# ALGEBRAIC SURFACES WITH $k^*$-ACTION

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Let $k$ be an arbitrary algebraically closed field and let $k^*$ denote the multiplicative group of $k$ considered as an algebraic group. We give a complete classification of all non-singular projective algebraic surfaces with $k^*$-action. Moreover, for singular surfaces we provide an algorithm for finding an equivariant resolution. Explicit computations are given for hypersurfaces in $P^3$.

The topologically significant case emerges when $k = \mathbf{C}$, the complex numbers, when non-singular surfaces are smooth orientable closed 4-manifolds with $\mathbf{C}^*$-action. This was the case studied in our earlier papers [16, 17, 18]. In order to facilitate the reading of the present paper for the reader whose main interest lies in these topological aspects, we have included geometric motivation for several of our algebraic constructions. It is clear from the naturality with which these constructions can be extended to the general case that it is appropriate to treat the problem from the point of view of algebraic geometry.

The paper is organized as follows. In section 1 the necessary tools are introduced both from transformation group theory and algebraic geometry. We prove a number of lemmas about actions in general, needed later in the paper. In section 2 we focus our attention on non-singular algebraic surfaces with $k^*$-action. The fixed points of a $k^*$-action are divided into three classes; elliptic, hyperbolic and parabolic (§ 2.3). A "topological" classification of non-singular surfaces with no elliptic fixed points is obtained in § 2.5. The classification theorem states that the fixed point set is $F = F^+ \cup F^- \cup \{x_1, ..., x_r\}$, where the $x_i$ are hyperbolic fixed points and $F^+$ and $F^-$ are isomorphic complete curves. More-

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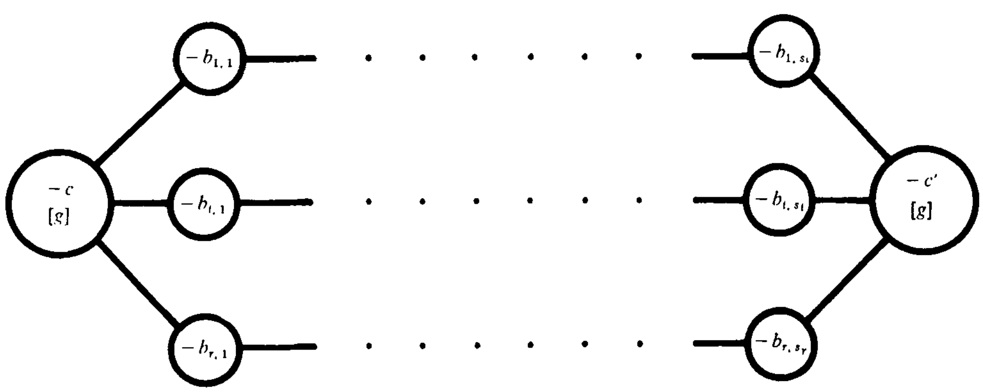
over the surface is obtained from a $P^1(k)$ bundle over $F^+$ with $k^*$ action in the fiber, by blowing up a suitable number of fixed points of the action. A complete set of "topological" invariants is included in a labeled graph $\Gamma$ associated to the surface $V$. Assign a vertex each to $F^+$ and $F^-$ with labels $[g]$ for their genus and $-c$ and $-c'$ respectively for their self-intersection in $V$. Assign a vertex to each 1-dimensional orbit whose closure does not meet both $F^+$ and $F^-$, with label $-b_{ij}$ for its self-intersection in $V$. Finally, let an edge connect two vertices if the corresponding curves meet (transversly with intersection number $+1$).

THEOREM 2.5. *Let $V$ be a non-singular, complete algebraic surface with $k^*$ action, so that* $\dim F^+ = \dim F^- = 1$. *Then we have:*

(i) *$F^+$ is a complete curve of genus $g$, isomorphic to $F^-$;*

(ii) *the graph $\Gamma$ of $(V, k^*)$ is of the form*




(iii) $b_{ij} \geq 1$ *and the continued fraction (see the Appendix for this notation)*

$$[b_{i,1}, \ldots, b_{i,s_i}] = 0 \quad \text{for} \quad i = 1, \ldots, r;$$

(iv) *if we define the integers $c_i$ by*

$$1/c_i = [b_{i,1}, \ldots, b_{i,s_i-1}] \quad \text{for} \quad i = 1, \ldots, r$$

*then the following equation holds*

$$c + c' = \sum_{i=1}^{r} c_i.$$

In section 3 we construct the canonical equivariant resolution of the singularities of a surface $V$ with $k^*$-action. By this we mean a non-singular surface $W$ with a $k^*$-action and a $k^*$ equivariant proper map $\pi$: $W \to V$ which is biholomorphic on the complement of the singular set of $V$ and so that there are no elliptic fixed points on $W$. Note that even if $V$ is nonsingular, $W$ will not be equal to $V$ if there are elliptic fixed points on $V$. We define in §3.2 a marked graph $\Gamma_V$, starting with $\Gamma_W$ and indicating which orbits in $W$ are identified to points in $V$ and which orbits in $W$ are identified to each other in $V$. This marked graph contains all the "topological" information about $V$. In particular if $k = \mathbf{C}$, $\Gamma_V$ determines $V$ up to equivariant homeomorphism. In §3.3 we define the Seifert orbit invariants $(\alpha, \beta)$ of an orbit of dimension 1. Here $\alpha$ is the order of the isotropy group of a point $v$ on the orbit and $\beta$ determines the action of the isotropy group on the tangent space at $v$. Section 3.4 is devoted to determining the behavior of orbit invariants under "blowing up" and we use this in §3.5 to prove a relation between the continued fraction expansion of $\alpha/\beta$ and the self-intersection of certain orbits on $V$. Explicit formulas for determining the orbit invariants and the graph of the resolution of an elliptic fixed point on a surface in $k^3$ are given. The corresponding action was called "good" in [16]. These results are an improvement over [16] since they are valid for a field of arbitrary characteristic. Moreover, the formulas are in terms of the weights and degree of the defining polynomial. In §3.7 we indicate how to find the minimal equivariant resolution of a hyperbolic or parabolic singular point, not treated in [16].

Section 4 is devoted to the diffeomorphism and algebraic classification problems. First we classify the (relatively) minimal surfaces with $k^*$-action §4.1 and apply this to determining which surfaces admit an essentially unique $k^*$-action and which $k^*$-actions extend to $k^* \times k^*$-actions §4.2. In §4.3 it is shown that a complete surface with algebraic $\mathbf{C}^*$-action is diffeomorphic to precisely one of the following

(1) $(R_g \times S^2) \# k\overline{\mathbf{C}P}^2 \quad g \geq 0, \quad k \geq 0,$

(2) $N_g$,

(3) $\mathbf{C}P^2$,

where $R_g$ is a compact Riemann surface of genus $g$ and $N_g$ is the non-trivial $S^2$ bundle over $R_g$, and $\overline{\mathbf{C}P}^2$ denotes $\mathbf{C}P^2$ with the opposite of the usual orientation. This is applied to give examples of non standard $S^1$ actions on $\mathbf{C}P^2 \# k\overline{\mathbf{C}P}^2$, $k \geq 3$ i.e. $S^1$ actions which do not extend to $S^1 \times S^1$ actions (§4.4). Finally, we give a complete list of invariants of $V$ up to equivariant algebraic isomorphism (§4.6).

Specific examples of the resolution of singularities of hypersurfaces $V$ in $P^3$ and calculation of $\Gamma_V$ are given in section 5. Some facts about continued fractions have been collected in an appendix.

## 1. Group actions on varieties

**1.1.** Let $G$ be an algebraic group. We shall not assume that $G$ is compact. In the applications $G$ will be $k^*$, the multiplicative group of a field $k$.

Let $V$ be a algebraic variety. An *action* of $G$ on $V$ is a morphism of algebraic varieties

$$\sigma: G \times V \to V$$

so that $\sigma(s, \sigma(t, v)) = \sigma(st, v)$ and $\sigma(1, v) = v$. If $g \in G$ and $v \in V$ it is customary to denote the element $\sigma(g, v)$ by $gv$. The action is *effective* if $gv = v \ \forall v \in V \Rightarrow g = e$, the identity element of $G$. For $v \in V$ $\sigma$ induces a morphism $\sigma_v: G \to V$ defined by $\sigma_v(g) = gv$. We define the isotropy subgroup $G_v$ of $v$ to be the scheme theoretic fiber $\sigma_v^{-1}(v)$. If $k$ is a field of characteristic 0 then $G_v = \{g \in G \mid gv = v\}$. The orbit of $v$ is the subvariety defined by $G(v) = \{w \in V \mid \exists g \in G, gv = w\}$. This induces a natural equivalence relation on $V$, $x \sim y \Leftrightarrow x \in G(y)$. The quotient $V^* = V/\sim$ equiped with the quotient topology is called the *orbit space* of the action. Even if $V$ is a complex manifold, $V^*$ can be rather unpleasant, as the examples below indicate, see also [8]. Let $F = \{v \in V \mid G_v = G\}$ denote the fixed point set and $E = \{v \in V \mid G_v$ is finite and $G_v \neq (1)\}$ denote the exceptional orbits, i.e. orbits with finite non-trivial isotropy group.

**1.2.** Let $V = \mathbf{C}^2$ with the metric topology, and $G = \mathbf{C}^*$ and $q_1, q_2$ integers. Consider the action $\sigma(t; v_1, v_2) = (t^{q_1} v_1, t^{q_2} v_2)$. First assume that $q_1, q_2 > 0$. Then $F = (0, 0)$ and $E$ consists of two orbits: $E_1 = \{v_1, v_2 \mid v_1 \neq 0, v_2 = 0\}$ with isotropy $\mathbf{Z}_{q_1}$ and $E_2 = \{v_1, v_2 \mid v_1 = 0, v_2 \neq 0\}$ with $\mathbf{Z}_{q_2}$. The orbit space $V^*$ fails to be $T_0$ at the image of the fixed point. However, if we let $V_0 = V - F$, then $V_0^*$ is Hausdorff, in fact by a theorem of Holmann [8] it has a natural complex structure and is easily identified as $P^1$. The situation is the same if $q_1, q_2 < 0$.

**1.3.** In the example above, if $q_1 \neq 0$, $q_2 = 0$, then $F = \{v_1, v_2 \mid v_1 = 0\}$ and $E = \{v_1, v_2 \mid v_1 \neq 0)\}$ so the action is not effective unless $q_1 = \pm 1$. In that case $E = \emptyset$. Again, $V^*$ fails to be $T_0$ at $F$, but $V_0 = V - F$ has Hausdorff orbit space, $V_0^* = \mathbf{C}$.

**1.4.** If we assume $q_1 > 0$, $q_2 < 0$, then $F$ and $E$ are the same as in § 1.2, but even the orbit space of $V_0$ fails to be Hausdorff. The images of $E_1$ and $E_2$ have no disjoint neighborhoods in $V_0^*$. (This holds even if $q_1 = 1$, $q_2 = -1$.) This phenomenon is indeed one of the crucial differences between the actions of compact and non-compact groups. The remarks above apply in the category of algebraic varieties if we replace "Hausdorff" by "separated".

**1.5.** A map $f: V \to W$ between $G$ spaces $V$ and $W$ is *equivariant* if $f(gv)=gf(v)$ for all $g \in G$, $v \in V$. A $G$-space is *linear* if it is equivariantly isomorphic to a vector space with linear $G$ action. Let $H$ be a closed subgroup of $G$ and $S$ a *linear* $H$-space. We form $G \times_H S$ by identifying in $G \times S$ $(g, s) \sim (gh^{-1}, hs)$ for all $h \in H$. Call the equivalence class of $(g, s)$, $[g, s]$. Given a $G$-space $V$ and $v \in V$ we call $S_v$ a *linear slice* at $v$ if it is a linear $G_v$-space and some $G$-invariant open neighborhood of the zero section of $G \times_{G_v} S_v$ is equivariantly isomorphic to a $G$-invariant neighborhood of the orbit $G(v)$ in $V$ by the map $[g, v] \to gv$, so that the zero section $G/G_v$ maps onto the orbit $G(v)$.

It is essential for the reader to keep in mind that the standard tools of transformation groups do not always apply in our context. For holomorphic or smooth action of a *compact* Lie group $G$, the existence of slices is a classical result. In our situation with $G=k^*$, a slice need not exist.

We refer to Holmann [8] and Luna [10] for results on the existence of slices. These will only be used in the present paper in §5.2.

**1.6.** A $k^*$-action on an affine variety $V=\operatorname{Spec}(A)$ is equivalent to a grading of the ring $A$. The correspondence is defined as follows. If $V$ is an affine variety with $k^*$ action we define a grading on $A$ by letting

$$A_i = \{f \in A \mid f(tz) = t^i f(z), \quad \text{for all } z \in V\}.$$

Then one can verify that $A = \oplus_{i \in Z} A_i$ and that $A_i A_j \subset A_{i+j}$. Conversely, if $A$ is graded we can find homogeneous generators of the $k$-algebra $A$, say $x_1, \ldots, x_n$. The homomorphism $\varphi: k[X_1, \ldots, X_n] \to A$ defined by $\varphi(X_i)=x_i$ defines an embedding of $V$ in $k^n$. Let $q_i =$ degree $x_i$. Then define an action of $k^*$ on $k^n$ by

$$t(z_1, \ldots, z_n) = (t^{q_1} z_1, \ldots, t^{q_n} z_n)$$

This action on $k^n$ leaves $V$ invariant.

We can use this correspondence to get some information about the structure of a $k^*$ variety.

LEMMA. *If $k^*$ acts effectively on $V$, then there exists an invariant Zariski open set $U$, equivariantly isomorphic to $W \times k^*$ where*

(i) *$W$ is affine, and*

(ii) *$k^*$ acts on $W \times k^*$ by translation on the second factor.*

*Proof.* We first consider the case where $V = A_k^n$ and the action is given by

$\sigma(t, (z_1, ..., z_n)) = (t^{m_1}z_1, ..., t^{m_n}z_n)$. In this situation let $U = \{(z_1, ..., z_n) | z_i \neq 0, \text{ for all } i\}$. We claim that $U$ is equivariantly isomorphic to $W \times k^*$ with $k^*$ acting on the second factor by multiplication. Now

$$U = \operatorname{Spec}(k[z_1, ..., z_n, z_1^{-1}, ..., z_n^{-1}]).$$

In this case the grading is defined by degree $z_i = m_i$. Since the action is effective, g.c.d. $(m_1, ..., m_n) = 1$. Therefore there exist integers $a_1, a_2, ..., a_n$ so that $\sum_{i=1}^n a_i m_i = 1$. Let $X = Z_1^{a_1} \dots Z_n^{a_n}$, a form of degree 1. Let $R$ be the subalgebra of forms of degree 0. Then $R$ is an algebra of finite type over $k$ and

$$k[z_1, ..., z_n, z_1^{-1}, ..., z_n^{-1}] = R[X, X^{-1}].$$

Letting $W = \operatorname{Spec}(R)$, we have proven our claim.

