ON SECTIONS OF SOME HOLOMORPHIC FAMILIES OF CLOSED RIEMANN SURFACES

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1. Introduction.

The study of holomorphic sections of the Teichmüller curves π_n: $V(p, n) \to T(p, n)$ was initiated by John Hubbard [8] for the case $n=0$. The existence of such sections would be important because each such section would allow us to choose a point on every Riemann surface of genus p in a way that depends holomorphically on the moduli. Unfortunately, Hubbard showed in [8] that π_0 has no holomorphic sections if $p \geq 3$.

In our paper [5] we studied the holomorphic sections of π_n for $n \geq 1$, but we were unable to obtain complete results. Now we are able to describe all the holomorphic sections of π_n for every genus $p \geq 2$. We also study sections of π_0: $V(p, 0) \to T(p, 0)$ over subspaces of $T(p, 0)$ that correspond to Riemann surfaces with automorphisms. We state our theorems in § 2, and prove them in §§ 5, 7, and 8. Since our proofs require some unfamiliar facts from Teichmüller theory, we develop the facts we need in §§ 3 and 4. Much material in these sections, especially in § 3, is expository in nature. Both of our main theorems have generalizations, which we give in §§ 10 and 11 with indications of their proofs. We have chosen to focus our attention in the body of the paper on the most important cases.

The remaining two sections of the paper deal with projections of norm one in certain Banach spaces. In § 6 we prove two general propositions about the existence of such projections. In § 9 we establish the non-existence of such projections in certain spaces of quadratic differentials. Most cases of Theorem 9.1 were proved already in [5] and [8], and we prove the remaining cases by the methods indicated in [5].

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2. Statement of results.

2.1. Recall that the Riemann surface X has *type* (p, n) if and only if there exist a closed Riemann surface $\bar{X}$ of genus p and n distinct points $x_1, \ldots, x_n$ on $\bar{X}$, called the "punctures of X", such that $X = \bar{X} \setminus \{x_1, \ldots, x_n\}$. Suppose

$$2p - 2 + n > 0. \tag{2.1}$$

Then the *Teichmüller space* $T(p, n)$ is a complex analytic manifold, of dimension $3p - 3 + n$, whose points represent the Riemann surfaces of type (p, n). The *Teichmüller curve* $V(p, n)$ is a complex manifold of dimension $3p - 2 + n$ with a holomorphic projection

$$\pi_n \colon V(p, n) \to T(p, n) \tag{2.2}$$

onto $T(p, n)$ such that for every τ in $T(p, n)$, $\pi_n^{-1}(\tau)$ is the closed surface $\bar{X}$ of genus p determined by the surface of type (p, n) represented by τ. We shall describe these spaces in more detail in § 3.

2.2. The map π_n in (2.2) has local holomorphic sections. The problem of describing the (global) holomorphic sections was first raised by John Hubbard. He showed in [8] that $\pi_0 \colon V(p, 0) \to T(p, 0)$ has no holomorphic sections if $p \geqslant 3$ and six if $p = 2$.

If $n \geqslant 1$, then every fiber $\pi_n^{-1}(\tau)$ in $V(p, n)$ contains n distinguished points, the punctures, and π_n has n *canonical holomorphic sections* $s_j \colon T(p, n) \to V(p, n)$, $1 \leqslant j \leqslant n$, such that $s_j(\tau)$ is a puncture for every τ in $T(p, n)$. We describe them more fully in § 3.5. In our earlier paper [5] we found all holomorphic sections $s \colon T(p, n) \to V(p, n)$ which are disjoint from the canonical ones; there are none if $2p + n \geqslant 5$. (Recall that two sections are called *disjoint* if their images are disjoint sets.) Our first theorem is a substantial improvement on our earlier results, since we describe *all* holomorphic sections of (2.2), provided that $p \geqslant 2$.

THEOREM. *The Teichmüller curve $\pi_n \colon V(p, n) \to T(p, n)$ has exactly n holomorphic sections if $p \geqslant 3$ and exactly $2n + 6$ holomorphic sections if $p = 2$.*

The n sections for $p \geqslant 3$ are the canonical sections. For $p = 2$ there are six *Weierstrass sections*, discovered by Hubbard [8] for $n = 0$. For $n \geqslant 1$ we shall describe them in § 4.7. In addition to the Weierstrass sections there are the canonical sections $s_1, \ldots, s_n$. The remaining sections have the form $J \circ s_1, \ldots, J \circ s_n$, where $J \colon V(2, n) \to V(2, n)$ is the holomorphic involution whose restriction to each fiber is the hyperelleptic involution (recall that each fiber is a hyperelliptic Riemann surface). We refer to § 8.3 for a fuller description of J and to §§ 5 and 7 for the proof of the theorem.

2.3. Our second theorem is about closed Riemann surfaces with automorphisms. Let X be a closed Riemann surface of genus $p \geqslant 2$, and let H be a non-trivial (finite) group of conformal automorphisms of X. The group H acts in a natural way on the Teichmüller space $T(p, 0)$ as a group of biholomorphic maps (see § 8). The fixed point set $T(p, 0)^H$ represents Riemann surfaces of genus p which admit H as a group of automorphisms. Indeed, H acts also as a group of fiber-preserving biholomorphic mappings of $V(p, 0)$, and the fiber over each point of $T(p, 0)^H$ is mapped onto itself by H (see § 8).

The action of H on $V(p, 0)$ allows us to distinguish certain points, the fixed points of non-trivial elements of H. In general these points account for all holomorphic sections of π_0: $V(p, 0) \to T(p, 0)$ over $T(p, 0)^H$.

THEOREM. *Let s: $T(p, 0)^H \to V(p, 0)$ be a holomorphic section of π_0: $V(p, 0) \to T(p, 0)$. Let p' be the genus of the closed surface X/H, and n' the number of points in X/H over which the projection from X to X/H is branched. If*

$$2p' + n' > 4,$$

then $s(\tau)$ is fixed by some non-trivial h in H for every τ in $T(p, 0)^H$.

We shall prove the theorem in § 8. As a special case of the theorem, take X to be hyperelliptic and H to be the group of order two generated by the hyperelliptic involution. Then $T(p, 0)^H$ is a branch of the hyperelliptic locus, and the theorem tells us that the only holomorphic sections over $T(p, 0)^H$ are Weierstrass sections. When $p = 2$, the hyperelliptic locus is the entire Teichmüller space $T(2, 0)$ and we recover Hubbard's theorem [8] about the sections of π_0: $V(2, 0) \to T(2, 0)$.

3. The Teichmüller curves $V(p, n)$.

3.1. In this section we shall review the definitions and some well known properties of the spaces $T(p, n)$ and $V(p, n)$. More details can be found in [2] or [5].

Let Γ be a Fuchsian group operating on the upper half plane U, hence also on the lower half plane L. We require Γ to have a compact fundamental domain, so that the quotient space U/Γ is compact. As usual we denote by $L^\infty(\Gamma)$ the space of *Beltrami differentials* for Γ. Recall that $L^\infty(\Gamma)$ consists of all μ in $L^\infty(U, \mathbf{C})$ satisfying

$$(\mu \circ \gamma)\bar{\gamma}'/\gamma' = \mu, \quad \text{all} \quad \gamma \in \Gamma. \tag{3.1}$$

The open unit ball $M(\Gamma)$ of $L^\infty(\Gamma)$ is the set of *Beltrami coefficients* for Γ.

3.2. For each μ in $M(\Gamma)$ there is a unique quasiconformal map w^μ of the plane onto itself which fixes zero and one, is conformal in L, and satisfies the Beltrami equation $w_{\bar{z}} = \mu w_z$ in U.

We say that μ and ν in $M(\Gamma)$ are *equivalent* (and write $\mu \sim \nu$) if and only if $w^\mu = w^\nu$ on the real axis. The *Teichmüller space* $T(\Gamma)$ is the set of equivalence classes in $M(\Gamma)$. We denote by Φ the projection of $M(\Gamma)$ onto $T(\Gamma)$, so that $\Phi(\mu)$ is the equivalence class of μ for each μ in $M(\Gamma)$.

THEOREM (Bers [2]). *$T(\Gamma)$ is a complex manifold and the map Φ: $M(\Gamma) \to T(\Gamma)$ is holomorphic with local holomorphic sections.*

Denote by $\Sigma_0(\Gamma)$ the group of all quasiconformal maps w of U onto itself such that

$$w \circ \gamma = \gamma \circ w, \quad \text{all} \quad \gamma \in \Gamma. \tag{3.2}$$

It is not hard to verify that $\mu \sim \nu$ in $M(\Gamma)$ if and only if $w^\nu = w^\mu \circ w$ for some w in $\Sigma_0(\Gamma)$.

3.3. Let U_Γ be the set of z in U which are not fixed by any elliptic element of Γ. We define the *type* of Γ to be the type of the Riemann surface U_Γ/Γ. There is a group of type (p, n) if and only if (p, n) satisfies (2.1). A theorem of Bers and Greenberg (see § 2.1 of [5]) says that $T(\Gamma)$ and $T(\Gamma')$ are biholomorphically equivalent if Γ and Γ' have the same type. The Teichmüller space $T(p, n)$ is defined to be $T(\Gamma)$ for some group of type (p, n).

3.4. The domain $w^\mu(U)$ depends only on the equivalence class $\Phi(\mu)$ of μ in $M(\Gamma)$, so we form the *Bers fiber space*

$$F(\Gamma) = \{(\Phi(\mu), z) \in T(\Gamma) \times \mathbf{C};\ \mu \in M(\Gamma) \text{ and } z \in w^\mu(U)\}.$$

$F(\Gamma)$ is a complex manifold on which the group Γ acts discontinuously as a group of biholomorphic mappings (see Bers [3]) by

$$\gamma(\Phi(\mu), z) = (\Phi(\mu), \gamma^\mu(z)) \tag{3.3}$$

where $\mu \in M(\Gamma)$, $z \in w^\mu(U)$, $\gamma \in \Gamma$, and

$$\gamma^\mu \circ w^\mu = w^\mu \circ \gamma. \tag{3.4}$$

(Notice that γ^μ depends only on $\Phi(\mu)$ because if $\mu \sim \nu$ then $w^\nu = w^\mu \circ w$ for some w in $\Sigma_0(\Gamma)$.) The quotient space $V(\Gamma) = F(\Gamma)/\Gamma$ has a canonical complex structure, and the map $(\Phi(\mu), z) \mapsto \Phi(\mu)$ induces a holomorphic projection of $V(\Gamma)$ onto $T(\Gamma)$. The inverse image of $\Phi(\mu)$ under that projection is the closed Riemann surface $w^\mu(U)/w^\mu\Gamma(w^\mu)^{-1}$. The *Teichmüller curve* $V(p, n)$ is defined to be the space $V(\Gamma)$ for some group Γ of type (p, n) with the above projection π_n: $V(p, n) \to T(p, n)$. For $p \geqslant 2$ we shall verify in § 4.6 that $V(\Gamma)$ depends only on the type of Γ, as it should.

3.5. The *canonical sections* s_j: $T(p, n) \to V(p, n)$ of π_n, $1 \leqslant j \leqslant n$, arise as follows. Let z_0 in U be fixed by the elliptic transformation γ in Γ. Then $w^\mu(z_0)$ is the fixed point of γ^μ in $w^\mu(U)$. The map $\Phi(\mu) \mapsto (\Phi(\mu), w^\mu(z_0))$ from $T(\Gamma)$ to $F(\Gamma)$ is well-defined and holomorphic, inducing a section s_j.

3.6. The considerations of § 3.5 help to prove the following.

PROPOSITION. $V(\Gamma)$, *with its canonical complex structure, is a complex manifold.*

Proof: Since $V(\Gamma)$ is the quotient of the complex manifold $F(\Gamma)$ by the discontinuous group Γ of biholomorphic selfmaps, it is a normal complex space. We need to show that it is a manifold. If γ in Γ is hyperbolic, we see from (3.3) that γ has no fixed points in $F(\Gamma)$. If γ in Γ is elliptic, the fixed point locus of γ in $F(\Gamma)$ is

$$\{(\Phi(\mu),\, w^{\mu}(z_0));\ \mu \in M(\Gamma)\},$$

where z_0 is the fixed point of γ in U. That locus is a closed complex submanifold of $F(\Gamma)$, of codimension one. It follows that $V(\Gamma)$ is a manifold (see [6, Satz 1]).

