ON THE NUMBER OF INVARIANT CLOSED GEODESICS

BY

KARSTEN GROVE and MINORU TANAKA

University of Copenhagen Denmark *Tokyo Institute of Technology Japan*

The study of closed (periodic) geodesics has a long and rich history. After Fet and Lyusternik [2] in 1951 proved that any compact riemannian manifold has at least one closed geodesic, the most outstanding problem has been whether such a manifold has actually infinitely many distinct closed geodesics. Here closed geodesics are always understood to be non-constant, and two geodesics are said to be distinct if one is not a reparametrization of the other. No real progress was made untill 1969 when Gromoll and Meyer [7] obtained the following celebrated result.

THEOREM. *Let M be a compact connected and simply connected riemannian manifold. Then M has infinitely many closed geodesics if the sequence of Betti numbers for the* (*rational*) *homology of the space of all maps $S^1 \to M$ is unbounded.*

Here a map is always understood to be continuous and the space of maps $S^1 \to M$ is endowed with the compact-open (uniform) topology. Recently Sullivan and Vigué [21] showed that the topological condition on M in the above theorem is satisfied if and only if the (rational) cohomology ring of M is not generated by one element.

Just very recently we have received the second revised and enlarged edition of a manuscript to a monograph on closed geodesics by W. Klingenberg [13]. In that manuscript a proof for the existence of infinitely many closed geodesics on any 1-connected compact riemannian manifold is offered. The proof involves new methods and ideas and is very complicated.

A related but more general theory than that of closed geodesics is the one of isometry-invariant geodesics developed by the first named author in [8] and [9]. A non-constant geodesic $c: \mathbf{R} \to M$ is said to be invariant under an isometry $A: M \to M$ if $A(c(t)) = c(t+1)$ for all $t \in \mathbf{R}$. Clearly an A-invariant geodesic with $A = id_M$ is simply a closed geodesic and vice versa. In contrast to the case of closed geodesics, there are examples of isometries which have no invariant geodesics (e.g. rotation on a flat torus). However, as a generalization of the theorem by Fet and Lyusternik, any isometry A which is homotopic to id_M has invariant geodesics [8]. Furthermore, there are isometries (even homotopic to the identity) which have no more than one invariant geodesic (e.g. rotation on the round sphere). As a main theorem of this paper we shall prove the following generalization of the Gromoll-Meyer theorem.

MAIN THEOREM. *Let M be a compact, 1-connected riemannian manifold and let $f: M \to M$ be an isometry of finite order. Then there are infinitely many f-invariant geodesics on M if the sequence of Betti numbers for the homology (any field as coefficients) of the space of all maps $\sigma: [0, 1] \to M$ with $\sigma(1) = f(\sigma(0))$ is unbounded.*

Note that any f-invariant geodesic is closed since f is of finite order. Furthermore, since the isometry group $I(M)$ of M is a compact Lie group, we have that the subgroup consisting of isometries with finite order is dense in $I(M)$. The theorem was announced in [12] and proved in particular cases in [10], [22] and [23].

In Grove, Halperin and Vigué [11] a necessary and sufficient condition is given (in terms of the action of f on the (rational) homotopy groups of M) in order for the space $\sigma: [0, 1] \to M$ with $f(\sigma(0)) = \sigma(1)$ to have an unbounded sequence of (rational) Betti numbers.

In dealing with isometry-invariant geodesics we apply the "modern" calculus of variations in the large i.e. critical point theory on infinite dimensional manifolds of maps. The A-invariant geodesics on M are precisely the critical points for the energy integral E^A (with positive energy) on a suitable space of "A-invariant curves" on M; $\Lambda(M, A)$. Now, Morse theory provides information about existence and number of critical points for E^A in terms of the topology of $\Lambda(M, A)$. This, however, does not immeditately give information about the number of distinct A-invariant geodesics. For each closed A-invariant geodesic all its multiple covers are also A-invariant, but no such two are of course distinct. As in the Gromoll-Meyer proof the theorem follows if for each closed A-invariant geodesic, the corresponding tower of critical points (orbits) in $\Lambda(M, A)$ contributes to the homology of $\Lambda(M, A)$ with at most a bounded amount. We are able to show this when A is of finite order. The proof makes use of equivariant degenerate Morse theory as developed by Gromoll and Meyer [6], [7] and a rather delicate study of indices and nullities related to the work of Bott [1]. In the special case where all the critical points (orbits) are non-degenerate i.e. all nullities are zero, the proof becomes much simpler and we need not assume that A is of finite order (cf. the discussion at the end of the paper).

We refer to [5], [3] and [18] for basic facts and tools in riemannian geometry, geometry of path-spaces and algebraic topology.

1. Preliminaries

Throughout the paper $(M, \langle\cdot,\cdot\rangle)$ shall denote a connected, compact riemannian manifold and $f: M\to M$ an isometry of finite order $s\in\mathbf{Z}^+$ i.e. $f^s=id_M$. Let $\Lambda(M,f)$ be the Hilbert manifold consisting of all absolutely continuous maps $\sigma: \mathbf{R}\to M$ with locally square integrable velocity field $\dot\sigma: \mathbf{R}\to TM$ and with $\sigma(t+1)=f(\sigma(t))$ for all $t\in\mathbf{R}$. The tangent space to $\Lambda(M,f)$ at σ consists of all absolutely continuous vector fields $X: \mathbf{R}\to TM$ along σ with locally square integrable covariant derivative X' and with $X(t+1)=f_*(X(t))$, where $f_*: TM\to TM$ denotes the differential of f. The restriction map $\sigma\to\sigma|[0,1]$ identifies $\Lambda(M,f)$ with the manifold $\Lambda_{G(f)}(M)$ introduced in [8]. $\Lambda(M,f)$ caries a natural complete riemannian metric $\langle\cdot,\cdot\rangle_1$ induced from the metric on M. If X and Y are tangent vectors to $\Lambda(M,f)$ at σ then

$$\langle X, Y\rangle_1 = \langle X, Y\rangle_0 + \langle X', Y'\rangle_0,$$

where $\langle X, Y\rangle_0 = \int_0^1 \langle X(t), Y(t)\rangle dt$ is the L^2-inner product.

The critical points for the energy integral $E^f: \Lambda(M,f)\to\mathbf{R}$ defined by

$$E^f(\sigma) = \tfrac{1}{2}\langle\dot\sigma, \dot\sigma\rangle_0$$

are precisely the geodesics $c: \mathbf{R}\to M$ with $c(t+1)=f(c(t))$, i.e. either closed f-invariant geodesics or a constant belonging to the fixed point set $\operatorname{Fix}(f)$ of f. Furthermore, E^f satisfies the important Palais-Smale condition (C) which is necessary in order to apply critical point theory (see [8]).