Now consider the case of general $V$. We may assume that $V$ is normal. Then a theorem of Sumihiro [23] asserts that an open subset of $V$ can be equivariantly embedded in $A_k^n$, with an action as above. Thus we may assume that $V \subset A_k^n$ and is invariant under $\sigma$. Let $I \subset R[X, X^{-1}]$ be the (prime) ideal defining $V$. The fact that $V$ is invariant under $\sigma$ means that $I$ is a graded ideal. Thus $I = \oplus_{j \in Z} I_j$ where $I_j$ consist of the elements of $I$ which are homogeneous of degree $j$. However $f \subset I_j \Leftrightarrow X^{-j}f \in I_0 \Leftrightarrow f = X^j(X^{-j}f) \in X^j I_0$. Thus $I = \oplus_j X^j I_0$ and hence $V = \operatorname{Spec}(R/I_0) \times \operatorname{Spec}(k[X, X^{-1}])$ which is the desired result.

**1.7. Lemma.** *Let $G$ be an algebraic group, $V$ and $W$ varieties with $G$ action and $U \subset V$ an invariant Zariski open set. If the map $f: V \to W$ is equivariant when restricted to $U$, then it is equivariant on $V$.*

*Proof.* The following diagram with 1 denoting the identity map

$$\begin{array}{ccc} G \times V & \xrightarrow{1 \times f} & G \times W \\ \downarrow \sigma_V & & \downarrow \sigma_W \\ V & \xrightarrow{f} & W \end{array}$$

is commutative because $G \times U$ is dense in $G \times V$ and the maps $\sigma \circ (1 \times f)$ and $f \circ \sigma$ agree on $G \times U$, hence they agree on $G \times V$.

**1.8.** Let $O_V$ denote the sheaf of rational functions on $V$ and $I \subset O_V$ an ideal sheaf. The *monoidal transform* with center $I$ is a pair $(\pi, V')$ with $\pi: V' \to V$ and

(i) $IO_{V'}$ is locally principal, i.e. $\forall w \in V'$ the stalk $(IO_{V'})_w$ is generated by one function,

(ii) for every $\pi_0\colon V_0 \to V$ satisfying "$IO_{V_0}$ is locally principal" there is a unique $\eta\colon V_0 \to V'$ with $\pi \circ \eta \to \pi_0$.

The monoidal transform exists by Hironaka [7] and it is unique by (ii). If $X$ is a subspace of $V$ and $I_X$ is the sheaf of functions vanishing on $X$, then the monoidal transform with center $X$ is defined to be the monoidal transform with center $I_X$. It is also called "blowing up of $X$". For a geometric description of the blowing up of a subvariety see [22].

LEMMA. *Let $G$ be an algebraic group acting on an algebraic variety $V$ and $(\pi, V')$ a monoidal transformation with center a $G$-invariant sheaf of ideals $I$. Then there is a unique extension of the action of $G$ to $V'$ so that $\pi$ is equivariant.*

*Proof.* We first show that there is a unique map $\sigma'\colon G \times V' \to V'$ so that the diagram below commutes.

$$\begin{array}{ccc} G \times V' & \xrightarrow{1 \times \pi} & G \times V \\ \Big\downarrow{\scriptstyle \sigma'} & & \Big\downarrow{\scriptstyle \sigma} \\ V' & \longrightarrow & V \end{array}$$

For this it is sufficient to show that $(\sigma \circ (1 \times \pi))^*(I)$ is a locally principal sheaf of ideals in $O_{G \times V'}$. Now $I$ is invariant under $\sigma$ so $\sigma^*(I) = O_G \otimes I$ and hence $(\sigma \circ (1 \times \pi))^*(I) = O_G \otimes \pi^*(I)$ which is locally principal because $\pi^*(I)$ is locally principal. To verify that $\sigma'$ is an action, let $U = V -$ support $(I)$ and $U' = \pi^{-1}(U)$. Since $U'$ is isomorphic to $U$, the following diagram is commutative

$$\begin{array}{ccc} G \times (G \times U') & \xrightarrow{m \times 1} & G \times U' \\ \Big\downarrow{\scriptstyle \sigma} & & \Big\downarrow{\scriptstyle \sigma} \\ G \times U' & \xrightarrow{\sigma} & U' \end{array}$$

where $m$ is the multiplication of $G$. An application of § 1.7 completes the proof.

**1.9.** PROPOSITION. *Suppose $V$ is a non-singular algebraic surface with $k^*$-action and $C \subset V$ is a complete curve with negative self-intersection number. Then $C$ is invariant under the action.*

*Proof.* Let $C_t = \{tx \mid x \in C\}$. Then $C_t$ is rationally equivalent (homologous if $k = \mathbf{C}$) to $C$ for all $t$. If $C$ is not invariant then $(C \cdot C) = (C \cdot C_t) > 0$, which is a contradiction.

## 2. Non-singular surfaces with k*-action

**2.1.** From now on we shall concentrate on the case $G=k^*$, the multiplicative group of a field, and $V$ an algebraic surface over that field. In this section we shall determine the topological type of a non-singular algebraic surface with $k^*$-action, provided it has no elliptic fixed points.

*Definition.* A morphism $f\colon X\to Y$ of algebraic varieties is said to be *birational* if there is a closed subvariety $Z\subset Y$ so that $f\colon X-f^{-1}(Z)\to Y-Z$ is an isomorphism.

LEMMA. *Suppose $f\colon X\to V$ is a $G$-equivariant birational proper morphism of non-sigular surfaces. Then there exist equivariant morphisms $f_i\colon X_i\to X_{i-1}$, $i=1,\dots,n$ so that $X_0=V$, $X_n=X$ and $f_i$ is a monoidal transform with center at a fixed point of $X_{i-1}$.*

*Proof.* By [9, § 26] the set of points $S$ where $f^{-1}$ is not defined is finite. Moreover if $y\in S$, then $y$ is a fixed point and $f^{-1}(y)$ is a divisor. Now let $S=\{y_0, y_1,\dots,y_r\}$ and define $h_1$ to be the blowing up of the points $y_0,\dots,y_r$. Then since $f^{-1}(\{y_0,\dots,y_r\})$ is locally principal we have a factorization $X\xrightarrow{g_1}X_1\xrightarrow{h_1}Y$. Repeating the process, by [9] we reach a point so that $f=h_1\circ h_2\circ\dots\circ h_r$, where $h_i\colon X_i\to X_{i-1}$ is a monoidal transform with center at a finite number of points. Factoring each $h_i$ gives the desired result.

**2.2.** *Definition.* $V$ is *normal* at $v\in V$ if the local ring $O_v$ at $v$ is integrally closed. For any variety $V$ there is a variety $\tilde V$ and a proper morphism $\pi\colon\tilde V\to V$ called the normalization of $V$ characterized by the fact that for every affine open subset $U$ of $V$, $\Gamma(\pi^{-1}(U), O_{\tilde V})$ is the integral closure of $\Gamma(IU, O_V)$ in its field of quotients [4, II. 6.3]. If $\sigma\colon V_1\to V_2$ is a birational morphism there exists a unique $\tilde\sigma\colon\tilde V_1\to\tilde V_2$ so that

$$\begin{array}{ccc} \tilde V_1 & \xrightarrow{\tilde\sigma} & \tilde V_2 \\ \downarrow & & \downarrow \\ V_1 & \xrightarrow{\sigma} & V_2 \end{array}$$

commutes.

*Definition.* $V$ is *complete* if the morphism $V\to\operatorname{Spec}(k)$ is proper. If $k=\mathbf{C}$ this is equivalent to $V$ being compact in the metric topology.

THEOREM. *Suppose $V$ is a complete, normal algebraic surface with $k^*$ action. Then there is a complete non-singular curve $C$, a complete surface $Z$ with $k^*$ action and morphisms*

$$V \xleftarrow{\varphi_1} Z \xrightarrow{\varphi_2} C \times P^1$$

*so that*

(i) *the action on $C \times P^1(k)$ is given by*

$$t(x, [z_0 : z_1]) = (x, [tz_0 : z_1]),$$

(ii) *$\varphi_1$ and $\varphi_2$ are equivariant,*

(iii) *$\varphi_2$ is a composite of monoidal transforms whose centers are fixed points of the $k^*$-action,*

(iv) *$\varphi_1$ is a composite of maps, each of which is either a normalization or a blowing up of a fixed point. If $V$ is non-singular, then each of the maps is a blowing up.*

*The variety of $Z$ constructed here is not canonical.*

*Proof.* It follows from § 1.6 that there is an open invariant subset $U \subset V$ equivariantly isomorphic to $C_0 \times k^*$, where $C_0$ is a non-singular curve. Thus there is an equivariant open embedding $f: U \to C \times P^1$ where $C$ is the (non-singular) completion of $C_0$. Now $f$ defines a rational map $f: V \to C \times P^1$ and it follows from [9] that the set $S$ of points where $f$ is not defined is finite. If $x \in X$ is not a fixed point then $f$ is defined at $x$, since for some $g \in k^*$, $f$ is defined at $gx$ and $f(x) = g^{-1}f(gx)$. Thus $S$ consists of fixed points. Let $V_0 = V$. Define inductively $f_i: V_i \to V_{i-1}$ to be the normalization of the monoidal transform with center at the singular points and points of indeterminacy of $f \circ f_1 \circ \ldots \circ f_{i-1}$. By [9, (26.2)] and § 2.1 there is an $s$ so that $V_s$ is non-singular and $f \circ f_1 \circ \ldots \circ f_s$ is defined on $V_s$. Let $Z = V_s$, $\varphi_1 = f_1 \circ \ldots \circ f_s$ and $\varphi_2 = f \circ \varphi_1$. The $f_i$ are equivariant by Lemma 1.8 and the functoriality of normalization. Moreover, $\varphi_2$ is equivariant by § 1.7. Assertion (iii) follows from § 2.1.

**2.3.** Let $V$ be a non-singular variety and $v \in V$ a fixed point of a $k^*$-action. The tangent space $T_v$ at $v$ has a local coordinate system in which the induced action of $k^*$ is linear and given by

$$t(z_n, \ldots, z_m) = (t^{q_1} z_1, \ldots, t^{q_m} z_m)$$

for integers $q_1, \ldots, q_m$ [2]. Denote by $N^+(v)$ the dimension of the positive eigenspace of this action $\# \{q_i > 0\}$, $N^-(v)$ the dimension of the negative eigenspace and $N^0(v)$ the dimension of the subspace fixed under the action in $T_v$.

We call $v$ an *elliptic fixed point* if $m = \dim T_v = N^+(v)$ or $m = N^-(v)$. It is a *source* in the former case and a *sink* in the latter.

We call $v$ a *parabolic fixed point* if $N^0(v)>0$ and either $m=N^+(v)+N^0(v)$ or $m=N^-(v)+N^0(v)$. The former is a parabolic source, the latter a parabolic sink.

All other fixed points are called *hyperbolic.*

We can extend the definition of an elliptic fixed point to the singular case. Assume $v \in V$ is normal. We say $v$ is an elliptic fixed point if there exists an invariant neighborhood $U$ containing $v$ so that $v$ is in the closure of every orbit in $U$. Recall that we denote by $F$ the fixed point set of $V$. Let $F^+=\{v \in F \mid N^-(v)=0\}$ and $F^-=\{v \in F \mid N^+(v)=0\}$. It follows from [2] that if $V$ is non-singular then $F^+$ and $F^-$ are irreducible, connected components of $F$. If $V$ is complete, then both $F^+$ and $F^-$ are non-empty.

2.4. Let $V$ be a non-singular, complete surface with $k^*$-action and assume that $\dim F^+=\dim F^-=1$. A 1-dimensional orbit $O$ is called *ordinary* if $\bar{O} \cap F^+ \neq \emptyset$ and $\bar{O} \cap F^- \neq \emptyset$, where $\bar{O}$ is the closure of $O$. According to Theorem 2.2 the odinary orbits form an open set in $V$ and since $V$ is compact, there are only a finite number of 1-dimensional orbits $E_1, E_2, \ldots, E_m$ which are *special,* i.e. not ordinary. We define the *weighted graph* $\Gamma$ of $(V, G)$ as follows:

(i) its vertices are $f^+$ for the curve $F^+$, $f^-$ for $F^-$ and $e_1, \ldots, e_m$ for the closures $\bar{E}_1, \ldots, \bar{E}_m$,

(ii) each vertex carries a weight $[g]$, referring to the genus of the curve it represents, (if $g=0$ this weight is omitted),

(iii) each vertex carries a weight $n$, representing the self-intersection in $V$ of the curve it represents,

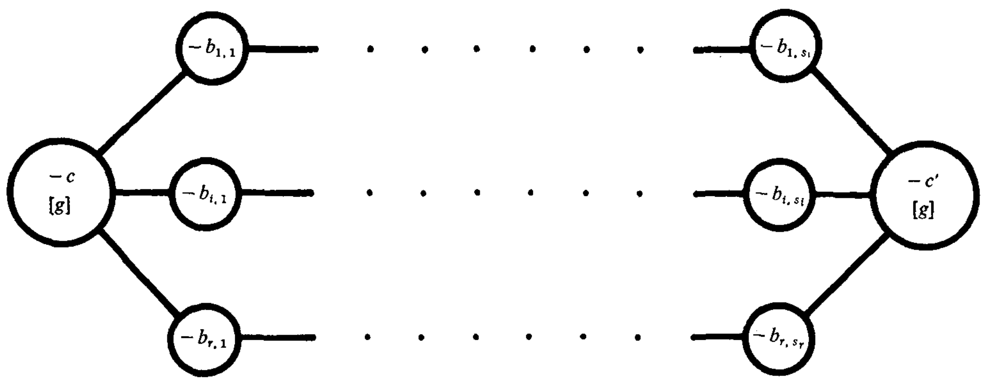
(iv) two vertices are connected by an edge if and only if the respective curves intersect in $V$.

Clearly each point of intersection between different orbit closures is fixed under the action. It follows from § 2.2 that if any two of the curves above intersect, they intersect transversely.