4. The map from $V(p,n)$ to $V(p,0)$.

4.1. There is a well-known holomorphic map f_n: $T(p, n) \to T(p, 0)$, for $p \geq 2$, which arises by "forgetting the punctures." We are going to construct that map and an analogous map g_n: $V(p, n) \to V(p, 0)$. These maps will lead us in § 4.6 to a useful alternate description of the spaces $V(p, n)$ for $p \geq 2$ and $n \geq 1$.

To begin, we choose a closed Riemann surface X of genus $p \geq 2$ and Fuchsian groups Γ' and Γ, of types (p, n) and $(p, 0)$ respectively, so that $U/\Gamma' = U/\Gamma = X$. Let π': $U \to X$ and π: $U \to X$ be the projection maps associated with the groups Γ' and Γ. Since Γ has type $(p, 0)$, π is an unbranched covering map, and there is a holomorphic map h: $U \to U$ such that

$$\pi' = \pi \circ h. \tag{4.1}$$

Hence there is a homomorphism θ: $\Gamma' \to \Gamma$ satisfying

$$h \circ \gamma = \theta(\gamma) \circ h \quad \text{for all} \quad \gamma \in \Gamma'. \tag{4.2}$$

LEMMA. *h: $U \to U$ and θ: $\Gamma' \to \Gamma$ are surjective.*

Proof. Set $D = h(U) \subset U$. It is easy to verify that U is the disjoint union of the open sets D and $\{\gamma(z); z \in D \text{ and } \gamma \in \Gamma \backslash \theta(\Gamma')\}$. Since U is connected, $D = U$ and $\theta(\Gamma') = \Gamma$.

4.2. For μ in $M(\Gamma')$ define $h_*(\mu)$ by

$$h_*(\mu) \circ h = \mu h' / \bar{h}'. \tag{4.3}$$

It is easy to verify that $h_*(\mu)$ is a well-defined member of $M(\Gamma)$ and that h_*: $M(\Gamma') \to M(\Gamma)$ is bijective. Moreover,

$$h^{\mu} = w^{\sigma} \circ h \circ (w^{\mu})^{-1}, \quad \sigma = h_*(\mu), \tag{4.4}$$

is a holomorphic map of $w^{\mu}(U)$ onto $w^{\sigma}(U)$. We need to study the dependence of h^{μ} on μ.

LEMMA. *For any fixed μ_0 in $M(\Gamma')$ and ζ in $w^{\mu_0}(U)$, $h^\mu(\zeta)$ depends holomorphically on μ in a neighborhood of μ_0.*

Proof. Choose an open disk $D\subset \mathbf{C}$ centered at ζ and an open ball $B\subset M(\Gamma')$ centered at μ_0 so that $D\subset w^\mu(U)$ if $\mu\in B$. That is possible because of the continuous dependence of w^μ on μ (see [1]). Formula (4.4) and the results of [1] show further that the functions h^μ, $\mu\in B$, are a normal family in D, and that h^ν tends to h^μ uniformly on compact sets in D if ν tends to μ in B.

Now fix μ in B and ν in $M(\Gamma')$. To prove the lemma we must show that $h^{\mu+t\nu}(\zeta)$ is a holomorphic function of the complex variable t at $t=0$. Set

$$w_t = w^{\mu+t\nu},\quad w_t^* = w^\sigma,\quad \sigma = h_*(\mu+t\nu),\quad h_t = h^{\mu+t\nu},\quad z = w_0^{-1}(\zeta).$$

By Theorem 10 of [1] (with $\dot{w}=\partial w/\partial t$),

$$w_t(z) = w_0(z)+t\dot{w}(z)+o(t) = \zeta+t\dot{w}(z)+o(t),$$
$$w_t^*(h(z)) = w_0^*(h(z))+t\dot{w}^*(h(z))+o(t).$$

Therefore, by (4.4),

$$\begin{aligned}h_t(w_t(z)) &= w_t^*(h(z)) = h_0(w_0(z))+t\dot{w}^*(h(z))+o(t)\\ &= h_0(\zeta)+t\dot{w}^*(h(z))+o(t).\end{aligned}$$

But the h_t are a normal family in D, so

$$h_t(w_t(z)) = h_t(\zeta+t\dot{w}(z)+o(t)) = h_t(\zeta)+t\dot{w}(z)h_t'(\zeta)+o(t),$$

and

$$t^{-1}(h_t(\zeta)-h_0(\zeta)) = \dot{w}^*(h(z))-\dot{w}(z)h_t'(\zeta)+o(1).$$

As $t\to 0$, the right hand side converges to

$$\dot{w}^*(h(z))-\dot{w}(z)h_0'(\zeta).$$

That proves the lemma.

4.3. Next we shall verify that h_* carries equivalence classes into equivalence classes and that h^μ depends only on the equivalence class of μ.

LEMMA. *Let w: $U\to U$ be a homeomorphism that commutes with Γ'. There is a unique homeomorphism w_*: $U\to U$ that commutes with Γ and satisfies $w_*\circ h=h\circ w$.*

Proof. Define f: $X\to X$ so that $\pi'\circ w=f\circ\pi'$, and define g: $U\to U$ so that $\pi\circ g=f\circ\pi$. Then f and g are homeomorphisms, and

$$\pi\circ g^{-1}\circ h\circ w = f^{-1}\circ\pi\circ h\circ w = f^{-1}\circ\pi'\circ w = \pi' = \pi\circ h.$$

Notice the use of (4.1). Since $\pi\circ h=\pi\circ g^{-1}\circ h\circ w$, there exists γ in Γ such that $\gamma\circ h=g^{-1}\circ h\circ w$. Put $w_*=g\circ\gamma$. Then $w_*\colon U\to U$ is a homeomorphism, and $w_*\circ h=h\circ w$. Further, for all γ in Γ', (4.2) gives

$$\theta(\gamma)\circ h=h\circ\gamma=w_*^{-1}\circ h\circ w\circ\gamma=w_*^{-1}\circ h\circ\gamma\circ w=w_*^{-1}\circ\theta(\gamma)\circ h\circ w=w_*^{-1}\circ\theta(\gamma)\circ w_*\circ h.$$

Since h and θ are surjective, w_* commutes with Γ. This completes the proof because w_* is clearly unique.

COROLLARY. *If $\mu\sim\nu$ in $M(\Gamma')$, then $h^\mu=h^\nu$ and $h_*(\mu)\sim h_*(\nu)$ in $M(\Gamma)$.*

Proof. $\mu\sim\nu$ means $w^\nu=w^\mu\circ w$ for some w in $\Sigma_0(\Gamma')$. The lemma gives us a homeomorphism $w_*\colon U\to U$ that commutes with Γ and satisfies $w_*\circ h=h\circ w$. We observe that w_* is quasiconformal in U, so $w_*\in\Sigma_0(\Gamma)$.

Now in U we have

$$h^\mu\circ w^\nu=h^\mu\circ w^\mu\circ w=w^\sigma\circ h\circ w=w^\sigma\circ w_*\circ h$$

where $\sigma=h_*(\mu)$. Differentiating both sides and comparing with (4.3) we find that $h_*(\nu)$ is the Beltrami coefficient of $w^\sigma\circ w_*$, so $h_*(\mu)\sim h_*(\nu)$ in $M(\Gamma)$. Put $\varrho=h_*(\nu)$. In $w^\nu(U)=w^\mu(U)$ we have

$$h^\nu=w^\varrho\circ h\circ(w^\nu)^{-1}=w^\sigma\circ w_*\circ h\circ w^{-1}\circ(w^\mu)^{-1}=w^\sigma\circ h\circ(w^\mu)^{-1}=h^\mu.$$

The proof is complete.

4.4. The following result is an easy consequence of the previous lemmas.

LEMMA. *Define $f\colon T(\Gamma')\to T(\Gamma)$ and $G\colon F(\Gamma')\to F(\Gamma)$ by*

$$f(\Phi(\mu))=\Phi(h_*(\mu)) \tag{4.5}$$

$$G(\Phi(\mu),\zeta)=(f(\Phi(\mu)),h^\mu(\zeta)). \tag{4.6}$$

Then f and G are well-defined surjective holomorphic maps.

Proof. The corollary to Lemma 4.3 says that f and G are well-defined. f is surjective because h_* is surjective and holomorphic because $f\circ\Phi=\Phi\circ h_*$ is. G is surjective because h^μ maps $h^\mu(U)$ onto $h^\sigma(U)$, $\sigma=h_*(\mu)$, by (4.4) and Lemma 4.1. Finally, $h^\mu(\zeta)$ is a holomorphic function of ζ for fixed μ and a holomorphic function of μ for fixed ζ by Lemma 4.2. It follows that G is holomorphic. The lemma is proved.

4.5. Now we are ready for the main result of this section.

THEOREM. *Let f_n: $T(p, n) \to T(p, 0)$ be the "forgetful map" defined by* (4.5). *There is a holomorphic map g_n: $V(p, n) \to V(p, 0)$ such that the diagram*

$$\begin{array}{ccc} V(p,n) & \xrightarrow{g_n} & V(p,0) \\ \Big\downarrow{\scriptstyle \pi_n} & & \Big\downarrow{\scriptstyle \pi_0} \\ T(p,n) & \xrightarrow{f_n} & T(p,0) \end{array} \tag{4.7}$$

commutes and g_n maps each fiber $\pi_n^{-1}(t)$, $t \in T(p, n)$, one-to-one onto the fiber $\pi_0^{-1}(f_n(t))$.

Proof. Represent $T(p, n)$ by $T(\Gamma')$ and $T(p, 0)$ by $T(\Gamma)$, as in §§ 4.1 to 4.4. The forgetful map f_n becomes the map f of Lemma 4.4. It is easy to verify using (3.4), (4.2), and (4.4), that the map G: $F(\Gamma') \to F(\Gamma)$ of Lemma 4.4 induces a well-defined holomorphic map $g(=g_n)$ of $V(\Gamma') = F(\Gamma')/\Gamma'$ onto $V(\Gamma) = F(\Gamma)/\Gamma$. g makes the diagram (4.7) commute.

It remains to prove that g maps each fiber one-to-one onto its image fiber. Let $\mu \in M(\Gamma')$ and let $\sigma = h_*(\mu) \in M(\Gamma)$. We need to show that h^μ: $w^\mu(U) \to w^\sigma(U)$ induces a bijective map between the closed Riemann surfaces

$$X_\mu = w^\mu(U)/w^\mu\Gamma'(w^\mu)^{-1}$$

and

$$X_\sigma = w^\sigma(U)/w^\sigma\Gamma(w^\sigma)^{-1}.$$

But $(w^\mu)^{-1}$ induces a homeomorphism of X_μ onto $X = U/\Gamma'$, h induces the identity map of $X = U/\Gamma'$ onto $X = U/\Gamma$, and w^σ induces a homeomorphism of U/Γ onto X_σ. The composite of these maps is the homemorphism induced by h^μ. The theorem is proved.

4.6. Our next result is an almost immediate corollary to Theorem 4.5.

THEOREM. *$\pi_n \times g_n$: $V(p, n) \to T(p, n) \times V(p, 0)$ maps $V(p, n)$ one-to-one onto the closed submanifold*

$$W = \{(t, x);\ f_n(t) = \pi_0(x)\}$$

of $T(p, n) \times V(p, 0)$.

Proof. Theorem 4.5 implies that $\pi_n \times g_n$ maps $V(p, n)$ one-to-one onto W. W is a closed submanifold of $T(p, n) \times V(p, 0)$ because the derivative of π_0: $V(p, 0) \to T(p, 0)$ has maximal rank at every point of $V(p, 0)$.

COROLLARY 1. *$\pi_n \times g_n$: $(Vp, n) \to W$ is biholomorphic.*

Proof. A bijective holomorphic map between complex manifolds is biholomorphic (see [7, p. 109]).

