C^∞ CONVEX FUNCTIONS AND MANIFOLDS OF POSITIVE CURVATURE

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Some years ago, Gromoll and Meyer [15] proved that if M is a complete noncompact Riemannian manifold with everywhere positive sectional curvature, then M is diffeomorphic to Euclidean space and the exponential map $\exp_p$: $TM_p \to M$ is for every point $p \in M$ a proper map. During our recent work on noncompact Kähler manifolds [9]–[11], we realized that these and other results on such Riemannian manifolds would follow quite readily from one existence theorem, namely: on a complete noncompact Riemannian manifold of positive curvature there is a C^∞ strictly convex exhaustion function τ, that is, a C^∞ function τ: $M \to [0, +\infty)$ which is proper and is such that all the eigenvalues of its second covariant differential are everywhere positive (Theorem 1(a)). The function τ can in fact be chosen to be (uniformly) Lipschitz continuous on all of M. The existence of a *continuous* strictly convex exhaustion function (see § 1 for the definition of strict convexity of continuous functions) was deduced in [12] from results in [3]. Therefore the main weight of the present existence theorem is the possibility of choosing the function to be C^∞: in fact, the existence theorem as stated is deduced in this paper from a general theorem that continuous strictly convex functions can be approximated by C^∞ strictly convex functions on any Riemannian manifold (Theorem 2). The purpose of this paper is thus to establish the existence theorem and to provide a systematic exposition of the consequences which flow from it.

In the terminology of classical analysis, Theorem 2 is a smoothing theorem for strictly convex functions on arbitrary Riemannian manifolds. It should be pointed out that the usual procedure of smoothing in euclidean space by convoluting with a spherically symmetric kernel does not carry over to this general situation. Moreover, the analogue of

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Theorem 2 for (not necessarily strictly) convex functions is thus far missing. For the corresponding theorems in the case of strictly plurisubharmonic functions on complex manifolds and subharmonic functions on Riemannian manifolds, see Richberg [25] and [13], [14].

The assertion of the existence of a C^∞ strictly convex exhaustion function on any complete noncompact Riemannian manifold of positive sectional curvature leads to simple proofs of the results of Gromoll and Meyer mentioned in the previous paragraph (Theorem 3 and Corollary (b) of Theorem 5). Moreover, this existence theorem also gives as an immediate corollary: a complete noncompact Kähler manifold of positive curvature is a Stein manifold (Theorem 10(b)). This result is the main theorem of [11] although as indicated there the method of [11] actually proves the more general theorem of [9, III]. A refinement (Theorem 1(b)) of the existence theorem is that if M is a complete noncompact Riemannian manifold whose curvature is positive outside some compact set then there is a C^∞ exhaustion function on M which is strictly convex and uniformly Lipschitz continuous outside some compact set. Using this fact, it will be shown that: If $f \not\equiv 0$ is a nonnegative subharmonic function on M then $\int_M f = +\infty$ (Theorem 7; originally proved in [12]). If the dimension of M is 4 and Θ denotes the Gauss-Bonnet integrand then $\int_M \Theta$ exists and is less than or equal to the Euler characteristic of M (Theorem 9; announced in [13]). If M is a Kähler manifold, then M is obtained from a Stein space by blowing up a finite number of points (Theorem 11b; see [11]). If M is a Kähler manifold and if (in addition to having sectional curvature positive outside some compact set) M has everywhere nonnegative sectional curvature then M is a Stein manifold (Theorem 12; this result is given for the first time here, not having occurred in [9] or [11]).

The scope of this paper is actually more general in two aspects than so far indicated. On the technical side, the assumption of positive curvature or positive curvature outside a compact set can in some instances be replaced by assuming the existence of a continuous exhaustion function which is strictly convex or strictly convex outside some compact set. In other instances, the positive curvature hypotheses can be replaced by assuming nonnegative curvature. In § 1–5, the results are stated and proved in the maximum generality consistent with keeping the conceptual character of the proofs unobscured by excessive technical detail. In § 6, there is a discussion of some more general results which can be obtained by merely technical modification of the arguments used in § 1–5. Some related results from other sources are also discussed in § 6.

On the methodological side, we hope to make a point that seems to have been overlooked until now: a knowledge of the function theory (of the geometrically interesting functions) on noncompact Riemannian manifolds is essential for the understanding of their topology and geometry. The fact that one can almost effortlessly derive so much geometrical and topological information from the existence of one C^∞ strictly convex exhaustion function is, we hope, sufficient to attest to the validity of this viewpoint.

The previously known C^∞ strictly convex exhaustion functions on Riemannian manifolds were all variants of the square of the geodesic distance from a fixed point of a simply connected complete Riemannian manifold of nonpositive curvature. In this case, the manifold is already known to be diffeomorphic to Euclidean space before the convex function is constructed. The problems which motivated this paper were first, how to construct a C^∞ strictly convex exhaustion function without relying on the geodesic distance function, and second, how to deduce geometric and topological information from the existence of such a function.

The completion of this paper depended in an essential way on a remark by Professor S. S. Chern. It gives us pleasure to record here our gratitude to him.

§ 1. Approximation of strictly convex functions

The principal goal of this section expressed in general terms is to show that a strictly convex function on a C^∞ Riemannian manifold M can be globally approximated by C^∞ strictly convex functions. A C^∞ function on M is called *strictly convex* if its second derivative along any geodesic is positive everywhere on the geodesic. An appropriate extension of the idea of strict convexity to arbitrary (continuous) functions on M is given in the following definition:

Definition 1. A function $f: M \to R$ is called strictly convex if for every $p \in M$ and every C^∞ strictly convex function φ defined in a neighborhood of p there is an $\varepsilon > 0$ such that $f - \varepsilon\varphi$ is convex in a neighborhood.

