ON THE CLASSIFICATION OF KLEINIAN GROUPS II—SIGNATURES

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Part of the classical theory of Fuchsian groups is a classification of finitely generated Fuchsian groups of the first kind. This classification proceeds roughly as follows. To each such Fuchsian group there is assigned a set of integers, called the signature; there is a simple set of conditions describing which sets of integers actually occur as signatures of Fuchsian groups. Two groups have the same signature if and only if one is a deformation of the other; i.e., there is a (quasiconformal) homeomorphism of the disc which conjugates one group into the other. The set of groups of a given signature has a real analytic structure, and can in a natural way be regarded as Euclidean n-space factored by a discontinuous group.

In this paper we give a similar classification for those Kleinian groups which have both an invariant region of discontinuity, and which, in their action on hyperbolic 3-space, have a finite-sided fundamental polyhedron. To each such group we assign a signature, which is basically a geometric object, but we also give an interpretation of this geometric object as a set of integers, similar to the signature of a Fuchsian group. We give a simple set of conditions which are necessary and sufficient for such a geometric object—or set of integers—to be the signature of a Kleinian group. Our main result in this paper is that two such Kleinian groups have the same signature if and only if one is a quasiconformal deformation of the other; there are also weaker results dealing with Kleinian groups which have an invariant component and which are assumed only to be finitely generated. Spaces of quasiconformal deformations of Kleinian groups in general have been discussed elsewhere by Bers [6], Kra [13], and Maskit [17]. The special cases arising from these particular Kleinian groups will be pursued in a subsequent paper in this series.

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The basic definitions appear in section 1; many of these appear in [18], but are repeated here for the convenience of the reader. The class $\mathcal{C}_0$ introduced there is the class of Kleinian groups described above; this is proved in section 10.

The signature of a Kleinian group is defined in sections 2 and 3. The conditions for a general "signature" to be that of a Kleinian group are given in section 4. The proof of our main theorem, and several related results, appear in sections 5–9. The main ingredients in this proof are the decomposition of Kleinian groups [19], and a version of Marden's isomorphism theorem [15], for which we give a purely 2-dimensional proof using an idea due to Koebe [12] (see also [18]), and the techniques of quasiconformal deformations due to Ahlfors and Bers [4, 7].

1. Definitions

1.1. In this section we give the basic definitions that are needed to define the signature.

1.2. We denote the class of finitely generated Kleinian groups which have an invariant component by $\mathcal{C}_1$; i.e., if $G \in \mathcal{C}_1$, then there is a connected component Δ of the set of discontinuity $\Omega(G)$, so that $g(\Delta) = \Delta$ for all $g \in G$.

We remark that the point in $\mathcal{C}_1$ is actually the pair (G, Δ); Fuchsian groups for example have two invariant components. We will however write $G \in \mathcal{C}_1$, and it is understood that we have chosen a particular invariant component Δ.

1.3. Let G and G^* be groups in $\mathcal{C}_1$ with invariant components Δ and Δ^*, respectively. We say that G and G^* are *weakly similar* if there is an orientation-preserving homeomorphism $\varphi: \Delta \to \Delta^*$, where $g \to \varphi \circ g \circ \varphi^{-1}$ defines an isomorphism Ψ of G onto G^*. The mapping φ is called a *weak similarity*, and Ψ is called the *induced isomorphism*.

1.4. An isomorphism $\Psi: G \to G^*$, between groups in $\mathcal{C}_1$ is called *type-preserving* if both Ψ and Ψ^{-1} preserve parabolic elements, and if Ψ preserves the square of the trace of every elliptic element.

1.5. If φ is a weak similarity between G and G^*, and if the induced isomorphism is type-preserving, then φ is a *similarity*, and we say that G and G^* are *similar*. If in addition, φ is quasiconformal or conformal, then we say that G and G^* are *quasiconformally* or *conformally similar*.

One easily sees, using Ahlfors' finiteness theorem [2] and Bers' approximation theorem [8], that G and G^* are similar if and only if they are quasiconformally similar.

1.6. A parabolic element g of a group $G \in \mathcal{C}_1$ is called *accidental* if there is a weak similarity φ so that $\varphi \circ g \circ \varphi^{-1}$ is not parabolic.

1.7. If $G \in \mathcal{C}_1$, with invariant component Δ, and H is a subgroup of G, then H has a *distinguished invariant component* $\Delta(H) \supset \Delta$.

1.8. A subgroup H of a group $G \in \mathcal{C}_1$ is called a *factor subgroup* if it satisfies

(i) $\Delta(H)$ is simply-connected,

(ii) H contains no accidental parabolic elements,

(iii) H contains every parabolic element of G whose fixed point lies in its limit set $\Lambda(H)$, and

(iv) H is a maximal subgroup of G satisfying properties (i)–(iii).

1.9. It was shown in [19] that every factor subgroup of a group in $\mathcal{C}_1$ again lies in $\mathcal{C}_1$.

1.10. A group $G \in \mathcal{C}_1$ whose invariant component is simply connected, and which contains no accidental parabolic elements, is called a *basic group*.

It was shown in [20] (see also Bers [9], and Kra and Maskit [14]), that every basic group H is either elementary (i.e., $\Lambda(H)$ is finite), or quasi-Fuchsian (i.e., H is a perhaps trivial quasiconformal deformation of a Fuchsian group), or degenerate (i.e., $\Delta = \Omega$).

1.11. A group in $\mathcal{C}_1$ is called *regular* if no factor subgroup is degenerate. The subclass of $\mathcal{C}_1$ consisting of regular groups is denoted by $\mathcal{C}_0$—other characterizations of $\mathcal{C}_0$ appear in [19]; we will show in section 10 that $\mathcal{C}_0$ consists of those groups in $\mathcal{C}_1$ which have a finite-sided fundamental polyhedron.

1.12. A group in $\mathcal{C}_0$ for which every factor subgroup is either elementary or Fuchsian is called a *Koebe group*. It was shown in [18] that every group in $\mathcal{C}_1$ is conformally similar to a unique Koebe group.

1.13. In general, if $A \subset \hat{C} = C \cup \{\infty\}$, and G is a Kleinian group, then *the stabilizer of A in G is* $\{g \in G \mid g(A) = A\}$.

If H is a subgroup of the Kleinian group G, and $A \subset \hat{C}$, then A *is precisely invariant under H in G* if $h(A) = A$ for all $h \in H$, and $g(A) \cap A = \emptyset$ for all $g \in G - H$.

2. Basic signatures

2.1. Throughout this section G will denote a basic group with invariant component Δ. In this section we recall the definition of the signature of a basic group, and we recall the relationship between the signature and the conformal type of the simply connected component Δ.

2.2. By mapping Δ onto the sphere, plane, or disc, we see that G is similar to an elementary group or to a finitely generated Fuchsian group of the first kind. In any case, Δ/G is a closed Riemann surface $\bar{S}$ from which a finite number of points have been removed, and the projection map π: $\Delta \to \Delta/G$ is branched over at most finitely many points. These points of $\bar{S}$ over which π is branched together with the points not in $\pi(\Delta)$ are called the *distinguished points* of $\bar{S}$.

To each distinguished point x we associate a *branch number* ν as follows. Let w be the boundary of a disc on $\bar{S}$ which contains x in its interior, and which contains no other distinguished point in its closure. A lifting of w determines an elliptic or parabolic element of G. We let ν be the order of that element.

Let g be the genus of $\bar{S}$, let n be the number of distinguished points of $\bar{S}$, and let $\nu_1, ..., \nu_n$ be the branch numbers of these points. Then the signature of G is the collection $(g, n; \nu_1, ..., \nu_n)$.

2.3. The signature is, of course, defined only up to permutation of the numbers $\nu_1,...,\nu_n$, and satisfies $g \geq 0$, $n \geq 0$, $2 \leq \nu_i \leq \infty$, $i = 1, ..., n$.

Not every collection of numbers $(g, n; \nu_1, ..., \nu_n)$ satisfying these inequalities is actually the signature of a basic group. The only possible signatures which do not occur are $(0, 1; \nu)$ and $(0, 2; \nu_1, \nu_2)$, $\nu_1 \neq \nu_2$.

2.4. If one knows the signature, then the conformal type of Δ is determined. Δ is elliptic, parabolic, or hyperbolic whenever $2(g-1) + \sum_{i=1}^{n}(1-(1/\nu_i))$ is negative, zero, or positive, respectively (here $1/\infty = 0$).

3. Signatures

3.1. In this section we define the signature of a group in $\mathfrak{C}_1$. We define the signature primarily as a geometric object; this is closely related to the definition given in [21] which unfortunately is not quite correct.

