all, that

$$S_{k,N_k+1} = S_{k1}.$$

We observe that the type (p_{kj}, m_{kj}) of S_{kj} satisfies $2p_{kj} - 2 + m_{kj} > 0$. Since $f^{N_k}(S_{k1}) = S_{k1}$ and $f^{N_k}|S_{k1}$ is irreducible, there is on S_{k1} an $(f^{N_k}|S_{k1})$-minimal conformal structure of the first kind σ_k and a mapping $F_k: S_{k1} \to S_{k1}$, isotopic to f^{N_k}, such that $\sigma_k \circ F_k \circ \sigma_k^{-1}$ is absolutely extremal. We set

$$A_k = K_{\sigma_k}(F_k)^{1/N_k} = K(\sigma_k \circ F_k \circ \sigma_k^{-1})^{1/N_k} \tag{7.6}$$

and assume, without loss of generality, that

$$A_1 \geqslant A_2 \geqslant \ldots \geqslant A_q. \tag{7.7}$$

Theorem 6 will follow from the following two assertions:

(A) *If σ is any conformal structure on S and f' is isotopic to f, then*

$$K_\sigma(f') > A_1. \tag{7.8}$$

(B) *For every $\varepsilon > 0$ there is a conformal structure σ on S and an f' isotopic to f, such that*

$$K_\sigma(f') < A_1 + \varepsilon. \tag{7.9}$$

(Note that f^* is parabolic if $A_1 = 1$, pseudohyperbolic if $A_1 > 1$.)

Proof of (A). Since f is not isotopic to a periodic mapping, (7.8) certainly holds if $A_1=1$. We assume now that $A_1>1$.

Without loss of generality we may assume that

$$\text{each } \sigma(C_j) \text{ is a geodesic on } \sigma(S). \tag{7.10}$$

Indeed, for each $j=1, ..., r$ there is a geodesic Jordan curve Γ_j on $\sigma(S)$ freely homotopic to $\sigma(C_j)$. The r Jordan curves $\hat{C}_j=\sigma^{-1}(\Gamma_j)$ are disjoint and each $\hat{C}_j$ is freely homotopic to C_j. There is a mapping h: $S\to S$, isotopic to the identity, with $h(C_j)=\hat{C}_j$, $j=1, ..., r$. The mapping $\hat{f}=h\circ f\circ h^{-1}$ is isotopic to f and is completely reduced by $\{\hat{C}_1, ..., \hat{C}_r\}$. Every component of $S-\{\hat{C}_1\cup ... \cup \hat{C}_r\}$ is of the form $\hat{S}_{kj}=h(S_{kj})$, and $\hat{f}(\hat{S}_{kj})=\hat{S}_{k,j+1}$. Since $\hat{f}^{N_k}|\hat{S}_{k1}=h\circ f^{N_k}\circ h^{-1}|\hat{S}_{k1}$, we conclude that we could have arrived to the same numbers (7.7) by starting with $\hat{f}$ and $\{\hat{C}_1, ..., \hat{C}_r\}$.

Now we assume (7.10) and represent the Riemann surface $\sigma(S)$ as U/G, G a torsion-free Fuchsian group. Let Δ be a component of the preimage of $\sigma(S_{11})$ under the canonical mapping π: $U\to U/G=\sigma(S)$ and let G_1 be the stabilizer of Δ in G. The mapping f^{N_1}: $S\to S$ can be lifted, via π, to a mapping W: $U\to U$ with $W(\Delta)=\Delta$. Clearly $WGW^{-1}=G$, $WG_1W^{-1}=G_1$.

Let f' be isotopic to f. Then $(f')^{N_1}$ is isotopic to f^{N_1} and hence can be lifted, via π, to a mapping W': $U\to U$ such that $W'\circ g\circ(W')^{-1}=W\circ g\circ W^{-1}$ for all $g\in G$. It follows that W' is the lift of a mapping F_1': $\Sigma\to\Sigma$ where $\Sigma=U/G_1$, that is, by Lemma 6, the Nielsen extension of $\Delta/G_1=\sigma(S_{11})$.