COROLLARY 2. *The holomorphic sections s: $T(p, n) \to V(p, n)$ of π_n are in bijective correspondance with the holomorphic maps h: $T(p, n) \to V(p, 0)$ such that $\pi_0 \circ h = f_n$.*

Proof. Given s, put $h = g_n \circ s$. Given h, notice that $t \mapsto (t, h(t)) = \varphi(t)$ is a holomorphic map of $T(p, n)$ into W. Put $s = (\pi_n \times g_n)^{-1} \circ \varphi$.

Remark. If Γ_1 and Γ_2 have type $(p, 0)$, then the fiber spaces $V(\Gamma_1)$ and $V(\Gamma_2)$ are biholomorphically equivalent in a fiber-preserving way (see [3]). Therefore the fiber space $V(p, 0)$ is well defined. Theorem 4.6 and Corollary 1 show that the spaces $V(p, n)$, $n \geqslant 1$, are also well defined, as we promised in § 3.4.

4.7. Let s: $T(2, 0) \to V(2, 0)$ be one of the six Weierstrass sections of Hubbard [8] (see § 4.5 of [5]). Then $h = s \circ f_n$: $T(2, n) \to V(2, 0)$ is holomorphic, and $\pi_0 \circ h = f_n$, so h determines a holomorphic section of π_n: $V(2, n) \to T(2, n)$. The sections obtained in this way are the *Weierstrass sections* of π_n.

4.8. *An unsolved problem.* Let B be any complex manifold, $p \geqslant 2$, and f: $B \to T(p, 0)$ a holomorphic mapping. As in § 4.6, form the complex manifold

$$W = \{(t, x) \in B \times V(p, 0);\ f(t) = \pi_0(x)\}.$$

Define π: $W \to B$ by $\pi(t, x) = t$. Then π is holomorphic, and for each t in B, $\pi^{-1}(t)$ is the closed Riemann surface represented by $f(t)$. As in § 4.6 the holomorphic sections of π are in bijective correspondence with the holomorphic maps h: $B \to V(p, 0)$ such that $\pi_0 \circ h = f$. This leads us to the following general problem:

Given f: $B \to T(p, 0)$ determine all holomorphic maps h: $B \to V(p, 0)$ such that $\pi_0 \circ h = f$.

In §§ 5 and 11 we solve this problem when $B = T(p, n)$, or $T(p, n)^H$ for certain groups H, and f is the forgetful map f_n. The general problem is open. If B is the unit disk Δ we conjecture that for every f there is at least one holomorphic h with $\pi_0 \circ h = f$. By a theorem of Grothendieck (see A. Grothendieck, Techniques de construction en géométrie analytique, Séminaire H. Cartan 1960/61), every holomorphic family π: $W \to \Delta$ of closed Riemann surfaces of genus $p \geqslant 2$ over Δ is obtained from a holomorphic map f: $\Delta \to T(p, 0)$. Thus, our conjecture asserts that every such family has a holomorphic section.

5. The linear version of Theorem 2.2.

5.1. If X is a Riemann surface of finite type, we denote by $Q(X)$ the linear space of holomorphic quadratic differentials φ on X with

$$\|\varphi\| = \tfrac{1}{2} \int_X |\varphi| < \infty. \tag{5.1}$$

The differentials φ in $Q(X)$ are meromorphic on the compactification $\bar{X}$ of X, with at worst simple poles at the punctures. $Q(X)$ is a finite dimensional Banach space, with norm (5.1).

Let τ belong to the Teichmüller space $T(p, n)$, and let X be the Riemann surface of type (p, n) represented by τ. Then the cotangent space to $T(p, n)$ at τ is canonically isomorphic to $Q(X)$ (see § 1.2 of [5]). By duality, the norms (5.1) induce norms on the tangent vectors of $T(p, n)$ and a Finsler metric on $T(p, n)$. A theorem of Royden [11] tells us that the induced metric on $T(p, n)$ is the "hyperbolic metric" of Kobayashi [9]. Royden states his theorem only for the spaces $T(p, 0)$, but his proof, in § 5 of [11], works equally well for $n \geqslant 1$.

5.2. Let Γ be a Fuchsian group of type $(p, 0)$, so that $T(\Gamma)$ is $T(p, 0)$ and $V(\Gamma)$ is $V(p, 0)$. As we saw in § 3, the universal covering space of $V(\Gamma)$ is the Bers fiber space $F(\Gamma)$. The isomorphism theorem of Bers [3] establishes a biholomorphic map of $F(\Gamma)$ onto the Teichmüller space $T(p, 1)$. Under that map the projection $(\Phi(\mu), z) \mapsto \Phi(\mu)$ of $F(\Gamma)$ to $T(\Gamma)$ is identified with the forgetful map $f_1: T(p,1) \to T(p,0)$ of § 4. Using the universal covering of $V(p,0)$ by $T(p,1)$, we obtain the following.

PROPOSITION. *Let π_0: $V(p,0) \to T(p,0)$ be the Teichmüller curve of genus $p \geqslant 2$. Let $x_0 \in V(p,0)$ and $\tau = \pi_0(x_0) \in T(p,0)$. Let $X = \pi_0^{-1}(\tau)$. Then the cotangent space to $V(p,0)$ at x_0 is $Q(X \backslash \{x_0\})$, the cotangent space to $T(p,0)$ at τ is $Q(X)$, and the map of cotangent spaces induced by π_0 is the inclusion map of $Q(X)$ in $Q(X \backslash \{x_0\})$. Further, the Finsler metric on $V(p,0)$ induced by the norm*

$$\|\varphi\| = \tfrac{1}{2} \int_X |\varphi| \quad \text{for all } \varphi \in Q(X \backslash \{x_0\})$$

and is the hyperbolic metric on $V(p,0)$.

The final statement of the proposition is true because the covering of $V(p,0)$ by $T(p,1)$ is a local isometry in the hyperbolic metrics (see [9, p. 48]). The map of cotangent spaces induced by the forgetful map from $T(p,1)$ to $T(p,0)$ is discussed in § 1.5 of [5]. The above proposition is the basis of Hubbard's discussion of the sections of π_0: $V(p,0) \to T(p,0)$ in [8], and it will play a basic role in our proof of Theorem 2.2.

Remark. In the final paragraph of § 4.2 of [5] we gave an incorrect sketch of the proof of Bers' isomorphism theorem. The correct reading of that paragraph, in the notation of § 1.5 of [5], is as follows. Define

$$\Psi: M(\Gamma') \to F(\Gamma) \quad \text{by} \quad \Psi(\mu) = (\Phi(\nu), w^{\nu}(a)),$$

where $\nu = h^*\mu$, $a \in U$, and $t(a) = x_0$. Then Ψ is holomorphic, and it projects to a biholomorphic map from $T(\Gamma')$ to $F(\Gamma)$.

5.3. Let s: $T(p, n) \to V(p, n)$ be a holomorphic section of π_n: $V(p, n) \to T(p, n)$. By Corollary 2 of Theorem 4.6, there is a holomorphic map $h = g_n \circ s$: $T(p, n) \to V(p, 0)$ such that $\pi_0 \circ h = f_n$ is the forgetful map from $T(p, n)$ to $T(p, 0)$.

Choose t in $T(p, n)$, and let $x_0 = h(t)$ and $\tau = \pi_0(x_0) = f_n(t)$. Let $X = \pi_0^{-1}(\tau)$ and let $X' = X \setminus \{x_0\}$. In addition, let $X'' = X \setminus \{y_1, ..., y_n\}$ be the surface of type (p, n) represented by t. The cotangent spaces to $T(p, n)$, $V(p, 0)$, and $T(p, 0)$ at t, x_0, and τ are $Q(X'')$, $Q(X')$, and $Q(X)$ respectively. By § 1.5 of [5], the map from $Q(X)$ to $Q(X'')$ induced by the forgetful map is the inclusion map. Similarly, by Proposition 5.2, the map π_0 induces the inclusion map of $Q(X)$ in $Q(X')$. Let L: $Q(X') \to Q(X'')$ be the map of cotangent spaces induced by h: $T(p, n) \to V(p, 0)$. Since $\pi_0 \circ h = f_n$, we have

$$L\varphi = L(\pi_0^*(x_0)\varphi) = f_n^*(t)\varphi = \varphi$$

for all φ in $Q(X)$. (Here $\pi_0^*(x_0)$: $Q(X) \to Q(X')$ and $f_n^*(t)$: $Q(X) \to Q(X'')$ are the (inclusion) maps of cotangent spaces induced by π_0 and f_n.) Further, since the holomorphic map h does not increase hyperbolic distances (see [9, p. 45]), L must satisfy

$$\|L\varphi\| \leqslant \|\varphi\| \quad \text{for all } \varphi \in Q(X').$$

5.4. Our stratefy is to determine the holomorphic sections s: $T(p, n) \to V(p, n)$ by studying the linear maps L obtained from s in § 5.3. The maps L can be rather completely described.

THEOREM. *Let X be a closed Riemann surface of genus $p \geqslant 2$, $X' = X \setminus \{x_0\}$, and $X'' = X \setminus \{y_1, ..., y_n\}$, where $x_0 \in X$, $y_1, ..., y_n \in X$, and the points y_j are distinct. Let L: $Q(X') \to Q(X'')$ be a linear map such that*

$$L\varphi = \varphi \quad \textit{for all } \varphi \in Q(X) \tag{5.2}$$

$$\|L\varphi\| \leqslant \|\varphi\| \quad \textit{for all } \varphi \in Q(X'). \tag{5.3}$$

If $p \geqslant 3$, then $x_0 = y_k$ for some k, and $L\varphi = \varphi$ for all φ in $Q(X')$. If $p = 2$, let j: $X \to X$ be the hyperelliptic involution of X. Then either $x_0 = y_k$ for some k, $j(x_0) = y_k$ for some k, or x_0 is a Weierstrass point of X.

We shall prove this theorem, which is the linear version of Theorem 2.2, in § 7.

5.5. In this section we shall prove Theorem 2.2, given Theorem 5.4. Let s: $T(p, n) \to V(p, n)$ be a holomorphic section of π_n, and let $t \in T(p, n)$. First suppose $p \geqslant 3$. Theorem 5.4 tells us that $x_0 = h(t) = g_n(s(t))$ is one of the points y_k, $1 \leqslant k \leqslant n$. But $y_k = g_n(s_k(t))$, where s_k is a canonical section of π_n. Since g_n is one-to-one on each fiber, we have $s(t) = s_k(t)$. Let

$$B_k = \{t \in T(p, n);\ s(t) = s_k(t)\}, \quad 1 \leqslant k \leqslant n.$$

Then $T(p, n)$ is the finite union of the sets B_k. Therefore, some B_k has interior, and the identity theorem for holomorphic maps implies that $s = s_k$ in all of $T(p, n)$. That proves the theorem for $p \geqslant 3$.

If $p = 2$, let $s_1, \ldots, s_{2n+6}$ be the holomorphic sections given in § 2.2. Theorem 5.4 again tells us that for each $t \in T(p, n)$, $s(t) = s_k(t)$ for some k, $1 \leqslant k \leqslant 2n+6$. Reasoning as above, we conclude that s is one of the given sections s_k. The proof is complete.

6. Projections of norm one.

6.1. We have seen in § 5 that the proof of Theorem 2.2 reduces to studying certain linear maps of norm one. In this section we shall prove two general propositions about projection operators of norm one. Both of them will be needed later.

6.2. Let V be any real Banach space whose norm is a differentiable function on $V \setminus \{0\}$, and set

$$A(v, w) = \lim_{t \to 0} \frac{\|v + tw\| - \|v\|}{t} \quad \text{for all } v \in V \setminus \{0\}, w \in V. \tag{6.1}$$

Then $w \mapsto A(v, w)$ is a bounded linear functional on V for any fixed nonzero v. There is a close relation between these linear functionals and the projections of norm one onto closed subspaces of V.

PROPOSITION. *Let W be a non-trivial closed subspace of V, and let W' be the closed subspace*

$$W' = \{v \in V;\, A(w, v) = 0 \quad \textit{for all } w \in W \setminus \{0\}\}. \tag{6.2}$$

There is a projection P of norm one from V onto W if and only if W' is a complementary subspace to W. Further, if P exists it is unique and its kernel is W'.