3.2. We start by recalling some of the results of [19].

Let G be a group in $\mathfrak{C}_1$. Then G contains only finitely many conjugacy classes of factor subgroups; each factor subgroup is finitely generated, and hence is a basic group.

On $S = \Delta/G$, there is a finite set of simple disjoint loops $w_1, ..., w_k$. Each connected component of the preimage of $\{w_1 \cup ... \cup w_k\}$ is stabilized either by a finite cyclic group, in which case it is a loop, or it is stabilized by a parabolic cyclic group, in which case it becomes a loop after we adjoin the parabolic fixed point. We will from here on regard the connected components of the preimage of $\{w_1 \cup ... \cup w_k\}$ as being loops; they are called *structure loops*.

Two structure loops may be tangent at a parabolic fixed point; otherwise they are simple and disjoint. They divide Δ into regions called *structure regions.*

For each structure region A, there is exactly one factor subgroup H, so that A is precisely invariant under H in G.

For each factor subgroup H, there is at least one structure region A so that A is precisely invariant under H in G.

Two factor subgroups of G are conjugate in G if and only if the structure regions they stabilize are equivalent under G.

Every structure loop W lies on the boundary of two structure regions A and A' stabilized by H and H', respectively. Then $J=H\cap H'$ is the stabilizer of W, and if J is non-trivial, then J is a maximal elliptic or parabolic cyclic subgroup of G.

The loop W bounds topological discs $B'\supset A$ and $B\supset A'$; the disc B is precisely invariant under J in H, and the disc B' is precisely invariant under J in H'.

3.3. We remark at this point that while the factor subgroups are intrinsically defined, the loops $w_1, \ldots, w_k$ are in general not even unique up to homology. The properties of these loops that we will use are all given above. It is clear that these properties are not affected by minor deformations, hence we can assume that every structure loop is smooth, except perhaps at a parabolic fixed point.

3.4. The signature of G is the collection $(g; K)$, where g is the genus of Δ/G (by Ahlfors' finiteness theorem [2], $g<\infty$), and K is the 2-complex described below.

Let $H_1, \ldots, H_s$ be a complete list of non-conjugate factor subgroups of G, and let $\sigma_i=(g_i, n_i; \nu_{i1}, \ldots, \nu_{in_i})$ be the signature of H_i. Let $K_1, \ldots, K_s$ be disjoint closed orientable surfaces, where K_i is of genus g_i, and K_i has n_i distinguished points on it, labelled with the branch numbers $\nu_{i1}, \ldots, \nu_{in_i}$. The surfaces $K_1, \ldots, K_s$ are called the *parts* of K. We say that H_i, or any conjugate of H_i, *lies over* K_i.

There are also at most k 1-cells in K. We fix i, and let W be a structure loop lying over w_i. If J, the stabilizer of W is trivial, then there is no 1-cell corresponding to w_i. If $|J|>1$, then let A and A' be the regions on either side of W, and let H and H', respectively, be their stabilizers, where H lies over K_j, and H' lies over some K_l. Let B be the topological disc bounded by W where $B\supset A'$, and let B' be the other topological disc bounded by W. Then B is precisely invariant under J in H, and so, after appropriate conjugation in G, B projects to a disc on $\Delta(H)/H$, which we identify with K_j, and this disc contains exactly one distinguished point with branch number $|J|$. Similarly B' projects to a disc on K_l, where this disc also contains exactly one distinguished point of order $|J|$. Thus w_i picks out a distinguished point on some K_j, and a distinguished point on some K_l, where both distinguished points have branch number $|J|$. In this case, K contains a 1-cell called a *connector*, where the end points of the connector are the two distinguished points picked out by w_i, and the connectors are otherwise disjoint from the parts of K, and from each other.

3.5. Looking at the orientation near W, one easily sees that the two endpoints of any connector are distinct.

It was also shown in [19] that G could be successively built up using the combination theorems [22, 23]. In the group $\hat{H}$ generated by H and H', not only does W not bound a disc which is precisely invariant under J in $\hat{H}$, but J does not correspond to a distinguished point in $\Delta(\hat{H})/\hat{H}$, and so there is no disc precisely invariant under J in $\hat{H}$, whose boundary lies in Δ. We conclude that each distinguished point of each part of K is the endpoint of at most one connector.

3.6. We remark next that the connectors establish a partial pairing among the distinguished points of the parts of K. This pairing satisfies the following:

(i) A distinguished point is paired with at most one other distinguished point.
(ii) A distinguished point cannot be paired with itself.
(iii) Two distinguished points can be paired only if they have the same branch number.

We also note that a connector can have its two endpoints on the same or on different parts of K; in particular, K may or may not be connected.

3.7. It is also worth remarking—here again we use the step by step construction of the group using the combination theorems (see [19]), that K can be constructed directly from Δ/G, once we know the branch points of the projection π: $\Delta \to \Delta/G$, and we know the loops $w_1, ..., w_k$ and the orders of their lifting.

Let $\bar{S}$ be the closed Riemann surface containing Δ/G. Let $x_1, ..., x_n$ be the distinguished points on $\bar{S}$, where x_i has branch number ν_i; i.e., x_i is either not in Δ/G, in which case $\nu_i = \infty$, or π is branched to order ν_i over x_i.

Let $w_1, ..., w_k$ be the loops on Δ/G; we regard these loops as lying on $\bar{S}$, and we remark that they do not pass through any of the distinguished points. Each loop w_i defines, by lifting to Δ, a conjugacy class of elements of G; let α_i be the order of any element in that class (note that this element is parabolic if $\alpha_i = \infty$).

We cut $\bar{S}$ along the loops $w_1, ..., w_k$, and pass discs through the resulting boundary loops as follows. If $\alpha_i = 1$ then we just cut S along w_i, and sew in two disjoint discs along the resultant boundary loops. If $\alpha_i > 1$, then we again cut and sew in disjoint discs, but now we consider the centers of these discs to be distinguished points of order α_i, and we adjoin a connector with its endpoints at these two distinguished points.

The resultant 2-complex is K.

3.8. The 2-complex K is of course defined only up to homeomorphism. We will regard two 2-complexes as being the same if there is a branch number preserving homeomorphism between them.

3.9. The signature can also be regarded as a set of numbers. We write $\sigma=(g; \sigma_1, ..., \sigma_s; P)$, where again g is the genus of Δ/G, σ_i is the signature of H_i—the σ_i are called the *factor signatures*, and P is a $\sum n_i \times \sum n_i$ symmetric incidence matrix, with at most one 1 in any row, which describes the pairing of 3.6.

Of course σ is defined only up to permutations of the σ_i, and permutations of the ν_{ij}, for fixed i. Each such permutation yields a different *partial pairing matrix P*.

We will interchangeably use the different versions of the signature σ.

3.10. As we have defined it, the signature appears to depend on the choice of the loops $w_1, ..., w_k$. In fact, as we will show in section 5, the signature depends only on the similarity class of the group in $\mathfrak{C}_1$.

3.11. One might try to define the connectors in terms of intersections of factor subgroups, as was unfortunately suggested in [21]. If there is a connector with one endpoint on K_i and the other on K_j, where the distinguished points have branch number ν, then there is a conjugate H' of H_j so that $H_i \cap H_j$ is a maximal elliptic or parabolic cyclic subgroup of order ν.

For Fuchsian groups, there is a one-to-one correspondence between distinguished points and conjugacy classes of maximal elliptic and parabolic cyclic subgroups. Hence if every factor subgroup of G is non-elementary, the partial pairing matrix is determined by the intersections of factor subgroups.

However, the odd dihedral groups (i.e., the finite basic groups with signature (0, 3; 2, 2, n), n odd) have only one conjugacy class of elements of order 2; likewise, the finite basic group with signature (0, 3; 2, 3, 3) has only one conjugacy class of elements of order 3. For groups which contain sufficiently many of these triangle groups as factor subgroups, the signature need not be determined by the intersections of factor subgroups. A particular example is given below.

Let S be a closed Riemann surface of genus 5, and let p_1: $\tilde{S}_1 \to S$ be the highest regular covering of S for which each of the loops indicated in figure 1, when raised to the indicated power lifts to a loop. We can realize this covering by a group G_1 in $\mathfrak{C}_1$ [24]. Then G_1 has signature (5; K_1), where K_1 is shown in figure 2.

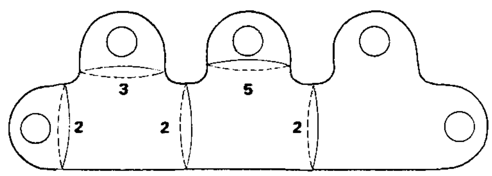

Fig. 1.