The $\mathbf{R}$-action on the parameter induces a continuous $S^1=\mathbf{R}/s\cdot\mathbf{Z}$-action by isometries on $\Lambda(M,f)$ under which E^f is clearly invariant [9]. Orbits of critical points with isotropy group S^1 correspond to fixed points of f, whereas orbits of critical points with finite cyclic isotropy group are embedded critical circles corresponding to oriented (unparametrized) f-invariant geodesics. By the index $\lambda(c,f)$ and nullity $\nu(c,f)$ of a critical point c for E^f in $\Lambda(M,f)$ we mean the index and nullity of the orbit $S^1\cdot c$ as a critical submanifold. The Hessian of E^f at a critical point c is given by

$$H(E^f)(X, Y) = \langle X', Y'\rangle_0 - \langle R(X,\dot c)\dot c, Y\rangle_0,$$

where R denotes the riemannian curvature tensor of M. It follows that the selfadjoint operator S defined by

$$H(E^f)(X, Y) = \langle SX, Y\rangle_1$$

admits a decomposition $S=id+k$, where k is given by

$$\langle kX, Y\rangle_1 = -\langle X + R(X,\dot c)\dot c, Y\rangle_0.$$

Since the inclusion of the Sobolev space L_1^2 into L^2 is compact so is k. In particular the index and nullity of c are finite. Furthermore, the eigenvectors X of S with eigenvalue λ are smooth, being solutions to the elliptic differential equation

$$(1-\lambda)X'' + R(X, \dot{c})\dot{c} + \lambda X = 0.$$

We conclude this paragraph by noting that $H(E^f)$ restricted to the dense subspace of all smooth "f-invariant" vector fields along c may be written as

$$H(E^f)(X, Y) = \langle LX, Y\rangle_0,$$

where L is an essentially selfadjoint elliptic differential operator defined by

$$LX = -X'' - R(X, \dot{c})\dot{c}.$$

2. Index and nullity

In this paragraph we shall study the sequences of indices and nullities of a tower of critical orbits determined by one f-invariant geodesic. In order to do this we extend our domain of study so as to contain the spaces Λ (Fix (f^n), f^m) for all n and m. Note that Fix (f^n) is a (collection of) closed totally geodesic submanifold(s) of M and that f^m (Fix $(f^n)) =$ Fix (f^n). Λ (Fix (f^n), f^m) is of course non-empty only if f^m preserves a component of Fix (f^n).

Let γ be an f-invariant geodesic, fixed throughout this paragraph. The following explicit expression for all the f-invariant geodesics with the same orientation as γ and geometrically coinciding with γ will be very important for us.

Let $c \in \Lambda(M, f)$ be a critical point of smallest E^f-value such that c and γ are equal up to a positive change in parameter. Then c is periodic of fundamental period s/m for some positive inter $m \leq s$. Now, $s/m = s_0/m_0$ where s_0 and m_0 are relatively prime positive integers. Then $s_0 \in \mathbf{Z}^+$ is the smallest positive integer with $c(\mathbf{R}) \subset$ Fix (f^{s_0}) and f "rotates" c by the fraction m_0/s_0 of its fundamental period. Since $(s_0, m_0) = 1$ we can find integers n_0 and k_0 such that $m_0 n_0 = 1 + s_0 k_0$. If we set $h = f^{n_0}$ and define $c^u\colon \mathbf{R} \to M$ for any $u \in \mathbf{R}$ by $c^u(t) = c(u \cdot t)$ for all $t \in \mathbf{R}$, then $\bar{c} = c^{1/m_0}$ is an h-invariant geodesic with fundamental period s_0 and $\bar{c}(\mathbf{R}) \subset$ Fix (f^{s_0}). Furthermore, for any pair of integers m and r with $ms_0 + rm_0 \neq 0$, $\bar{c}^{ms_0+rm_0}$ is f^r-invariant and the set of all f-invariant geodesics coinciding with c (and hence γ) up to a positive change of parameter is given by the tower of S^1-orbits $S^1 \cdot \bar{c}^{ms_0+m_0}$, $m \in \mathbf{Z}^+ \cup \{0\}$.

In order to derive the desired formulas for $\lambda(\bar{c}^{ms_0+m_0}, f)$ and $\nu(\bar{c}^{ms_0+m_0}, f)$ we need formulas for the index and nullity of $\bar{c}^{ms_0+rm_0}$ in $\Lambda(M, f^r)$ for all r.

Fix m and r and set $\bar{m}=ms_0+rm_0$. Let $\mathfrak{V}_{\bar{c}}$ be the vector space of all C^∞ vector fields along $\bar{c}$ which are orthogonal to $\bar{c}$. From § 1 we see that

$$(2.1)\qquad \begin{aligned}\lambda(\bar{c}^{\bar{m}}, f^r) &= \sum_{\mu<0} \dim\{X\in\mathfrak{V}_{\bar{c}}\,|\,LX=\mu X,\, X(t+\bar{m})=f^r_*(X(t))\;\forall t\in\mathbf{R}\}\\ \nu(\bar{c}^{\bar{m}}, f^r) &= \dim\{X\in\mathfrak{V}_{\bar{c}}\,|\,LX=0,\, X(t+\bar{m})=f^r_*(X(t))\;\forall t\in\mathbf{R}\}.\end{aligned}$$

Let us equivalently consider the complexification $V_{\bar{c}}=\mathfrak{V}_{\bar{c}}\otimes\mathbf{C}$ of $\mathfrak{V}_{\bar{c}}$ and let L and f_* denote also the $\mathbf{C}$-linear extensions of L: $\mathfrak{V}_{\bar{c}}\to\mathfrak{V}_{\bar{c}}$ and f_*: $TM\to TM$ respectively. For each real number μ, any non-zero integer m and every complex number ω of absolute value 1, we introduce the complex vector space

$$S_{\bar{c}}[\mu, m, \omega h^m_*] = \{X\in V_{\bar{c}}\,|\,LX=\mu X,\; X(t+m)=\omega h^m_*(X(t))\;\forall t\in\mathbf{R}\}.$$

Note that from (2.1)

$$(2.2)\qquad \lambda(\bar{c}^{\bar{m}}, f^r)=\sum_{\mu<0}\dim_{\mathbf{C}} S_{\bar{c}}[\mu, \bar{m}, f^r_*] \qquad \nu(\bar{c}^{\bar{m}}, f_r)=\dim_{\mathbf{C}} S_{\bar{c}}[0, \bar{m}, f^r_*].$$

In the next lemma we reduce the study of $S_{\bar{c}}[\mu, \bar{m}, f^r_*]$ to subspaces with boundary conditions imposed only at 1. Here we may consider $f^{s_0}_*$ also as a linear map of $V_{\bar{c}}$ since $\bar{c}$ is fixed by f^{s_0}.