Proof. First suppose a projection P of norm one exists. For any v in V and $w \neq 0$ in W consider the function

$$f(t) = \|w + tv\| - \|P(w + tv)\| = \|w + tv\| - \|w + tPv\|.$$

$f(t) \geqslant 0 = f(0)$ for all real t, so

$$0 = f'(0) = A(w, v) - A(w, Pv). \tag{6.3}$$

If $Pv \neq 0$, we take $w = Pv$ in (6.3) and find, using (6.1), that

$$A(w, v) = A(w, w) = \|w\| \neq 0.$$

Hence $v \notin W'$ if $Pv \neq 0$. If $Pv = 0$, then $v \in W'$ because $A(w, v) = 0$ for all $w \neq 0$ in W, by (6.3). Therefore W' is the kernel of P. It follows immediately that P is unique and that W' is a complementary subspace to W.

Conversely, suppose W' is a complement to W. Let $P: V \to W$ be the unique projection with image W and kernel W'. We claim P has norm one. That is,

$$\|w\| \leqslant \|w+v\|, \quad \text{all } w \in W,\ v \in W' \tag{6.4}$$

If $w=0$ there is nothing to prove. If $w \neq 0$, set $f(t) = \|w+tv\|$ for t in $\mathbf{R}$. Then f is a convex function of t, and

$$f'(0) = A(w, v) = 0$$

by hypothesis, so $f(t) \geqslant f(0)$ for all t. Setting $t=1$ we obtain (6.4). The proof is complete.

COROLLARY. *If V has finite dimension, a projection of norm one from V onto W exists if and only if*

$$\dim W + \dim W' = \dim V.$$

Proof. $W \cap W' = \{0\}$, since $A(w, w) = \|w\| \neq 0$ for all $w \neq 0$ in W.

Remark. Suppose V is a complex Banach space and the closed subspace W is a complex subspace. Then the projection P of norm one from V onto W, if it exists, is complex linear. That can be verified easily by noticing that W', the kernel of P, is a complex subspace. Alternatively, notice that $P'v = -iP(iv)$ defines a projection of norm one from V onto W. Since P is unique, $P = P'$.

6.3. The next proposition will be used in the proof of Theorem 5.4.

PROPOSITION. *Let $L: V \to V$ be a linear map of norm one. If V has finite dimension and the (closed) subspace*

$$W = \{v \in V;\ Lv = v\}$$

is non-trivial, there is a projection P of norm one from V onto W.

Proof. Choose any $w \neq 0$ in W and v in V. Since

$$f(t) = \|w+tv\| - \|w+tLv\|$$

has a minimum at $t=0$, we have

$$0 = f'(0) = A(w, v) - A(w, Lv) = A(w, v - Lv).$$

Hence $v - Lv \in W'$ for all v in V, so

$$\dim W' \geqslant \dim \text{image } (I-L).$$

But the kernel of $I-L$ is W, so

$$\dim V \geqslant \dim W + \dim W' \geqslant \dim \ker (I-L) + \dim \text{image } (I-L) = \dim V.$$

By the corollary to Proposition 6.2, P exists.

Remark. It is not hard to show directly that

$$P = \lim_{n\to\infty} \frac{1}{n+1} \quad (I + L + \ldots + L^n).$$

7. Proof of Theorem 5.4.

7.1. Let φ and ψ be L^1 functions on the unit disk Δ, holomorphic and nonzero for $z \neq 0$, and bounded except possibly in a small neighborhood of $z=0$. Let ν and μ be the orders of φ and ψ at zero, and note that $\nu, \mu \geqslant -1$. Define for real t

$$f(t) = \iint_\Delta |\varphi(z) + t\psi(z)|\, dx\, dy.$$

The following lemma, from [5], is a straightforward generalization of Lemma 1 of Royden [11].

LEMMA. *$f(t)$ is a differentiable function of t near $t=0$, and*

$$f'(0) = \iint_\Delta \operatorname{Re}[\psi(z)\,\bar{\varphi}(z)/|\varphi(z)|]\, dx\, dy.$$

Further, the second derivative $f''(0)$ exists if $\nu \leqslant 2\mu+1$. If $\nu > 2\mu+1$, there is a positive number c such that

$$f(t) = f(0) + tf'(0) + c\varepsilon(t) + o(\varepsilon(t)), \tag{7.1}$$

where

$$\varepsilon(t) = \begin{cases} t^2 \log(1/|t|) & \text{if} \quad \nu = 2\mu+2, \\ |t|^{1+(2+\mu)/(\nu-\mu)} & \text{if} \quad \nu > 2\mu+2. \end{cases}$$

The lemma implies that the norm (5.1) on $Q(X)$ is differentiable on $Q(X)\setminus\{0\}$ for any Riemann surface X of finite type, so the results of § 6 are applicable to the spaces $Q(X)$.

7.2. Now we turn to the proof of Theorem 5.4. Recall that X is a closed Riemann surface of genus $p \geqslant 2$, $X' = X\setminus\{x_0\}$, and $X'' = X\setminus\{y_1, \ldots, y_n\}$, where $y_1, \ldots, y_n$ are distinct points on X. We are given a linear map $L: Q(X') \to Q(X'')$ of norm one such that $L\varphi = \varphi$ if $\varphi \in Q(X)$.

7.3. Suppose $p \geqslant 3$. Let $\psi \in Q(X')$, and let y_k be a pole of $L\psi$. We shall prove that $y_k = x_0$. Choose φ in $Q(X)$ with a zero of order $m \geqslant 3p-4 > p$ at y_k, and notice that all other zeros of φ have order $\leqslant p < m$. Put

$$f_1(t) = \|\varphi + t\psi\| \quad \text{and} \quad f_2(t) = \|L(\varphi + t\psi)\| = \|\varphi + tL\psi\|.$$

Then Lemma 7.1 gives

$$f_2(t) = f_2(0) + tf_2'(0) + c_2|t|^{1+1/(m+1)} + o(|t|^{1+1/(m+1)}), \tag{7.2}$$

with $c_2 > 0$. Similarly, if $y_k \neq x_0$ we obtain

$$f_1(t) = f_1(0) + tf_1'(0) + O(|t|^{1+s}) \tag{7.3}$$

with $s = \min\{2/m, 1/(p+1)\}$. But $f_1(t) \geqslant f_2(t)$ for all t, and $f_1(0) = f_2(0)$. Hence $f_1'(0) = f_2'(0)$ and it follows from (7.2) and (7.3) that $s \leqslant 1/(m+1)$. That is impossible because $m > p$, so $y_k = x_0$ as we claimed.

We have proved that L maps $Q(X')$ into itself. Since $Q(X)$ has codimension one in $Q(X')$, the subspace of $Q(X')$ on which $L\varphi=\varphi$ is either $Q(X')$ or $Q(X)$. If L is not the identity on $Q(X')$, then Proposition 6.3 gives us a projection P of norm one from $Q(X')$ onto $Q(X)$. But X has genus $p\geqslant 3$, so no such projection exists, by Lemma 2 of Hubbard [8] (see [5, Lemma 4.9] for another proof). Therefore $L\varphi=\varphi$ for all φ in $Q(X')$, and $Q(X')\subset Q(X'')$.

7.4. Now suppose X has genus two. Suppose x_0 is not a Weierstrass point of X, since otherwise there is nothing to prove. Again Lemma 2 of Hubbard [8] says there is no projection $P: Q(X')\to Q(X)$ of norm one (see § 9.5 of this paper for another proof). It follows that there is a ψ in $Q(X')$ such that $L\psi$ has a pole at some point y_k. If y_k is a Weierstrass point, choose φ in $Q(X)$ with a zero of order four at y_k and no other zeros. Put $f_1(t)=\|\varphi+t\psi\|$ and $f_2(t)=\|\varphi+tL\psi\|$. Lemma 7.1 gives

$$f_1(t)=f_1(0)+tf_1'(0)+O(|t|^{3/2}),$$
$$f_2(t)=f_2(0)+tf_2'(0)+c_2|t|^{6/5}+o(|t|^{6/5})$$

with $c_2>0$. Again $f_1(0)=f_2(0)$, $f_1(t)\geqslant f_2(t)$ for all t, and $f_1'(0)=f_2'(0)$, so we arrive at a contradiction. We conclude that y_k is not a Weierstrass point of X.

Let $j: X\to X$ be the hyperelliptic involution. Choose φ in $Q(X)$ with double zeros at y_k and $j(y_k)$ and no other zeros. If x_0 is neither y_k nor $j(y_k)$, then defining $f_1(t)$ and $f_2(t)$ as above we have

$$f_1(t)=f_1(0)+tf_1'(0)+O(t^2\log(1/|t|)), \tag{7.4}$$
$$f_2(t)=f_2(0)+tf_2'(0)+c_2|t|^{4/3}+o(|t|^{4/3}) \tag{7.5}$$

with $c_2>0$. That is again impossible, so x_0 is either y_k or $j(y_k)$. The proof is complete.

8. Proof of Theorem 2.3.

8.1. Let X be a closed Riemann surface of genus $p\geqslant 2$, and H a non-trivial group of conformal automorphisms of X. Choose a holomorphic covering map $\pi: U\to X$. Let Γ be the group of cover transformations, and let Γ' be the group of lifts to U of the maps in H. That is,

$$\Gamma'=\{g: U\to U; \pi\circ g=h\circ\pi \quad \text{for some } h\in H\}.$$

Then Γ and Γ' are Fuchsian groups, Γ is a normal subgroup of Γ', and $\Gamma'/\Gamma=H$.

Let Y be the closed Riemann surface $U/\Gamma'=X/H$, and let Y' be Y minus the points over which the projection from X to Y is branched. Then (recall the definitions in § 3.3)

$$Y'=U_{\Gamma'}/\Gamma',$$

so Y' and Γ' have the same type (p', n'). We assume

$$2p' + n' > 4. \tag{8.1}$$

Since $\pi: U \to X$ is a covering map, the group Γ has type $(p, 0)$.

8.2. We define a right action of the group Γ' on the space $M(\Gamma)$ of Beltrami coefficients for Γ by

$$\mu \cdot g = (\mu \circ g)\bar{g}'/g', \quad \text{all} \quad \mu \in M(\Gamma), \quad g \in \Gamma'. \tag{8.2}$$

Γ' carries equivalence classes to equivalence classes, and the subgroup Γ acts trivially on $M(\Gamma)$. Therefore (8.2) induces a right action of H on the Teichmüller space $T(\Gamma)$ by

$$\Phi(\mu) \cdot \alpha(g) = \Phi(\mu \cdot g), \quad \text{all} \quad \mu \in M(\Gamma), \quad g \in \Gamma'. \tag{8.3}$$

Here $\alpha: \Gamma' \to H$ is the natural quotient homomorphism.

8.3. The action (8.3) of H on $T(\Gamma)$ is an easy special case of the action on $T(\Gamma)$ of the Teichmüller modular group (see § 3.1 of [5]). Bers has lifted the action of the modular group to the fiber space $V(\Gamma)$ in [3]. Here we shall describe how to lift the action of H. Our formulas differ slightly from those of Bers [3] because we are making H act from the *right*. First we convert the left action (3.3) of Γ on $F(\Gamma)$ to a right action by the usual device:

$$(\Phi(\mu), z) \cdot \gamma = \gamma^{-1}(\Phi(\mu), z) = (\Phi(\mu), (\gamma^{\mu})^{-1}(z)).$$

Next we extend that action to a right action of Γ' on $F(\Gamma)$ by

$$(\Phi(\mu), z) \cdot g = (\Phi(\mu \cdot g), (g^{\mu})^{-1}(z)). \tag{8.4}$$

Here $\mu \in M(\Gamma)$, $z \in w^{\mu}(U)$, $g \in \Gamma'$, and

$$g^{\mu} \circ w^{\mu \cdot g} = w^{\mu} \circ g.$$

The action (8.4) of Γ' on $F(\Gamma)$ induces an action of the quotient group $H = \Gamma'/\Gamma$ on the quotient space $V(\Gamma) = F(\Gamma)/\Gamma$. H acts as a group of biholomorphic maps (see [3]). Comparing (8.3) and (8.4) we see that h in H maps the fiber over τ onto the fiber over $\tau \cdot h$ for each τ in $T(\Gamma)$. If τ in $T(\Gamma)$ is fixed by the group H, then H acts as a group of conformal automorphisms of the fiber over τ. In particular, H acts on the fiber $X = U/\Gamma$ over $\Phi(0)$ by $x \cdot h = h^{-1}(x)$ for x in X and h in H, as it should.