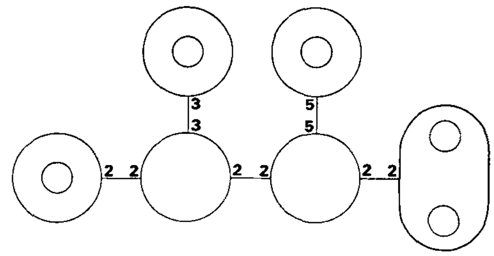

Fig. 2.

We similarly define G_2 by the system of loops and powers given in figure 3, and observe that G_2 has signature (5; K_2), where K_2 is shown in figure 4.

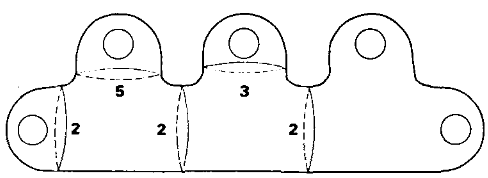

Fig. 3.

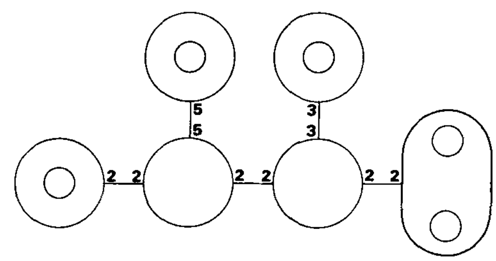

Fig. 4.

The two groups G_1 and G_2 have the same factor signatures, and the same intersection of factor subgroups—in both G_1 and G_2 every element of order 2 is in the intersection of four non-conjugate factor subgroups, but the groups have different signatures.

4. Admissable signatures

4.1. Not every pair (g; K), where $g \geq 0$, and K is as in 3.4–3.6 is the signature of a group in $\mathfrak{C}_1$. In this section we give a set of necessary and sufficient conditions for this to be so.

4.2. We already have two conditions for factor signatures:

(i) $\sigma_i \neq (0, 1; \nu)$, $i = 1, ..., s$, and

(ii) $\sigma_i \neq (0, 2; \nu_1, \nu_2)$, $\nu_1 \neq \nu_2$, $i = 1, ..., s$.

4.3. Since each factor subgroup is maximal, if any one of them is trivial, it must be the only one. Hence,

(iii) if for some i, $\sigma_i = (0, 0)$, then $1 = i = s$.

4.4. If a factor subgroup H_i is cyclic, then its factor signature is $\sigma_i = (0, 2; \nu, \nu)$. If a distinguished point on the corresponding part K_i is paired with some other distinguished point, then there is another factor subgroup H', so that $H_i \cap H'$ is a non-trivial maximal cyclic subgroup in G. We conclude that $H' = H_i$ and so the connector with one endpoint on K_i also has its other endpoint on K_i.

Let A_i be a structure region stabilized by H_i, and let W be a structure loop on the boundary of A_i, where W is also stabilized by H_i (the existence of the connector guarantees that there is such a loop). Then as we observed above, the structure region A' on the other side of W is also stabilized by H_i, and so there is another structure loop W' on the boundary of A_i, where W' is stabilized by H_i, and there is an element $g \in G - H_i$, which commutes with H. We note that if H is parabolic, then the maximality of H precludes the existence of such an element g.

We summarize the above in the following two conditions:

(iv) If some part has factor signature $(0, 2; \nu, \nu)$, then any connector with one endpoint on this part also has its other endpoint on this part.

(v) If some part has factor signature $(0, 2; \infty, \infty)$, then no connector has an endpoint on this part.

4.5. Similar to the above, we observe that if H and H' are distinct factor subgroups of G, each having signature $(0, 3; 2, 2, \infty)$, then H and H' cannot have a common parabolic element. For if they did, the subgroup generated by H and H' would be the basic group of signature $(0, 4; 2, 2, 2, 2)$ and this would contradict the maximality of both H and H'.

We restate the above in terms of the signature.

(vi) If some part K_i has factor signature $(0, 3; 2, 2, \infty)$ and some other part K_j has factor signature $(0, 3; 2, 2, \infty)$, then there is no connector connecting the distinguished point with branch number ∞ on K_i, with the distinguished point with branch number ∞ on K_j.

4.6. We let p be the number of connected components of K, and we let m be the number of connectors in K. Let $g(K)$ be the genus of K; i.e., if we replace each connector by a tube, then we get a disjoint union of closed orientable surfaces; $g(K)$ is the sum of the genera of these surfaces.

One easily sees that

$$g(K) = \sum_{i=1}^{s} g_i + m - (s - p).$$

It was shown in [19] that $g \geq g(K)$. This yields the last condition:

(vii) $t = g - g(K) \geq 0$.

4.7. A signature satisfying (i)–(vii) is called *admissable*. We have proven half of

THEOREM 1. *A signature* $\sigma=(g; K)$ *is the signature of a group in* $\mathcal{C}_1$ *if and only if it is admissible.*

With minor adaptations, the proof of the second half of this theorem is given by the constructions in [20] and [24]. One needs to verify, as one builds up the group using the combination theorems, that the admissability conditions rule out all cases of two basic groups being combined to yield a larger basic group, and that the factor subgroups of the combined groups are precisely the conjugates of the factor subgroups of the smaller groups.

The first of these facts is easy, and the second was verified in [19].

5. Independence of signature

5.1. In this section we prove that the signature depends only on the similarity class of a group in $\mathcal{C}_1$, and we will prove the converse in section 6. We state the result as

THEOREM 2. *Two groups* G *and* G^* *in* $\mathcal{C}_1$ *are similar if and only if they have the same signature.*

5.2. For the remainder of this section we assume that G and G^* are similar groups in $\mathcal{C}_1$, where G has signature $(g; K)$, and G^* has signature $(g^*; K^*)$.

5.3. The fact that G and G^* are similar gives us at once that $g^*=g$. Further, it was shown in [18] that similarities preserve factor subgroups; hence K and K^* have the same parts, with the same factor signatures.

The only thing remaining is to show that the connectors are the same. If there is a connector in K, then it is defined by a structure loop W in Δ, whose stabilizer J is non-trivial. Let A_1 and A_2 be the structure regions on either side of W, and for $i=1, 2$, let H_i be the factor subgroup which stabilizes A_i.

Let φ: $\Delta\rightarrow\Delta^*$ be the similarity, and let Ψ: $G\rightarrow G^*$ be the isomorphism induced by φ. For $i=1, 2$, let $H_i^*=\Psi(H_i)$.

5.4. We first take up the case that $H_1=H_2$. This can occur only if $H=H_1=H_2$ is cyclic; i.e., $H=J$. In this case, A_1 has two structure loops W and W' on its boundary which are both invariant under H. The element of G which maps W onto W' commutes with H. Thus H^* is finite and there is a loxodromic element of G^* which commutes with H^*; hence the fixed points of H^* are not in Δ^*. Thus each of the two distinguished points of $\Delta(H^*)/H^*$

are paired with something, but as observed in 4.4, they can only be paired with each other.

5.5. We now assume that $H_1 \neq H_2$. In this case, since J is a non-trivial common maximal cyclic subgroup, neither H_1 nor H_2 can be cyclic.

One easily observes that each non-cyclic factor subgroup stabilizes a unique structure region. Let A_i^* be the structure region stabilized by H_i^*, $i=1, 2$.

5.6. We know that $H_1^* \cap H_2^* = J^*$ is non-trivial; it suffices to show that there is an element $g^* \in G^*$, so that $\tilde{A}_1^* = g^*(A_1^*) \neq A_2^*$, and so that there is a structure loop W^* on the common boundary of $\tilde{A}_1^*$ and A_2^*, where W^* is stabilized by J^*.

Let W_1^* be the structure loop on the boundary of A_1^* which separates A_1^* from A_2^*; similarly, let W_2^* be the structure loop on the boundary of A_2^* which separates A_1^* from A_2^*.

Since A_1^* and A_2^* are both invariant under J^*, it follows that W_1^*, W_2^*, and any structure region lying between them, are all invariant under J^*.

We assume that there are one or more structure regions lying between W_1^* and W_2^*. If necessary, we replace A_1^* by some transform of it, so we can assume that none of these structure regions are equivalent to A_1^* under G^*.

Let A_3^* be the structure lying between W_1^* and W_2^*, where W_1^* lies on the boundary of A_3^*. Let W_3^* be the structure loop on the boundary of A_3^*, where W_3^* separates A_3^* from A_2^*.