LEMMA 2.3. *For all m, $r\in\mathbf{Z}$ with $ms_0+rm_0\neq 0$ and any $\mu\in\mathbf{R}$*

$$S_{\bar{c}}[\mu, \bar{m}, f^r_*] = \bigoplus_{\alpha^{s/s_0}=1}\;\bigoplus_{\omega^{\bar{m}}=\alpha}\;\bigoplus_{z^{\bar{n}}=\alpha^{-1}} S_{\bar{c}}[\mu, 1, \omega h_*]\cap\ker(f^{s_0}_*-z),$$

where $\bar{m}=ms_0+rm_0$ and $\bar{n}=mn_0+rk_0$.

Proof. We first observe that

$$S_{\bar{c}}[\mu, \bar{m}, f^r_*]\subset S_{\bar{c}}[\mu, \bar{m}\cdot s, h^{\bar{m}\cdot s}_*] = \bigoplus_{\omega^{\bar{m}\cdot s}=1} S_{\bar{c}}[\mu, 1, \omega h_*].$$

The first inclusion is trivial since $f^s=id$ and the second is essentially the same as Theorem I in Bott [1]. Every $Y\in S_{\bar{c}}[\mu, \bar{m}\cdot s, h^{\bar{m}\cdot s}_*]$ admits a unique expansion

$$Y=\sum_{\omega^{\bar{m}s}=1}\omega Y_\omega, \text{ with } Y_\omega\in S_{\bar{c}}[\mu, 1, \omega h_*]$$

given by

$$Y_\omega(t)=1/|\bar{m}s|\cdot\sum_{q=0}^{|\bar{m}s|-1}\omega^{-q}h_*^{-q+1}(Y(t+q-1)).$$

Since for $X\in S_{\bar{c}}^-[\mu, \bar{m}, f_*^r]$ clearly $X_\omega\in S_{\bar{c}}^-[\mu, \bar{m}, f_*^r]$ we get

$$S_{\bar{c}}^-[\mu, \bar{m}, f_*^r]=\bigoplus_{\omega^{\bar{m}s}=1} S_{\bar{c}}^-[\mu, 1, \omega h_*]\cap S_{\bar{c}}^-[\mu, \bar{m}, f_*^r].$$

A straightforward computation shows that

$$S_{\bar{c}}^-[\mu, 1, \omega h_*]\cap S_{\bar{c}}^-[\mu, \bar{m}, f_*^r]=S_{\bar{c}}^-[\mu, 1, \omega h_*]\cap \ker(f_*^{s_0\bar{n}}-\omega^{-\bar{m}})$$

and when $\bar{n}\neq 0$ an expansion argument as above for

$$Y\in\ker(f_*^{s_0\bar{n}}-\alpha^{-1}) \text{ with } Y_z=1/|\bar{n}|\cdot\sum_{q=0}^{|\bar{n}|-1} z^{-q}f_*^{s_0(q-1)}(Y)\in\ker(f_*^{s_0}-z)$$

where z ranges over $\bar{n}$-roots of $\alpha^{-1}=\omega^{-\bar{m}}$, proves that

$$S_{\bar{c}}^-[\mu, 1, \omega h_*]\cap\ker(f_*^{s_0\bar{n}}-\alpha^{-1})=\bigoplus_{z^{\bar{n}}=\alpha^{-1}} S_{\bar{c}}^-[\mu, 1, \omega h_*]\cap\ker(f_*^{s_0}-z)$$

In the case $\bar{n}=0$ a direct computation shows that the above equality holds. Thus

$$S_{\bar{c}}^-[\mu, \bar{m}, f_*^r]=\bigoplus_{\alpha^s=1}\ \bigoplus_{\omega^{\bar{m}}=\alpha}\ \bigoplus_{z^{\bar{n}}=\alpha^{-1}} S_{\bar{c}}^-[\mu, 1, \omega h_*]\cap\ker(f_*^{s_0}-z)$$

and we are through because every z with $\ker(f_*^{s_0}-z)\neq\{0\}$ satisfies $z^{s/s_0}=1$.

For each complex number z of absolute value 1, we define non-negative integer valued functions Λ^z and N^z on the unit circle $S^1\subset\mathbf{C}$ by

$$\Lambda^z(\omega)=\sum_{\mu<0}\dim_{\mathbf{C}}\{S_{\bar{c}}^-[\mu, 1, \omega h_*]\cap\ker(f_*^{s_0}-z)\}$$

and

$$N^z(\omega)=\dim_{\mathbf{C}}\{S_{\bar{c}}^-[0, 1, \omega h_*]\cap\ker(f_*^{s_0}-z)\}$$

for all $\omega\in S^1$. From (2.2) and Lemma 2.3 we obtain the desired formulas

$$\lambda(\bar{c}^{ms_0+rm_0}, f^r)=\sum_{\alpha^{s/s_0}=1}\ \sum_{\omega^{ms_0+rm_0}=\alpha}\ \sum_{z^{mn_0+rk_0}=\alpha^{-1}}\Lambda^z(\omega) \tag{2.4}$$
$$\nu(\bar{c}^{ms_0+rm_0}, f^r)=\sum_{\alpha^{s/s_0}=1}\ \sum_{\omega^{ms_0+rm_0}=\alpha}\ \sum_{z^{mn_0+rk_0}=\alpha^{-1}}N^z(\omega)$$

Note that Λ^z and N^z are identically zero unless $\ker(f_*^{s_0}-z)\neq\{0\}$. In particular there are only finitely many non-zero functions Λ^z and N^z. We obtain further properties of these functions from the following observation. The complexification of the normal bundle $\bar{c}(\mathbf{R})^\perp$ to $\bar{c}(\mathbf{R})$ in M admits a Whitney sum decomposition into "eigenbundles" for $f_*^{s_0}$ considered as a bundle map. If $\bar{c}(\mathbf{R})^\perp(z)$ denotes the eigenbundle for $f_*^{s_0}$ with the eigenvalue z, and if $V_{\bar{c}}(z) \subset V_{\bar{c}}$ is the complex vector fields along $\bar{c}$ in $\bar{c}(\mathbf{R})^{\perp}(z)$ then clearly $\ker(f_*^{s_0} - z) = V_{\bar{c}}(z)$. Furthermore L preserves $V_{\bar{c}}(z)$ since it commutes with $f_*^{s_0}$. Let L^z be the restriction of L to $V_{\bar{c}}(z)$, then

$$S_{\bar{c}}[\mu, 1, \omega h_*] \cap \ker(f_*^{s_0} - z) = \{X \in V_{\bar{c}}(z) \mid L^z X = \mu X,\ X(t+1) = \omega h_*(X(t))\}.$$

If we identify $\Lambda(M, h)$ in the canonical way with L_1^2-sections of the mapping torus bundle $M \times_h I = M \times \mathbf{R}/\mathbf{Z} \to \mathbf{R}/\mathbf{Z} = S^1$ of h, then the boundary condition $X(t+1) = \omega h_*(X(t))$ is exchanged with $X(t+1) = \omega X(t)$. Here $\mathbf{Z}$ acts on $M \times \mathbf{R}$ by $(n, (p, t)) \to (h^n(p), t+n)$ (see [10]). From this we see that Λ^z and N^z are functions of the type Λ and N introduced in Bott [1]. In particular Λ^z and N^z have the following important properties.

