FINDING A BOUNDARY FOR A HILBERT CUBE MANIFOLD

BY

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1. Introduction

In [21] Siebenmann considers the problem of putting a boundary on an open smooth manifold. A necessary condition is that the manifold have a finite number of ends, that the system of fundamental groups of connected open neighborhoods of each end be "essentially constant" and that there exist arbitrarily small open neighborhoods of ∞ homotopically dominated by finite complexes. When the manifold has dimension greater than five and has a single such end, there is an obstruction $\sigma(\infty)$ to the manifold having a boundary; it lies in $\tilde{K}_0\pi_1(\infty)$, the projective class group of the fundamental group at ∞. When the manifold does admit a connected boundary, and is therefore the interior of a compact smooth manifold, such compactifications are conveniently classified relative to a fixed one by certain torsions τ in Wh $\pi_1(\infty)$, the Whitehead group of $\pi_1(\infty)$. In other words, σ is the obstruction to putting a boundary on the manifold and τ then classifies the different ways in which this can be done. One can deal with manifolds having a finite number of ends by treating each one in the above manner.

In this paper we carry out a similar program for the problem of putting boundaries on non-compact Q-manifolds, where a *Q-manifold M* is a separable metric manifold modeled on the Hilbert cube Q (the countable-infinite product of closed intervals).[2] The first problem is to decide upon a suitable definition of a boundary for a Q-manifold; for example $B^n \times Q$ is a perfectly good Q-manifold and $(\partial B^n) \times Q$ has every right to be called its boundary, but unfortunately there exist homeomorphisms of $B^n \times Q$ onto itself taking $(\partial B^n) \times Q$ into its complement. To see this just write Q as $[0, 1] \times Q$ and note that there exists a homeomorphism of $B^n \times [0, 1]$ onto itself taking $(\partial B^n) \times [0, 1]$ into its complement. In the

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(2) It is, for example, conjectured that Q-manifolds are precisely those ANR's that *locally* are compact ∞-dimensional and homogeneous.

absence of any intrinsic notion of a boundary we adopt the following rather general definition. A compact metric space Z is a *boundary* for M is there exists a compact Q-manifold $N \supset M$ such that $N - M = Z$ and such that Z is contained in some closed collared subset of N. Such a compactum Z is also known as a *$\mathcal{Z}$-set* in N and is equally-well characterized by the existence, for every $\varepsilon > 0$, of an ε-map of N into $N - Z$. (The notion of a $\mathcal{Z}$-set is an important tool in the study of Q-manifolds.) The above definition of boundary makes sense in finite-dimensional manifolds; Z will simply be a compact subset of the usual intrinsic boundary ∂N of N. Thus our problem is more ambitious than the finite-dimensional one solved in [21]; indeed it strongly suggests that the thesis [21] admits a generalization along the lines of this article, cf. [24].

It is easy to find examples of Q-manifolds which do not admit boundaries, for any which does must have finite homotopy type; indeed using the notation above we have $M = N - Z \simeq N =$ (finite complex) $\times Q$ by [7]. An example with finite type is given by Whitehead's example of a contractible open subset W of R^3 which is not homeomorphic to R^3; the contractible Q-manifold $W \times Q$ does not admit a boundary.

We find that if a Q-manifold M satisfies certain minimal necessary homotopy theoretic conditions (finite type and tameness at ∞), there exists a unified obstrction $\beta(M)$ to M having a boundary; it lies in an algebraically-defined abelian group $S_\infty(M)$, which is none other than the quotient of the group $S(M)$ of all infinite simple homotopy types on M by the image of the Whitehead group $\operatorname{Wh} \pi_1(M)$. This group depends only on the inverse system $\{\pi_1(M - A) \mid A \subset M \text{ compact}\}$, where the homomorphisms are inclusion-induced. Secondly we determine that the different boundaries that can be put on M constitute a whole shape class and we classify the different ways of putting boundaries on M by elements of the group $\operatorname{Wh} \pi_1 E(M) = \varprojlim \{\operatorname{Wh} \pi_1(M - A) \mid A \subset M \text{ compact}\}$.

Here is an attractive way to describe the obstruction $\beta(M) \in S_\infty(M)$ to finding a boundary. If M is of finite type and tame at ∞ (cf. § 2), one readily forms a homotopy commutative ladder

$$\simeq \quad \begin{array}{ccccccc} M & \hookleftarrow & U_1 & \hookleftarrow & U_2 & \hookleftarrow & \cdots \\ \uparrow & \swarrow & \uparrow & \swarrow & \uparrow & & \\ X_0 & \leftarrow & X_1 & \leftarrow & X_2 & \leftarrow & \cdots \end{array}$$

where $\{U_i\}$ is a basis of neighborhoods of ∞ in M and each X_i is a finite complex. Letting Map (σ) denote the infinite mapping cylinder of the inverse sequence

$$\sigma\colon X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \ldots \quad \text{(cf. § 2)},$$

the homotopy commutativity mentioned assures that one can form a proper map

f: Map $(\sigma) \to M$. With some further care f can be chosen to be a proper equivalence([1]); this is carried out explicitly in Appendix II. Then $\beta(M)$ is none other than the class of f in $S_\infty(M)$.

The invariant $\beta(M) \in S_\infty(M)$ is an absolute torsion invariant just as are the Reidemeister torsions of lens spaces; and it enjoys the corresponding naturality properties. We will leave the reader to seek (if he wishes) a more algebraic definition of $\beta(M)$ in the spirit of [12].

The reader will find in this article two essentially independent proofs that a Q-manifold M of finite type and tame at ∞ admits a boundary precisely if $\beta(M)=0 \in S_\infty(M)$. The one suggested by the above definition requires above all a proof that $\beta(M)$ as described above is well-defined. This amounts to proving that a proper equivalence of two infinite mapping cylinders Map $(\sigma) \to$ Map (σ') has its torsion zero in S_∞. (This proof is marred only by the tedium of the "further care" required in the construction of f: Map $(\sigma) \to M$.) To pick out this proof the reader should read § 8 and Appendix 2.

The second proof is somewhat longer, but at the same time richer since we define and interpret geometrically, two partial obstructions $\sigma_\infty(M)$ and $\tau_\infty(M)$ analogous to obstructions σ_∞ and τ_∞ of infinite simple homotopy theory [22]. Since the theory was only sketched in [22] this article is intended to offer instruction in the theory that has become the basis of the classification of non-compact Q-manifolds [23]. For this proof the reader should read straight through, avoiding only § 8.

In § 2 we give more detailed statements of our results. The remaining sections are devoted to proofs. Here is a list of contents.

([1]) If dim X_i is uniformly bounded, this f is automatically a proper equivalence by a criterion $(\pi_*)^\flat$ of [22]. Whether f as constructed is always a proper homotopy equivalence is a special case of the open *problem:* Is a weak proper homotopy equivalence of infinite-dimensional polyhedra a (genuine) proper homotopy equivalence? (Cf. Appendix 2.)

2. Statements of results

One of the most useful tools in this paper is the notion of the *infinite mapping cylinder* of an inverse sequence of spaces. If $\sigma = \{X_i, f_i\}_{i=1}^{\infty}$ is an inverse sequence of compact metric spaces, $X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} X_3 \leftarrow \dots$, then we use Map (σ) to denote the locally compact space formed by sewing together the usual mapping cylinders $M(f_i)$ along their naturally-identified ends. With these identifications we have Map $(\sigma) = M(f_1) \cup M(f_2) \cup \dots$, which is naturally compactified by adding on the inverse limit $\varprojlim \sigma$ of σ. It is obvious that $\varprojlim \sigma$ is a $\mathcal{Z}$-set in Map $(\sigma) \cup \varprojlim \sigma$. It follows from [28] that if each X_i is a compact polyhedron, then Map $(\sigma) \times Q$ is a Q-manifold. Here is a sharper statement.

THEOREM 1: CYLINDER COMPLETION (see § 4). *If σ is an inverse sequence of compact Q-manifold factors, then* (Map $(\sigma) \cup \varprojlim \sigma) \times Q$ *is a compact Q-manifold homeomorphic to $X_1 \times Q$ and therefore $\varprojlim \sigma \times Q$ is a boundary for* Map $(\sigma) \times Q$.

By a *Q-manifold factor* we mean a space which yields a Q-manifold upon multiplication by Q. The class of Q-manifold factors includes at least all locally-finite CW complexes [28], (and may perhaps include *all* locally compact ANR's).(¹) We therefore obtain a large class of Q-manifolds which admit boundaries by considering Map $(\sigma) \times Q$, for σ any inverse sequence of compact Q-manifold factors. The next result shows that this characterizes all such Q-manifolds.

THEOREM 2: GEOMETRIC CHARACTERIZATION (see § 4). *A Q-manifold admits a boundary if and only if it is homeomorphic to* Map $(\sigma) \times Q$, *for some inverse sequence σ of compact polyhedra.*

The basic necessary condition for a Q-manifold M to admit a boundary is that M be tame at ∞, where *tame at* ∞ means that for each compactum $A \subset M$ there exists a larger compactum $B \subset M$ such that the inclusion $M - B \hookrightarrow M - A$ factors up to homotopy through some finite complex. Note that Whitehead's example cited in § 1 fails to be tame at ∞. It

(¹) Added 1976: This has been proved by R. D. Edwards [11], [9].

follows from the *Geometric characterization* above that this condition *is* necessary for M to admit a boundary, and it is easy to see that it is an invariant of proper homotopy type. (Recall that a *proper* map or *proper* homotopy is one for which pre-images of compacta are compact.) If M is a Q-manifold which is tame at ∞, we will first define two obstructions to M having a boundary. Later on we will combine them into a single obstruction.

For any Q-manifold M we say that a Q-manifold $M' \subset M$ is *clean* provided that M' is closed and the topological frontier $\delta M'$ of M' in M is a Q-manifold which is collared in M' and in the closure of $M - M'$. It follows from the triangulation of Q-manifolds [7] that every Q-manifold has arbitrarily large compact and clean submanifolds. Observe also that triangulability implies that if $M' \subset M$ is compact and clean and M admits a boundary, then $M - M'$ must have finite type. Our first obstruction is the obstruction to all such $M - M'$ having finite type; using Wall's finiteness obstruction this turns out to be just an element $\sigma_\infty(M)$ of

$$\tilde{K}_0 \pi_1 E(M) = \varprojlim \{\tilde{K}_0 \pi_1(M - A) \mid A \subset M \text{ compact}\}.$$

Here $\tilde{K}_0 \pi_1$ is the projective class group functor and the homomorphisms are inclusion-induced. $\tilde{K}_0 \pi_1 E(M)$ clearly depends only on the inverse system $\{\pi_1(M - A) \mid A \subset M$ compact$\}$.

THEOREM 3: CLASS GROUP OBSTRUCTION (see § 5). *If M is a Q-manifold which is tame at ∞, the obstruction $\sigma_\infty(M) \in \tilde{K}_0 \pi_1 E(M)$, an invariant of (infinite) simple homotopy type, is zero if and only if there exist arbitrarily large clean compact M' in M such that the inclusion $\delta(M') \hookrightarrow M - \operatorname{Int}(M')$ is a homotopy equivalence.*(1)

It is somewhat suprising that if $\sigma_\infty(M) = 0$, then there is yet a further obstruction to M having a boundary. This is identified in our next theorem.

THEOREM 4: RESIDUAL OBSTRUCTION (see § 6). *If M is a Q-manifold which is tame at ∞ and for which $\sigma_\infty(M) = 0$, then there is an obstruction $\tau_\infty(M) \in \mathrm{Wh}\, \pi_1 E'(M)$ which vanishes if and only if M admits a boundary. It is an invariant of simple homotopy type.*

The abelian group $\mathrm{Wh}\, \pi_1 E'(M)$ is the first derived limit of the inverse system

$$\{\mathrm{Wh}\, \pi_1(M - A) \mid A \subset M \text{ compact}\},$$

where $\mathrm{Wh}\, \pi_1$ is the Whitehead group functor. (The article [18] is a good reference for the derived limit construction, but in § 6 we clearly state the definition.) To show the necessity of the obstruction τ_∞ we give in § 7 an example (based on [2]) of a Q-manifold which is tame at ∞ and for which $\sigma_\infty(M) = 0$, yet M does not admit a boundary.

(1) The symbols δ and Int indicate frontier and interior respectively (in M).

The obstruction $\tau_\infty(M)$ occurs as follows. Using the *Class group obstruction* theorem choose a sequence $M_1 \subset M_2 \subset \ldots$ of compact clean submanifolds of M such that M_i lies in the topological interior of M_{i+1} and $\delta(M_i) \hookrightarrow M - \text{Int}(M_i)$ is a homotopy equivalence. If τ_i denotes the Whitehead torson of the inclusion

$$\delta(M_i) \hookrightarrow M_{i+1} - \text{Int}(M_i)$$

$i=1, 2, \ldots$, experts will recognize that the sequence $(\tau_1, \tau_2, \ldots)$ determines an element of $\text{Wh}\,\pi_1 E'(M)$. It is $\tau_\infty(M)$, and we will show in § 6 that $\tau_\infty(M)=0$ if and only if the M_i can be rechosen so that the τ_i are all 0, i.e. if and only if M admits a boundary.

Essential in our definition of the total boundary obstruction is the following result, which is of independent interest.

THEOREM 5: HOMOTOPY BOUNDARY CRITERION (see § 5). *A Q-manifold M is proper homotopy equivalent to one which admits a boundary if M has finite type and is tame at* ∞.

We give two proofs of this result. The first one relies upon Theorem 3 plus the realization of elements of the class group at infinity $\tilde{K}_0\pi_1 E(M)$ by proper homotopy equivalences, as explained in [22]; since the argument was only sketched there we given a full discussion in Appendix 1. The second proof relies upon Appendix 2 to show that we can choose a proper homotopy equivalence Map $(\sigma) \simeq M$, for some σ (cf. § 1).

By the Hauptvermutung and Triangulation results for Q-manifolds [8], [23], the infinite simple type theory [22] extends canonically from locally compact polyhedra to Q-manifolds. If $f: N \to M$ is a proper equivalence of Q-manifolds the infinite torsion $\tau(f)$ in $S(M)$, the group of simple types on M, vanishes if and only if f is proper homotopic to a homeomorphism. Also one easily shows that the image $\beta(f)$ of $\tau(f)$ in $S(M)/\text{Image}$ $\text{Wh}\,\pi_1(M)$ is zero if and only if f is proper homotopic to a map that is a homeomorphism *near infinity.* (All this is explained in § 2.)

THEOREM 6: TOTAL OBSTRUCTION (see § 8 and Appendix 2). *If M is a Q-manifold which has finite type and is tame at* ∞, *then there is an element* $\beta(M) \in S_\infty(M) = S(M)/\text{Wh}\,\pi_1(M)$ *which vanishes if and only if M admits a boundary. It can be defined unambigously as the residue of the infinite torsion* $\tau(f) \in S(M)$, *where* $f: N \to M$ *is a proper homotopy equivalence and N admits a boundary.*

The surprising feature of this result is isolated in the following statement that we take time to prove both geometrically and using naturality properties of the obstructions σ_∞ and τ_∞ above.