But W is also a lift of a mapping w: $\Sigma\to\Sigma$ with the property: $w|\sigma(S_{11})=\sigma\circ f^{N_1}\circ\sigma^{-1}|\sigma(S_{11})$. We conclude that F_1' can be written as $F_1'=\sigma'\circ w'\circ(\sigma')^{-1}$ where the mapping σ': $S_{11}\to\Sigma$ is a conformal structure on S_{11} and w' is isotopic to $f^{N_1}|S_{11}$. Since $A_1>0$, the mapping $f^{N_1}|S_{11}$ is not isotopic to a periodic one and neither is F_1'. Since the Riemann surface Σ has boundary curves, F_1' cannot be absolutely extremal (cf. Theorem 2) so that $K(F_1')>K(F_1)=A_1^{N_1}$. But $K(F_1')=K(W')=K((f')^{N_1})$, and the inequality $K((f')^{N_1})>A_1^{N_1}$ implies (7.8).

Proof of (B). Using the conformal structures $\sigma_1, ..., \sigma_q$ and the mappings $F_1, ..., F_q$ constructed above, we define conformal structures σ_{kj} on S_{kj} and mappings $\hat{f}_{kj}$: $S_{kj}\to S_{k,j+1}$ as follows.

In all cases $\sigma_{k1}=\sigma_k$.

If $N_k=1$, then $\hat{f}_{k1}=F_k$. Note that in this case $\hat{f}_{k1}$ is isotopic to $f|S_{k1}$, and $\sigma_{k1}\circ\hat{f}\circ\sigma_{k1}^{-1}$ is either conformal (if $A_k=1$) or a Teichmüller mapping of dilatation $A_k>1$ whose initial and terminal quadratic differentials coincide.

If $N_k > 1$, we set $\mathring{f}_{kj} = f \mid S_{kj}$ for $j = 1, ..., N_k - 1$ and

$$\mathring{f}_{N_k} = F_k \circ f^{1-N_k} \mid S_{k, N_k}. \tag{7.11}$$

Then $\mathring{f}_{kj}$ is isotopic to $f \mid S_{kj}$.

If $A_k = 1$, then we define the conformal structures $\sigma_{k2}, ..., \sigma_{k, N_k}$ by the requirement that the mappings $\sigma_{k, j+1} \circ \mathring{f}_{kj} \circ \sigma_{kj}^{-1}$ be conformal for $j = 1, ..., N_k - 1$. It follows that the mapping

$$\sigma_{k1} \circ \mathring{f}_{k, N_k} \circ \sigma_{k, N_k}^{-1} = (\sigma_{k1} \circ F_k \circ \sigma_{k1}^{-1}) \circ (\sigma_{k2} \circ \mathring{f}_{k1} \circ \sigma_{k1})^{-1} \circ \ldots \circ (\sigma_{k, N_k} \circ \mathring{f}_{k, N_k - 1} \circ \sigma_{k, N_k - 1}^{-1})^{-1} \tag{7.12}$$

is also conformal.

If $A_k > 1$, then F_k is a Teichmüller mapping of dilatation $A_k^{N_k}$ whose initial quadratic differential Φ_{k1}, defined on S_{k1}, is also its terminal differential. In this case we determine the conformal structures $\sigma_{k2}, ..., \sigma_{k, N_k}$ and quadratic differentials $\Phi_{k2}, ..., \Phi_{k, N_k}$ by requiring that each mapping $\sigma_{k, j+1} \circ \mathring{f}_{kj} \circ \sigma_{kj}^{-1}$ be a Teichmüller mapping of dilatation A_k, with initial quadratic differential Φ_{kj} and terminal quadratic differential $\Phi_{k, j+1}$, $j = 1, ..., N_k - 1$. A simple calculation, utilizing natural parameters, shows that the mapping (7.12) is a Teichmüller mapping of dilatation A_k, with initial and terminal quadratic differentials Φ_{k, N_k} and Φ_{k1}, respectively.