Remark. Suppose Γ has type $(2, 0)$. Then $X = U/\Gamma$ is hyperelliptic, and the hyperelliptic involution generates a group H, of order two, of conformal automorphisms of X. Since all surfaces of genus two are hyperelliptic, H acts trivially on the Teichmüller space $T(2, 0)$. Its generator J acts as above on the fiber space $V(2, 0)$. J is a fiber preserving holomorphic involution of $V(2, 0)$. Let J operate on $T(2, n)\times V(2, 0)$ by

$$(t, x)\cdot J = (t, x\cdot J).$$

Then J maps the submanifold W, of Theorem 4.6, onto itself, so it defines a holomorphic involution of $V(2, n)$. That is the involution referred to in the final paragraph of § 2.2.

8.4. Since Γ is a subgroup of Γ', $M(\Gamma')$ is a subspace of $M(\Gamma)$ and $T(\Gamma')\subset T(\Gamma)$. Formula (8.2) shows that Γ' acts trivially on $M(\Gamma')$, so H fixes every point of $T(\Gamma')$.

PROPOSITION (Kravetz [10]). *The set of points in* $T(\Gamma)$ *left fixed by every member of* H *is precisely* $T(\Gamma')$.

The proposition is a simple consequence of Teichmüller's theorem about extremal quasiconformal maps.

8.5. For any Fuchsian group G we denote by $Q(G)$ the space of holomorphic quadratic differentials for G. Thus, $Q(G)$ consists of all holomorphic functions φ on U satisfying

$$\varphi(g(z))g'(z)^2 = \varphi(z), \quad \text{all} \quad g\in G, \quad z\in U.$$

The cotangent spaces to $T(\Gamma)$ and $T(\Gamma')$ at $\Phi(0)$ are $Q(\Gamma)$ and $Q(\Gamma')$ respectively. We want to find the map $\theta\colon Q(\Gamma)\to Q(\Gamma')$ between cotangent spaces induced by the inclusion map $i\colon T(\Gamma')\to T(\Gamma)$. According to § 1.2 of [5], θ is determined by

$$(\theta\varphi, \mu)_{\Gamma'} = (\varphi, \mu)_{\Gamma}, \quad \text{all} \quad \varphi\in Q(\Gamma), \quad \mu\in L^\infty(\Gamma'). \tag{8.5}$$

Here

$$(\varphi, \mu)_\Gamma = \iint_{U/\Gamma} \varphi(z)\,\mu(z)\,dx\,dy$$

and $(\theta\varphi, \mu)_{\Gamma'}$ has similar meaning.

Choose a fundamental polygon $D\subset U$ for Γ' and a complete set of inequivalent coset representatives $\{\gamma_1, ..., \gamma_N\}$ for Γ in Γ'. Then (8.5) gives

$$\begin{aligned}(\theta\varphi, \mu)_{\Gamma'} = (\varphi, \mu)_\Gamma &= \sum_{j=1}^{N} \iint_{\gamma_j(D)} \varphi(z)\,\mu(z)\,dx\,dy \\ &= \sum_{j=1}^{N} \iint_D \varphi(\gamma_j(z))\,\mu(\gamma_j(z))\,|\gamma_j'(z)|^2\,dx\,dy \\ &= \iint_{U/\Gamma'} \left(\sum_{j=1}^{N} \varphi(\gamma_j(z))\,\gamma_j'(z)^2\right)\mu(z)\,dx\,dy\end{aligned}$$

for all φ in $Q(\Gamma)$ and μ in $L^\infty(\Gamma')$. Therefore $\theta\colon Q(\Gamma)\to Q(\Gamma')$ is the relative Poincaré series

$$\theta\varphi(z) = \sum_{j=1}^{N} \varphi(\gamma_j(z))\,\gamma_j'(z)^2, \quad \text{all} \quad \varphi\in Q(\Gamma). \tag{8.6}$$

8.6. It is desirable to interpret the map (8.6) on the Riemann surfaces $X = U/\Gamma$ and $Y = U/\Gamma' = X/H$. Let $f: X \to Y$ be the quotient map, and let Y', as in § 8.1, be Y with the branch set deleted. Then $Q(\Gamma)$ is just the lift of $Q(X)$ to U, and $Q(\Gamma')$ is the lift of $Q(Y')$. The inclusion map of $Q(\Gamma')$ in $Q(\Gamma)$ corresponds to the map

$$\varphi \mapsto f^*\varphi$$

from $Q(Y')$ into $Q(X)$, and the map $\theta: Q(X) \to Q(Y')$ of (8.6) is given by

$$f^*\theta\varphi = \Sigma_{h \in H} h^*\varphi, \quad \text{all } \varphi \in Q(X). \tag{8.7}$$

Suppose $\varphi = f^*\psi$ for some ψ in $Q(Y')$. Then $h^*\varphi = \varphi$ for all h in H, so we have

$$\theta f^*\psi = N\psi, \quad \text{all } \psi \text{ in } Q(Y'), \tag{8.8}$$

where N is the order of the group H. Since N is the degree of the map f, we also have

$$\|f^*\psi\| = N\|\psi\|, \quad \text{all } \psi \text{ in } Q(Y'). \tag{8.9}$$

8.7. Now we are ready to begin the proof of Theorem 2.3. Let the group H operate on $T(p, 0)$ as in § 2.3, and let $s: T(p, 0)^H \to V(p, 0)$ be a holomorphic section of $\pi_0: V(p, 0) \to T(p, 0)$. Choose any τ in $T(p, 0)^H$, and let X be the Riemann surface $\pi_0^{-1}(\tau)$. Let $x_0 = s(\tau) \in X$. We must show that x_0 is fixed by a non-trivial member of H.

Choose a group Γ of type $(p, 0)$ so that $X = U/\Gamma$, and form the group Γ' of type (p', n') as in § 8.1. We represent $T(p, 0)$, $V(p, 0)$, and $T(p, 0)^H$ by $T(\Gamma)$, $V(\Gamma)$, and $T(\Gamma') = T(p', n')$, as in §§ 8.1 to 8.5. Let $i: T(p', n') \to T(p, 0)$ be the inclusion map. The cotangent spaces to $T(p', n')$, $V(p, 0)$, and $T(p, 0)$ at τ, $s(\tau)$, and $i(\tau)$ are $Q(Y')$, $Q(X')$, and $Q(X)$ respectively. (Of course $X' = X \setminus \{x_0\}$.) Let $L: Q(X') \to Q(Y')$ be the map of cotangent vectors induced by s. Then $\|L\varphi\| \leq \|\varphi\|$ for all φ in $Q(X')$, by Royden [11] and Kobayashi [9]. Further, since $\pi_0 \circ s = i$, and π_0 induces the inclusion of $Q(X)$ in $Q(X')$, we have $L\varphi = \theta\varphi$ for all φ in $Q(X)$, where θ is defined by (8.7). Hence Theorem 2.3 is an immediate corollary of the following.

THEOREM. *Let X, H, Y, and Y' be as above. Let $x_0 \in X$ and $X' = X \setminus \{x_0\}$. Suppose the linear map $L: Q(X') \to Q(Y')$ satisfies*

$$\|L\varphi\| \leq \|\varphi\|, \quad \textit{all } \varphi \textit{ in } Q(X') \tag{8.10}$$

and

$$L\varphi = \theta\varphi, \quad \textit{all } \varphi \textit{ in } Q(X), \tag{8.11}$$

where $\theta: Q(X) \to Q(Y')$ is defined by (8.7). *Then x_0 is fixed by a non-trivial element of H, provided that the type (p', n') of Y' satisfies* (8.1).

8.8. It remains to prove Theorem 8.7. We shall assume that x_0 is not fixed by any non-trivial element of H, and we shall reach a contradiction. Let $y_0 = f(x_0) \in Y$. Our assumption on x_0 means that y_0 belongs to Y'. Set $Y'' = Y' \setminus \{y_0\}$.

For any φ in $Q(X')$, define $\theta\varphi$ by (8.7). Then $\theta\varphi \in Q(Y'')$, and

$$\|\theta\varphi\| = \tfrac{1}{2}\int_Y |\theta\varphi| = \frac{1}{2N}\int_X |f^*\theta\varphi| \leqslant \frac{1}{2N}\sum_{h\in H}\int_X |h^*\varphi| = \|\varphi\|.$$

(Recall that N is the order of H.)

Choose φ_0 in $Q(X')$ with a pole at x_0. Then $L\varphi_0 \in Q(Y')$ and $f^*L\varphi_0 \in Q(X)$. Hence $\varphi_1 = \varphi_0 - N^{-1}f^*L\varphi_0$ is in the kernel of L, by (8.8) and (8.11). Further, φ_1 has a pole at x_0. It follows, by (8.7), that $\psi_1 = \theta\varphi_1$ has a pole at y_0; in particular, ψ_1 is different from zero.

Choose any $\psi \neq 0$ in $Q(Y')$ and put $g_1(t) = \|N^{-1}f^*\psi + t\varphi_1\|$. Lemma 7.1 guarantees that $g_1'(0)$ exists. From (8.10), (8.11), (8.8) and (8.9) we obtain

$$g_1(t) \geqslant \|L(N^{-1}f^*\psi + t\varphi_1)\| = \|\theta(N^{-1}f^*\psi)\| = \|\psi\| = g_1(0)$$

for all t, so $g_1'(0) = 0$.

Next put $g_2(t) = \|\psi + t\psi_1\|$. Then $g_1'(0)$ exists, and

$$g_2(t) = \|\theta(N^{-1}f^*\psi + t\varphi_1)\| \leqslant g_1(t)$$

for all t. But $g_2(0) = \|\psi\| = g_1(0)$, so $g_2'(0) = g_1'(0) = 0$.

We have just proved, in the notation of § 6.2, that ψ_1 belongs to the complementary subspace $Q(Y')'$ of $Q(Y')$ in $Q(Y'')$. Since ψ_1 has a pole at y_0, ψ_1 and $Q(Y')$ together span all of $Q(Y'')$. Hence, by the Corollary to Proposition 6.2, there is a projection P of norm one from $Q(Y'')$ onto $Q(Y')$. But, by Theorem 9.1 (a), no such projection P exists when the type (p', n') of Y' satisfies (8.1). We have reached the desired contradiction, and Theorem 8.7 is proved.

Remark. See § 11.1 for a generalization of Theorem 8.7.

8.9. We outline here a short alternate proof of Theorem 2.3, valid when $Y = X/H$ has genus at least three. In § 8.3 we saw that H acts on the fiber space $V(p, 0) = V(\Gamma)$. In $\pi_0^{-1}(T(p,0)^H) = \pi_0^{-1}(T(\Gamma'))$ we have

$$\pi_0^{-1}(T(\Gamma'))/H = F(\Gamma')/\Gamma' = V(p', n').$$

Let $I: \pi_0^{-1}(T(p,0)^H) \to V(p', n')$ be the holomorphic quotient mapping, and $i: T(p', n') \to T(p, 0)^H$ the inclusion map. If $s: T(p,0)^H \to V(p, 0)$ is a holomorphic section of π_0 over $T(p,0)^H$, then $I \circ s \circ i: T(p', n') \to V(p', n')$ is a holomorphic section of $\pi_{n'}: V(p', n') \to T(p', n')$. If $p' \geqslant 3$, then $I \circ s \circ i$ must be a canonical section, by Theorem 2.2. Theorem 2.3 follows at once.