THEOREM 7: PERIPHERAL HOMEOMORHPISM PARADOX (see § 8). *If* $f: M \to N$ *is a (merely!) proper homotopy equivalence of Q-manifolds which admit boundaries, then f is proper* *homotopic to a homeomorphism near* ∞. *If* M *and* N *are contractible, then* f *is proper homotopic to a homeomorphism.*

It is worth noting that the obstruction $\beta(M) \in S_\infty(M)$ can still be defined if we drop the requirement that M have finite type and only assume it to be tame at ∞. For such an M there is a compact polyhedron which homotopically dominates M (see § 5, Lemma 5.1) and the mapping cone of such a dominating map yields, upon multiplication by Q, a Q-manifold N which is simply connected (and hence has finite type) and which agrees with M off some compactum. Then N has finite type and is tame at ∞, and we can define $\beta(M) = \beta(N)$, which is well-defined.

We also remark that for every Q-manifold M there is an exact sequence

$$0 \to \mathrm{Wh}\, \pi_1 E'(M) \to S_\infty(M) \to \tilde{K}_0 \pi_1 E(M) \to \tilde{K}_0 \pi_1(M),$$

which comes from amalgamating the two exact sequences of [22]. If M has finite type and is tame at ∞, we will observe (see § 8, Proposition 8.1) that

(1) $\sigma_\infty(M)$ is the image of $\beta(M)$ in $\tilde{K}_0 \pi_1 E(M)$ and
(2) if $\sigma_\infty(M) = 0$, then $\beta(M)$ is the image of $\tau_\infty(M) \in \mathrm{Wh}\, \pi_1 E'(M)$.

Here are some applications of the above results.

THEOREM 8: PRATICAL BOUNDARY THEOREM (see § 9). *If* M *is a* 1*-ended* Q*-manifold which is tame at* ∞ *such that* π_1 *is essentially constant at* ∞, *with* $\pi_1(\infty)$ *free or free abelian, then* $S_\infty(M) = 0$ *and therefore* M *admits a boundary.*

COROLLARY. *If* M *is a* Q*-manifold which is* LC^1 *at* ∞ *and for which the homology* $H_*(M)$ *is finitely generated, then* M *admits a boundary.*

Concerning the classification of possible boundaries we prove the following result. Its proof relies on the main result of [6], which classifies shapes of compact $\mathcal{Z}$-sets in Q in terms of the homeomorphism types of their complements (see [24] for an alternate proof in the spirit of this article).

THEOREM 9: BOUNDARY CLASSIFICATION (see § 10). *If* Z *is a boundary for* M, *then a compact metric space* Z' *is also a boundary for* M *if and only if* Z' *is shape equivalent to* Z (*in the sense of* [4]).

If N is a compactification of M, then we say that another compactification N' of M is *equivalent* to N if for every compactum $A \subset M$ there exists a homeomorphism of N onto N' fixing A pointwise. The following result classifies these equivalence classes of compactifications of M.

THEOREM 10: COMPACTIFICATION CLASSIFICATION (see § 11). *If M admits a compactification, then the equivalence classes of compactifications of M are in $1-1$ correspondence with the elements of* $\mathrm{Wh}\,\pi_1 E(M)$.

Here $\mathrm{Wh}\,\pi_1 E(M)$ is the inverse limit $\varprojlim \{\mathrm{Wh}\,\pi_1(M-A)\,|\,A\subset M \text{ compact}\}$.

3. Simple homotopy and Q-manifold preliminaries

The purpose of this section is to recall some basic facts from simple homotopy theory and Q-manifold theory. Our basic references for simple homotopy theory are [10] and [22], and our basic references for Q-manifold theory are [7], [8], [9] and [23].

A proper homotopy equivalence (p.h.e.) $f: X\to Y$ of (locally compact) polyhedra is a *simple homotopy equivalence* (s.h.e.) provided that there exists a polyhedron Z and proper, contractible PL surjections $\alpha: Z\to X$, $\beta: Z\to Y$ such that $f\alpha$ is proper homotopic to β. (A *contractible* map is one for which all inverse images of points are contractible in themselves.)

For a given polyhdron X we let $\mathcal{S}(X)$ be the set of all equivalence classes $[Y, X]$ of pairs (Y, X) where X is a subpolyhedron of Y and $X\hookrightarrow Y$ is a p.h.e. We define $(Y', X)\in [Y, X]$ provided that there exists a s.h.e. $f: Y\to Y'$ fixing X pointwise. Addition in $\mathcal{S}(X)$ is defined by

$$[Y, X]+[Z, X]=[Y\cup_X Z, X],$$

where $Y\cup_X Z$ is the polyhedron formed by sewing Y and Z together along X. Then with this operation $\mathcal{S}(X)$ becomes an abelian group. It is the group of all simple types on X. For X compact it is canonically isomorphic to $\mathrm{Wh}\,\pi_1(X)$.

If $f: X_1\to X_2$ is a proper map of polyhedra and $g: X_1\to X_2$ is any PL map which is proper homotopic to f, then we get an induced homomorphism $f_*: \mathcal{S}(X_1)\to \mathcal{S}(X_2)$ by setting

$$f_*([Y, X])=[Y\cup_X M(g), X_2],$$

where $M(g)$ is the polyhedral mapping cylinder of g and X_1, X_2 are naturally identified subpolyhedra of $M(g)$. The torsion of a p.h.e. $f: X\to Y$ is the element $\tau(f)=f_*([M(g), X])\in \mathcal{S}(Y)$, where $g: X\to Y$ is any PL map which is proper homotopic to f.

The algebraically-defined Whitehead group $\mathrm{Wh}\,\pi_1(X)$ is naturally isomorphic to the direct limit

$$\varinjlim \{\mathcal{S}(X_1)\,|\,X_1\subset X \text{ is a compact subpolyhedron}\},$$

where the homomorphisms are inclusion-induced. This gives a homomorphism $\mathrm{Wh}\,\pi_1(X)\to \mathcal{S}(X)$ and we define $\mathcal{S}_\infty(X)=\mathcal{S}(X)/\mathrm{Image}\,(\mathrm{Wh}\,\pi_1(X))$. This homomorphism is natural in the sense that if $f: X\to Y$ is a proper map, then the following diagram commutes:

$$\begin{array}{ccc} S(X) & \xrightarrow{f_*} & S(Y) \\ \uparrow & & \uparrow \\ \mathrm{Wh}\,\pi_1(X) & \longrightarrow & \mathrm{Wh}\,\pi_1(Y) \end{array}$$

(Here $\mathrm{Wh}\,\pi_1(X)\to\mathrm{Wh}\,\pi_1(Y)$ is induced by f.) Thus S_∞ is functorial for such f.

If $f: X\to Y$ is a p.h.e., then we use $\beta(f)$ for the image of $\tau(f)$ in $S_\infty(Y)$ and we call it the *torsion of f near* ∞. We say that f is a *s.h.e. near* ∞ if $\beta(f)=0$. The naturality diagram above implies that the formulas for the torsion of a composition [22, p. 481] [10, p. 72] and the Sum theorem [22, p. 482] [10, p. 76] translate into corresponding formulas for β.

We now recall some facts from Q-manifold theory. All spaces here and in the sequel will be locally compact, separable and metric. (Closed) $\mathcal{Z}$-sets in Q-manifolds are important because of the following approximation, unknotting and collaring results. If $f: X\to M$ is a proper map of a space X into a Q-manifold M, then f can be arbitrarily closely approximated by *$\mathcal{Z}$-embeddings* $g: X\to M$, i.e. $g(X)$ is a $\mathcal{Z}$-set in M. For $\mathcal{Z}$-embeddings $f, g: X\to M$ which are proper homotopic, there exists an ambient isotopy $h_t: M\to M$ such that $h_0=\mathrm{id}$ and $h_1 f=g$. If M is a $\mathcal{Z}$-set in the Q-manifold N, then M is *collared in* N, i.e. there exists an open embedding $g: M\times[0, 1)\to N$ such that $f(m, 0)=m$, for all $m\in M$.

If $f: X\to Y$ is a map between compact polyhedra, then $M(f)\times Q$ is a Q-manifold. In fact $r\times\mathrm{id}_Q: M(f)\times Q\to Y\times Q$ can be arbitrarily closely approximated by homeomorphisms, where $r: M(f)\to Y$ is the collapse of $M(f)$ to its base Y obtained by retracting along the rays of $M(f)$. (We call $r\times\mathrm{id}$ a *near homeomorphism*.) It is also true that $X\times Q$ is a Q-manifold, for any polyhedron X. The "Hauptvermutung" for Q-manifolds asserts that a p.h.e. f: $X\to Y$ of polyhedra is a s.h.e. iff $f\times\mathrm{id}: X\times Q\to Y\times Q$ is proper homotopic to a homeomorphism.

All Q-manifolds can be *triangulated* (i.e. are homeomorphic to $X\times Q$, for some polyhedron X). In fact we have the following relative version. If M is a Q-manifold which is a $\mathcal{Z}$-set in a Q-manifold N and $\alpha: M\to X\times Q$ is a triangulation of M, then there exists a triangulation $\beta: N\to Y\times Q$ such that X is a subpolyhedron of Y and β extends α.

If $f: M\to N$ is a p.h.e. of Q-manifolds, then we can define a torsion $\tau(f)\in S(N)$ which vanishes if and only if f is proper homotopic to a homeomorphism. This is done by choosing triangulations $M\cong X\times Q$, $N\cong Y\times Q$ and a p.h.e. $f_0: X\to Y$ which makes the following diagram proper homotopy commute:

$$\begin{array}{ccc} M\cong X\times Q & \xrightarrow{\mathrm{proj}} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_0} \\ N\cong Y\times Q & \xrightarrow{\mathrm{proj}} & Y \end{array}$$

Then put $\tau(f) = \tau(f_0)$, where we recall from § 2 that by definition $S(M) = S(X)$ and $S(N) = S(Y)$. It easily follows from the "Hauptvermutung" that $S(M)$ and $S(N)$ are well defined up to canonical isomorphism independent of the triangulations chosen, and that $\tau(f) = 0$ if and only if f is proper homotopic to a homeomorphism.

In analogy with the definition of $\tau(f)$ above we could also define $\beta(f) = \beta(f_0) \in S_\infty(Y) = S_\infty(N)$. We assert that $\beta(f) = 0$ iff f is proper homotopic to a homeomorphism near ∞. [To verify this first suppose that $\beta(f) = 0$. This means that f_0 factorizes, $X \overset{\varphi}{\hookrightarrow} X' \xrightarrow{\psi} Y$, where $X' - X$ consists of finitely many cells and $X' \to Y$ is a simple proper equivalence. As both $\varphi \times (\mathrm{id}\,|\,Q)$ and $\psi \times (\mathrm{id}\,|\,Q)$ are proper homotopic to homeomorphisms near ∞, the same is true of their composition $f_0 \times (\mathrm{id}\,|\,Q)$ and so of f. Conversely, supposing f proper homotopic to f', a homeomorphism near ∞, we can artificially choose the triangulations $M \cong X \times Q$ and $N \cong Y \times Q$ (by using the relative triangulation theorem) so that $X = Y$ near ∞ and

$$X \times Q \cong M \xrightarrow{f'} N \cong Y \times Q$$

is the identity map near ∞. Then clearly $\beta(f_0) = 0$.]

4. Infinite mapping cylinders

In this section we will prove the *Cylinder completion* and *Geometric characterization* theorems of § 2 concerning infinite mapping cylinders. First we will introduce some additional notation which will clarify the paragraph preceding the statement of the *Cylinder completion* theorem.

Let σ be the inverse sequence $\{X_i, f_i\}_{i=1}^\infty$ and for each $i \geqslant 1$ let $M(f_i)$ denote the mapping cylinder of $f_i: X_{i+1} \to X_i$. We regard $M(f_i)$ as the disjoint union $X_{i+1} \times [0, 1) \cup X_i$ along with an appropriate topology. The *source* of $M(f_i)$ is $X_{i+1} = X_{i+1} \times \{0\}$ and the *target* of $M(f_i)$ is X_i. For each $i \geqslant 1$ let $M_i(\sigma)$ denote the compact space formed by sewing together the mapping cylinders $M(f_1), \ldots, M(f_i)$ along their naturally identified sources and targets. Then we have $M_{i+1}(\sigma) = M_i(\sigma) \cup M(f_{i+1})$ and $\mathrm{Map}\,(\sigma) = \bigcup_{i=1}^\infty M_i(\sigma)$. There is also a natural map $g_i: M_{i+1}(\sigma) \to M_i(\sigma)$ which is the identity on $M_i(\sigma)$ and on $M(f_{i+1})$ it is just the collapse to the base X_i. The compactification of Map (σ) by $\varprojlim \sigma$ that was mentioned in § 2 is just the inverse limit $X_\sigma = \varprojlim \{M_i(\sigma), g_i\}_{i=1}^\infty$. Clearly we may write $X_\sigma = \mathrm{Map}\,(\sigma) \cup \varprojlim \sigma$ and we note that $\varprojlim \sigma$ is a $\mathcal{Z}$-set in X_σ.

Proof of the Cylinder completion theorem. The Cylinder completion theorem asserts that if $\sigma = \{X_i, f_i\}_{i=1}^\infty$ is an inverse sequence of compact Q-manifold factors, then $X_\sigma \times Q$ is a Q-manifold which is homeomorphic to $X_1 \times Q$. Consider the sequence $\{M_i(\sigma), g_i\}_{i=1}^\infty$, which is an inverse sequence of compact Q-manifold factors. Since each $g_i: M_{i+1}(\sigma) \to M_i(\sigma)$ is just the collapse of a mapping cylinder to its base, it follows that each $g_i \times \mathrm{id}\colon M_{i+1}(\sigma) \times Q \to M_i(\sigma) \times Q$ is a near homeomorphism. It is shown in [5] (see Lemma 4.1 below) that if $\{Y_i, h_i\}_{i=1}^{\infty}$ is an inverse sequence of compact metric spaces and the h_i's are near homeomorphisms, then $\varprojlim \{Y_i, h_i\}_{i=1}^{\infty}$ is homeomorphic to Y_1. Applying this result to $\{M_i(\sigma) \times Q, g_i \times \mathrm{id}\}_{i=1}^{\infty}$ we get our desired result. ■

In the following result we sketch a proof of the main result of [5] which was used above.

LEMMA 4.1. *If* $X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} \ldots$ *is an inverse sequence of compact metric spaces, then we can choose neighborhoods* $\mathcal{U}_i$ *of* f_i *in* $C(X_{i+1}, X_i)$ *such that if* $g_i \in \mathcal{U}_i$, *then*

$$\varprojlim \{X_i, f_i\} \cong \varprojlim \{X_i, g_i\}.$$

($C(X_{i+1}, X_i)$ is the space of continuous functions from X_{i+1} to X_i.)

