THE FUNDAMENTAL SERIES OF REPRESENTATIONS OF A REAL SEMISIMPLE LIE ALGEBRA

BY

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1. Introduction

Let $\mathfrak{g}_0$ be a semisimple Lie algebra over the real numbers. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be a Cartan decomposition. Suppose that $\mathfrak{k}_0$ contains a Cartan subalgebra of $\mathfrak{g}_0$. In [7], [18] a family of representations $W_{P,\Lambda}$ of $\mathfrak{g}_0$ was constructed. These representations were shown to be intimately related with Harish–Chandra's discrete series [10] for connected Lie groups with Lie algebra $\mathfrak{g}_0$ in [16], [18] (see also [4]). In this paper we extend the construction of the $W_{P,\Lambda}$ to the case when $\mathfrak{k}_0$ does not necessarily contain a Cartan subalgebra of $\mathfrak{g}_0$. The $W_{P,\Lambda}$ constitute the fundamental series of the title of this paper. We then begin an analysis of these representations and give two different characterizations of them. In [5] a criterion for the irreducibility of these representations will be given. Using the irreducibility criterion and a certain exact sequence (generalizing the Bernstein, Gelfand, Gelfand resolution of a finite dimensional representation), a new proof of Blattner's conjecture for the discrete series will be given in [6]. In [8] we will show that the representations $W_{P,\Lambda}$ are equivalent to the analytic continuation of the fundamental series of representations as defined in [11]. Using the irreducibility criterion of [5] a new proof of the irreducibility of the fundamental series (in the sense of Harish–Chandra) will be given. In [8] we will also lay the groundwork for the determination of the composition series for the analytic continuation of the fundamental series (cf. also [17]).

We now give a more detailed description of the contents of this paper. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote respectively the complexifications of $\mathfrak{g}_0$ and $\mathfrak{k}_0$. In sections 2 and 3 the results of

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[7] and [18] are extended to the case where $\mathfrak{g}$ and $\mathfrak{k}$ do not necessarily have the same rank. Actually, just as in [18], only certain properties of the pair $(\mathfrak{g}, \mathfrak{k})$ are necessary for the construction of sections 2 and 3 (these are given in section 1).

The main result in sections 2 and 3 is given in Theorems 3.2 and 3.3 and is described as follows: Let $\mathfrak{h}_1$ (resp. $\mathfrak{h}$) be a Cartan subalgebra of $\mathfrak{k}$ (resp. $\mathfrak{g}$). We may assume that $\mathfrak{h}_1 \subseteq \mathfrak{h}$. Let $\Delta_{\mathfrak{k}}$ (resp. Δ) denote the roots of $(\mathfrak{k}, \mathfrak{h}_1)$ (resp. $(\mathfrak{g}, \mathfrak{h})$) and fix an admissible positive system P of roots for Δ (see § 2 for the definition of admissible). Let $P_{\mathfrak{k}}$ be the corresponding positive system of roots for $\Delta_{\mathfrak{k}}$ obtained from P and let t_0 denote the unique element of the Weyl group of $\Delta_{\mathfrak{k}}$ such that $t_0 \cdot P_{\mathfrak{k}} = -P_{\mathfrak{k}}$. For each dominant integral μ in $\mathfrak{h}_1^*$ (the dual of $\mathfrak{h}_1$), let V^{μ} denote the irreducible $\mathfrak{k}$-module with highest weight μ. We can now define the fundamental series of representations. Assume that $\Lambda \in \mathfrak{h}^*$ (dual of $\mathfrak{h}$) is such that $\lambda = t_0(\Lambda|_{\mathfrak{h}_1} + \delta_{\mathfrak{k}}) - \delta_{\mathfrak{k}}$ is $P_{\mathfrak{k}}$-dominant integral. Then there exists an admissible $\mathfrak{g}$-module $W_{P,\Lambda}$ with the following properties:

(i) $\dim \operatorname{Hom}_{\mathfrak{k}}(V^{\lambda}, W_{P,\Lambda}) = 1$

(ii) If $\mu \in \mathfrak{h}_1^*$ is $P_{\mathfrak{k}}$-dominant integral and if $\operatorname{Hom}_{\mathfrak{k}}(V^{\mu}, W_{P,\Lambda}) \neq \{0\}$, then $\mu = \lambda + Q|_{\mathfrak{h}_1}$ where Q is a sum of elements of P.

(iii) The $\mathfrak{k}$-component of $W_{P,\Lambda}$ corresponding to V^{λ} is $\mathfrak{g}$-cyclic for $W_{P,\Lambda}$

(iv) $\mathcal{G}^{\mathfrak{k}}$, the centralizer of $\mathfrak{k}$ in the universal enveloping algebra of $\mathfrak{g}$, acts on the V^{λ}-component by the scalar action: $x \mapsto \eta_{P,\Lambda}(x)\,\mathrm{Id}$ for $x \in \mathcal{G}^{\mathfrak{k}}$, where $\eta_{P,\Lambda}$ is a homomorphism of $\mathcal{G}^{\mathfrak{k}}$ into the complex numbers (see Theorem 3.2 for the definition of $\eta_{P,\Lambda}$). The modules $W_{P,\Lambda}$ all contain unique maximal proper submodules $M_{P,\Lambda}$ and we set $D_{P,\Lambda}$ equal to $W_{P,\Lambda}/M_{P,\Lambda}$. The modules $W_{P,\Lambda}$ are called the fundamental series of representations. The dependence is on two parameters, the admissible positive system P and the linear functional Λ in $\mathfrak{h}^*$ subject to the condition that $\lambda = t_0(\Lambda|_{\mathfrak{h}_1} + \delta_{\mathfrak{k}}) - \delta_{\mathfrak{k}}$ is $P_{\mathfrak{k}}$-dominant integral. The first characterization theorem is actually a theorem about the unique maximal quotients $D_{P,\Lambda}$. This result (Theorem 3.3) asserts that within the set of equivalence classes of admissible irreducible $\mathfrak{g}$-modules the equivalence class of $D_{P,\Lambda}$ is uniquely determined by the condition:

If M is an element of the equivalence class, then there exists a nonzero element A in $\operatorname{Hom}_{\mathfrak{k}}(V^{\lambda}, M)$ such that $\mathcal{G}$ acts on $A(V^{\lambda})$ by the formula

$$x|_{A(V^{\lambda})} \equiv \eta_{P,\Lambda}(x)\,\mathrm{Id}, \quad x \in \mathcal{G}^{\mathfrak{k}} \text{ and } \eta_{P,\Lambda} \text{ as defined in Theorem 3.2.}$$

In section 4 a resolution of $\mathfrak{g}$-Verma modules is given in terms of certain modules induced up from $\mathfrak{k}$ to $\mathfrak{g}$. This resolution is then used in sections 5 and 6 to prove two results. The first asserts that most of the modules $W_{P,\Lambda}$ possess a certain universal mapping prop erty. In order to be precise, more notation is necessary. Let $\mathfrak{p}$ be the orthogonal complement to $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form of $\mathfrak{g}$, and let P be an admissible positive system for Δ which gives $P_{\mathfrak{k}}$. $\mathfrak{h}_1$ acts semisimply on $\mathfrak{p}$ and thus for any $\mu \in \mathfrak{h}_1^*$ let $\mathfrak{p}[\mu]$ denote the μ weight subspace of $\mathfrak{p}$. Set $\mathfrak{p}^+ = \sum_{\beta \in P} \mathfrak{p}[\beta|_{\mathfrak{h}_1}]$. Now if $\lambda \in \mathfrak{h}_1^*$, then λ is said to be strongly $P_{\mathfrak{k}}$-dominant integral relative to P, if λ and $\lambda - \mu$ are $P_{\mathfrak{k}}$-dominant integral for all weights μ of $\wedge \mathfrak{p}^+$ ($\wedge$ denotes the exterior algebra). Note that this condition is a type of "sufficiently" regular condition with respect to the compact roots and also that it depends on the choice of P. Let M be an admissible $\mathfrak{g}$-module, and let $\lambda \in \mathfrak{h}_1^*$ be $P_{\mathfrak{k}}$-dominant integral, then M is said to have V^λ as a weak minimal $\mathfrak{k}$-type relative to P if

(i) There is an element A in $\mathrm{Hom}_{\mathfrak{k}}(V^\lambda, M)$ such that $M = \mathcal{G} \cdot A(V^\lambda)$.

(ii) If $\beta \in P$ and $\mu = \beta|_{\mathfrak{h}_1}$, then $\mathrm{Hom}_{\mathfrak{k}}(V^{\lambda-\mu}, \mathfrak{p} \cdot A(V^\lambda))\{0\}$ (here $\mathfrak{p} \cdot A(V^\lambda)$ denotes the $\mathfrak{k}$-submodule of M spanned by elements of the form $x \cdot a$, $x \in \mathfrak{p}$ and $a \in A(V^\lambda)$).

(iii) $\dim \mathrm{Hom}_{\mathfrak{k}}(V^\lambda, A(V^\lambda) + \mathfrak{p} \cdot A(V^\lambda)) = 1$.

The universal mapping property (Theorem 6.2) asserts that if M is an admissible $\mathfrak{g}$-module with weak minimal $\mathfrak{k}$-type V^λ relative to P and if λ is strongly $P_{\mathfrak{k}}$-dominant integral relative to P then there exists $\Lambda \in \mathfrak{h}^*$ such that $\lambda = t_0(\Lambda|_{\mathfrak{h}_1} + \delta_{\mathfrak{k}}) - \delta_{\mathfrak{k}}$ and a surjective $\mathfrak{g}$-module homomorphism from $W_{P,\Lambda}$ onto M.

The second result in section 6 (Theorem 6.3) gives a classification of the modules $D_{P,\Lambda}$ in terms of a somewhat different minimal $\mathfrak{k}$-type criterion. Let P be an admissible positive system giving $P_{\mathfrak{k}}$ and let $\lambda \in \mathfrak{h}_1^*$ be strongly $P_{\mathfrak{k}}$-dominant integral relative to P. Assume that M is an admissible irreducible $\mathfrak{g}$-module such that $\mathrm{Hom}_{\mathfrak{k}}(V^\lambda, M) \neq \{0\}$ and $\mathrm{Hom}_{\mathfrak{k}}(V^{\lambda-\mu}, M) = 0$ where $\mu = \beta|_{\mathfrak{h}_1}$ for all $\beta \in P$ with $\mathfrak{p}[\mu] \neq \{0\}$. Then there exists $\Lambda \in \mathfrak{h}^*$ such that $t_0(\Lambda|_{\mathfrak{h}_1} + \delta_{\mathfrak{k}}) - \delta_{\mathfrak{k}} = \lambda$ and $D_{P,\Lambda}$ is isomorphic to M. An obvious corollary worth stating is the following multiplicity-one theorem: Assume that λ is strongly $P_{\mathfrak{k}}$-dominant integral relative to the admissible positive system P. If M is an admissible irreducible $\mathfrak{g}$-module such that $\mathrm{Hom}_{\mathfrak{k}}(V^\lambda, M) \neq \{0\}$ and $\mathrm{Hom}_{\mathfrak{k}}(V^{\lambda-\mu}, M) = \{0\}$ where $\mu = \beta|_{\mathfrak{h}_1}$ for all $\beta \in P$ with $\mathfrak{p}[\mu] \neq \{0\}$, then $\dim \mathrm{Hom}_{\mathfrak{k}}(V^\lambda, M) = 1$ (i.e. V^λ occurs with multiplicity one in M).

Although the intent of this paper is to construct and study the properties of certain admissible $\mathfrak{g}$-modules, one interesting result about $\mathfrak{g}$-Verma modules naturally emerges. For $\alpha \in P_{\mathfrak{k}}$ let $\mathfrak{k}_\alpha$ denote the α weight space in $\mathfrak{k}$ relative to $\mathfrak{h}_1$ and set $\mathfrak{n}_{\mathfrak{k}} = \sum_{\alpha \in P_{\mathfrak{k}}} \mathfrak{k}_\alpha$. If L is any $\mathfrak{n}_{\mathfrak{k}}$-module set $L^{\mathfrak{n}_{\mathfrak{k}}}$ equal to the submodule of L of elements u such that $\mathfrak{n}_{\mathfrak{k}} \cdot u = \{0\}$. $L^{\mathfrak{n}_{\mathfrak{k}}}$ is called the subspace of $\mathfrak{n}_{\mathfrak{k}}$-invariants. Assume as above that P is an admissible positive system of roots for Δ which gives $P_{\mathfrak{k}}$. Set $\mathfrak{p}^- = \sum_{\alpha \in P} \mathfrak{p}[-\alpha|_{\mathfrak{h}_1}]$ and let $S(\mathfrak{p}^-)$ denote the symmetric tensor algebra of $\mathfrak{p}^-$. For any $\Lambda \in \mathfrak{h}^*$ let $V_{\mathfrak{g},P,\Lambda}$ denote the $\mathfrak{g}$-Verma module with P-highest weight Λ. In section 5 (Theorem 5.15), we obtain the following result:

Assume that $\lambda = t_0(\Lambda|_{\mathfrak{h}_1} + \delta_{\mathfrak{k}}) - \delta_{\mathfrak{k}}$ is strongly $P_{\mathfrak{k}}$-dominant integral, that $\mu \in \mathfrak{h}_1^*$ is $P_{\mathfrak{k}}$-dominant integral and that r is an element of the Weyl group of $\Delta_{\mathfrak{k}}$, then

$$\dim (V_{\mathfrak{g}, P, \Lambda})^{\mathfrak{n}_{\mathfrak{k}}}[r(\mu+\delta_{\mathfrak{k}}) - \delta_{\mathfrak{k}}] = \dim S(\mathfrak{p}^-)[r(\mu+\delta_{\mathfrak{k}}) - (\Lambda|_{\mathfrak{h}_1} + \delta_{\mathfrak{k}})].$$

2. Structural preliminaries

In this section, we introduce a class of pairs $(\mathfrak{g}, \mathfrak{k})$, where $\mathfrak{g}$ is a semi-simple Lie algebra over $\mathbf{C}$ (the complex numbers) and $\mathfrak{k} \subset \mathfrak{g}$ a reductive subalgebra, which will be studied throughout this paper. The pairs $(\mathfrak{g}, \mathfrak{k})$ will include the pairs of the introduction (see Lemma 2.2).

Definition 2.1. Let $\mathfrak{g}$ be a semi-simple Lie algebra over $\mathbf{C}$. Let $\mathfrak{k} \subset \mathfrak{g}$ be a reductive subalgebra of $\mathfrak{g}$. Then $\mathfrak{k}$ is said to be regular in $\mathfrak{g}$ ($(\mathfrak{g}, \mathfrak{k})$ is said to be a regular pair) if the following two conditions are satisfied:

(a) Let $\mathfrak{h}_1 \subset \mathfrak{k}$ be a Cartan subalgebra of $\mathfrak{k}$. Then $C_{\mathfrak{g}}(\mathfrak{h}_1) = \{X \in \mathfrak{g} \mid [X, \mathfrak{h}_1] = 0\} = \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$.

(b) Let $\mathfrak{h}_1$, $\mathfrak{h}$ be as in (a) and let Δ be the root system of $(\mathfrak{g}, \mathfrak{h})$. If $\mathfrak{h}_{\mathbf{R}} = \{H \in \mathfrak{h} \mid \alpha(H) \in \mathbf{R}$ for all $\alpha \in \Delta\}$, then $\mathfrak{h}_{\mathbf{R}} \cap \mathfrak{h}_1 = (\mathfrak{h}_1)_{\mathbf{R}}$ is a real form of $\mathfrak{h}_1$.

LEMMA 2.2. *Let $\mathfrak{g}_0$ be a semi-simple Lie algebra over $\mathbf{R}$. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be a Cartan decomposition of $\mathfrak{g}_0$ with Cartan involution θ. Let $\mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$ be a parabolic subalgebra of $\mathfrak{g}_0$ with Langlands decomposition as indicated (in particular, $\mathfrak{a}_0 \subset \mathfrak{p}_0$, $\theta\mathfrak{m}_0 = \mathfrak{m}_0$ and $\mathfrak{n}_0$ is the unipotent radical). Set $\mathfrak{u}_0 = (\mathfrak{m}_0 \cap \mathfrak{k}_0) \oplus \mathfrak{a}_0$. If $\mathfrak{g}$ is the complexification of $\mathfrak{g}_0$ and if $\mathfrak{u}$ is the complexification of $\mathfrak{u}_0$ then $(\mathfrak{g}, \mathfrak{u})$ is a regular pair.*

Notes. 1. $\mathfrak{g}_0 = \mathfrak{m}_0$, $\mathfrak{a}_0 = (0)$, $\mathfrak{n}_0 = (0)$ is a parabolic subalgebra of $\mathfrak{g}_0$.

2. If $\mathfrak{u}_1 \subset \mathfrak{u}_2 \subset \mathfrak{g}$, $\mathfrak{g}$ a semi-simple Lie algebra over $\mathbf{C}$ and $\mathfrak{u}_j$, $j = 1, 2$ reductive subalgebras and if $(\mathfrak{g}, \mathfrak{u}_1)$ is a regular pair, then if rank $\mathfrak{u}_1 =$ rank $\mathfrak{u}_2$, it is clear that $(\mathfrak{g}, \mathfrak{u}_2)$ is a regular pair.

Proof. Let $\mathfrak{h}_*$ be a Cartan subalgebra of $\mathfrak{m}_0$ so that $\mathfrak{h}_* \cap \mathfrak{k}_0$ is maximal abelian in $\mathfrak{m}_0 \cap \mathfrak{k}_0$ and $\theta\mathfrak{h}_* = \mathfrak{h}_*$. Let $\mathfrak{h}_*^- = \mathfrak{h}_* \cap \mathfrak{k}_0$, $\mathfrak{h}_*^+ = \mathfrak{h}_* \cap \mathfrak{p}_0$. Now, $\mathfrak{h}_0 = \mathfrak{h}_* + \mathfrak{a}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$ and $\mathfrak{h}_0 = \mathfrak{h}_*^- \oplus \mathfrak{h}_*^+ \oplus \mathfrak{a}_0$. Also, if $\mathfrak{h}_1$ is the complexification of $\mathfrak{h}_*^- \oplus \mathfrak{a}_0$, then $\mathfrak{h}_1$ is a Cartan subalgebra of $\mathfrak{u}$. Let $\mathfrak{h}$ be the complexification of $\mathfrak{h}_0$. Let Δ be the root system of $(\mathfrak{g}, \mathfrak{h})$. Clearly $\mathfrak{h}_{\mathbf{R}} \cap \mathfrak{h}_1 = i\mathfrak{h}_*^- \oplus \mathfrak{a}_0$. Since $i\mathfrak{h}_*^- \oplus \mathfrak{a}_0$ is clearly a real form of $\mathfrak{h}_1$, it is enough to show that $C_{\mathfrak{g}}(\mathfrak{h}_1) = \mathfrak{h}$. For $\alpha \in \Delta$, let $\mathfrak{g}_\alpha \subset \mathfrak{g}$ be the root space. If $X \in \mathfrak{g}$, then $X = h + \sum_{\alpha \in \Delta} X_\alpha$, $X_\alpha \in \mathfrak{g}_\alpha$, $h \in \mathfrak{h}$. If X is also in $C_{\mathfrak{g}}(\mathfrak{h}_1)$ then if $X_\alpha \neq 0$, $\alpha(\mathfrak{h}_1) = 0$. Thus it is enough to show that if $\alpha \in \Delta$, then $\alpha|_{\mathfrak{h}_1} \neq 0$.

If $\alpha|_{\mathfrak{h}_1}=0$, then $\mathfrak{g}_\alpha \subset \mathfrak{m}$ ($\mathfrak{m}$ is the complexification of $\mathfrak{m}_0$). Since $\alpha(\mathfrak{h})\neq 0$, we see that $\alpha(\mathfrak{h}_*^+)\neq 0$. Hence $\mathfrak{g}_\alpha \subset [\mathfrak{m}, \mathfrak{m}]$. Since $\alpha(\mathfrak{h}_*^-)=0$, we may choose $Y \in \mathfrak{g}_\alpha \cap \mathfrak{m}_0$, $Y\neq 0$. Now $Y+\theta Y \in \mathfrak{m}_0 \cap \mathfrak{k}$ and $[\mathfrak{h}_*^-, Y+\theta Y]=0$ (this is because $\theta Y \in \mathfrak{g}_{-\alpha}$). But then $Y+\theta Y\neq 0$ and $\mathfrak{h}_*^- + \mathbf{R}(Y+\theta Y)$ is an abelian subalgebra contained in $\mathfrak{m}_0 \cap \mathfrak{k}_0$. Since $\mathfrak{h}_*^-$ is maximal abelian in $\mathfrak{m}_0 \cap \mathfrak{k}_0$, we have a contradiction. This completes the proof of the lemma.

For the remainder of this section, we fix $\mathfrak{g}$ a complex semi-simple Lie algebra and $\mathfrak{k}\subset\mathfrak{g}$ a *reductive* Lie subalgebra of $\mathfrak{g}$ so that $(\mathfrak{g}, \mathfrak{k})$ is a regular pair. Fix $\mathfrak{h}_1\subset\mathfrak{k}$ a Cartan subalgebra of $\mathfrak{k}$ and set $\mathfrak{h}=C_{\mathfrak{g}}(\mathfrak{h}_1)$, a Cartan subalgebra of $\mathfrak{g}$ (see Definition 2.1). Let $\mathfrak{h}_{\mathbf{R}}$ and $(\mathfrak{h}_1)_{\mathbf{R}}$ be as in Definition 2.1.

Let Δ be the root system of $(\mathfrak{g}, \mathfrak{h})$ and let $\Delta_{\mathfrak{k}}$ be the root system of $(\mathfrak{k}, \mathfrak{h}_1)$. Set $W=W(\Delta)$ (resp. $W_{\mathfrak{k}}=W(\Delta_{\mathfrak{k}})$) the Weyl group of Δ (resp. $\Delta_{\mathfrak{k}}$).