9. Projections in $Q(Y'')$.

9.1. We shall prove the following theorem, most cases of which were proved in [5] and [8].

THEOREM. *Let Y be a closed Riemann surface of genus p, $Y' = Y\setminus\{y_1, ..., y_n\}$ a surface of type (p, n), and $Y'' = Y'\setminus\{y_0\}$ a surface of type $(p, n+1)$.*

(a) *If $2p+n>4$, there is no projection P of norm one from $Q(Y'')$ onto $Q(Y')$.*

(b) *If $(p, n)=(2, 0)$ or $(1, 2)$ there is no projection P of norm one from $Q(Y'')$ onto $Q(Y')$ unless y_0 is a Weierstrass point of Y'.*

We shall define the Weierstrass points of Y' in §§ 9.5 and 9.6, when we prove (b).

9.2. Lemma 4.9 of [5] proves the above theorem if the type (p, n) of Y' satisfies $3p+n>6$. To complete the proof of (a) we must consider types $(p, n)=(1, 3)$, $(0, 6)$, and $(0, 5)$.

Let Y' have type $(1, 3)$, and suppose a projection $P\colon Q(Y'')\to Q(Y')$ of norm one exists. Choose $\psi\neq 0$ in the kernel of P. The torus Y is the quotient of the complex plane $\mathbf{C}$ by a lattice subgroup L, so Y is an abelian group. The equation

$$3y = y_1+y_2+y_3 \tag{9.1}$$

has nine solutions on Y, and ψ has at most four zeros in Y''. Therefore (9.1) has a solution y in Y'' such that ψ is not zero at y. By Abel's theorem there is φ in $Q(Y')$ with a triple zero at y, simple poles at y_1, y_2, and y_3, and no other zeros or poles in Y.

Choose $\psi_1\neq 0$ in $Q(Y)$ and notice that ψ_1 has no zeros or poles. Finally, choose a complex number α so that $\psi_2=\psi_1+\alpha\psi$ is zero at y.

Set $f_1(t)=\|\varphi+t\psi_1\|=\|P(\varphi+t\psi_2)\|$ and $f_2(t)=\|\varphi+t\psi_2\|$. Using Lemma 7.1 we find

$$f_1(t) = f_1(0)+tf_1'(0)+c_1|t|^{5/3}+o(|t|^{5/3}),$$
$$f_2(t) = f_2(0)+tf_2'(0)+O(t^2\log(1/|t|)),$$

with $c_1>0$. But $f_1(0)=f_2(0)$, and $f_1(t)\leq f_2(t)$ for all t, so $f_1'(0)=f_2'(0)$. This leads to a contradiction and we conclude that the projection P cannot exist.

9.3. Now suppose Y' has type $(0, 6)$. If P exists, choose $\psi\neq 0$ in its kernel. ψ has at least four poles on the Riemann sphere Y, so it has a simple pole at some y_k, $1\leq k\leq 6$. Choose φ in $Q(Y')$ with a simple zero at y_k, simple poles at the other punctures of Y', and no other zeros or poles. Next choose ψ_1 in $Q(Y')$ with a simple pole at y_k, and choose a complex number α so that $\psi_2=\psi_1+\alpha\psi$ is regular at y_k. Define $f_1(t)$ and $f_2(t)$ as in § 9.2.

Again $f_1(0)=f_2(0)$, $f_1(t)\leqslant f_2(t)$ for all t, and $f_1'(0)=f_2'(0)$. But Lemma 7.1 gives

$$f_1(t)=f_1(0)+tf_1'(0)+c_1|t|^{3/2}+o(|t|^{3/2}),$$
$$f_2(t)=f_2(0)+tf_2'(0)+O(t^2\log(1/|t|)).$$

Again we have a contradiction, so P cannot exist.

9.4. Now let Y' have type $(0, 5)$. In this case arguments based on Lemma 7.1 fail, and we use the method, based on ideas of Hubbard, that we sketched in § 4.10 of [5]. Assume that $P\colon Q(Y'')\to Q(Y')$ exists, and choose $\psi\neq 0$ in its kernel. Then $\psi\notin Q(Y')$, so ψ has a pole at y_0, the extra puncture of Y''. Choose φ_1 in $Q(Y')$ with a simple zero at y_0, and φ_2 in $Q(Y')$ with no zeros in Y.

By formula (4.10.1) of [5], we have

$$f(t)=\int_Y \psi[(\bar\varphi_1+t\bar\varphi_2)/|\varphi_1+t\varphi_2|]=0 \tag{9.2}$$

for all real t. Of course it follows that $f'(0)$ exists and equals zero. We shall obtain a contradiction by showing that $f'(0)$ does not exist.

The function φ_1/φ_2 serves as a local coordinate on Y at y_0. We choose a closed neighborhood D of y_0 in Y' such that φ_1/φ_2 maps D homeomorphically onto the closed disk $\{z\in\mathbf{C};\ |z|\leqslant r\}$. Write $f(t)=f_1(t)+f_2(t)$ where f_1 is the integral over D and f_2 is the integral over $Y\backslash D$. Since φ_2/φ_1 is bounded in $Y\backslash D$, $f_2'(0)$ exists. In fact differentiation under the integral sign is easily justified by the dominated convergence theorem, with the help of the inequality

$$\left|\frac{z+w}{|z+w|}-\frac{z}{|z|}\right|\leqslant 2|w/z|,\quad \text{all } z,\ w\in\mathbf{C} \text{ with } z\neq 0,\ w\neq -z. \tag{9.3}$$

In terms of the local coordinate $z=\varphi_1/\varphi_2$ on D, we can write ψ as $\psi(z)\,dz^2$, φ_2 as $\theta(z)\,dz^2$, and φ_1 as $z\theta(z)\,dz^2$, where $\theta(z)$ is holomorphic and nonzero in $\{z;\ |z|\leqslant r\}$, and $\psi(z)$ is meromorphic with a simple pole at $z=0$. The integral over D becomes

$$f_1(t)=\iint_{|z|\leqslant r}\psi(z)[(\bar z+t)/|z+t|]\,[\bar\theta(z)/|\theta(z)|]\,dx\,dy. \tag{9.4}$$

The dominated convergence theorem and inequality (9.3) imply that

$$g(t)=\iint_{|z|\leqslant r}\varrho(z)[(\bar z+t)/|\bar z+t|]\,dx\,dy$$

is differentiable at $t=0$ if $|\varrho(z)/z|$ is integrable in $\{z;\ |z|\leqslant r\}$. Hence, the differentiability at $t=0$ of (9.4) is not altered if we replace $\theta(z)$ by $\theta(0)$ and $\psi(z)$ by its principal part at $z=0$. Define $h(t)$ by

$$h(t)=\iint_{|z|\leqslant r}z^{-1}[(\bar z+t)/|\bar z+t|]\,dx\,dy\quad \text{all } |t|<r. \tag{9.5}$$

Then $f'(0)$ exists if and only if $h'(0)$ does. The following lemma gives the desired contradiction.

LEMMA. *Define $h(t)$ by (9.5). Then $h'(0)$ does not exist.*

Proof. Let $w(z)=\frac{2}{3}|z+t|^3/(z+t)^2z$. Then $h(t)$ is the integral of $w_{\bar z}$ over $\{z;\ |z|\leqslant r\}$. By Stokes' theorem

$$h(t)= -\tfrac{2}{3}\pi|t|+\frac{1}{3i}\int_{|z|=r}[|z+t|^3/(z+t)^2z]\,dz.$$

The line integral is a smooth function of t in the interval $-r<t<r$. Hence $h'(0)$ does not exist. The proof of the lemma, and of Theorem 9.1 (a), is complete.

9.5. It remains to prove Theorem 9.1 (b). First we consider type (2, 0). It is convenient to change our notation. Let X be a closed Riemann surfaces of genus two, and $X'=X\setminus\{x_0\}$. Let $j\colon X\to X$ be the hyperelliptic involution, and H the group of order two generated by j. The Weierstrass points of X are the six fixed points of j. Let X'' be X minus the six Weierstrass points. Set $Y'=X''/H$. Then Y' has type (0, 6).

Suppose $P\colon Q(X')\to Q(X)$ is a projection of norm one. Define $\theta\colon Q(X)\to Q(Y')$ by (8.7) and $L\colon Q(X')\to Q(Y')$ by $L\varphi=\theta P\varphi$. Then L satisfies (8.10) and (8.11), so Theorem 8.7 says that x_0 is a fixed point of j, as required. We may use Theorem 8.7 here because only Theorem 9.1 (a) was used in its proof.

9.6. Now let X have type (1, 2). Represent $\bar X$ as $\mathbf{C}$ modulo a lattice subgroup L, generated by 1 and τ with $\operatorname{Im}\tau>0$. Without loss of generality we take 0 to be one puncture of X and $a\notin L$ the other. The map $z\mapsto -z+a$ on $\mathbf{C}$ induces an involution j on X. Let X'' be X minus the four fixed points of j, and let H be the group of order two generated by j. The quotient surface $Y'=X''/H$ has type (0, 5). By analogy with the (2, 0) situation we call j the *hyperelliptic involution* of X and its four fixed points the *Weierstrass points* of X. It is clear that j is the unique involution of X with four fixed points.

Let $P\colon Q(X')\to Q(X)$ be a projection of norm one. Again define $\theta\colon Q(X)\to Q(Y')$ by (8.7) and $L\colon Q(X')\to Q(Y')$ by $L\varphi=\theta P\varphi$. Our proof of Theorem 8.7 remains valid in this situation, and we conclude once more that x_0 is fixed by j. The proof of Theorem 9.1 is complete.

10. Generalization of Theorem 2.2.

10.1 In this section we shall indicate the proof of a generalized version of Theorem 2.2, stated in § 10.3. First we extend Theorem 5.4 to surfaces of finite type. A Riemann surface and its type (p, n) are called *exceptional* if $2p+n\leqslant 4$.

THEOREM. *In Theorem 5.4, replace the assumption that X is a closed surface of genus $p \geqslant 2$ by the assumption that X is of finite type (p, n) satisfying* (2.1).

(a) *If X is non-exceptional, then $x_0 = y_k$ for some k, and $L\varphi = \varphi$ all φ in $Q(X')$.*

(b) *If X is of type* (2, 0) *or* (1, 2), *then either $x_0 = y_k$ for some k, $j(x_0) = y_k$ for some k, or x_0 is a Weierstrass point (where j is the hyperelliptic involution on X).*

Proof. Assume $2p+n>4$. Let $\psi \in Q(X')\backslash Q(X)$, and let y_k be a pole of $L\psi$ with $y_k \in X$. We must prove $y_k = x_0$. Choose a $\varphi \in Q(X)$ with a zero of order $m \geqslant 3p-4+n>0$ at y_k, and notice that all other zeros of φ have order $\leqslant p<m$. Define functions f_1 and f_2 as in § 7.3. The function f_2 is given by (7.2); and if $y_k \neq x_0$, then

$$f_1(t) = f_1(0) + tf_1'(0) + O(|t|^{1+s} \log (1/|t|))$$

with $s = \min \{2/m, 1/(p+1), 1\}$. (The log term is needed only when $s=1$. It is harmless in general.) As before, $f_1(t) \geqslant f_2(t)$, and we conclude $s \leqslant 1/(m+1)$. This contradiction establishes the claim that $y_k = x_0$.

The rest of the arguments proceed as in § 7.3 using Theorem 9.1 instead of Lemma 2 of Hubbard [8].

10.2. Now assume that X is of type (1, 2) and x_0 is not a Weierstrass point of X. Theorem 9.1 says there is no projection $P: Q(X') \to Q(X)$ of norm one. It follows that there is a $\psi \in Q(X')$ such that $L\psi$ has a pole at some point y_k. If y_k is a Weierstrass point, choose $\varphi \in Q(X)$ with a double zero at y_k (and no other zeros and poles in X). Define f_1 and f_2 as in § 7.4, and conclude that f_1 and f_2 satisfy (7.4) and (7.5) respectively.

As in § 7.4, we arrive at a contradiction; so that y_k is not a Weierstrass point of X.

Now we choose $\varphi \in Q(S)$ with simple zeros at y_k and $j(y_k)$. If $x_0 \neq y_k$ and $x_0 \neq j(y_k)$, then with the same definitions of f_1 and f_2 we have f_1 satisfying (7.4) and f_2 satisfying

$$f_2(t) = f_2(0) + tf_2'(0) + c_2|t|^{3/2} + o(|t|^{3/2})$$

with $c_2 > 0$. This is again impossible, completing the proof.