Sketch of proof. For any choice of the $\mathcal{U}_i$ and $g_i \in \mathcal{U}_i$ we have inverse sequences

$$\sigma_0\colon X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} \ldots$$

$$\sigma_1\colon X_1 \xleftarrow{g_1} X_2 \xleftarrow{f_2} X_3 \xleftarrow{f_3} \ldots$$

$$\sigma_2\colon X_1 \xleftarrow{g_1} X_2 \xleftarrow{g_2} X_3 \xleftarrow{f_3} X_4 \xleftarrow{f_4} \ldots$$

$$\vdots$$

Let A_i denote the inverse limit of σ_i and define $\varphi_i\colon A_i \to A_{i+1}$ by

$$\varphi_i(x_1, x_2, \ldots) = (g_1 g_2 \ldots g_{i+1}(x_{i+2}), g_2 \ldots g_{i+1}(x_{i+2}), g_{i+1}(x_{i+2}), x_{i+2}, x_{i+3}, \ldots).$$

Then φ_i is a homeomorphism and, if $\mathcal{U}_i$ is chosen sufficiently close to f_i, the sequence of embeddings

$$\varphi_0\colon A_0 \to \prod_{i=1}^{\infty} X_i,$$

$$\varphi_1 \varphi_0\colon A_0 \to \prod_{i=1}^{\infty} X_i,$$

$$\vdots$$

is Cauchy in the complete space of embeddings of A_0 into $\prod_{i=1}^{\infty} X_i$. Therefore $\varphi = \lim_{n \to \infty} \varphi_n \varphi_{n-1} \ldots \varphi_2 \varphi_1\colon A_0 \to \prod_{i=1}^{\infty} X_i$ gives a homeomorphism of $\varprojlim \{X_i, f_i\}$ onto $\varprojlim \{X_i, g_i\}$. ■

Proof of the Geometric characterization theorem. The Geometric characterization theorem asserts that a Q-manifold admits a boundary iff it is homeomorphic to $\mathrm{Map}(\sigma) \times Q$, for some inverse sequence σ of compact polyhedra. The "if" part follows from the *Cylinder completion* theorem. For the other part let M be a Q-manifold which admits a boundary Z and let $N = M \cup Z$ be a compactification of M. Since Z is a $\mathcal{Z}$-set we may replace N by $N \times [0, 1]$ and assume that $Z \subset N \times \{0\}$. To see this we first recall that N is homeomorphic to $N \times [0, 1]$ and then use $\mathcal{Z}$-set unknotting. With this notation we must prove that $N \times [0, 1] - Z$ is homeomorphic to some $\operatorname{Map}(\sigma) \times Q$.

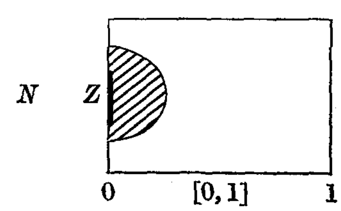
Since N can be triangulated we can write $Z = \bigcap_{i=1}^{\infty} M_i$, where

$$N \times \{0\} \supset M_1 \supset M_2 \supset \ldots$$

is a basis of compact and clean neighborhoods of Z in $N \times \{0\}$. By "poking-in" along the $[0, 1]$-direction we can enlarge the M_i's to obtain $Z = \bigcap_{i=1}^{\infty} N_i$, where

$$N \times [0, 1] \supset N_1 \supset N_2 \supset \ldots$$

is a basis of compact and clean neighborhoods of Z in $N \times [0, 1]$ such that each N_i is a collar on $\delta(N_i)$, the frontier of N_i.



In the picture above the shaded region represents N_i. Its intersection with $N \times \{0\}$ is just M_i.

Choose compact polyhedra $X_0, X_1, \ldots$ and homeomorphisms

$$\begin{aligned} &h_0: X_0 \times Q \to N \times [0, 1] - \operatorname{Int}(N_1), \\ &h_1: X_1 \times Q \to \delta(N_1), \\ &h_2: X_2 \times Q \to \delta(N_2), \\ &\vdots \end{aligned}$$

We will construct maps $f_i: X_{i+1} \to X_i$ such that if $\sigma = \{X_i, f_i\}_{i=0}^{\infty}$, then $\operatorname{Map}(\sigma) \times Q \cong N \times [0, 1] - Z$.

Define $f_0: X_1 \to X_0$ and $f_i: X_{i+1} \to X_i$, $i \geq 1$, so that the following rectangles homotopy commute:

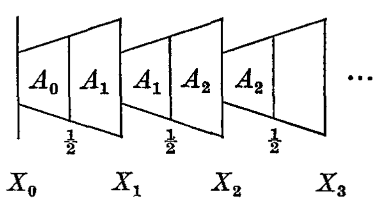
$$\begin{array}{ccc} X_1 \times Q & \xrightarrow{h_1} & \delta(N_1) \\ {\scriptstyle f_0 \times \mathrm{id}} \downarrow & & \downarrow \\ X_0 \times Q & \xrightarrow{h_0} & N \times [0,1] - \operatorname{Int}(N_i) \end{array} \qquad \begin{array}{ccc} X_{i+1} \times Q & \xrightarrow{h_{i+1}} & \delta(N_{i+1}) \\ {\scriptstyle f_i \times \mathrm{id}} \downarrow & & \downarrow \\ X_i \times Q & \xrightarrow{h_i} & \delta(N_i) \end{array}$$

Here $\delta(N_{i+1}) \to \delta(N_i)$ is a composition

$$\delta(N_{i+1}) \hookrightarrow N_i - \operatorname{Int}(N_{i+1}) \to \delta(N_i),$$

where the last arrow is a homotopy inverse of inclusion.

Consider the infinite mapping cylinder Map (σ) and write Map $(\sigma) = A_0 \cup A_1 \cup \ldots$, where in the picture of Map (σ) below we have indicated A_0, A_1 and A_2.



Using our notation for mapping cylinders we have

$$A_0 = (X_1 \times [\tfrac{1}{2}, 1)) \cup X_0,$$
$$A_1 = (X_2 \times [\tfrac{1}{2}, 1)) \cup (X_1 \times [0, \tfrac{1}{2}]),$$
$$\vdots$$

Now $X_0 \times Q \hookrightarrow A_0 \times Q$ is homotopic to a homeomorphism, thus using h_0 we get a homeomorphism g_0: $A_0 \times Q \to N \times [0, 1] - \operatorname{Int}(N_1)$ which, by $\mathcal{Z}$-set unknotting, can be adjusted so that $g_0 | X_1 \times \{\frac{1}{2}\} \times Q$ is given by h_1, i.e. $g_0(x, \frac{1}{2}, q) = h_1(x, q)$ for all $x \in X_1$. This uses the commutativity of the first rectangle above. Using the second rectangle we can similarly construct homeomorphisms

$$g_i \colon A_i \times Q \to N_i - \operatorname{Int}(N_{i+1}), \quad i \geqslant 1,$$

such that $g_i(x, \frac{1}{2}, q) = h_{i+1}(x, q)$, for all $x \in X_{i+1}$, and $g_i(x, \frac{1}{2}, q) = h_i(x, q)$, for all $x \in X_i$. Then the g_i's clearly piece together to give a homeomorphism of Map $(\sigma) \times Q$ onto $N \times [0, 1] - Z$. ∎

The following consequence of the above proof will be useful in the sequel.

COROLLARY 4.2. *If M is a Q-manifold which is written as $M = \bigcup_{i=1}^{\infty} M_i$, where the M_i's are compact and clean, $M_i \subset \operatorname{Int}(M_{i+1})$, and each $\delta(M_i) \hookrightarrow M_{i+1} - \operatorname{Int}(M_i)$ is a s.h.e., then M admits a boundary.*

5. The class group obstruction

In this section we prove the *Class group obstruction* and *Homotopy boundary criterion* theorems. In a further result (*Naturality*) we relate σ_∞ as defined here to a homomorphism σ_∞: $S(M) \to \tilde{K}_0 \pi_1 E(M)$ defined in [22]. We will first need the following result.

DOMINATION LEMMA 5.1. *If X is a polyhedron which is tame at ∞, then X is dominated by a compact polyhedron.*

Proof. Since X is tame at ∞ we can write $X = X_1 \cup X_2$, where X_1 and X_2 are subpolyhedra such that X_1 is compact and $X_2 \hookrightarrow X$ factors up to homotopy through some compact polyhedron Y. Then we have a homotopy commutative diagram

$$\begin{array}{ccc} X_2 & \hookrightarrow & X \\ & \alpha\searrow \quad \nearrow\beta & \\ & Y & \end{array}$$

Let $F_t\colon X_2 \to X$ be a homotopy such that $F_0 = \beta\alpha$ and $F_1\colon X_2 \hookrightarrow X$. Using the homotopy extension theorem we can find a homotopy $G_t\colon X_1 \to X$ such that $G_t = F_t$ on $X_1 \cap X_2$ and $G_1\colon X_1 \hookrightarrow X$. Define $H_t\colon X \to X$ by setting $H_t = F_t$ on X_2 and $H_t = G_t$ on X_1. Then we have a homotopy commutative diagram

$$\begin{array}{ccc} X & \hookrightarrow & X \\ & H_0\searrow \quad \nearrow & \\ & Z, & \end{array}$$

where Z is any compact subpolyhedron of X containing $H_0(X)$. This means that Z dominates X. ■

The following corollary makes the notion of tameness much more concrete.

COROLLARY. *If M is a Q-manifold which is tame at ∞ and $M' \subset M$ is compact and clean, then $M - M'$ is finitely dominated.*

Proof. By the triangulation of Q-manifolds we have $M - M'$ proper homotopy equivalent to some polyhedron, which must be tame at ∞. ■

Proof of the Class group obstruction theorem. Let M be a Q-manifold which is tame at ∞. We want to define an element $\sigma_\infty(M) \in \tilde{K}_0 \pi_1 E(M)$ which vanishes iff $M - M'$ has finite type, for each compact and clean $M' \subset M$. Write $M = \bigcup_{i=1}^\infty M_i$, where the M_i's are compact and clean and $M_i \subset \operatorname{Int}(M_{i+1})$. The above Corollary implies that each $M - M_i$ is finitely dominated. Thus there is an element $\sigma(M - M_i) \in \tilde{K}_0 \pi_1(M - M_i)$ (the Wall obstruction [27]) which vanishes iff $M - M_i$ has finite type. This makes good sense even if the $M - M_i$'s are not connected [21]. If $j > i$, then the fact that $M_j - M_i$ has finite type implies that the inclusion-induced homomorphism of $\tilde{K}_0 \pi_1(M - M_j)$ to $\tilde{K}_0 \pi_1(M - M_i)$ sends $\sigma(M - M_j)$ to $\sigma(M - M_i)$. Thus we have an element $\{\sigma(M - M_i)\}_{i=1}^\infty$ of $\varprojlim \{\tilde{K}_0 \pi_1(M - M_i)\}_{i=1}^\infty = \tilde{K}_0 \pi_1 E(M)$ which we call $\sigma_\infty(M)$. Clearly $\sigma_\infty(M) = 0$ iff $M - M'$ has finite type, for each compact and clean $M' \subset M$.

To see that $\sigma_\infty(M)$ is an invariant of infinite simple homotopy type let $f: M \to N$ be an infinite simple homotopy equivalence to another Q-manifold N. Then f induces an isomorphism $f_*: \tilde{K}_0\pi_1 E(M) \to \tilde{K}_0\pi_1 E(N)$ and we need to check that $f_*\sigma_\infty(M) = \sigma_\infty(N)$. We can replace f by any map in its proper homotopy class, so we may assume that f is a homeomorphism. If $M = \bigcup_{i=1}^\infty M_i$ and $\sigma_\infty(M)$ is represented by $\{\sigma(M - M_i)\}_{i=1}^\infty$ above, then $N = \bigcup_{i=1}^\infty f(N_i)$ and $\sigma_\infty(N)$ may be represented by $\{\sigma(f(M) - f(M_i))\}_{i=1}^\infty$. But f_* clearly sends $\{\sigma(M - M_i)\}$ to $\{\sigma(f(M) - f(M_i))\}$ because the Wall obstruction is an invariant of homotopy type. ∎

We now state and prove the *Naturality theorem*. Recall from [22] that for each polyhedron X there exists a homomorphism $\sigma_\infty: \mathcal{S}(X) \to \tilde{K}_0\pi_1 E(X)$. Since Q-manifolds can be triangulated this naturally defines a homomorphism $\sigma_\infty: \mathcal{S}(M) \to \tilde{K}_0\pi_1 E(M)$, for each Q-manifold M.

THEOREM 5.2. NATURALITY. *If $f: M \to N$ is a p.h.e. of Q-manifolds which are tame at ∞, then*

$$\sigma_\infty(N) = \sigma_\infty(f) + f_*\sigma_\infty(M),$$

where $f_*: \tilde{K}_0\pi_1 E(M) \to \tilde{K}_0\pi_1 E(N)$ is induced by f.

Proof. Writing $M = X \times Q$ and $N = Y \times Q$ it will suffice to consider a p.h.e. $f: X \to Y$, of polyhedra which are tame at ∞ such that f is the inclusion $f: X \hookrightarrow Y$, where $\sigma_\infty(X) = \sigma_\infty(M)$ and $\sigma_\infty(Y) = \sigma_\infty(N)$ could be defined in analogy with $\sigma_\infty(M)$ and $\sigma_\infty(N)$.

Write $Y = \bigcup_{i=1}^\infty Y_i$, where the Y_i's are compact subpolyhedra such that $\delta(Y_i)$ is PL bicollared and $Y_i \subset \mathrm{Int}(Y_{i+1})$. Then $\sigma_\infty(X \hookrightarrow Y)$ is defined to be the element of $\tilde{K}_0\pi_1 E(Y)$ which is represented by

$$\{\sigma(Y - Y_i, (Y - Y_i) \cap X)\}_{i=1}^\infty,$$

where $\sigma(Y - Y_i, (Y - Y_i) \cap X) \in \tilde{K}_0\pi_1(Y - Y_i)$ is the relative finiteness obstruction of [27].

Note that $Y - Y_i$ and $(Y - Y_i) \cap X$ are finitely dominated. Using [27, p. 138] we have

$$\sigma(Y - Y_i, (Y - Y_i) \cap X) + (j_i)_*\sigma((Y - Y_i) \cap X) = \sigma(Y - Y_i),$$

where $(j_i)_*$ is inclusion-induced. This gives

$$\sigma_\infty(Y) = \sigma_\infty(f) + f_*\sigma_\infty(X)$$

as we wanted. ∎

We now turn to the proof of the *Homotopy boundary criterion* theorem. The following result, which is crucial for its proof, will also be needed for the *Residual obstruction* theorem of § 6.