Definition 2.3. A system of positive roots, P, for Δ is said to be admissible if $(\mathfrak{h}_1)^+_{\mathbf{R}} = \{h\in(\mathfrak{h}_1)_{\mathbf{R}} \mid \beta(h)>0 \text{ for all } \beta\in P\}\neq\emptyset$.

LEMMA 2.4. (i) *There exists an admissible system of positive roots for Δ.*

(ii) *If P is an admissible system of positive roots for Δ and if $P_{\mathfrak{k}}=\{\alpha\in\Delta_{\mathfrak{k}} \mid \alpha=\beta|_{\mathfrak{h}_1}$ for some $\beta\in P\}$ then $P_{\mathfrak{k}}$ is a system of positive roots for $\Delta_{\mathfrak{k}}$. Furthermore, all systems of positive roots for $\Delta_{\mathfrak{k}}$ can be obtained in this manner.*

(iii) *If $s\in W_{\mathfrak{k}}$, then there exists a unique $\tilde{s}\in W$ so that $\tilde{s}\mathfrak{h}_1=\mathfrak{h}_1$ and $\tilde{s}|_{\mathfrak{h}_1}=s$. We will abuse notation and identify $W_{\mathfrak{k}}$ with a subgroup of W. That is, we write s for $\tilde{s}$.*

Proof. Let $(\mathfrak{h}_1)_{\mathbf{R}}=\mathfrak{h}_1\cap\mathfrak{h}_{\mathbf{R}}$ as usual. Set $(\mathfrak{h}_1)'_{\mathbf{R}}=\{h\in(\mathfrak{h}_1)_{\mathbf{R}} \mid \alpha(h)\neq 0 \text{ for all } \alpha\in\Delta\}$. (a) and (b) of Definition 2.1 easily imply that $(\mathfrak{h}_1)'_{\mathbf{R}}$ is open and dense in $(\mathfrak{h}_1)_{\mathbf{R}}$. Let $h\in(\mathfrak{h}_1)'_{\mathbf{R}}$. Set $P=\{\beta\in\Delta \mid \beta(h)>0\}$. Then P is a system of positive roots for Δ and $h\in(\mathfrak{h}_1)^+_{\mathbf{R}}$ (see Definition 2.3). This proves (i).

To prove (ii), we fix P an admissible system of positive roots for Δ. Let $h\in(\mathfrak{h}_1)^+_{\mathbf{R}}$ (see Definition 2.3). If $P_{\mathfrak{k}}=\{\alpha\in\Delta_{\mathfrak{k}} \mid$ there is $\beta\in P$, $\beta|_{\mathfrak{h}_1}=\alpha\}$ then $P_{\mathfrak{k}}\subset\{\alpha\in\Delta_{\mathfrak{k}} \mid \alpha(h)>0\}$. If $\alpha\in\Delta_{\mathfrak{k}}$ and $\alpha(h)>0$, then if $\beta\in\Delta$ is such that $\beta|_{\mathfrak{h}_1}=\alpha$, then $\beta(h)>0$, hence $\beta\in P$. This says $P_{\mathfrak{k}}=\{\alpha\in\Delta_{\mathfrak{k}} \mid \alpha(h)>0\}$. Therefore, we see that $P_{\mathfrak{k}}$ is a system of positive roots for $\Delta_{\mathfrak{k}}$.

If $P_{\mathfrak{k}}$ is a system of positive roots for $\Delta_{\mathfrak{k}}$, let $C=\{h\in(\mathfrak{h}_1)_{\mathbf{R}} \mid \alpha(h)>0 \text{ for all } \alpha\in P_{\mathfrak{k}}\}$. Then C is open in $(\mathfrak{h}_1)_{\mathbf{R}}$. Since $(\mathfrak{h}_1)'_{\mathbf{R}}$ is open and dense in $(\mathfrak{h}_1)_{\mathbf{R}}$, we see that $(\mathfrak{h}_1)'_{\mathbf{R}}\cap C\neq\emptyset$. Fix $h\in(\mathfrak{h}_1)'_{\mathbf{R}}\cap C$. Let $P=\{\alpha\in\Delta \mid \alpha(h)>0\}$. Then $P_{\mathfrak{k}}=\{\alpha\in\Delta_{\mathfrak{k}} \mid$ there exists $\beta\in P$ so that $\beta|_{\mathfrak{h}_1}=\alpha\}$. This completes the proof of (ii).

Let $\mathrm{Ad}_{\mathfrak{g}}(\mathfrak{g})$ (resp. $\mathrm{Ad}_{\mathfrak{g}}(\mathfrak{k})$) be the group of automorphisms of $\mathfrak{g}$ generated by the automorphisms of the form $\exp(\mathrm{ad}\, X)$, $X\in\mathfrak{g}$ (resp. $X\in\mathfrak{k}$). The map $\mathrm{Ad}_{\mathfrak{g}}(\mathfrak{k})\to\mathrm{Ad}_{\mathfrak{k}}(\mathfrak{k})$ given by $g\mapsto g|_{\mathfrak{k}}$ is clearly surjective. Thus if $s\in W_{\mathfrak{k}}$, there exists $g\in\mathrm{Ad}_{\mathfrak{g}}(\mathfrak{k})$ so that $g\mathfrak{h}_1=\mathfrak{h}_1$ and

$g|_{\mathfrak{h}_1}=s$. Now, $C_{\mathfrak{g}}(\mathfrak{h}_1)=\mathfrak{h}$. Set $\tilde{s}=g|_{\mathfrak{h}}$. Clearly $g\cdot\mathfrak{h}\subset\mathfrak{h}$. Then $\tilde{s}\in W$ and $\tilde{s}|_{\mathfrak{h}_1}=s$. If $s_1, s_2\in W$ are such that $s_i|_{\mathfrak{h}_1}=s$, $i=1, 2$, then $s_2^{-1}s_1|_{\mathfrak{h}_1}=I$. But then if P is an admissible system of positive roots for Δ, $s_2^{-1}s_1P=P$. Hence $s_2^{-1}s_1=I$. This proves (iii).

If M is a $\mathfrak{g}$-module (resp. N a $\mathfrak{k}$-module) and if $\mu\in\mathfrak{h}^*$ (resp. $\nu\in\mathfrak{h}_1^*$) set $M(\mu)$ (resp. $N[\nu]$) equal to the μ (resp. ν) weight space for M (resp. N) relative to $\mathfrak{h}$ (resp. $\mathfrak{h}_1$). If M is a $\mathfrak{g}$-module, it is clearly a $\mathfrak{k}$-module by restriction and we have:

$$M[\nu]=\sum_{\substack{\mu\in\mathfrak{h}^*\\ \mu|\mathfrak{h}_1=\nu}} M(\mu) \tag{2.1}$$

If $\Lambda\in\mathfrak{h}^*$ we denote by $V_{\mathfrak{g},P,\Lambda}$ the $\mathfrak{g}$-Verma module with highest weight Λ relative to P.

LEMMA 2.5. *Set $V=V_{\mathfrak{g},P,\Lambda}$. Suppose that P is admissible.*

(i) *If $\nu\in\mathfrak{h}_1^*$, then* $\dim\ V[\nu]<\infty$.

(ii) *If $\nu=\Lambda|_{\mathfrak{h}_1}$, then* $\dim\ V[\nu]=1$.

Proof. Let $h_0\in(\mathfrak{h}_1)_{\mathbf{R}}$ be such that $\alpha(h_0)\geqslant 1$ for all $\alpha\in P$; this is possible since P is admissible (see Definition 2.3). By (2.1) we know $\dim\ V[\nu]=\sum_{\mu\in\mathfrak{h}^*,\ \mu|\mathfrak{h}_1=\nu}\dim\ V(\mu)$. Since $\dim\ V(\mu)<\infty$, we need only show that there are only a finite number of $\mu\in\mathfrak{h}^*$ so that $V(\mu)\neq 0$ and $\mu|_{\mathfrak{h}_1}=\nu$. If $V(\mu)\neq 0$, then $\mu=\Lambda-Q$ with Q a sum of elements of P. Thus, if $V(\mu)\neq 0$ and $\mu|_{\mathfrak{h}_1}=\nu$, then $\Lambda(h_0)-\mu(h_0)=\Lambda(h_0)-\nu(h_0)$ and $\Lambda(h_0)-\mu(h_0)=Q(h_0)$. This says that the number of elements of P in the expression of Q must be bounded by $\Lambda(h_0)-\nu(h_0)$. This implies that there are only a finite number of such Q. If $\nu=\Lambda|_{\mathfrak{h}_1}$ and if $\mu\in\mathfrak{h}^*$ is such that $\mu=\Lambda-Q$, Q a sum of elements of P and $\mu|_{\mathfrak{h}_1}=\nu$, then $Q(h_0)=0$. Hence $Q=0$. We have thus proved the lemma.

Let $\mathcal{G}$ denote the universal enveloping algebra of $\mathfrak{g}$ (we will also sometimes use the notation $U(\mathfrak{g})$). Let $\mathcal{G}^{\mathfrak{h}_1}$ denote the subalgebra of elements of $\mathcal{G}$ that commute with $\mathfrak{h}_1$.

Let P be an admissible system of positive roots for Δ. Set $\mathfrak{n}=\mathfrak{n}_P=\sum_{\alpha\in P}\mathfrak{g}_\alpha$. Set $\mathfrak{n}^-=\mathfrak{n}_{-P}$. The Poincare–Birkhoff–Witt theorem (P–B–W) implies that if $U(\mathfrak{h})$ denotes the universal enveloping algebra of $\mathfrak{h}$ then $\mathcal{G}=U(\mathfrak{h})\oplus(\mathfrak{n}^-\mathcal{G}+\mathcal{G}\mathfrak{n})$. Let Q_P: $\mathcal{G}\to U(\mathfrak{h})$ be the corresponding projection. Since P is admissible, it is easy to see that Q_P: $\mathcal{G}^{\mathfrak{h}_1}\to U(\mathfrak{h})$ is an algebra homomorphism (argue as in, e.g., [3], 7.4.2 Lemme). If $\Lambda\in\mathfrak{h}^*$ we can thus define $Q_{P,\Lambda}$: $\mathcal{G}^{\mathfrak{h}_1}\to\mathbf{C}$ by $Q_{P,\Lambda}(g)=\Lambda(Q_P(g))$. (Here Λ also denotes the extension of Λ to $U(\mathfrak{h})$.)

LEMMA 2.6. *Let P be an admissible system of positive roots for Δ. Set $V=V_{\mathfrak{g},P,\Lambda}$. If $g\in\mathcal{G}^{\mathfrak{h}_1}$, then $g|_{V(\Lambda)}=Q_{P,\Lambda}(g)\,\mathrm{Id}$.*

Proof. The result is obvious if we note that $\mathcal{G}^{\mathfrak{h}_1}\subset U(\mathfrak{h})\oplus\mathcal{G}\mathfrak{n}$ (cf. [3], 7.4.2 Lemme).

The following lemma will be quite useful in the next section. The argument used in its proof is due to Nasaki Kashiwara. It was used in the context of the next section by Donald King in a seminar at M.I.T. We are grateful to Michele Vergne for having pointed it out to us. A simple proof of this lemma has also been given by Kostant using the Ore condition satisfied by universal enveloping algebras. We give the Kashiwara proof since it is quite elementary.

LEMMA 2.7. *Let $\mathfrak{a}$ be a Lie algebra over* $\mathbf{C}$, *and let $\mathfrak{b}\subset\mathfrak{a}$ be a subalgebra of $\mathfrak{a}$. Let M be an $\mathfrak{a}$-module. Set $M_1=\{m\in M \mid$ there exists $b\in U(\mathfrak{b})$, $b\neq 0$ and $b\cdot m=0\}$. Then M_1 is an $\mathfrak{a}$-submodule of M.*

Proof. Let $\{U_N(\mathfrak{b})\}$ denote the standard filtration of $U(\mathfrak{b})$ (that is, $U_N(\mathfrak{b})$ is the subspace of $U(\mathfrak{b})$ spanned by m-fold products of elements of $\mathfrak{b}$ with $m\leqslant N$).

(a) $\dim U_N(\mathfrak{b})=\binom{N+r}{r}$ where $r=\dim\mathfrak{b}$. This is a simple consequence of the Poincare–Birkhoff–Witt theorem.

(b) $U_N(\mathfrak{b})\mathfrak{a}=\mathfrak{a}U_N(\mathfrak{b})$.

We prove (b) by induction on N. It is clear if $N=0$. Suppose true for $0\leqslant N-1$. Let $X_1, ..., X_N\in\mathfrak{b}$ and let $Y\in\mathfrak{a}$. Then $X_1 \dots X_N Y=YX_1\dots X_N-\sum_{i=1}^N X_1\dots[Y, X_i]\dots X_N$. Now by the induction hypothesis $X_1\dots X_{i-1}[Y, X_i]\in\mathfrak{a}U_{i-1}(\mathfrak{b})$. Thus $X_1\dots X_{i-1}[Y,X_i]X_{i+1}\dots X_N\in\mathfrak{a}U_{i-1}(\mathfrak{b})$. $U_{N-i}(\mathfrak{b})\subset\mathfrak{a}U_{N-1}(\mathfrak{b})$. This proves (b).

Using (a), (b), we prove the lemma. We note that it is enough to show that if $X\in\mathfrak{a}$, $m\in M_1$ and N is large, then $\dim U_N(\mathfrak{b})Xm_1<\dim U_N(\mathfrak{b})$. But $\dim U_N(\mathfrak{b})Xm\leqslant\dim U_N(\mathfrak{b})\mathfrak{a}\cdot m=\dim\mathfrak{a}\cdot U_N(\mathfrak{b})m$. Since $m\in M_1$ there exists $u_0\in U_{j_0}(\mathfrak{b})$ so that $u_0\neq 0$ and $u_0\cdot m=0$. If $N>j_0$ then $U_{N-j_0}(\mathfrak{b})\cdot u_0\subset U_N(\mathfrak{b})$ and $U_{N-j_0}(\mathfrak{b})u_0\cdot m=0$. Hence $\dim U_N(\mathfrak{b})Xm\leqslant\dim\mathfrak{a}\cdot(\dim U_N(\mathfrak{b})-\dim U_{N-j_0}(\mathfrak{b}))$. But $\dim U_N(\mathfrak{b})=\binom{N+r}{r}=(1/r!)(N+r)\dots(N+1)=1/r!\,N^r+$lower order terms in N. $\dim U_{N-j_0}(\mathfrak{b})=(1/r!)(N-j_0)^r+$(lower order terms in $N-j_0)=(1/r!)N^r+$lower order terms in N. This implies that $\dim U_N(\mathfrak{b})Xm$ is dominated by a polynomial of degree at most $r-1$ in N. Thus, if N is sufficiently large $\dim U_N(\mathfrak{b})\times Xm<\dim U_N(\mathfrak{b})$. Q.E.D.

3. The modules $W_{P,\Lambda}$

In this section we extend the construction of the modules of [7], [18] to the case where the ranks of $\mathfrak{k}$ and $\mathfrak{g}$ are not necessarily the same. Many details are proved in exactly the same way as the equal rank case. For these details the reader is referred to [18].

Let $(\mathfrak{g},\mathfrak{k})$ be a regular pair (see Definition 2.1). Let $\mathfrak{h}_1\subset\mathfrak{k}$ be a Cartan subalgebra of $\mathfrak{k}$, $\mathfrak{h}=C_{\mathfrak{g}}(\mathfrak{h}_1)$ is then a Cartan subalgebra of $\mathfrak{g}$. Let Δ and $\Delta_{\mathfrak{k}}$ be respectively the root systems of $(\mathfrak{g},\mathfrak{h})$, $(\mathfrak{k},\mathfrak{h}_1)$. We fix $P\subset\Delta$ an admissible system of positive roots (see Definition 2.3).

Let $P_{\mathfrak{k}}$ be as in Lemma 2.4 (ii). Then $P_{\mathfrak{k}}$ is a system of positive roots for $\mathfrak{k}$. Set $\delta_{\mathfrak{k}} = \frac{1}{2}\sum_{\alpha \in P_{\mathfrak{k}}} \alpha$. If $s \in W_{\mathfrak{k}}$, $\lambda \in \mathfrak{h}_{\mathrm{I}}^*$ set $s'\lambda = s(\lambda + \delta_{\mathfrak{k}}) - \delta_{\mathfrak{k}}$. We also recall (see Lemma 2.4 (iii)) that $W_{\mathfrak{k}}$ can (and will) be identified with a subgroup of W. Let $t_0 \in W_{\mathfrak{k}}$ be such that $t_0 P_{\mathfrak{k}} = -P_{\mathfrak{k}}$. Then $-t_0 P$ is an admissible system of positive roots for Δ.

If $\mu \in \mathfrak{h}_{\mathrm{I}}^*$ (resp. $\Lambda \in \mathfrak{h}^*$) and $Q_{\mathfrak{k}} \subset \Delta_{\mathfrak{k}}$, (resp. $Q \subset \Delta$) is a system of positive roots for $\Delta_{\mathfrak{k}}$ (resp. Δ). Then $V_{\mathfrak{k}, Q_{\mathfrak{k}}, \mu}$ (resp. $V_{\mathfrak{g}, Q, \Lambda}$) denotes the Verma module for $\mathfrak{k}$ (resp. $\mathfrak{g}$) with highest weight μ (resp. Λ) relative to $Q_{\mathfrak{k}}$ (resp. Q).

Returning to $P_{\mathfrak{k}}$ and P, we will use the notation $V_{\mathfrak{k}, P_{\mathfrak{k}}, \mu} = V_\mu$ and $V_{\mathfrak{g}, -t_0 P, \Lambda} = M_\Lambda$.

As is well known (see [3], Theorémè 7.6.6, p. 237), dim $\mathrm{Hom}_{\mathfrak{k}}(V_{\mu_1}, V_{\mu_2}) = 0$ or 1, and if $A \in \mathrm{Hom}_{\mathfrak{k}}(V_{\mu_1}, V_{\mu_2})$ then A is either zero or injective. If $\lambda \in \mathfrak{h}_{\mathrm{I}}^*$ and λ is $P_{\mathfrak{k}}$-dominant integral, then $\mathrm{Hom}_{\mathfrak{k}}(V_{s'\lambda}, V_\lambda) \neq (0)$ for each $s \in W_{\mathfrak{k}}$. Furthermore, there is a partial order on the Weyl group, $W_{\mathfrak{k}}$, (depending only on $P_{\mathfrak{k}}$) so that $\mathrm{Hom}_{\mathfrak{k}}(V_{s_1'\lambda}, V_{s_2'\lambda}) \neq (0)$ if and only if $s_1 < s_2$ (see [3], § 7.7 and Theorémè 7.7.7, p. 253). We recall the definition of the order on $W_{\mathfrak{k}}$. If $s \in W_{\mathfrak{k}}$ we define $l(s)$ as the minimal number, r, of $P_{\mathfrak{k}}$-simple reflections, $s_{i_1}, \ldots, s_{i_r}$ so that $s = s_{i_1} \ldots s_{i_r}$. If $s, t \in W_{\mathfrak{k}}$ and $\gamma \in P_{\mathfrak{k}}$ then we say $s \xrightarrow{\gamma} t$ if $s = s_\gamma t$ and $l(s) = l(t) + 1$. We say that $s < t$ if there exist $s_0, \ldots, s_p \in W_{\mathfrak{k}}$ and $\gamma_1, \ldots, \gamma_p \in P_{\mathfrak{k}}$ so that

$$s = s_0 \xrightarrow{\gamma_1} s_1 \xrightarrow{\gamma_2} \ldots \xrightarrow{\gamma_p} S_p = t.$$

We make our first notational convention:

Convention 1. If λ is in $\mathfrak{h}_{\mathrm{I}}^*$ and if there exists $s \in W_{\mathfrak{k}}$ so that $s'\lambda$ is $P_{\mathfrak{k}}$-dominant integral, then the notation V_λ will denote the subspace $\{Av \mid A \in \mathrm{Hom}_{\mathfrak{k}}(V_\lambda, V_{s'\lambda}), v \in V_\lambda\}$.

With this convention we see that if λ is $P_{\mathfrak{k}}$-dominant integral and if $s_1, s_2 \in W_{\mathfrak{k}}$, $V_{s_1'\lambda} \subset V_{s_2'\lambda}$ if and only if $s_1 < s_2$ (the inclusion is both set theoretic and module theoretic).

Let $\mathcal{G}$ (resp. $\mathcal{K}$) denote the universal enveloping algebra of $\mathfrak{g}$ (resp. $\mathfrak{k}$). Let $\lambda \in \mathfrak{h}_{\mathrm{I}}^*$ be $P_{\mathfrak{k}}$-dominant integral. Then $V_{s'\lambda} \subset V_\lambda$ for each $s \in W_{\mathfrak{k}}$. Set $U_{1,\lambda} = \mathcal{G} \otimes_{\mathcal{K}} V_\lambda$. Let $j : \mathcal{G} \otimes_{\mathcal{K}} V_{s'\lambda} \to \mathcal{G} \otimes_{\mathcal{K}} V_\lambda = U_{1,\lambda}$ be the canonical inclusion corresponding to $V_{s'\lambda} \subset V_\lambda$. Set $U_{s,\lambda} = j(\mathcal{G} \otimes_{\mathcal{K}} V_{s'\lambda}) \subset U_{1,\lambda}$. Then we see that $U_{s_1,\lambda} \subset U_{s_2,\lambda}$ if $s_1 < s_2$.

Fix $\Lambda \in \mathfrak{h}^*$ so that $\Lambda|_{\mathfrak{h}_{\mathrm{I}}} = t_0'\lambda$. There is a canonical, surjective, $\mathfrak{g}$-module homomorphism $\mathcal{G} \otimes_{\mathcal{K}} V_{t_0'\lambda} \to M_\Lambda = V_{\mathfrak{g}, -t_0 P, \Lambda}$ given by extending the isomorphism between $V_{t_0'\lambda}$ and the $\mathcal{K}$-cyclic space of the highest weight vector of M_Λ. Let I'_Λ be the kernel of this homomorphism. Set $I_\Lambda = j(I'_\Lambda) \subset U_{t_0,\lambda} \subset U_{s,\lambda}$ for all $s \in W_{\mathfrak{k}}$.