2.5. THEOREM. *Let $V$ be a non-singular, complete algebraic surface with $k^*$-action so that* $\dim F^+=\dim F^-=1$. *Then we have*

(i) *$F^+$ is a complete curve of genus $g$, isomorphic to $F^-$,*

(ii) *the graph $\Gamma$ of $(V, k^*)$ is of the form*



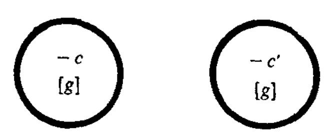

(iii) $\{b_{i,1}, b_{i,2}, ..., b_{i,s_i}\}$ *is admissible* (*see Appendix*) *and the continued fraction*

$$[b_{i,1}, b_{i,2} ..., b_{i,s_i}] = 0 \quad \text{for} \quad i = 1, ..., r,$$

(iv) *with* $c_i = 1/[b_{i,1}, ..., b_{i,s_i-1}]$ *we have*

$$c + c' = \sum_{i=1}^{r} c_i.$$

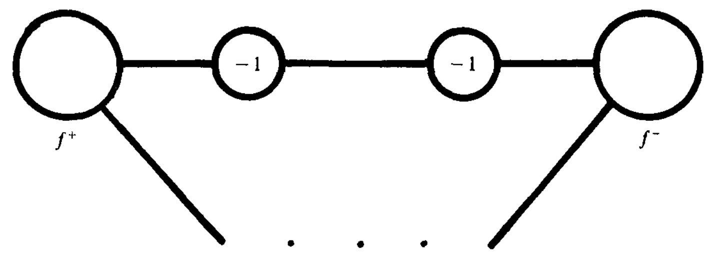
*Proof*. Applying Theorem 2.2 we see that $V$ can be obtained from $C \times P^1$ by blowing up fixed points and then blowing down curves. No fixed curves are blown down since $\dim F^+ = \dim F^- = 1$. The theorem clearly holds for $V = C \times P^1(k)$, since $\Gamma$ is of the form



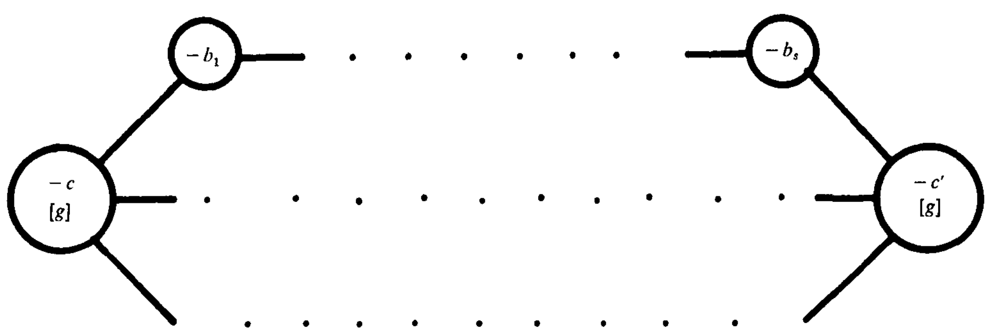

where $c = c' = 0$, and $F^+ = F^- = C$. Hence it is sufficient to show that if $V$ is a surface with $\dim F^+ = \dim F^- = 1$ and $\pi$: $V \to W$ is a monoidal transform with center $x$ at a hyperbolic or parabolic fixed point of $V$, then the theorem holds for $V$ if and only if it holds for $W$. This is obvious for (i) and (ii). For (iii) we have

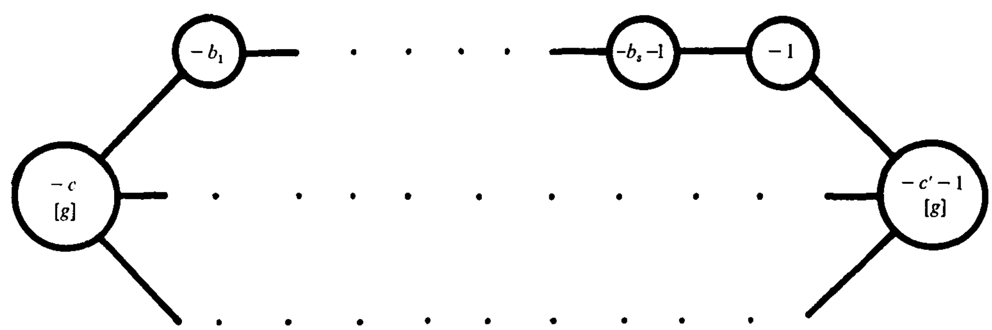
$$[b_1, ..., b_i, b_{i+1}, ..., b_s] = [b_1, ..., b_{i-1}, b_i + 1, 1, b_{i+1} + 1, b_{i+2}, ..., b_s]$$

according to A.7. Finally, the expressions of the formula of (iv) remain unchanged if $x$ is not on $F_V^+$ or $F_V^-$. Assume without loss of generality that the center of $\pi$ is on $F_V^-$. Then $c_W = c_V$ and $c'_W = c'_V + 1$. If $x$ is in the closure of an ordinary orbit, then $\Gamma_W$ is obtained from $\Gamma_V$ by adding one new arm [1, 1] as follows




This clearly increases both sides of (iv) by 1, as required. If $x$ is not on the closure of an ordinary orbit, then $W$ is obtained from $V$ by blowing up a point on $F^-$ which is in the closure of a special orbit. This corresponds to adding a new vertex to the $i$th arm of $\Gamma_V$. Omitting the first index for convenience, it changes as shown:




Finally, $1+1/[b_1, \ldots, b_{s-1}] = 1/[b_1, \ldots, b_{s-1}, b_s+1]$ follows from Lemma (A.8) to complete the argument.

**2.6.** PROPOSITION. *Suppose $k$ is an algebraically closed field. Any graph $\Gamma$ satisfying* (ii)–(iv) *of* § 2.5 *arises from a complete algebraic surface with $k^*$-action.*

*Proof.* Let $V_0 = C \times P^1$ where $C$ is a non-singular complete curve of genus $g$ and let $k^*$ act on the second factor. By A.3 and A.7 we can obtain a surface $V_1$ with the desired weights on the arms by a sequence of monoidal transforms with centers at fixed points. The graph of $V_1$ may not have the desired weights $c$ and $c'$ for $f^-$ and $f^+$. Let elm$(V_1)$ be the variety obtained from $V_1$ by blowing up a point $x$ of $F^+$ which lies in the closure of an ordinary orbit $\bar{O}$, and then blowing down the transform of $\bar{O}$. This surface has the same graph as $V_1$ except that the weight of $f^-$ is increased by 1 and the weight of $f^+$ is decreased by 1. Similarly the weight of $f^-$ can be decreased. Thus in a finite number of steps we can obtain a surface $V_2$ with $F^+$ having self intersection $-c$. Now $c'$ is determined by the relation (iv) hence $V_2$ has the desired graph.

## 3. Resolution of singularities

**3.1.** We shall now consider *singular* surfaces $V$ with $k^*$-action. By an *equivariant resultion* of the singularities of $V$ we mean a non-singular surface $W$ and a $k^*$ equivariant map $\pi$: $W \to V$ which is proper and birational. The existence of an equivariant resolution was demonstrated in § 2.2. In this section we construct a *canonical* equivariant resolution.

**3.2.** *The canonical equivariant resolution.* Suppose $V$ is an irreducible normal surface. We construct the canonical equivariant resolution locally and then put the pieces together. Since $V$ is normal, the singular set consists of a finite number of points, each of which must be fixed. Suppose $v$ is an elliptic fixed point (singular or not). By [23], the normality of $V$ implies there is an affine invariant neighborhood $U$ of $v$ and an equivariant embedding $f$: $U \to k^n$, where the action on $k^n$ is given by

$$t(z_1, \ldots, z_n) = (t^{q_1} z_1, \ldots, t^{q_n} z_m).$$

Moreover we can assume $q_i > 0$, for all $i$, if $v$ is a source and $q_i < 0$, for all $i$ if $v$ is a sink. Assume that $v$ is a source. Now $U = \text{Spec}\,(A)$ and $A = k[X_1, \ldots, X_n]/I$, where $I$ is the ideal of functions vanishing on $U$. The invariance of $U$ under the action is equivalent to the fact that $A$ is a graded ring, the grading being induced by letting degree $X_i = q_i$, § 1.6. The quotient space $X = U - \{v\}/k^*$ is the algebraic variety Proj $(A)$ [4, II, § 2]. We shall construct a canonical variety $U_X$ and morphisms $h$: $U_X \to U$ and $\nu$: $U_X \to X$ .Geometrically $\nu$: $U_X \to X$ is the Seifert $k^*$ bundle with fiber $k$ associated to the principal Seifert $k^*$-bundle $U - \{v\} \to X$. The map $h$ collapes the zero section to the point $v$.

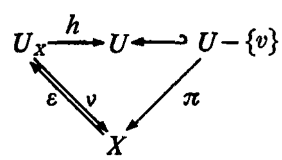
**Let**

$$A_{[n]} = \bigoplus_{m \geq n} A_m \quad \text{and} \quad A^\natural = \bigoplus_{n \geq 0} A_{[n]}.$$

If we consider $A$ as an ungraded ring then $A^\natural$ is a graded $A$ algebra under the multiplication which makes

$$A_{[n]}\,A_{[m]} \subset A_{[n+m]}.$$

Let $U_X = \text{Proj}\,(A^{\natural})$. Then by [4, II, 8.3 and 8.6.2] $U_X$ is a projective variety over $U = \text{Spec}\,(A)$ and there are natural morphisms




so that

(1) $U_X - \varepsilon(X)$ is isomorphic to $U - \{v\}$ and $\nu$ restricted to $U_X - \varepsilon(X)$ is the orbit map $U - \{v\} \to X$.

(2) $\nu \circ \varepsilon = id_X$.

If $v \in V$ is a sink and $U = \text{Spec}\,(A)$ is a neighborhood of $v$ as above, then $A$ is generated by forms of negative degree. We can define a new grading on $A$ by letting new degree $x = -(\text{old degree } x)$. Then we can perform the same construction as above and $h\colon U_X \to U$ will again be equivariant.

Now there is a unique proper birational morphism $\pi_1\colon V_1 \to V$ so that $\pi_1$ agrees with the above in the neighborhood of every elliptic fixed point and $\pi_1$ is an isomorphism elsewhere. Moreover one can show that $V_1$ is normal (A.10). The only fixed points on $V_1$ are hyperbolic or parabolic. Note that the definition of $\pi\colon V_1 \to V$ makes sense for a variety with $k^*$-action of any dimension.

LEMMA. *If $v \in V$ is a singular point on a normal surface with $k^*$-action and $\pi\colon \tilde{V} \to V$ is the minimal resolution of the singularity, then there is a unique action $\tilde{\sigma}$ on $\tilde{V}$ so that $\pi$ is equivariant.*

*Proof.* By §2.2 we know that there is an equivariant resolution $p\colon W \to V_0$. The minimal resolution is obtained stepwise by collapsing rational curves on $W$ having self-intersection $-1$. By §1.8 these curves are invariant under the action, hence there is an induced action on the minimal resolution.

Now define $\pi_2\colon \tilde{V} \to V_1$ to be the minimal resolution of the singularities of $V_1$. The composite $\pi = \pi_2 \circ \pi_1\colon \tilde{V} \to V$ is called the *canonical equivariant resolution of* $V$.

If $V$ is not normal we let $\pi_0\colon V_0 \to V$ be the normalization of $V$ and $\pi_1\colon \tilde{V} \to V_0$ the canonical equivalent resolution of $V_0$. Then $\pi = \pi_0 \circ \pi_1$ is the *canonical equivariant resolution of* $V$.

**3.3.** *Orbit invariants.* Suppose V is a variety with $k^*$-action and $v \in V$ is a simple point. If $v$ is not a fixed point then $G_v$ is a proper subgroup scheme of $k^*$. All such subgroups are of the form $\mu_\alpha = \operatorname{Spec}(k[T]/T^\alpha - I)$, for some $\alpha \geqslant 1$. If the characteristic $p$ of $k$ does not divide $\alpha$, then this is the group of $\alpha$th roots of unity. If $p$ divides $\alpha$ then the scheme $\mu_\alpha$ has nilpotent elements. The action of $k^*$ on $V$ induces an action of $G_v$ on the tangent space $T_v$. Every such action is linear and by [3, III 8.4] we can choose coordinates $(x_1, x_2, ..., x_n)$ so that the tangent space to the orbit is the $x_n$ axis and the action is defined by

$$t(x_1, \ldots, x_{n-1}, x_n) = (t^{\gamma_1}x_1, \ldots, t^{\gamma_{n-1}}x_{n-1}, x_n).$$

If $V$ is a *surface* the integer $\gamma_1$ is well defined.

*Definition.* The Seifert invariant of the point $v$ is the pair $(\alpha, \beta)$ where $G_v = \mu_\alpha$ and

$$\gamma_1 \beta \equiv 1 \pmod{\alpha}$$

$0 < \beta < \alpha$, g.c.d. $(\alpha, \beta) = 1$.

*Warning.* The definition of $\beta$ adopted here corresponds to [13] and is the negative of that given in [16].

In [16] we calculated the graph of the resolution of an elliptic point from the Seifert invariants and other data. We shall generalize that result in our current context. For this we must first analyze how the orbit invariants are affected by "blowing up".

**3.4.** PROPOSITION. *Suppose $V$ is a non-singular surface with $k^*$-action and $v \in V$ is a hyperbolic fixed point.*

*Suppose the action of $k^*$ on $T_v$ is given by*

$$t(x, y) = (t^{-a}x, t^{b}y)$$

*where $a, b \geqslant 0$ and $(a, b) = 1$. Let $\pi: V' \to V$ be the monoidal transform with center $v$ and $X = \pi^{-1}(v)$. Then*

(i) *$X$ is invariant under the action and there are two fixed points $x_0$ and $x_\infty$ on $X$, where $x_0$ is a sink and $x_\infty$ is a source of the action on $X$,*

(ii) *if $x \in X$ and $x \neq x_0, x_\infty$ then the Seifert invariants $(\alpha, \beta)$ of $x$ are given by $\alpha = a + b$ and $b\beta \equiv 1 \pmod{\alpha}$,*

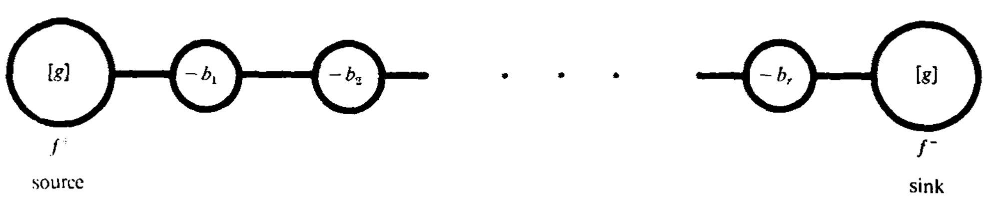
(iii) *there are coordinates $x$, $y$ for $T_{V, x_0}$ (resp. $T_{V, x_\infty}$) so that the action of $k^*$ is given by*

$$t(x, y) = (t^{-(a+b)}x, t^{b}y) \quad (\textit{resp. } t(x, y) = (t^{-a}x, t^{a+b}y)).$$