Here, as usual, a function $f: M \to R$ is called convex if its restriction to every geodesic is convex in the one variable sense. The fact that a function on the line is convex if and only if it is convex in a neighborhood of each point of the line implies that a function on a Riemannian manifold is convex if and only if it is convex in a neighborhood of each point.

The following lemma gives a necessary and sufficient condition for strict convexity in terms of difference quotients along geodesics. This lemma shows in particular that a strictly convex function is convex and thus necessarily continuous. The condition for strict convexity given in the lemma was used as the definition of strict convexity in [12] and [13]; the lemma states that the present terminology is equivalent to that in [12] and [13].

LEMMA 1. *A function $f: M \to R$ has the following property* (1) *if and only if it has property* (2):

(1) *f is strictly convex.*

(2) *For every compact set $K \subset M$, there is a $\delta > 0$ such that for every arc-length parameter geodesic segment C: $[-\lambda, \lambda] \to M$ with $0 < \lambda < \delta$ and $C(0) \in K$:*

$$f(C(\lambda)) + f(C(-\lambda)) - 2f(C(0)) > \delta\lambda^2.$$

Proof that (1) *implies* (2). For each point p of M, there exists a positive number ξ_p such that on the open ball of radius ξ_p about p there is a C^∞ strictly convex function defined. For instance, if $(x_1, ..., x_n)$ is a Riemannian normal coordinate system at p (with p corresponding to $(0, ..., 0)$) then $\Sigma_{i=1}^n x_i^2$ is C^∞ and strictly convex on a sufficiently small open ball about p. That this function is strictly convex near p follows from the fact that the second derivative at p along any arc-length parameter geodesic through p is 2.

Now suppose that statement (2) is false for a compact set $K \subset M$. Then there exists a sequence $\{p_i \mid i = 1, 2, ...\}$ of points of K, a sequence $\{\lambda_i \mid i = 1, 2, ...\}$ of positive real numbers converging to 0 and a sequence $\{C_i: [-\lambda_i, \lambda_i] \to M \mid i = 1, 2, ...\}$ of arc-length parameter geodesic segments such that $C_i(0) \in K$ for all i and

$$\liminf_i \frac{1}{\lambda_i^2}\{f(C_i(\lambda_i)) + f(C(\lambda_i)) - 2f(C)0))\} \leqslant 0.$$

By passage to a subsequence if necessary, it may be assumed that the sequence $\{p_i\}$ converges to a point p of K. On the open ball about p of radius ξ_p, there exists a C^∞ strictly convex φ_p. And by the definition of strict convexity, there exists an $\varepsilon > 0$ such that $f - \varepsilon\varphi_p$ is convex in a neighborhood U_p of p. Let $g = f - \varepsilon\varphi_p$. Now suppose that i_0 is so large that for all $i > i_0$, the geodesic segment C_i lies in the intersection of U_p with the ball around p of radius $\xi_p/2$. Such an i_0 exists because $p_i \to p$ and $\lambda_i \to 0$. Then for all $i > i_0$

$$\frac{1}{\lambda_i^2}[g(C_i(\lambda_i)) + g(C_i(-\lambda_i)) - 2g(C_i(0))] \geqslant 0,$$

because g is convex on U_p. Also,

$$\begin{aligned}\frac{1}{\lambda_i^2}\,\varepsilon[\varphi_p(C_i(\lambda_i)) + \varphi_p(C_i(-\lambda_i)) - 2\varphi_p(C_i(0))]& \\ \geqslant \{2\varepsilon \times \text{ the infimum over the closed ball of radius } \xi_p/2& \\ \text{of the derivative of } \varphi_p \text{ along arc-length parameter geodesics}\}.&\end{aligned}$$

This is positive because φ_p is strictly convex in a neighborhood of this closed ball. Denote this infimum by η. Then for all $i > i_0$

$$\frac{1}{\lambda_i^2}\{f(C_i(\lambda_i))+f(C_i(-\lambda_i))-2f(C_i(0))\}$$

$$=\frac{1}{\lambda_i^2}\{\varepsilon[\varphi_p(C_i(\lambda_i))+\varphi_p(C_i(-\lambda_i))-2\varphi_p(C_i(0))]+g(C_i(\lambda_i))+g(C_i(-\lambda_i))-2g(C_i(0))\}\geqslant\varepsilon\eta.$$

Hence

$$\liminf_i \frac{1}{\lambda_i^2}\{f(C_i(\lambda_i))+f(C_i(-\lambda_i))-2f(C_i(0))\}\geqslant\varepsilon\eta>0.$$

This contradiction shows that statement (2) must hold if f is strictly convex.

Proof that (2) *implies* (1). Let f be a function satisfying (2) and p be a point of M. Suppose that φ: $U\to R$ is any C^∞ strictly convex function defined on a neighborhood U of p. Let V be a neighborhood of p with the closure cl V of V compact and cl $V\subset U$. Take $K=\text{cl } V$ in statement 2). Choose $\varepsilon>0$ such that the second derivatives of φ along arc-length parameter geodesics at p are $<\frac{1}{4}\delta/\varepsilon$, $\delta>0$ being obtained from statement (2). Then for all geodesic segments C: $[-\lambda,\lambda]\to M$ with $\lambda>0$ sufficiently small and $C(0)$ sufficiently near p:

$$0<\varphi(C(\lambda))+\varphi(C(-\lambda))-2\varphi(C(0))<\frac{\delta}{2\varepsilon},$$

so that

$$\frac{1}{\lambda^2}\{f(C(\lambda))+f(C(-\lambda))-2f(C(0))-\varepsilon[\varphi(C(\lambda))+\varphi(C(-\lambda))-2\varphi(C(0))]\}\geqslant\delta-\tfrac{1}{2}\delta>0.$$

Thus $f-\varepsilon\varphi$ has nonnegative second difference quotients along geodesics near p so that $f-\varepsilon\varphi$ is a convex function on a (sufficiently small) neighborhood of p. □

A special role is played in geometric considerations by functions which are closely related to the Riemannian distance on M. One such relationship is Lipschitz continuity in the sense of the following definition:

Definition 2. A function f: $M\to R$ is Lipschitz continuous if there exists a real number B such that

$$|f(p)-f(q)|\leqslant B\,\text{dis}_M(p,q)\quad\text{for all } p,q\in M.$$

(Here $\text{dis}_M=$ the Riemannian distance function on M). Any such constant B is called a *Lipschitz constant* for f.