Set $M_{s,\Lambda} = U_{s,\lambda}/I_\Lambda$ for each $s \in W_{\mathfrak{k}}$. Then we have:

(I) If $s_1 < s_2$ then $M_{s_1,\Lambda} \subset M_{s_2,\Lambda}$. (Notice: all our inclusions are *set theoretic and module theoretic.*)

(II) $j(1\otimes V_{s'\lambda})+I_\Lambda\subset M_{s,\Lambda}$ is $\mathfrak{k}$-isomorphic with $V_{s'\lambda}$. Set $B_s=j(1\otimes V_{s'\lambda})+I_\Lambda\subset M_{s,\Lambda}$. If $s_i\in W_{\mathfrak{k}}$, $i=1, 2$ and $s_1<s_2$ then $B_{s_1}\subset B_{s_2}$. Finally, $M_{s,\Lambda}=\mathcal{G}\cdot B_s$.

This is proved in exactly the same way as the analogous result in [18].

Set $M'_{s,\Lambda}=\{m\in M_{s,\Lambda}\mid$ there exists $u\in U(\mathfrak{n}^-)$, $u\neq 0$ so that $u\cdot m=0\}$. Here $\mathfrak{n}^-=\sum_{\alpha\in t_0P}\mathfrak{g}_\alpha$. Then $M'_{s,\Lambda}$ is a $\mathfrak{g}$-submodule of $M_{s,\Lambda}$ (Lemma 2.7). Set $W_{s,\Lambda}=M_{s,\Lambda}/M'_{s,\Lambda}$. From the definitions, we see that if ε: $M_{1,\Lambda}\to W_{1,\Lambda}$ is the canonical map, then $\operatorname{Ker}\varepsilon|_{M_{s,\Lambda}}=M'_{s,\Lambda}$. Thus we can (and do) identify $W_{s,\Lambda}$ with $\varepsilon(M_{s,\Lambda})\subset W_{1,\Lambda}$. We have

(III) If $s_1<s_2$ then $W_{s_1,\Lambda}\subset W_{s_2,\Lambda}$.

It is not hard to see that if $A_s=\varepsilon(B_s)$, then

(IV) A_s is isomorphic to $V_{s'\lambda}$. If $s_1<s_2$, s_1, $s_2\in W_{\mathfrak{k}}$ then $A_{s_1}\subset A_{s_2}$. Finally, $W_{s,\Lambda}=\mathcal{G}\cdot A_s$.

The following are also clear.

(V) If $w\in W_{s,\Lambda}$ and $u\in U(\mathfrak{n}^-)-\{0\}$ then $u\cdot w=0$ implies $w=0$.

(VI) $W_{t_0,\Lambda}$ is $\mathcal{G}$-isomorphic with M_Λ. (This follows since $M_{t_0,\Lambda}\equiv M_\Lambda$ hence $M'_{t_0,\Lambda}=(0)$).

The next result shows that properties (III), (IV), (V), (VI) of the $W_{s,\Lambda}$ completely characterize the $W_{s,\Lambda}$.

THEOREM 3.1. *Let $\Lambda\in\mathfrak{h}^*$ be such that $t_0'(\Lambda|_{\mathfrak{h}_1})=\lambda$ is $P_{\mathfrak{k}}$-dominant integral. Let $\{Z_s\}_{s\in W_{\mathfrak{k}}}$ be a family of $\mathfrak{g}$-modules satisfying the following four properties:*

(i) *For each $s\in W_{\mathfrak{k}}$ there exists an injective homomorphism α_s: $V_{s'\lambda}\to Z_s$ so that $Z_s=\mathcal{G}\cdot\alpha_s(V_{s'\lambda})$.*

(ii) *If $\alpha\in P_{\mathfrak{k}}$ is $P_{\mathfrak{k}}$-simple and if $s\in W_{\mathfrak{k}}$ is such that $s_\alpha s<s$ then $Z_{s_\alpha s}\subset Z_s$ (as a submodule) and $\alpha_{s_\alpha s}(V_{(s_\alpha s)'\lambda})\subset\alpha_s(V_{s'\lambda})$.*

(iii) *If $\alpha\in P_{\mathfrak{k}}$ is $P_{\mathfrak{k}}$-simple and if $X\in\mathfrak{k}_{-\alpha}$ (the $-\alpha$ root space for $(\mathfrak{k},\mathfrak{h}_1)$), $X\neq 0$, then the action of X on Z_s is injective.*

(iv) $Z_{t_0}\equiv M_\Lambda=V_{\mathfrak{g},-t_0P,\Lambda}$.

Then there exist bijective $\mathfrak{g}$-module isomorphisms β_s: $W_{s,\Lambda}\to Z_s$, $s\in W_{\mathfrak{k}}$ such that if s and α are as in (ii), *then $\beta_{s_\alpha s}=\beta_s|_{W_{s_\alpha s,\Lambda}}$. In particular, $\beta_s=\beta_1|_{W_{s,\Lambda}}$.*

Proof. Using (i) and the universal mapping property of the tensor product we see that there is a surjective homomorphism τ: $\mathcal{G}\otimes_{\mathcal{K}}V_\lambda\to Z_1$ given by $\tau(g\otimes v)=g\alpha_1(v)$. Since $U_{1,\lambda}=\mathcal{G}\otimes_{\mathcal{K}}V_\lambda$ we have τ: $U_{1,\lambda}\to Z_1$ a surjective homomorphism. Using (ii), we see that $\tau(U_{s,\lambda})=Z_s$ for each $s\in W_{\mathfrak{k}}$. In particular, $\tau(U_{t_0,\Lambda})=Z_{t_0}$. Since $Z_{t_0}\equiv M_\Lambda$ we see that $\operatorname{Ker}\tau\supset I_\Lambda$. We, therefore, find that we have a surjective homomorphism, μ: $M_{1,\Lambda}\to Z_1$. Furthermore, $\mu(M_{s,\Lambda})=Z_s$ for each s, and μ: $M_{t_0,\Lambda}\to Z_{t_0}$ is bijective.

Set for each $s \in W_{\mathfrak{k}}$, $Z'_s = \{z \in Z_s \mid$ there exists $u \in U(\mathfrak{n}^-) - \{0\}$ so that $u \cdot z = 0\}$.

(A) $Z'_s = (0)$ for each $s \in W_{\mathfrak{k}}$.

To prove this result we need a bit of notation. Fix for each $\alpha \in P_{\mathfrak{k}}$, $X_{-\alpha} \in \mathfrak{k}_{-\alpha} X_{-\alpha} \neq 0$. Set $z_s = \alpha_s(v_{s'\lambda})$ where $v_{s'\lambda}$ is a non-zero highest weight vector for $V_{s'\lambda}$. If $s \in W_{\mathfrak{k}}$ and if $\alpha \in P_{\mathfrak{k}}$ is $P_{\mathfrak{k}}$-simple and if $l(s_\alpha s) = l(s) + 1$ set $2\langle s'\lambda + \delta_{\mathfrak{k}}, \alpha \rangle / \langle \alpha, \alpha \rangle = n$, then $n \geqslant 0$ and $X^n_{-\alpha} v_{s'\lambda} = \bar{c} v_{(s_\alpha s)'\lambda}$ for some constant $\bar{c}$, $\bar{c} \neq 0$. (Here we use our convention.) Thus $X^n_{-\alpha} z_s = c z_{s_\alpha s}$, for some constant, c, $c \neq 0$. This implies

(a) If α and s are as above and if $z \in Z_s$, there exists $m \geqslant 0$ so that $X^m_{-\alpha} z \in Z_{s_\alpha s}$.

(iii) implies that if $z \neq 0$, then $x^m_{-\alpha} z \neq 0$. We prove (A) by induction on $l(t_0) - l(s)$.

If $l(t_0) - l(s) = 0$, then $Z_s = Z_{t_0} \equiv M_\Lambda$. Hence $Z'_s = (0)$. Suppose that we have proved (A) for all $s \in W_{\mathfrak{k}}$ with $0 \leqslant l(t_0) - l(s) < m$ and that $l(t_0) - l(s) = m$. Then there exists $\alpha \in P_{\mathfrak{k}}$, with α $P_{\mathfrak{k}}$-simple so that $l(s_\alpha s) = l(s) + 1$. Thus $l(t_0) - l(s_\alpha s) = m - 1$. If $z \in Z'_s$ then $X^p_{-\alpha} z \in Z_{s_\alpha s}$ for some $p \geqslant 0$, $p \in \mathbf{Z}$. Hence $X^p_{-\alpha} z \in Z'_{s_\alpha s}$ (Lemma 2.7). But the inductive hypothesis implies that $Z'_{s_\alpha s} = (0)$. Thus $X^p_{-\alpha} z = 0$. Hence $z = 0$. We have proved (A).

(A) implies that $\operatorname{Ker} \mu \supset M'_{1,\Lambda}$. Hence μ induces a surjective $\mathfrak{g}$-module homomorphism β: $W_{1,\Lambda} \to Z_1$ and as before we see that $\beta(W_{s,\Lambda}) = Z_s$.

To complete the proof of the theorem we need only show that β is injective. We prove that $\beta|_{Ws,\Lambda}$ is injective by induction on $l(t_0) - l(s)$. If $l(t_0) - l(s) = 0$, then $s = t_0$ and $W_{t_0,\Lambda} \equiv Z_{t_0} \equiv M_\Lambda$, and since $\beta(W_{t_0,\Lambda}) = Z_{t_0}$, we see that $\beta|_{W_{t_0,\Lambda}}$ is injective. Suppose that we have shown that $\beta|_{W_{s,\Lambda}}$ is injective for $0 \leqslant l(t_0) - l(s) \leqslant m$. If $l(t_0) - l(s) = m$, choose $\alpha \in P_{\mathfrak{k}}$, α simple in $P_{\mathfrak{k}}$ so that $l(s_\alpha s) = l(s) + 1$. Let $z \in W_{s,\Lambda}$, suppose $\beta(z) = 0$. Let $p \geqslant 0$, $p \in \mathbf{Z}$ be so that $X^p_{-\alpha} z \in W_{s_\alpha s,\Lambda}$ (this is possible since the $W_{s,\Lambda}$ satisfy (i), (ii), (iii), (iv)). Then $\beta(X^p_{-\alpha} z) = X^p_{-\alpha} \beta(z) = 0$. But then the inductive hypothesis implies $X^p_{-\alpha} z = 0$. Hence $z = 0$. The proof of the theorem is now complete.

If $\mu \in \mathfrak{h}_1^*$ is $P_{\mathfrak{k}}$-dominant integral, set V^μ equal to the irreducible $\mathfrak{k}$-module with highest weight μ. (The realization we use of V^μ is $V_\mu / \sum_{s<1} V_{s'\mu}$.) If m is a non-negative integer, we use the notation mV^μ for any $\mathfrak{k}$-module isomorphic with the direct sum of m copies of V^μ.

In light of the results of this section and those of § 1, the following theorem is proved in exactly the same way as Theorem 2.4 of [18].

THEOREM 3.2. *Let* $\Lambda \in \mathfrak{h}^*$ *be such that* $\lambda = t'_0(\Lambda|_{\mathfrak{h}_1})$ *is* $P_{\mathfrak{k}}$*-dominant integral. Set* $W_{P,\Lambda} = W_{1,\Lambda} / \sum_{s<1} W_{s,\Lambda}$.

(i) *As a* $\mathfrak{k}$*-module* $W_{P,\Lambda} = \sum_\mu m_\Lambda(\mu) V_\mu$ *with* $0 \leqslant m_\Lambda(\mu) < \infty$ *and* $m_\Lambda(\mu) \in \mathbf{Z}$, *the sum taken over all* $P_{\mathfrak{k}}$*-dominant integral* μ. *Furthermore,*

(a) $m_\Lambda(\lambda) = 1$

(b) *If* $m_\lambda(\mu)\neq 0$ *and* $\mu\neq\lambda$, *then there exist* $\beta_1, \ldots, \beta_m \in P$ *(not necessarily distinct) so that* $\mu=\lambda+\beta_1|_{\mathfrak{h}_1}+\ldots+\beta_m|_{\mathfrak{h}_1}$.

(ii) *Let* $\mathcal{G}^{\mathfrak{k}}$ *denote the centralizer of* $\mathfrak{k}$ *in* $\mathcal{G}$. *Let* $\eta_{P,\Lambda}=Q_{-t_0P,\Lambda}|_{\mathcal{G}^{\mathfrak{k}}}$ $(\mathcal{G}^{\mathfrak{k}}\subset\mathcal{G}^{\mathfrak{h}_1}$, *see Lemma 2.6). Then looking upon* V^λ *as a* $\mathfrak{k}$*-submodule of* $W_{P,\Lambda}$ *(see (i) (a)), the action of* $\mathcal{G}^{\mathfrak{k}}$ *on* V^λ *is given by*

$$g|_{V^\lambda}=\eta_{P,\Lambda}(g)\,\mathrm{Id}$$

for $g\in\mathcal{G}^{\mathfrak{k}}$.

In general, the modules $W_{P,\Lambda}$ are not irreducible. Let $M_{P,\Lambda}$ be the sum of all $\mathfrak{g}$-submodules $M\subseteq W_{P,\Lambda}$ such that $\mathrm{Hom}_{\mathfrak{k}}(V^\lambda, M)=\{0\}$ and set $D_{P,\Lambda}=W_{P,\Lambda}/M_{P,\Lambda}$. The argument which yields Theorem 2 in [7] now proves:

THEOREM 3.3. *Let* $\Lambda\in\mathfrak{h}^*$ *be such that* $\lambda=t_0'(\Lambda|_{\mathfrak{h}_1})$ *is* $P_{\mathfrak{k}}$*-dominant integral, then* $M_{P,\Lambda}$ *is the unique maximal proper submodule of* $W_{P,\Lambda}$ *and thus* $D_{P,\Lambda}$ *is a non-zero irreducible* $\mathfrak{g}$*-module. Within the set of equivalence classes of* $\mathfrak{k}$*-finite irreducible* $\mathfrak{g}$*-modules, the equivalence class containing* $D_{P,\Lambda}$ *is uniquely determined by the condition:*

If M *is an element of the equivalence class, then there exists a non-zero element* A *in* $\mathrm{Hom}_{\mathfrak{k}}(V^\lambda, M)$ *such that* $\mathcal{G}^{\mathfrak{k}}$ *acts on* $A(V^\lambda)$ *by the formula*

$$x|_{A(V^\lambda)}\equiv\eta_{P,\Lambda}(x)\,\mathrm{Id}$$

for $x\in\mathcal{G}^{\mathfrak{k}}$.

4. Resolutions of Verma modules

The purpose of this section is to prove the existence of an exact sequence of $\mathfrak{g}$-modules whose last term is a $\mathfrak{g}$-Verma module and whose other terms are induced modules from a Borel subalgebra of $\mathfrak{k}$. We will use this exact sequence in § 5 to give an analogous resolution for most $W_{s,\Lambda}$. In particular, we will see in § 6 that for Λ "sufficiently regular", $W_{P,\Lambda}$ has a universal mapping property.

We will use the following lemma several times in this section.

LEMMA 4.1. *Let* $\mathfrak{a}$ *be a Lie algebra over* $\mathbf{C}$ *and let* $\mathfrak{b}\subset\mathfrak{a}$ *be a subalgebra. Let* M *be an* $\mathfrak{a}$*-module and let* N *be a* $\mathfrak{b}$*-module. Let* M' *denote* M *as a* $\mathfrak{b}$*-module. Define* $j\colon U(\mathfrak{a})\otimes_{U(\mathfrak{b})}(M'\otimes N)\to M\otimes(U(\mathfrak{a})\otimes_{U(\mathfrak{b})}N)$ *by*

$$j(a\otimes(m\otimes n))=a\cdot(m\otimes(1\otimes n))$$

for $a\in U(\mathfrak{a})$, $m\in M=M'$, $n\in N$. *Then* j *is a surjective isomorphism of* $\mathfrak{a}$*-modules.*

This lemma seems to be well known. However, the only proof of it in the literature is in [9], Proposition 1.7.

We now use Lemma 4.1 to give a slight generalization of the relative homology resolution of $\mathbf{C}$ (see B–G–G [1]).

LEMMA 4.2. *Let $\mathfrak{a}$ be a Lie algebra over $\mathbf{C}$ and let $\mathfrak{m}\subset\mathfrak{a}$ be a subalgebra. Let P: $\mathfrak{a}\to\mathfrak{a}/\mathfrak{m}$ be the natural $\mathfrak{m}$-module projection* ($\mathfrak{m}$ *acts on* $\mathfrak{a}$ *and* $\mathfrak{a}/\mathfrak{m}$ *by the restriction of* ad). *If $\Lambda\in\mathfrak{a}^*$ and $\Lambda|_{[\mathfrak{a},\mathfrak{a}]}=0$, let $\mathbf{C}_\Lambda$ denote the corresponding one-dimensional $\mathfrak{a}$-module. Let $\Lambda|_{\mathfrak{m}}=\lambda$ and let $\mathbf{C}_\lambda$ denote the corresponding one-dimensional $\mathfrak{m}$-module. We define*

$$\partial_\Lambda : U(\mathfrak{a}) \underset{U(\mathfrak{m})}{\otimes} (\wedge^j(\mathfrak{a}/\mathfrak{m})\otimes\mathbf{C}_\lambda) \longrightarrow U(\mathfrak{a}) \underset{U(\mathfrak{m})}{\otimes} (\wedge^{j-1}(\mathfrak{a}/\mathfrak{m})\otimes\mathbf{C}_\lambda)$$

as follows: If $a\in U(\mathfrak{a})$, $X_1, \ldots, X_j\in\mathfrak{a}/\mathfrak{m}$ $Y_j\in\mathfrak{a}$ so that $PY_j=X_j$, then

$$\begin{aligned}
&\partial_\Lambda(a\otimes X_1\wedge\ldots\wedge X_j\otimes 1)\\
&\quad=\sum_{i=1}^{j}(-1)^{i+1}a(Y_i-\Lambda(Y_i)1)\otimes X_1\wedge\ldots\wedge\hat{X}_i\wedge\ldots\wedge X_j\otimes 1\\
&\qquad+\sum_{1\leqslant r<s\leqslant j}(-1)^{r+s}a\otimes P[Y_r,Y_s]\wedge X_1\wedge\ldots\wedge\hat{X}_r\wedge\ldots\wedge\hat{X}_s\wedge\ldots\wedge X_j\otimes 1.
\end{aligned}$$

Then ∂_Λ is a well defined $\mathfrak{a}$-module homomorphism. Furthermore, if $g=\dim\mathfrak{a}/\mathfrak{m}$, then the following sequence of $\mathfrak{a}$-module homomorphisms is exact

$$\begin{gathered}
0\longrightarrow U(\mathfrak{a})\underset{U(\mathfrak{m})}{\otimes}(\wedge^g(\mathfrak{a}/\mathfrak{m})\otimes\mathbf{C}_\lambda)\xrightarrow{\partial_\Lambda}U(\mathfrak{a})\underset{U(\mathfrak{m})}{\otimes}(\wedge^{g-1}(\mathfrak{a}/\mathfrak{m})\otimes\mathbf{C}_\lambda)\\
\xrightarrow{\partial_\Lambda}\ldots\xrightarrow{\partial_\Lambda}U(\mathfrak{a})\underset{U(\mathfrak{m})}{\otimes}(\mathfrak{a}/\mathfrak{m}\otimes\mathbf{C}_\lambda)\xrightarrow{\partial_\Lambda}U(\mathfrak{a})\underset{U(\mathfrak{m})}{\otimes}\mathbf{C}_\lambda\xrightarrow{\varepsilon}\mathbf{C}_\Lambda\longrightarrow 0
\end{gathered}\tag{4.1}$$

here $\varepsilon(a\otimes 1)=\Lambda(a)$.

For each non-negative integer j, set $U_j(\mathfrak{a})$ equal to the subspace of $U(\mathfrak{a})$ spanned by 1 *and all i-fold products of elements of $\mathfrak{a}$ where $i\leqslant j$. For negative integers j, set $U_j(\mathfrak{a})=\{0\}$. Then for any integer $j\geqslant -g$, the following sequence is exact:*

$$\begin{gathered}
0\longrightarrow U_j(\mathfrak{a})\underset{U(\mathfrak{m})}{\otimes}(\wedge^g(\mathfrak{a}/\mathfrak{m})\otimes\mathbf{C}_\lambda)\xrightarrow{\partial_\Lambda}\\
U_{j+1}(\mathfrak{a})\underset{U(\mathfrak{m})}{\otimes}(\wedge^{g-1}(\mathfrak{a}/\mathfrak{m})\otimes\mathbf{C}_\lambda)\xrightarrow{\partial_\Lambda}\ldots\xrightarrow{\partial_\Lambda}U_{j+g}(\mathfrak{a})\underset{U(\mathfrak{m})}{\otimes}\mathbf{C}_\lambda\xrightarrow{\varepsilon}\mathbf{C}_\Lambda\longrightarrow 0
\end{gathered}\tag{4.2}$$

Proof. In [1] it was observed that the sequence (4.1) with $\Lambda=0$ and hence $\lambda=0$ is exact. The same technique also shows that sequence (4.2) is exact when $\Lambda=0$. Now for $0\leqslant i\leqslant g$, let $J_{i,\Lambda}$ be the bijective homomorphism from $U(\mathfrak{a})\otimes_{U(\mathfrak{m})}(\wedge^i(\mathfrak{a}/\mathfrak{m})\otimes\mathbf{C}_\lambda)$ onto $(U(\mathfrak{a})\otimes_{U(\mathfrak{m})}\wedge^i(\mathfrak{a}/\mathfrak{m}))\otimes\mathbf{C}_\Lambda$ given by Lemma 4.1. Now tensoring an exact sequence with the finite-dimensional module $\mathbf{C}_\Lambda$ yield another exact sequence. Thus for any integer $j \geqslant -g$, we obtain the exact sequence:

$$0 \longrightarrow U_j(\mathfrak{a}) \underset{U(\mathfrak{m})}{\otimes} (\wedge^g(\mathfrak{a}/\mathfrak{m}) \otimes \mathbf{C}_\lambda) \xrightarrow{J_{g-1,\Lambda}^{-1} \circ (\partial_0 \otimes 1) \circ J_{g,\Lambda}} \cdots$$

$$\xrightarrow{J_{0,\Lambda}^{-1} \circ (\partial_0 \otimes 1) \circ J_{1,\Lambda}} U_{j+g}(\mathfrak{a}) \underset{U(\mathfrak{m})}{\otimes} \mathbf{C}_\lambda \xrightarrow{\ \varepsilon\ } \mathbf{C}_\Lambda \longrightarrow 0$$

Set ∂_Λ equal to $J_{i-1,\Lambda}^{-1} \circ (\partial_0 \otimes 1) \circ J_{i,\Lambda}$. A straightforward computation shows that ∂_Λ is given by the formula in the lemma (note that it is sufficient to check the formula on the subspaces $1 \otimes \wedge^i(\mathfrak{a}/\mathfrak{m}) \otimes \mathbf{C}_\lambda$).