SPLITTING PROPOSITION 5.3. *If M is a Q-manifold which is tame at ∞ and for which $\sigma_\infty(M)=0$, then we can write $M=\bigcup_{i=1}^{\infty} M_i$ such that the M_i's are compact and clean, $M_i \subset \operatorname{Int}(M_{i+1})$, and each inclusion $\delta(M_i) \hookrightarrow M_{i+1}-\operatorname{Int}(M_i)$ is a homotopy equivalence.*

Proof. Choose any compact and clean $M' \subset M$. It will suffice to construct a compact and clean $M'' \supset M'$ such that $\delta(M'') \hookrightarrow M-\operatorname{Int}(M'')$ is a homotopy equivalence. Since $\sigma_\infty(M)=0$ it follows that $M-\operatorname{Int}(M')$ has finite type. Thus there is a compact polyhedron X and a homotopy equivalence $f: X \times Q \to M-\operatorname{Int}(M')$. Let $g: X \times Q \to M-\operatorname{Int}(M')$ be a $\mathcal{Z}$-embedding which is homotopic to f. Using $\mathcal{Z}$-set unknotting we can find a homeomorphism h of $M-\operatorname{Int}(M')$ onto itself such that $h(\delta(M')) \subset g(X \times Q)$. Then $h^{-1}g: X \times Q \to M-\operatorname{Int}(M')$ is a $\mathcal{Z}$-embedding which is a homotopy equivalence and whose image contains $\delta(M')$. Let $N=h^{-1}g(X \times Q)$, which is a Q-manifold. Since N is a $\mathcal{Z}$-set in $M-\operatorname{Int}(M')$ we can find a collaring $\theta: N \times [0,1) \to M-\operatorname{Int}(M')$ of N. Then

$$M'' = M' \cup \theta(N \times [0, \tfrac{1}{2}])$$

fulfills our requirements. ∎

COROLLARY. *If M is as above, then M is p.h.e. to* Map$(\sigma) \times Q$, *for some inverse sequence σ of compact polyhedra.*

Proof. This is similar to the proof of the *Geometric characterization* theorem of § 4. If we knew that each inclusion $\delta(M_i) \hookrightarrow M_{i+1}-\operatorname{Int}(M_i)$ were a s.h.e., then by the Corollary of the *Geometric characterization* theorem we would have a homeomorphism $M \cong \operatorname{Map}(\sigma) \times Q$. As each inclusion $\delta(M_i) \hookrightarrow M_{i+1}-\operatorname{Int}(M_i)$ is only known to be a homotopy equivalence, the same argument gives a p.h.e. of M with $\operatorname{Map}(\sigma) \times Q$. ∎

First proof of the Homotopy boundary criterion theorem. We are given a Q-manifold M which has finite type and is tame at ∞ and we want to prove that M is p.h.e. to a Q-manifold which admits a boundary.

Note that $\sigma_\infty(M)$ is an element of $\tilde{K}_0\pi_1 E(M)$ which is sent to 0 by the homomorphism $\tilde{K}_0\pi_1 E(M) \to \tilde{K}_0\pi_1(M)$ induced by inclusions. In Appendix 1 we prove that there exists a Q-manifold N and a p.h.e. $f: N \to M$ such that $\sigma_\infty(f)=\sigma_\infty(M)$. By Naturality we have

$$\sigma_\infty(M) = \sigma_\infty(f) + f_* \sigma_\infty(N),$$

and therefore $\sigma_\infty(N)=0$. Then apply the above Corollary. ∎

Second proof of the Homotopy boundary criterion theorem. It follows from Appendix 2 that if M has finite type and is tame at ∞, then M is proper homotopy equivalent to $\operatorname{Map}(\sigma)$, for some inverse sequence σ of compact polyhedra. Then apply the *Geometric characterization* theorem. ∎

6. The residual obstruction

Proposition 5.3 above tells us precisely when a Q-manifold M can be filtered as $M = \bigcup_{i=1}^{\infty} M_i$, where the M_i's are compact and clean, $M_i \subset \operatorname{Int}(M_{i+1})$, and each $\delta(M_i) \hookrightarrow M_{i+1} - \operatorname{Int}(M_i)$ is a homotopy equivalence. Supposing this done, in this section we define an obstruction to improving such a filtration so that in addition each $\delta(M_i) \hookrightarrow M_{i+1} - \operatorname{Int}(M_i)$ is a s.h.e. By the Corollary of the *Geometric characterization* theorem we know this means that M admits a boundary. This will complete our proof of the *Residual obstruction* theorem. In a further result (*Naturality*) we relate our residual obstruction to a similar obstruction of [22].

First it will be convenient to recall the definition of $\operatorname{Wh} \pi_1 E'(X)$ for X any locally compact ANR. Write $X = \bigcup_{i=1}^{\infty} C_i$, where the C_i's are compact and $C_i \subset \operatorname{Int}(C_{i+1})$, let $G_i = \operatorname{Wh} \pi_1(X - C_i)$ and consider the sequence

$$G_1 \xleftarrow{p_1} G_2 \xleftarrow{p_2} G_3 \xleftarrow{p_3} \dots,$$

where the p_i's are inclusion-induced. Define the shift operator Δ from $\prod_{i=1}^{\infty} G_i$ to itself by $\Delta(\tau_1, \tau_2, \dots) = (\tau_1 - p_1(\tau_2), \tau_2 - p_2(\tau_3), \dots)$, and let

$$\varprojlim{}^1 \{\operatorname{Wh} \pi_1(X - C_i)\} = \prod_{i=1}^{\infty} G_i / \operatorname{Image}(\Delta).$$

If $(\tau_1, \tau_2, \dots) \in \prod_{i=1}^{\infty} G_i$, we use $\langle \tau_1, \tau_2, \dots \rangle$ for its image in $\varprojlim{}^1 \{\operatorname{Wh} \pi_1(X - C_i)\}$. If $X = \bigcup_{i=1}^{\infty} C_i'$ is a similarly-defined filtration of X, then $\varprojlim{}^1 \{\operatorname{Wh} \pi_1(X - C_i)\}$ and $\varprojlim{}^1 \{\operatorname{Wh} \pi_1(X - C_i')\}$ are canonically isomorphic. To see this first note that there exists a filtration $X = \bigcup_{i=1}^{\infty} D_i$ such that $\{C_i\}_{i=1}^{\infty}$ and $\{D_i\}_{i=1}^{\infty}$ have subsequences in common and such that $\{D_i\}_{i=1}^{\infty}$ and $\{C_i'\}_{i=1}^{\infty}$ have subsequences in common. It therefore suffices to regard $\{C_i'\}_{i=1}^{\infty}$ as a subsequence, $\{C_{i(n)}\}_{n=1}^{\infty}$, of $\{C_i\}_{i=1}^{\infty}$. Then an isomorphism of $\varprojlim{}^1 \{\operatorname{Wh} \pi_1(X - C_i)\}$ onto $\varprojlim{}^1 \{\operatorname{Wh} \pi_1(X - C_i')\}$ is defined by sending $\langle \tau_1, \tau_2, \dots \rangle$ to

$$\langle \tau_{i(1)} + \tau_{i(1)+1} + \dots + \tau_{i(2)-1}, \tau_{i(2)} + \tau_{i(2)+1} + \dots + \tau_{i(3)-1}, \dots \rangle,$$

where we have avoided writing down compositions of p_i's. Thus we can unambiguously define

$$\operatorname{Wh} \pi_1 E'(X) = \varprojlim{}^1 \{\operatorname{Wh} \pi_1(N - C_i)\},$$

which is an invariant of proper homotopy type.

Proof of the Residual obstruction theorem. Let M be a Q-manifold which is tame at ∞ and for which $\sigma_\infty(M) = 0$. We want to define an element $\tau_\infty(M) \in \operatorname{Wh} \pi_1 E'(M)$ which vanishes iff M admits a boundary. We divide the proof into convenient steps.

(i) *Definition of* $\tau_\infty(M)$. Using Proposition 5.3 write $M = \bigcup_{i=1}^{\infty} M_i$, where the M_i's are compact and clean, $M_i \subset \operatorname{Int}(M_{i+1})$, and each $j(i)\colon \delta(M_i) \hookrightarrow M_{i+1} - \operatorname{Int}(M_i)$ is a homotopy equivalence. Then each $\delta(M_i)\hookrightarrow M_{i+1}-\mathrm{Int}(M_i)$ determines a torsion in $\mathrm{Wh}\,\pi_1(M_{i+1}-\mathrm{Int}(M_i)$ and we let τ_i denote its image in $\mathrm{Wh}\,\pi_1(M-\mathrm{Int}(M_i))$. We define $\tau_\infty(M)$ to be the element of $\mathrm{Wh}\,\pi_1 E'(M)$ represented by the element

$$\langle\tau_1,\tau_2,\dots\rangle\in \underleftarrow{\lim}^1\{\mathrm{Wh}\,\pi_1(M-\mathrm{Int}(M_i))\}. \quad \blacksquare$$

(ii) *$\tau_\infty(M)$ is well-defined.* If $\{M_i\}_{i=1}^\infty$ is replaced by a subsequence then the formula for the torsion of a composition implies that $\langle\tau_1,\tau_2,\dots\rangle$ is replaced by

$$\langle\tau_{i(1)}+\tau_{i(1)+1}+\dots+\tau_{i(2)-1},\ \tau_{i(2)}+\tau_{i(2)+1}+\dots+\tau_{i(3)-1},\ \dots\rangle,$$

where we have suppressed inclusion-induced homomorphisms. But this sequence represents the same element of $\mathrm{Wh}\,\pi_1 E'(M)$. $\blacksquare$

(iii) *If M admits a boundary, then $\tau_\infty(M)=0$.* If M admits a boundary, then the Geometric characterisation theorem implies that $M\cong \mathrm{Map}(\sigma)\times Q$, and therefore we can write $M=\bigcup_{i=1}^\infty M_i$ such that $\delta(M_i)\hookrightarrow M_{i+1}-\mathrm{Int}(M_i)$ is a s.h.e. Thus $\tau_\infty(M)=0$. $\blacksquare$

(iv) *If $\tau_\infty(M)=0$, then M admits a boundary.* If $\tau_\infty(M)=0$, then there is an element $(\mu_1,\mu_2,\dots)\in\prod_{i=1}^\infty \mathrm{Wh}\,\pi_1(M-\mathrm{Int}(M_i))$ such that $\Delta(\mu_1,\mu_2,\dots)=(\tau_1,\tau_2,\dots)$. We will use this to construct a new filtration $M=\bigcup_{i=1}^\infty M_i'$ such that each $\delta(M_i')\hookrightarrow M_{i+1}'-\mathrm{Int}(M_i')$ is a s.h.e. Recall that the Corollary of the *Geometric characterization* theorem will then imply that M admits a boundary. Before giving this modification of the M_i's we will need a lemma.

LEMMA 6.1. *If M is a compact Q-manifold and $\mu\in\mathrm{Wh}\,\pi_1(M)$, then there is a decomposition $M\times[0,1]=M_1\cup M_2$ such that*

(1) *the M_i's are compact Q-manifolds and $M_1\cap M_2$ is a bicollared Q-manifold,*

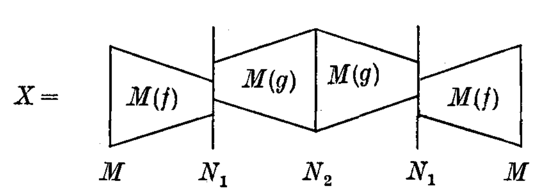
(2) *$M\times\{0\}\subset\mathrm{Int}(M_1)$ and $M\times\{1\}\subset\mathrm{Int}(M_2)$, (interiors taken in $M\times[0,1]$),*

(3) *$M\times\{0\}\hookrightarrow M_1$ is a homotopy equivalence and $\mu=\tau(M\times\{0\}\hookrightarrow M_1)$,*

(4) *$\delta M_2)\hookrightarrow M_2$ is a homotopy equivalence and $\tau(\delta(M_2)\hookrightarrow M_2)=-\mu$.*

(All Whitehead groups in sight are identified naturally to $\mathrm{Wh}\,\pi_1(M)$.)

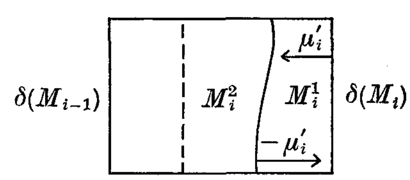
Proof. Let $f\colon M\to N_1$ be a homotopy equivalence, where N_1 is a compact Q-manifold and $\tau(f)\equiv\tau(M)\hookrightarrow M(f))$ is μ, and similarly let $g\colon N_2-N_1$ be a homotopy equivalence where $\tau(g)=-\mu$. Then sew two copies of $M(f)$ to two copies of $M(g)$ to get $X=M(f)\cup_{N_1}M(g)\cup_{N_2} M(g)\cup_{N_1}M(f)$ as pictured.



Note that $\tau(M \hookrightarrow X)=0$, so multiplying everything by Q we get a homeomorphism h of $X \times Q$ onto $M \times [0, 1]$ which takes each $M \times Q$ onto one of $M \times \{0\}$ or $M \times \{1\}$. Then put

$$M_1 = h((M(f) \cup_{N_1} M(g)) \times Q), \quad M_2 = h((M(g) \cup_{N_1} M(f)) \times F). \quad \blacksquare$$

Now let us return to the proof of step (iv) above. Using the fact that $\delta(M_i) \hookrightarrow M - \operatorname{Int}(M_i)$ is a homotopy equivalence we can choose $\mu_i' \in \operatorname{Wh} \pi_1(\delta(M_i))$ such that μ_i' is sent to μ_i by the inclusion-induced isomorphism. Let $\delta(M_i) \times [0, 1] \subset M_i - M_{i-1}$ be a collar on $\delta(M_i) \equiv \delta(M_i) \times \{0\}$ and use Lemma 6.1 to decompose $\delta(M_i) \times [0, 1]$ as $M_i^1 \cup M_i^2$ so that, if j denotes inclusion, $j_*(\mu_i') + \tau(\delta M_i x \{1\} \hookrightarrow M_i^2)$ and $\tau(M_i^1 \cap M_i^2 \hookrightarrow M_i^1) = -j_*(\mu_i')$, Here is a picture of $M_i - \operatorname{Int}(M_{i-1})$.



If we define $M_i' = \text{closure } (M_i - M_i^1)$, then it is clear that $\delta(M_i') \hookrightarrow M_{i+1}' - \operatorname{Int}(M_i')$ is a homotopy equivalence. But using the relationship $\Delta(\mu_1, \mu_2, \ldots) = (\tau_1, \tau_2, \ldots)$ and the formula for the torsion of a composition we now have $\delta(M_i') \hookrightarrow M_{i+1}' - \operatorname{Int}(M_i')$ a s.h.e. $\blacksquare$

This completes the proof of step (iv). Finally we remark that τ_∞ is an invariant of infinite simple homotopy type by the *Naturality theorem* below. $\blacksquare$

We now state and prove the *Naturality theorem*. Recall from [22] that if X is a polyhedron and σ_∞: $S(X) \to \tilde{K}_0 \pi_1 \subset E(X)$ is as mentioned in § 5, then there is a homomorphism τ_∞: $\operatorname{Ker}(\sigma_\infty) \to \operatorname{Wh} \pi_1 E'(X)$. In view of our remarks in § 3 this is equally true if X is a Q-manifold.

THEOREM 6.2. NATURALITY. *If f: $M \to N$ is a p.h.e. of Q-manifolds which are tame at ∞ and if $\sigma_\infty(M) = \sigma_\infty(N) = 0$, then $\tau_\infty(f)$ is defined and*

$$\tau_\infty(N) = \tau_\infty(f) + f_* \tau_\infty(M).$$

Proof of Theorem 6.2. Using the *Naturality* theorem for σ_∞ we have $\sigma_\infty(f) = 0$; which shows that $\tau_\infty(f)$ is defined.