The next lemma shows that Lipschitz continuity on a Riemannian manifold (with a particular Lipschitz constant) is a local property.

LEMMA 2. *A function $f: M \to R$ is Lipschitz continuous with Lipschitz constant B if and only if: f is Lipschitz continuous with Lipschitz constant B in a neighborhood of each point of M, i.e. for each point $q \in M$, there exists a neighborhood U_q of q such that for every $q_1, q_2 \in U_q$*

$$|f(q_1) - f(q_2)| \leqslant B \operatorname{dis}_M(q_1, q_2).$$

Proof. Lipschitz continuity on M implies the local condition since one may take $U_q = M$ for every $q \in M$. To show the converse, recall that for any $p_1, p_2 \in M$ $\operatorname{dis}_M(p_1, p_2) = \inf_C l(C)$, where C ranges over all rectifiable arcs $C: [0, 1] \to M$ with $C(0) = p_1$ and $C(1) = p_2$ and $l(C) =$ length of C. Thus to establish Lipschitz continuity of f with Lipschitz constant B one need only show that for any such arc C

$$|f(p_1) - f(p_2)| \leqslant Bl(C).$$

Choose a finite subdivision of $[0, 1]$ by points $\lambda_0 = 0 < \lambda_1 < \lambda_2 < \ldots < \lambda_l = 1$ such that for all $i = 0, \ldots, l-1$ $C([\lambda_i, \lambda_{i+1}])$ is contained in a neighborhood U_q satisfying the condition of the lemma for some $q \in M$. Such a choice is possible because the U_q form an open cover of M. Then

$$|f(p_1) - f(p_2)| \leqslant \sum_{i=0}^{l-1} |f(C(\lambda_i)) - f(C(\lambda_{i+1}))| \leqslant B \sum_{i=0}^{l-1} \operatorname{dis}_M(C(\lambda_i), C(\lambda_{i+1})) \leqslant Bl(C).$$

The final definition needed to state the theorems of this section is the definition of an exhaustion function.

Definition 3. A function $f: M \to R$ is an *exhaustion function* if, for every $\lambda \in R$, $f^{-1}((-\infty, \lambda])$ is a compact subset of M.

THEOREM 1. (a) *If M is a complete noncompact Riemannian manifold of everywhere positive sectional curvature, then there exists on M a C^∞ Lipschitz continuous strictly convex exhaustion function.*

(b) *If M is a complete noncompact Riemannian manifold and if there is a compact subset K_1 of M such that $M - K_1$ has everywhere positive section curvature, then there exists a compact subset K_2 of M and a C^∞ Lipschitz continuous exhaustion function $\varphi: M \to R$ on M such that φ is strictly convex on $M - K_2$.*

Theorem 1 will be deduced from certain results in [12] together with the following theorem:

THEOREM 2. (a) *If M is a Riemannian manifold, if there is a strictly convex function $\psi: M \to R$ on M, and if ε is any positive real number, then there is a C^∞ strictly convex function $\varphi: M \to R$ with $|\varphi - \psi| < \varepsilon$ everywhere on M. Moreover, if ψ is Lipschitz continuous with Lip-*

schitz constant B then there exists such a function φ which is Lipschitz continuous with Lipschitz constant $B+\varepsilon$.

(b) *If M is a Riemannian manifold, if $\psi: M\to R$ is a function with the property that there is a compact subset K_1 of M such that ψ is strictly convex on $M-K_1$, and if ε is any positive number and K_2 is any compact subset of M with $K_1\subset \overset{0}{K}_2$, then there exists a C^∞ function $\varphi: M\to R$ such that $|\psi-\varphi|<\varepsilon$ on $M-K_2$ and φ is strictly convex on $M-K_2$. Moreover, if ψ is Lipschitz continuous on $M-K_1$ with Lipschitz constant B then there is such a function φ with φ Lipschitz continuous on $M-K_2$ with Lipschitz constant $B+\varepsilon$.*

In the second statement of Theorem 2(b), the function φ, being C^∞ on M and Lipschitz continuous outside a compact subset of M_2 is necessarily Lipschitz continuous on M, but perhaps with a larger constant than $B+\varepsilon$.

Proof of Theorem 1 *from Theorem* 2.

1(a). It was shown in [12] that if M is a complete Riemannian manifold of positive sectional curvature then there is a strictly convex Lipschitz continuous exhaustion function $\psi: M\to R$ on M. Let φ be a function satisfying the requirements (including Lipschitz continuity) of the conclusion of Theorem 2(a) with $\varepsilon=1$. Then φ is necessarily an exhaustion function because, for any $\lambda\in R$, $\{p\in M\,|\,\varphi(p)\leqslant\lambda\}$ is a closed subset of the compact set $\{p\in M\,|\,\psi(p)\leqslant\lambda+1\}$. Thus φ satisfies the requirements of Theorem 1(a).