(2.5) For each z, $N^z(\omega) = 0$ except for at most $2 \dim \bar{c}(\mathbf{R})^{\perp}(z)$ $(\leqslant 2(\dim M - 1))$ points, the so called Poincaré points of L^z.

(2.6) For each z, Λ^z is locally constant except possibly at the Poincaré points of L^z.

For each z, the inequality

(2.7) $\lim\limits_{\omega \to \omega_0} \Lambda^z(\omega) \geqslant \Lambda^z(\omega_0)$

holds for any ω_0.

Remark. The above properties for Λ^z and N^z as well as (2.1) can also be derived by the classical methods of Morse involving quadratic forms on finite dimensional approximations of our spaces (se in particular Theorem IV, 3.1, IV 3.2 and III 2.3 in [15] and compare with [22]).

We should also like to remark that the Poincaré points of L^z can be described by means of the geodesic flow in the unit tangent bundle of $M \times_h I$. Let $P_{\bar{c}}$ denote the differential of the Poincaré map for the closed orbit $t \to (\dot{\bar{c}}(t), \partial/\partial t)/\|\cdot\|$ of the geodesic flow for $M \times_h I$. If we consider f^{s_0} as an isometry on $M \times_h I$ (identity on I) then $P_{\bar{c}}$ commutes with $f_{**}^{s_0}$. Hence, (the complexification of) $P_{\bar{c}}$ preserves the eigenspaces of $f_{**}^{s_0}$. The Poincaré points of L^z can now be described as the set of eigenvalues of norm 1 for $P_{\bar{c}}$ restricted to the z-eigenspace for $f_{**}^{s_0}$. Note that in the horizontal and vertical splitting of the double tangent bundle $f_{**}^{s_0} = (f_*^{s_0}, f_*^{s_0})$.

We are now in position to derive a growth estimate for the sequence

$$\{\lambda(\bar{c}^{ms_0+m_0}, f)\}_{m \in \mathbf{Z}^+ \cup \{0\}}$$

analogous to Lemma 1 in [7].

LEMMA 2.8. *Either $\lambda(\bar{c}^{ms_0+m_0}, f)=0$ for all $m\in\mathbf{Z}^+\cup\{0\}$ or there exist numbers $\varepsilon, a\in\mathbf{R}^+$ such that*

$$\lambda(\bar{c}^{m_1s_0+m_0}, f)-\lambda(\bar{c}^{m_2s_0+m_0}, f)\geqslant(m_1-m_2)\,s_0\cdot\varepsilon-a$$

for all integers $m_1\geqslant m_2\geqslant 0$.

Proof. Fix a z with $\ker(f_*^{s_0}-z)\neq\{0\}$, in particular $z^{s/s_0}=1$. Let $0<\varrho_1(z)<\ldots<\varrho_{r(z)}(z)\leqslant 1$ be the Poincaré exponents with respect to z i.e. $e^{2\pi i\varrho_1(z)},\ldots,e^{2\pi i\varrho_{r(z)}(z)}$ are the Poincaré points of L^z (see (2.5)). Set $\varrho_0(z)=0$, $\varrho_{r(z)+1}(z)=1$, and let $a_j(z)=\Lambda^z(\omega)$ for $\omega=e^{2\pi i\alpha}$ with $\alpha\in]\varrho_{j-1}(z),\varrho_j(z)[$ and $1\leqslant j\leqslant r(z)+1$; compare (2.6). In case $\varrho_{r(z)}(z)=1$ set $a_{r(z)+1}(z)=0$. By simple angle comparison we get

$$\sum_{\omega^{m_1s_0+m_0}=\alpha_1}\Lambda^z(\omega)\geqslant\sum_{j=1}^{r(z)+1}\left[(m_1s_0+m_0)\left(\varrho_j(z)-\varrho_{j-1}(z)\right)-1\right]a_j(z)$$

with $\alpha_1=z^{-m_1n_0-k_0}$, and

$$\sum_{\omega^{m_2s_0+m_0}=\alpha_2}\Lambda^z(\omega)\leqslant\sum_{j=1}^{r(z)+1}\left[(m_2s_0+m_0)\left(\varrho_j(z)-\varrho_{j-1}(z)\right)+1\right]a_j(z)$$

with $\alpha_2=z^{-m_2n_0-k_0}$. Thus from (2.4) we get

$$\lambda(\bar{c}^{m_1s_0+m_0}, f)-\lambda(\bar{c}^{m_2s_0+m_0}, f)\geqslant\sum_z\sum_{j=1}^{r(z)+1}\left((m_1-m_2)\,s_0(\varrho_j(z)-\varrho_{j-1}(z))-2\right)a_j(z)$$

Now, if $\lambda(\bar{c}^{ms_0+m_0}, f)\neq 0$ for some m, then by (2.4), (2.6) and (2.7) there is a z_0 and a j_0 such that $\varrho_{j_0-1}(z_0)<\varrho_{j_0}(z_0)$ and $a_{j_0}(z_0)>0$. Hence

$$\lambda(\bar{c}^{m_1s_0+m_0}, f)-\lambda(c^{m_2s_0+m_0}, f)\geqslant(m_1-m_2)\,s_0\cdot\varepsilon-a,$$

where
$$\varepsilon=\left(\varrho_{j_0}(z_0)-\varrho_{j_0-1}(z_0)\right)a_{j_0}(z_0)\quad\text{and}\quad a=\sum_z\sum_{j=1}^{r(z)+1}2a_j(z).$$

The next lemma is a crucial generalization of Lemma 2 in [7]. Before stating it, note that if c is an f-invariant geodesic fixed by f^m for some m, then c is also critical for the restriction of E^f to $\Lambda(\mathrm{Fix}(f^m), f)$. We denote the nullity of c in $\Lambda(\mathrm{Fix}(f^m), f)$ by $\nu(c, f\,|\,\mathrm{Fix}(f^m))$.