*Proof.* This is easy to see in the case $k = \mathbf{C}$. Near $x_0$ we have coordinates $(x, y)$ so that $\pi(x, y) = (xy, y)$ and near $x_\infty$ we have coordinates $x, y$ so that $\pi(x, y) = (x, xy)$. This implies (iii) and hence (ii). To consider the general (algebraic) case we replace $V$ by a suitable affine neighborhood of $v$ so that we may assume that $V = \operatorname{Spec}(A)$ and $m$, the maximal ideal of $v$, is generated by a homogeneous system of parameters $x, y$ i.e. $x$ and $y$ are homogeneous and generate $m/m^2$. Then $V' = \operatorname{Spec}(B_1) \cup \operatorname{Spec}(B_2)$ where $B_1 = A[x/y]$ and $B_2 = A[y/x]$. Let $x_1 = x/y$, $y_1 = y$, $x_2 = x$, $y_2 = y/x$. By (1.6) the action on $V'$ is determined by the grading of the rings $B_i$. These rings are graded in the natural way i.e. degree $x_1$ = degree $x$ − degree $y = -a-b$, degree $y_1$ = degree $y = b$, degree $x_2$ = degree $x = -a$, degree $y_2$ = degree $y$ − degree $x = b+a$. Now $X$ is invariant since $v$ is a fixed point. Moreover $x_0$ is the point in $\operatorname{Spec}(B_1)$ given by $x_1 = y_1 = 0$ and $x_\infty$ is the point in $\operatorname{Spec}(B_2)$ given by $x_2 = y_2 = 0$. Assertion (iii) follow from the grading given above and in addition we see that $x_0$ is the source on $X$ and $x_\infty$ the sink. To verify (ii) we note that the $(\alpha, \beta)$ given above are the Seifert orbit invariants of any point on the line $y_1 = 0$ in $T_{V, x_0}$. The inclusion $k[x_1, y_1] \to B_1$ induces an equivariant map $f$: $\operatorname{Spec}(B_1) \to \operatorname{Spec}(k[x_1, y_1]) \approx T_{V, x_0}$. This map is etale at $x_0$ since $x_1, y_1$ is a system of parameters and hence it is étale in a neighborhood of $x_0$. If we choose a point $x$ on $X$ near $x_0$, then the Seifert invariants of $x$ and $f(x)$ are the same hence $x$, and therefore any point in $G(x)$, has invariants $(\alpha, \beta)$

**3.5.** *Isotropy groups and the isotropy representation.* (1)

PROPOSITION. *Suppose $V$ is a non-singular surface with graph $\Gamma$. Given an "arm" of the graph*




*let $X_i$ be the curve corresponding to the $i$th vertex on the arm, i.e. $(X_i, X_i) = -b_i$ and let*

$$[b_1, \ldots, b_i] = p_i/q_i, \text{ where } (p_i, q_i) = 1.$$

(1) *If $v \in X_i \cap X_{i+1}$, $0 \leqslant i \leqslant r$, then we can choose coordinates $(x, y)$ in $T_v$ so that the induced action of $k^*$ on $T_v$ is of the form*

$$(t^{-p_{i-1}}x, t^{p_i}y).$$

(1) For the terminology of this section, see the appendix.

(2) *If* $v \in X_{i+1}$ *is not a fixed point, then the Seifert orbit invariants at* $v$ *are* $(\alpha, \beta)$, *where* $\alpha = p_i$ *and* $\beta p_{i-1} \equiv -1 \pmod{p_i}$, $0 \leqslant i \leqslant r-1$.

(3) *Suppose* $b_1, b_2, \ldots, b_i \geqslant 2$. *If the Seifert orbit invariants of a non-fixed point* $v \in X_{i+1}$ *are* $(\alpha, \beta)$, *then the equation* $[b_1, \ldots, b_i] = \alpha/(\alpha-\beta)$ *determines the* $b_i$-*uniquely. Moreover* $[b_r, b_{r-1}, \ldots, b_{i+2}] = \alpha/\beta$.

*Proof.* By induction on $r$. If $r=1$ then $b_1=0$, $p_{-1}=0$, $p_0=1$ and $p_1=0$. On the other hand $X_0=F^+$, $X_1$ is an ordinary principal orbit and $X_2=F^-$. The assertions are easily verified. Now suppose the statements are true for an arm with fewer than $r$ vertices. By (A.3) and (2.5) one of the integers $b_{j+1}$ must be equal to 1. Let $\pi: V \to V_0$ be the map which collapses $X_{j+1}$. Then $\pi$ maps $X_i$ isomorphically onto a curve $X_i'$ for $i \neq j+1$. Let

$$p_i'/q_i' = [b_1, \ldots, b_j - 1, b_{j+2} - 1, \ldots, b_{i+1}]$$

as in the appendix. Then by A.6 $p_i'=p_i$, $q_i'=q_i$ for $i \leqslant j-1$ and $p_i'=p_{i+1}$, $q_i'=q_{i+1}$ for $i \geqslant j$. Now one can easily verify that (1) holds for $i \neq j, j+1$ and (2) holds for $i \neq j$. Let $v$ be the point of intersection of $X_j'$ and $X_{j+2}'$. By (1) we may choose coordinates $(x, y)$ in $T_v$ so that the action of $k^*$ on $T_v$ is of the form

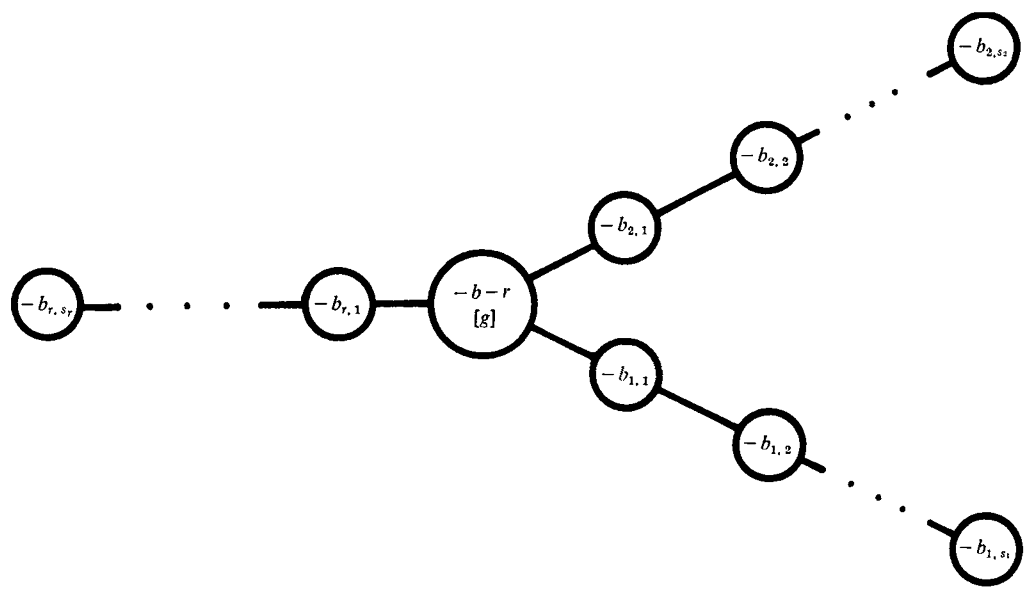
$$t(x, y) = (t^{-p_{j-1}'}x, t^{p_j'}y)$$

and $p_{j-1}'=p_{j-1}$ and $p_j'=(b_j-1)p_{j-1}-p_{j-2}=p_j-p_{j-1}=p_{j+1}$. Now by §3.4, $X_i$ has isotropy group $\mu_\alpha$ where $\alpha=p_{j-1}'+p_j'=p_j$. Again by §3.4 there are coordinates at $X_j \cap X_{j+1}$ and $X_{j+1} \cap X_{j+2}$ so that the action is of the form in (1). By §3.4 (ii) we have that for any $x \in X_{j+1}$, the Seifert invariants of $x$ are $(\alpha, \beta)$ where $p_{j+1}\beta \equiv 1 \pmod{p_j}$. Now $p_{j+1}=p_j-p_{j-1}$ so we get the desired result for (2). To verify (3) first note that $\alpha-\beta=q_i$ since both are between 0 and $\alpha$ and $(\alpha-\beta)p_{i-1} \equiv 1 \pmod{p_i}$ by (2) and $q_i p_{i-1} \equiv 1 \pmod{p_i}$ by A.1. The uniqueness when $b_1, \ldots, b_i \geqslant 2$ is easily verified. Finally, the last equation follows from considering the inverse action and using the above argument.

**3.6.** *Invariants of an elliptic singular point.*

The determination of the invariants of an affine variety $V$ with $k^*$-action ($k=\mathbf{C}$), having an elliptic singular point $v \in V$ was the main object of [16]. The orbit space $V^* = V - \{v\}/k^*$ of the action is a non-singular complete algebraic curve of genus $g$ (i.e. a Riemann surface if $k=\mathbf{C}$). It has a finite number of orbits with non-trivial isotropy and Seifert invariants $(\alpha_i, \beta_i)$, $i=1, \ldots, r$. There is an additional integer invariant $b$, related to the "Chern class" of the Seifert bundle $V-\{v\} \to V^*$. When $k=\mathbf{C}$ the invariants

$\{b; g; (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r)\}$ determine the neighborhood boundary $K$ of $v$ up to orientation preserving $S^1$ equivariant diffeomorphism. The main result of [16] states that the canonical equivariant resolution of the singularity has weighted graph




where the $b_{i,j} \geq 2$ are obtained from the continued fractions

$$\alpha_i/(\alpha_i - \beta_i) = [b_{i,1}, \ldots, b_{i,s_i}], \quad i = 1, \ldots, r.$$

Note that the definition of $b$ here is that of [13] and differs from [16].

Thus in order to find the resolution of $V$ at this fixed point we have to find all orbits with finite isotropy and obtain their Seifert invariants $(\alpha_i, \beta_i)$ from the action of the isotropy group in the tangent space; the genus of the orbit space, and the integer $b$.

This was described in principle for hypersurfaces and complete intersections in [16, 18, 13]. If $V = \operatorname{Spec}(A)$ is an affine variety with $k^*$-action, there is a $k^*$-action on $k^n$ defined by

$$t(z_1, \ldots, z_n) = (t^{q_1} z_1, \ldots, t^{q_n} z_n)$$

and an embedding of $V$ in $k^n$ so that $V$ is invariant under this action. If $v \in V$ is an elliptic fixed point we can choose the embedding so that $q_i > 0$ for all $i$. The ideal of functions $I$ vanishing on $V$ is generated by polynomials which are homogeneous with respect to the grading given by degree $(X_i) = q_i$. Thus we can choose generators $f_i$ for $I$ so that

$$f_i(t^{q_1} z_1, \ldots, t^{q_n} z_n) = t^{d_i} f_i(z_1, \ldots, z_n).$$

We say that a polynomial satisfying the above is *weighted homogeneous of degree* $d_i$ relative to the exponents $q_1, \ldots, q_n$. The rational numbers $w_{ij} = d_i/q_j$ are called the *weights* of $f_i$.

An explicit computation of the Seifert invariants of an isolated elliptic singularity of a hypersurface in $\mathbf{C}^3$ defined by a single weighted homogeneous polynomial was given in [16, 18, 13]. Unfortunately, there is an omission in [16], where it is claimed that if the singularity of a weighted homogeneous polynomial in three variables is isolated, then it is essentially one of six classes:

(i) $z_0^{a_0} + z_1^{a_1} + z_2^{a_2}$,

(ii) $z_0^{a_0} + z_1^{a_1} + z_1 z_2^{a_2}$, $\quad a_1 > 1$

(iii) $z_0^{a_0} + z_1^{a_1} z_2 + z_2^{a_2} z_1$, $\quad a_1 > 1, a_2 > 1$

(iv) $z_0^{a_0} + z_0 z_1^{a_1} + z_1 z_2^{a_2}$, $\quad a_0 > 1$

(v) $z_0^{a_0} z_1 + z_1^{a_1} z_2 + z_0 z_2^{a_2}$,

(vi) $z_0^{a_0} + z_1 z_2$.

The correct description of weighted homogeneous polynomials in general [14] shows that there are two more classes to consider (see also Arnold [1]):

(vii) $z_0^{a_0} + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2}$, $\quad (a_0 - 1)(a_1 b_2 + a_2 b_1)/a_0 a_1 a_2 = 1$

(viii) $z_0^{a_0} z_1 + z_0 z_1^{a_1} + z_0 z_2^{a_2} + z_1^{b_1} z_2^{b_2}$, $\quad (a_0 - 1)(a_1 b_2 + a_2 b_1)/a_2(a_0 a_1 - 1) = 1$.

We could amend the discussion of [16, § 3] to give the Seifert invariants of the corresponding classes. However, we now have a new method of obtaining these Seifert invariants *directly from d and the* $q_i$, making it unnecessary to list all the classes above.

For integers $a_1, \ldots, a_r$ let $\langle a_1, \ldots, a_r \rangle$ denote their least common multiple and $(a_1, \ldots, a_r)$ their greatest common divisor.

Let $f(z_0, z_1, z_2)$ be a weighted homogeneous polynomial whose locus is $V$ in $k^3$. In order for a point $(z_0, z_1, z_2) \in V$ to have non-trivial isotropy, at least one of the $z_i$ must be zero. The number of orbits so that say $z_0 = 0$ is equal to the number of factors of the polynomial $f(0, z_1, z_2)$. The following lemma lets us determine this number easily.

LEMMA. *Suppose* $g(z_1, z_2)$ *is a weighted homogeneous polynomial so that* $g(t^{q_1} z_1, t^{q_2} z_2) = t^r g(z_1, z_2)$. *Then the irreducible factors* $g_i$ *of* $g$ *satisfy* $g_i(t^{q_1} z_1, t^{q_2} z_2) = t^{n_i} g_i(z_1, z_2)$ *and each* $g_i$ *is one of the following forms*

(i) $g_i(z_1, z_2) = c z_1$, $\quad c \neq 0$

(ii) $g_i(z_1, z_2) = c z_2$, $\quad c \neq 0$

(iii) $g_i(z_1, z_2) = c_1 z_1^{p_2} + c_2 z_2^{p_1}$ *where* $c_1 c_2 \neq 0$ *and* $p_i = q_i/(q_1, q_2)$.

*In particular, if* $z_1$ *and* $z_2$ *do not divide* $g$, *then* $g$ *has* $r(q_1, q_2)/q_1 q_2 = r/\langle q_1, q_2 \rangle$ *factors.*

*Proof.* The fact that the $g_i$ are weighted homogeneous is a general fact about factorization in graded rings. If $g_i$ is irreducible then the variety $\{g_i(z_1, z_2)=0\}$ must be the closure of an orbit of the action $t(z_1, z_2)=(t^{q_1}z_1, t^{q_2}z_2)$. This action has the same orbits as the effective action $t(z_1, z_2)=(t^{p_1}z_1, t^{p_2}z_2)$. The closures of the orbits are precisely $\{z_1=0\}$, $\{z_2=0\}$ and $\{z_1^{p_2}+cz_2^{p_1}=0\}$, where $c\neq 0$.

We apply the lemma above to show that in the case of isolated singularities the orbit invariants are determined by $d$ and the $q_i$. Let $w_i=d/q_i=u_i/v_i$ where g.c.d. $(u_i, v_i)=1$ and $u_i, v_i \geq 1$.