We now return to the notation of § 3. That is, $(\mathfrak{g}, \mathfrak{k})$ is a regular pair. Let $\mathfrak{h}_1$, $\mathfrak{h} = C_\mathfrak{g}(\mathfrak{h}_1)$, Δ, $\Delta_\mathfrak{k}$ be as in § 3, and let P be an admissible system of positive roots for Δ. Let $P_\mathfrak{k}$ be the corresponding positive roots for $\Delta_\mathfrak{k}$. Set $\mathfrak{b}_\mathfrak{k} = \mathfrak{h}_1 + \sum_{\alpha \in P_\mathfrak{k}} \mathfrak{k}_\alpha$.

Set $\mathfrak{p} = \mathfrak{k}^\perp$ relative to the Killing form of $\mathfrak{g} \cdot \mathfrak{k}^\perp \cap \mathfrak{k} = (0)$ by the definition of regular pair. Set $\mathfrak{h}_2 = \{X \in \mathfrak{p} \mid [\mathfrak{h}_1, X] = 0\}$. Then $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$. As an $\mathfrak{h}_1$-module $\mathfrak{p} = \mathfrak{h}_2 \oplus \sum_{\lambda \neq 0} \mathfrak{p}[\lambda]$, $(\mathfrak{p}[\lambda] = \{X \in \mathfrak{p} \mid [H, X] = \lambda(H) X \text{ for } H \in \mathfrak{h}_1\})$. Set $\mathfrak{p}_{t_0}^- = \sum_{\alpha \in -t_0 P} \mathfrak{p}[\alpha \mid \mathfrak{h}_1] \oplus \mathfrak{h}_2$. Set $\mathfrak{n} = \sum_{\alpha \in -t_0 P} \mathfrak{g}_\alpha$. Hence $\mathfrak{b}_\mathfrak{k} = \mathfrak{h}_1 \oplus \mathfrak{n}_\mathfrak{k}$ $(\mathfrak{n}_\mathfrak{k} = \sum_{\alpha \in P_\mathfrak{k}} \mathfrak{k}_\alpha)$ and let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$. Then $\mathfrak{b} = \mathfrak{b}_\mathfrak{k} \oplus \mathfrak{p}_{t_0}^-$. Clearly $[\mathfrak{b}_\mathfrak{k}, \mathfrak{p}_{t_0}^-] \subset \mathfrak{p}_{t_0}^-$. Let π: $\mathfrak{b} \to \mathfrak{p}_{t_0}^-$ be the corresponding $\mathfrak{b}_\mathfrak{k}$-module projection.

Lemma 4.3. *Let* $\Lambda \in \mathfrak{h}^*$ *and set* $M_\Lambda = V_{\mathfrak{g}, -t_0 P, \Lambda}$. *Let* $\lambda = t_0'(\Lambda \mid \mathfrak{h}_1)$. *Define* ∂_Λ: $\mathcal{G} \otimes_{U(\mathfrak{b}_\mathfrak{k})} (\wedge^j \mathfrak{p}_{t_0}^- \otimes \mathbf{C}_{t_0'\lambda}) \longrightarrow \mathcal{G} \otimes_{U(\mathfrak{b}_\mathfrak{k})} (\wedge^{j-1} \mathfrak{p}_{t_0}^- \otimes \mathbf{C}_{t_0'\lambda})$ *as follows: Extend* Λ *to* $\mathfrak{b}$ *by* $\Lambda(\mathfrak{n}) = 0$. *If* $g \in \mathcal{G}$, $X_1, \ldots, X_j \in \mathfrak{p}_{t_0}^-$ *then*

$$\partial_\Lambda(g \otimes X_1 \wedge \ldots \wedge X_j \otimes 1)$$
$$= \sum_{i=1}^{j} (-1)^{i+1} g(X_i - \Lambda(X_i)) \otimes X_1 \wedge \ldots \wedge \hat{X}_i \wedge \ldots \wedge X_j \otimes 1$$
$$+ \sum_{1 \leqslant r < s \leqslant j} (-1)^{r+s} g \otimes \pi[X_r, X_s] \wedge X_1 \wedge \ldots \wedge \hat{X}_r \wedge \ldots \wedge \hat{X}_s \wedge \ldots \wedge X_j \otimes 1.$$

Then ∂_Λ *is a well defined* $\mathfrak{g}$*-module homomorphism. Furthermore*

(i) M_Λ *is* $\mathfrak{g}$*-isomorphic with*

$$\mathcal{G} \underset{U(\mathfrak{b}_\mathfrak{k})}{\otimes} \mathbf{C}_{t_0'\lambda} / \partial_\Lambda(\mathcal{G} \underset{U(\mathfrak{b}_\mathfrak{k})}{\otimes} (\mathfrak{p}_{t_0}^- \otimes \mathbf{C}_{t_0'\lambda})),$$

(ii) *The following sequence of* $\mathfrak{g}$*-module homomorphisms is exact* ($m = \dim \mathfrak{p}_{t_0}^-$)

$$0 \longrightarrow \mathcal{G} \underset{U(\mathfrak{b}_\mathfrak{k})}{\otimes} (\wedge^m \mathfrak{p}_{t_0}^- \otimes \mathbf{C}_{t_0'\lambda}) \xrightarrow{\partial_\Lambda} \cdots \xrightarrow{\partial_\Lambda} \mathcal{G} \underset{U(\mathfrak{b}_\mathfrak{k})}{\otimes} (\mathfrak{p}_{t_0}^- \otimes \mathbf{C}_{t_0'\lambda}) \xrightarrow{\partial_\Lambda} \mathcal{G} \underset{U(\mathfrak{b}_\mathfrak{k})}{\otimes} \mathbf{C}_{t_0'\lambda} \xrightarrow{\ \varepsilon\ } M_\Lambda \longrightarrow 0,$$

here ε *is the projection onto* M_Λ *identified as in* (i).

Proof. We note that as a $\mathfrak{b}_{\mathfrak{k}}$-module $\mathfrak{p}_{t_0}^- = \mathfrak{b}/\mathfrak{b}_{\mathfrak{k}}$. Let π: $\mathfrak{b} \to \mathfrak{b}/\mathfrak{b}_{\mathfrak{k}}$ be the corresponding projection.

Let $m = \dim \mathfrak{p}_{t_0}^- = \dim \mathfrak{b}/\mathfrak{b}_{\mathfrak{k}}$ and let

$$\begin{aligned} 0 \longrightarrow U(\mathfrak{b}) \underset{U(\mathfrak{b}_{\mathfrak{k}})}{\otimes} (\wedge^m(\mathfrak{b}/\mathfrak{b}_{\mathfrak{k}}) \otimes \mathbf{C}_{t_0'\lambda}) &\xrightarrow{\partial_\Lambda} U(\mathfrak{b}) \underset{U(\mathfrak{b}_{\mathfrak{k}})}{\otimes} (\wedge^{m-1}(\mathfrak{b}/\mathfrak{b}_{\mathfrak{k}}) \otimes \mathbf{C}_{t_0'\lambda}) \xrightarrow{\partial_\Lambda} \cdots \\ \longrightarrow U(\mathfrak{b}) \underset{U(\mathfrak{b}_{\mathfrak{k}})}{\otimes} (\mathfrak{b}/\mathfrak{b}_{\mathfrak{k}} \otimes \mathbf{C}_{t_0'\lambda}) &\xrightarrow{\partial_\Lambda} U(\mathfrak{b}) \underset{U(\mathfrak{b}_{\mathfrak{k}})}{\otimes} \mathbf{C}_{t_0'\lambda} \longrightarrow \mathbf{C}_\Lambda \longrightarrow 0 \end{aligned} \tag{4.3}$$

be as in Lemma 4.2. (Here $\mathfrak{b}$ plays the role of $\mathfrak{a}$ and $\mathfrak{b}_{\mathfrak{k}}$ that of $\mathfrak{m}$.)

Tensoring (4.3) with $\mathcal{G} = U(\mathfrak{g})$ over $U(\mathfrak{b})$ and "cancelling" the $U(\mathfrak{b})$'s which appear twice gives

$$\begin{aligned} 0 \longrightarrow \mathcal{G} \underset{U(\mathfrak{b}_{\mathfrak{k}})}{\otimes} (\wedge^m(\mathfrak{b}/\mathfrak{b}_{\mathfrak{k}}) \otimes \mathbf{C}_{t_0'\lambda}) \xrightarrow{\partial_\Lambda} \cdots \xrightarrow{\partial_\Lambda} \mathcal{G} \underset{U(\mathfrak{b}_{\mathfrak{k}})}{\otimes} ((\mathfrak{b}/\mathfrak{b}_{\mathfrak{k}}) \otimes \mathbf{C}_{t_0'\lambda}) \\ \xrightarrow{\partial_\Lambda} \mathcal{G} \underset{U(\mathfrak{b}_{\mathfrak{k}})}{\otimes} \mathbf{C}_{t_0'\lambda} \longrightarrow \mathcal{G} \underset{U(\mathfrak{b})}{\otimes} \mathbf{C}_\Lambda = M_\Lambda \longrightarrow 0 \end{aligned} \tag{4.4}$$

The formulas for the ∂_Λ are as before (with the obvious change of coefficients from $U(\mathfrak{b})$ to $\mathcal{G}$).

Now identify $\mathfrak{b}/\mathfrak{b}_{\mathfrak{k}}$ with $\mathfrak{p}_{t_0}^-$ as a $\mathfrak{b}_{\mathfrak{k}}$-module and note that (4.4) remains exact since $\mathcal{G}$ is a free $U(\mathfrak{b}_{\mathfrak{k}})$-module under right multiplication. This completes the proof.

For $0 \leqslant i \leqslant m$ set $N_{t_0,i} = \mathcal{K} \otimes_{U(\mathfrak{b}_{\mathfrak{k}})} (\wedge^i \mathfrak{p}_{t_0}^- \otimes \mathbf{C}_{t_0'\lambda})$. (Here $\mathcal{K} = U(\mathfrak{k})$ the universal enveloping algebra of $\mathfrak{k}$.)

Then $\mathcal{G} \otimes_{U(\mathfrak{b}_{\mathfrak{k}})} (\wedge^i \mathfrak{p}_{t_0}^- \otimes \mathbf{C}_{t_0'\lambda}) \equiv \mathcal{G} \otimes_{\mathcal{K}} N_{t_0,i}$. We can rephrase Lemma 4.3 as:

LEMMA 4.4. *There exists an exact sequence*

$$\begin{aligned} 0 \longrightarrow \mathcal{G} \underset{\mathcal{K}}{\otimes} N_{t_0,m} \xrightarrow{\partial_\Lambda} \mathcal{G} \underset{\mathcal{K}}{\otimes} N_{t_0,m-1} \xrightarrow{\partial_\Lambda} \cdots \xrightarrow{\partial_\Lambda} \mathcal{G} \underset{\mathcal{K}}{\otimes} N_{t_0,1} \xrightarrow{\partial_\Lambda} \\ \mathcal{G} \underset{\mathcal{K}}{\otimes} V_{t_0'\lambda} \longrightarrow M_\Lambda \longrightarrow 0 \quad (\textit{here } V_{t_0'\lambda} = V_{\mathfrak{k}, P_{\mathfrak{k}}, t_0'\lambda} = N_{t_0,0}). \end{aligned}$$

Let $\mathcal{G}_0 = \mathbf{C}1 \subset \mathcal{G}_1 \subset \mathcal{G}_2 \subset \ldots$ be the canonical filtration of $\mathcal{G}$ ($\mathcal{G}_j$ is the subspace of $\mathcal{G}$ spanned by j or less products of elements of $\mathfrak{g}$). The following lemma is an easy consequence of the definition of the ∂_Λ.

LEMMA 4.5.

(i) $\partial_\Lambda(\mathcal{G}_j \cdot (1 \otimes N_{t_0,i}) \subset \mathcal{G}_{j+1}(1 \otimes N_{t_0,i-1})$ *for* $j = 0, 1, \ldots$.

(ii) ∂_Λ: $1 \otimes N_{t_0,i} \to \mathcal{G}_1 \cdot (1 \otimes N_{t_0,i-1})$ *is injective.*

Note. $\mathcal{G}_1 \cdot (1 \otimes N_{t_0,i-1})$ as a $\mathfrak{k}$-module is isomorphic with $N_{t_0,i-1} \oplus (\mathfrak{p} \otimes N_{t_0,i-1})$ $\mathfrak{k}$ acting on $\mathfrak{p}$ by ad.

5. Resolutions for the $W_{s,\Lambda}$

In this section we show that if Λ is sufficiently regular, the resolution of $W_{t_0,\Lambda}$ can be lifted to a resolution of $W_{s,\Lambda}$ for each $s \in W_{\mathfrak{k}}$. This in particular will give a new definition of the $W_{s,\Lambda}$. In order to carry out the lifting of resolutions, we need a few more results about $\mathfrak{k}$-Verma modules ($\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{h}_1$, $\mathfrak{h}$ as in § 4).

If M is a $\mathfrak{k}$-module and $\nu \in \mathfrak{h}_1^*$ set $M[\nu] = \{w \in M \mid h \cdot w = \nu(h) w$ for all $h \in \mathfrak{h}_1\}$. Set $M^{\mathfrak{n}_{\mathfrak{k}}} = \{w \in M \mid \mathfrak{n}_{\mathfrak{k}} \cdot w = 0\}$ and set $M^{\mathfrak{n}_{\mathfrak{k}}}[\nu] = M^{\mathfrak{n}_{\mathfrak{k}}} \cap M[\nu]$. Set as usual $V_\lambda = V_{\mathfrak{k}, P_{\mathfrak{k}}, \lambda}$.

LEMMA 5.1. *Let $\nu \in \mathfrak{h}_1^*$ and suppose that V_ν is irreducible* (*this is equivalent to* $\dim V_\nu^{\mathfrak{n}_{\mathfrak{k}}} = 1$). *Let F be a finite dimensional $\mathfrak{k}$-module. If $\mu \in \mathfrak{h}_1^*$ then*

$$\dim (V_\nu \otimes F)^{\mathfrak{n}_{\mathfrak{k}}}[\mu] \leqslant \dim F[\mu - \nu].$$

Proof. Put a lexicographic order on $(\mathfrak{h}_1)^*_{\mathbf{R}}$ which gives $P_{\mathfrak{k}}$ as the positive roots. Let $f_1, \ldots, f_d$ be a basis of F so that $f_i \in F[\xi_i]$ and $\xi_1 \geqslant \xi_2 \geqslant \ldots \geqslant \xi_d$. If $v \in (V_\nu \otimes F)^{\mathfrak{n}_{\mathfrak{k}}}[\mu]$ then $v = \sum_{i=1}^d v_i \otimes f_i$ and $v_i \in V_\nu[\mu - \xi_i]$. Let i_0 be the largest i so that $v_i \neq 0$. If $\alpha \in P_{\mathfrak{k}}$ and $X_\alpha \in \mathfrak{k}_\alpha$, then

$$0 = X_\alpha v = \sum X_\alpha v_i \otimes f_i + \sum v_i \otimes X_\alpha f_i.$$

We note that for each i, $X_\alpha f_i \in \sum_{j \leqslant i'} \mathbf{C} f_j$ where $i' < i$ and $\xi_{i'} \geqslant \xi_i + \alpha$.

Let $\lambda \in V_\nu^*$. Then

$$0 = (\lambda \otimes I)(X_\alpha v) = \sum \lambda(X_\alpha v_i) f_i + \sum \lambda(v_i) X_\alpha f_i.$$

Therefore, we see that if J is the set of indices j so that $\xi_j = \xi_{i_0}$ then $\sum_{j \in J} \lambda(X_\alpha v_j) f_j = 0$. Since the f_j are linearly independent, we see that $\lambda(X_\alpha v_j) = 0$ for $j \in J$, $\lambda \in V_\nu^*$. This implies that for each $j \in J$, $\alpha \in P_{\mathfrak{k}}$, $X_\alpha v_j = 0$. Let 1_ν be the fundamental generator of V_ν. The irreducibility of V_ν now implies that $v_j = c_j(v) 1_\nu$. This implies that $\xi_{i_0} = \mu - \nu$. Set $q(v) = \sum_{\xi_i = \mu - \nu} c_i(v) f_i = \sum_{j \in J} c_j(v) f_j$. Then q: $(V_\nu \otimes F)^{\mathfrak{n}_{\mathfrak{k}}}[\mu] \to F[\mu - \nu]$ is linear and injective.

Set $\mathfrak{n}_{\mathfrak{k}}^- = \sum_{\alpha \in P_{\mathfrak{k}}} \mathfrak{k}_{-\alpha}$.

LEMMA 5.2. *Let $\lambda \in \mathfrak{h}_1^*$ be $P_{\mathfrak{k}}$-dominant integral. If $s \in W_{\mathfrak{k}}$ we have by convention $V_{s' \cdot \lambda} \subseteq V_\lambda$. Let F be a finite dimensional $\mathfrak{k}$-module and assume that $\mu \in \mathfrak{h}_1^*$ is $P_{\mathfrak{k}}$-dominant integral. Then*

(i) $\dim (F \otimes V_{s' \cdot \lambda})^{\mathfrak{n}_{\mathfrak{k}}}[s' \cdot \mu] = \dim (F \otimes V_\lambda)^{\mathfrak{n}_{\mathfrak{k}}}[\mu] = \dim F[\mu - \lambda]$

(ii) *If $r, s \in W_{\mathfrak{k}}$ with $r > s$, then there exists an element $d_{s,r}(\mu)$ in $U(\mathfrak{n}_{\mathfrak{k}}^-)$ depending only on μ, r and s which induces a bijection from $(F \otimes V_{r' \cdot \lambda})^{\mathfrak{n}_{\mathfrak{k}}}[r' \cdot \mu]$ onto $(F \otimes V_{s' \cdot \lambda})^{\mathfrak{n}_{\mathfrak{k}}}[s' \cdot \mu]$, and*

(iii) *If $r, s \in W_{\mathfrak{k}}$ with $r > s$, then $d_{sr^{-1},1}(\mu)$ induces a bijection from $(F \otimes V_{r' \cdot \lambda})^{\mathfrak{n}_{\mathfrak{k}}}[\mu]$ onto $(F \otimes V_{s' \cdot \lambda})^{\mathfrak{n}_{\mathfrak{k}}}[(sr^{-1})' \cdot \mu]$. Also the dimension o these spaces is* $\dim F[\mu - r' \cdot \lambda]$.

Proof. We first note that if $s \in W_{\mathfrak{k}}$, if α is a $P_{\mathfrak{k}}$-simple element of $P_{\mathfrak{k}}$ and if $l(s_\alpha s) = l(s)+1$, then there is $n \geqslant 0$, $n \in \mathbf{Z}$ so that $X_{-\alpha}^n 1_{s'\lambda} = c 1_{(s_\alpha s)'\lambda}$, $c \neq 0$ (here 1_ν denotes the canonical generator for V_ν). Thus under this condition if $w \in F \otimes V_{s'\lambda}$, then there is $m \in \mathbf{Z}$, $m \geqslant 0$ so that $X_{-\alpha}^m w \in F \otimes V_{(s_\alpha s)'\lambda}$.

Using this observation we see that if s, $s_\alpha s$ are as above, and μ is as in the statement of the lemma and if $n = 2\langle s'\mu + \delta_{\mathfrak{k}}, \alpha\rangle / \langle \alpha, \alpha\rangle$ then

$$X_{-\alpha}^n \colon (F \otimes V_{s'\lambda})^{\mathfrak{n}_{\mathfrak{k}}}[s'\mu] \to (F \otimes V_{(s_\alpha s)'\lambda})^{\mathfrak{n}_{\mathfrak{k}}}[(s_\alpha s)'\mu]$$

is injective. We, therefore, see (by recurrance on this result) that

$$\dim (F \otimes V_\lambda)^{\mathfrak{n}_{\mathfrak{k}}}[\mu] \leqslant \dim (F \otimes V_{s'\lambda})^{\mathfrak{n}_{\mathfrak{k}}}[s'\mu] \leqslant \dim (F \otimes V_{t_0'\lambda})^{\mathfrak{n}_{\mathfrak{k}}}[t_0'\mu] \tag{5.1}$$

for all $s \in W_{\mathfrak{k}}$ and $\mu \in \mathfrak{h}_1^*$, $P_{\mathfrak{k}}$-dominant integral.