One verifies easily that all conformal structures σ_{kj} are of the first kind. We set $X_{kj} = \sigma_{kj}(S_{kj})$, $X_{k, N_k+1} = X_{k1}$. Let us call those punctures on X_{kj} which correspond to curves C_l *inner punctures*, and let us note that to each of the r curves C_l there belong exactly two inner punctures. We also set

$$g_{kj} = \sigma_{k, j+1} \circ \mathring{f}_{kj} \circ \sigma_{kj}^{-1}.$$

Applying to each of the q sequences of mappings $g_{k1}, ..., g_{k, N_k}$ Lemma 7 we obtain mappings g'_{kj} isotopic to g_{kj} and satisfying

$$K(g'_{kj}) < K(g_{k1}) + \varepsilon = A_k + \varepsilon \leqslant A_1 + \varepsilon \tag{7.13}$$

and for each inner puncture P, a Jordan arc α_P emanating from P and a local parameter $z_P = x_P + iy_P$ defined near and vanishing at P such that the following conditions hold.

The domain of definition of z_P contains α_P, and in terms of z_P the arc α_P is given by $0 \leqslant x_P \leqslant 1$, $y_P = 0$. If P and Q are two inner punctures, then α_P and α_Q intersect only if $P = Q$, and if $g_{kj}(P) = Q$, then $g'_{kj}(P) = Q$, $g'_{kj}(\alpha_P) = \alpha_Q$, and mapping $g'_{kj} \mid \alpha_P$ is given by $x_Q = x_P$.

We denote by X'_{kj} the complement in X_{kj} of the arcs α_P emanating from the inner punctures $P \in X_{kj}$, and we note that $g'_{kj}(X'_{kj}) = X'_{k,j+1}$, with $X'_{k,N_k+1} = X'_{k1}$. The two banks of each α_P, $P \in X_{kj}$, form an ideal boundary curve of X'_{kj}.

Now we construct a Riemann surface X_0 by identifying, for each curve C_l on S, the upper bank of one of the two arcs α_P belonging to C_l with the lower bank of the other such arc α_Q, and vice versa, taking care that a point with $x_P = t$ is identified with a point with $x_Q = t$, $0 \leqslant t \leqslant 1$.

There is a mapping g_0: $X_0 \to X_0$ such that $g_0 | X'_{kj} = g'_{kj}$ for all k, j. The $2r$ segments α_P give rise to r Jordan curves $\Delta_1, \ldots, \Delta_r$ on X_0 such that $g_0(\Delta_j) = \Delta_{\pi(j)}$, π being a permutation of $(1, \ldots, r)$. Furthermore, for every Δ_j, there is a local parameter ζ_j defined in a domain containing Δ_j, such that $\zeta_j(\Delta_j)$ is the unit circle, and the mapping $g_0 | \Delta_j$ is given by $\zeta_j = \zeta_{\pi(j)} = e^{i\theta}$, $0 \leqslant \theta \leqslant 2\pi$. Clearly,

$$K(g_0) < A_1 + \varepsilon, \tag{7.14}$$

in view of (7.13).

Let $R > 0$, and let Z denote the product of the unit circle $\zeta = e^{i\theta}$, θ real, with the segment $-R < t < R$. This is a Riemann surface of type $(0, 2)$ with two ideal boundary curves, on which $t + i\theta$ is a (multiple-valued) holomorphic function. Let X_R denote that Riemann surface obtained by cutting X_0 along each curve Δ_j and inserting into each cut a copy Z_l of Z; the ideal boundary curves of Z_l are attached to the banks of each cut by identifying the points $\zeta_l = e^{i\theta}$ with the points $(\pm R, \theta)$. There is a mapping g_R: $X_R \to X_R$ such that $g_R | X'_{kj} = g'_{kj}$, $g_R(Z_j) = Z_{\pi(j)}$, and $g_R | Z_j$ respects the coordinates (t, θ). Clearly, g_R being conformal on each Z_j,