We are at liberty to replace f: $M \to N$ by a p.h.e. f: $X \to Y$ of polyhedra (see § 3), a minor convenience letting us apply [22] more directly. We can assume f is an inclusion $X \hookrightarrow Y$. Recall from [22] that an inclusion $X \hookrightarrow Y$ is called "bumpy" if $\overline{Y - X}$ is a disjoint union $\bigcup_{i=1}^\infty B_i$ of compact subpolyhedra B_i of Y such that, for each i, the inclusion $B_i \cap X \hookrightarrow B_i$ is a homotopy equivalence. It is shown in [22] that $\sigma_\infty(f) = 0$ implies that, after expansion of Y, the inclusion f: $X \hookrightarrow Y$ is a composition $X \hookrightarrow Z \hookrightarrow Y$ of two bumpy inclusions;

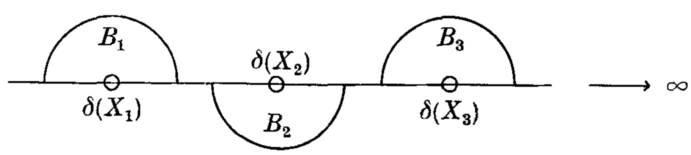
and we note that σ_∞ of everything in view is still zero. If we can verify the additivity equation for each of these two inclusions, then by adding up we deduce it for $f: X \hookrightarrow Y$. Hence we may assume with no loss of generality that $f: X \hookrightarrow Y$ is bumpy as described above.

Since $\sigma_\infty(X)=0$, by expansion of X[1] we can arrange that X has a filtration $X_1 \subset X_2 \subset \ldots$ by subpolyhedra with $X_i \subset \operatorname{Int} X_{i+1}$, so that $\delta X_i \hookrightarrow (X - \operatorname{Int} X_i)$ is a homotopy equivalence.

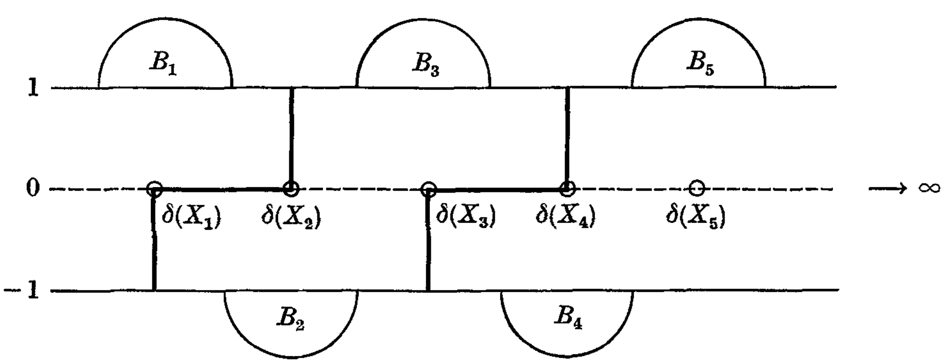
Now $f: X \hookrightarrow Y = X \cup \{\bigcup_{i=1}^\infty B_i\}$ is still bumpy; what is more, by amalgamating the bumps B_i and refining the filtration of X we can arrange that

$$B_j \subset \operatorname{Int} X_i \quad \text{for} \quad j<i \quad \text{and}$$
$$B_j \subset X - X_i \quad \text{for} \quad j>i.$$

Then we get the following picture of Y.



Let Y' be the space formed from $X' \equiv X \times [-1, 1]$ by sewing the odd bumps onto $X \times \{1\}$ and the even bumps onto $X \times \{-1\}$. Then we get the following picture of Y'.



It will suffice to prove that

$$\tau_\infty(Y') = \tau_\infty(X \hookrightarrow Y') + (X \hookrightarrow Y')_* \tau_\infty(X),$$

where X is identified with $X \times \{0\}$ in Y'.

For our computation of $\tau_\infty(X)$ we use the filtration $X_2 \subset X_4 \subset X_6 \subset \ldots$. For our computation of $\tau_\infty(Y')$ we use the filtration $Y_1' \subset Y_3' \subset Y_5' \subset \ldots$, where

$$Y_i' = (X_i \times [-1, 0]) \cup (X_{i+1} \times [0, 1]) \cup (B_1 \cup B_2 \cup \ldots \cup B_i).$$

[1] By abstractly amalgamating with Y along the old X this gives a simultaneous expansion of Y.

It is easy to see that $\delta(Y_i') \hookrightarrow Y' - \text{Int } Y_i'$ is a homotopy equivalence. In the picture above we have $X \equiv X \times \{0\}$ represented by the dotted line and the heavy solid lines mark $\delta(Y_1')$ and $\delta(Y_3')$.

Let b_i be the torsion of $B_i \cap X^+ \hookrightarrow B_i$; let τ_i be the torsion of $\delta X_i \hookrightarrow X - \text{Int } X_i$. Using the basic composition and sum theorems for torsion it is easy to see that the torsion τ_i' of the inclusion $\delta(Y_i') \hookrightarrow Y_{i+2}' - \text{Int } Y_i'$ is a sum

$$\tau_i' = (\tau_i + \tau_{i+1}) + (b_{i+1} + b_{i+2})$$

where we as usual suppress canonical inclusion induced maps to $\text{Wh}\,\pi_1(Y' - \text{Int } Y_i') = G_i$. Then in $\text{Wh}\,\pi_1 E'(Y') = \varprojlim{}^1 \{G_1 \leftarrow G_3 \leftarrow G_5 \leftarrow \ldots\}$ we get

$$\begin{aligned} \langle \tau_1', \tau_3', \ldots \rangle &= \langle \tau_1 + \tau_2, \tau_3 + \tau_4, \ldots \rangle + \langle b_2 + b_3, b_4 + b_5, \ldots \rangle \\ &= (X \hookrightarrow Y')_* \tau_\infty(X) + \tau_\infty(X \hookrightarrow Y'), \end{aligned}$$

where the equality $\langle b_2 + b_3, b_4 + b_5, \ldots \rangle = \tau_\infty(X \hookrightarrow Y')$ follows from the definition given in [22]. ■

7. Realization of the obstructions

The naturality theorems for the obstructions σ_∞, τ_∞ and β show that when the Q-manifold M varies in a fixed proper homotopy type, say running through all of $\mathcal{S}(M_0)$, the obstructions $\sigma_\infty(M)$, $\tau_\infty(M)$ and $\beta(M)$ assume all conceivable values in their respective groups

$$\text{Kernel}\,\{\tilde{K}_0 \pi_1 E(M_0) \to \tilde{K}_0 \pi_1(M_0)\},\ \text{Wh}\,\pi_1 E'(M_0),\ \mathcal{S}_\infty(M_0).$$

(Of course τ_∞ is not defined until $\sigma_\infty = 0$.)

Since the system $\{\pi_1(M_0 - A) \mid A \text{ compact}\}$ is in our case filtered by finitely presented groups (by tameness of M_0) the standard examples, say as given in [22], might suggest that $\text{Wh}\,\pi_1 E'(M_0)$ is always zero in our case.

This is not so, and we propose to give a counterexample in this section. To motivate the algebra we first recall how to pass from the nontrivial group to nontrivial geometric examples.

Geometric examples. Let G be the group $Z \times Z \times Z_6$ and let $\alpha: G \to G$ be given by $\alpha(a, b, c) = (2a, b, c)$. In Proposition 7.1 below we will prove that the first derived limit of the induced inverse sequence of Whitehead groups,

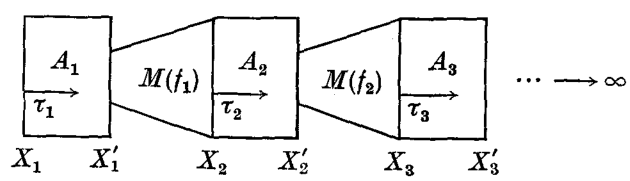
$$\text{Wh}\,(G) \xleftarrow{\alpha_*} \text{Wh}\,(G) \xleftarrow{\alpha_*} \ldots,$$

is non-zero.

For each $i \geq 1$ let X_i be a compact connected polyhedron such that $\pi_1(X_i) \approx G$ and let f_i: $X_{i+1} \to X_i$ induce the map α above on the fundamental groups. Then the first derived limit

$$\varprojlim{}^1 \{\mathrm{Wh}\, \pi_1(X_i), (f_i)_*\} = \varprojlim{}^1 \{\mathrm{Wh}\,(G) \xleftarrow{\alpha_*} \mathrm{Wh}\,(G) \xleftarrow{\alpha_*} \dots\}$$

is non-zero so we may choose an element $(\tau_1, \tau_2, \dots) \in \prod_{i=1}^{\infty} \mathrm{Wh}\, \pi_1(X_i)$ such that $\langle \tau_1, \tau_2, \dots \rangle \neq 0$. Consider the non-compact space X pictured below.



It is the union of compact polyhedra $A_1, A_2, \dots$ and the mapping cylinders $M(f_i)$ (where $X_1', X_2', \dots$ are copies of $X_1, X_2, \dots$) and

(1) there is a homeomorphism of X_i onto X_i' which is homotopic to the identity of X_i, with the homotopy taking place in A_i,

(2) $X_i \hookrightarrow A_i$ is a homotopy equivalence and $\tau(X_i \hookrightarrow A_i)$ equals the image of τ_i. The A_i's are constructed just like the W_i's in step (iv) of the proof of the *Compactification classification* theorem in § 11 below.

Clearly X is tame at ∞ and $\sigma_\infty(X) = 0$. Since $\tau(X_i \hookrightarrow A_i \cup M(f_i))$ equals the image of τ_i we have $\tau_\infty(X) \neq 0$. Then $M = X \times Q$ is our example. ∎

It remains to verify

PROPOSITION 7.1. *Let $G = Z \times Z \times Z_6$ and let α: $G \to G$ be given by $\alpha(a, b, c) = (2a, b, c)$. Then the first derived limit of*

$$\mathrm{Wh}\,(G) \xleftarrow{\alpha_*} \mathrm{Wh}\,(G) \xleftarrow{\alpha_*} \dots$$

is non-zero.

Proof. Here are the main steps in the proof. For convenience we write $G = Z \times H$, where $H = Z \times Z_6$.

(i) We first recall that Wh (G) contains $\tilde{K}_0(H)$ as a direct summand.

(ii) Next we show that α_*: Wh $(G) \to$ Wh (G) is a direct sum of the homomorphism μ_2: $\tilde{K}_0(H) \to \tilde{K}_0(H)$ (multiplication by 2) with some other homomorphism.

(iii) Finally we observe that $\tilde{K}_0(H)$ contains an infinite cyclic summand J.

Before these three steps let us show how they imply that our required first derived limit is non-zero. Steps (i) and (iii) give us

$$\mathrm{Wh}\,(G) = J \oplus K,$$

for some K, and step (ii) implies that α_*: Wh $(G) \to$ Wh (G) is given by $\alpha_* = \mu_2 \oplus k$, where μ_2: $J \to J$ is multiplication by 2. It is easy to see that a non-trivial element of the above first derived limit is given by

$$\langle 1 \oplus 0, -1 \oplus 0, 1 \oplus 0, -1 \oplus 0, \ldots \rangle,$$

where 1 is a generator of J. (Note that $\langle 1 \oplus 0, 1 \oplus 0, \ldots \rangle$ will not work.). In fact such a $\lim^1$ is always uncountable [15].

Proof of (i). The Fundamental theorem of algebraic K-Theory [3], [1, p. 663], (cf [25, p. 15]) asserts that

$$\mathrm{Wh}\,(G) \approx \tilde{K}_0(H) \oplus R,$$

where the injection and projection to $\tilde{K}_0(H)$,

$$\mathrm{Wh}\,(G) \underset{j}{\overset{p}{\rightleftarrows}} \tilde{K}_0(H),$$

are defined as follows. (We will need descriptions of p and j for step (ii)).

(a) Given $[P] \in \tilde{K}_0(H)$ we can naturally write

$$Z[G] \otimes P = P[t, t^{-1}] = \ldots \oplus t^{-1}P \oplus P \oplus tP \oplus \ldots,$$

where the tensor product is taken over $Z[H]$ and t is a generator of Z in G. A shift automorphism of $P[t, t^{-1}]$ is given by multiplication by t and it represents $j[P] \in$ Wh (G).

(b) Let $[\varphi] \in$ Wh (G) be represented by a $Z[G]$-linear automorphism φ: $Z[G]^m \to Z[G]^m$. Naturally writing $Z[G]^m = Z[H]^m[t, t^{-1}]$ we let $B = Z[H]^m[t] \subset Z[G]^m$. Without changing the class $[\varphi]$ we can arrange it so that $\varphi(B) \subset B$ and $B/\varphi(B)$ is a f.g. projective $Z[H]$ module. Then

$$p[\varphi] = [B/\varphi(B)] \in \tilde{K}_0(H).$$

Proof of (ii). We will first show that $\alpha_* | j\tilde{K}_0(H)$: $j\tilde{K}_0(H) \to j\tilde{K}_0(H)$ is μ_2. If $[P] \in \tilde{K}_0(H)$, then $j[P]$ is represented by the shift automorphism on $P[t, t^{-1}]$ and $\alpha_* j[P]$ comes from substituting t^2 for t, then extending canonically to retrieve an automorphism of $P[t, t^{-1}]$. What we retrieve is the 2-fold shift automorphism $x \to t^2 x$ which certainly represents $2j[P]$.

It remains only to show that α_* maps the "remainder" R to itself. Since $R =$ Kernel (p) it suffices a fortiori to prove that

$$p\alpha_*[\varphi] = 2p[\varphi],$$

for all $[\varphi] \in$ Wh (G). If $[\varphi]$ is given by a $Z[G]$-linear automorphism φ: $Z[G]^m \to Z[G]^m$, then $\alpha_*[\varphi]$ is represented by the automorphism $\alpha_\# \varphi$ obtained by substituting t^2 for t and then extending canonically to a $Z[G]$-linear automorphism of $Z[G]^m$. (Again we use $Z[G]^m = Z[H]^m[t, t^{-1}]$.) As a $Z[H]$-linear automorphism, our $\alpha_{\#}\varphi$ is thus a sum of 2 copies of

$$\varphi': \ldots \oplus t^{-2}Z[H]^m \oplus Z[H]^m \oplus t^2 Z[H]^m \oplus \ldots \hookleftarrow,$$

namely φ' and $t\varphi'$. Hence $B/\alpha_{\#}\varphi(B)$ is isomorphic to a sum of 2 copies of $B_2/\varphi'(B_2)$, where B_2 is B with t^2 substituted for t. But clearly $B_2/\varphi'(B_2)$ is $Z[H]$-isomorphic to $B/\varphi(B)$. Thus $p\alpha_*[\varphi] = 2p[\varphi]$ as required.

Proof of (iii). For this we refer to [2, § 8.10], where it is shown that if $H = T \times F$, where T is torsion abelian with order divisible by two distinct primes and F is free of rank $\geqslant 1$, then $\tilde{K}_0(H)/T$orsion is free of rank $\geqslant 1$ (This rank comes from $\tilde{K}_{-1}(T) = \tilde{K}_{-1}(Z[T])$ via periodicity.) Certainly $T = Z_6$ and $F = Z$ fulfill these requirements. ■

8. The total obstruction

For M a Q-manifold which has finite type and is tame at ∞, we will show how to define an obstruction $\beta(M) \in \mathcal{S}_\infty(M)$ to M having a boundary and thereby prove the *Total obstruction* theorem. In a further result (*Naturality*) we relate our obstruction to β as defined in § 3 for infinite simple homotopy theory. As mentioned in § 2 the essential ingredient of our construction is the *Peripheral homeomorphism paradox* which we prove first. We will give two proofs of this result. The first is a short argument based upon the exact sequences of [22]; the second is longer but is completely geometric in nature.