PROPOSITION 1. *Suppose $V$ has an isolated singularity at 0. Assume $1\leq v_0\leq v_1\leq v_2$. Then the table below indicates the number of orbits of each type.*

| | $\alpha=(q_1, q_2)$ | $\alpha=(q_0, q_2)$ | $\alpha=(q_0, q_1)$ | $\alpha=q_2$ | $\alpha=q_1$ | $\alpha=q_0$ |
|---|---|---|---|---|---|---|
| $1=v_0=v_1=v_2$ | $d/\langle q_1, q_2\rangle$ | $d/\langle q_0, q_2\rangle$ | $d/\langle q_0, q_1\rangle$ | | | |
| $1=v_0=v_1<v_2$ | $(d-q_1)/\langle q_1, q_2\rangle$ | $(d-q_0)/\langle q_0, q_2\rangle$ | $d/\langle q_0, q_2\rangle$ | 1 | | |
| $1=v_0<v_1\leq v_2$ | $(d-q_1-q_2)/\langle q_1, q_2\rangle$ | $(d-q_0)/\langle q_0, q_2\rangle$ | $(d-q_0)/\langle q_0, q_1\rangle$ | 1 | 1 | |
| $1<v_0\leq v_1\leq v_2$ | $(d-q_1-q_2)/\langle q_1, q_2\rangle$ | $(d-q_0-q_2)/\langle q_0, q_2\rangle$ | $(d-q_0-q_1)/\langle q_0, q_1\rangle$ | 1 | 1 | 1 |

*The blank entries are zero if $q_i$ does not divide $q_j$ for $j\neq i$. If $q_i|q_j$, then $(q_i, q_j)=q_i$ and we list those orbits under the column headed $(q_i, q_j)$.*

*Proof.* The exceptional orbits are in the hyperplanes $z_i=0$. Suppose $v_0=v_1=v_2=1$. The exceptional orbits contained in $\{z_0=0\}$ are given by $\{f(0, z_1, z_2)=0\}$ We write $f(0, z_1, z_2)=z_1^{\varepsilon_1} z_2^{\varepsilon_2}\prod_{i=1}^{s} g_i(z_1, z_2)$ a product of irreducible factors. The factors must be distinct since otherwise the curve $z_0=g_i(z_1, z_2)=0$ would be a singular curve on V. By Lemma 3.6; each $g_i$ has weighted degree $q_1q_2/(q_1, q_2)=\langle q_1, q_2\rangle$ so we have

$$\varepsilon_1 q_1+\varepsilon_2 q_2+s\langle q_1, q_2\rangle = d \tag{1}$$

where $\varepsilon_i=0$ or 1. If $\varepsilon_1=\varepsilon_2=0$ then $s=d/\langle q_1, q_2\rangle$ which is the desired result. If say $\varepsilon_1=1$ then $q_1\equiv 0 \bmod q_2$ so that $q_2=(q_2, q_1)$. One can easily verify that the number of orbits in $\{z_0=0\}$ is $d/q_1=d/\langle q_1, q_2\rangle$.

Suppose $1=v_0=v_1<v_2$. Then $q_0|d$, $q_1|d$ and $q_2|d$. If we write $f(0,z_1,z_2)$ as a product of irreducible factors as above, then the equation (1) gives us $\varepsilon_1\not\equiv 0 \bmod q_2$, hence $\varepsilon_1=1$. Thus there must be an orbit $\{z_0=z_2=0\}$ with isotropy $q_2>(q_1, q_2)$. If $\varepsilon_2=1$ then $q_2\equiv 0 \pmod{q_1}$ so $\langle q_1, q_2\rangle=q_2$. Thus the orbits contained in $\{z_0=0\}$ with isotropy precisely $(q_1, q_2)$ correspond to the factors of $z_2^{\varepsilon_2}\prod_{i=1}^{s} g_i(z_1, z_2)$. If $\varepsilon_2=0$ then $s=(d-q_1)/\langle q_1, q_2\rangle$. If $\varepsilon_2=1$ then $s+1=(d-q_1)/q_2$ which is the desired result.

Suppose $1=v_0<v_1\leqslant v_2$. Then again we have the factorization of $f(0, z_1, z_2)$ which gives us the equation (1). Now

$$\varepsilon_1 q_1 \equiv d \not\equiv 0 \bmod q_2$$

$$\varepsilon_2 q_2 \equiv d \not\equiv 0 \bmod q_1$$

hence $\varepsilon_1=\varepsilon_2=1$. This gives us $s=(d-q_1-q_2)/\langle q_1, q_2\rangle$. The calculation of exceptional orbits contained in $\{z_i=0\}$, $i=1, 2$ is similar to those above.

PROPOSITION 2. *Under the hypotheses of Proposition* 1, *the values of* $\beta$ *are computed as follows.*

*For* $\alpha=(q_1, q_2)$, $q_0\beta\equiv 1 \bmod \alpha$, $0<\beta<\alpha$.

*For an orbit of the form* $\{z_0=z_1=0\}$ *we have* $\alpha=q_2$. *In this case*

(i) *if there is an* $x\in Z^+$ *so that*

$$\frac{1}{w_0}+\frac{x}{w_2}=1,$$

then $q_1\beta\equiv 1 \pmod{\alpha}$

(ii) *if there is a* $y\in Z^+$ *so that*

$$\frac{1}{w_1}+\frac{y}{w_2}=1,$$

*then* $q_0\beta\equiv 1 \pmod{\alpha}$.

*Cyclic permutation of the indices* 0, 1, 2 *permits calculation of all the required* $\beta$.

*Proof.* For $\alpha=(q_1, q_2)$ the action of $\mu_\alpha$ on $k^3$ is $\xi(z_0, z_1, z_2)=(\xi^{q_0}z_0, z_1, z_2)$. This implies that the action of $\mu_\alpha$ on the tangent plane, an affine subspace of $k^3$, must have $\nu\equiv q_0 \pmod{\alpha}$.

For an orbit of the form $\{z_0=z_1=0\}$ we calculate the action of $\mu_\alpha$ on the tangent space $T_v$, with $v=(0, 0, 1)$. The action of $\mu_\alpha$ on $k^3$ is $\xi(z_0, z_1, z_2)=(\xi^{q_0}z_0, \xi^{q_1}z_1, z_2)$. The tangent plane at $v$ is $(\partial f/\partial z_0)(v)z_0+(\partial f/\partial z_1)(v)z_1=0$, since $(\partial f/\partial z_2)(v)=0$. If $(\partial f/\partial z_0)(v)\neq 0$ then the action of $\mu_\alpha$ on the tangent plane must have $\nu\equiv q_1 \pmod{\alpha}$. If $(\partial f/\partial z_1)(v)\neq 0$ then the action of $\mu_\alpha$ on the tangent plane must have $\nu\equiv q_0 \pmod{\alpha}$. Thus $q_1\beta\equiv 1 \pmod{\alpha}$ in the former case and $q_0\beta\equiv 1 \pmod{\alpha}$ in the latter. Now if $(\partial f/\partial z_0)(v)\neq 0$ there must be a monomial of the form $z_0 z_2^x$ in $f$, hence $(1/w_0)+(x/w_2)=1$. If $(\partial f/\partial z_1)(v)\neq 0$ then there is a monomial of the form $z_1 z_2^y$ in $f$ so $(1/w_1)+(y/w_2)=1$. Finally, if both $(1/w_0)+(x/w_2)=1$ and $(1/w_1)+(y/w_2)=1$ then we claim that $q_0\equiv q_1 \pmod{\alpha}$. From the two equations we obtain

$$x=(u_0-v_0)u_2/u_0v_2 \quad \text{hence} \quad u_0\equiv v_0 \bmod v_2,$$

$$y=(u_1-v_1)u_2/u_1v_2 \quad \text{hence} \quad u_1\equiv v_1 \bmod v_2.$$

Also, $u_0$ and $u_1$ divide $u_2$ and $d=\langle u_0, u_1, u_2\rangle$ so $d=u_2$ and $\alpha=q_2=v_2$. By definition $q_0=dv_0/u_0$, $q_1=dv_1/u_1$ so $q_0\equiv q_1 \bmod v_2 \Leftrightarrow u_0v_1\equiv u_1v_0 \bmod v_2$ and the latter follows from the above congruences.

COROLLARY. *For an orbit of the form* $\{z_0=z_1=0\}$ *we have* $\alpha=q_2$. *In this case*

(i) *if*

$$\frac{\partial f}{\partial z_0}(0,0,1)\neq 0 \quad \textit{then } q_1\beta\equiv 1 \bmod \alpha.$$

(ii) *if*

$$\frac{\partial f}{\partial z_0}(0,0,1)\neq 0 \quad \textit{then } q_0\beta\equiv 1 \bmod \alpha.$$

PROPOSITION 3. *The formulas for g and b are given by*

$$2g=\frac{d^2}{q_0q_1q_2}-\frac{d(q_0,q_1)}{q_0q_1}-\frac{d(q_1,q_2)}{q_1q_2}-\frac{d(q_2,q_0)}{q_2q_0}+\frac{(d,q_0)}{q_0}+\frac{(d,q_1)}{q_1}+\frac{(d,q_2)}{q_2}-1$$

$$b=\frac{d}{q_0q_1q_2}-\sum_{j=1}^{r}\frac{\beta_j}{\alpha_j}.$$

*Proof.* A formula for the genus is given in [17, 5.3]. The proof there is valid under the hypotheses of this paper. The formula reduces to the formula above once we verify that for any hypersurface in $k^3$ with an elliptic *isolated* singular point, for each $i$ either $d\equiv 0 \pmod{q_i}$ or $d\equiv q_j \pmod{q_i}$ and $(q_i, q_k)=1$ for some $j\neq k\neq i$, see [17, 5.4]. To verify this for $i=0$ we first note that $(q_i, q_j)\mid d$ for $i\neq j$ because there must be some monomial in $f$ which does not involve $z_k$. Moreover there must be come monomial $M=\alpha z_0^{i_0}z_1^{i_1}z_2^{i_2}$ in $f$ so that $\alpha\neq 0$, $i_1+i_2\leq 1$, since otherwise $z_1=z_2=0$ would be a singular line on the surface. If $i_1+i_2=0$ we get $d\equiv 0 \pmod{q_0}$ and if, say $i_1=1$ we get $i_0q_0+q_1=d$ so $d\equiv q_1 \pmod{q_0}$ and $(q_0, q_2)=1$. By symmetry, we get the result for $i=1, 2$. The formula for $b$ was proven in [16, 3.6.1]. The proof there generalizes to our case.

This completes the computation of the weighted graph of the resolution of $V$ from the integers $d, q_0, q_1, q_2$. In the case that $k=\mathbf{C}$, $\{b; g; (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r)\}$ are the Seifert invariants of the neighborhood boundary $K$, and hence $K$ is determined up to $S^1$-equivariant diffeomorphism by these invariants [13].

3.7. *Non-elliptic singular points.* Suppose $v\in V$ is a normal non-elliptic singular point on a variety $V$ with $k^*$-action. If $\pi\colon \tilde V\to V$ is the canonical equivariant resolution (=minimal resolution) of $V$, then none of the curves in $\pi^{-1}(v)$ can be fixed curves, otherwise $v$ would be elliptic. The graph $\Gamma$ of the resolution must be a connected subgraph of the graph in §2.5, since $\tilde{V}$ has a non-singular equivariant completion [23]. Thus $\Gamma$ must be of the form


Let $A$ (resp. $B$) be the orbit on $\tilde{V}$ (and $V$) which is not contained in $\pi^{-1}(v)$ so that $\bar{A}$ intersects the closure of the orbit corresponding to the first (resp. $r$th) vertex. The Seifert orbit invariants of the orbits, $(\alpha_1, \beta_1)$ for $A$ and $(\alpha_2, \beta_2)$ for $B$ determine the $n_i$ as follows.

With the notation in the appendix, $\alpha_1/\beta_1 = [m_1, m_2, ..., m_s]$, $m_i \geq 2$. Let $m_{s+1}$ be the self intersection of $\bar{A}$. By §3.5, $\alpha_2/\beta_2 = [m_1, m_2, ..., m_s, m_{s+1}, n_1, ..., n_t]$. Now $n_i \geq 2$ and $m_{s+1} \geq 1$, hence by A.9 the $n_i$ and $m_{s+1}$ are uniquely determined.

## 4. Diffeomorphism and algebraic classification

**4.1.** *Minimal models.* A non-singular surface $V$ with $k^*$-action is said to be relatively minimal if there is no curve $X \subset V$ so that $(X \cdot X) = -1$ and $X$ is isomorphic to $P^1$. If such a curve does exist it must be invariant under the action (1.8) and hence there is a contraction of $X$, $\pi: V \to V_0$ which is equivariant.

PROPOSITION. *If $V$ is a non-singular complete relatively minimal surface with $k^*$-action, then $V$ is equivariantly isomorphic to one of the following*

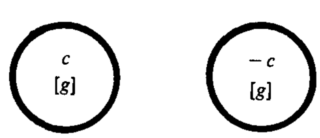
(i) *a $P^1$-bundle of degree $d$ over a complete curve of genus $g$ (the degree is the self-intersection of the fixed curve $F^+$), $g > 0$.*

(ii) *a $P^1$-bundle $E$ of degree $d \neq \pm 1$ over $P^1$ with an action subordinate to the standard $k^* \times k^*$ action $\sigma_1 \times \sigma_2$ on $V$ i.e. $\sigma_1$ acts in the fiber and $\sigma_2$ acts on the base.*

(iii) *$P^2$ with an action subordinate to the standard $k^* \times k^*$ action on $V$*

$$\sigma((t_1, t_2), (z_0 : z_1 : z_2)) = (z_0 : t_1 z_1 : t_2 z_2).$$

*Proof.* If there is no elliptic point on $V$ then §2.5, A.3 and the minimality of $V$ imply that $\Gamma_V$ is of the form




Hence $V$ is a $P^1$ bundle over a complete curve of genus $g$, and if $g = 0$ then $c$ and $-c \neq -1$.

If there is an elliptic point on $V$, then the non-singularity of $V$ implies $g=0$. By a theorem of Nagata [6] a relatively minimal rational surface is isomorphic to $P^2$ or a $P^1$ bundle $E_d$ of degree $d \neq \pm 1$ over $P^1$. It remains to show that any $k^*$-action on $P^2$ or $E_d$ can be extended to the standard $k^* \times k^*$-action on $P^2$ (resp. $E_d$). Now a $k^*$-action on $V$ is equivalent to an injective algebraic group homomorphism $k^* \to \mathrm{Aut}(V)$. In our case $\mathrm{Aut}(P^2) = PGL(3, k)$ and $\mathrm{Aut}(E_d) = PGL(2) \times k^*$. Each of these (linear algebraic groups) has a maximal torus of dimension 2, i.e. every subgroup isomorphic to $k^*$ is contained in a subgroup isomorphic to $k^* \times k^*$ and any two subgroups isomorphic to $k^* \times k^*$ are conjugate [3, IV. 11.3]. This implies the desired result.