$$K(g_R) < A_1 + \varepsilon. \tag{7.15}$$

Let Γ_j denote the Jordan curve on Z_j given by $t = 0$, $0 \leqslant \theta \leqslant 2\pi$. By a Dehn twist δ_j about Γ_j we shall mean the mapping δ_j: $X_R \to X_R$ such that $\delta_j | X_R - Z_j = \mathrm{id}$, and on Z_j the mapping δ_j is given by

$$(t, \theta) \mapsto \begin{cases} (t, \theta) & \text{for } -R < t \leqslant 0 \\ (t, \theta + 2\pi t/R) & \text{for } 0 \leqslant t < R. \end{cases}$$

Note that $K(\delta_j)$ depends only on R and

$$\lim_{R \to +\infty} K(\delta_j) = 1. \tag{7.16}$$

Now our construction shows that there is a mapping σ: $S \to X_R$ with $\sigma(C_l) = \Gamma_l$, $l = 1, \ldots, r$, and such that for the mapping

$$f'_0 = \sigma^{-1} \circ g_R \circ \sigma$$

we have that $f_0'|S_{kj}$ is isotopic to $f|S_{kj}$ for $k=1, \ldots, q$, $j=1, \ldots, N_k$. A simple topological argument shows that there is a product of Dehn twists

$$\delta=\delta_1^{n_1}\circ\delta_2^{n_2}\circ\ldots\circ\delta_r^{n_r}$$

such that

$$f'=\sigma^{-1}\circ\delta\circ g_R\circ\sigma$$

is isotopic to f. Now, by (7.15) and (7.16),

$$K_\sigma(f')=K(\delta\circ g)\leqslant K(\delta)\,K(g_R)<K(\delta_1)^{n_1+\ldots+n_r}K(g_R)<A_1+\varepsilon$$

if R is large enough. Thus (7.8) holds.

Remark. The collection of conformal structures σ_{kj} and mappings $\hat{f}_{kj}$, $k=1, \ldots, q$, $j=1, \ldots, N_k$, constructed during the proof, is the generalized solution of our extremal problem alluded to in § 1. This will become clearer from a reinterpretation of our results in the next section.

§ 8. Restatement of the results

It is instructive to restate the results obtained above using the concept of a Riemann surface with nodes.

A *Riemann surface with nodes* X is a connected one-dimensional complex space such that every point $P\in X$ has a fundamental system of neighborhoods each of which is isomorphic either to the disc $|z|<1$ in $\mathbf{C}$ or to the set $z_1z_2=0$, $|z_1|<1$, $|z_2|<1$ in $\mathbf{C}^2$; in the second case P is called a *node*. Let N be the set of nodes of X. Every component of $X-N$ is called a *part* of X; it is an ordinary (non-singular) Riemann surface. Note that we do not require N to be non-empty. Hence a non-singular Riemann surface is a special case of a Riemann surface with nodes.

We assume that all Riemann surfaces with nodes X occurring below are of *finite type* and *stable*. This means that X has finitely many parts, and each part X_0 is of some finite type (p_0, m_0) with $2p_0-2+m_0>0$. We call X *of the first kind* if no part of X has ideal boundary curves.

Let X and Y be Riemann surfaces with nodes. A topological bijection f: $X\to Y$ will be called *orientation preserving* if the restriction of f to every part of X is. The *dilatation* $K(f)$ of such a map is defined as the largest of the numbers $K(f|X_0)$, where X_0 runs over all parts of X.