1(b). It was also shown in [12] that if M is a complete Riemannian manifold whose sectional curvatures are positive outside some compact set K then there exists a compact set K' and an exhaustion function $\psi: M\to R$ which is Lipschitz continuous and strictly convex on $M-K'$. If φ satisfies the requirements (including Lipschitz continuity) of Theorem 2(b) with $\varepsilon=1$, $K_1=K'$ and $K_2=$any compact set whose interior contains K', then φ is an exhaustion function because for any $\lambda\in R$, $\{p\in M\,|\,\varphi(p)\leqslant\lambda\}$ is a closed subset of the compact set $K\cup\{p\in M\,|\,\psi(p)\leqslant\lambda+1\}$. Thus φ satisfies the requirements of Theorem 1(b). □

Theorem 2(a) implies Theorem 2(b): For let K_3 be a compact set in M satisfying $K_1\subset\overset{0}{K}_3\subset K_3\subset\overset{0}{K}_2$, where K_1 and K_2 are as in 2(b). Then apply 2(a) to $\psi|(M-K_3)$ to obtain a C^∞ function $\varphi_1: M-K_3\to R$ which is strictly convex and satisfies $|\psi-\varphi_1|<\varepsilon$ on $M-K_3$ (and is Lipschitz continuous with Lipschitz constant $B+\varepsilon$ on $M-K_3$, in the case that ψ is Lipschitz continuous with constant B on $M-K_1$). Then by a standard extension process, there is a C^∞ function $\varphi: M\to R$ with $\varphi=\varphi_1$ on $M-K_2$. This function φ satisfies the requirements of 2(b). Thus to establish Theorem 2 (and hence Theorem 1), it remains only to establish Theorem 2(a). For this purpose, the following lemmas will be used:

LEMMA 3. *Let $\tau: M\to R$ be a continuous function and A_1 and A_2 be compact subsets of M with $A_1\subset A_2$. Suppose that τ is strictly convex in a neighborhood of A_2 and C^∞ in a neigh-*

borhood of A_1. *Then there exists a neighborhood* U *of* A_2 *and a family of* C^∞ *functions* $\{\tau_\varepsilon\colon U\to R\,|\,\varepsilon\in(0,\varepsilon_0)\}$, $\varepsilon_0>0$, *defined on* U *such that*

(1) *for each* $\varepsilon\in(0,\varepsilon_0)$, τ_ε *has positive second derivatives along geodesics at points of* A_2.

(2) $\tau_\varepsilon\to\tau$ *uniformly on* A_2 *as* $\varepsilon\to0^+$.

(3) *for any positive integer* r *the derivatives of* τ_ε *of order* r *converge uniformly to the corresponding derivatives of* τ *on* A_1 *as* $\varepsilon\to0^+$.

Moverover, if τ *is Lipschitz continuous with Lipschitz constant* B *and if* η *is any positive number, then the* τ_ε *may be chosen to be Lipschitz continuous on* U *(relative to Riemannian distance* dis_U *on* U*) with Lipschitz constant* $B+\eta$.

Proof of Lemma 3. The Riemannian convolution smoothing approximations of [10] have the properties required. These are constructed (in summary; see [10] for details) as follows: Let $\mathcal{K}\colon R\to R$ be a nonnegative C^∞ function which has support contained in $[-1,1]$, is constant in a neighborhood of 0, and has $\int_{v\in R^n}\mathcal{K}(\|v\|)=1$, where $n=\dim M$. Define

$$\tau_\varepsilon(p)=\frac{1}{\varepsilon^n}\int_{v\in TM_p}\mathcal{K}\left(\frac{\|v\|}{\varepsilon}\right)\tau(\exp_p v),$$

where the integration is with respect to the measure induced on the tangent space TM_p at p by the Riemannian metric of M. There is a neighborhood of A_2 on which the τ_ε are defined and C^∞ for all sufficiently small ε. Properties (2) and (3) follow from standard arguments (see [10; p. 646 ff.]).

Now suppose that property (1) fails for every $\varepsilon_0>0$. Then there exist sequences $\{p_i\,|\,p_i\in A_2,\ i=1,2,\ldots\}$ and $\{\varepsilon_i\,|\,\varepsilon_i>0,\ i=1,2,\ldots\}$ with $\varepsilon_i\to0^+$ such that for each $i=1,2,\ldots$ there exists an arc-length parameter geodesic C_i with $C_i(0)=p_i$ and the second derivative $(d^2/dt^2)\tau_{\varepsilon_i}(C(t))\,|_{t=0}\leqslant0$. By passage to a subsequence if necessary, it may be assumed that the sequence $\{p_i\}$ converges to some point p in A_2. Let ϱ be a C^∞ strictly convex function defined in a neighborhood of p such that $\tau-\varrho$ is convex in a neighborhood of p: the existence of C^∞ strictly convex functions defined in a neighborhood of p was demonstrated in the proof of Lemma 1 and the required ϱ can be obtained according to the definition of strict convexity by multiplying such a function by a sufficiently small positive number. Now $\tau_\varepsilon=(\tau-\varrho)_\varepsilon+\varrho_\varepsilon$ at all points of M for all $\varepsilon>0$ for which all terms are defined. Let $\eta=$ infimum of the second derivatives at p of ϱ along arc-length parameter geodesics through p. Then $\eta>0$ and on a sufficiently small closed ball $\bar{B}$ (of positive radius) about p the second derivatives of ϱ along arc-length parameter geodesics are $>\eta/2$ since ϱ is C^∞. The second derivatives of ϱ_ε on $\bar{B}$ converge uniformly to the second derivatives of ϱ as $\varepsilon\to0^+$. Thus for all sufficiently small $\varepsilon>0$ the second derivatives along arc-length parameter geodesics at points of $\bar{B}$ are $>\eta/4$. Now according to [10; p. 644] the convexity of $\tau-\varrho$ implies that

$$\liminf_{\varepsilon\to 0^+}\left(\inf_C \frac{d^2}{dt^2}(\tau-\varrho)_\varepsilon(C(t))\big|_{t=0}\right)\geqslant 0,$$