LEMMA 2.9. *There exist positive integers $k_1,\ldots,k_q$ and sequences $\{m_j^i\}$, $i>0$, $j=1,\ldots,q$ such that the numbers $m_j^ik_j$ are mutually distinct, $\{m_j^ik_j\}=\{ms_0+m_0\,|\,m\in\mathbf{Z}^+\cup\{0\}\}$ and*

$$\nu(\bar{c}^{m_j^ik_j}, f)=\nu(\bar{c}^{m_j^ik_j}, f\,|\,\mathrm{Fix}(f^{s_0s_j^i}))=\nu(\bar{c}^{k_j}, f^r\,|\,\mathrm{Fix}(f^{s_0s_j^i})),$$

where s_j^i is the maximal integer relatively prime to m_j^i and dividing s/s_0, and where r is an integer with the property $rm_j^i \equiv 1 \bmod s_0 s_j^i$.

Proof. For each $l \in \{1, ..., s/s_0\}$, each $\alpha = e^{2\pi i^u/v}$ with $(u, v) = 1$ and $v | s/s_0$ let P_l^α be the collection of Poincaré points for L^z with $z^l = \alpha^{-1}$ i.e.

$$P_l^\alpha = \{\omega \mid \sum_{z^l = \alpha^{-1}} N^z(\omega) > 0\}$$

and let

$$Q_l^\alpha = \{q \in \mathbf{Z}^+ \mid \exists b \in \mathbf{Z}^+ \text{ s.t. } (b, qv) = 1,\ e^{2\pi i^b/qv} \in P_l^\alpha\}.$$

If we set

$$Q_l = \bigcup_{\alpha^{s/s_0} = 1} Q_l^\alpha \quad \text{and} \quad Q = Q_1 \cup \ldots \cup Q_{s/s_0} \cup \{1\}$$

then Q is a finite set by (2.5). Note that if $\nu(\bar{c}^{ms_0+m_0}, f) \neq 0$ for some $m \in \mathbf{Z}^+ \cup \{0\}$, then by (2.4) there exist l, $\alpha = e^{2\pi i^u/v}$ and $\omega = e^{2\pi i^p/q}$ with $(p, q) = (u, v) = 1$ such that $\omega^{ms_0+m_0} = \alpha$ and $\omega \in P_l^\alpha$. The property $\omega^{ms_0+m_0} = \alpha$ implies that v divides q i.e. $q/v \in Q_l^\alpha$.

For each subset $D \subset Q$, let $k(D)$ denote the least common multiple of all the element in D. Choose distinct numbers $\bar{k}_1, ..., \bar{k}_t$ such that $\{\bar{k}_1, ..., \bar{k}_t\} = \{k(D) \mid D \subset Q\}$. For each $j \in \{1, ..., t\}$ we select from the sequence $m\bar{k}_j$, $m \in \mathbf{Z}^+$ the greatest subsequence $\bar{m}_j^i \bar{k}_j$ with the property that whenever $q \in Q$ and $q | \bar{m}_j^i \bar{k}_j$ then $q | \bar{k}_j$. Then the numbers $\bar{m}_j^i \bar{k}_j$ are mutually distinct and $\{\bar{m}_j^i \bar{k}_j \mid i > 0,\ j = 1, ..., t\} = \mathbf{Z}^+$. Let now $\{\bar{k}_{j_1}, ..., \bar{k}_{j_q}\}$ be the maximal subset of $\{\bar{k}_1, ..., \bar{k}_t\}$ such that $\{\bar{m}_{j_r}^i \bar{k}_{j_r} \mid i > 0\} \cap \{ms_0 + m_0 \mid m \in \mathbf{Z}^+ \cup \{0\}\} \neq \emptyset$ for every $r \in \{1, ..., q\}$. Choose subsequences $\{m_r^i\}$, $i > 0$, $r \in \{1, ..., q\}$ from the sequences $\{\bar{m}_{j_r}^i\}$ so that $\{m_r^i \bar{k}_{j_r} \mid i > 0,\ r \in \{1, ..., q\}\} = \{ms_0 + m_0 \mid m \in \mathbf{Z}^+ \cup \{0\}\}$. If we set $k_r = \bar{k}_{j_r}$, then we claim that the positive integers $k_1, ..., k_q$ and sequences $\{m_j^i\}$, $i > 0$, $j = 1, ..., q$ have the required properties.

Let us fix a k_j and an m_j^i. Then there is a unique integer m such that $ms_0 + m_0 = m_j^i k_j$. Let $l \in \{1, ..., s/s_0\}$ be determined by $l \equiv mn_0 + k_0 \bmod s/s_0$. Suppose that for some $\alpha = e^{2\pi i^u/v}$ with $(u, v) = 1$

$$\sum_{\omega^{m_j^i k_j} = \alpha} \sum_{z^l = \alpha^{-1}} N^z(\omega) \neq 0$$

i.e. there is an $\omega = e^{2\pi i^b/qv} \in P_l^\alpha$ with $(b, qv) = 1$, so that $\omega^{m_j^i k_j} = \alpha$. Hence m_j^i and v are relatively prime. Thus from (2.4) we get

$$\nu(\bar{c}^{m_j^i k_j}, f) = \sum_{\alpha^{s_j^i} = 1} \sum_{\omega^{m_j^i k_j} = \alpha} \sum_{z^l = \alpha^{-1}} N^z(\omega),$$

where s_j^i is the maximal integer which satisfies $(s_j^i, m_j^i) = 1$ and $s_j^i | s/s_0$. Let $s_1^{n_1} \cdot \ldots \cdot s_p^{n_p}$ be the decomposition of $s/s_0 s_j^i$ into prime factors. From $m_j^i k_j n_0 \equiv l s_0 + 1 \bmod s/s_0$ and $s_1, ...,$

$s_p|m_j^i$ we see that l and $s/s_0s_j^i$ are relatively prime. Thus if z satisfies $z^{s/s_0}=1$ and $z^l=\alpha^{-1}$ where $\alpha^{s_j^i}=1$ then $z^{s_j^i}=1$. In the above expression for $\nu(\bar{c}^{m_j^ik_j}, f)$ this means that the sum over z is taken only for z satisfying $\ker(f_*^{s_0}-z)\neq\{0\}$ and $z^{s_j^i}=1$ i.e.

$$\nu(\bar{c}^{m_j^ik_j}, f)=\nu(\bar{c}^{m_j^ik_j}, f|\operatorname{Fix}(f^{s_0s_j^i})).$$