Let H_3^* be the stabilizer of A_3^*. Since A_3^* has two structure loops on its boundary which are invariant under $J^* \subset H_3^*$, but which are not equivalent under H_3^*, we must have either that H_3^* has signature $(0, 3; 2, 2, n)$, n odd, $|J^*| = 2$, or that H_3^* has signature $(0, 3; 2, 3, 3)$, $|J^*| = 3$.

5.7. Now let $H_3 = \Psi^{-1}(H_3^*)$, and let A_3 be the structure region stabilized by H_3. Note that $H_3 \cap H_1 = H_3 \cap H_2 = J$, and so A_3 has at least one structure loop on its boundary which is stabilized by J.

There are two possibilities as to the relative positions of A_1, A_2 and A_3, but the argument is the same in both cases; we assume that A_1 lies between A_2 and A_3. Then A_1 has two structure loops on its boundary stabilized by J; we conclude that H_1 is also a finite non-cyclic group.

5.8. We remark at this point that the statement that W bounds a disc which is precisely invariant under J in say H_1 is equivalent to the statement that W is precisely invariant under J in H_1 and that all the translates of W under $H_1 - J$ lie on the same side of W.

We note next that there is a structure loop W_4 on the boundary of A_1 which is invariant under J and which separates the elliptic fixed points of $H_3 - J$ from the elliptic fixed points and limit points of $H_1 - J$ and $H_2 - J$.

We conclude that $\varphi(W_4)$ is invariant under J^* and separates the fixed points of $H_3^* - J^*$ from the fixed points and limit points of $H_1^* - J^*$ and $H_2^* - J^*$.

Let B^* be the topological disc bounded by $\varphi(W_4)$ which does not contain any of the fixed points of $H_3^* - J^*$. Moving only inside B^*, we can deform $\varphi(W_4)$ to a new loop W^*, where W^* projects to a power of a simple loop, W^* is invariant under J^*, and W^* intersects both W_1^* and W_3^*. Then, looking only at $\Delta(H_3^*)$, we can extend $W^* \cap A_3^*$ to obtain a path V^* connecting the fixed points of J^*, where V^* projects to a simple path on $\Delta(H_3^*)/H_3^*$. One easily sees that this is not possible for any of the possibilities listed in 5.6 for H_3^* and J^*.

6. Signatures and similarities

6.1. In this section we prove the second half of Theorem 2. We assume throughout this section that G and G^* are given groups in $\mathfrak{C}_1$, with the same signature $\sigma = (g; K)$; our goal is to prove that G and G^* are similar. The proof procedes in a step-by-step fashion, with the similarity φ defined first on one structure region, then on neighboring ones.

6.2. We call two structure regions, A_1 and A_2 *immediately connected* if there is a structure loop W lying on the boundary of both A_1 and A_2, where the stabilizer of W is non-trivial.

Two structure regions are *connected* if they can be connected by a finite chain of immediately connected regions.

Modulo the action of G, there are exactly p connectedness classes of structure regions, where p is the number of connected components of K.

6.3. If H is any subgroup of G, a *decent fundamental domain D for H* is a connected set bounded by a finite number of smooth arcs, where no two points of D are equivalent under H; every point of $\Delta(H)$ is equivalent to some point of D; every structure loop which intersects D either lies in the interior of D, or it intersects $\partial D \cap \Delta(H)$ in exactly two points; if two structure loops are equivalent under H, then at most one of them intersects D.

It is clear that every factor subgroup of G has a decent fundamental domain.

6.4. We first remark that if $H \subset G$ and $H^* \subset G^*$ are factor subgroups lying over the same part of K, then there is a similarity φ: $\Delta(H) \to \Delta(H^*)$, where φ preserves the endpoints of connectors.

6.5. Our first goal is to define a similarity on $\Delta(H)$, where H is the stabilizer of a connected class of structure regions.

Let $K_1, \ldots, K_r$ be the parts of a connected component L_1 of K, where the ordering $K_1, \ldots, K_r$ has been chosen so that there is a connector c_1 connecting K_2 to K_1, there is a connector c_2 connecting K_3 to $K_1 \cup K_2$, etc. Let $d_1, \ldots, d_q$ be the connectors of L_1 other than $c_1, \ldots, c_{r-1}$.

6.6. We choose some structure region A_1, with stabilizer H_1, where H_1 lies over K_1. Similarly for G^*, we choose some structure region A_1^*, with stabilizer H_1^*, also lying over K_1. We have already observed that there is a similarity φ_1: $\Delta(H_1) \to \Delta(H_1^*)$.

We choose a decent fundamental domain D_1 for H_1.

Corresponding to the connector c_1 which joins K_2 to K_1, there is a structure loop W_1 which intersects D_1, where the structure region A_2 on the other side of W_1 lies over K_2, and where the order of J_1, the stabilizer of W_1 is equal to the branch number of either endpoint of c_1.

As above, we choose a decent fundamental domain D_2 for H_2 where W_1 is one of the structure loops intersecting D_2, and so that $D_1 \cap W_1 = D_2 \cap W_1$.

We observe that $\langle H_1, H_2 \rangle$, the group generated by H_1 and H_2 is formed from H_1 and H_2 by using combination theorem I [22]; in particular, $D_1 \cap D_2$ is a fundamental domain for the action of $\langle H_1, H_2 \rangle$ on $\Delta(\langle H_1, H_2 \rangle)$.

It is clear that we can invariantly deform φ_1, so that $D_1^* = \varphi_1(D_1)$ is a decent fundamental domain for H_1^*. Then there is a structure loop W_1^* which intersects D_1^*, where the stabilizer H_2^* of the structure region A_2^*, on the other side of W_1^*, lies over K_2, and where the stabilizer of W_1^* has the same order as J_1.

Since H_2 and H_2^* both lie over K_2, there is a similarity φ_2: $\Delta(H_2) \to \Delta(H_2^*)$, and this similarity can be chosen so that $\varphi_2 \circ J_1 \circ \varphi_2^{-1} = J_1^*$. We invariantly deform φ_2 so that $\varphi_2(W_2) = W_1^*$, $\varphi_2 | W_1 = \varphi_1 | W_1$, and so that $D_2^* = \varphi_2(D_2)$ is a decent fundamental domain for G_2^*.

We now define the similarity φ_{12}: $\Delta(\langle H_1, H_2 \rangle) \to \Delta(\langle H_1^*, H_2^* \rangle)$. If $z \in D \cap A_i$, then we set $\varphi_{12}(z) = \varphi_i(z)$; if $z \in W \cap D$, then we set $\varphi_{12}(z) = \varphi_1(z) = \varphi_2(z)$. As defined, φ_{12} is continuous in D. If $z \in j(D)$, $j \in J$, then we set $\varphi_{12}(z) = j^* \circ \varphi_{12} \circ j^{-1}(z)$, where $j^* = \varphi_1 \circ j \circ \varphi_1^{-1} = \varphi_2 \circ j \circ \varphi_2^{-1}$. Every other point $z \in \Delta(\langle H_1, H_2 \rangle)$ lies in some $g(D) = g_n \circ g_{n-1} \circ \ldots \circ g_1(D)$, where the g_i alternately belong to $H_1 - J$ and $H_2 - J$; for such a point, we set

$$\varphi_{12}(z) = g_n^* \circ g_{n-1}^* \circ \ldots \circ g_1^* \circ \varphi_{12} \circ g_1^{-1} \circ \ldots \circ g_n^{-1}(z),$$

where g_i^* is either $\varphi_1 \circ g_i \circ \varphi_1^{-1}$, or $\varphi_2 \circ g_i \circ \varphi_2^{-1}$, whichever is defined.

Using combination theorem 1 again, we see that $D^* = D_1^* \cap D_2^*$ is a fundamental domain for the action of $\langle H_1^*, H_2^* \rangle$ on its invariant component, and that $\langle H_1^*, H_2^* \rangle$ is the free product of H_1^* and H_2^* with amalgamated subgroup J^*. Hence φ_{12} is a weak similarity.

The admissability criteria for the signature guarantee that $\langle H_1, H_2\rangle$ is not elementary. An easy adaptation of the argument in [25] (see [18]) shows that every parabolic element of $\langle H_1, H_2\rangle$ is conjugate to some element of H_1 or of H_2; similarly every parabolic element of $\langle H_1^*, H_2^*\rangle$ is conjugate to some element of H_1^* or of H_2^*. Hence φ_{12} is a similarity.