4.2. *Definition*. Call a $k^*$-action $\sigma$ on $V$ *essentially unique* if the only other $k^*$-action on $V$ is $\tau(t, v) = \sigma(t^{-1}, v)$.

PROPOSITION. *Suppose $V$ is a non-singular surface with $k^*$-action $\sigma_1$. Then the following are equivalent.*

(i) *$g=0$ and any fixed curve intersects at most two invariant curves which have negative self-intersection.*

(ii) *the action $\sigma_1$ extends to an effective $k^* \times k^*$-action $\sigma_1 \times \sigma_2$.*

(iii) *the action $\sigma_1$ is not essentially unique.*

*Proof*. The group of automorphisms of a projective variety $\mathrm{Aut}(V)$, is a reduced algebraic group [11], and a $G$-action on $V$ is equivalent to a homomorphism $\sigma: G \to \mathrm{Aut}(V)$. Hence it is sufficient to show that if (i) holds the maximal torus of $\mathrm{Aut}(V)$ is $k^* \times k^*$, and if (i) does not hold then $\mathrm{Aut}_0(V) = k^*$. Here $\mathrm{Aut}_0(V)$ denotes the component of the identity in $\mathrm{Aut}(V)$.

We may assume $V \neq P^2$. Then there is a birational equivariant morphism $f: V \to V_0$ so that $V_0$ is a $P^1$ bundle over a curve $X$ (4.1). By an argument analogous to that in § 1.9 we can show that any element of $\mathrm{Aut}_0(V)$ leaves a curve with negative self-intersection invariant. Hence there is an injective homomorphism $f_*: \mathrm{Aut}_0(V) \to \mathrm{Aut}_0(V_0)$. Moreover $\mathrm{Aut}(V_0) = \mathrm{Aut}(X) \times k^*$. Now if $g > 0$, then $\mathrm{Aut}(X)$ is finite and therefore $\mathrm{Aut}_0(V) = \mathrm{Aut}_0(V_0) = k^*$. If $g=0$ then any element of $\mathrm{Aut}_0(V_0)$ which is in the image of $\mathrm{Aut}_0(V)$, leaves the points of $X$ which lie on curves with negative self-intersection, fixed. Thus $\mathrm{Aut}_0(V) = k^*$ if there are more than two such points, and $\mathrm{Aut}_0(V)$ has a maximal torus isomorphic to $k^* \times k^*$ otherwise.

4.3. THEOREM. *A complex algebraic surface $V$ with $\mathbf{C}^*$ action is diffeomorphic to one, and only one, of the following*

(i) $M_{g,k} = (R_g \times S^2) \# k\overline{CP}^2 \quad g \geq 0,\ k \geq 0,$

(ii) $N_g$, *the non-trivial* $S^2$ *bundle over* $R_g$ $g \geq 0$,

(iii) $CP^2$,

*where* $R_g$ *is a compact Riemann surface of genus* $g$ *and* $\overline{CP}^2$ *denotes* $CP^2$ *with the opposite orientation.*

*Proof.* Blowing up a point on a non-singular complex surface is equivalent to taking connected sum with $\overline{CP}^2$. Hence the proposition implies that $V$ is diffeomorphic to $W \# k\overline{CP}^2$ where $W = CP^2$ or a $CP^1$ bundle $E_d$ of degree $d$ over a Riemann surface $R_g$. Now $E_d \# \overline{CP}^2$ is diffeomorphic to $E_{d+1} \# \overline{CP}^2$. In fact, one can construct an equivariant diffeomorphism by blowing up a point on $F^+$ in $E_d$ and a point on $F^-$ in $E_{d+1}$. Both give us a surface with graph


We can conclude that if $W = E_d$ and $k > 0$ then $V$ is diffeomorphic to $M_{g,k}$. Now suppose $V = E_d$. There are two $S^2$ bundles over a compact Riemann surface, the trivial one and a non-trivial one. Thus $E_d$ is diffeomorphic to $E_{\bar{d}}$ where $0 \leq \bar{d} \leq 1$, is the residue of $d$ modulo 2. $E_0$ is not diffeomorphic to $E_1$ since the former has an even quadratic form (intersection form) and the latter does not. Finally suppose $V = CP^2 \# k - \overline{CP}^2$. It is well known that $CP^2 \# \overline{CP}^2$ is diffeomorphic to the $CP^1$ bundle of degree 1 over $CP^1$.

Thus $V = M_{0,k}$, if $k \geq 2$ and $V = N_0$ if $k = 1$. The manifolds $M_{g,k}$, $N_g$ and $CP^2$ can be shown to be distinct by comparing the ranks of their homology groups, $H_1$ and $H_2$, and the parity of the quadratic form given by cup product.

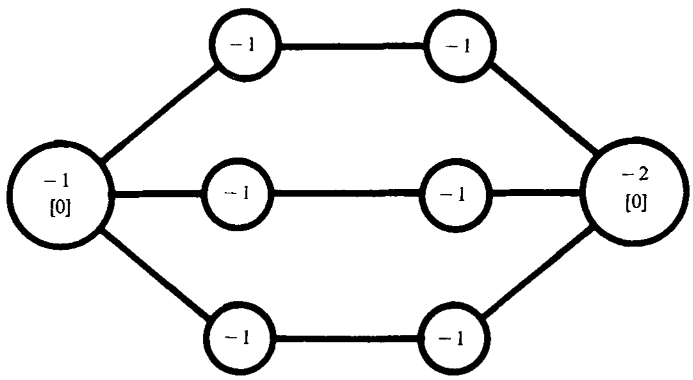
**4.4.** *$S^1$-actions on 4-manifolds.* It is an open question whether all $S^1$-actions on $CP^2$ are orthogonal i.e. of the form

$$t(z_0 : z_1 : z_2) = (t^{q_0} z_0 : t^{q_1} z_1 : t^{q_2} z_2)$$

for $t \in S^1$. Orlik and Raymond [15] classified $T^2 = S^1 \times S^1$ actions on 4-manifolds and have proven that every smooth $T^2$ action on $CP^2$ is orthogonal. Thus we can rephrase the question to ask whether every smooth effective $S^1$ action extends to a smooth effective $T^2$ action. We have the following related non-extension result.

THEOREM. *If* $k \geq 3$ *there is a smooth effective* $S^1$ *action on* $CP^2 \# k\overline{CP}^2$ *which does not extend to a smooth effective* $S^1 \times S^1$ *action.*

*Proof.* It follows from [15, § 5] that if the $S^1$ action extends to a $T^2$ action the graph $\Gamma_V$ must have $\leqslant 2$ arms. The surface $V_1$ with graph




is diffeomorphic to $\mathbf{C}P^2 \# 4\mathbf{C}P^2$ and has 3 arms, thus the $\mathbf{C}^*$-action induces an $S^1$-action $\sigma_1$ which does not extend to a $T^2$ action. If we collapse the curve corresponding to the left hand vertex we get a surface $V_2$ diffeomorphic to $\mathbf{C}P^2 \# 3\overline{\mathbf{C}P}^2$. If the $S^1$ action $\sigma_1$ extends to $\sigma_1 \times \sigma_2$, $\sigma_2$ must leave the fixed points of $\sigma_1$ fixed and hence $\sigma_1 \times \sigma_2$ extends to $V_1$. This is a contradiction, hence the action does not extend. We can get non-extendable actions on $\mathbf{C}P^2 \# k\overline{\mathbf{C}P}^2$, $k \geqslant 4$, by blowing up fixed points on $V_1$.

We do not know if such actions can be obtained for $k \leqslant 2$.

**4.5.** We can now list invariants which give an equivariant algebraic classification of normal surfaces $V$ with $k^*$-action. Let $\pi: \tilde{V} \to V$ be the canonical equivariant resolution of $V$ and $\Gamma_{\tilde{V}}$, the graph of $\tilde{V}$. We would like to describe the topology of $V$ using $\Gamma_{\tilde{V}}$ and some additional data. The map $\pi$ induces an equivalence relation in $\tilde{V}$. There are two kinds of identifications; certain curves are identified to points and certain curves are identified to each other. If a collection of curves is mapped to a single point by $\pi$ we enclose the corresponding vertices of $\Gamma$ by a solid line. If curves $X_1, \ldots, X_r$ on $\tilde{V}$ are all mapped onto the same curve $Y$ in $V$ and $d_i$ is the degree of $\pi | X_i$ then we enclose the corresponding vertices with a dotted line and indicate the degree $d_i$ above the vertex. This diagram is denoted by $\Gamma_V$. See § 5 for examples.

Suppose $V$ is a complete non-singular surface with $k^*$-action and no elliptic fixed points, $\Gamma_V$ is the graph of $V$, and the arms of $\Gamma_V$ are numbered $1, 2, \ldots, r$. Define the contraction of $V$, $f$: $V \to \operatorname{cont}(V)$, to be the contraction of that orbit $X$ which lies on the $r$th arm, has a self-intersection $-1$ and is the orbit closest to $F^+$ having the first two properties (i.e. the vertex corresponding to $X$ is closest to $f^+$). If we apply the contraction operator repeatedly we finally arrive at a surface $B_V$ whose graph has no arms, i.e. $B_V$ is a $P^1$ bundle over $F^+$. Let

$$\varphi: V \to B_V$$

be the contraction map. We say $(V, B_V, \varphi)$ is the *canonical relatively minimal model* of $V$. One can easily see that $(V, B_V, \varphi)$ is independent of the numbering of the arms of $\Gamma$. Let $p: B_V \to F^+$ be the structure morphism of the bundle. If $V$ is an arbitrary complete normal surface we define $B_V = B_{\tilde{V}}$.

Let $x_1, ..., x_r$ be the points of $F^+$ which are in the closures of non-ordinary orbits. The *algebraic invariants* of $(V, \sigma)$ are $\{F^+; x_1, ..., x_r; \Gamma_V; (B_V, p)\}$ together with a numbering of the arms of $\Gamma_V$ so that $x_i$ is in the closure of an orbit on the $i$th arm. Two sets of invariants $\{F_i^+; x_1^{(i)}, ..., x_{r_i}^{(i)}; \Gamma_i; (B_{V_i}, p_i)\}$ $i = 1, 2$ are *equivalent* if $r_1 = r_2$ and there is an isomorphism $\sigma: F_1^+ \to F_2^+$ and a permutation $\pi$ so that $\sigma(x_i^{(1)}) = x_{\pi(i)}^{(2)}$, $\sigma^*(B_{V_2}) = B_{V_1}$ and $\Gamma_2 = \Gamma_1$ with its arms reordered by $\pi$.

**4.6. Theorem.** *Suppose $V_1$ and $V_2$ are complete normal surfaces with $k^*$-action. Then $V_1$ and $V_2$ are equivariantly isomorphic if and only if their respective invariants*

$$\{F^+; x_1, ..., x_r; \Gamma_V; (B_V, p)\}$$

*are equivalent.*

*Proof.* Since $V_1$ and $V_2$ are normal, $V_i$ is determined by $\tilde{V}_i$ and that part of the graph $\Gamma_i$ which indicates which curves are contracted. Thus we may assume $V_i = \tilde{V}_i$. The equivalence of invariants is certainly a necessary condition. Finally the sufficiency is easily seen by induction on the number of vertices of $\Gamma$ and the uniqueness of blowing up (down).

## 5. Examples and remarks

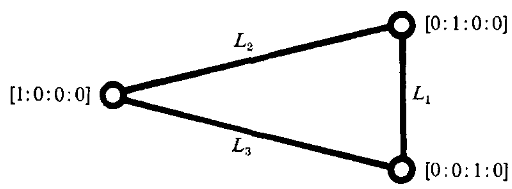
**5.1.** Consider the homogeneous polynomial

$$f(z) = z_0 z_1 z_2 + z_3^3$$

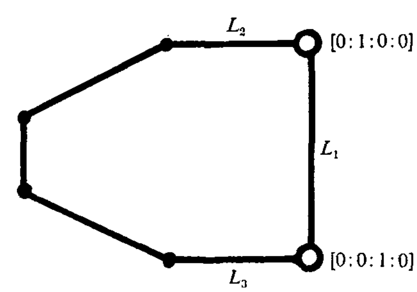
which defines a hypersurface $V$ in $P^3$, invariant under the $k^*$-action

$$t[z_0 : z_1 : z_2 : z_3] = [z_0 : t^3 z_1 : t^3 z_2 : t^2 z_3].$$

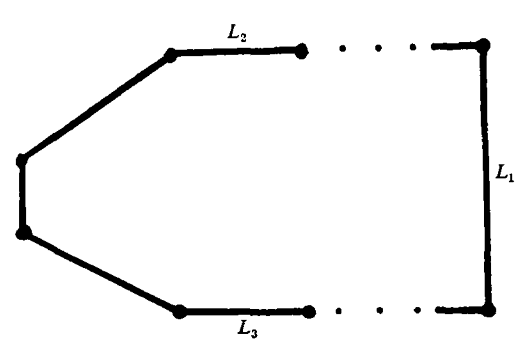
The partials of $f$ show that $V$ is singular at $[0:0:1:0]$, $[0:1:0:0]$ and $[1:0:0:0]$. The closures of the exceptional orbits are $L_1 = \{z_0 = z_3 = 0\}$ with isotropy $k^*$, $L_2 = \{z_2 = z_3 = 0\}$ and $L_3 = \{z_1 = z_3 = 0\}$ with isotropy $\mu_3$. In the diagram below a line indicates a curve; a non-singular point of intersection is denoted by a dot, a singular one by a small circle. We have therefore



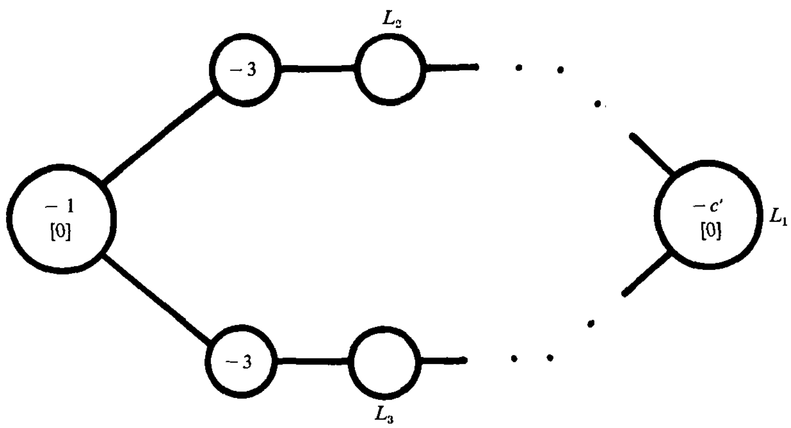

*The elliptic singularity* [1: 0: 0: 0]. The affine piece of $V$ where $z_0 \neq 0$ is the surface defined by the weighted homogeneous polynomial $z_1 z_2 + z_3^3 = 0$ with $q_1 = q_2 = 3$, $q_3 = 2$, $d = 6$. We see from (3.6.2) that the Seifert orbit invariants of the orbits $L_2$ and $L_3$ are (3,2). Now by (3.6.3) $g = 0$ and $b = -1$. Thus if we resolve the singular point at [1: 0: 0: 0] we get