Since $(s_0, m_j^ik_j)=1$ and $(s_j^i, m_j^i)=1$ we can pick an integer r so that $rm_j^i\equiv 1 \bmod s_0s_j^i$. Therefore for each α with $\alpha^{s_j^i}=1$ we have

$$\{\omega\in P_l^\alpha|\,\omega^{m_j^ik_j}=\alpha\}=\{\omega\in P_l^\alpha|\,\omega^{k_j}=\alpha^r\}$$

and hence

$$\nu(\bar{c}^{m_j^ik_j}, f|\operatorname{Fix}(f^{s_0s_j^i}))=\sum_{\alpha^{s_j^i}=1}\ \sum_{\omega^{k_j}=\alpha^r}\ \sum_{z^l=\alpha^{-1}} N^z(\omega).$$

On the other hand $k_j\equiv(rm)s_0+rm_0 \bmod s_0s_j^i$ and $lr\equiv(rm)n_0+rk_0 \bmod s/s_0$. Thus we can find an integer n so that $k_j=ns_0+rm_0$ and $nn_0+rk_0\equiv lr \bmod s_j^i$. Therefore $\bar{c}^{k_j}\in\Lambda(\operatorname{Fix}(f^{s_0s_j^i}), f^r)$ and from (2.4) we have

$$\nu(\bar{c}^{k_j}, f^r|\operatorname{Fix}(f^{s_0s_j^i}))=\sum_{\alpha^{s_j^i}=1}\ \sum_{\omega^{k_j}=\alpha}\ \sum_{z^{lr}=\alpha^{-1}} N^z(\omega)$$

Since $(r, s_j^i)=1$ we are done.

3. Local and characteristic invariants

In finite dimensions Morse [15] associated to any isolated critical point a local homological invariant. His construction was modified and generalized to infinite dimensions by Gromoll and Meyer [6], in the case where the involved function satisfies condition (C) and the hessian operator is of the form $S=id+k$, where k is a compact operator (see § 1).

Consider an isolated critical orbit $S^1\cdot c$ in $\Lambda(M, f)$. The S^1-action on $\Lambda(M, f)$ induces an isometric S^1-action on the normal bundle $\mathcal{N}$ of $S^1\cdot c$. Let Ψ: $\mathcal{N}\to\Lambda(M, f)$ be an arbitrary equivariant smooth map which is the identity on the zero-section and of maximal rank there. The image by Ψ of a sufficiently small discbundle of $\mathcal{N}$ defines an equivariant tubular neighbourhood $\mathcal{D}=S^1\cdot\mathcal{D}_c$ of $S^1\cdot c$, where $\mathcal{D}_c$ is the fiber over c. By the so called splitting lemma of Gromoll and Meyer [6] any function with an isolated critical point (and which has the above properties) splits locally in a non-degenerate part and a completely degenerate part. From an orbit version of that lemma it follows that for $\mathcal{D}_c$ sufficiently small E^f restricted to $\mathcal{D}_c$ satisfies condition (C) and has only c as a critical point (compare [7] and [22]).

We can now define the local invariant for $E^f | \mathcal{D}_c$ at c. For $\delta>0$ we let $d_\delta: \mathcal{D}_c \to \mathbf{R}$ be given by

$$d_\delta(\sigma) = 2\delta^{-1}(\varrho/5)^2 E(\sigma) + \|\Psi^{-1}(\sigma)\|_1^2$$

for all $\sigma \in \mathcal{D}_c$, where $E = E^f | \mathcal{D}_c - E^f(c)$. Then for δ sufficiently small

$$W_c(\delta) = E^{-1}[-\delta, \delta] \cap d_\delta^{-1}(-\infty, 17\cdot(\varrho/5)^2],\ W_c(\delta)^- = E^{-1}(-\delta) \cap W_c(\delta)$$

is a pair of so called admissible regions and

$$\mathcal{H}(E^f, c) = H_*(W_c(\delta),\ W_c(\delta)^-)$$

is a well defined local homological invariant of c (see [6]; for simplicity we have chosen ϱ_1 and ϱ_0 there to be $\varrho_1 = \varrho$ and $\varrho_0 = (3/5)\cdot\varrho$ respectively). Here we take homology with coefficients in an arbitrary field. If we set $W(\delta) = S^1 \cdot W_c(\delta)$ and $W(\delta)^- = S^1 \cdot W_c(\delta)^-$ then for δ sufficiently small

$$\mathcal{H}(E^f, S^1\cdot c) = H_*(W(\delta),\ W(\delta)^-)$$

is a well defined local homological invariant of the orbit $S^1\cdot c$ (see also Klingenberg [13] for a different approach). The crucial property of this invariant is contained in the following lemma, which is proved by deformation and excision arguments exactly as Lemma 4 in Gromoll and Meyer [7].

LEMMA 3.1. *If b is the only critical value of E^f in $[b-\varepsilon, b+\varepsilon]$ for some $\varepsilon>0$ and if $S^1\cdot c_1, \ldots, S^1\cdot c_n$ are the only critical orbits with E^f-value b then*

$$H_*(\Lambda(M,f)^{b+\varepsilon}, \Lambda(M,f)^{b-\varepsilon}) = \bigoplus_{i=1}^{n} \mathcal{H}(E^f, S^1\cdot c_i),$$

where as usual $\Lambda(M,f)^a = (E^f)^{-1}(-\infty, a]$.

Observe now that $(W(\delta),\ W(\delta)^-)$ can be considered as a pair of bundles over the circle S^1 with fiber $(W_c(\delta),\ W_c(\delta)^-)$. If we write S^1 as the union of two intervals and apply the relative Mayer-Vietoris sequence [18] to the corresponding two pairs of trivial bundles, we get

$$\dim \mathcal{H}_k(E^f, S^1\cdot c) \leqslant 2\ (\dim \mathcal{H}_k(E^f, c) + \dim \mathcal{H}_{k-1}(E^f, c))$$

for all k. Hence it is sufficient to study the local invariants $\mathcal{H}(E^f, c)$.