where C ranges over all arc-length parameter geodesics with $C(0)\in\bar{B}$. For such geodesics C

$$\frac{d^2}{dt^2}\tau_\varepsilon(C(t))\big|_{t=0}=\frac{d^2}{dt^2}(\tau-\varrho)_\varepsilon(C(t))\big|_{t=0}+\frac{d^2}{dt^2}\varrho_\varepsilon(C(t))\big|_{t=0}>\frac{d^2}{dt^2}(\tau-\varrho)_\varepsilon(C(t))\big|_{t=0}+\eta/4,$$

so that $(d^2/dt^2)\tau_\varepsilon(C(t))\big|_{t=0}>0$ for all ε sufficiently small (uniformly with respect to variation of C). This inequality contradicts the combined properties of the sequences $\{p_i\}$, $\{\varepsilon_i\}$ and $\{C_i\}$. Thus property (1) must hold.

That the τ_ε are Lipschitz continuous in the neighborhood U of A_2 with Lipschitz constant $B+\delta$ is established in [12; p. 285]. □

LEMMA 4. *Let σ: $M\to R$ be a strictly convex function and L_1 and L_2 compact subsets of M with $L_1\subset L_2$. Suppose that σ is C^∞ in a neighborhood of L_1. Then there exists a family $\{\sigma_\varepsilon: M\to R\,|\,\varepsilon\in(0,\varepsilon_0)\}$, $\varepsilon_0>0$, of strictly convex functions on M such that*

(1) *for each $\varepsilon\in(0,\varepsilon_0)$ σ_ε is C^∞ in a neighborhood of L_2.*

(2) $\sup_M|\sigma-\sigma_\varepsilon|\to 0$ *as* $\varepsilon\to 0^+$.

(3) *for any positive integer r, the derivatives of σ_ε of order r converge uniformly on L_1 to the corresponding derivatives of σ.*

Moreover, if σ is Lipschitz continuous with Lipschitz constant B and if δ is a positive number then each σ_ε, $\varepsilon\in(0,\varepsilon_0)$, may be taken to be Lipschitz continuous with Lipschitz constant $B+\delta$.

Proof of Lemma 4. Let L_3 be a compact subset of M with $L_2\subset\overset{0}{L}_3$. Then let ϱ: $M\to R$ be a nonnegative C^∞ function which is 1 on L_2 and 0 in a neighborhood of the closure of $M-L_3$. The existence of such a function ϱ is a consequence of the standard partition of unity result for the open cover $\{M-L_2,\overset{0}{L}_3\}$. Set $\varrho_1=\varrho-\frac{1}{2}$. Then ϱ_1 is C^∞ on M, $=\frac{1}{2}$ on L_2 and $=-\frac{1}{2}$ on a neighborhood of the closure of $M-L_3$. Now let $\{\tilde{\sigma}_\varepsilon\}$ be the family of functions given by Lemma 3 with τ of that lemma $=\sigma$, $A_1=L_1$, and $A_2=L_3$. Define $\eta_\varepsilon=3\sup_{L_3}|\tilde{\sigma}-\sigma_\varepsilon|$. Then $\eta_\varepsilon\to 0$ as $\varepsilon\to 0^+$. Also $\tilde{\sigma}_\varepsilon+\eta_\varepsilon\varrho_1$ is $>\sigma$ on L_2 and $<\sigma$ on a neighborhood of $L_3-\overset{0}{L}_3$. Now define σ_ε: $M\to R$ by

$$\begin{aligned}\sigma_\varepsilon(p)&=\max(\tilde{\sigma}_\varepsilon(p)+\eta_\varepsilon\varrho_1,\,\sigma(\varrho)) && \text{for } p\in L_3\\ \sigma_\varepsilon(p)&=\sigma(p) && \text{for } p\in M-L_3.\end{aligned}$$

Then $\sigma_\varepsilon(p)=\tilde{\sigma}_\varepsilon(p)+\eta_\varepsilon\varrho_1$ in a neighborhood of L_2; and, thus σ_ε is C^∞ on a neighborhood of L_2. Moreover, since $\eta_\varepsilon\to 0$ as $\varepsilon\to 0^+$ and $\tilde{\sigma}_\varepsilon$ satisfies (3) of Lemma 3, σ_ε satisfies (3) of Lemma 4. Also $\sup_M|\sigma_\varepsilon-\sigma|=\sup_{L_3}|\sigma_\varepsilon-\sigma|\leqslant\sup_{L_3}|\tilde{\sigma}_\varepsilon-\sigma|+\eta_\varepsilon\sup_{L_3}|\varrho_1|$. Thus $\sup_M|\sigma_\varepsilon-\sigma|\to 0$ as $\varepsilon\to 0^+$.

To show that each σ_ε is strictly convex, provided that ε is sufficiently small, it is enough to verify strict convexity in a neighborhood of each point p of M: Let V be a neighborhood of $M-\overset{0}{L}_3$ on which $\sigma_\varepsilon=\sigma$; then if $p\in V$, σ_ε is strictly convex in a neighborhood of p (e.g., V istself) because σ is strictly convex. If $p\in\overset{0}{L}_3$, then in a neighborhood of p, σ_ε is the maximum of σ and $\tilde{\sigma}_\varepsilon+\eta_\varepsilon\varrho_1$. The second derivatives along arc-length parameter geodesics of $\tilde{\sigma}_\varepsilon$ are, for all sufficiently small ε, positive and bounded away from zero uniformly as $\varepsilon\to 0^+$ on L_3, as shown in the proof of Lemma 3. Since $\eta_\varepsilon\to 0$ as $\varepsilon\to 0^+$, $\tilde{\sigma}_\varepsilon+\eta_\varepsilon\varrho_1$ is strictly convex on $\overset{0}{L}_3$ for sufficiently small ε. Since the maximum of two strictly convex functions is strictly convex, σ_ε is strictly convex in a neighborhood (e.g. $\overset{0}{L}_3$) of each point p of $\overset{0}{L}_3$ when the positive number ε is sufficiently small.