An orientation preserving topological bijection f: $X\to X$ will be called *absolutely extremal* if, for every mapping f': $X\to X$ isotopic to f, for every orientation preserving topological bijection σ: $X \to Y$ onto another Riemann surface with nodes, and for every part X_0 of X, the following condition is satisfied. Let n be smallest positive integer with $f^n(X_0) = X_0$, and set $f^n(X_0) = X_j$, $j = 1, ..., n-1$. Then

$$\max_{0 \leqslant j \leqslant n-1} (K(f|X_j) \leqslant \max_{0 \leqslant j \leqslant n-1} K(\sigma \circ f' \circ \sigma^{-1}|\sigma(X_j)). \tag{8.1}$$

This condition implies, but is not implied by, the inequality $K(f) \leqslant K(\sigma \circ f' \circ \sigma^{-1})$. If X is nonsingular, however, the present definition coincides with the one given in § 1.

THEOREM 8. *Let X be a Riemann surface with nodes. A topological orientation preserving bijection f of X onto itself is absolutely extremal if and only if, for every part X_0 of X, either $f|X_0$ and $f|f(X_0)$ are conformal or $f|X_0$ and $f|f(X_0)$ are Teichmüller mappings with the same dilatation and the terminal quadratic differential of $f|X_0$ coincides with the initial quadratic differential of $f|f(X_0)$.*

This follows from Theorem 6 and from the proof of Theorem 7.

THEOREM 9. *Given a self-mapping f of a surface S, there exists a mapping f' isotopic to f, a continuous surjection σ of S onto a Riemann surface $\sigma(S)$ with nodes and of the first kind and an absolutely extremal self-mapping g of $\sigma(S)$ such that*

(i) *the inverse image under σ of the set of nodes of $\sigma(S)$ is a set of disjoint Jordan curves, and the restriction of σ to the complement of these curves is an orientation preserving homeomorphism,*

(ii) *we have that*

$$\sigma \circ f' = g \circ \sigma,$$

and

(iii) *if f'' is any map isotopic to f, σ_1 any continuous surjection of S onto a Riemann surface with nodes, having property* (i), *and such that*

$$\sigma_1 \circ f'' = g_1 \circ \sigma_1$$

for some orientation preserving topological self-mapping g_1 of $\sigma_1(S)$, then

$$K(g) \leqslant K(g_1).$$

Furthermore, $\sigma(S)$ may be chosen as nonsingular if f is isotopic to a periodic mapping, must be so chosen if f is irreducible, and must have at least one node if f is reducible and not isotopic to a periodic mapping.

The proof follows from Theorem 1, Theorem 4, Theorem 6 and the statement and proof of Theorem 7. In the case where f is reducible the map σ is obtained by combining the structures σ_{kj} used in that proof. The details are left to the reader.

§ 9. Thurston's theorem

In order to state the theorem one needs the concept of a *pseudo-Anosov diffeomorphism.* For our purposes the following definition is convenient. A pseudo-Anosov diffeomorphism of a surface S of finite type is a mapping which can be written as $\sigma^{-1} \circ g \circ \sigma$ where σ is a mapping of S onto a Riemann surface X, without boundary curves, and $g: X \to X$ is a Teichmüller mapping whose initial and terminal quadratic differentials coincide.

The equivalence of this definition with that by Thurston follows from the following observations.

Let X be a compact Riemann surface of genus p and $P_1, ..., P_m$ distinct points on X. Let Φ be an integrable meromorphic quadratic differential on X, whose only singularities are at $P_1, ..., P_m$; these singularities can be only simple poles. Then the horizontal and vertical trajectories of Φ, that is, the curves along which $\Phi > 0$ or $\Phi < 0$, are the leaves of two transversal foliations on $X - \{P_1, ..., P_m\}$, with only n-pronged singularities, $n > 2$. These foliations are made into *measured foliations* by using the metric $ds = |\Phi|^{1/2}$ to measure the distance between leaves. A Teichmüller mapping $g: X \to X$ takes the two foliations into themselves, but it multiplies the distance between horizontal trajectories by $K^{1/2}$ and that between vertical ones by $K^{-1/2}$, where $K = K(g)$ is the dilatation.