In Gromoll and Meyer [6] there was also introduced a characteristic invariant $\mathcal{H}^0$, which will play a very important role in this paper. This invariant may be defined as the local invariant of the degenerate part of the function. Since $\nu(c,f) \leqslant 2\ (\dim M - 1)$

for any critical point $c \in \Lambda(M, f)$ we get in particular that the characteristic invariant of $E^f | \mathcal{D}_c$ at c satisfies

$$\mathcal{H}_k^0(E^f, c) = 0 \quad \text{for} \quad k \geq 2 \dim M - 1. \tag{3.2}$$

Furthermore, according to the shifting theorem of [6]

$$\mathcal{H}_{k+\lambda}(E^f, c) = \mathcal{H}_k^0(E^f, c) \quad \text{for all } k,$$

where $\lambda = \lambda(c, f)$. In particular, if we set

$$B_k(c, f) = \dim \mathcal{H}_k(E^f, S^1 \cdot c) \quad \text{and} \quad B_k^0(c, f) = \dim \mathcal{H}_k^0(E^f, c)$$

then

$$B_k(c, f) \leq 2(B_{k-\lambda}^0(c, f) + B_{k-\lambda-1}^0(c, f)) \tag{3.3}$$

for all k, when $\lambda = \lambda(c, f)$ as above. We will use $B_k(c)$ instead of $B_k(c, f)$ when there is no danger of confusion.

For the behavior of characteristic invariants we recall also the following very useful lemma of [6].

LEMMA 3.4. *Suppose c is an isolated critical point for $E: \Lambda \to \mathbf{R}$ and let $\hat{\Lambda}$ be a closed Hilbert submanifold of Λ through c. If* grad *E restricted to $\hat{\Lambda}$ is tangent to $\hat{\Lambda}$ and if the null space of the Hessian of E at c is contained in $T_c \hat{\Lambda}$ then $\mathcal{H}^0(E, c) = \mathcal{H}^0(E | \hat{\Lambda}, c)$.*

From this lemma we shall now derive the following two impotant properties for the characteristic invariants of isometry-invariant geodesics.

PROPOSITION 3.5. *Let N be a totally geodesic submanifold of M with $f(N) = N$. Let $c: \mathbf{R} \to N$ be an f-invarint geodesic such that $S^1 \cdot c$ is an isolated critical orbit in $\Lambda(M, f)$. Then $\mathcal{H}^0(E^f, c) = \mathcal{H}^0(E^f | \Lambda(N, f), c)$ if $\nu(c, f) = \nu(c, f | N)$.*

Proof. Since N is totally geodesic it is intuitively obvious that grad E^f is tangent to the closed submanifold $\Lambda(N, f)$ of $\Lambda(M, f)$. In order to prove it we pick a $\sigma \in \Lambda(N, f)$. Since the set $C^\infty(N, f)$ of smooth f-invariant curves is dense in $\Lambda(N, f)$ we can assume that σ is C^∞. Let $X \in T_\sigma \Lambda(M, f)$ be orthogonal to $\Lambda(N, f)$. Then X is pointwise orthogonal to N because N is totally geodesic. Hence

$$\mathrm{d}E^f(X) = \langle X', \dot{\sigma} \rangle_0 = -\langle X, \dot{\sigma}' \rangle = 0$$

since σ is smooth and $\dot{\sigma}'$ is tangent to N.

Let now $S^1 \cdot \mathcal{D}_c^N$ be an equivariant tubular neighbourhood of $S^1 \cdot c$ in $\Lambda(N, f)$ and let D be an equivariant tubular neighbourhood of $\Lambda(N, f)$ in $\Lambda(M, f)$. Then $\mathcal{D} = D | S^1 \cdot \mathcal{D}_c^N$ is an equivariant tubular neighbourhood of $S^1 \cdot c$ in $\Lambda(M, f)$ of the form $S^1 \cdot \mathcal{D}_c$ where $\mathcal{D}_c = D | \mathcal{D}_c^N$. Since grad E^f is tangent to $\Lambda(N, f)$ we see from this that gard $(E^f | \mathcal{D}_c)$ restricted to $\mathcal{D}_c^N$ is tangent to $\mathcal{D}_c^N$. Moreover, the null space of the Hessian of $E^f | \mathcal{D}_c$ at c consists of f-invariant Jacobi fields orthogonal to c. From N totally geodesic in M and $\nu(c, f) = \nu(c, f | N)$ it follows that the above null space is contained in $T_c \mathcal{D}_c^N$. Hence $\mathcal{H}^0(E^f, c) = \mathcal{H}^0(E^f | \Lambda(N, f), c)$ by Lemma 3.4

PROPOSITION 3.6. *Assume that $S^1 \cdot c$ is a critical orbit of E^f such that $S^1 \cdot c^m$ for some m is an isolated critical orbit for E^{f^m} in $\Lambda(M, f^m)$. Then $S^1 \cdot c$ is also isolated and $\mathcal{H}^0(E^f, c) = \mathcal{H}^0(E^{f^m}, c^m)$ if $\nu(c, f) = \nu(c^m, f^m)$.*

Proof. By arguing on the mapping torus $M \times_f I$ for f (compare the remark in § 2) this proposition follows from Theorem 3 in [7]. For the sake of completeness we proceed as follows:

Let m: $\Lambda(M, f) \to \Lambda(M, f^m)$ be the iteration map defined by $m(\sigma) = \sigma^m$ for all $\sigma \in \Lambda(M, f)$. If we endow $\Lambda(M, f^m)$ with the following equivalent riemannian metric

$$\langle X, Y \rangle_{1,m} = \langle X, Y \rangle_0 + m^{-2} \langle X', Y' \rangle_0$$

then m is an isometric embedding. Furthermore $E \circ m = m^2 \cdot E$ and hence $\mathcal{H}^0(c, f) = \mathcal{H}^0(E^{f^m} | m(\Lambda(M, f)), c^m)$.

Note that the action of f on M as well as the translation by m^{-1} on $\mathbf{R}$ induces two isometric $\mathbf{Z}$-actions on $\Lambda(M, f^m)$. Moreover, $m(\Lambda(M, f))$ is exactly the submanifold of $\Lambda(M, f^m)$ on which these two actions coincide. Since both actions leave E^{f^m} invariant we see from this that grad E^{f^m} is tangent to $m(\Lambda(M, f))$. We can now argue as in Proposition 3.5 so as to obtain $\mathcal{H}^0(E^{f^m} | m(\Lambda(M, f)), c^m) = \mathcal{H}^0(E^{f^m}, c^m)$.

Combining the above propositions with Lemma 2.9 we see in particular that there are only finitely many characterisic invariants associated with a single (isolated tower of) f-invariant geodesic(s).

4. Existence of infinitely many invariant geodesics

Let M be a compact, connected and simply connected riemannian manifold and let A: $M \to M$ be an isometry. Then the Banach manifold $C^0(M, A)$ of continuous curves σ: $[0, 1] \to M$ satisfying $\sigma(1) = A(\sigma(0))$ with the compact-open topology is connected. Using the evaluation fibration "at 0", $C^0(M, A) \to M$ with fiber the ordinary loop space Ω (up to homotopy), it follows from Serre [17] that the Betti numbers $b_k(C^0(M, A)) = \dim H_k(C^0(M, A))$ are all finite. Hence our main theorem is a consequence of the next theorem.