6.7. If L_1 has a third part K_3, then the connector c_2 has an endpoint lying on $K_1 \cup K_2$. Hence, there is a structure loop W_2, which intersects $D_1 \cap D_2$ where J_2, the stabilizer of W_2 is non-trivial, and where H_3, the stabilizer of the structure region A_3 lying on the other side of W_2, lies over K_3. We procede exactly as above and find H_3^*, which also lies over K_3, and we find decent fundamental domains D_3 for H_3, D_3^* for H_3^*, and we find a similarity $\varphi_{123}: \Delta(\langle H_1, H_2, H_3\rangle) \to \Delta(\langle H_1^*, H_2^*, H_3^*\rangle)$, where $\varphi_{123}(D_1 \cap D_2 \cap D_3) = D_1^* \cap D_2^* \cap D_3^*$.

Continuing as above, after $r-1$ steps we obtain similar subgroups $H_{r+1} = \langle H_1, ..., H_r\rangle$, and $H_{r+1}^* = \langle H_1^*, ..., H_r^*\rangle = \varphi_{r+1}(H_{r+1})$, where $\varphi_{r+1} = \varphi_{1,...,r}$. We also have decent fundamental domains $D_{r+1} = D_1 \cap ... \cap D_r$ for H_{r+1}, and $D_{r+1}^* = \varphi_{r+1}(D_{r+1})$, for H_{r+1}^*.

Corresponding to the connector d_1, we can find structure loops U_1 and V_1, where U_1 and V_1 both intersect the boundary of D_{r+1}, and U_1 and V_1 are equivalent under the action of G, but not under the action of H_{r+1}. Then there is an element $f_1 \in G - H_{r+1}$, where $f_1(U_1) = V_1$.

There are in fact several such elements in G; we choose f_1 and deform D_{r+1} in the neighborhood of V_1, so that $f_1(U_1 \cap D_{r+1}) = V_1 \cap D_{r+1}$.

Since $\varphi_1, ..., \varphi_r$ have been chosen to preserve the endpoints of connectors, there are structure loops U_1^*, V_1^* intersecting D_{r+1}^*, where φ_{r+1} conjugates the stabilizer I_1 of U_1 onto the stabilizer I_1^* of U_1^*, and φ_{r+1} conjugates the stabilizer J_1 of V_1 onto the stabilizer J_1^* of V_1^*. There is also an element f_1^* in $G^* - H_{r+1}^*$ which maps U_1^* onto V_1^*. We deform D_{r+1}^* near V_1^* so that $f_1^*(U_1^* \cap D_{r+1}^*) = V_1^* \cap D_{r+1}^*$.

We now invariantly deform φ_{r+1} so that $\varphi_{r+1}(U_1) = U_1^*, \varphi_{r+1}(V_1) = V_1^*, \varphi_{r+1}(D_{r+1}) = D_{r+1}^*$, and so that $\varphi_{r+1} | U_1$ conjugates f_1 into f_1^*.

We observe that H_{r+2}, the group generated by H_{r+1} and f_1 is formed using combination theorem II [23]. Similarly, H_{r+2}^*, the group generated by H_{r+1}^* and f_1^* is formed using combination theorem II. We conclude that D_{r+2}, the connected part of D_{r+1} cut out by U_1 and V_1 is a decent fundamental domain for H_{r+2}; the connected part D_{r+2}^* of D_{r+1}^* cut out by U_1^* and V_1^* is a decent fundamental domain for H_{r+2}^*; there is an isomorphism Ψ of H_{r+2} onto H_{r+2}^*, where

$$\Psi(g) = \varphi_{r+1} \circ g \circ \varphi_{r+1}^{-1} \quad \text{for } g \in H_{r+1}, \text{ and } \Psi(f_1) = f_1^*.$$

We define φ_{r+2} in the obvious fashion by restricting φ_{r+1} to D_{r+2}, and then using the isomorphism Ψ. As in the preceding case, the argument of [25] is easily adapted to show that φ_{r+2} is actually a similarity.

6.8. We repeat the above argument q times until we arrive at a group, which we now call G_1, a decent fundamental domain E_1 for G_1, and a similarity ψ_1 between G_1 and G_1^*, where $\psi_1(E_1)$ is a decent fundamental domain for G_1^*.

We note that G_1 lies over all of L_1; more precisely, every factor subgroup contained in G_1 lies over some part of L_1, and G_1 stabilizes an entire connectedness class of structure regions lying over L_1. Let B_1 be the relative interior in Δ of the union of the closures of these structures regions, so that B_1 is precisely invariant under G_1 in G. We note that every structure loop which intersects E_1, and which lies on the boundary of B_1, is contained in the interior of E_1. Such a structure loop is of course stabilized only by the identity.

From here on we will consider only regions such as B_1, which essentially are connectedness classes of structure regions; we call such a region a *super structure region*, and its stabilizer is called a *super factor subgroup*. The structure loops on the boundary of a super structure region are called *super structure loops*.

Repeating the argument of 6.6–6.8, we have shown that if G_i and G_i^* are super factor subgroups of G and G^*, respectively, where G_i and G_i^* lie over the same connected component L_i of K, then there is a similarity, which we now call ψ_i: $\Delta(G_i)\to\Delta(G_i^*)$.

6.9. Before preceding with the construction of the similarity, we need an auxilliary construction in which we change some of the super structure loops.

There are a total of p connected components of K. If $p>1$, then there is a super structure loop in the interior of E_1, where B_2, the super structure region on the other side of this loop lies over say $L_2 \neq L_1$. Call the super structure loop W_1; let G_2 be the stabilizer of B_2, and let E_2 be a decent fundamental domain for G_2, where W_1 lies in the interior of E_2.

Let W_1, $U_1, \ldots, U_u$ be an enumeration of the super structure loops contained in the interior of E_2. We choose non-intersecting paths V_i connecting W_1 to U_i, $i=1, \ldots, u$, where except for their endpoints, the V_i are disjoint from all super structure loops. Let W_1' be the boundary of a small neighborhood of $W_1 \cup U_1 \cup \ldots \cup U_u \cup V_1 \cup \ldots \cup V_u$, where W_1' is homologous to $W_1 \pm U_1 \pm \ldots \pm U_u$.

We replace W_1 and its translates under G, by W_1' and its translates.

We observe that the new set of super structure loops are mutually disjoint, just as the old ones were, and so they also divide Δ into super structure regions, where, except for B_1 and B_2, these new regions are the same as the old ones. For $i=1, 2$, G_i is the stabilizer of both B_i and B_i', so the relationship between super structure regions and super factor subgroups remains unchanged. The essential difference is that modulo G_2, B_2' has only one class of super structure loops on its boundary; the loops $U_1, ..., U_u$ now lie on the boundary of B_1'. We note also that E_i is still a decent fundamental domain for G_i.

We repeat the above process as often as necessary until we have altered $B_1, ..., B_p$ to new super structure regions, which we again call $B_1, ..., B_p$, so that for $2 \leqslant i \leqslant p$, B_i has only one class modulo G_i of super structure loops on its boundary.

We perform the same operations with the super structure loops B_i^* (B_1^* has already been chosen, B_2^* is chosen so that B_1^* and B_2^* have a common super structure loop on their boundary, after changing B_1^* and B_2^*, B_3^* and B_1^* have a common super structure loop on their boundary, etc.) so that for $2 \leqslant i \leqslant p$, each B_i^* has only one equivalence class under G_i^* of super structure loops on its boundary; there is a super structure loop on the common boundary of B_i^* and B_1^*, and this common super structure loop lies in E_1^*.

6.10. Having performed the above operations, we relabel $B_2^*, ..., B_p^*$, together with $G_2^*, ..., G_p^*$, so that for $1 \leqslant i \leqslant p$, G_i and G_i^* both lie over the same connected component of K.

6.11. We return now to the construction of the similarity.

Let W_1 be the super structure loop lying in E_1 which separates B_1 from B_2. We observe that $G_{12} = \langle G_1, G_2 \rangle$ is formed from G_1 and G_2 using combination theorem I [22, 25], that G_{12} is the free product of G_1 and G_2, and that $E_{12} = E_1 \cap E_2$ is a fundamental domain—in fact a decent fundamental domain—for G_{12}.

Similarly, let W_1^* be the super structure loop in E_1^* which separates B_1^* from B_2^*. We note again that $G_{12}^* = \langle G_1^*, G_2^* \rangle$ is formed using combination theorem I, it is the free product of G_1^* and G_2^*, and $E_{12}^* = E_1^* \cap E_2^*$ is a decent fundamental domain for G_{12}^*.

We deform ψ_1 inside E_1 so that $\psi_1(W_1) = W_1^*$; we also deform ψ_2 inside E_2 so that $\psi_2(W_1) = W_1^*$, and so that $\psi_1 | W_1 = \psi_2 | W_1$.

We define the similarity ψ_{12}: $\Delta(G_{12}) \to \Delta(G_{12}^*)$ by setting $\psi_{12} | (E_{12} \cap E_i) = \psi_i$, $i = 1, 2$, and then using the natural isomorphism between G_{12} and G_{12}^* to define ψ_{12} on all of $\Delta(G_{12})$.