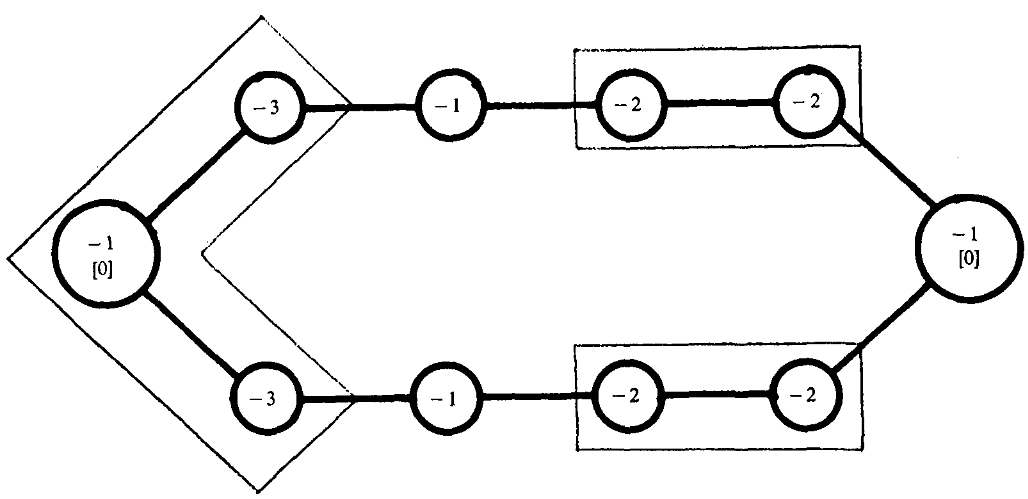

The points [0: 1: 0: 0] and [0: 0: 1: 0] are parabolic singular points. The minimal resolution of these two points gives us the canonical equivariant resolution $\tilde{V}$. The corresponding diagram is of the form below




The dual graph (as defined in 2.4) with weights is the following




Now by §3.6 the missing vertices are given by the continued fraction expansion of $\alpha/\beta = 3/2$. Finally, the self-intersection of $L_2$ and $L_3$ must by $-1$ by A.3 and hence $c' = 1$ by §2.5, (iv). Thus the graph associated to $V$ is:




Note that we have made no assumptions about the characteristic of $k$. We leave it an exercise to find the canonical equivariant resolution of this singularity under the action $t[z_0 : z_1 : z_2 : z_3] = [tz_0 : t^{-1}z_1 : z_2 : z_3]$.

**5.2.** Let us now turn to an example with singular lines. Consider the hypersurface in $P^3$ defined by the zeros of the homogeneous polynomial

$$f(z) = z_0^4 z_1 z_3^3 + z_1^3 z_2^2 z_3^3 + z_0^7 z_2$$

with action

$$\sigma(t; [z_0 : z_1 : z_2 : z_3]) = [t^2 z_0 : t^5 z_1 : t^{-1} z_2 : z_3].$$

Computing the partials of $f$ shows that:

$L_1 = \{z_0 = z_3 = 0\}$ is a singular line with isotropy $\mu_6$,

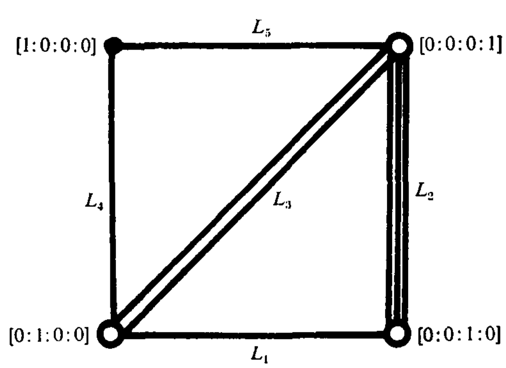
$L_2 = \{z_0 = z_1 = 0\}$ is a singular line without isotropy,

$L_3 = \{z_0 = z_2 = 0\}$ is a singular line with isotropy $\mu_5$,

$L_4 = \{z_2 = z_3 = 0\}$ is a non-singular line with isotropy $\mu_3$,

$L_5 = \{z_1 = z_2 = 0\}$ is a non-singular line with isotropy $\mu_2$.

As above we draw a diagram with line segments representing orbits with isotropy. A double (resp. triple) line segment represents a singular orbit with two (resp. three) sheets passing through it i.e. two (resp. three) orbits on $\tilde{V}$ are identified to one on $V$.




In the following we shall assume that $k = \mathbf{C}$ so that we can use slices. For details of the following analysis consult [17]. The same result is valid for arbitrary $k$ by a slight additional analysis.

*The elliptic singularity at* [0: 0: 1: 0]

At $z_2 = 1$ the equation is $z_0^4 z_1 z_3^3 + z_1^3 z_3^3 + z_0^7 = 0$ with $q_0 = 3$, $q_1 = 6$, $q_3 = 1$ and $d = 21$. Call $V'$ the corresponding affine hypersurface.

$L_1$ is singular with slice at $z_1 = 1$:

$$z_0^4 z_3^3 + z_3^3 + z_0^7 = 0$$

which is locally irreducible, hence $L_1$ is covered by one orbit. With $\xi = \exp(2\pi/6)$ the slice action on $V'$ is $\xi(z_0, z_3) = (\xi^3 z_0, \xi z_3)$. If $t$ is a local coordinate for the slice on $\tilde{V}$ then $z_0 = t^3$, $z_3 = t^7$ so $\sigma(\xi, t) = \xi t$ is the slice action on $\tilde{V}$. So $1 \cdot \beta \equiv 1(6)$. Thus $L_1$ gives rise to one exceptional orbit of type $(6, 1)$.

$L_2$ is singular with slice at $z_3=1$: $z_0^4z_1+z_1^3+z_0^7=0$. One can see that this curve has 3 branches through 0 (for example by blowing up once), thus $L_2$ is covered by three principal orbits.

From (3.6.3) we get $b=21/3\cdot 6-1/6=1$ and we use [17, § 5] to compute the genus of the resolution of the curve $X'=(V'-O)/\mathbf{C}^*$.

Its arithmetic genus is given by the formula

$$p_a = 1+\frac{d^2}{2q_0q_1q_2}-\frac{d(q_0,q_1)}{2q_0q_1}-\frac{d(q_1,q_2)}{2q_1q_2}-\frac{d(q_2,q_0)}{2q_2q_0}$$

$$-\varrho(q_0;q_1,q_2;d)-\varrho(q_1;q_2,q_0;d)-\varrho(q_2;q_0,q_1;d),$$

where by definition

$$\varrho(k;n_1,n_2;d)=\frac{1}{k}\sum_{j\in I}\frac{1-\xi^{jd}}{(1-\xi^{jn_1})(1-\xi^{jn_2})}$$

for $\xi=\exp(2\pi i/k)$ and

$$I=\{j\,|\,0<j<k,\ jn_1\not\equiv 0 \pmod k),\ jn_2\not\equiv 0 \pmod k)\}.$$

We compute directly that

$$\varrho(3;6,1;21)=\varrho(1;6,3;21)=0 \quad\text{and}\quad \varrho(6;3,1;21)=1/4,$$

so from the formula above, $p_a=6$. Finally,

$$g=p_a-\sum_{x\in X'}\delta_x$$

where $\delta_x$ is an invariant of the point $x$ defined in [12, § 10] (see also [17]).

The image of $L_1$, say $x_1\in X$ has $\delta_{x_1}=0$, since $X$ is non-singular at $x_1$. The image of $L_2$, say $x_2\in X$ has $\delta_{x_2}=6$ by [12, p. 93]. Thus $g=0$ and we have that the neighborhood boundary at [0: 0: 1: 0] has Seifert invariants

$$K[0:0:1:0]=\{1;0;(6,1)\sim 6;(1,0)\approx(1,0)\approx(1,0)\sim 1\}.$$

See Orlik-Wagreich [17, 3.5] for this notation.

*The elliptic singularity at* [0: 1: 0: 0]

At $z_1=1$ we have $z_0^4z_3^3+z_2^2z_3^3+z_0^7z_2=0$ with $q_0=-3$, $q_2=-6$, $q_3=-5$ and $d=-27$. We impose the inverse action by letting

$$q_0=3,\quad q_2=6,\quad q_3=5,\quad d=27.$$

$L_1$ is singular with slice at $z_2=1$: $z_0^4z_3^3+z_3^3+z_0^7=0$ covered by one orbit of type (6, 5).

$L_3$ is singular with slice at $z_3=1$: $z_0^4+z_2^2+z_0^7z_2=0$. This has two non-singular branches each of which is tangent to the line $\{z_2=0\}$. The action in the tangent space is $\xi(z_0, z_2)=(\xi^3z_0, \xi z_2)$ so $3\beta\equiv 1$ (5), $\beta=2$, and $L_3$ is covered by two orbits of type (5, 2).

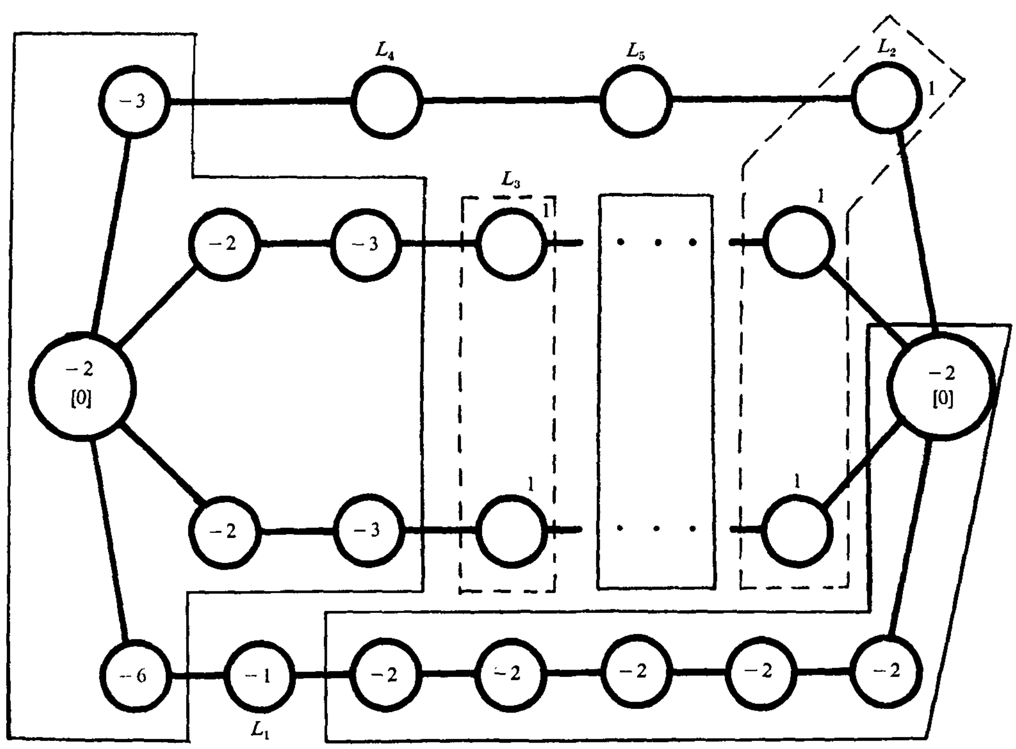
$L_4$ is non-singular. The action in the tangent space at $[1: 1: 0: 0]\in L_4$ is $\xi(z_2, z_3)=(z_2, \xi^2z_3)$. Thus $2\beta\equiv 1$ (3) and $\beta=2$.

From (3.6.3) we get $b=27/3\cdot 6\cdot 5-5/6-2/5-2/5-2/3=-2$. We know that $g=0$ since $g$ is zero at the other elliptic singular point. Hence

$$K[0:1:0:0]=\{-2;0;\ (6,5)\sim 6,\ (5,2)\approx(5,2)\sim 5,\ (3,2)\}.$$

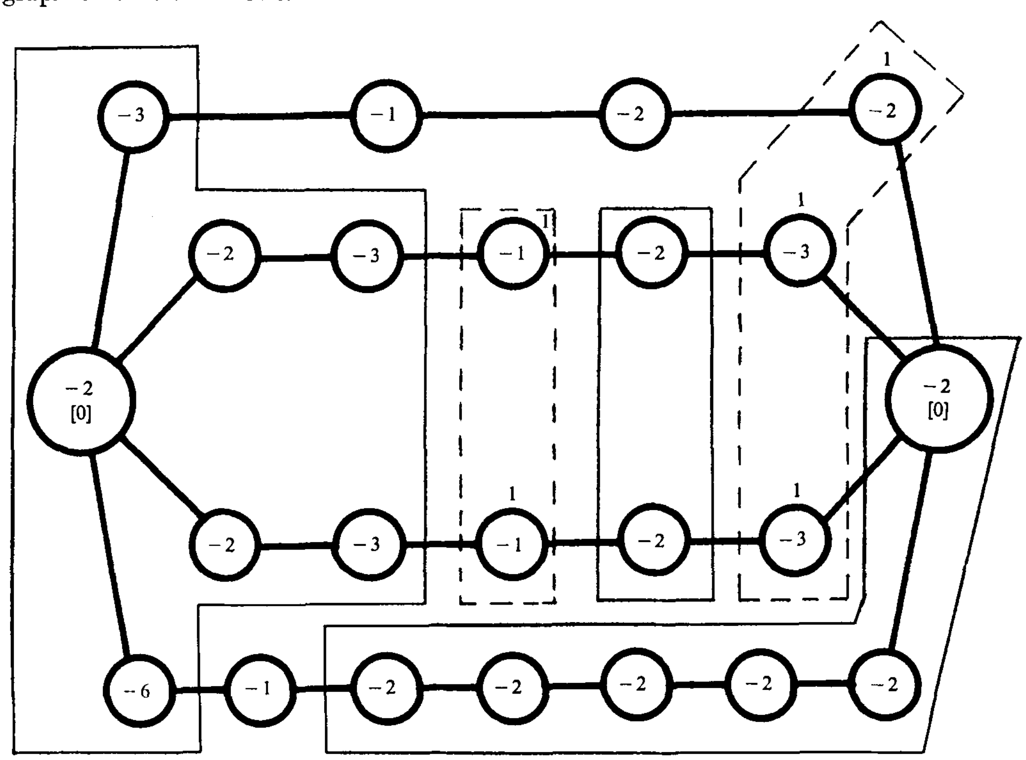
Comparing orientations in $W$ shows that the orientation given here for $K[0: 1: 0: 0]$ is correct.

We now know that the (dual) graph $\Gamma$ of $V$ (as defined in 2.4) is of the form below:




Recall here that the integer above a vertex enclosed by dotted lines denotes the degree of the map $\pi$: $\tilde{V}\to V$ on the corresponding curve, see §4.5.