First proof of the Peripheral homeomorphism paradox. We have a p.h.e. $f: M \to N$ between Q-manifolds which admit boundaries and we want to prove that f is proper homotopic to a homeomorphism near ∞. If $\beta(f)$ is the torsion of f in $\mathcal{S}_\infty(N)$, then by (§ 3) all we have to do is prove that $\beta(f) \in \mathcal{S}_\infty(N)$ vanishes. There exists an exact sequence from [22] as we will explain again below:

$$0 \to \mathrm{Wh}\,\pi_1 E'(N) \to \mathcal{S}_\infty(N) \to \tilde{K}_0\,\pi_1 E(N)$$

Since M and N admit boundaries we have $\sigma_\infty(M) = \sigma_\infty(N) = \tau_\infty(M) = \tau_\infty(N) = 0$. By Naturality this gives $\sigma_\infty(f) = 0$ and $\tau_\infty(f) = 0$, hence $\beta(f) = 0$ and therefore f is proper homotopic to a homeomorphism near ∞. ■

If $\mathrm{Wh}\,\pi_1(N) = 0$, which happens for example when M and N are contractible, then $\tau(f) = 0$ and therefore f is proper homotopic to a homeomorphism. ■

The second proof requires repeated use of the

TRANSVERSALITY LEMMA. *Let N be a Q-manifold and let $N_1 \subset N$ be a clean compact submanifold. Suppose $M \subset N$ is a $\mathcal{Z}$-set that is a Q-manifold. Then there exists a homeomor-*

phism h: $N \to N$ arbitrarily close to $\mathrm{id}|N$ *such that $h(M) = M'$ is a Q-manifold that cuts $\delta(N_1)$ transversally in the sense that $\delta(N_1)$ has a bicollaring in N restricting to a bicollaring of $M' \cap \delta(N_1)$ in M', while $M' \cap \delta(N_1)$ is itself a Q-manifold.*

This follows easily from a PL transversality lemma using triangulations and $\mathcal{Z}$-set properties. We leave the proof as an exercise.

Second proof of the Peripheral homeomorphism paradox. We are given a p.h.e. f: $M \to N$ of Q-manifolds which admit boundaries and without loss of generality we may assume that M is a $\mathcal{Z}$-set in N and f is the inclusion map. Since M and N admit boundaries we can clearly write $M = \bigcup_{i=1}^{\infty} M_i$ and $N = \bigcup_{i=1}^{\infty} N_i$ such that

(1) the M_i's and N_i's are compact and clean,

(2) $M_i \subset \mathrm{Int}(M_{i+1})$ and $N_i \subset \mathrm{Int}(N_{i+1})$,

(3) the inclusions $\delta(M_i) \hookrightarrow M_{i+1} - \mathrm{Int}(M_i)$ and $\delta(N_i) \hookrightarrow N_{i+1} - \mathrm{Int}(N_i)$ are s.h.e.'s. (Here $\delta(M_i)$ and $\mathrm{Int}(M_i)$ are computed relative to M.)

What is more, applying the Transversality lemma above we can assure that

(4) M crosses $\delta(N_i)$ *transversally.*

At this point, after refining and reindexing we have $M_i \subset N_i \subset M$ and the simple trick of subtracting from N_i the (interior of) a suitable collar on $M - \mathrm{Int}\, M_i$ will even assure that

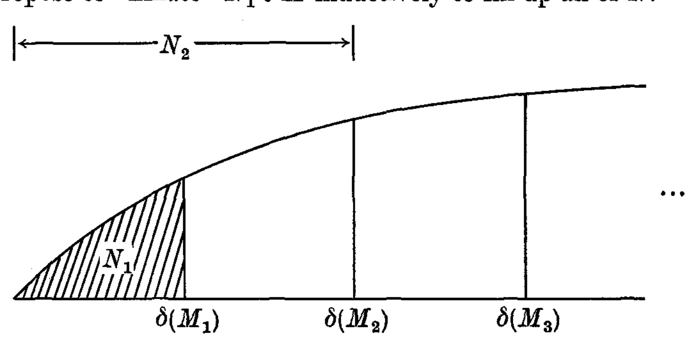
(5) $M_i = N_i \cap M$,

The inclusion $M \hookrightarrow N$ factors into the composition

$$M \hookrightarrow M \cup N_1 \hookrightarrow N,$$

where the first inclusion is clearly a homeomorphism near ∞. So all we have to do is prove that $M \cup N_1 \hookrightarrow N$ is proper homotopic to a homeomorphism. This will also take care of the case in which M and N are contractible, for it then follows that M_1, N_1 are Hilbert cubes and therefore $M \hookrightarrow M \cup N_1$ is proper homotopic to a homeomorphism.

We now propose to "inflate" $N_1 \cup M$ inductively to fill up all of N.



Observe first that for any $k \geqslant 2$ the inclusion

$$j: \delta N_1 \cup (M_k - \operatorname{Int} M_1) \hookrightarrow (N_k - \operatorname{Int} N_1)$$

is a simple homotopy equivalence (s.h.e.) because $\delta M_1 \hookrightarrow (M_k - \operatorname{Int} M_1)$ and $\delta N_1 \hookrightarrow (N_k - \operatorname{Int} N_1)$ are s.h.e.'s. Using the Hauptvermutung of [7] together with $\mathcal{Z}$-set unknotting we deduce a homeomorphism

$$\theta: N_1 \cup M_k \to N_k$$

fixing δM_k and also all of N_1 except a small collar C of δN_1.

Assertion: If k is sufficiently large, θ can be corrected so that $\theta^{-1} \delta N_k \subset M_k - \operatorname{Int} M_1$.

Proof: Using the fact that M is a strong proper deformation retract of N together with homotopy extension properties it is not difficult to construct, *for k large,* a deformation retraction

$$\gamma: \delta N_1 \cup (M_k - \operatorname{Int} M_1) \leftarrow (N_k - \operatorname{Int} N_1)$$

such that $\gamma(\delta N_k) \subset M_k - \operatorname{Int} M_1$. Now $\theta^{-1} | \delta N_k$ is homotopic to $\gamma | \delta N_k$, as a map into $C \cup (M_k - \operatorname{Int} M_1)$; indeed θ^{-1} and γ both give a homotopy inverse to $j \cup (\mathrm{id} | C)$. Thus $\mathcal{Z}$-set principles applied to this homotopy of $\theta^{-1} | \delta M_k$ let us correct θ as desired. ■

Extend the corrected θ by the identity to a homeomorphism

$$h_1: N_1 \cup M \to N_{k_1} \cup M \quad (k = k_1)$$

Repeat the construction of h_1 with N_{k_1} in place of N_1 to produce a homeomorphism h_2: $N_{k_1} \cup M \to N_{k_2} \cup M$ and iterate to produce a sequence $h_1, h_2, h_3, \ldots$ of homeomorphisms h_i: $N_{k_{i-1}} \cup M \to N_{k_i} \cup M$ such that h_i is supported on a small neighborhood of $\delta N_{k_{i-1}} \cup (M_{k_i} - \operatorname{Int} M_{k_{i-1}})$ disjoint from $h_{i-1}(N_{k_{i-2}})$, which (see our assertion) lies in $\operatorname{Int}(N_{k_{i-1}})$. The limit

$$h(x) = \lim_{i \to \infty} h_i h_{i-1} \ldots h_1(x)$$

is a homeomorphism $h: N_1 \cup M \to N$ since every point x in $N_1 \cup M$ has a neighborhood U_x such that the sequence of restrictions to U_x of the maps $h_1, h_2 h_1, h_3 h_2 h_1, \ldots$ moves at most twice to reach $h | U_x$. ■

Proof of the Total obstruction theorem. Let M be a Q-manifold which has finite type and is tame at ∞. Using the *Homotopy boundary criterion* theorem there exists a Q-manifold N which has a boundary and a p.h.e. $f: N \to M$. We define $\beta(M) = \beta(f) \in S_\infty(M)$.

By § 3 we see that the vanishing of $\beta(M)$ is a sufficient condition for M to admit a boundary. To show that it is also necessary assume that M also admits a boundary. Then the *Peripheral homeomorphism paradox* implies that $\beta(M) = \beta(f) = 0$. It is clear that $\beta(M)$ is an invariant of infinite simple homotopy type. ■

THEOREM 8.1: NATURALITY. *If $f: M \to N$ is a p.h.e. of Q-manifolds which have finite type and are tame at ∞, then*

$$\beta(N) = \beta(f) + f_*\beta(M).$$

Proof. Let P be a Q-manifold which admits a boundary and let $g: P \to M$ be a p.h.e. Then $\beta(M) = \beta(g)$ and $\beta(N) = \beta(fg)$. The formula for the torsion of a composition gives

$$\begin{aligned}\beta(N) &= \beta(fg) = \beta(f) + f_*\beta(g)\\ &= \beta(f) + f_*\beta(M). \quad \blacksquare\end{aligned}$$

Finally we establish a result which connects $\beta(M)$, $\sigma_\infty(M)$ and $\tau_\infty(M)$. The exact sequences of [22] give

$$S(M) \xrightarrow{\sigma_\infty} \tilde{K}_0\pi_1 E(M) \longrightarrow \tilde{K}_0\pi_1(M),$$

$$\mathrm{Wh}\,\pi_1 E(M) \longrightarrow \mathrm{Wh}\,\pi_1(N) \longrightarrow \mathrm{Ker}\,(\sigma_\infty) \xrightarrow{\tau_\infty} \mathrm{Wh}\,\pi_1 E'(M) \longrightarrow 0,$$

for any Q-manifold M, where σ_∞ is as described in § 5 of this paper and τ_∞ is as described in § 6. If we mod out $\mathrm{Wh}\,\pi_1(M)$ we get an induced exact sequence

$$0 \longrightarrow \mathrm{Wh}\,\pi_1 E'(M) \longrightarrow S_\infty(M) \xrightarrow{\sigma_\infty} \tilde{K}_0\pi_1 E(M) \longrightarrow \tilde{K}_0\pi_1(M).$$

PROPOSITION 8.2. *If M is a Q-manifold which has finite type and is tame at ∞, then $\sigma_\infty(\beta(M)) = \sigma_\infty(M)$. If $\sigma_\infty(M) = 0$, then $\tau_\infty(\beta(M)) = \tau_\infty(M)$.*

Proof. Choose a p.h.e. $f: N \to M$, where N admits a boundary, and use *Naturality* to get

$$\sigma_\infty(M) = \sigma_\infty(f) + f_*\sigma_\infty(N).$$

But $\sigma_\infty(N) = 0$ (since N admits a boundary), hence $\sigma_\infty(M) = \sigma_\infty(f) = \sigma_\infty(\beta(M))$. If $\sigma_\infty(M) = 0$, then again using Naturality we get $\tau_\infty(M) = \tau_\infty(f) = \tau_\infty(\beta(M))$. ■

9. A practical boundary theorem

In this section we prove the *Practical boundary theorem* and its Corollary. A non-compact Q-manifold M is said to be 1-*ended* provided that for each compactum $A \subset M$, $M - A$ has exactly one unbounded component. This permits us to find a basis $U_1 \supset U_2 \supset \dots$ of connected open neighborhoods of ∞. We say that π_1 is *essentially constant at* ∞ if $\{U_i\}$ can be chosen so that the sequence

$$\pi_1(U_1) \xleftarrow{\varphi_1} \pi_1(U_2) \xleftarrow{\varphi_2} \dots$$

induces isomorphisms

$$\mathrm{Image}\,(\varphi_1) \xleftarrow{\approx} \mathrm{Image}\,(\varphi_2) \xleftarrow{\approx} \dots$$

where $\varphi_1, \varphi_2, \ldots$ are inclusion-induced. Then $\pi_1(\infty) = \varprojlim \{\text{Image}\ (\varphi_1)\}$ is well-defined up to isomorphism.

Proof of the Practical boundary theorem. We are given a 1-ended Q-manifold M which is tame at ∞ such that π_1 is essentially constant at ∞, with $\pi_1(\infty)$ free or free abelian. We need to show that $S_\infty(M) = 0$. It follows from [22] that $S_\infty(M) \approx \tilde{K}_0 \pi_1(\infty)$ and for $\pi_1(\infty)$ free or free abelian we have $\tilde{K}_0 \pi_1(\infty) = 0$ by [3]. ∎

A non-compact Q-manifold J is said to be LC^1 at ∞ provided that M is 1-ended and for every compactum $A \subset M$ there exists a larger compactum $B \subset M$ such that every loop in $M - B$ is null-homotopic in $M - A$.

Proof of the Corollary. We are given a Q-manifold M which is LC^1 at ∞ and for which $H_*(M)$ is f.g. It is easy to see that π_1 is essentially constant at ∞, with $\pi_1(\infty) = 0$. Using the above result it suffices to prove that M is tame at ∞.

It follows from the techniques of [22] that we can write $M = \bigcup_{i=1}^{\infty} M_i$, where the M_i's are compact and clean, $M_i \subset \text{Int}(M_{i+1})$, and $\delta(M_i)$, $M - \text{Int}(M_i)$ are 1-connected. It will suffice to prove that each $M - \text{Int}(M_i)$ has finite type. It follows from the homology exact sequence of the pair (M, M_i) that $H_*(M, M_i)$ is f.g. Using excision we get $H_*(M - \text{Int}(M_i), \delta(M_i))$ f.g. and using the homology sequence of $(M - \text{Int}(M_i), \delta(M_i))$ it follows that $H_*(M - \text{Int}(M_i))$ is f.g. Thus $M - \text{Int}(M_i)$ has finite type [26, p. 420]. ∎

10. Classification of boundaries

The purpose of this section is to prove the *Boundary classification* theorem. For its proof we have to use the $\mathcal{Z}$-set classification result of [6].

Proof of the Boundary classification theorem. For the first half of this result let Z and Z' be boundaries for M. We want to prove that Z and Z' have the same shape. Let $N = M \cup Z$ and $N' = M \cup Z'$, and replace N by $N \times [0, 1]$ so that $Z \subset N \times \{0\}$. There is a homeomorphism $h: (N \times [0, 1]) - Z \to N' - Z'$. Let $N \times \{1\}$ be regarded as a $\mathcal{Z}$-set in Q and put

$$Q_1 = Q \cup (N \times [0, 1]) \quad \text{(sewn along } N \times \{1\}),$$

$$Q_2 = Q \cup N' \quad \text{(sewn along } h(N \times \{1\})).$$

Then Q_1 is clearly a copy of Q and Q_2 is a copy of Q because it is a compact contractible Q-manifold. Also Z and Z' are $\mathcal{Z}$-sets in Q_1 and Q_2, respectively. Since $Q_1 - Z \cong Q_2 - Z'$ we have $\text{Shape}(Z) = \text{Shape}(Z')$ by the $\mathcal{Z}$-set classification result of [6] cf. [9], [24].