Now $V_{t_0'\lambda}$ is irreducible. Thus Lemma 5.1 implies $\dim (F \otimes V_{t_0'\lambda})^{\mathfrak{n}_{\mathfrak{k}}}[t_0'\mu] \leqslant \dim F[t_0(\mu - \lambda)] = \dim F[\mu - \lambda]$. Also $\dim (F \otimes V_\lambda)^{\mathfrak{n}_{\mathfrak{k}}}[\mu] = \dim F[\mu - \lambda]$ since μ is $P_{\mathfrak{k}}$-dominant integral (we leave this to the reader, it follows easily from 7.6.14, p. 241 of [3] and infinitesimal character considerations). But then $\dim F[\mu - \lambda] \leqslant \dim (F \otimes V_{s'\lambda})^{\mathfrak{n}_{\mathfrak{k}}}[s'\mu] \leqslant \dim F[\mu - \lambda]$ for all $s \in W_{\mathfrak{k}}$ by (5.1). We have thus proven (i).

To prove (ii), we choose for $r > s$ and μ $P_{\mathfrak{k}}$-dominant integral the unique element (up to scalar multiple) of $U(\mathfrak{n}_{\mathfrak{k}}^-)$, $d_{s,r}(\mu)$, so that $d_{s,r}(\mu) \cdot 1_{r'\mu} = 1_{s'\mu}$. Then $d_{s,r}(\mu) \cdot (F \otimes V_{r'\lambda})^{\mathfrak{n}_{\mathfrak{k}}}[r'\mu] \subset (F \otimes V_{r'\lambda})^{\mathfrak{n}_{\mathfrak{k}}}[s'\mu]$.

Now it is clear by the proof of (i) that

$$d_{t_0,r}(\mu) \cdot (F \otimes V_{r'\lambda})^{\mathfrak{n}_{\mathfrak{k}}}[r'\mu] = (F \otimes V_{t_0'\lambda})^{\mathfrak{n}_{\mathfrak{k}}}[t_0'\mu].$$

It is also easily seen that $d_{t_0,s}(\mu) d_{s,r}(\mu) = c d_{t_0,r}(\mu)$ for some $c \neq 0$. Thus if $W = d_{s,r}(\mu)(F \otimes V_{r'\lambda})^{\mathfrak{n}_{\mathfrak{k}}}[r'\mu]$, we see that $d_{t_0,s}(\mu)(W) = (F \otimes V_{t_0'\lambda})^{\mathfrak{n}_{\mathfrak{k}}}[t_0'\mu]$. But then $d_{t_0,s}(\mu)(F \otimes V_{s'\lambda})^{\mathfrak{n}_{\mathfrak{k}}}[s'\mu] = d_{t_0,s}(\mu)(W)$. Since $d_{t_0,s}(\mu)$ acts injectively on $F \otimes V_\lambda$, we see that

$$W = (F \otimes V_{s'\lambda})^{\mathfrak{n}_{\mathfrak{k}}}[s'\mu].$$

This proves (ii). A similar dimension argument proves (iii).

LEMMA 5.3. *Let N be a finite dimensional $\mathfrak{b}_{\mathfrak{k}}$-module which is semi-simple as an $\mathfrak{h}_1$-module. Let $N = \sum_\mu N[\mu]$ ($N[\mu]$ is as usual the μ-weight space for N relative to $\mathfrak{h}_1$). Set $\delta_{\mathfrak{k}} = \frac{1}{2}\sum_{\alpha \in P_{\mathfrak{k}}} \alpha$. Suppose that there is a system of positive roots $Q_{\mathfrak{k}}$ for $\Delta_{\mathfrak{k}}$ so that if $N[\mu] \neq 0$, then $\langle \mu + \delta_{\mathfrak{k}}, \alpha\rangle \geqslant 0$ for all $\alpha \in Q_{\mathfrak{k}}$ ($Q_{\mathfrak{k}}$ is not necessarily $P_{\mathfrak{k}}$). Then $\mathcal{K} \otimes_{U(\mathfrak{b}_{\mathfrak{k}})} N$ splits into a direct sum of Verma modules V_μ each counted with multiplicity equal to* $\dim N[\mu]$.

Proof. Using the exactness of the tensor product over $U(\mathfrak{b}_\mathfrak{k})$ we see that $\mathcal{K}\otimes_{U(\mathfrak{b}_\mathfrak{k})}N$ has a composition series

$$\mathcal{K}\underset{U(\mathfrak{b}_\mathfrak{k})}{\otimes} N = M_1\supset M_2\supset \ldots \supset M_d\supset M_{d+1}=(0)$$

where $M_i/M_{i+1}\equiv V_{\mu_i}$ $i=1, ..., d$ and the number of i such that $\mu_i=\mu$ is exactly equal to $\dim N[\mu]$. Let η_μ be the infinitesimal character of V_μ for $\mu\in\mathfrak{h}_1^*$. Then we see that $\mathcal{K}\otimes_{U(\mathfrak{b}_\mathfrak{k})}N=M_1=\sum (M_1)_{\eta_\mu}$ the sum over all $\mu\in\mathfrak{h}_1^*$ so that $N[\mu]\neq 0$. Here $(M_1)_{\eta_\mu}$ is the subspace of M_1 where if $z\in\mathfrak{Z}_\mathfrak{k}$, the center of $\mathcal{K}$, $z-\eta_\mu(z)$ acts nilpotently. Next we show:

(a) If μ_1, $\mu_2\in\mathfrak{h}_1^*$ if $N[\mu_i]\neq 0$, $i=1, 2$ and if $\mu_1\neq\mu_2$, then $\eta_{\mu_1}\neq\eta_{\mu_2}$. If $\eta_{\mu_1}=\eta_{\mu_2}$ then $\mu_2=s'\mu_1$ for some $s\in W_\mathfrak{k}$. Thus $\mu_2+\delta_\mathfrak{k}=s(\mu_1+\delta_\mathfrak{k})$. But $\mu_2+\delta_\mathfrak{k}$ and $\mu_1+\delta_\mathfrak{k}$ are $Q_\mathfrak{k}$-dominant. This implies $\mu_1=\mu_2$.

(a) implies:

(b) If $\mu_j\neq\mu$, then $(M_j)_{\eta_\mu}=(M_{j+1})_{\eta_\mu}$.

(c) If $\mu_j\neq\mu$ for $j\geqslant i$, then $(M_j)_{\eta_\mu}=0$.

Using (b) and (c), we see that $(M_1)_{\eta_\mu}$ has a composition series $H_1\supset H_2\supset \ldots\supset H_q\supset H_{q+1}=(0)$ with $H_i/H_{i+1}\equiv V_\mu$ and $q=\dim N[\mu]$. But then $\dim H_1[\mu]=q$, and μ is the highest weight of H_1. Hence $\mathfrak{n}_\mathfrak{k}\cdot H_1[\mu]=0$. But then $\mathcal{K}\cdot H_1[\mu]$ is a sum of q copies of V_μ since $H_i/H_{i+1}\cong V_\mu$, $1\leqslant i\leqslant q$, $\mathcal{K}\cdot H_1[\mu]=H_1$ and $H_1[\mu]$ contains a basis of vectors which are linearly independent over the ring $U(\mathfrak{n}_\mathfrak{k}^-)$. From this we see that the sum mentioned above must be direct. The lemma now follows.

We now assume that $(\mathfrak{g}, \mathfrak{k})$ is a regular pair. We let $\mathfrak{h}_1$, $\mathfrak{h}$, Δ and $\Delta_\mathfrak{k}$ have their usual meaning. Fix $P\subset\Delta$ an admissible system of positive roots. Let $P_\mathfrak{k}$ be the corresponding system of positive roots for $\Delta_\mathfrak{k}$.

Let $\mathfrak{p}$ be the orthogonal complement to $\mathfrak{k}$ in $\mathfrak{g}$ relative to the Killing form of $\mathfrak{g}$. If $\mu\in\mathfrak{h}_1^*$ let $\mathfrak{p}[\mu]=\{X\in\mathfrak{p}\,|\,\mathrm{ad}\,h\cdot X=\mu(h)X,\ h\in\mathfrak{h}_1\}$. Then $\mathfrak{p}[0]=\mathfrak{h}_2$ in the notation of § 4 and $\mathfrak{h}=\mathfrak{h}_1\oplus\mathfrak{h}_2$. Let P_n be the set of $\mu\in\mathfrak{h}_1^*$, $\mu=\beta|_{\mathfrak{h}_1}$, $\beta\in P$ and $\mathfrak{p}[\mu]\neq 0$ counted with multiplicity equal to $\dim\mathfrak{p}[\mu]$. Set $\mathfrak{p}^+=\sum\mathfrak{p}[\mu]$ the sum over all $\beta|_{\mathfrak{h}_1}$, $\beta\in P$.

Let $\Lambda\in\mathfrak{h}^*$ and suppose that $t_0'(\Lambda|_{\mathfrak{h}_1})=\lambda$ is $P_\mathfrak{k}$-dominant integral.

Definition 5.4. λ is said to be *strongly* $P_\mathfrak{k}$-dominant integral relative to P if λ and $\lambda-\mu$ are $P_\mathfrak{k}$-dominant integral for all weights μ of $\wedge\mathfrak{p}^+=\sum_j\wedge^j\mathfrak{p}^+$.

LEMMA 5.5. *Let $N_{t_0,i}$ be defined as in the material preceding Lemma 4.4. Suppose that λ is strongly $P_\mathfrak{k}$-dominant integral. Then*

$$N_{i,t_0} \cong \sum_{\mu} \oplus V_{t_0'(\lambda-\mu)}$$

the sum over all weights, μ, of $\wedge^i(\mathfrak{h}_2 \oplus \mathfrak{p}^+)$ with μ multiplicity equal to dim $\wedge^i(\mathfrak{h}_2 \oplus \mathfrak{p}^+)[\mu]$.

Proof. This is an immediate consequence of Lemma 5.3 (use $Q_{\mathfrak{k}} = t_0 P_{\mathfrak{k}} = -P_{\mathfrak{k}}$).

In what follows, we will use two more notational conventions in addition to Convention 1 of § 3.

Convention 2. If N is a $\mathfrak{k}$-module and $M \subset N$ is a $\mathfrak{k}$-submodule, then we identify $\mathcal{G} \otimes_{\mathfrak{K}} M$ with its image in $\mathcal{G} \otimes_{\mathfrak{K}} N$.

Convention 3. When an arrow is not labeled it will be an inclusion in the sense of Convention 1 or 2. (In particular, it corresponds to a set theoretic inclusion.)

Let $\mu_1, ..., \mu_{r_i}$, $0 \leq i \leq m$ equal to the weights of $\wedge^i(\mathfrak{h}_2 \oplus \mathfrak{p}^+)$ counting multiplicity (in particular, $r_i = \dim \wedge^i(\mathfrak{h}_2 \oplus \mathfrak{p}^+)$). Then Lemma 5.5 says that

$$N_{t_0,i} \cong \sum_{i=1}^{r_i} \oplus V_{t_0'(\lambda-\mu_i)}.$$

We assume that $\Lambda \in \mathfrak{h}^*$, and $\lambda = t_0'(\Lambda|_{\mathfrak{h}_1})$ is strongly $P_{\mathfrak{k}}$-dominant integral. We then have for each $s \in W_{\mathfrak{k}}$, $1 \leq i \leq r_i$,

$$V_{t_0'(\lambda-\mu_i)} \subset V_{s'(\lambda-\mu_i)}.$$

Set $N_{s,i} = \sum_{i=1}^{r_i} \oplus V_{s'(\lambda-\mu_i)}$. We *identify* $N_{t_0,i}$ with $\sum_{i=1}^{r_i} V_{t_0'(\lambda-\mu_i)}$. Then we have $N_{t_0,i} \subset N_{s,i}$ for each $s \in W_{\mathfrak{k}}$. Also if $r, s \in W_{\mathfrak{k}}$ and $r < s$, then $N_{r,i} \subset N_{s,i}$.

LEMMA 5.6. *For each $s \in W_{\mathfrak{k}}$ and $1 \leq i \leq m$, there exists a unique $\mathfrak{k}$-module homomorphism d_Λ: $N_{s,i} \to \mathcal{G}_1 \cdot (1 \otimes N_{s,i-1}) \subset \mathcal{G} \otimes_{\mathfrak{K}} N_{s,i-1}$ so that if $r < s$, $r, s \in W_{\mathfrak{k}}$ the following diagram commutes*

$$\begin{tikzcd} N_{s,i} \arrow[r, "d_\Lambda"] & \mathcal{G}_1 \cdot (1 \otimes N_{s,i-1}) \\ N_{r,i} \arrow[u] \arrow[r, "d_\Lambda"] & \mathcal{G}_1 \cdot (1 \otimes N_{r,i-1}) \arrow[u] \end{tikzcd}$$

(see Convention 3).

Proof. Set $d_{\Lambda,t_0}(v) = \partial_\Lambda(1 \otimes v)$ for $v \in N_{t_0,i}$. Using Lemma 4.5 (and the note after it) and Lemma 5.2, we have for each $1 \leq j \leq r_i$

$$d_{t_0,s}(\lambda-\mu_j) \cdot (\mathcal{G}_1 \cdot (1 \otimes N_{s,i-1}))^{\mathfrak{n}_{\mathfrak{k}}}[s'(\lambda-\mu_j)] = (\mathcal{G}_1 \cdot (1 \otimes N_{t_0,i-1}))^{\mathfrak{n}_{\mathfrak{k}}}[t_0'(\lambda-\mu_j)].$$

In fact $d_{t_0,s}(\lambda-\mu_j)$ gives a bijection between these two sets.

Let $d_{\Lambda,s}$: $V_{s'(\lambda-\mu_j)} \to \mathcal{G}_1 \cdot (1 \otimes N_{s,i-1})$ be defined so that

(a) $d_{t_0,s}(\lambda-\mu_j) \cdot d_{\Lambda,s}(V_{s'(\lambda-\mu_j)})^{\mathfrak{n}_\mathfrak{k}}[s'(\lambda-\mu_j)] = d_{\Lambda,t_0}(V_{t_0'(\lambda-\mu_j)})^{\mathfrak{n}_\mathfrak{k}}[t_0'(\lambda-\mu_j)]$ and

(b) The restriction of $d_{\Lambda,s}$ from $V_{s'(\lambda-\mu_j)}$ to $V_{t_0'(\lambda-\mu_j)}$ is the original d_{Λ,t_0} restricted to $V_{t_0'(\lambda-\mu_j)}$.

From this we get

$$d_{\Lambda,s}\colon N_{s,i} \to \mathcal{G}_1 \cdot (1 \otimes N_{s,i-1})$$

and (a) and (b) guarantee that

(c)
$$\begin{array}{ccc} N_{s,i} & \xrightarrow{d_{\Lambda,s}} & \mathcal{G}_1 \cdot (1 \otimes N_{s,i-1}) \\ \uparrow & & \uparrow \\ N_{t_0,i} & \xrightarrow{d_{\Lambda,t_0}} & \mathcal{G}_1 \cdot (1 \otimes N_{t_0,i-1}) \end{array}$$

commutes for all $s \in W_\mathfrak{k}$, and the maps $d_{\Lambda,s}$ are injections.

Suppose now that $r < s$, $s \in W_\mathfrak{k}$. Using the fact that $d_{t_0,s}(\lambda-\mu_i) = c d_{t_0,r}(\lambda-\mu_i) \cdot d_{r,s}(\lambda-\mu_i)$, $c \neq 0$, we see that

$$d_{\Lambda,s}(N_{s,i}) \supset d_{\Lambda,s}(N_{r,i}) \supset d_\Lambda(N_{t_0,i}).$$

If $v \in N_{r,i}$, there is $u \in U(\mathfrak{n}_\mathfrak{k}^-)$, $u \neq 0$ so that $u \cdot v \in N_{t_0,i}$. Thus if $v \in N_{r,i}$ then $d_{\Lambda,s}(u \cdot v) = d_{\Lambda,t_0}(u \cdot v)$ by (c) and $d_{\Lambda,r}(u \cdot v) = d_{\Lambda,t_0}(u \cdot v)$ by (c). Hence $u \cdot d_{\Lambda,s}(v) = u \cdot d_{\Lambda,r}(v)$. But u acts injectively on $\mathcal{G} \otimes_\mathfrak{K} N_{1,i} \supset \mathcal{G} \otimes_\mathfrak{K} N_{s,i} \supset \mathcal{G} \otimes_\mathfrak{K} N_{r,i} \supset \mathcal{G} \otimes_\mathfrak{K} N_{t_0,i}$ for each i. Thus $d_{\Lambda,s}(v) = d_{\Lambda,r}(v)$. An identical argument shows that $d_{\Lambda,s}$ is unique.

We can thus set $d_\Lambda = d_{\Lambda,1}$ and $d_\Lambda|_{N_{s,i}} = d_{\Lambda,s}$. The lemma now follows.

Define for each $1 \leq i \leq m$, a $\mathfrak{g}$-homomorphism ∂_Λ:

$$\partial_\Lambda : \mathcal{G} \otimes_\mathfrak{K} N_{s,i} \longrightarrow \mathcal{G} \otimes_\mathfrak{K} N_{s,i-1}$$

by letting ∂_Λ be the canonical extension from $N_{s,i}$ to $\mathcal{G} \otimes_\mathfrak{K} N_{s,i}$ of the $\mathfrak{k}$-homomorphism

$$d_\Lambda\colon N_{s,i} \to \mathcal{G}_1 \cdot (1 \otimes N_{s,i-1}).$$

Set $E_{s,i,\lambda} = \mathcal{G} \otimes_\mathfrak{K} N_{s,i}$. We can now state the main result of this section.

THEOREM 5.7. *Let $\Lambda \in \mathfrak{h}^*$, $\lambda = t_0'(\Lambda|_{\mathfrak{h}_1})$, and assume λ is strongly $P_\mathfrak{k}$-dominant integral. Then*

(i) *$W_{s,\Lambda}$ is equivalent as a $\mathfrak{g}$-module with*

$$E_{s,0,\lambda}/\partial_\Lambda(E_{s,1,\lambda})$$

for each $s \in W_{\mathfrak{k}}$.

(ii) *Identifying $E_{s,0,\lambda}/\partial_\Lambda(E_{s,1,\lambda})$ with $W_{s,\Lambda}$ then the following sequence is exact*

$$0 \longrightarrow E_{s,m,\mu} \xrightarrow{\partial_\Lambda} E_{s,m-1,\lambda} \xrightarrow{\partial_\Lambda} \cdots \xrightarrow{\partial_\Lambda} E_{s,0,\lambda} \xrightarrow{\varepsilon} W_{s,\Lambda} \longrightarrow 0$$

Here ε: $E_{s,0,\lambda} \to E_{s,0,\lambda}/\partial_\Lambda(E_{s,1,\lambda})$ is the canonical map.

(iii) *For each i, $1 \leqslant i \leqslant m$, $r < s$, the following diagram is commutative*

$$\begin{tikzcd} E_{s,i,\lambda} \arrow[r, "\partial_\Lambda"] & E_{s,i-1,\lambda} \\ E_{r,i,\lambda} \arrow[r, "\partial_\Lambda"] \arrow[u] & E_{r,i-1,\lambda} \arrow[u] \end{tikzcd} \tag{5.2}$$

and this diagram induces the inclusions among the $W_{s,\Lambda}$.

Note. The $E_{s,i,\lambda}$ depend only on λ, the ∂_Λ depend on Λ.

We will need two simple lemmas for the proof of Theorem 5.7.

LEMMA 5.8. *Let V be a d-dimensional vector space over $\mathbf{C}$ ($d < \infty$), and let $A \in \operatorname{End}(V)$ be nilpotent. Let W be a vector space over $\mathbf{C}$, and let $B \in \operatorname{End}(W)$ be surjective. Define C: $V \otimes W \to V \otimes W$ by $C(v \otimes w) = Av \otimes w + v \otimes Bw$. Then C is surjective.*

Proof. By induction on d. If $d = 1$, then $A = 0$, and the result is clear. Suppose that $d > 1$, and that the result is true for $d-1$. Then $AV \neq V$. Hence there is a subspace V_1 of V so that $V_1 \supset AV$ and $\dim V_1 = d-1$.

$AV_1 \subset V_1$. Hence the inductive hypothesis implies that $C(V \otimes W) \supset V_1 \otimes W$. If $v \in V$ and $w \in W$, then $w = Bu$, $u \in W$. $C(v \otimes u) = Av \otimes u + v \otimes w$. Since $Av \in V_1$, $v \otimes w \in \operatorname{Im} C$. Since the $v \otimes w$, $v \in V$, $w \in W$ span $V \otimes W$, the lemma follows.

LEMMA 5.9. *If $\lambda \in \mathfrak{h}_{\mathrm{I}}^*$ and if $\alpha \in P_{\mathfrak{k}}$ is such that $2\langle\lambda, \alpha\rangle/\langle\alpha, \alpha\rangle = -n$, $n > 0$ $n \in \mathbf{Z}$, and if $X \in \mathfrak{k}_\alpha$, $X \neq 0$, then X is surjective on $\mathcal{G} \otimes_{U(\mathfrak{b}_{\mathfrak{k}})} \mathbf{C}_\lambda$.*

Proof. Let $v_0 = 1 \otimes 1 \in \mathcal{G} \otimes_{U(\mathfrak{b}_{\mathfrak{k}})} \mathbf{C}_\lambda$. Let $Y \in \mathfrak{k}_{-\alpha}$ be so that $[X, Y] = H$, $\alpha(H) = 2$. If $\mathfrak{a} = \mathbf{C}H + \mathbf{C}X + \mathbf{C}Y$ then $U(\mathfrak{a})v_0 = W$ is the Verma module for $\mathfrak{a}$ with highest weight $-n$. It is easy to see that X acts surjectively on W. Let $U_j(\mathfrak{g}) \subset U_{j+1}(\mathfrak{g})$ be the standard filtration of $U(\mathfrak{g})$. Let X act on $U(\mathfrak{g})$ by $X \cdot u = [X, u]$. Then $X \cdot U_j(\mathfrak{g}) \subset U_j(\mathfrak{g})$ for all j and X is a nilpotent transformation. Clearly $U_j(\mathfrak{g})\cdot v_0 \subset U_{j+1}(\mathfrak{g})v_0$ and $\bigcup_{j=0}^{\infty} U_j(\mathfrak{g})v_0 = \mathcal{G}\otimes_{U(\mathfrak{b}_{\mathfrak{k}})}\mathbf{C}_\lambda$. Let X act on $U_j(\mathfrak{g})\otimes W$ by the tensor product action. Then Lemma 5.8 implies that $X\cdot(U_j(\mathfrak{g})\otimes W) = U_j(\mathfrak{g})\otimes W$. But if $\psi(g\otimes w) = gw$, $g\in U(\mathfrak{g})$, $w\in W$, then $\psi(X\cdot(g\otimes w)) = Xgw$. If $u\in \mathcal{G}\otimes_{U(\mathfrak{b}_{\mathfrak{k}})}\mathbf{C}_\lambda$, then $u\in U_j(\mathfrak{g})v_0$ for some j. Hence $u=\psi(h)$, $h\in U_j(\mathfrak{g})\otimes W$. $h = X\cdot h_1 h_1 \in U_j(\mathfrak{g})\otimes W$. Thus $u=\psi(X\cdot h_1) = X\psi(h_1)$. Hence $u\in \operatorname{Im} X$. Q.E.D.