We again observe, using [25], that every parabolic element of G_{12} is conjugate to an element of either G_1 or of G_2. Similarly, every parabolic element of G_{12}^* is conjugate to some element of G_1^* or G_2^*; hence the isomorphism induced by ψ_{12} is type-preserving, and so ψ_{12} is a similarity.

If $p > 2$, we repeat the above process until we arrive at a subgroup $G_{1...p} = G_{p+1}$, which covers all of K. The construction also yields a decent fundamental domain E_{p+1} for G_{p+1}, and a similarity

$$\psi_{p+1}: \Delta(G_{p+1}) \to \Delta(G^*_{p+1}), \quad \text{where } E^*_{p+1} = \psi_{p+1}(E_{p+1})$$

is a decent fundamental domain for G^*_{p+1}.

We observe that since G_{p+1} lies over all of K, the super structure loops lying in the interior of E_{p+1} are necessarily pairwise identified by elements of G. Similarly, the super structure loops lying in the interior of G_{p+1} are pairwise identified by elements of G^*.

6.12. We pause at this point to remark that if G were a Schottky group, then K would consist only of a sphere with no distinguished points, and so in this case, all our constructions up to this point, will have been vacuous.

6.13. There are $2t$ super structure loops in the interior of E_{p+1}; one easily sees that $t = g - g(K)$, as in 4.6. These $2t$ loops are paired by elements of G which we label as $f_1, \ldots, f_t$; we label these super structure loops as $U_1, V_1, \ldots, U_t, V_t$ where $f_i(U_i) = V_i$.

We observe that these super structure loops bound a region E_{p+2}, which is a fundamental domain for the Schottky group G_{p+2} generated by $f_1, \ldots, f_t$.

The group $\tilde{G} = \langle G_{p+1}, G_{p+2} \rangle$ is formed from these groups using combination theorem I [22, 25]. Hence $\tilde{G}$ is the free product of G_{p+1} and G_{p+2}, and $\tilde{E} = E_{p+1} \cap E_{p+2}$ is a decent fundamental domain for $\tilde{G}$.

We observe next that $\tilde{E} \subset \Delta$, and so $\Delta(\tilde{G}) \subset \Delta$. We conclude that $\Delta(\tilde{G}) = \Delta$. Since the elements of G permute the super structure regions, and since each super structure region has the same stabilizer in both G and $\tilde{G}$, we conclude that $G = \tilde{G}$.

The remarks above hold equally well for G^*. That is, E^*_{p+1} contains $2t$ super structure loops $U^*_1, V^*_1, \ldots, U^*_t, V^*_t$; for each $i = 1, \ldots, t$, there is an element f^*_i with $f^*_i(U^*_i) = V^*_i$; the region E^*_{p+2} bounded by $U^*_1, \ldots, V^*_t$ is a fundamental domain for the Schottky group G^*_{p+2}; G^* is the free product, formed using combination theorem I, of G^*_{p+1} and G^*_{p+2}.

We have a similarity $\psi_{p+1}: \Delta(G_{p+1}) \to \Delta(G^*_{p+1})$, where $\psi_{p+1}(E_{p+1}) = E^*_{p+1}$. We deform ψ_{p+1} inside E_{p+1} to obtain a new map $\varphi: E_{p+1} \to E^*_{p+1}$, where $\varphi(U_i) = U^*_i$, $\varphi(V_i) = V^*_i$, and $\varphi \circ f_i | U_i = f^*_i \circ \varphi | U_i$, $i = 1, \ldots, p$. Then using the isomorphism between G and G^*, we can extend φ to a weak similarity between G and G^*. That φ is in fact a similarity follows from combination theorem I [25].

7. Extensions of maps

7.1. In this section we restrict our attention to groups in $\mathfrak{C}_0$, and we show that two groups in $\mathfrak{C}_0$ have the same signature if and only if one is a topological deformation of the other.

7.2. THEOREM 3. *Let G and G^* in $\mathfrak{C}_0$ have the same signature. Then there is a quasiconformal homeomorphism $\varphi: \Omega(G) \to \Omega(G^*)$, where $\varphi \circ g \circ \varphi^{-1}$ defines a type-preserving isomorphism of G onto G^*.*

Proof. From Theorem 2 it follows that G and G^* are similar; we denote the similarity by φ_0.

It was shown in [19] that if $\Delta_i \neq \Delta$ is a component of G, then there is a quasi-Fuchsian factor subgroup H_i of G so that Δ_i and $\Delta(H_i)$ are the two components of H_i. It was also shown in [19] that if H_i is a quasi-Fuchsian factor subgroup of G, with components $\Delta(H_i)$ and Δ_i, then Δ_i is a component of G.

Let $H_1, ..., H_q$ be a complete list of non-conjugate quasi-Fuchsian factor subgroups of G, and let $\Delta_i \neq \Delta(H_i)$ be the other component of H_i, $i=1, ..., q$. As we have already observed (see also [18]), $H_1^* = \varphi_0 \circ H_1 \circ \varphi_0^{-1}, ..., H_q^* = \varphi_0 \circ H_q \circ \varphi_0^{-1}$ is a complete list of non-conjugate quasi-Fuchsian factor subgroups of G^*. Let $\Delta_i^* = \Delta(H_i^*)$ be the other component of H_i^*, $i=1, ..., q$.

By the Nielsen realization theorem [10] (for proof, see Marden [16], or Zieschang [27]), for each $i=1, ..., q$, there is a homeomorphism φ_i: $\Delta_i \to \Delta_i^*$, where $\varphi_i \circ h \circ \varphi_i^{-1} = \varphi_0 \circ h \circ \varphi_0^{-1}$ for every $h \in H_i$.

Using Bers' approximation theorem [8], we can assume that $\varphi_0, \varphi_1, ..., \varphi_q$ are all quasiconformal. We define φ by $\varphi | \Delta = \varphi_0$, $\varphi | \Delta_i = \varphi_i$, and we define φ on the rest of Ω by using the action of G, and the isomorphism between G and G^*.

7.3. A homeomorphism φ: $\hat{C} \to \hat{C}$ is called a *global* homeomorphism.

THEOREM 4. *Let G and G^* be groups in $\mathfrak{C}_0$, and let φ: $\Omega(G) \to \Omega(G^*)$ be a homeomorphism, where $g \to \varphi \circ g \circ \varphi^{-1}$ defines an isomorphism of G onto G^*. Then φ is the restriction of a global homeomorphism.*

Proof. We first extend φ to the limit sets of factor subgroups. Let H be a factor subgroup of G, and let A be the structure region stabilized by H. Let $W_1, ..., W_u$ be a complete list of inequivalent—under H—structure loops on the boundary of A. Each W_i bounds a topological disc B_i which is precisely invariant under the cyclic group J_i in H. We replace φ inside B_i by some homeomorphism which agrees with φ on W_i, which maps B_i onto the appropriate topological disc bounded by $\varphi(W_i)$, and which conjugates J_i into $\varphi \circ J_i \circ \varphi^{-1}$. We then use the action of H and $\varphi \circ H \circ \varphi^{-1}$ to replace φ in the translates of the B_i.

After the above replacements, we have a new homeomorphism φ', where φ' maps $\Omega(H)$ onto $\Omega(\varphi \circ H \circ \varphi^{-1})$, and $\varphi \circ h \circ \varphi^{-1} = \varphi' \circ h \circ (\varphi')^{-1}$, for all $h \in H$. If H is elementary then φ' trivially extends to $\Lambda(H)$; if H is quasi-Fuchsian, then so is $\varphi \circ H \circ \varphi^{-1}$, and we can approximate φ' by a quasiconformal homeomorphism [8], and then use [14] to extend φ' to $\Lambda(H)$.

For $z \in \Lambda(H)$, we set $\varphi(z) = \varphi'(z)$, and we observe that φ is continuous across $\Lambda(H)$.

If $z \in \Lambda(G)$, but z is not a limit point of any factor subgroup of G, then [19] there is a structure loop W, and a sequence $\{g_n\}$ of elements of G, so that $g_n(W)$ nests about z; i.e., for each $n > 1$, $g_n(W)$ separates z from $g_{n-1}(W)$, and

$$\lim_{n\to\infty} g_n(W) = z.$$

The images $\varphi \circ g_n(W)$ have the same separation property, and it was shown in [22] (see also [25] and [18]) that the loops $\varphi \circ g_n(W)$ accumulate to a single point w. Set $\varphi(z) = w$.