For the other half let Z be a boundary for M and let Z' be shape equivalent to Z. We want to prove that Z' is also a boundary for M. Let $N \times [0, 1] = M \cup Z$ be a compactification of M, where $Z \subset N \times \{0\}$, and form

$$Q_1 = Q \cup (N \times [0, 1])$$

as above. Then let $Z' \subset Q_1$ be embedded as a $\mathcal{Z}$-set and use [6] to get a homeomorphism h of $Q_1 - Z$ onto $Q_1 - Z'$. Clearly

$$h((N \times [0, 1]) - Z) \cup Z'$$

gives us a compactification of M with Z' as the boundary. ∎

11. Classification of compactifications

In this section we classify the different ways in which a Q-manifold can be compactified. Let $N = M \cup Z$ be a fixed compactification of the Q-manifold M and let $N' = M \cup Z'$ be any other one. We will define an element $\tau(N, N') \in \mathrm{Wh}\, \pi_1 E(M)$ which vanishes iff N' is equivalent to N (as defined in § 2). This defines a $1-1$ correspondence between $\mathrm{Wh}\, \pi_1 E(M)$ and the different compactifications of M.

Proof of the Compactification classification theorem. We have divided the proof into four steps.

(i) *Construction of* $\tau(N, N')$. Write $N - Z = \bigcup_{i=1}^{\infty} M_i$, where the M_i's are compact and clean, $M_i \subset \mathrm{Int}(M_{i+1})$, and $\delta(M_i) \hookrightarrow N - \mathrm{Int}(M_i)$ is a s.h.e. Let $N' - Z' = \bigcup_{i=1}^{\infty} M_i'$ be a similar filtration and arrange it so that

$$M_1 \subset \mathrm{Int}(M_1') \subset M_1' \subset \mathrm{Int}(M_2) \subset M_2 \subset \mathrm{Int}(M_2') \subset M_2' \subset \dots$$

Then $\delta(M_i) \hookrightarrow M_i' - \mathrm{Int}(M_i)$ is a homotopy equivalence and we use x_i for the image of $\tau(\delta(M_i) \hookrightarrow M_i' - \mathrm{Int}(M_i))$ in $\mathrm{Wh}\, \pi_1(M - \mathrm{Int}(M_i))$. It is not hard to see that the inclusion-induced homomorphism of $\mathrm{Wh}\, \pi_1(M - \mathrm{Int}(M_{i+1}))$ to $\mathrm{Wh}\, \pi_1(M - \mathrm{Int}(M_i))$ takes x_{i+1} to x_i. Then $(x_1, x_2, \dots)$ defines an element of $\mathrm{Wh}\, \pi_1 E(M)$ which we denote $\tau(N, N')$.

(ii) $\tau(N, N')$ *is well-defined*. It is only necessary to show that if $\{M_i\}_{i=1}^{\infty}$ and $\{M_i'\}_{i=1}^{\infty}$ are replaced by subsequences $\{M_{k_i}\}_{i=1}^{\infty}$ and $\{M_{k_i}'\}_{i=1}^{\infty}$, then we get the same definition of $\tau(N, N')$. Let x_{k_i}' denote the image of $\tau(\delta(M_{k_i}) \hookrightarrow M_{k_i}' - \mathrm{Int}(M_{k_i}))$ in $\mathrm{Wh}\, \pi_1(M - \mathrm{Int}(M_{k_i}))$. All we need to do is show that $(x_1, x_2, \dots)$ and $(x_{k_1}', x_{k_2}', \dots)$ give the same element of $\mathrm{Wh}\, \pi_1 E(M)$, and for this it suffices to note that the image of x_{k_i}' in $\mathrm{Wh}\, \pi_1(M - \mathrm{Int}(M_i))$ is x_i.

(iii) N' *is equivalent to* N *iff* $\tau(N, N') = 0$. First assume that $\tau(N, N') = 0$. Let $\{M_i\}$, $\{M_i'\}$ be chosen to define $\tau(N, N')$ and let $A \subset M$ be compact. Then for some i we have $A \subset \mathrm{Int}(M_i)$. Since $\delta(M_i) \hookrightarrow N - \mathrm{Int}(M_i)$ is a s.h.e. we have a homeomorphism of N onto M_i fixing A. We also have a homeomorphism of N' onto M_i' fixing A. Then since $\delta(M_i) \hookrightarrow$

$M_i' - \text{Int}(M_i)$ is a s.h.e. (which follows from $\tau(N, N') = 0$) we have a homeomorphism of M_i onto M_i' which is fixed on A.

On the other hand assume that N' is equivalent to N and let $\{M_i\}$, $\{M_i'\}$ be chosen to define $\tau(N, N')$. Then there is a homeomorphism $h: N \to N'$ which is fixed on M_1 (pointwise). This implies that $\delta(M_1) \hookrightarrow N' - \text{Int}(M_1)$ is a s.h.e. and as $\delta(M_1') \hookrightarrow N' - \text{Int}(M_1')$ is a s.h.e. it follows that $\delta(M_1) \hookrightarrow M_1' - \text{Int}(M_1)$ is a s.h.e. In like manner we can prove that $\delta(M_i) \hookrightarrow M_i' - \text{Int}(M_i)$ is a s.h.e., for each i. Thus $\tau(N, N') = 0$.

Now we define θ to be the function from the equivalence classes of compactifications of M to $\text{Wh}\,\pi_1 E(M)$ defined by the rule $N' \to \tau(N, N')$. The following step will finish our proof.

(iv) *θ is a $1-1$ correspondence.* If N' and N'' are other compactifications of M, then it is easy to see that

$$\tau(N, N'') = \tau(N, N') + \tau(N', N'').$$

This therefore implies that θ is well-defined and $1-1$. All we have left to do is prove that θ is onto. Choose $M = \bigcup_{i=1}^{\infty} M_i$ as in step (i) and choose $\{x_i\} \in \varprojlim \{\text{Wh}\,\pi_1(M - \text{Int}(M_i))\} = \text{Wh}\,\pi_1 E(M)$. We must construct a compactification N' of N such that $\tau(N, N') = \{x_i\}$.

Since $\delta(M_i) \hookrightarrow M - \text{Int}(M_i)$ is a homotopy equivalence we can find an element $y_i \in \text{Wh}\,\pi_1(\delta(M_i))$ which is sent to x_i by the inclusion-induced homomorphism. Let $\delta(M_i) \times [0, 1] \subset \text{Int}(M_{i+1}) - \text{Int}(M_i)$ be a closed collar on $\delta(M_i) \equiv \delta(M_i) \times \{0\}$. It is possible to find a clean $W_i \subset \delta(M_i) \times [0, 1)$ containing $\delta(M_i)$ in its interior such that

(1) $\delta(M_i) \hookrightarrow W_i$ is a homotopy equivalence and $\tau(\delta(M_i) \hookrightarrow W_i)$ equals the image of y_i,

(2) there exists a homeomorphism of $\delta(M_i)$ onto $\delta(W_i)$ which is homotopic to the identity on $\delta(M_i)$ with the homotopy taking place in W_i.

To see this consider the following picture of a Q-manifold

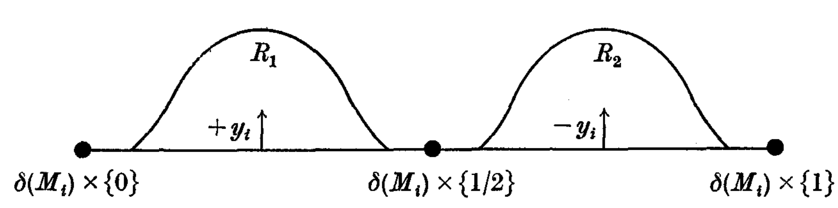
$$X = (\delta(M_i) \times [0, 1]) \cup R_1 \cup R_2$$

obtained from $\delta(M_i) \times [0, 1]$ by adding on compact Q-manifolds R_1 and R_2 such that

(a) $R_k \cap (\delta(M_i) \times [0, 1]) \hookrightarrow R_k$ is a homotopy equivalence for $k = 1, 2$,

(b) $\tau(\delta(M_i) \times \{0\} \hookrightarrow (\delta(M_i) \times (0, \tfrac{1}{2}]) \cup R_1)) =$ image of y_i,

(c) $\tau(\delta(M_i) \times \{1\} \hookrightarrow (\delta(M_i) \times [\tfrac{1}{2}, 1] \cup R_2)) =$ image of $-y_i$.



Then $\delta(M_i)\times\{0\}\hookrightarrow X$ is a s.h.e. and we can construct a homeomorphism of X onto $\delta(M_i)\times[0, 1]$ so that our required W_i is the image of $(\delta(M_i)\times[0, \frac{1}{2}])\cup R_1$.

Using (1) and (2) above it is easy to construct a homeomorphism h of $M-\mathrm{Int}(M_1)$ onto $M-\mathrm{Int}(M_1\cup W_1)$ such that $h(\delta(M_i))=\delta(W_i)$, for each i. Let N' be the compactification of M obtained by sewing M_1 to $h(M-\mathrm{Int}(M_1))$ by h. It is then clear that $\tau(N, N')=\{x_i\}$. ■

Appendix 1. The realization theorem for σ_∞

The purpose of this section is to give a proof of the Realization Theorem of [22], since only the barest outline of a proof was given there. Recall the proof of the *Naturality Theorem* 5.1 of this paper, where $\sigma_\infty(X\hookrightarrow Y)\in\tilde{K}_0\pi_1E(Y)$ was defined for a p.h.e. $X\hookrightarrow Y$ of polyhedra.

REALIZATION THEOREM. *Let X be a connected polyhedron and choose $x\in\tilde{K}_0\pi_1E(X)$ which is sent to 0 by the inclusion-induced homomorphism $\tilde{K}_0\pi_1E(X)\to\tilde{K}_0\pi_1(X)$. Then there exists a polyhedron Y containing X as a subpolyhedron such that $X\hookrightarrow Y$ is a p.h.e. and $\sigma_\infty(X\hookrightarrow Y)$ equals $(X\hookrightarrow Y)_*(x)$, the image of x in $\tilde{K}_0\pi_1E(Y)$. Furthermore $Y-X$ can consist of cells of dimension n and $n+1$ only, the n-cells trivially attached ($n\geq 2$).*

Needless to say, this result can be reformulated for locally finite CW complexes or for Q-manifolds.

Proof of the Realization theorem. We will construct Y so that $X\hookrightarrow Y$ is a p.h.e. *near* ∞ and $\sigma_\infty(X\hookrightarrow Y)=(X\hookrightarrow Y)_*(x)$. Recall from [22, p. 491] that this was the only missing part of the proof. *For this we will not have to suppose that x goes to 0 in $\tilde{K}_0\pi_1(X)$.*

We will assume for the moment that X has only one end. Thus we can write

$$X=X_1\hookleftarrow X_2\hookleftarrow\ldots,$$

a basis of connected open neighborhoods of ∞ in X. Note that $x\in\tilde{K}_0\pi_1E(X)$ gives a rule assigning to each X_i an element $x_i\in\tilde{K}_0\pi_1(X_i)$ such that x_{i+1} goes to x_i under the inclusion-induced homeomorphism. Let x_i be represented by $[P_i]$, where P_i is a f.g. projective module over $\Lambda_i=Z[\pi_1(X_i)]$. Since $x_{i+1}\mapsto x_i$ we have P_i stably isomorphic to

$$P'_{i+1}=\Lambda_i\otimes_{\Lambda_{i+1}}P_{i+1},$$

where Λ_i may be regarded as a f.g. projective Λ_{i+1}-module because of the inclusion-induced homeomorphism $\pi_1(X_{i+1})\to\pi_1(X_i)$. Therefore we can inductively choose f.g. projective Λ_i-modules Q_i, F_i, G_i, where F_i and G_i are free, and Λ_i-isomorphisms

$$\varphi_i: P_i \oplus Q_i \to F_i,$$
$$\psi_i: Q_i \oplus P'_{i+1} \to G_i.$$

We will now add 2- and 3-cells to X to obtain our required Y. First wedge a collection B_i of 2-spheres onto X_i, one for each basis element of F_i. Do this for all i and set

$$Z_i = X_i \cup B_i \cup B_{i+1} \cup \dots .$$

Using $\sim$ to indicate universal covers we see that $H_*(\tilde{Z}_i, \tilde{X}_i)$ (regarded as a Λ_i-module) is isolated in dimension 2 and

$$H_2(\tilde{Z}_i, \tilde{X}_i) \cong F_i \oplus F'_{i+1} \oplus F'_{i+2} \oplus \dots .$$

(Primes indicate a module converted by tensoring with Λ_i to become a f.g. projective Λ_i-module.) Using the isomorphisms φ_i we have

$$H_2(\tilde{Z}_i, \tilde{X}_i) \cong P_i \oplus Q_i \oplus P'_{i+1} \oplus Q'_{i+1} \oplus \dots$$

Using the isomorphism φ_i^{-1} we get a homomorphism

$$\Psi_i: G_i \to H_2(\tilde{Z}_i, \tilde{X}_i) = \pi_2(Z_i, X_i)$$

mapping isomorphically onto $Q_i \oplus P'_{i+1}$, where the last equality here follows from the Hurewicz isomorphism theorem. Since we have a retraction $r_i: Z_i \to X_i$ it follows from the homotopy sequence of (Z_i, X_i) that

$$\pi_2(Z_i, X_i) = \text{Kernel}((r_i)_*) \subset \pi_2(Z_i),$$

where $(r_i)_*: \pi_2(Z_i) \to \pi_2(X_i)$. Thus for each basis element of G_i we have a homotopy class $S^2 \to Z_i$. For each base element we now attach a 3-cell and call these 3-cells

$$E_i = e_{i,1} \cup \dots \cup e_{i,n}.$$

Make these additions for each i and set

$$Y_i = Z_i \cup E_i \cup E_{i+1} \cup \dots, \; i \geq 1.$$

Then put $X = Y_1$ to complete our construction.

It is clear that $H_*(\tilde{Y}_i, \tilde{X}_i)$ is a f.g. Λ_i-module isolated in dimension 2 and

$$H_2(\tilde{Y}_i, \tilde{X}_i) \approx P_i$$

Thus to conclude that $\sigma_\infty(X \hookrightarrow Y) = (X \hookrightarrow Y)_*(x)$ all we need to do is prove that $X \hookrightarrow Y$ is a p.h.e. near ∞.

As indicated in [22] a convenient way to do this is to verify the two conditions of [22] called $(\pi_1)_\infty$ and $(H_*)_\infty$, which together imply that $X \hookrightarrow Y$ is a p.h.e. near ∞. It is essential here that $\dim(Y-X)<\infty$. The condition $(\pi_1)_*$ is satisfied since $\pi_k(X_i)\to\pi_k(Y_i)$ is an isomorphism for $k=0$ and 1. (If we had added 1- and 2-cells this might not be so.)

The condition $(H_*)_\infty$ is that for each i there exists a $j>i$ so that

$$H_*(\tilde{Y}_i, \tilde{X}_i) \leftarrow H_*((Y_j \cup X_i)^\sim, \tilde{X}_i) \tag{†}$$

is zero. We will show that this is the case for $j \geq i+1$. The cellular complex for $H_*(\tilde{Y}_i, \tilde{X}_i)$ is

$$C_i(\tilde{Y}_i, \tilde{X}_i): 0 \to Q_i \oplus P'_{i+1} \oplus Q'_{i+1} \oplus \ldots \to P_i \oplus Q_i \oplus P'_{i+1} \oplus Q'_{i+1} \oplus \ldots \to 0,$$

the differential being inclusion. Now $C((Y_j \cup X_i)^\sim, \tilde{X}_i)$ is the *subcomplex* similarly described with j in place of i (but primes everywhere). Hence the homology map (†) is the zero map: $Pi \xleftarrow{0} P'_j$. This completes the proof for the case in which X has only one end.