That for all sufficiently small ε, σ_ε is Lipschitz continuous with Lipschit constant $B+\delta$ (if σ is Lipschitz continuous with Lipschitz constant B) follows similarly (cf. [12]): According to Lemma 2, it is enough to verify Lipschitz continuity with constant $B+\delta$ in a neighborhood of each point of M. On U, the Lipschitz continuity of σ_ε follows with constant $B+\delta$ immediately from that of σ (with constant $B<B+\delta$). On $\overset{0}{L}_3$, $\tilde{\sigma}_\varepsilon$ is, for all sufficiently small ε, Lipschitz continuous with Lipschitz constant $B+\frac{1}{2}\delta$. Also, for ε sufficiently small, $\eta_\varepsilon\varrho_1$ is Lipschitz continuous on M with Lipschitz constant $\frac{1}{2}\delta$ since $\eta_\varepsilon\to 0$ as $\varepsilon\to 0^+$ and ϱ_1 is C^∞ with compact support. Thus, again for all sufficiently small ε, $\tilde{\sigma}_\varepsilon+\eta_\varepsilon\varrho_1$ is Lipschitz continuous on $\overset{0}{L}_3$ with Lipschitz constant $B+\delta$. Since the maximum of two Lipschitz continuous functions with a certain Lipschitz constant is itself Lipschitz continuous with that Lipschitz constant, σ_ε is Lipschitz continuous with Lipschitz constant $B+\delta$ on a neighborhood of every point of $\overset{0}{L}_3$ and hence on M with Lipschitz constant $B+\delta$, provided that ε is sufficiently small. □

The completion of the proof of Theorem (2a) will depend on some standard function space topology concepts, which will now be summarized. For further details, one can consult [21], for instance.

Let K be a compact subset of M; let $C^\infty(K)$ denote the set of pairs (U,f) where U is an open subset of M containing K and f is a C^∞ function on U. Choose a fixed covering of K by a finite number of (open) coordinate systems, say $x^{(\lambda)}: V_\lambda\to R^n$, $n=\dim M$, $\lambda\in\Lambda$, where Λ is a finite set. Choose then for each $\lambda\in\Lambda$ an open set V'_λ having compact closure contained in V_λ in such a way that $K\subset\bigcup_{\lambda\in\Lambda}V'_\lambda$. These choices are possible by the compactness of K. Then for each positive integer i and each $f\in C^\infty(K)$ the supremum

$$\sup_{\lambda\in\Lambda}\left[\sup_{p\in V'_\lambda\cap K}\begin{pmatrix}\text{maximum of the } x^{(\lambda)}\text{-coordinate system}\\ \text{partial derivatives of } f \text{ of order } i \text{ at } p\end{pmatrix}\right]$$

is finite. This supremum will be denoted by $\|f\|_{K,i}$. Define $\|f\|_{K,0}$ for $f\in C^\infty(K)$ to be $\sup_{p\in K}|f(p)|$. The function d_K: $C^\infty(K)\times C^\infty(K)\to R$ defined by

$$d_K(f,g)=\|f-g\|_{K,0}+\sum_{i=1}^{+\infty}2^{-i}\min(1,\|f-g\|_{K,i}),$$

is a (finite-valued) pseudo-metric on $C^\infty(K)$. The topology on $C^\infty(K)$ that it determines is independent of the choices made in defining the pseudometric d_K even though d_K itself is not independent of these choices. In the following discussions, the notation d_K will be used without explicitly noting the assumption that appropriate choices of Λ, the V_λ's, and the V'_λ's have to be made. In all cases, these choices may be made arbitrarily except for the conditions already given. Finally, for $f\in C^\infty(K)$ set $c_K(f)=\inf_C (d^2/dt^2)f(C(t))|_{t=0}$ where the infinimum is taken over all arc-length parameter geodesics $C(t)$ having $C(0)\in K$.

To complete the proof of Theorem 2(a), let $\{K_i|i\in\mathbf{Z}^+\}$ be a sequence of compact subsets of M with $\bigcup_i K_i=M$ and $K_i\subset K^0_{i+1}$ for all $i\in\mathbf{Z}^+$. Then define iteratively a sequence $\{\psi_i|i=0,1,2,\ldots\}$ of functions on M as follows:

$\psi_0=\psi$.

$\psi_1=$ a function which is strictly convex on M; is C^∞ in a neighborhood of K_1; and satisfies $|\psi_0(p)-\psi_1(p))<\varepsilon/4$ for all $p\in M$.

$\vdots$

$\psi_i=$ a function which is strictly convex on M; is C^∞ in a neighborhood of K_i; satisfies $d_{K_i}(\psi_i,\psi_{i-1})<\varepsilon 2^{-(i+1)}$ and $|\psi_i(p)-\psi_{i-1}(p)|<\varepsilon 2^{-(i+1)}$ for every $p\in M$; and has

$$c_{K_{i-1}}(\psi_i)\geq(1-\tfrac{1}{3})\,c_{K_{i-1}}(\psi_{i-1})$$
$$c_{K_{i-2}}(\psi_i)\geq\left(1-\frac{1}{3}-\frac{1}{3^2}\right)c_{K_{i-2}}(\psi_{i-2})$$
$$\vdots$$
$$c_{K_1}(\psi_i)\geq\left(1-\frac{1}{3}-\ldots-\frac{1}{3^{i-1}}\right)c_{K_1}(\psi_1)$$

Lemma 4 guarantees the possibility at each stage of carrying out this construction.