Since Ω is dense in $\hat{C}$, it suffices to check that φ is continuous from inside Ω, and this is immediate. Since the above construction can also be used to define φ^{-1}, φ is one-to-one, and hence a homeomorphism.

7.4. We remark that in the statement of Theorem 4, we did not require the isomorphism to be type-preserving. We obtain, as a corollary to Theorem 4, that such an isomorphism is necessarily type-preserving.

8. Uniqueness for Koebe groups

8.1. In [18] we showed that a Koebe group is uniquely determined—as a Koebe group—by its similarity class and by the conformal structure on Δ. In this section we show that a Koebe group is uniquely determined—as a group in $\mathfrak{C}_0$, by its similarity class and by the conformal structure on Ω.

THEOREM 5. *Let G be a Koebe group, and let φ be a global homeomorphism where $\varphi \circ G \circ \varphi^{-1} = G^*$ is a Kleinian group, and where $\varphi \mid \Omega(G)$ is conformal. Then φ is a fractional linear transformation.*

The remainder of this section is devoted to the proof of Theorem 5.

8.2. We remark first that the maximality conditions given in [19] imply that $G^* \in \mathfrak{C}_0$.

We denote the spherical diameter of any set A by dia (A). Let $W_1, W_2, \ldots$ be a complete listing of the structure loops of G, then it was shown in [18] that

$$\sum_i \text{dia}^2 (W_i) < \infty.$$

LEMMA 1. *Let G^* be any group in $\mathfrak{C}_0$, and let $W_1^*, W_2^*, \ldots$ be a complete listing of the structure loops of G^*. Then*

$$\sum_i \text{dia}^2 (W_i^*) < \infty.$$

Proof. Let $W_1^*, \ldots, W_k^*$ be a complete list of inequivalent—under G^*—structure loops.

We observe that if the stabilizer J_j^* of W_j^* is finite, then there are fundamental domains E for J_j^*, and D for G^*, and there is a neighborhood N of W_j^*, so that $E \cap N = D \cap N$. Then by Koebe's Theorem (see [25]),

$$\sum_i \operatorname{dia}^2 g_i(W_j^*) < \infty$$

where the g_i range over a complete list of left coset representatives of J_j in G^*.

We now assume that J_j^* is parabolic.

Let A and A' be the structure regions on either side of W_j^*, and let H and H' respectively be their stabilizers.

If H and H' are both quasi-Fuchsian, then H and H' have non-invariant components, and so the hypotheses for Koebe's theorem [25] are valid.

If, say, H is elementary, then the signature of H is $(0, 3; 2, 2, \infty)$ and H' cannot be elementary. Hence H' is quasi-Fuchsian, and so by passing to a subgroup of index 2, we reduce this case to the preceding one.

8.3. We now normalize G so that $\infty \in \Delta$, ∞ lies in the interior of some structure region, and ∞ is not an elliptic fixed point.

We also normalize G^* so that near ∞,

$$\varphi(z) = z + O(|z|^{-1}). \tag{1}$$

We fix some number R so large that $|z| > R$ is precisely invariant under the identity in G, and so that $|z| > R$ is contained in the interior of some structure region for G.

8.4. LEMMA 2. *For* $|z| > 2R$,

$$\sum_i \left| \int_{W_i} \frac{\varphi(\zeta)\, d\zeta}{\zeta - z} \right| < \infty.$$

Proof. We denote the length of W_i by $L(W_i)$ and we use $D(A)$ to denote the Euclidean diameter of A.

For each i, we choose a point ζ_i on W_i, and observe that

$$\left| \int_{W_i} \frac{\varphi(\zeta)\, d\zeta}{\zeta - z} \right| = \left| \int_{W_i} \frac{(\varphi(\zeta) - \varphi(\zeta_i))\, d\zeta}{\zeta - z} \right| \leqslant R^{-1} L(W_i)\, D(\varphi(W_i)). \tag{2}$$

It was shown in [18] that there is a constant $k > 0$ so that

$$L(W_i) \leqslant k D(W_i). \tag{3}$$

Combining (2) and (3) with Lemma 1, we obtain

$$\sum_i \left| \int_{W_i} \frac{\varphi(\zeta)\,d\zeta}{\zeta - z} \right| \leqslant R^{-1} k \sum_i D(W_i) D(\varphi(W_i))$$

$$\leqslant R^{-1} k (\sum_i D^2(W_i))^{1/2} (\sum_i D^2(\varphi(W_i)))^{1/2} < \infty.$$

8.5. For the structure region containing the point at ∞, we call $|z| = R$ the *outer structure loop*; all other loops on its boundary are called *inner structure loops*. For any other structure region A, the outer structure loop is that loop on the boundary of A which separates A from ∞; all other structure loops on the boundary of A are *inner*.

LEMMA 3. *Let A be a structure region for G with outer structure loop U, and inner structure loops $V_1, V_2, \ldots$. Then for $|z| > 2R$,*

$$\int_U \frac{\varphi(\zeta)\,d\zeta}{\zeta - z} = \sum_i \int_{V_i} \frac{\varphi(\zeta)\,d\zeta}{\zeta - z}.$$

Proof. Let H be the stabilizer of A. If H is finite, then the sum on the right is finite, and our lemma reduces to Cauchy's theorem.

If H has one limit point, then we can assume without loss of generality that this limit point is the origin. For r sufficiently small, the circle $|z| = r$ intersects only inner structure loops; for each such intersection, we deform $|z| = r$ so that it remains in the closure of A, and so that it runs along the shorter arc of the inner structure loops. We call this deformed loop Y_r. For each r,

$$\int_U \frac{\varphi(\zeta)\,d\zeta}{\zeta - z} = \sum_{i=1}^{N_r} \int_{V_i} \frac{\varphi(\zeta)\,d\zeta}{\zeta - z} + \int_{Y_r} -\frac{\varphi(\zeta)\,d\zeta}{\zeta - z}, \tag{4}$$

where $V_1, \ldots, V_{N_r}$ are the inner structure loops lying between U and V_r.

It was shown in [18] that for Koebe groups the ratio of chord to shorter length of arc of structure loop is uniformly bounded from below. Hence,

$$L(Y_r) \leqslant kr. \tag{5}$$

Our result in this case follows from Lemma 2, together with (4) and (5).

If H is Fuchsian, then we can assume without loss of generality that the limit set of H is the unit circle. For $r > 1$ and sufficiently small, we define Y_r exactly as above. Equality (4) and inequality (5) hold, exactly as above, and so we can conclude that

$$\int_U \frac{\varphi(\zeta)\,d\zeta}{\zeta - z} - \sum_i \int_{V_i} \frac{\varphi(\zeta)\,d\zeta}{\zeta - z} = \int_{|z|=1} \frac{\varphi(\zeta)\,d\zeta}{\zeta - z},$$

where we have used the continuity of φ up to $\Lambda(H)$. Since φ is holomorphic in $|z|<1$, the integral on the right is zero.

8.6. Since every structure loop is the outer structure loop of exactly one structure region, we can use Lemmas 2 and 3 to conclude that for $|z|>2R$,

$$\int_{|\zeta|=R} \frac{\varphi(\zeta)\,d\zeta}{\zeta - z} = 0. \tag{6}$$

We combine (1) and (6) to obtain that for $|z|>2R$, $\varphi(z)=z$. Hence φ is the identity.

9. Uniqueness

9.1. In this section, we prove our main result—that if two groups in $\mathfrak{C}_0$ have the same signature, then one is a quasi-conformal deformation of the other, and we also derive several other consequences of Theorem 5.

Several of the results in this and the next section were announced in [21].

9.2. LEMMA 4. *Let G and G^* be groups in $\mathfrak{C}_0$, where G is a Koebe group. Let φ: $\Omega(G)\to\Omega(G^*)$ be a quasiconformal homeomorphism, where $\varphi \circ g \circ \varphi^{-1}$ defines an isomorphism of G onto G^*. Then φ is the restriction of a global quasiconformal homeomorphism.*

Proof. Using the existence of global homeomorphic solutions to the Beltrami equation, due to Ahlfors and Bers [4] (see also Bers [7, 8]), there is a global quasiconformal homeomorphism ψ, where $\psi \circ G^* \circ \psi^{-1} = G'$ is a group in $\mathfrak{C}_0$, so that $\psi \circ \varphi$ is conformal on all of Ω. By Theorem 4, $\psi \circ \varphi$ is the restriction of a global homeomorphism. Then by Theorem 5, $\psi \circ \varphi$ and ψ are both global quasiconformal homeomorphisms, and hence φ is also.