We now begin the proof of Theorem 5.7. We note that the first part of (iii) is obvious from the definitions. If $v\in E_{s,i,\lambda}$, then there is $u\in U(\mathfrak{n}_{\mathfrak{k}}^-)$, $u\neq 0$ so that $u\cdot v\in E_{t_0,i,\lambda}$. The sequence at the t_0 level is exact, hence if $i\geqslant 2$, $0=\partial_\Lambda^2(u\cdot v) = u\cdot\partial_\Lambda^2(v)$. Now u acts injectively, hence $\partial_\Lambda^2(v)=0$. This implies that the sequence

$$0 \longrightarrow E_{s,m,\lambda} \xrightarrow{\partial_\Lambda} E_{s,m-1,\lambda} \xrightarrow{\partial_\Lambda} \cdots \xrightarrow{\partial_\Lambda} E_{s,0,\lambda} \tag{5.3}$$

is a complex.

Let $m_0 = l(t_0) = |P_{\mathfrak{k}}|$. We prove the exactness of (5.3) by induction on $l'(s) = m_0 - l(s)$.

If $l'(s)=0$, then $s=t_0$, and the sequence is exact by Lemma 4.4. Suppose that (5.3) has been shown to be exact for $0\leqslant l'(s) < p$. Let $s\in W_{\mathfrak{k}}$, $l'(s)=p$. Let α be a $P_{\mathfrak{k}}$-simple root so that $l(s_\alpha s) = l(s)+1$. Thus $l'(s_\alpha s) = p-1$. We have the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & E_{s,m,\lambda} \arrow[r, "\partial_\Lambda"] & E_{s,m-1,\lambda} \arrow[r, "\partial_\Lambda"] & \cdots \arrow[r, "\partial_\Lambda"] & E_{s,0,\lambda} \\ 0 \arrow[r] & E_{s_\alpha s,m,\lambda} \arrow[r, "\partial_\Lambda"] \arrow[u] & E_{s_\alpha s,m,\lambda} \arrow[r, "\partial_\Lambda"] \arrow[u] & \cdots \arrow[r, "\partial_\Lambda"] & E_{s_\alpha s,0,\lambda} \arrow[u] \end{tikzcd} \tag{5.4}$$

with the bottom row exact.

Let $\mathfrak{a} = \mathfrak{k}_\alpha + \mathfrak{k}_{-\alpha} + [\mathfrak{k}_\alpha, \mathfrak{k}_{-\alpha}]$. Then $E_{s,j,\lambda}/E_{s_\alpha s,j,\lambda}$ consists of $\mathfrak{a}$-finite vectors. Fix $X_\alpha\in\mathfrak{k}_\alpha$, $X_{-\alpha}\in\mathfrak{k}_{-\alpha}$ so that $[X_\alpha, X_{-\alpha}] = H_\alpha$ and $\alpha(H_\alpha)=2$. Set $\mathfrak{L} = \mathfrak{a}+\mathfrak{b}_{\mathfrak{k}}$. Then $\mathfrak{L}$ is a parabolic subalgebra of $\mathfrak{k}$. Let $\tilde{N}_{s,j}$ be the $\mathfrak{L}$-submodule of $N_{s,j}$ generated over $\mathfrak{L}$ by the canonical generators of the Verma modules $V_{s'(\lambda-\mu_i)}$, $i\subseteq 1, \ldots, r_j$ (see the definition of $N_{s,j}$). Then $\tilde{N}_{s,j}$ is a direct sum of $\mathfrak{a}$-Verma modules.

There is a natural isomorphism as $\mathfrak{g}$-modules between

$$\mathcal{G}\otimes_{U(\mathfrak{L})}\tilde{N}_{s,j} \quad\text{and}\quad \mathcal{G}\otimes_{\mathcal{K}} N_{s,j} = E_{s,j,\lambda}$$

Set $M_j = \mathcal{G}_j U(\mathfrak{L})$. Then as an $\mathfrak{a}$-module

$$M_j\otimes_{U(\mathfrak{L})}\tilde{N}_{i,s} = S_j(\mathfrak{g}/\mathfrak{L})\otimes\tilde{N}_{i,s}$$

(here $S^i(V)$ is the ith homogeneous component of the symmetric algebra on V and $S_j(V) = \sum_{i\leqslant j} S^i(V)$).

From this we see, using Lemma 5.2, that

(a) If $v \in E_{s_\alpha s, j, \lambda}$ and $H_\alpha v = -(n+2)v$, $X_\alpha v = 0$ for some $n \geqslant 0$, $n \in \mathbf{Z}$, then there is $v_1 \in E_{s,j,\lambda}$ so that $X_{-\alpha}^{n+1} v_1 = v$ and $H_\alpha v_1 = nv_1$, $X_\alpha v_1 = 0$.

Using the $\mathfrak{a}$-finiteness of any element of $E_{s,j,\lambda}/E_{s_\alpha s,j,\lambda}$, the top sequence in (5.3) will be exact if we can show that if $v \in E_{s,j,\lambda}$, $H_\alpha v = nv$, $n \geqslant 0$, $n \in Z$, $X_\alpha v = 0$ and $\partial_\Lambda v = 0$, then $v = \partial w$ for some $w \in E_{s,j-1,\lambda}$.

To prove this, we note that if $X_{-\alpha}^{n+1} v = u$ then $H_\alpha u = -(n+2)u$, $X_\alpha u = 0$ and $u \in E_{s_\alpha s, j, \lambda}$. Furthermore, $\partial_\Lambda u = 0$. The exactness of the bottom sequence of (5.4) implies that $u = \partial_\Lambda w_1$, $w_1 \in E_{s_\alpha s, j+1, \lambda}$. We may assume $H_\alpha w_1 = -(n+2)w_1$. Now $\partial_\Lambda(X_\alpha w_1) = X_\alpha \partial_\Lambda(w_1) = X_\alpha u = 0$. Thus $X_\alpha w_1 = \partial_\Lambda z$, $z \in E_{s_\alpha s, j+2, \lambda}$. Lemma 5.9 implies that X_α acts surjectively on $E_{s_\alpha s, j+2, \lambda}$. Hence $z = X_\alpha z_1$, $z_1 \in E_{s_\alpha s, j+2, \lambda}$. We may assume $H_\alpha z = -nz$ and $H_\alpha z_1 = -(n+2)z_1$. Set $w_2 = w_1 - \partial_\Lambda z_1$. Then $X_\alpha w_2 = X_\alpha w_1 - X_\alpha \partial_\Lambda z_1 = X_\alpha w_1 - \partial_\Lambda X_\alpha z_1 = X_\alpha w_1 - \partial_\Lambda z = 0$. Thus $H_\alpha w_2 = -(n+2)w_2$, $X_\alpha w_2 = 0$. But then (a) implies $w_2 = X_{-\alpha}^{n+1} w$ with $w \in E_{s,j+1,\lambda}$, $H_\alpha w = nw$, $X_\alpha w = 0$. But now we see $X_{-\alpha}^{n+1} \partial w = X_{-\alpha}^{n+1} v$. Hence $X_{-\alpha}^{n+1}(\partial w - v) = 0$. Since $X_{-\alpha}$ acts injectively, $\partial w = v$. We have, therefore, proved the exactness of (5.3).

We next prove that if $\tilde{W}_{s,\Lambda} = E_{s,0,\lambda}/\partial_\Lambda(E_{s,1,\lambda})$ then $W_{s,\Lambda}$ is isomorphic with $W_{s,\Lambda}$; in the course of the proof of this, we will also prove the last part of (iii).

We note that $\tilde{W}_{t_0,\Lambda} \equiv M_\Lambda \equiv W_{t_0,\Lambda}$. We prove by induction on $l'(s) = l(t_0) - l(s)$ that $\tilde{W}'_{s,\Lambda} = \{w \in \tilde{W}_{s,\Lambda} \mid$ there exists $u \in U(\mathfrak{n}_{\mathfrak{k}}^-) - \{0\}$, $u \cdot w = 0\}$ is (0). If $l'(s) = 0$, then $s = t_0$ and the result is proved. Suppose that $s \in W_{\mathfrak{k}}$ $l'(s) = p$ and the result is known for $0 \leqslant l'(t) < p$. Let α be a $P_{\mathfrak{k}}$-simple root so that $l(s_\alpha s) = l(s) + 1$. Then $l'(s_\alpha s) = p - 1$. The commutivity of the diagram (5.2) implies that there is a homomorphism

$$\psi : \tilde{W}_{s_\alpha s, \Lambda} \longrightarrow \tilde{W}_{s,\Lambda}$$

so that

$$\begin{tikzcd}
{} \arrow[r, "\partial_\Lambda"] & E_{s,1,\lambda} \arrow[r, "\partial_\Lambda"] & E_{s,0,\lambda} \arrow[r, "\varepsilon"] & \tilde{W}_{s,\Lambda} \arrow[r] & 0 \\
{} \arrow[r, "\partial_\Lambda"] & E_{s_\alpha s,1,\lambda} \arrow[r, "\partial_\Lambda"] \arrow[u] & E_{s_\alpha s,0,\lambda} \arrow[r, "\varepsilon"] \arrow[u] & \tilde{W}_{s_\alpha s,\Lambda} \arrow[r] \arrow[u] & 0 \\
& 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd} \tag{5.5}$$

commutes, the rows are exact, the unlabeled arrows are (as per our conventions) inclusions. If $w \in \tilde{W}_{s_\alpha s,\Lambda}$ and $\psi(w) = 0$, then $w = \varepsilon(w_1)$, $w_1 \in E_{s_\alpha s,0,\lambda}$ and $w_1 = \partial_\Lambda(w_2)$, $w_2 \in E_{s,1,\lambda}$. There is

$n>0$, $n\in\mathbf{Z}$, so that $X_{-\alpha}^n w_2 \in E_{s_\alpha s,1,\lambda}$. Hence $\partial_\Lambda(X_{-\alpha}^n w_2)=X_{-\alpha}^n\partial_\Lambda w_2 = X_{-\alpha}^n w_1$. But then $0=\varepsilon(X_{-\alpha}^n w_1)=X_{-\alpha}^n w$. The inductive hypothesis implies $w=0$. We, therefore, see

(b) ψ: $\tilde{W}_{s_\alpha s,\Lambda}\to\tilde{W}_{s,\Lambda}$ is injective.

Using the fact that every element of $E_{s,0,\lambda}/E_{s_\alpha s,0,\lambda}$ is $\mathfrak{a}=\mathfrak{k}_\alpha+\mathfrak{k}_{-\alpha}+[\mathfrak{k}_\alpha,\mathfrak{k}_{-\alpha}]$ finite, we see that $\tilde{W}_{s,\Lambda}/\psi(\tilde{W}_{s_\alpha s,\Lambda})$ consists of $\mathfrak{a}$-finite vectors. Let X_α, $X_{-\alpha}$, H_α be as above.

Suppose that $w\in\tilde{W}'_{s,\Lambda}$, $w\neq 0$. We may assume $H_\alpha w=mw$. Let $X_\alpha^q w\neq 0$, $X_\alpha^{q+1}w=0$. Set $w_0=X_\alpha^q w$, $n=(m+2q)$. Then $H_\alpha w_0=nw_0$ and $X_\alpha w_0=0$. Then $w_0\in\tilde{W}'_{s,\Lambda}$. If n is not a non-negative integer then $w_0\in\psi(\tilde{W}_{s_\alpha s,\Lambda})$. But ψ is injective. Hence $w_0\in\psi(\tilde{W}'_{s_\alpha s,\Lambda})$ ($\tilde{W}'_{s,\Lambda}$ is a $\mathfrak{g}$-submodule of $\tilde{W}_{s,\Lambda}$). But then $w_0=0$. Hence we may assume $n\geqslant 0$, $n\in\mathbf{Z}$. There exists $w_1\in E_{s,0,\lambda}$ so that $H_\alpha w_1=nw_1$, $X_\alpha w_1=0$ and $\varepsilon(w_1)=w_0$. $X_{-\alpha}^{n+1}w_1\in E_{s_\alpha s,0,\lambda}$ and $\varepsilon(X_{-\alpha}^{n+1}w_1)\in\tilde{W}'_{s_\alpha s,\Lambda}=\{0\}$. Using (5.5) we see that $X_{-\alpha}^{n+1}w_1=\partial_\Lambda(w_2)$, $w_2\in E_{s_\alpha s,1,\lambda}$. We may assume $H_\alpha w_2=-(n+2)w_2$. Now $\partial_\Lambda(X_\alpha w_2)=X_\alpha\partial_\Lambda(w_2)=X_\alpha X_{-\alpha}^{n+1}w_1=0$. Thus $X_\alpha w_2=\partial_\Lambda(w_3)$, $w_3\in E_{s_\alpha s,2,\lambda}$ and $H_\alpha w_3=-nw_3$. Using Lemma 5.9, we see that $w_3=X_\alpha w_4$ with $w_4\in E_{s_\alpha s,2,\lambda}$ and $H_\alpha w_4=-(n+2)w_4$. Set $w_2'=w_2-\partial_\Lambda(w_4)$. Then $X_\alpha w_2'=X_\alpha w_2-X_\alpha\partial_\Lambda(w_4)=X_\alpha w_2-\partial_\Lambda(X_\alpha w_4)=X_\alpha w_2-\partial_\Lambda(w_3)=0$. Using (a) above, we see that $w_2'=X_{-\alpha}^{n+1}w_5$, $w_5\in W_{s,1,\lambda}$ and $H_\alpha w_5=nw_5$, $X_\alpha w_5=0$. But then $\partial_\Lambda(w_2')=X_{-\alpha}^{n+1}w_1$ and $\partial_\Lambda(w_2')=X_{-\alpha}^{n+1}\partial_\Lambda(w_5)$. This implies that $X_{-\alpha}^{n+1}(w_1-\partial_\Lambda(w_5))=0$. Since $X_{-\alpha}$ acts injectively on $E_{s,0,\lambda}$, we see $w_1=\partial_\Lambda(w_5)$. Hence $\varepsilon(w_1)=w_0=0$. This contradiction completes the induction.

We have proven

(c) If $u\in U(\mathfrak{n}_\mathfrak{k}^-)-\{0\}$ then u acts injectively on each $\tilde{W}_{s,\Lambda}$ and if $s\in W_\mathfrak{k}$, $\alpha\in P_\mathfrak{k}$, α simple in $P_\mathfrak{k}$ and if $l(s_\alpha s)=l(s)+1$, then the natural $\mathfrak{g}$-module homomorphism ψ: $\tilde{W}_{s_\alpha s,\Lambda}\to\tilde{W}_{s,\Lambda}$ is injective.

Set $Z_1=\tilde{W}_{1,\Lambda}$ and for $s\in W_\mathfrak{k}$ let ψ_s: $\tilde{W}_{s,\Lambda}\to\tilde{W}_{1,\Lambda}$ be the homomorphism coming from the commutative diagram 3. Using the fact that if $v\in E_{1,0,\lambda}$ then there exists $u\in U(\mathfrak{n}_\mathfrak{k}^-)-\{0\}$ so that $u\cdot v\in E_{s,0,\lambda}$ and the fact that $\tilde{W}'_{t,\Lambda}=(0)$ for all $t\in W_\mathfrak{k}$ we see that ψ_s is injective. Set $Z_s=\psi_s(\tilde{W}_{s,\Lambda})$. If r, $s\in W_\mathfrak{k}$ and $r<s$ then arguing as above, there is a $\mathfrak{g}$-module homomorphism $\psi_{s,r}$: $\tilde{W}_{r,\Lambda}\to\tilde{W}_{s,\Lambda}$. Using the commutativity of the following diagram

$$\begin{tikzcd}
E_{1,1,\lambda} \arrow[r, "\partial_\Lambda"] & E_{1,0,\lambda} \arrow[r, "\varepsilon"] & \tilde{W}_{1,\lambda} \arrow[r] & 0 \\
E_{s,1,\lambda} \arrow[u] \arrow[r, "\partial_\Lambda"] & E_{s,0,\lambda} \arrow[u] \arrow[r, "\varepsilon"] & \tilde{W}_{s,\lambda} \arrow[u, "\psi_s"'] \arrow[r] & 0 \\
E_{r,1,\lambda} \arrow[u] \arrow[r, "\partial_\Lambda"] & E_{r,0,\lambda} \arrow[u] \arrow[r, "\varepsilon"] & \tilde{W}_{r,\lambda} \arrow[u, "\psi_{s,r}"'] \arrow[r] & 0
\end{tikzcd}$$

we see that $\psi_s \circ \psi_{s,r} = \psi_r$. Hence if $r < s$, $Z_r \subset Z_s$. It is now clear that the family $\{Z_s\}_{s \in W_{\mathfrak{k}}}$ satisfies (i), (ii), (iii), (iv) of Theorem 3.1. This completes the proof of Theorem 5.7.

The result stated as Theorem 5.7 is not the strongest possible. Since the $\mathfrak{h}_1$ weight spaces of the modules $E_{s,i,\lambda}$, $s \in W_{\mathfrak{k}}$, $0 \leqslant i \leqslant m$ are infinite dimensional, in order to compute the dimensions of any special subspaces of $W_{s,\Lambda}$, it will be useful to have a filtered version of Theorem 5.7. For any non-negative integer j set $\mathcal{G}_j$ equal to the subspace of $\mathcal{G}$ spanned by 1 and all i-fold products of elements in $\mathfrak{g}$ with $i \leqslant j$. For negative integers j set $\mathcal{G}_j = \{0\}$. $\mathcal{G}_j \subseteq \mathcal{G}_{j+1}$ and this family of subspaces will be called the *standard filtration* of $\mathcal{G}$. For any integers i, j with $0 \leqslant i \leqslant m$ and any $s \in W_{\mathfrak{k}}$, set $E^j_{s,i,\lambda} = \mathcal{G}_j \cdot \mathcal{K} \otimes_{\mathcal{K}} N_{s,i}$. Let λ denote the symmetrizer map and if A is a vector space set $S(A)$ equal to the symmetric tensor algebra of A. For any integer k set $S_k(A)$ equal to the subspace of $S(A)$ of all homogeneous tensors of degree k and then set $S^k(A) = \sum_{0 \leqslant j \leqslant k} S_j(A)$. Since $\mathcal{G}_j \mathcal{K} = \mathcal{K} \mathcal{G}_j$, $E^j_{s,i,\lambda}$ is a $\mathfrak{k}$-module and in fact $\boldsymbol{\lambda} \otimes 1$ gives a $\mathfrak{k}$-module isomorphism from $S^j(\mathfrak{p}) \otimes N_{s,i}$ onto $E^j_{s,i,\lambda}$. We now prove the filtered version of Theorem 5.7 for $s = t_0$.

LEMMA 5.10. *Let $\Lambda \in \mathfrak{h}^*$, $\lambda = t_0'(\Lambda|_{\mathfrak{h}_1})$ and assume that λ is strongly $P_{\mathfrak{k}}$-dominant integral. Let j be an integer with $j \geqslant -m$. Then the following sequence of $\mathfrak{k}$-modules is exact:*

$$0 \longrightarrow E^j_{t_0,m,\lambda} \xrightarrow{\partial_\Lambda} E^{j+1}_{t_0,m-1,\lambda} \xrightarrow{\partial_\Lambda} \cdots \xrightarrow{\partial_\Lambda} E^{j+m}_{t_0,0,\lambda} \xrightarrow{\varepsilon} W^{j+m}_{t_0,\Lambda} \longrightarrow 0$$

where $W^{j+m}_{t_0,\Lambda}$ is by definition the image of $E^{j+m}_{t_0,0,\lambda}$.