For each $j\in Z^+$, the functions in the sequence $\{\psi_{i+j}|i\in\mathbf{Z}^+\}$ are C^∞ on $\overset{0}{K}_{j+1}$. Moreover, this sequence is a Cauchy sequence in the d_{K_j} pseudo-metric. It follows that the sequence converges to a C^∞ function on $\overset{0}{K}_j$. Thus the sequence $\{\psi_i\}$ converges uniformly on compact subsets of M to a C^∞ function on M. Call this function $\varphi: M \to R$. Clearly $|\varphi - \psi| \leqslant \varepsilon \sum_{i=1}^{+\infty} 2^{-(i+1)} < \varepsilon$ everywhere on M. Also, since $1 - \sum_{i=1}^{+\infty} 3^{-i} = \frac{1}{2} > 0$, $c_{K_i}(\varphi) > 0$ for every i and so φ is strictly convex on M.

Finally, if φ is Lipschitz continuous with Lipsclitz constant B and if $\delta > 0$, then, by virtue of the last statement of Lemma 4, each ψ_i may be taken successively to be Lipschitz continuous with Lipschitz constant $B + \delta(1 - 2^{-i})$. Then the limit function φ will be Lipschitz continuous with Lipschitz constant $B + \delta$. □

§ 2. The topology and exponential map of manifolds of positive curvature

In this section, it will be shown how the existence on a Riemannian manifold of a strictly convex exhaustion function (or of an exhaustion function which is strictly convex outside some compact set) implies certain topological properties of the manifold and certain characteristics of the exponential map on the manifold. The first step in the investigation of such implications is to observe that according to Theorem 1 there is on such a manifold a C^∞ exhaustion function which is strictly convex or strictly convex outside some compact set: the same observation applies in the case of Lipschitz continuous exhaustion functions which are strictly convex or strictly convex outside some compact set. Thus without loss of generality, one may consider only the C^∞ exhaustion function case. The theorems of this section will be stated for the general case, but in the proofs it is as noted necessary to consider only the C^∞ case. Theorem 1 combined with Theorem 2 will show that these results apply in particular to complete manifolds whose sectional curvature is positive or positive outside some compact set; the assumptions of this type appropriate in each case will be stated explicitly in the theorems.

THEOREM 3. (a) *A Riemannian manifold on which there exists a strictly convex exhaustion function is diffeomorphic to euclidean space.* (b) (Gromoll-Meyer) *A complete noncompact Riemannian manifold of everywhere positive sectional curvature is diffeomorphic to euclidean space.*

Proof of Theorem 3. The statement 3(a) combined with Theorem 2(a) immediately implies statement 3(b). To establish 3(a), let M be a Riemannian manifold with a strictly convex exhaustion function, and let $\varphi: M \to R$ be a C^∞ strictly convex exhaustion function on M, the existence of which is guaranteed by Theorem 1(a). The only possible critical points of a C^∞ strictly convex function are nondegenerate local minima, because the Hessian of such a function (at a critical point) is clearly positive definite (see [19] for the terminology and results from Morse theory used in the present and in later arguments). M thus has the homotopy type of a CW-complex containing only 0-cells, a 0-cell corresponding to each (local) minimum of φ. Since M is connected, there can be only one minimum and thus only one critical point of φ, the point at which φ attains its (globally) minimum value. Let $p \in M$ be this minimum point. Now by the Lemma of Morse ([19; p. 6]) there exists a coordinate system $x: U \to R^n$, where U is a neighborhood of p, with $x(p) = (0, ..., 0)$ and $\varphi(q) = \varphi(p) + \Sigma_{i=1} (x_i(q))^2$ for all $q \in U$, where $x(q) = (x_1(q), x_2(q), ..., x_n(q))$. Let λ be a positive number which is small enough that $\{(x_1, ..., x_n) \in R^n \mid \Sigma_{i=1}^n x_i^2 < \lambda\} \subset x(U)$. Construct a metric G on M with $G(\partial/\partial x_i, \partial/\partial x_j) = \delta_{ij}$ on $\{q \in U \mid \Sigma_{i=1}^n (x_i(q))^2 < \lambda/2\}$; such a metric can be constructed by the standard partition-of-unity extension process.

Let $S = \{(t_1, ..., t_n) \in R^n \mid \Sigma_{i=1}^n t_i^2 = \lambda/2\}$. Define $F_1: S \times (0, +\infty) \to R^n - \{(0, ..., 0)\}$ by $F_1((t_1, ..., t_n), t) = (st_1, ..., st_n)$ where $s = t^{\frac{1}{2}}/(t_1^2 + ... + t_n^2)^{\frac{1}{2}}$. F_1 is a diffeomorphism. Define $F_2: S \times (0, +\infty) \to M - \{p\}$ by $F_2((t_1, ..., t_n), t) = C(t - \frac{1}{2}\lambda)$ where C = the integral curve of grad $\varphi/\|\text{grad } \varphi\|^2$ with $C(0)$ = the point of U having x coordinates $(t_1, ..., t_n)$. F_2 is also a diffeomorphism: F is injective because of the uniqueness of integral curves, and F is surjective because through every point of $M - \{p\}$ there is a maximal integral curve of grad $\varphi/\|\text{grad } \varphi\|^2$ on $M - \{p\}$ and this integral curve necessarily intersects the set $x(S)$: For, if the curve is $C: (\alpha, \beta) \to M$ then $\beta = +\infty$; and as $t \to \alpha^+$, $C(t) \to p$ while as $t \to \beta(= +\infty)$, $\varphi(C(t)) \to +\infty$; since $M - x(S)$ has two components, one compact and containing p and the other noncompact, there must exist $\gamma \in (\alpha, \beta)$ such that $C(\gamma) \in x(S)$. Thus $F_2: S \times (0, +\infty) \to M$ is bijective. That F_2 and F_2^{-1} are C^∞ follows from the standard results on the C^∞ character of the flow generated by a C^∞ vector field ([19; p. 10]).