9.3. Our next result is the quasiconformal version of Theorem 4.

THEOREM 6. *Let G and G^* be groups in $\mathfrak{C}_0$, and let φ: $\Omega(G)\to\Omega(G^*)$ be a quasiconformal homeomorphism, where $\varphi \circ g \circ \varphi^{-1}$ defines an isomorphism of G onto G^*. Then φ is the restriction of a global quasiconformal homeomorphism.*

Proof. It was shown in [18] that there is a Koebe group G' similar to G. Using Theorems 2 and 3, there is a quasiconformal homeomorphism ψ: $\Omega(G')\to\Omega(G)$, where $\psi \circ g' \circ \psi^{-1}$ defines an isomorphism of G' onto G. By Lemma 4, ψ and $\varphi \circ \psi$ are both restrictions of global quasiconformal homeomorphisms. Hence φ is the restriction of a global quasiconformal homeomorphism.

9.4. THEOREM 7. *Two groups G and G^* in $\mathcal{C}_0$ have the same signature if and only if G^* is a quasiconformal deformation of G (i.e., there is a global quasiconformal homeomorphism φ so that $G^* = \varphi \circ G \circ \varphi^{-1}$).*

Proof. If G^* is a quasiconformal deformation of G, then they are similar, and so by Theorem 2, they have the same signature.

If G and G^* have the same signature, then by Theorems 3 and 6, G^* is a quasiconformal deformation of G.

9.5. Our final application is a conformal version of Theorems 4 and 6.

THEOREM 8. *Let G and G^* be groups in $\mathcal{C}_0$ and let φ: $\Omega(G) \to \Omega(G^*)$ be a conformal homeomorphism, where $\varphi \circ g \circ \varphi^{-1}$ defines an isomorphism of G onto G^*. Then φ is the restriction of a fractional linear transformation.*

Proof. It is classical that a quasiconformal homeomorphism which is conformal a.e. is in fact conformal. Hence, it suffices to show that for $G \in \mathcal{C}_0$, $\Lambda(G)$ has 2-dimensional measure 0; we prove this in section 10.

10. Finite sided fundamental polyhedra

10.1. Every Kleinian group can be regarded as a group of isometries of hyperbolic 3-space, and so every Kleinian group has at least one convex fundamental polyhedron. It was shown in [5] (see also Marden [15]) that if one convex fundamental polyhedron for G has finitely many sides, then they all do.

Our main result in this section is:

THEOREM 9. *A group $G \in \mathcal{C}_1$ lies in $\mathcal{C}_0$ if and only if G has a finite sided fundamental polyhedron.*

This theorem was announced in [26], and a proof for B-groups was given by Abikoff [1].

10.2. The proof of this theorem makes essential use of the criterion of Beardon and Maskit [5].

Let x be a fixed point of a parabolic element of the Kleinian group G, and let J be the stabilizer of x in G. We say that x is a *cusped parabolic fixed point* if either (i) J has rank 2, or (ii) there are two disjoint open circular discs (with boundaries tangent at x), whose union is precisely invariant under J in G.

Let y be a limit point of G. We say that y is a *point of approximation* if there is a sequence $\{g_n\}$ of distinct elements of G, and there is a point $z \in \Omega$, so that the spherical distance $d(g_n(y), g_n(z))$ does not converge to 0.

THEOREM (Beardon and Maskit). *A Kleinian group G has a finite sided fundamental polyhedron if and only if every limit point of G either is a cusped parabolic fixed point, or is a point of approximation.*

10.3. LEMMA 5. *Let x be the fixed point of the parabolic element $g \in G \in \mathcal{C}_0$. Then x is a cusped parabolic fixed point.*

Proof. It was shown in [19] that g lies in at least one factor subgroup H.

Let J be the stabilizer of x in G. If J has rank 2, there is nothing to prove, so we assume from here on that J has rank 1.

If H is cyclic, then H corresponds to two distinguished points on Δ/G; liftings of neighborhoods of these points yield the required discs.

If H is elementary, but not cyclic, and g goes not lie in any other factor subgroup, then the maximal cyclic subgroup containing g represents one distinguished point on Δ/G; lifting a neighborhood of that point, and then applying an element of order 2 in J to the resultant disc, yields a pair of discs with the required property.

If H is elementary, but not cyclic, and g also lies in some other factor subgroup H', then H' is necessarily quasi-Fuchsian. Let $\Delta' \neq \Delta(H')$ be the other component of H', and let U be a circular disc in Δ' which is precisely invariant under J in H'. Let $j \in J$ have order 2; then $U \cup j(U)$ has the required property.

If H is non-elementary, then it is quasi-Fuchsian. If g does not lie in any other factor subgroup, then J_0, the maximal cyclic subgroup containing g, represents two distinguished points on $\Omega(G)$; one in Δ/G, and the other coming from the other component of H. If g lies in H and H', then J_0 represents two distinguished points, one each coming from the other components of H and H'. In either case, appropriate liftings of neighborhoods of the two distinguished points yield the required pair of discs.

10.4. We now prove half of Theorem 9. Let $G \in \mathcal{C}_0$, and let $x \in \Lambda(G)$. By Lemma 5, we can assume that x is not a parabolic fixed point.

It was shown by Marden [15] (it also follows easily from [5] and standard facts about Fuchsian groups) that quasi-Fuchsian groups have finite-sided fundamental polyhedra. Hence, if $x \in \Lambda(H)$ for some factor subgroup H, then x is a point of approximation for H, and hence for G.

If x is not a limit point of any factor subgroup, then [19] there is a structure loop W, and there is a sequence $\{g_n\}$ of elements of G so that $g_n(W)$ nests about x (i.e., $g_{n+1}(W)$ separates x from $g_n(W)$, and $g_n(W) \to x$). We assume without loss of generality that $g_1 = 1$; then W separates $g_n^{-1}(x)$ from $g_n^{-1}(W)$. We choose a point $z \in W \cap \Omega$, and we observe that if J, the stabilizer of W is finite, then the points $\{g_n^{-1}(x)\}$ are bounded away from W, and so

$$d(g_n^{-1}(x),\, g_n^{-1}(z)) \geqslant k > 0.$$

If J is parabolic, then we choose a fundamental domain D for J, where D is bounded by two tangent circles, and we choose $j_n \in J$, so that $j_n \circ g_n^{-1}(x) \in D$. Then by Lemma 5, $j_n \circ g_n^{-1}(x)$ is bounded away from W, and so $d(j_n \circ g_n^{-1}(x),\, j_n \circ g_n^{-1}(z)) \geqslant k > 0$.

We have shown that if $G \in \mathcal{C}_0$, then G has a finite sided fundamental polyhedron.

10.5. We now assume that $G \in \mathcal{C}_1 - \mathcal{C}_0$. Then there is a degenerate factor subgroup H in G.

It was shown by Greenberg [11] that degenerate groups do not have finite-sided fundamental polyhedra, hence there is a point $x \in \Lambda(H)$, where x is not a cusped parabolic fixed point of H, and x is not a point of approximation for H.

Since parabolic fixed points cannot be points of approximation, we can assume that x is not a parabolic fixed point.

Suppose there were a sequence $\{g_n\}$ of distinct elements of G, and a point $\zeta \in \Omega$, so that

$$d(g_n(x),\, g_n(\zeta)) \geqslant k > 0.$$

After passing to an appropriate subsequence, there are two possibilities to consider: either the sets $\{g_n(\Lambda(H))\}$ are all equal, or they are all distinct.

Since x is not a point of approximation for H, the sets $\{g_n(\Lambda(H))\}$ cannot all be equal.

We suppose the sets $\{g_n(\Lambda(H))\}$ are all distinct. We choose some point $z_0 \in \Delta$, and observe that for each n there are a finite positive number of structure loops which separate z_0 from $g_n(\Lambda(H))$; let W_n be the one which lies closest to $g_n(\Lambda(H))$. Then the $\{W_n\}$ are all distinct. It was shown in [22] that under these circumstances, the spherical diameter of W_n converges to 0. Hence

$$d(g_n(x),\, g_n(z)) \to 0, \quad \text{for every } z \in \Lambda(H).$$

We choose a subsequence, which we again call $\{g_n\}$, so that $g_n(z_0) \to \zeta_0$. Then [22], there is a subsequence, which we again call $\{g_n\}$ so that $g_n(z) \to \zeta_0$, for all $z \in \hat{C}$, with at most one exception. The one exception must be x, which cannot be, for the diameter of $g_n(\Lambda(H)) \to 0$, and $\Lambda(H)$ contains more than one point.

This concludes the proof of Theorem 9.

10.6. We combine Theorem 9 with a Theorem of Ahlfors [3] (see also Beardon and Maskit [5]), and obtain the following:

COROLLARY. *If $G \in \mathcal{C}_0$, then $\Lambda(G)$ has 2-dimensional measure* 0.

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