Proof. Set $\mathfrak{p}^-_{t_0} = \mathfrak{h}_2 \oplus \sum_{\alpha \in t_0 P} \mathfrak{p}[-\alpha|_{\mathfrak{h}_1}]$, $\mathfrak{p}_{t_0} = \sum_{\alpha \in t_0 P} \mathfrak{p}[\alpha|_{\mathfrak{h}_1}]$. As usual for any integer i, $1 \leqslant i \leqslant m = \dim \mathfrak{p}^-_{t_0}$, let $\wedge^i(\mathfrak{p}^-_{t_0})$ denote the elements in the exterior algebra $\wedge(\mathfrak{p}^-_{t_0})$ of degree i. For $i = 0$ set $\wedge^i(\mathfrak{p}^-_{t_0}) = \mathbf{C}$, the trivial $\mathfrak{b}_{\mathfrak{k}}$-module. For $i = 1$ define a map ∂ from $S(\mathfrak{p}) \otimes \wedge^1(\mathfrak{p}^-_{t_0})$ into $S(\mathfrak{p})$ by extending linearly the map $f \otimes x \mapsto f \cdot x$, with $\cdot$ denoting multiplication of symmetric tensors. If $m \geqslant i \geqslant 2$, then define a map also called ∂ from $S(\mathfrak{p}) \otimes \wedge^i(\mathfrak{p}^-_{t_0})$ to $S(\mathfrak{p}) \otimes \wedge^{i-1}(\mathfrak{p}^-_{t_0})$ by extending linearly the map $f \otimes x_1 \wedge \ldots \wedge x_i \mapsto \sum_{1 \leqslant j \leqslant i} (-1)^{j+1} f \cdot x_j \otimes x_1 \wedge \ldots \wedge \hat{x}_j \wedge \ldots \wedge x_i$ where $f \in S(\mathfrak{p})$, $x_i \in \mathfrak{p}^-_{t_0}$ and $\hat{}$ denotes omission of the term. It is well-known, [2], that the maps ∂ are all $\mathfrak{b}_{\mathfrak{k}}$-module homomorphisms, $\partial^2 \equiv 0$, and the following sequence is exact for any integer j, $j \geqslant -m$:

$$0 \longrightarrow S^j(\mathfrak{p}) \otimes \wedge^m(\mathfrak{p}^-_{t_0}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} S^{j+m-1}(\mathfrak{p}) \otimes \mathfrak{p}^-_{t_0} \xrightarrow{\partial} S^{j+m}(\mathfrak{p}) \xrightarrow{\varepsilon} S^{j+m}(\mathfrak{p}_{t_0}) \longrightarrow 0$$

where we identify $\mathfrak{p}_{t_0}$ with $\mathfrak{p}/\mathfrak{p}^-_{t_0}$ and ε is the algebra map which extends the $\mathfrak{b}_{\mathfrak{k}}$-module projection $\mathfrak{p} \to \mathfrak{p}/\mathfrak{p}^-_{t_0}$. If $\nu \in \mathfrak{h}_1$ then let $\mathbf{C}_\nu$ be the one-dimensional module for $\mathfrak{b}_{\mathfrak{k}}$ corresponding to ν. Now tensoring $\mathbf{C}_\nu$ with the above exact sequence yields another exact sequence of

$\mathfrak{b}_{\mathfrak{k}}$-modules. Now tensor the resulting sequence on the left by $\mathcal{K}$ over the ring $U(\mathfrak{b}_{\mathfrak{k}})$ and note that since $\mathcal{K}$ is a free $U(\mathfrak{b}_{\mathfrak{k}})$-module under right multiplication, we obtain the exact sequence:

$$0 \longrightarrow \mathcal{K} \underset{U(\mathfrak{b}_{\mathfrak{k}})}{\otimes} S^j(\mathfrak{p}) \otimes \wedge^m(\mathfrak{p}_{t_0}^-) \otimes \mathbf{C}_\nu \xrightarrow{1\otimes\partial} \mathcal{K} \underset{U(\mathfrak{b}_{\mathfrak{k}})}{\otimes} S^{j+1}(\mathfrak{p}) \otimes \wedge^{m-1}(\mathfrak{p}_{t_0}^-) \otimes \mathbf{C}_\nu \xrightarrow{1\otimes\partial} \cdots$$
$$\xrightarrow{1\otimes\partial} \mathcal{K} \underset{U(\mathfrak{b}_{\mathfrak{k}})}{\otimes} S^{j+m}(\mathfrak{p}) \otimes \mathbf{C}_\nu \longrightarrow \mathcal{K} \underset{U(\mathfrak{b}_{\mathfrak{k}})}{\otimes} S^{j+m}(\mathfrak{p}_{t_0}) \otimes \mathbf{C}_\nu \longrightarrow 0$$

For any integer i with $0 \leqslant i \leqslant m$ let Φ be the $\mathfrak{k}$-module homomorphism from $\mathcal{K} \otimes_{U(\mathfrak{b}_{\mathfrak{k}})} S^j(\mathfrak{p}) \otimes \wedge^i(\mathfrak{p}_{t_0}^-) \otimes \mathbf{C}_{t_0' \cdot \Lambda}$ into $E^j_{t_0, i, \lambda}$ which extends the $\mathfrak{b}_{\mathfrak{k}}$-homomorphism of $S(\mathfrak{p})^j \otimes \wedge^i(\mathfrak{p}_{t_0}^-) \otimes \mathbf{C}_{t' \cdot \lambda}$ into $E^j_{t_0, i, \lambda}$ given by $f \otimes e \otimes 1 \mapsto \boldsymbol{\lambda}(f) \otimes_{U(\mathfrak{b}_{\mathfrak{k}})} e \otimes 1 \in \mathcal{G}_j \mathcal{K} \otimes_{U(\mathfrak{b}_{\mathfrak{k}})} \wedge^i(\mathfrak{p}_{t_0}^-) \otimes \mathbf{C}_{t_0' \cdot \lambda}$. Now Lemma 4.1 asserts that $\mathcal{K} \otimes_{U(\mathfrak{b}_{\mathfrak{k}})} S^j(\mathfrak{p}) \otimes \wedge^i(\mathfrak{p}_{t_0}^-) \otimes \mathbf{C}_{t_0' \cdot \lambda}$ is $\mathfrak{k}$-isomorphic with $S^j(\mathfrak{p}) \otimes \mathcal{K} \otimes_{U(\mathfrak{b}_{\mathfrak{k}})} \wedge^i(\mathfrak{p}_{t_0}^-) \otimes \mathbf{C}_{t_0' \cdot \lambda}$ and hence isomorphic with $E^j_{t_0, i, \lambda}$. From the definition of the map we see easily that Φ is surjective and hence in fact an isomorphism since the $\mathfrak{h}_1$ weight spaces of both image and range are finite dimensional and these dimensions are equal.

For integers i, j with $0 \leqslant i \leqslant m$ set $D^j_{i, \lambda} = \mathcal{K} \otimes_{U(\mathfrak{b}_{\mathfrak{k}})} S^j(\mathfrak{p}) \otimes \wedge^i(\mathfrak{p}_{t_0}^-) \otimes \mathbf{C}_{t_0' \cdot \lambda}$ and $U^j_\lambda = \mathcal{K} \otimes_{U(\mathfrak{b}_{\mathfrak{k}})} S^j(\mathfrak{p}_{t_0}) \otimes \mathbf{C}_{t' \cdot \lambda}$. We now have the following diagram which does not commute:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & E^j_{t_0, m, \lambda} & \xrightarrow{\partial_\Lambda} & \cdots & \xrightarrow{\partial_\Lambda} & E^{j+m}_{t_0, 0, \lambda} & \xrightarrow{\varepsilon} & W^{j+m}_{t_0, \Lambda} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \Phi} & & & & \uparrow{\scriptstyle \Phi} & & & & \\
0 & \longrightarrow & D^j_{m, \lambda} & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & D^{j+m}_{0, \lambda} & \xrightarrow{\varepsilon} & U^{j+m}_\lambda & \longrightarrow & 0
\end{array}$$

By comparing the definitions of ∂_Λ and ∂ on generators, we find that the following diagram is commutative

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & E^j_{t_0, m, \lambda}/E^{j-1}_{t_0, m, \lambda} & \xrightarrow{\bar\partial_\Lambda} & \cdots & \xrightarrow{\bar\partial_\Lambda} & E^{j+m}_{t_0, 0, \lambda}/E^{j+m-1}_{t_0, 0, \lambda} & \xrightarrow{\bar\varepsilon} & W^{j+m}_{t_0, \Lambda}/W^{j+m-1}_{t_0, \Lambda} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \bar\Phi} & & & & \uparrow{\scriptstyle \bar\Phi} & & & & \\
0 & \longrightarrow & D^j_{m, \lambda}/D^{j-1}_{m, \lambda} & \xrightarrow{\bar\partial} & \cdots & \xrightarrow{\bar\partial} & D^{j+m}_{0, \lambda}/D^{j+m-1}_{0, \lambda} & \xrightarrow{\bar\varepsilon} & U^{j+m}_\lambda/U^{j+m-1}_\lambda & \longrightarrow & 0
\end{array}$$

(here the $^-$ denotes the induced map). The lower sequence is an exact complex and all the maps Φ are $\mathfrak{k}$-module isomorphisms which implies that there exists a $\mathfrak{k}$-module isomorphism from $U^{j+m}_\lambda/U^{j+m-1}_\lambda$ to $W^{j+m}_{t_0, \Lambda}/W^{j+m-1}_{t_0, \Lambda}$ which gives an equivalence between these two sequences and thus the top sequence is exact. It now follows easily that the sequence of the $E^j_{t_0, k, \lambda}$ must itself be an exact complex. This completes the proof.

Using Lemma 5.10 as the starting point then the arguments which prove Theorem 5.7 give the following filtered version of that theorem.

Lemma 5.11. *Let $\Lambda \in \mathfrak{h}^*$, $\lambda = t_0'(\Lambda|_{\mathfrak{h}_1})$ and assume that λ is strongly $P_{\mathfrak{k}}$-dominant integral. Let $s \in W_{\mathfrak{k}}$ and let j be an integer with $j \geqslant -m$, then the following sequence of $\mathfrak{k}$-modules is an exact complex:*

$$0 \longrightarrow E^j_{s,m,\lambda} \xrightarrow{\partial_\Lambda} E^{j+1}_{s,m-1,\lambda} \xrightarrow{\partial_\Lambda} \dots \xrightarrow{\partial_\Lambda} E^{j+m}_{s,0,\lambda} \xrightarrow{\varepsilon} W^{j+m}_{s,\Lambda} \longrightarrow 0$$

(where we define $W^{j+m}_{s,\Lambda}$ to be the image of $E^{j+m}_{s,0,\lambda}$).

The $\mathfrak{g}$-module $E_{s,i,\lambda}$ is isomorphic as a $\mathfrak{k}$-module to $S(\mathfrak{p}) \otimes N_{s,i}$ and thus Lemma 5.2 applies to $E_{s,i,\lambda}$ and yields:

Lemma 5.12. *Let $\Lambda \in \mathfrak{h}^*$, $\lambda = t_0'(\Lambda|_{\mathfrak{h}_1})$. Assume that λ is strongly $P_{\mathfrak{k}}$-dominant integral. If $r, s \in W_{\mathfrak{k}}$ with $r > s$, and if $\mu \in \mathfrak{h}_1^*$ is $P_{\mathfrak{k}}$-dominant integral, then the element $d_{s,r}(\mu)$ in $U(\mathfrak{n}_{\mathfrak{k}}^-)$ induces a bijection from $E^{\mathfrak{n}_{\mathfrak{k}}}_{r,i,\lambda}[r' \cdot \mu]$ onto $E^{\mathfrak{n}_{\mathfrak{k}}}_{s,i,\lambda}[s' \cdot \lambda]$. Also the element $d_{sr^{-1},1}(\mu)$ induces a bijection from $E^{\mathfrak{n}_{\mathfrak{k}}}_{r,i,\lambda}[\mu]$ onto $E^{\mathfrak{n}_{\mathfrak{k}}}_{s,i,\lambda}[(sr^{-1})' \cdot \mu]$.*

Proposition 5.13. *Let $\Lambda \in \mathfrak{h}^*$, $\lambda = t_0'(\Lambda|_{\mathfrak{h}_1})$. Assume that λ is strongly $P_{\mathfrak{k}}$-dominant integral, and that $\mu \in \mathfrak{h}_1^*$ is $P_{\mathfrak{k}}$-dominant integral. Then if $r \in W_{\mathfrak{k}}$, the following sequence is an exact complex:*

$$\begin{aligned} 0 \longrightarrow E^{\mathfrak{n}_{\mathfrak{k}}}_{t_0,m,\lambda}[r' \cdot \mu] &\xrightarrow{\partial_\Lambda} E^{\mathfrak{n}_{\mathfrak{k}}}_{t_0,m-1,\lambda}[r' \cdot \mu] \xrightarrow{\partial_\Lambda} \dots \\ &\xrightarrow{\partial_\Lambda} E^{\mathfrak{n}_{\mathfrak{k}}}_{t_0,0,\lambda}[r' \cdot \mu] \xrightarrow{\varepsilon} W^{\mathfrak{n}_{\mathfrak{k}}}_{t_0,\Lambda}[r' \cdot \mu] \longrightarrow 0 \end{aligned} \tag{5.6}$$

For integers i, k with $i \geqslant -m$ and $0 \leqslant k \leqslant m$, set $B^i_k = (E^i_{t_0,k,\lambda})^{\mathfrak{n}_{\mathfrak{k}}}$. Then B^i_k is a semisimple $\mathfrak{h}_1$-module and the following sequence is an exact complex:

$$0 \longrightarrow B^i_m[r' \cdot \mu] \xrightarrow{\partial_\Lambda} B^{i+1}_{m-1}[r' \cdot \mu] \xrightarrow{\partial_\Lambda} \dots \xrightarrow{\partial_\Lambda} B^{i+m}_0[r' \cdot \mu] \xrightarrow{\varepsilon} (W^{i+m}_{t_0,\Lambda})^{\mathfrak{n}_{\mathfrak{k}}}[r' \cdot \mu] \longrightarrow 0 \tag{5.7}$$

Note. This last exact sequence will be used to compute the dimension of $W^{\mathfrak{n}_{\mathfrak{k}}}_{t_0,\Lambda}[r' \cdot \mu] = V^{\mathfrak{n}_{\mathfrak{k}}}_{\mathfrak{g},-t_0P,\Lambda}[r' \cdot \mu]$.

Proof. Lemma 7 in [7] shows that if $r = 1$, then the exactness statement in Lemma 5.10 implies the exactness of sequences (5.6) and (5.7). If $r \neq 1$, then set $s = r^{-1}t_0$. Lemma 7 in [7] now applies to Lemma 5.11 to give the exact sequence:

$$0 \longrightarrow (E^i_{s,m,\lambda})^{\mathfrak{n}_{\mathfrak{k}}}[\mu] \xrightarrow{\partial_\Lambda} \dots \xrightarrow{\partial_\Lambda} (E^{i+m}_{s,0,\lambda})^{\mathfrak{n}_{\mathfrak{k}}}[\mu] \xrightarrow{\varepsilon} (W^{i+m}_{s,\Lambda})^{\mathfrak{n}_{\mathfrak{k}}}(\mu) \longrightarrow 0 \tag{5.8}$$

Using Lemma 5.12 and noting that the maps ∂_Λ are $\mathfrak{g}$-homomorphisms, we obtain the commutative diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow & (E^i_{s,m,\lambda})^{\mathfrak{n}_\mathfrak{k}}[\mu] & \xrightarrow{\partial_\Lambda} & \cdots & \xrightarrow{\partial_\Lambda} & (E^{i+m}_{s,0,\lambda})^{\mathfrak{n}_\mathfrak{k}}[\mu] \\ & \Big\downarrow d_{r,1}(\mu) & & & & \Big\downarrow d_{r,1}(\mu) \\ 0 \longrightarrow & B^i_m[r'\cdot\mu] & \xrightarrow{\partial_\Lambda} & \cdots & \xrightarrow{\partial_\Lambda} & B^{i+m}_0[r'\cdot\mu] \end{array} \tag{5.9}$$

The vertical maps are bijections by Lemma 5.12, and thus the exactness of the top sequence implies the exactness of the lower. It remains only to check the exactness of the sequence:

$$B^{i+m-1}_1[r'\cdot\mu] \xrightarrow{\partial_\Lambda} B^{i+m}_0[r'\cdot\mu] \xrightarrow{\varepsilon} (W^{i+m}_{t_0,\Lambda})^{\mathfrak{n}_\mathfrak{k}}[r'\cdot\mu] \longrightarrow 0 \tag{5.10}$$

Since $d_{r,1}(\mu)$ acts injectively on $W_{s,\Lambda}$, the exactness of (5.8) implies that sequence (5.10) is exact at $B^{i+m}_0[r'\cdot\mu]$. We now prove that ε maps $B^{i+m}_0[r'\cdot\mu]$ surjectively onto $(W^{i+m}_{t_0,\Lambda})^{\mathfrak{n}_\mathfrak{k}}[r'\cdot\mu]$. This follows if we show that $d_{r,1}(\mu)$ induces a bijection from $W^{\mathfrak{n}_\mathfrak{k}}_{s,\Lambda}[\mu]$ onto $W^{\mathfrak{n}_\mathfrak{k}}_{t_0,\Lambda}[r'\cdot\mu]$. The case $r=1$ is trivial. Assume that $r\neq 1$, and then choose a $P_\mathfrak{k}$-simple root α in $P_\mathfrak{k}$ such that $l(s_\alpha s)=l(s)+1$. Let $\mathfrak{a}$ equal the one-dimensional subalgebra of $\mathfrak{k}$, $\mathfrak{a}=\mathfrak{k}_\alpha$. Choose $X_{-\alpha}\in\mathfrak{k}_{-\alpha}$, $X_{-\alpha}\neq 0$. During the proof of Theorem 5.7, the following fact was established: Let $\nu\in\mathfrak{h}_1^*$ be such that $n=2\langle\nu+\delta_\mathfrak{k},\alpha\rangle/\langle\alpha,\alpha\rangle$ is a positive integer, then $X^n_{-\alpha}$ induces a bijection from $(W_{s,\Lambda})^{\mathfrak{a}}[\nu]$ onto $(W_{s_\alpha s,\Lambda})^{\mathfrak{a}}[s'_\alpha\cdot\nu]$. Now if $u\in(W_{s_\alpha s,\Lambda})^{\mathfrak{n}_\mathfrak{k}}[s'_\alpha\cdot\nu]$ then choose $\bar u\in(W_{s,\Lambda})^{\mathfrak{a}}[\nu]$ such that $X^n_{-\alpha}\cdot\bar u=u$. If β is a $P_\mathfrak{k}$-simple root, $\alpha\neq\beta$, and if $Y\in\mathfrak{k}_\beta$ then $[Y, X_{-\alpha}]=0$ and thus $0=X^n_{-\alpha}\cdot Y\cdot\bar u$. But $X^n_{-\alpha}$ acts injectively on $W_{s,\Lambda}$ and thus $Y\cdot\bar u=0$, and in turn $\bar u\in(W_{s,\Lambda})^{\mathfrak{n}_\mathfrak{k}}[\nu]$. This shows that if $m=2\langle\mu+\delta_\mathfrak{k},\alpha\rangle/\langle\alpha,\alpha\rangle$ then X^m induces a bijection from $(W_{s,\Lambda})^{\mathfrak{n}_\mathfrak{k}}[\mu]$ onto $(W_{s_\alpha s,\Lambda})^{\mathfrak{n}_\mathfrak{k}}[s'_\alpha\cdot\mu]$. We continue this process in an iterative fashion. Let $t_0s^{-1}=s_1\ldots s_l$ be a minimal expression for t_0s^{-1} where each s_i is a simple reflection in $W_\mathfrak{k}$ corresponding to the simple root γ_i, $1\leqslant i\leqslant l$. Choose $Y_i\in\mathfrak{k}_{-\gamma_i}$, $Y_i\neq 0$ and set $n_l=2\langle\mu+\delta_\mathfrak{k},\gamma_l\rangle/\langle\gamma_l,\gamma_l\rangle$ and for $1\leqslant i<l$, $n_i=2\langle s_{i+1}\ldots s_l(\mu+\delta_\mathfrak{k}),\gamma_i\rangle/\langle\gamma_i,\gamma_i\rangle$. The minimality of the expression implies that n_i is a positive integer $1\leqslant i\leqslant l$, and thus if we repeat the above argument l times, we find that $Y_1^{n_1}\ldots Y_l^{n_l}$ induces a bijection from $(W_{s,\Lambda})^{\mathfrak{n}_\mathfrak{k}}[\mu]$ onto $(W_{t_0,\Lambda})^{\mathfrak{n}_\mathfrak{k}}[r'\cdot\mu]$ since $r=t_0s^{-1}$. $Y_1^{n_1}\ldots Y_l^{n_l}$ is a scalar multiple of $d_{r,1}(\mu)$, and thus the proof of the proposition is complete.

For purposes of later reference we state the structural fact just proved as a lemma.

LEMMA 5.14. *Let Λ, λ, and μ be as in Proposition 5.13. Let $r\in W_\mathfrak{k}$ and set $s=r^{-1}t_0$. Then $d_{r,1}(\mu)$ induces a bijection from $(W_{s,\Lambda})^{\mathfrak{n}_\mathfrak{k}}[\mu]$ onto $(W_{t_0,\Lambda})^{\mathfrak{n}_\mathfrak{k}}[r'\cdot\mu]$.*

We can now prove the main theorem on $\mathfrak{n}_\mathfrak{k}$-invariants in $W_{s,\Lambda}$.

THEOREM 5.15. *Let* $\Lambda \in \mathfrak{h}^*$, $\lambda = t_0'(\Lambda|_{\mathfrak{h}_1})$ *and assume that* λ *is strongly* $P_{\mathfrak{k}}$*-dominant integral. Let* $\mu \in \mathfrak{h}_1^*$ *be* $P_{\mathfrak{k}}$*-dominant integral, then if* $r \in W_{\mathfrak{k}}$ *and* $s = r^{-1} t_0$, *the following formula holds:*

$$\dim (W_{s,\Lambda})^{\mathfrak{n}_{\mathfrak{k}}}[\mu] = \dim (W_{t_0,\Lambda})^{\mathfrak{n}_{\mathfrak{k}}}[r' \cdot \mu] = \dim S(\mathfrak{p}_{t_0})[r' \cdot \mu - t_0' \cdot \lambda]$$

(*where* $\mathfrak{p}_{t_0} = \sum_{\alpha \in t_0 P} \mathfrak{p}[\alpha|_{\mathfrak{h}_1}]$).

Proof. Lemma 5.14 gives the first equality. We now consider the second. For any integer i it is easy to check that the $\mathfrak{k}$-module $E^i_{t_0,k,\lambda}$ is isomorphic to the $\mathfrak{k}$-module $S^i(\mathfrak{p}) \otimes N_{t_0,k}$.