Applying §3.5 we see that the fact that $L_5$ has isotropy $\mu_2$ implies that $L_4$ has self intersection $-1$ and the branch of $L_2$ intersecting $L_5$ has self intersection $-2$. But then $L_5$ must have self intersection $-2$. In order to calculate the missing part of the middle two arms we note that the orbit invariants of $L_3$ are $(5,2)$. The vertices are determined by the continued fraction expansion of $5/2=[3,2]$ if we know that there are no vertices with self intersection $-1$. Now the vertices inside the curve come from a minimal resolution, so they do not have self intersection $-1$. On the other hand, if one of the branches of $L_2$ had self intersection $-1$ the relation §2.5 (iv) would not hold. Thus, the graph of $V$ is as follows.



**5.3.** Let $V$ be a complex surface with $\mathbf{C}^*$ action and with only isolated singular points. Let $V_0$ denote the compact 4-manifold with boundary obtained by removing an open $S^1$-invariant tubular neighborhood of the fixed point set. Then $V_0$ may be viewed as a fixed-point-free $S^1$-cobordism between its various boundary components. This relates our paper to the results of Ossa [19]. As an example, consider the hypersurface $V$ in $\mathbf{C}P^3$ defined by the zeros of the homogeneous polynomial

$$f(z)=z_0z_1^2+z_2^3+z_3^3$$

with $\mathbf{C}^*$ action $t[z_0: z_1: z_2: z_3]=[t^{-2}z_0: tz_1: z_2: z_3]$. It has five fixed points. $[1: 0: 0: 0]$, $[0: 1: 0: 0]$ and $z_0=z_1=0$, $z_2^3+z_3^3=0$. The first one is singular.

We leave it to the reader to show that

$$K[1\colon 0\colon 0\colon 0] = \{-1; 0; (2,1), (2,1), (2,1)\}$$
$$K[0\colon 1\colon 0\colon 0] = \{1; 0\}$$
$$K[0\colon 0\colon z_2^3 + z_3^3 = 0] = \{0; 0; (2,1)\}$$

so $V_0$, the complement of open $S^1$-invariant tubular neighborhoods of the fixed points, is a fixed point free $S^1$-cobordism between $K[1\colon 0\colon 0\colon 0]$ and its decomposition as a linear combination of the generators in the cobordism group $O_3(\infty)$, see [19].

## Appendix

We recall a few elementary facts about continued fractions, see e.g. [20]. Let $a_1, \ldots, a_n$ be positive integers and let $[a_1, \ldots, a_n]$ denote the continued fraction

$$[a_1, \ldots, a_n] = a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \cfrac{1}{\ddots - \cfrac{1}{a_n}}}}$$

provided none of the denominators equals zero. Define

$$p_{-1}=0, \quad p_0=1, \quad p_1=a_1, \quad p_i=a_i p_{i-1}-p_{i-2}, \quad i\geq 2$$
$$q_0=0, \quad q_1=1, \quad q_i=a_i q_{i-1}-q_{i-2}, \quad i\geq 2.$$

*Definition.* Call $\{a_1, \ldots, a_n\}$ admissible if $p_i>0$ for $i=0, \ldots, n-1$.

Consider the symmetric matrix

$$\begin{pmatrix} a_1 & -1 & & & \\ -1 & a_2 & -1 & & \\ & -1 & a_3 & & \\ & & & \ddots & -1 \\ & & & -1 & a_n \end{pmatrix}$$

If we diagonalize the matrix over the rational numbers starting in the upper left hand corner we get a matrix with diagonal elements $[a_1]$, $[a_2, a_1]$, $[a_3, a_2, a_1]$ ....

$$\begin{bmatrix} a_1 & & & \\ & a_2-\frac{1}{a_1} & & \\ & & a_3-\cfrac{1}{a_2-\frac{1}{a_1}} & \\ & & & \ddots \end{bmatrix}$$

One can easily prove by induction that the $i$th entry is $p_i/p_{i-1}$ and $\{a_1, ..., a_n\}$ is admissible if and only if the matrix is positive semi-definite of rank $\geqslant n-1$. From this one can see that if $\{a_1, ..., a_n\}$ is admissible then $\{a_n, ..., a_1\}$ is admissible and $\{a_i, a_{i+1}, ..., a_j\}$ is admissible for any $1 \leqslant i \leqslant j \leqslant n$. If we diagonalize the matrix starting at the lower right hand corner we can see that if $\{a_1, ..., a_n\}$ is admissible then the denominators in the expression

$$a_1-\cfrac{1}{a_2-\cfrac{1}{\ddots-\cfrac{1}{a_n}}}$$

are not zero. Hence $[a_1, ..., a_n]$ makes sense if $\{a_1, ..., a_n\}$ is admissible.

Henceforth we will always assume that $\{a_1, ..., a_n\}$ is *admissible*.

It is shown in [20], that if $p_i$ and $q_i$ are defined as above then $[a_1, ..., a_i] = p_i/q_i$.

LEMMA A1. $p_iq_{i-1}-p_{i-1}q_i=-1$ *for* $0<i\leqslant n$.

*Proof*. Use induction. By definition $p_1q_0-p_0q_1=-1$. Assuming the statement for $i=r$ we have $p_{r+1}q_r-p_rq_{r+1}=(a_{r+1}p_r-p_{r-1})q_r-p_r(a_{r+1}q_r-q_{r-1})=p_rq_{r-1}-p_{r-1}q_r=-1$.

From this we see that $(p_i, q_i)=1$, $(p_i, p_{i-1})=1$ and $(q_i, q_{i-1})=1$. In particular the fraction $p_i/q_i$ is in reduced form for all $i$. Also, $p_{i-1}q_i\equiv 1(p_i)$.

LEMMA A2. *If for each* $i$, $a_i\geqslant 2$, *then* $p_i>p_{i-1}$, $q_i>q_{i-1}$ *and* $p_i>q_i$, *so in particular* $p_i/q_i>0$ *for all* $i$.

*Proof*. We shall only prove the first assertion using induction: $p_1=a_1>p_0=1$. Assuming $p_i>p_{i-1}$ we have $p_{i+1}=a_{i+1}p_i-p_{i-1}>(a_{i+1}-1)p_i\geqslant p_i$.

COROLLARY A3. *If* $[a_1, ..., a_n]=0$, *then for at least one* $i$, $a_i=1$.

LEMMA A4. *Let* $\{a_1, ..., a_n\}$ *be admissible and* $p_i$, $1\leqslant i\leqslant n$, *as defined above. Then the continued fraction* $[a_i, a_{i-1}, ..., a_1]=p_i/p_{i-1}$.

*Proof.* From $p_i = a_i p_{i-1} - p_{i-2}$ we have $p_i/p_{i-1} = a_i - (1/(p_{i-1}/p_{i-2}))$ and repeated application gives the desired result.

LEMMA A5. *Let* $[a_1, ..., a_n] = 0$. *Consider the two continued fractions* $[a_{r-1}, ..., a_1] = u/v$ *and* $[a_{r+1}, ..., a_n] = u'/v'$. *Then we have that* $u = u'$ *and* $v + v' = a_r u$.

*Proof.* Since $[a_1, ..., a_n] = 0$, also $[a_n, ..., a_1] = 0$. By a repeated application of the following calculation

$$a_n = \frac{1}{p_{n-1}/p_{n-2}}$$

$$\frac{a_{n-1}p_{n-2} - p_{n-3}}{p_{n-2}} = \frac{1}{a_n}$$

$$a_{n-1} - \frac{1}{a_n} = \frac{1}{p_{n-2}/p_{n-3}}$$

we obtain

$$u'/v' = [a_{r+1}, \dots, a_n] = \frac{1}{p_r/p_{r-1}}$$

$$u'/v' = \frac{1}{a_r - \dfrac{1}{p_{r-1}/p_{r-2}}}.$$

Using Lemma 4 we have $p_{r-1}/p_{r-2} = u/v$ so

$$(u'/v')\left(a_r - \frac{v}{u}\right) = 1$$

and hence

$$a_r = \frac{v}{u} + \frac{v'}{u'}.$$

Since both $(u, v) = 1$ and $(u', v') = 1$, we conclude that $u = u'$ and $v + v' = a_r u$ as desired.

LEMMA A6. *Let* $[a_1, ..., a_{r-1}, 1, a_{r+1}, ..., a_n]$ *be a continued fraction with* $p_i$ *and* $q_i$ *defined above. Let* $[a_1, ..., a_{r-1} - 1, a_{r+1} - 1, a_{r+2}, ..., a_n]$ *have corresponding values* $p'_i$ *and* $q'_i$. *Then*

$$p'_i = p_i,\ q'_i = q_i \qquad \text{for} \quad i \leqslant r-2$$

$$p'_i = p_{i+1},\ q'_i = q_{i+1} \quad \text{for} \quad i \geqslant r-1.$$

*Proof.* The first half of the assertion is obvious. Now $p'_{r-1} = (a_{r-1}-1)p'_{r-2} - p'_{r-3} = (a_{r-1}-1)p_{r-2} - p_{r-3} = 1(a_{r-1}p_{r-2} - p_{r-3}) - p_{r-2} = a_r p_{r-1} - p_{r-2} = p_r$.

The argument is the same for $q_{r-1}$.

$$\begin{aligned} p'_r &= (a_{r+1}-1)p'_{r-1} - p'_{r-2} = (a_{r+1}-1)p_r - p_{r-2} \\ &= a_{r+1}p_r - p_r - p_{r-2} = a_{r+1}p_r - (1p_{r-1} - p_{r-2}) - p_{r-2} \\ &= a_{r+1}p_r - p_{r-1} = p_{r+1} \end{aligned}$$

Again, the same holds for $q'_r$ and clearly, for $i > r$ the conclusion is trivially true.

*Definition.* Given $\{a_1, ..., a_{r-1}, a_r, a_{r+1}, ..., a_r\}$ and $a_r = 1$ we say that we have "blown down $a_r$" to obtain $\{a_1, ..., a_{r-2}, a_{r-1}-1, a_{r+1}-1, a_{r+2}, ..., a_n\}$. The reverse process is called "blowing up".

*Definition.* $\{a_1, ..., a_n\}$ and $\{b_1, ..., b_m\}$ are called *equivalent* if a series of blowing up and blowing down of $\{a_1, ..., a_n\}$ makes it equal to $\{b_1, ..., b_m\}$. This is clearly an equivalence relation.

LEMMA A7. *$\{a_1, ..., a_n\}$ and $\{b_1, ..., b_m\}$ are equivalent if and only if $[a_1, ...., a_n] = [b_1, ..., b_m]$.*

*Proof.* Repeated application of A3 blows both sequences down to $[1, 1]$ (if $[a_1, ..., a_n] = 0$) or a sequence with no ones (otherwise). But every positive rational number has a unique representation in the form $[c_1, ..., c_r]$, $c_i \geqslant 2$.

LEMMA A8. *Let $[a_1, ..., a_s] = 0$. Then*

$$[a_1, ..., a_{s-1}] = 1/q_{s-1}$$

*Moreover $\{a_1, ..., a_{s-1}, a_s+1, 1\}$ is also admissible, and*

$$[a_1, ..., a_{s-1}, a_s+1] = 1/(q_{s-1}+1).$$

*Proof.* The first part is obvious. Let

$$[a_1, ..., a_{s-1}, a_s+1] = u/v.$$

Now $q_s = 1$ by A.1 hence we have

$$\begin{aligned} u &= (a_s+1)p_{s-1} - p_{s-2} = a_s p_{s-1} - p_{s-2} + p_{s-1} = p_s + p_{s-1} = p_{s-1} = 1 \\ v &= (a_s+1)q_{s-1} - q_{s-2} = a_s q_{s-1} - q_{s-2} + q_{s-1} = q_s + q_{s-1} = 1 + q_{s-1} \end{aligned}$$

LEMMA A9. *Suppose* $\alpha > \beta \geqslant 1$ *are integers and* $\alpha/\beta = [m_1, ..., m_s, m_{s+1}, n_1, ..., n_t] = [m_1, ..., m_s, m_{s+1}, n'_1, ..., n'_t]$, where $m_i \geqslant 2$, *for* $i = 1, ..., s$, $n_i$ *and* $n_i \geqslant 2$ *for all* $i$, $m_{s+1} \geqslant 1$, $m'_{s+1} \geqslant 1$ *and both sequences are admissible. Then* $m_{s+1} = m'_{s+1}$, *and* $n'_i = n_i$, *for all* $i$.

*Proof.* It follows easily from the definition of continued fractions that we may assume $s = 0$. Proceeding by induction on $t$, if $t = 0$ the assertion is trivial. If $t > 0$ and $m_{s+1}$ and $m'_{s+1} \geqslant 2$ the lemma follows from the uniqueness of continued fraction expansions with entries $\geqslant 2$. If $m_{s+1} = 1$ then the first sequence can be blown down to a shorter sequence. Thus $m'_{s+1} = 1$ and the lemma follows by blowing down and applying the inductive hypothesis.

PROPOSITION A10. *Suppose* $A$ *is a graded ring and define*

$$A^{\natural} = \bigoplus_{n \geqslant 0} A_{[n]}$$

*as in* (3.2). *If* $A$ *is an integrally closed domain then* $A^{\natural}$ *is an integrally closed domain.*

*Proof.* We define a grading on the polynomial ring $A[t]$ by defining degree $t = 1$ and the degree of the coefficients to be 0. Then $A^{\natural}$ is isomorphic as a graded ring to

$$\bigoplus_{n \geqslant 0} A_{[n]} t^n \subset A[t].$$

Clearly $A^{\natural}$ is a domain and if we let $B$ be the integral closure of $A^{\natural}$ then it follows from Bourbaki (Commutative Algebra, V, § 1, no. 8, Prop. 20 and no. 3, Prop. 13, Cor. 2) that B is a graded subring of $A[t]$.

Suppose $b \in B_n$. Then $b = at^n$, where $a \in A$. It is sufficient to show $a \in A_{[n]}$. Since $b$ is integral over $A^{\#}$ there exist $a_i \in A_{[ni]}$ so that

$$(at^n)^m + a_1 t^n (at^n)^{m-1} + \ldots + a_m t^{nm} = 0$$

Thus

$$a^m = -(a_1 a^{m-1} + \ldots + a_m) \tag{2}$$

in $A$. Suppose $a \notin A_{[n]}$. Then we can write

$$a = \alpha_i + \alpha_{i+1} + \ldots$$

where $\alpha_j \in A_j$ for all $j$, $\alpha_i \neq 0$ and $i < n$. Now $a_j a^{m-j} \in A_{[jn+(m-j)i]}$, for $j = 1, ..., m$, and hence the right hand side of equation (2) lies in $A_{[mi+n-i]}$. This contradicts the fact that $a^m = (\alpha_i)^m + y$ where $y$ is a sum of forms of higher degree.

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