THEOREM 4.1. *Assume that M is a compact connected riemannian manifold and that $f: M \to M$ is an isometry of finite order. If there are at most finitely many f-invariant geodesics on M, then the sequence $b_k(C^0(M, f))$, $k \geq 2 \dim M$ is bounded.*

Proof. Since the inclusion $\Lambda(M, f) \subset C^0(M, f)$ is a homotopy equivalence [8] we may just as well prove the theorem for $\Lambda(M, f)$.

Suppose f has only finitely many invariant geodesics represented in $\Lambda(M, f)$ by the orbits $S^1 \cdot \bar{c}_1^{ms_1+m_1}, \ldots, S^1 \cdot \bar{c}_r^{ms_r+m_r}$, $m \in \mathbf{Z}^+ \cup \{0\}$, where $\bar{c}_i$ are f^{n_i}-invariant as in § 2. Since all these orbits are isolated we can apply the results of the previous paragraphs. By Lemma 2.9, Proposition 3.5 and 3.6 there is a constant $B > 0$ such that

$$B_k^0(\bar{c}_i^{ms_i+m_i}) \leq B \quad \text{for all} \quad k,\ i = 1, \ldots, r \quad \text{and} \quad m \in \mathbf{Z}^+ \cup \{0\}.$$

From this, (3.2), (3.3) and Lemma 2.8

$$B_k(\bar{c}_i^{ms_i+m_i}) \leq 4B \quad \text{for all} \quad k, i \quad \text{and} \quad m$$

and the number of orbits with $B_k(\bar{c}_i^{ms_i+m_i}) \neq 0$ is bounded by a constant $C > 0$ for all $k \geq 2 \dim M$. Thus from Lemma 3.1 together with an exact sequence argument we get (Morse inequalities) for all regular values $0 < a < b$

$$b_k(\Lambda(M, f)^b, \Lambda(M, f)^a) \leq 4BC \quad \text{for} \quad k \geq 2 \dim M.$$

For a sufficiently small $a > 0$, Fix $(f) \subset \Lambda(M, f)$ is a strong deformation retract of $\Lambda(M, f)^a$ [8]. Since furthermore dim Fix $(f) \leq \dim M$ we see that

$$b_k(\Lambda(M, f)^b) \leq 4BC \quad \text{for} \quad k \geq 2 \dim M$$

and all regular values b. Fix now a $k \geq 2 \dim M$ and choose b so large that $B_k(c) = B_{k+1}(c) = 0$ for all critical orbits $S^1 \cdot c$ with $E^f(c) > b$. Then, again by Lemma 3.1 and an exact sequence argument $b_k(\Lambda(M, f)) = b_k(\Lambda(M, f)^b)$. Hence $\sup \{b_k(\Lambda(M, f)) \mid k \geq 2 \dim M\} \leq 4BC$.

Note that if M is not simply connected then $C^0(M, f)$ is not connected and we can very well have $b_k(C^0(M, f)) = \infty$ if $k < 2 \dim M$ even in the case where f has only finitely many invariant geodesics. This may happen if there is an f-invariant geodesic all of whose iterates have index zero.

As an immediate application of our theorem and the Sullivan-Vigué theorem [21] mentioned in the introduction we have:

COROLLARY 4.2. *Let M be a compact, 1-connected manifold whose rational cohomology ring is not generated by one element, and let $f: M \to M$ be a finite order smooth map which is homotopic to id_M. Then there are infinitely many f-invariant geodesics on M in any metric for which f is an isometry.*

Let us discuss the general case where $A: M \to M$ is an arbitrary isometry. One might still hope that the condition $\{b_k(\Lambda(M, A))\}_{k \geqslant 0}$ unbounded would ensure the existence of infinitely many A-invariant geodesics.

Consider the example $A = A_1 \times A_2: S^2 \times S^2 \to S^2 \times S^2$, where A_1 and A_2 are rotations on the 2-sphere of constant curvature 1. A is clearly homotopic to $id_{S^2 \times S^2}$ and hence $\Lambda(S^2 \times S^2, A)$ has unbounded sequence of Betti numbers. From the geometry it is clear that any such A has infinitely many invariant geodesics in the product metric on $S^2 \times S^2$. Note however, that if the ratio between the rotations of A_1 and A_2 is irrational, then A has only four closed invariant geodesics. Hence as far as closed A-invariant geodesics is concerned our theorem seems to be optimal.

Note also that if in the above example $A_1 = A_2$ is an irrational rotation on S^2 then the isometric S^1-action on $S^2 \times S^2$ generated by A has only finitely many (in fact five) geodesic orbits.

In general it is true, that if A has a non-closed invariant geodesic then it has uncountably many invariant geodesics (see [9]). Thus if A has only finitely many invariant geodesics $c_1, \ldots, c_r$ then they must all be closed. Here again we are faced with the problem of the iterates of each c_i, $i = 1, \ldots, r$. Fix $i \in \{1, \ldots, r\}$ and let c be an A-invariant geodesic with minimal E^A-value having the same image as c_i. Then the fundamental period of c is α for some $\alpha \geqslant 1$, and all the iterates of c are described by $c^{m\alpha+1}$, $m \in \mathbf{Z}^+ \cup \{0\}$. Using a general index theorem by Klingmann [14] together with Lemma 2.8 (in the special case $f = id_M$) it is possible to derive a growth estimate for the sequence $\lambda(c^{m\alpha+1}, A)$ $m \geqslant 0$ exactly of the type in Lemma 2.8 (see also [16]). Hence from our arguments in the proof of Theorem 4.1 we see that the main theorem holds for an arbitrary isometry if there are only finitely many different characteristic invariants among $\mathcal{H}^0(E^A, c^{m\alpha+1})$. As we observed, this follows from Lemma 2.9, Proposition 3.5 and 3.6 in the case where $A = f$ is of finite order, and it seems to us that the general case will have to be treated in quite a different manner.

An interesting problem in connection with this paper is to find necessary and sufficient conditions on A and on M for $\Lambda(M, A)$ to have an unbounded sequence of Betti numbers. Here one can assume that $A = f$ is of finite order. As mentioned in the introduction the problem has been solved completely for rational coefficients in the case $f = id_M$ [21]. In that case it is not difficult to find the so called minimal model for $\Lambda(M, id_M)$, which contains all the information about the rational homotopy theory of the space (see [19], [4], [20], [21] and also [13]). The general case is more subtle and will be treated in a subsequent paper by the first named author, S. Halperin and M. Vigué [11].

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