Lemma 5.5 states that $N_{t_0,k}$ splits as the direct sum of irreducible $\mathfrak{k}$-Verma modules where V_ν occurs as a summand with multiplicity equal to $\dim (\wedge^k(\mathfrak{p}_{t_0}^-) \otimes \mathbf{C}_{t_0' \cdot \lambda})[\nu]$. Now applying Lemma 5.2 gives:

$$\dim (E^i_{t_0,k,\lambda})^{\mathfrak{n}_{\mathfrak{k}}}[\nu] = \dim (S^i(\mathfrak{p}) \otimes \wedge^k(\mathfrak{p}_{t_0}^-) \otimes \mathbf{C}_{t_0',\lambda}([\nu]$$

Set $\delta(i, k, \nu)$ equal to this dimension.

Proposition 5.13 implies that $\dim (W^i_{t_0,\Lambda})^{\mathfrak{n}_{\mathfrak{k}}}[r' \cdot \mu] = \sum_{0 \leqslant k \leqslant m} (-1)^k \delta(i-k, k, r' \cdot \mu)$. Now return to the Koszul complex defined at the beginning of the proof of Lemma 5.10 and we find that the alternating sum in the above equation equals $\dim S^i(\mathfrak{p}_{t_0})[r' \cdot \mu - t_0' \cdot \lambda]$. This completes the proof.

Remarks. 1. The reader should note that although the aim of this paper is to use the theory of Verma modules to construct other $\mathfrak{g}$-modules, Theorem 5.15 contains a non-trivial structural fact about $\mathfrak{n}_{\mathfrak{k}}$-invariants in certain $\mathfrak{g}$-Verma modules (i.e., $W_{t_0,\Lambda} = V_{\mathfrak{g}, -t_0 P, \Lambda}$).

2. Theorem 5.15 will be used in [6] to prove that with Λ, λ, and μ as in the theorem, then the multiplicity of V^μ in $W_{P,\Lambda}$ is given by:

$$\sum_{s \in W_{\mathfrak{k}}} \det(st_0) \dim(S(\mathfrak{p}_{t_0})[s' \cdot \mu - t_0' \cdot \lambda])$$

In [6] we shall also write this as an alternating sum of certain partition functions, and in the case where $\mathfrak{h}_1 = \mathfrak{h}$, we obtain precisely Blattner's formula.

A lowest $\mathfrak{k}$-type theorem

If $\lambda \in \mathfrak{h}_1^*$ is $P_{\mathfrak{k}}$-dominant integral, we denote by V^λ the irreducible finite dimensional representation with highest weight λ. If $\lambda \in \mathfrak{h}_1^*$ is not $P_{\mathfrak{k}}$-dominant integral, we set $V^\lambda = (0)$.

A $\mathfrak{g}$-module, M, is said to be *admissible* if the following two conditions are satisfied:

(i) If $m \in M$, $\dim \mathcal{K}\cdot m < \infty$

(ii) If $\lambda \in \mathfrak{h}_1^*$, then $\dim \operatorname{Hom}_{\mathfrak{k}}(V^\lambda, M) < \infty$ (here M is looked upon as a $\mathfrak{k}$-module).

If M is admissible, then it is clear that as a $\mathfrak{k}$-module

$$M \cong \sum_\lambda V^\lambda \otimes \operatorname{Hom}_{\mathfrak{k}}(V^\lambda, M).$$

If $n_\lambda = \dim \operatorname{Hom}_{\mathfrak{k}}(V^\lambda, M)$, we write $M = \sum n_\nu V^\lambda$. n_λ is called the multiplicity of V^λ in M.

Definition 6.1. Let M be an admissible $\mathfrak{g}$-module. Let $\lambda \in \mathfrak{h}_1^*$ be $P_{\mathfrak{k}}$-dominant integral. Let P be an admissible system of positive roots giving $P_{\mathfrak{k}}$. Then M is said to have V^λ as a weak minimal $\mathfrak{k}$-type relative to P if

(a) There is an element A in $\operatorname{Hom}_{\mathfrak{k}}(V^\lambda, M)$, $A \neq 0$ so that $M = \mathcal{G}\cdot A(V^\lambda)$.

(b) If $\beta \in P$ and $\mu = \beta|_{\mathfrak{h}_1}$, then $\operatorname{Hom}_{\mathfrak{k}}(V^{\lambda-\mu}, \mathfrak{p}\cdot A(V^\lambda)) = 0$.

(c) $\dim \operatorname{Hom}_{\mathfrak{k}}(V^\lambda, A(V^\lambda) + \mathfrak{p}\cdot A(V^\lambda)) = 1$.

THEOREM 6.2. *Let M be an admissible $\mathfrak{g}$-module. Let P be an admissible system of positive roots. Suppose that V^λ is a weak minimal $\mathfrak{k}$-type for M relative to P and that λ is strongly $P_{\mathfrak{k}}$-dominant integral (see Definition 5.4). Then there exists a $\Lambda \in \mathfrak{h}^*$ so that $\Lambda|_{\mathfrak{h}_1} = t_0'\lambda$ and a surjective $\mathfrak{g}$-module homomorphism of $W_{P,\Lambda}$ onto M.*

Proof. Then Definition 6.1 (a) states that $M = \mathcal{G}\cdot A(V^\lambda)$. Using the universal mapping property of the tensor product, we see that there is a surjective $\mathfrak{g}$-module homomorphism ψ: $\mathcal{G}\otimes_{\mathcal{K}} V_\lambda \to M$ which extends the $\mathfrak{k}$-module homomorphism A: $1\otimes V_\lambda \to A(V^\lambda)$.

Let $\mathfrak{p}^+ = \sum_{\alpha\in P}\mathfrak{p}[\alpha|_{\mathfrak{h}_1}]$. In § 5, we found that ∂_Λ maps $N_{1,1}$ injectively into $\mathcal{G}_1(1\otimes V_\nu)\,(=E^1_{1,0,\lambda})$. Let $N'_{1,1}$ be the $\mathfrak{k}$-submodule of $N_{1,1}$ where $V_{\nu-\mu}$ occurs as a summand with multiplicity equal to $\dim \mathfrak{p}^+[\mu]$. Definition 6.1 (b) implies that $\psi(\partial_\Lambda(N'_{1,1})) \equiv 0$.

$E^1_{1,0,\lambda}$ is isomorphic to $S^1(\mathfrak{p})\otimes V_\nu$ as a $\mathfrak{k}$-module and thus V_ν occurs in $E^1_{1,0,\lambda}$ with multiplicity $\dim \mathfrak{h}_2 + 1$ ($\mathfrak{h}_2 = \mathfrak{p}[0]$). Let M_1 equal the $\mathcal{K}$-module generated by $\operatorname{Ker}\psi \cap (E^1_{1,0,\lambda})^{\mathfrak{n}_{\mathfrak{k}}}[\lambda]$ and set $\tilde{M} = M_1 \oplus \partial_\Lambda(N'_{1,1})$. By construction we have $\tilde{M} \subseteq E^1_{1,0,\lambda} \cap \operatorname{Ker}\psi$. Lemma 5.12 implies that $d_{t_0,1}(\lambda-\mu)$ induces a bijection from $(E^1_{1,0,\lambda})^{\mathfrak{n}_{\mathfrak{k}}}[\lambda-\mu]$ onto $(E^1_{t_0,0,\lambda})^{\mathfrak{n}_{\mathfrak{k}}}[t_0'(\lambda-\mu)]$ for any $\mu \in \mathfrak{h}_1^*$ such that $\mathfrak{p}^+[\mu] \neq \{0\}$. Now if we set M_{t_0} equal to the $\mathfrak{k}$-submodule of $\tilde{M}$ generated by $d_{t_0,1}(\lambda-\mu)(M^{\mathfrak{n}_{\mathfrak{k}}}[\lambda-\mu])$ with μ as above, then $M_{t_0} \subseteq E^1_{t_0,0,\lambda}$. $E^1_{t_0,0,\lambda}$ is isomorphic with $S^1(\mathfrak{p})\otimes V_{t_0'\cdot\lambda}$ and thus by comparing $\mathfrak{h}$ weight spaces of M_{t_0} and $E^1_{t_0,0,\lambda}$, we find that if ν a non-zero sum of elements in $-t_0P$, then $M_{t_0}[t_0'\cdot\lambda+\nu] \equiv E^1_{t_0,0,\lambda}[t_0'\cdot\lambda+\nu]$. Set as usual $\mathfrak{n} = \sum_{\alpha\in -t_0P}\mathfrak{g}_\alpha$, then if 1 denotes the canonical cyclic vector in

$V_{t_0' \cdot \lambda}$ then $\mathfrak{n}\cdot(1\otimes 1)\subseteq M_{t_0}$. Again by comparing the $\mathfrak{h}_1$ weight space dimensions of M_{t_0} and $E^1_{t_0,0,\lambda}$ we find that $M_{t_0}[t_0'\cdot\lambda]$ has codimension one in $E^1_{t_0,0,\lambda}[t_0'\cdot\lambda]$, and thus there exists an element $\Lambda\in\mathfrak{h}^*$ so that the action of $\mathfrak{h}$ on $(E^1_{t_0,0,\lambda}/M_{t_0})[t_0'\cdot\lambda]$ is equivalent to $\mathbf{C}_\Lambda$. It is obvious that $\Lambda|_{\mathfrak{h}_1}=t_0'\cdot\lambda$ and thus since $t_0^{-1}=t_0$ that $\lambda=t_0'(\Lambda|_{\mathfrak{h}_1})$.

Now using the results of § 4 with $N_{t_0,1}$ and ∂_Λ as defined there, we obtain $M_{t_0}=\partial_\Lambda(N_{t_0,1})$. Recalling the definitions of § 5, we see that $\partial_\Lambda(1\otimes N_{1,1})=\tilde{M}$ and thus by Theorem 5.7 that ψ induces a map ψ_1 from $W_{1,\Lambda}$ onto M. $\psi_1(V_\nu)=A(V_\nu)=V^\lambda$ and thus by Theorem 3.1 (i), $\psi_1|W_{s,\Lambda}=0$ for all $s\in W_{\mathfrak{k}}$, $s\neq 1$. In turn, this shows that ψ_1 induces a surjective $\mathfrak{g}$-homomorphism ψ_2 from $W_{P,\Lambda}$ onto M. This completes the proof.

For any $\Lambda\in\mathfrak{h}^*$ such that $\lambda=t_0'(\Lambda|_{\mathfrak{h}_1})$ is $P_{\mathfrak{k}}$-dominant integral, we know by Theorem 3.2 that $\dim \operatorname{Hom}_{\mathfrak{k}}(V^\lambda, W_{P,\Lambda})=1$. Let $M_{P,\Lambda}$ be the inique maximal $\mathfrak{g}$-submodule of $W_{P,\Lambda}$ such that $\operatorname{Hom}_{\mathfrak{k}}(V^\lambda, M_{P,\Lambda})=\{0\}$, and set $D_{P,\Lambda}=W_{P,\Lambda}/M_{P,\Lambda}$. $D_{P,\Lambda}$ is the unique irreducible quotient of $W_{P,\Lambda}$. Set $\mathcal{H}_{P,\lambda}$ equal to the set of equivalence classes of the irreducible representations $D_{P,\Lambda}$ where $\Lambda\in\mathfrak{h}^*$ and $\lambda=t_0'(\Lambda|_{\mathfrak{h}_1})$. This set of equivalence classes can be characterized in several ways.

THEOREM 6.3. *Let $\lambda\in\mathfrak{h}_1^*$ be $P_{\mathfrak{k}}$-dominant integral and let M be any admissible irreducible $\mathfrak{g}$-module. If $\{M\}$ denotes the equivalence class of M, then the following two statements are equivalent:*

(i) $\{M\}\in\mathcal{H}_{P,\lambda}$

(ii) *There exists a $\mathfrak{k}$-submodule $L\subseteq M$ which is isomorphic to V^λ, $\mathcal{G}^{\mathfrak{k}}$-stable and on which $\mathcal{G}^{\mathfrak{k}}$ acts by the formula: $x|_L=\eta_{P,\Lambda}(x)\,\mathrm{Id}$, where $x\in\mathcal{G}^{\mathfrak{k}}$, $\Lambda\in\mathfrak{h}^*$ and $\lambda=t_0'(\Lambda|_{\mathfrak{h}_1})$ (see Theorem 3.2 for the definition of $\eta_{P,\Lambda}$).*

If in addition λ is strongly $P_{\mathfrak{k}}$-dominant integral, then (i) *and* (ii) *are equivalent to either of the following two equivalent statements:*

(iii) *V^λ is a weak minimal $\mathfrak{k}$-type for M*

(iv) $\dim \operatorname{Hom}_{\mathfrak{k}}(V^\lambda, M)\geqslant 1$ *and* $\operatorname{Hom}_{\mathfrak{k}}(V^{\lambda-\mu}, M)=\{0\}$ *for any $\mu\in\mathfrak{h}_1^*$ where $\mu=\beta|_{\mathfrak{h}_1}$ for some $\beta\in P$ and $\mathfrak{p}[\mu]\neq\{0\}$.*

Proof. Let M be an admissible $\mathfrak{g}$-module and if $\mu\in\mathfrak{h}_1^*$ is $P_{\mathfrak{k}}$-dominant integral then set $M\{\mu\}$ equal to the sum of all $\mathfrak{k}$-submodules of M isomorphic with V^μ. The definition of admissible implies that M is the direct sum of the $\mathfrak{k}$-submodules $M\{\mu\}$. Note also that each $M\{\mu\}$ is stable under the action of $\mathcal{G}^{\mathfrak{k}}$. One of the fundamental theorems for admissible representations, [14], asserts:

(A) If M and M' are admissible irreducible $\mathfrak{g}$-modules, then M and M' are $\mathfrak{g}$-isomorphic if and only if for some $P_{\mathfrak{k}}$-dominant integral λ in $\mathfrak{h}_1^*$, $M\{\lambda\}\neq\{0\}$ and $M\{\lambda\}$ and $M'\{\lambda\}$ are isomorphic as $\mathcal{G}^{\mathfrak{k}}\otimes\mathcal{K}$-modules.

Theorem 3.2 gives the implications (i) $\Rightarrow$ (ii), (i) $\Rightarrow$ (iii), and (i) $\Rightarrow$ (iv). Assume that M is an admissible irreducible $\mathfrak{g}$-module which satisfies (ii). Since M is irreducible, then for any $P_{\mathfrak{k}}$-dominant integral element μ in $\mathfrak{h}_1^*$, we know [14] that $M\{\mu\}$ is an irreducible $\mathcal{G}^{\mathfrak{k}}\otimes\mathcal{K}$-module. Let L be as in statement (ii), then since L is $\mathcal{G}^{\mathfrak{k}}\otimes\mathcal{K}$-stable $V^{\lambda}\cong L\equiv M\{\lambda\}$. If $\mathcal{G}^{\mathfrak{k}}$ acts on L by the formula $x|_L\equiv\eta_{P,\Lambda}\,\mathrm{Id}$, then (A) and Theorem 3.2 imply that M is isomorphic with $D_{P,\Lambda}$. This proves that (ii) $\Rightarrow$ (i).

Assume that λ is strongly $P_{\mathfrak{k}}$-dominant integral. If M is an admissible irreducible $\mathfrak{g}$-module which satisfies (iii), then Theorem 6.2 gives a surjective $\mathfrak{g}$-homomorphism σ from $W_{P,\Lambda}$ onto M for some $\Lambda\in\mathfrak{h}^*$ such that $\lambda=t_0'(\Lambda\,|_{\mathfrak{h}_1})$. $W_{P,\Lambda}$ has a unique irreducible quotient $D_{P,\Lambda}$ and thus σ induces an isomorphism of $D_{P,\Lambda}$ onto M. This gives the implication (iii) $\Rightarrow$ (i).

We now complete the proof by showing that (iv) $\Rightarrow$ (iii). Clearly it will be sufficient to show that $\dim \mathrm{Hom}_{\mathfrak{k}}(V^{\lambda}, M)=1$. $\dim \mathrm{Hom}_{\mathfrak{k}}(V^{\lambda}, M)=\dim M^{\mathfrak{n}_{\mathfrak{k}}}[\lambda]$ and thus we shall actually prove that $\dim M^{\mathfrak{n}_{\mathfrak{k}}}[\lambda]=1$. The work of § 4 and § 5 is based on Lemma 4.2 applied in the case where $\mathfrak{a}=\mathfrak{h}\oplus\mathfrak{n}=\mathfrak{b}$ and $\mathfrak{m}=\mathfrak{h}_1\oplus\mathfrak{n}_{\mathfrak{k}}=\mathfrak{b}_{\mathfrak{k}}$. The exact same procedure can be carried out in the case $\mathfrak{a}=\mathfrak{h}_1\oplus\mathfrak{n}$ and $\mathfrak{m}=\mathfrak{b}_{\mathfrak{k}}$. In this case the maps ∂_Λ have no dependence on $\mathfrak{h}_2$ action, and thus we write them as ∂_λ. Write $N'_{s,i}$ in place of $N_{s,j}$ where in this case $0\leqslant i\leqslant \dim\Lambda(\mathfrak{n}/\mathfrak{n}_{\mathfrak{k}})=m'$ and write $E'_{s,i,\lambda}$ in place of $E_{s,i,\lambda}$. Set $W'_{s,\lambda}=E_{s,0,\lambda}/\partial_\Lambda(E'_{s,1,\lambda})$ and recall that Lemma 5.12 states that $d_{t_0,1}(\lambda)$ gives a bijection from $(W'_{1,\lambda})^{\mathfrak{n}_{\mathfrak{k}}}[\lambda]$ onto $(W'_{t_0,\lambda})^{\mathfrak{n}_{\mathfrak{k}}}[t_0'\cdot\lambda]$. In this case $W'_{t_0,\lambda}$ is not a $\mathfrak{g}$-Verma module; however, it is easy to see that $t_0'\cdot\lambda$ is a $P_{\mathfrak{k}}$-highest weight and also that $\mathfrak{h}_1$-weight spaces are in fact all infinite dimensional.

Set $B_1=(W'_{1,\lambda})^{\mathfrak{n}_{\mathfrak{k}}}[\lambda]$ and $B_{t_0}=(W'_{t_0,\lambda})^{\mathfrak{n}_{\mathfrak{k}}}[t_0'\cdot\lambda]\equiv W'_{t_0,\lambda}[t_0'\cdot\lambda]$. Let σ be any non-zero element in $\mathrm{Hom}_{\mathfrak{k}}(V_\lambda, M)$ and let σ also denote the unique extension to a $\mathfrak{g}$-homomorphism of $\mathcal{G}\otimes_{\mathcal{K}}V_\lambda=E'_{1,0,\lambda}$. Our assumptions on M imply that $\sigma|_{\partial_\Lambda(E'_{1,1,\lambda})}\equiv 0$ and thus σ induces a homomorphism σ_1 of $W'_{1,\lambda}$ onto M. Set $C_1=B_1\cap\mathrm{Ker}\,\sigma_1$; then since λ is $P_{\mathfrak{k}}$-dominant integral, B_1/C_1 and $M^{\mathfrak{n}_{\mathfrak{k}}}[\lambda]$ are isomorphic $\mathcal{G}^{\mathfrak{k}}$-modules. Since $d_{t_0,1}(\lambda)$ commutes with $\mathcal{G}^{\mathfrak{k}}$, if we set $C_{t_0}=d_{t_0,1}(\lambda)C_1$ then as $\mathcal{G}^{\mathfrak{k}}$-modules we have:

$$M^{\mathfrak{n}_{\mathfrak{k}}}[\lambda]\cong B_1/C_1\cong B_{t_0}/C_{t_0}.$$

Since M is by assumption irreducible, these $\mathcal{G}^{\mathfrak{k}}$-modules are all irreducible.

B_{t_0} is the $P_{\mathfrak{k}}$-highest weight space for $W'_{t_0,\lambda}$ and thus $\mathfrak{n}\cdot B_{t_0}=\{0\}$. This implies that the commutator subalgebra of $\mathcal{G}^{\mathfrak{k}}$ acts trivially on B_{t_0} (the action of $\mathcal{G}^{\mathfrak{k}}$ factors through the map $\mathcal{G}\xrightarrow{\iota}\mathcal{G}/\mathcal{G}\cdot\mathfrak{n}$ and $\iota(\mathcal{G}^{\mathfrak{k}})\subset\iota(\mathcal{G}^{\mathfrak{h}_1})$ which is abelian). Hence the image of $\mathcal{G}^{\mathfrak{k}}$ in $\mathrm{End}\,(B_{t_0}/C_{t_0})$ is commutative and acts irreducibly. Thus $\dim B_{t_0}/C_{t_0}=1$.

COROLLARY 6.4. *Let $\lambda \in \mathfrak{h}_1^*$ be strongly $P_{\mathfrak{k}}$-dominant integral, and let M be an admissible irreducible $\mathfrak{g}$-module. If $\operatorname{Hom}_{\mathfrak{k}}(V^\lambda, M) \neq \{0\}$ and $\operatorname{Hom}_{\mathfrak{k}}(V^{\lambda-\mu}, M) = \{0\}$ for any $\mu \in \mathfrak{h}_1^*$ where $\mu = \beta|_{\mathfrak{h}_1}$, $\beta \in P$ and $\mathfrak{p}[\mu] \neq \{0\}$, then*

$$\dim \operatorname{Hom}_{\mathfrak{k}}(V^\lambda, M) = 1$$

(i.e., V^λ occurs with multiplicity one in M).

Proof. This follows by Theorem 6.3 using the equivalence (i)$\Leftrightarrow$(iv), and the fact that if $\Lambda \in \mathfrak{h}^*$ is such that $\lambda = t_0' \cdot (\Lambda|_{\mathfrak{h}_1})$ then $\dim \operatorname{Hom}_{\mathfrak{k}}(V^\lambda, W_{P,\Lambda}) = 1$.

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