The generalization for many ends runs as follows. We choose a basis $X_1 \hookleftarrow X_2 \hookleftarrow \ldots$ of open neighborhoods of ∞, each component of which is unbounded. Since X is connected, it is well-known that each X_i has only finitely many components. Collapsing each to a point we get the set $\pi_0(X_i)$. The inclusion-induced sequence

$$\sigma: \pi_0(X_1) \xleftarrow{f_1} \pi_0(X_2) \xleftarrow{f_2} \ldots,$$

thought of as a sequence of compacta, has an infinite mapping cylinder $\operatorname{Map}(\sigma)$. We map this onto X,

$$F: \operatorname{Map}(\sigma) \to X,$$

so that by restriction we have

(i) $\pi_0(X_i) \to X_i$ that is inverse to the quotient map and
(ii) $\operatorname{Map}(f_i) \to X_i$.

F is none other than a choice of base points and connecting base paths.

Define $\pi_1(X_i)$ to be the collection of π_1's of the components of X_i taken at the above base points. We have an inclusion-induced sequence using F,

$$\pi_1(X_1) \leftarrow \pi_1(X_2) \leftarrow \ldots,$$

which can be thought of as a functor from the diagram(1) σ to groups. Similarly for $\pi_*(Z_i)$, $\pi_*(Z_i, X_i)$, $H_*(\tilde{Y}_i, \tilde{X}_i)$, $C_*(\tilde{Y}_i, \tilde{X}_i)$, etc.

By a projective module P_i over $Z[\pi_1(X_i)] = \Lambda_i$ is meant a collection

$$\{P_C \mid C \text{ a component of } X_i\}$$

(1) An object of σ is a component of some X_i, and an arrow of σ corresponds to an inclusion of components.

(really a function $C \mapsto P_C$), where P_C is a projective module over $Z[\pi_1(C)]$. A projective P_{i+1} over Λ_{i+1} yields one called $P'_{i+1} = \Lambda_i \otimes \Lambda_{i+1} P_{i+1}$ over Λ_i; we just work component by component. When two more more components of X_{i+1} fall into the same component of X_i we add up (using $\oplus$) the projectives obtained by tensoring. With these conventions, the reader will perceive that the above proof can be repeated verbatim in the general case. ■

Appendix 2. An alternate description of the total obstruction

The purpose of this section is to show how to carry out the description of the total obstruction $\beta(M)$ which was outlined in the introduction § 1. For this it will be convenient to use the language of the weak proper homotopy category (cf. [6]). We say that proper maps $f, g: X \to Y$ are *weakly proper homotopic* provided that for every compactum $B \subset Y$ there exists a compactum $A \subset X$ and an ordinary homotopy from f to g, each level of which takes $X - A$ into $Y - B$. Using this, one then defines *weak p.h.e.* in the obvious manner.

We are given a Q-manifold M which has finite type and is tame at ∞ and we want to prove that M is p.h.e. to $\mathrm{Map}(\sigma)$, where σ is some inverse sequence of compact polyhedra which is defined as in § 1. Here are the main steps in the proof.

(i) M is weakly p.h.e. to $\mathrm{Map}(\sigma)$.

(ii) If M, N are weakly p.h.e. Q-manifolds (or polyhedra) which have finite type and are tame at ∞, then M, N are p.h.e.

Remark. (ii) is still true if the assumption "tame at ∞" is dropped; see D. A. Edwards and H. M. Hastings, Trans. Amer. Math. Soc., 221 (1976), 239–248. For the question if every weak p.h.e., itself is a genuine p.h.e., compare H. M. Hastings, On weak and strong equivalences in pro-homtopy (to appear).

Proof of (i). Using the Corollary to Lemma 5.1 we can write $M = \bigcup_{i=1}^{\infty} M_i$, where the M_i's are compact and clean, $M_i \subset \mathrm{Int}(M_{i+1})$ and such that there exist compact polyhedra $X_i \subset \mathrm{Int}(M_{i+1}) - M_i$ and maps $\alpha^i: M - \mathrm{Int}(M_i) \to X_i$ for which $\alpha^i \simeq \mathrm{id}$ (with the homotopy taking place in $M - \mathrm{Int}(M_i)$). Using the fact that M has finite type we can choose M_1 large enough so that there exists a compact polyhedron $X_0 \subset \mathrm{Int}(M_1)$ and a retraction $\alpha^0: M \to X_0$ such that $\alpha^0 \simeq \mathrm{id}$. Let $\alpha_t^0: M \to M$ be such a homotopy with $\alpha_0^0 = \mathrm{id}$ and $\alpha_1^0 = \alpha^0$. Similarly let $\alpha_t^i: M - \mathrm{Int}(M_i) \to M - \mathrm{Int}(M_i)$ be a homotopy such that $\alpha_0^i = \mathrm{id}$ and $\alpha_1^i = \alpha^i$. Define $f_i: X_{i+1} \to X_i$ by $f_i = \alpha^i | X_{i+1}$, for all $i \geq 0$, and let σ be the inverse sequence $\{X_i, f_i\}$ $i \geq 0$.

Our next step is to define a proper map $f: \mathrm{Map}(\sigma) \to M$. To do this it suffices to define a map $\bar{f}_i: M(f_i) \to M - \mathrm{Int}(M_i)$, for all $i \geq 0$ (where $M_0 = \phi$), which extends the inclusions on X_i and X_{i+1}. For $x \in X_i \cup X_{i+1}$ we therefore define $\bar{f}_i(x) = x$ and for $(x, t) \in X_{i+1} \times [0, 1)$ we put $\bar{f}_i(x, t) = \alpha_t^i(x)$. (See § 4 for mapping cylinder notation.) Then we piece the $\bar{f}_i$'s together to obtain our required $\bar{f}$.

To show that f is a weak p.h.e. we will now define a weak proper homotopy inverse $g: M \to \mathrm{Map}(\sigma)$ of f. To define g it suffices to define a map $g_i: M_{i+1} - \mathrm{Int}(M_i) \to M(f_i)$, for all $i \geq 0$, which agrees with α^i on $\delta(M_i)$ and α^{i+1} on $\delta(M_{i+1})$. Let $\delta(M_i) \times [0, 2] \subset M_i - (\mathrm{Int}(M_{i-1}) \cup X_{i-1})$ be a collar on $\delta(M_i)$, for each $i \geq 1$, where $\delta(M_i) \equiv \delta(M_i) \times \{0\}$. Without loss of generality we may assume that α_t^i is defined on $(M - \mathrm{Int}(M_i)) \cup (\delta(M_i) \times [0, 2]$, for all $i \geq 1$. Define $g_i = \alpha^i$ on $M_{i+1} - [(\delta(M_{i+1}) \times [0, 2)) \cup \mathrm{Int}(M_i)]$, for all $i \geq 0$, and on $\delta(M_{i+1}) \times [0, 2]$ we define

$$g(x,t) = \begin{cases} \alpha^i \alpha_{2-t}^{i+1}(x), & \text{for } 1 \leq t \leq 2 \\ (\alpha^{i+1}(x), t) \in X_{i+1} \times [0,1), & \text{for } 0 \leq t < 1 \end{cases}$$

We then piece the g_i's together to get $g: M \to \mathrm{Map}(\sigma)$. We leave it as a manageable exercise to show that g is a weak proper homotopy inverse of f. ∎

Proof of (ii). Let $f: M \to N$ be a weak p.h.e. If $M' \subset M$ is compact and clean, then we know from the Corollary of Lemma 5.1 that $M - M'$ is finitely dominated. Using [20] we see that $(M - M') \times S^1$ has finite type. Applying *Proposition* 5.3 it follows that $M \times S^1$ is p.h.e. to $\mathrm{Map}(\sigma)$(¹), for some inverse sequence σ of compact polyhedra. Similarly $N \times S^1$ is p.h.e. to some $\mathrm{Map}(\tau)$. Thus we have a weakly proper homotopy commutative diagram

$$\begin{array}{ccc} M \times S^1 & \xrightarrow{f \times \mathrm{id}} & N \times S^1 \\ \uparrow & & \uparrow \\ \mathrm{Map}(\sigma) & \xrightarrow{g} & \mathrm{Map}(\tau), \end{array}$$

where the horizontal arrows are weak p.h.e.'s and the vertical arrows are genuine p.h.e.'s. By the proof of Proposition B of [24] we can find a genuine p.h.e. $\mathrm{Map}(\sigma) \to \mathrm{Map}(\tau)$ which is homotopic(²) to g. *Thus we observe that there is a genuine* p.h.e. $h: M \times S^1 \to N \times S^1$ *for which the following diagram homotopy commutes:*

$$\begin{array}{ccc} M \times S^1 & \xrightarrow{h} & N \times S^1 \\ & \mathrm{proj} \searrow \quad \swarrow \mathrm{proj} & \\ & S^1 & \end{array}$$

We will prove that the composition

$$h_0: M \xrightarrow{i} M \times S^1 \xrightarrow{h} N \times S^1 \xrightarrow{\mathrm{proj}} N$$

$(i(m) = (m, *))$ is a genuine p.h.e.

(¹) No knowledge of Wall's finiteness obstruction is required here!

(²) In fact weakly proper homotopic to g.

Let $h': N \times S^1 \to M \times S^1$ be a proper homotopy inverse of h and let $h_0': N \to M$ be defined in analogy with h_0, i.e. $h_0' = (\text{proj})h'i$. Since h commutes with projection to S^1 (up to homotopy) it is easy to see that the composition i(proj)h i,

$$M \xrightarrow{i} M \times S^1 \xrightarrow{h} N \times S^1 \xrightarrow{\text{proj}} N \xrightarrow{i} N \times S^1,$$

is proper homotopic to $hi: M \to N \times S^1$. Thus we have proper homotopies

$$\text{id}\,|\,M \simeq (\text{proj})h'hi \simeq (\text{proj})h'i(\text{proj})hi = h_0'h_0$$

Similarly we can prove that $h_0h_0': N \to N$ is proper homotopic to $\text{id}\,|\,N$. ■

Appendix 3. What does it mean to have a boundary?(1)

There are, to be sure, other notions of boundary. Here we point out that some are essentially equivalent to ours (which was set out in the introduction § 1).

A first alternative was suggested by B. Rushing. Say that the Q-manifold M admits a *globally-$\mathcal{Z}$ boundary* B if there exists a compact Q-manifold N such that M is open in N while $B = N - M$ and B is *globally-$\mathcal{Z}$* in N in the following sense: for any neighborhood U of B in N the inclusion $(U - B) \hookrightarrow U$ is a homotopy equivalence.

PROPOSITION. *Suppose M admits, a globally-$\mathcal{Z}$ boundary. Then M admits a boundary (as in § 1).*

First we establish

Assertion (for above data). *If V is any neighborhood of B in N, there exists a smaller clean compact neighborhood $W \subset V$ such that W is a collar on its frontier δW in N.*

Proof of assertion. For convenience we can assume V is a compact and clean Q-submanifold of N. Since $V - B \hookrightarrow V$ is a homotopy equivalence, we can find a homotopy of $\text{id}\,|\,V$ *fixing* δV to a map $f: V \to V - B$. (To see this use for instance the fact $(V - B) \times 0$ is a strong deformation retract of $V \times [0, 1]$, see [26, p. 31].) We can then arrange that f is an embedding onto a $\mathcal{Z}$-set in $V - B$. Then $f(V)$ has a clean collaring $f(V) \times [0, 1]$ in $V - B$, with $f(V) \times 0 = f(V)$. Defining $W = V - f(V) \times [0, 1]$. We observe that $\delta W = f(V) \times 1$, and that W is a collar on δW since $f(V) \hookrightarrow V$ is simple homotopy equivalence (indeed homotopic to a homeomorphism). ■

Proof of Proposition: The assertion shows that B *is a nested intersection of compact clean neighborhoods W_i, with W_i and also $W_i - \text{Int } W_{i+1}$ a collar on δW_i*. Then criterion 4.2 shows that M admits a boundary. ■

(1) Added Jan., 1976.

As a second alternative, say that the Q-manifold M *admits a boundary B in a compactum* if there is a metric compactum N so that M is open in N while $B = N - M$ with B a *$\mathcal{Z}$-set in N* in the following sense: there exists an ε-homotopy of $\mathrm{id}|N$ to a map into $N - B = M$.

In this situation, S. Ferry [14] has shown (using [11]) that N is a Q-manifold provided it is an ANR. But S. Kozlowski has observed (cf. [14]) that N is here necessarily an ANR, presumably by verifying that N is ε-dominated by locally finite complexes (because M is); which implies N is an ANR (Hanner's criterion [16]).

Thus this apparently more general notion of boundary is really identical to ours.

Finally we note that the problem of finding a boundary can reasonably be posed for locally compact ANR's. Say that a locally compact ANR M *admits a boundary B in a compactum N* (if just as above) $N = M \cup B$ with B a compact $\mathcal{Z}$-set in N. As above, N is necessarily an *ANR*.

For a locally compact ANR M to admit such a boundary it is certainly *necessary* that the Q-manifold $M \times Q$ (cf. [11]) admit a boundary (as in § 1). *Question:* Is it also sufficient?

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The last article treats in passing the case of our boundary theorem for isolated ends with "constant" (="moveable") shape.

We take this opportunity to correct an error in the last sentence of [30] preceeding the appendix; it was recently noticed by T. Chapman. Replace the sentence by:

« L'application f: $M \to S^1$ est homotope à une fibration localement triviale si et seulement si: (*i*) $\tau(M, f) = 0$, (ii) $\bar{M}$ est de type fini, et (iii) la torsion $p_* \tau(T) \in \mathrm{Wh}(\pi_1 M)$ est zéro, p: $\bar{M} \to M$. Cette torsion $p_* \tau(T)$ est indépendante du type fini imposé sur $\bar{M}$ vue la suite exacte $\mathrm{Wh}(\pi_1 \bar{M}) \xrightarrow{\mathrm{id.} - T_*} \mathrm{Wh}(\pi_1 \bar{M}) \xrightarrow{p_*} \mathrm{Wh}(\pi_1 M)$, par [25, Chap. III]. Supposant vérifiées ces trois conditions nécessaires, on peut choisir $F \simeq \bar{M}$ d'après (i), et choisir ensuite ϱ: $F \to F$ une équivalence *simple* d'après (iii), et même un homéomorphisme PL. Finalement on a $(M, f) \sim (F_\varrho \times Q, \text{proj.})$ d'après (i), où visiblement proj.: $F_\varrho \times Q \to S^1$ est un fibré de fibre $F \times Q$ ».

This corrected triple condition is equivalent to $\tau(M, f) = 0 = \tau(M, \bar{f})$ in $\mathrm{Wh}(\pi_1 M)$, where $\bar{f}$ is f followed by complex conjugation.

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