Conversely, if we are given two transversal measured foliations on a surface S, of type (p, m), satisfying certain conditions at the boundary continua, there is a mapping $\sigma: S \to X$ onto a Riemann surface X of genus p, compact but for m punctures, which takes the leaves of the foliations into the horizontal and vertical trajectories of an integrable holomorphic quadratic differential.

THEOREM 10 (Thurston [21]). *A self-mapping of a surface which is not isotopic to a periodic one is either isotopic to a pseudo-Anosov diffeomorphism or is reducible, but not both.*

Proof. Let f be the map in question and f^* the induced modular transformation. Then f^* is not elliptic, since it must be of infinite order, hence it is either hyperbolic (if and only if f is irreducible, see Theorem 4 and 7) or not. But Proposition 2 and Theorem 6 show that f is isotopic to a pseudo-Anosov diffeomorphism if and only if f^* is hyperbolic.

References

[1]. AHLFORS, L. V., On quasiconformal mappings. *J. Analyse Math.*, 3 (1953–54), 1–58.
[2]. —— *Lectures on Quasiconformal Mappings.* Van Nostrand, Princeton (1966).

[3]. Bers, L., Quasiconformal mappings and Teichmüller's theorem. *Analytic Functions.* Princeton University Press, Princeton (1960), 89–119.
[4]. —— Fiber spaces over Teichmüller spaces. *Acta Math.*, 130 (1973), 89–126.
[5]. —— A remark on Mumford's compactness theorem. *Israel J. Math.*, 12 (1972), 400–407.
[6]. —— Uniformization, moduli and Kleinian groups. *Bull. London Math. Soc.*, 4 (1972), 257–300.
[7]. —— Nielsen extensions of Riemann surfaces. *Ann. Acad. Sci. Fenn.*, 2 (1976), 17–22.
[8]. Busemann, H., *Metric Methods in Finsler Geometry and in the Foundations of Geometry.* Ann. of Math. Studies, 8. Princeton Univ. Press, Princeton (1942).
[9]. Earle, C. J., Teichmüller theory. *Discrete Groups and Automorphic Functions* (edited by W. J. Harvey). Academic Press, London–New York–San Francisco (1977), 143–162.
[10]. —— Teichmüller distance is differentiable. *Duke Math. J.*, 44 (1977), 389–398.
[11]. Epstein, D. B., Curves on two-manifolds and isotopies. *Acta Math.*, 115 (1965), 83–107.
[12]. Fenchel, W., Estensioni di gruppi e trasformazioni periodiche delle superficie. *Rend. Lincei*, 5 (1948), 326–329.
[13]. Kravetz, S., On the geometry of Teichmüller spaces and the structures of their modular groups. *Ann. Acad. Sci. Fenn.*, 278 (1959), 1–35.
[14]. Linch, M. R., On metrics in Teichmüller space. Ph.D. Thesis, Columbia University (1971).
[15]. Mangler, W., Die Klassen topologischer Abbildungen einer geschlossenen Fläche auf sich. *Math. Z.*, 44 (1939), 541–554.
[16]. Matelski, J. P., A compactness theorem for Fuchsian groups of the second kind. *Duke Math. J.*, 43 (1976), 829–840.
[17]. Mori, A., On an absolute constant in the theory of quasi-conformal mappings. *J. Math. Soc. Japan*, 8 (1956), 156–166.
[18]. Mumford, D., A remark on Mahler's compactness theorem. *Proc. Amer. Math. Soc.*, 28 (1971), 289–294.
[19]. Teichmüller, O., Extremale quasikonforme Abbildungen und quadratische Differentiale. *Abh. Preuss. Akad. Wiss.*, 22 (1939).
[20]. —— Bestimmung der extremalen quasikonformen Abbildungen bei geschlossenen orientierten Riemannschen Flächen. Ibid., 4 (1943).
[21]. Thurston, W. P., On the geometry and dynamics of diffeomorphisms of surfaces, I. To appear.
[22]. Wolpert, S., The length spectrum of a compact Riemann surface, I. To appear.

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