10.3. The previous theorem has, of course, applications to the problem of cross sections of $\pi_n: V(p, n) \to T(p, n)$, for $p \leqslant 2$. Using the methods of § 3–7 we can establish the following result.

THEOREM. (a) *The Teichmüller curve $\pi_n: V(2, n) \to T(2, n)$, $n \geqslant 1$, has precisely $n-1$ holomorphic sections disjoint from s_1.*

(b) *The Teichmüller curve $\pi_n: V(1, n) \to T(1, n)$, $n \geqslant 3$, has precisely $n-3$ holomorphic sections disjoint from s_1, s_2, s_3.*

(b′) *The Teichmüller curve:* π_n: $V(1, n) \to T(1, n)$, $n \geq 2$, *has precisely* $2(n-2)+4$ *holomorphic sections disjoint from* s_1, s_2.

(c) *The Teichmüller curve* π_n: $V(0, n) \to T(0, n)$, $n \geq 5$, *has precisely* $n-5$ *holomorphic sections disjoint from* s_1, s_2, s_3, s_4, s_5.

Outline of Proof. Let us define the spaces

$$V(p,n)_k = V(p,n) \setminus \bigcup_{j=1}^{k} s_j(T(p,n)) \quad \text{for } 0 \leq k \leq n.$$

Thus, $V(p, n)_0 = V(p, n)$ and $V(p, n)_n = V(p, n)'$ in the notation of [5]. We are interested in finding all holomorphic sections of

$$\pi_n\colon V(p, n)_k \to T(p, n).$$

We, of course, have the $n-k$ canonical sections $s_{k+1}, \ldots, s_n$.

10.4. Let Γ be a Fuchsian group of type (p, n). We assume that Γ has precisely k conjugacy classes of parabolic elements and $(n-k)$ conjugacy classes of elliptic elements. The constructions of § 3 apply to this situation and we obtain the space $V(\Gamma)$. We show in the next section that $V(\Gamma)$ is $V(p, n)_k$.

10.5. Choose a compact Riemann surface X of genus $p \geq 0$. Choose n distinct points on X: $x_1, \ldots, x_n$. Assume (2.1) holds. Choose Fuchsian groups Γ and Γ' of type (p, n) such that $U/\Gamma = X$ and $U/\Gamma' = X' = X \setminus \{x_1, \ldots, x_k\}$, $k \leq n$. Let π: $U \to X$ and π': $U \to X'$ be the projection maps associated with the groups Γ and Γ'. Furthermore, we assume that $\pi^{-1}(x_j)$ has the same ramification number as $\pi'^{-1}(x_j)$ for $j = k+1, \ldots, n$. Because of this last assumption, there is a holomorphic mapping h: $U \to U$ such that

$$\pi' = \pi \circ h.$$

As in § 4.1, there is a homomorphism θ: $\Gamma' \to \Gamma$ satisfying (4.2).

LEMMA. $h(U) = \pi^{-1}(X')$ *(and is thus dense and open in* U*) and* θ *is surjective.*

Proceeding as in § 4.2–4.4, we obtain holomorphic mappings

$$f\colon T(\Gamma') \to T(\Gamma) \quad \text{and} \quad G\colon F(\Gamma') \to F(\Gamma).$$

f is the Bers–Greenberg [4] isomorphism, and the mapping G induces a holomorphic injective mapping g of $V(\Gamma') = F(\Gamma')/\Gamma'$ into $V(\Gamma) = F(\Gamma)/\Gamma$. It is easy to check that the image of g is precisely $V(\Gamma) \setminus \bigcup_{j=1}^{k} s_j(T(\Gamma))$, where s_j is the section determined by the puncture x_j. Therefore $V(\Gamma')$ is indeed $V(p, n)_k$.

10.6. The straightforward generalization of Theorem 4.5 is the following result.

THEOREM. *Let* (p, n) *satisfy* (2.1). *Choose* k *so that* $0 \leqslant k \leqslant n$ *and* $2p+k>2$. *There exist a "forgetful map"* $f: T(p, n) \to T(p, k)$ *and a holomorphic mapping* $g: V(p, n)_k \to V(p, k)'$ *such that the diagram*

$$\begin{tikzcd} V(p,n)_k \arrow[r, "g"] \arrow[d, "\pi_n"] & V(p,k)' \arrow[d, "\pi_k"] \\ T(p,n) \arrow[r, "f"] & T(p,k) \end{tikzcd}$$

commutes and g *maps each fiber* $\pi_n^{-1}(t)$, $t \in T(p, n)$, *one-to-one onto the fiber* $\pi_k^{-1}(f(t))$.

10.7. As in § 4.6 we find that holomorphic sections $s: T(p, n) \to V(p, n)_k$ of π_n correspond to holomorphic maps $h: T(p, n) \to V(p, k)'$ such that $\pi_k \circ h = f$. The isomorphism theorem of Bers [3] implies that the holomorphic universal covering space of $V(p, k)'$ is $T(p, k+1)$, so the arguments of § 5 can be repeated to obtain Theorem 10.3 from Theorem 10.1. The extra sections in Theorem 10.3 (b′) are obtained from the Weierstrass points and hyperelliptic involution for type (1, 2), just as in genus two.

11. Generalization of Theorem 2.3.

11.1. We begin with a theorem that includes both Theorem 8.7 and Theorem 10.1 as special cases. Let X be a Riemann surface of type (p, n) satisfying (2.1), and let H be a (necessarily finite) group of conformal automorphisms of X. (H is a group of automorphisms of $\bar{X}$ which permute the punctures of X.) As in § 8, set $Y = X/H$, let $f: X \to Y$ be the quotient map, and let Y' be Y with the branch set deleted. Recall that the inverse image of the branch set is the set of points in X fixed by non-trivial elements of H. Define $\theta: Q(X) \to Q(Y')$ by (8.7).

THEOREM. *Let* X, H, Y, *and* Y' *be as above. Let* $x_0 \in X$ *and* $X' = X \setminus \{x_0\}$. *Let* $Y'' = Y' \setminus \{y_1, ..., y_m\}$. *Suppose the linear map* $L: Q(X') \to Q(Y'')$ *satisfies*

$$\|L\varphi\| \leqslant \|\varphi\|, \quad \text{all } \varphi \text{ in } Q(X')$$

and

$$L\varphi = \theta\varphi, \quad \text{all } \varphi \text{ in } Q(X).$$

(a) *If* Y' *is not exceptional, then* $f(x_0) \notin Y''$.

(b) *If* Y' *has type* (2, 0) *or* (1, 2), *then* $f(x_0)$ *is a Weierstrass point of* Y', $f(x_0) \notin Y''$, *or* $j(f(x_0)) \notin Y''$, *where* $j: Y' \to Y'$ *is the hyperelliptic involution.*

To obtain Theorem 10.1 let H be the trivial group. To obtain Theorem 8.7 let X be a closed surface and $\{y_1, ..., y_m\}$ the empty set.

11.2. Since θ maps $Q(X)$ onto $Q(Y')$, and $Q(X)$ has codimension one in $Q(X')$, the image of L must either equal $Q(Y')$ or contain $Q(Y')$ as a subspace of codimension one. In the first case, Theorem 11.1 is proved by repeating the argument of § 8.8 verbatim, except that if Y' has type $(2, 0)$ or $(1, 2)$ Theorem 9.1 (b) is used in place of Theorem 9.1 (a). The second case is more difficult. We shall outline the proof briefly, under the assumption that the surface Y' (of type (p', n')) is not exceptional.

Choose a $\psi_0 \in Q(X')$ such that $L\psi_0 \notin Q(Y')$. Let $y_k \in Y'$ be a pole of $L\psi_0$. Then $L\psi$ has a pole at y_k for every $\psi \in Q(X')\backslash Q(X)$. We shall prove that $f(x_0) = y_k (\notin Y'')$. As usual our method is to assume $f(x_0) \neq y_k$ and look for a contradiction.

Choose a $\varphi \in Q(Y')$ such that φ has a zero of order $m' \geqslant 3p' - 4 + n' > p' \geqslant 0$ at y_k. (All the other zeros of φ have order $\leqslant p'$.) Note that $f^*\varphi \in Q(X)$ and that $f^*\varphi$ has a zero of order m' at the N distinct points (N = order H) in $f^{-1}(y_k)$. Write $z_1 = y_k$, $z_2 = f(x_0)$, and let $z_3, z_4, \ldots$ be the (finitely many) remaining points in $\overline{Y}$ which are either punctures of Y'' or zeros or poles of φ. Let R_j be the order of φ at z_j. Then the order r_j of $f^*\varphi$ at each point in $f^{-1}(z_j)$ is

$$r_j = \nu_j(R_j + 2) - 2,$$

where ν_j is the order of the subgroup of H fixing the point. Note that the points $f^{-1}(z_j)$ account for all the zeros and poles of $f^*\varphi$.

By the Riemann-Roch theorem, there is a $\psi \in Q(X')$ such that

(a) ψ has a pole at x_0, and

(b) the order of ψ at each point in $\mathrm{f}^{-1}(z_j)$, $j \neq 1, 2$, is $\geqslant \frac{1}{2}(r_j - 1)$.

By (a), $L\psi$ has a pole at y_k. Now set $f_2(t) = \|\varphi + tL\psi\|$ and $f_1(t) = \|N^{-1}f^*\varphi + t\psi\|$, so that $f_2(t) = \|L(N^{-1}f^*\varphi + t\psi)\| \leqslant f_1(t)$, and $f_2(0) = f_1(0)$. As usual, Lemma 7.1 leads to the desired contradiction.

11.3. Next we extend the considerations of §§ 8.1–8.4 to surfaces with punctures. Let X be a closed Riemann surface of genus $p \geqslant 2$, and let Γ be a (finitely generated) Fuchsian group of type (p, n) such that $U/\Gamma = X$. As usual we denote by U_Γ the complement in U of the set of elliptic fixed points of Γ. Then

$$X'' = U_\Gamma/\Gamma$$

is a surface of type (p, n). We require all elliptic transformations in Γ to have the same order, say two. Let H be a non-trivial group of conformal automorphisms of X''. Since every $h \in H$ can be extended to an automorphism of X that permutes the punctures, it can be lifted to U, and we may form the group Γ' as in § 8.1. Let

$$Y = X/H = U/\Gamma'$$

and let $f: X \to Y$ be the quotient map. We let Y' be Y minus the branch set of f, and we set

$$Y'' = Y' \cap f(X'') = U_{\Gamma'}/\Gamma'.$$

We denote the type of Y'' and Γ' by (p', n''). As in §§ 8.2–8.4, the group H acts on $T(\Gamma) = T(p, n)$ and $V(\Gamma) = V(p, n)$, and the fixed point set of H in $T(\Gamma)$ is $T(\Gamma') = T(p', n'')$.

11.4. Let f_n: $T(p, n) \to T(p, 0)$ be the forgetful map of § 4, and i: $T(p', n'') \to T(p, n)^H$ the (surjective) inclusion map. The following result extends the results of § 4.6.

PROPOSITION. *For $p \geq 2$, the holomorphic sections s: $T(p, n)^H \to V(p, n)$ of π_n are in bijective correspondence with the holomorphic maps h: $T(p', n'') \to V(p, 0)$ such that $\pi_0 \circ h = f_n \circ i$.*

The proposition follows immediately from Theorem 4.6, since we can identify $\pi_n^{-1}(T(p, n)^H)$ with the complex manifold

$$\{(t, x) \in T(p', n'') \times V(p, 0);\ f_n(i(t)) = \pi_0(x)\}.$$

It is important to notice that the group H acts on $T(p, 0)$ as well as $T(p, n)$ since H is a group of automorphisms of the closed surface X as well as the punctured surface X''. The image of $T(p, n)^H$ under f_n is precisely $T(p, 0)^H$. As in § 8, $T(p, 0)^H$ can be identified with $T(p', n')$, where (p', n') is the type of the punctured surface Y' defined in § 11.3. The connections among all these spaces are displayed in the following commutative diagram:

$$\begin{tikzcd} & \pi_n^{-1}(T(p,n)^H) \arrow[r, "g_n"] \arrow[d, "\pi_n"] & \pi_0^{-1}(T(p,0)^H) \arrow[d, "\pi_0"] & \\ T(p',n'') \arrow[r, "i"] & T(p,n)^H \arrow[r, "f_n"] & T(p,0)^H & T(p',n') \arrow[l, "i_0"'] \end{tikzcd} \tag{11.1}$$

11.5. We shall now describe some holomorphic sections s: $T(p, n)^H \to V(p, n)$ of π_n. We shall assume that the type (p', n') of Y' satisfies

$$3p' - 3 + n' \geq 2.$$

Notice that $3p' - 3 + n'$ is the complex dimension of the fixed point set $T(p, 0)^H$.

As in § 11.3, represent $T(p, n)$ and $T(p, n)^H$ by $T(\Gamma)$ and $T(\Gamma')$ respectively. Each elliptic transformation in Γ' determines a holomorphic section s, as in § 3.5. We call these the *fixed point sections*. If $n > 0$ the restrictions of the canonical sections s_j: $T(p, n) \to V(p, n)$ to $T(p, n)^H$ are among the fixed point sections.

If Y' has type (2, 0) or (1, 2) there are additional sections. We describe these in §§ 11.6 and 11.7.

11.6. Suppose first that Y'' (and Γ') have type (2, 0). Then there is a Fuchsian group $\Gamma''\supset\Gamma'$, of type (0, 6), such that $T(p, n)^H\cong T(\Gamma')=T(\Gamma'')$. The elliptic transformations in Γ'' determine holomorphic sections as above. We call the new ones (those not already determined by elliptic elements of Γ') *Weierstrass sections.* More generally, let Y' have type (2, 0). Since Y' plays the same role for the closed surface X that Y'' plays for X'', we obtain Weierstrass sections s: $T(p, 0)^H\to V(p, 0)$. For every such sections, the map $h=s\circ f_n\circ i$: $T(p', n'')\to V(p, 0)$ determines a Weierstrass section of π_n: $\pi_n^{-1}(T(p, n)^H)\to T(p, n)^H$, by Proposition 11.4. Our two methods of construction agree if Y'' (and therefore Y') has type (2, 0). There is precisely one Weierstrass section for each point $x_0\in X$ such that $f(x_0)$ is a Weierstrass point of Y'.

Finally, if Y' has type (2, 0), then in many instances (for example, with commutative groups H), the hyperelliptic involution of Y' lifts to X. As in § 8.3 we obtain, in this case, a holomorphic involution J of $V(p, 0)$ that maps each fiber over $T(p, 0)^H$ onto itself. The map $(t, x)\to(t, J(x))$ of $T(p', n'')\times V(p, 0)$ onto itself defines a holomorphic involution J of $\pi_n^{-1}(T(p, n)^H)$. If s: $T(p, n)^H\to V(p, n)$ is a holomorphic section of π_n, then so is $J\circ s$. If s is a Weierstrass section, then $J\circ s=s$, but if s is a fixed point section, $J\circ s$ is a new section.

C. H. Sah (oral communication to one of the authors) has constructed a wide class of examples where the hyperelliptic involution of Y' (of type (2, 0)) does not lift to X. In these cases the extra sections of the preceding paragraph must be constructed in a slightly different manner. Lemma 4.4 showed that over each Teichmüller space $T(\Gamma)$ we have a two fiber spaces: $V(\Gamma)$ and $F(\Gamma)$. The manifold $F(\Gamma)$ depends not only on the type of Γ but on the signature of Γ. For our purposes (unless otherwise indicated) this dependence on signature may be ignored, and we denote by $F(p, n)$, a Bers fiber space over $T(p, n)$. corresponding to a group Γ of type (p, n). In addition to the commutative diagram (4.7), we also have the commutative diagram

$$\begin{tikzcd} F(p,n) \arrow[r, "G_n"] \arrow[d, "\varrho_n"'] & F(p,0) \arrow[d, "\varrho_0"] \\ T(p,n) \arrow[r, "f_n"] & T(p,0) \end{tikzcd} ; \tag{11.2}$$

here ϱ_n, ϱ_0 are the canonical projections and G_n is holomorphic and surjective but *not* injective on the fibers if $n>0$. Since $\varrho_0^{-1}(T(p, 0)^H)$ is a holomorphic universal cover of $\pi_0^{-1}(T(p, 0)^H)$, it follows easily from (11.1) and (11.2) that

$$W = \{(t, x)\in T(p, n)^H\times\varrho_0^{-1}(T(p, 0)^H);\ f_n(t)=\varrho_0(x)\}$$

is a holomorphic universal cover of $\pi_n^{-1}(T(p, n)^H)$. The hyperelliptic involution of Y' certainly lifts to an involution J of $\varrho_0^{-1}(T(p, 0)^H)$ that acts in a fiber preserving way. Thus J also acts in the usual manner on W: $(t, x) \to (t, J(x))$. Every section s: $T(p, n)^H \to V(p, n)$ of π_n lifts to a mapping from $T(p, n)^H$ into W. This mapping may be composed with J and projected to a section "$J \circ s$" of π_n. If s is a Weierstrass section, "$J \circ s$"(τ) projects to a Weierstrass point on $\pi_n^{-1}(\tau)$ modulo H. Further, it could happen that for some fixed point sections s, "$J \circ s$" is again a fixed point section. The sections $J \circ s$ and "$J \circ s$" will be described henceforth as the section s *composed* with J.

11.7. Suppose now that Y'' has type $(1, 2)$. The Fuchsian group Γ' (also of type $(1, 2)$) need not be contained in any Fuchsian group of type $(0, 5)$. Such a normal extension never exists if Γ' has signature $(1, 2; \mu, \nu)$ with $\mu \neq \nu$ (see Singerman [13]).

This signature for Γ' can occur even when Γ has type $(p, 0)$, since every finitely generated Fuchsian group of the first kind contains a torsion free normal subgroup of finite index (see Selberg [12] or Zieschang–Vogt–Coldewey [14]).

Thus even the construction of the Weierstrass sections of § 11.6 does not work in this case. To get around this difficulty, we represent $T(p, n)^H \cong T(\Gamma') = T(1, 2)$ by $T(\tilde{\Gamma}')$ for a group $\tilde{\Gamma}'$ of signature $(1, 2; \infty, \infty)$. This is possible by the Bers–Greenberg isomorphism theorem [4]. Now there will be a group $\tilde{\Gamma}'' \supset \tilde{\Gamma}'$ of type $(0, 5)$. We must recall a construction of the group $\tilde{\Gamma}'$. Let h: $U \to U_{\Gamma'}$, be a holomorphic universal covering map. As in § 4.1, there then exists a surjective homomorphism θ: $\tilde{\Gamma}' \to \Gamma'$ satisfying (4.2) for all $\gamma \in \tilde{\Gamma}'$. Let $\tilde{\Gamma} = \theta^{-1}(\Gamma)$. Then $\tilde{\Gamma}$ is a normal subgroup of $\tilde{\Gamma}'$. Further, $U/\tilde{\Gamma} \cong U_{\Gamma'}/\Gamma$. From this it follows that $F(\tilde{\Gamma}')$ is a holomorphic universal covering space of $\pi_n^{-1}(T(p, n)^H)$ minus the images of the fixed point sections (those constructed in § 11.5). (This construction is analogous to those in §§ 10.4 and 10.5.) Since $\tilde{\Gamma}''$ acts in a fiber preserving way on $F(\tilde{\Gamma}')$, the elliptic elements of $\tilde{\Gamma}''$ not in $\tilde{\Gamma}'$ produce *Weierstrass sections* as in § 11.6. If Y' has type $(1, 2)$, then the Weierstrass sections of π_0: $\pi_0^{-1}(T(p, 0)^H) \to T(p, 0)^H$ induce Weierstrass sections of π_n: $\pi_n^{-1}(T(p, n)^H) \to T(p, n)^H$ by Proposition 11.4.

The considerations of the above paragraph have also shown that the hyperelliptic involution of Y' (of type $(1, 2)$) lifts to an involution J of a universal covering space of $\pi_0^{-1}(T(p, 0)^H)$ minus the images of the fixed point sections, and to the corresponding space over $T(p, n)^H$. This allows us to construct the section J *composed* with s for every section s: $T(p, n)^H \to V(p, n)$, except for those fixed point sections whose images were deleted. (Using Proposition 11.4 we can describe these exceptional sections as the ones which are determined by holomorphic maps $h = s \circ f_n \circ i$: $T(p', n'') \to V(p, 0)$, where s: $T(p, 0)^H \to V(p, 0)$ is a fixed point section.)

11.8. Let us return briefly to the commutative diagram (11.1). The mapping f of X onto Y induces a surjective holomorphic mapping $f: \pi_n^{-1}(T(p, n)^H) \to V(p', n'')$ such that

$$\begin{array}{ccc} V(p', n'') & \xleftarrow{\ f\ } & \pi_n^{-1}(T(p,n)^H) \\ \downarrow{\scriptstyle \pi_{n''}} & & \downarrow{\scriptstyle \pi_n} \\ T(p', n'') & \xrightarrow{\ i\ } & T(p,n)^H \end{array}$$

commutes. It is important to notice that we have constructed a section s: $T(p,n)^H \to V(p,n)$ of π_n: $V(p, n) \to T(p, n)$ with $s(\tau_0) = x_0$ for all $\tau_0 \in T(p, n)^H$ and $x_0 \in V(p, n)$ whenever

(i) the type (p', n') is non-exceptional, and $f(x_0)$ is a puncture on $\pi_{n''}^{-1}(i^{-1}(\tau_0))$,

or

(ii) the type (p', n') is exceptional and $f(x_0)$ is a Weierstrass point on $\pi_{n''}^{-1}(i^{-1}(\tau_0))$, or $f(x_0)$ is a puncture on this surface, or $j(f(x_0))$ has this property, where j is hyperelliptic involution on this surface.

11.9. Now we are ready to generalize Theorem 2.3.

THEOREM. *Let s: $T(p, n)^H \to V(p, n)$ be a holomorphic section of π_n: $V(p, n) \to T(p, n)$, $p \geq 2$. If $2p' + n' > 4$, then s is one of the fixed point sections. If $(p', n') = (2, 0)$ or $(1, 2)$, then s is a fixed point section, a Weierstrass section, or one of these sections composed with J.*

We outline the proof for the case $2p' + n' > 4$. By the methods of § 5.5, it suffices to show that s agrees with a fixed point section at each point in $T(p, n)^H$.

Let g_n: $V(p, n) \to V(p, 0)$ be the map of Theorem 4.5, and let $h = g_n \circ s \circ i$: $T(p', n'') \to V(p, 0)$. Then h satisfies the condition $\pi_0 \circ h = f_n \circ i$ of Proposition 11.4. Let $t \in T(p', n'')$. Set $x_0 = h(t) \in V(p, 0)$ and $\tau = \pi_0(x_0) = f_n(i(t)) \in T(p, 0)^H$. Let $X = \pi_0^{-1}(\tau)$ be the surface represented by τ, and put $X' = X \setminus \{x_0\}$. Finally, let $X'' = X \setminus \{x_1, ..., x_n\}$ be the surface represented by $i(t)$. The group H is a group of automorphisms of X and X'', so we form the surfaces Y, Y', and Y'' as in § 11.3. Y'' is the surface represented by t. The section s agrees with a fixed point section at $i(t)$ if and only if the projection $f(x_0)$ of x_0 to $Y = X/H$ is a puncture of Y''.

The cotangent spaces to $T(p', n'')$, $V(p, 0)$, and $T(p, 0)$ at t, x_0, and τ are $Q(Y'')$, $Q(X')$, and $Q(X)$ respectively. The map on cotangent spaces induced by $f_n \circ i$ is θ: $Q(X) \to Q(Y'')$. The map π_0 induces the inclusion map from $Q(X)$ to $Q(X')$. Let L: $Q(X') \to Q(Y'')$ be the map on cotangent spaces induced by h. Then L satisfies the conditions of Theorem 11.1, so $f(x_0)$ is a puncture of Y'', as required.

The exceptional cases are handled similarly.

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