Now define a mapping $F: R^n \to M$ as follows

(i) $F((0, ..., 0)) = p$,
(ii) $F \mid (R^n - \{(0, ..., 0)\}) = F_2 \circ F_1^{-1}$.

Clearly F is bijective and $F \mid (R^n - \{(0, ..., 0)\})$ is a diffeomorphism onto $M - \{p\}$ since F_1 and F_2 are diffeomorphisms. To show that F is a diffeomorphism, it is thus necessary to show only that F and F^{-1} are C^∞ in a neighborhood of $(0, ..., 0)$ and p, respectively. Now the integral curves of grad $\varphi/\|\text{grad } \varphi\|^2$ near p are, when expressed in the x-coordinate system, just the straight lines emanating from $(0, ..., 0)$ parametrized by $\Sigma_{i=1}^n x_i^2$. And in the definition of F_1, $\Sigma_{i=1}^n (sx_i)^2 = t$ if $F_1((t_1, ..., t_n), t) = (st_1, ..., st_n)$. Thus $x(F((x_1, ..., x_n))) = (x_1, ..., x_n)$. Hence near p, $F = x^{-1}$ and near $(0, ..., 0)$, $F^{-1} = x$ so that F and F^{-1} are C^∞ as required. □

THEOREM 4. (a) *If M is a Riemannian manifold on which there exists an exhaustion function which is strictly convex outside a compact subset of M, then M is diffeomorphic to the interior of a compact manifold with boundary. In fact, there exists a compact submanifold-with-boundary M_1 of M such that there is an isotopy of the identity map of M to M with a diffeomorphism of M onto $\overset{0}{M}_1$.* (b) *If M is a complete Riemannian manifold whose sectional curvature is positive everywhere outside some compact subset of M, then M is diffeomorphic to the interior of a compact manifold with boundary. In fact, there exists a compact submanifold M_1 with boundary of M such that the identity map of M is isotopic to a diffeomorphism of M onto $\overset{0}{M}_1$.*

Proof of Theorem 4. The statement (a) implies statement (b) in view of Theorem 2(b). To prove statement (a), and hence (b) also, suppose that a Riemannian manifold M satisfies the hypotheses of (a) and let $\varphi: M \to R$ be a C^∞ exhaustion function on M which is strictly convex on $M-K_1$, where K_1 is a compact subset of M; the existence of such a function φ is guaranteed by Theorem 1(b). Let K_2 be a compact subset of M with $K_1 \subset \overset{0}{K}_2$. Since φ is convex on $M-K_1$, it has only nondegenerate critical points there (see the proof of Theorem 3). By a result of Morse theory ([20; pp. 12–16]) there exists a C^∞ function τ: $M \to R$ which has no degenerate critical points on M and which equals φ on $M-K_2$. Since $\tau=\varphi$ on $M-K_2$ and φ is an exhaustion function, τ is an exhaustion function, also. Another result of Morse theory ([19; p. 20]) is that if an exhaustion function on a manifold has no degenerate critical points then the manifold has the homotopy type of a CW-complex with one cell of dimension k for each critical point of the function of index k. Since every critical point of τ of index $\geqslant 1$ is contained in K_2 and nondegenerate critical points are isolated, τ has only finitely many critical points of index $k \geqslant 1$. Since M is connected, τ has then only finitely many critical points of index 0, as well: for a connected CW-complex cannot contain infinitely many 0-cells if it contains only finitely many cells of dimension greater than 0 (in fact, if it contains only finitely many cells of dimension 1). Thus τ has only finitely many critical points.

Let λ be a real number which is larger than any of the (finitely many) critical values of τ. Set $M_1=\tau^{-1}((-\infty, \lambda+2])$. Then M_1 is a compact submanifold-with-boundary of M: compact because τ is an exhaustion function and a submanifold-with-boundary because $\lambda+2$ is not a critical value of τ.

To construct an isotopy of the identity map of M to itself with a diffeomorphism of M onto $\overset{0}{M}_1$, note that if $M_2=\{q\in M \mid \tau(q)=\lambda+1\}$ then M_2 is a compact submanifold (without boundary) of M since $\lambda+1$ is not a critical value of τ. Moreover, there is a diffeomorphism of $M_2\times(0,+\infty)$ onto $M-\tau^{-1}((-\infty,\lambda])=\tau^{-1}((\lambda,+\infty))$ which maps $M_2\times(0,1)$ onto

$\tau^{-1}((\lambda, \lambda+1))$: such a diffeomorphism $F: M_2 \times (0, +\infty) \to \tau^{-1}((\lambda, +\infty))$ is given by setting $F(q, t) = C_q(t-1)$, where C_q = the integral curve of $\operatorname{grad} \varphi / \|\operatorname{grad} \varphi\|^2$ with $C_q(0) = q$. F is injective because of the uniqueness of integral curves. F is surjective because the integral curve of $\operatorname{grad} \varphi / \|\operatorname{grad} \varphi\|^2$ through a point of $\tau^{-1}((\lambda, +\infty))$ of $\tau^{-1}((\lambda, +\infty))$ necessarily intersects M_2. That F and its inverse are C^∞ follows from the standard results on the C^∞ character of the flow generated by a vector field ([19; p. 10]).

Let $h: [0, 1] \times (0, +\infty) \to (0, +\infty)$ be an isotopy with $h(0, .)$ = the identity map of $(0, +\infty)$ to $(0, +\infty)$ and $h(1, .)$ = a diffeomorphism of $(0, +\infty)$ onto $(0, 2)$. The isotopy h can and will be assumed to be chosen so that $h(t, s) = s$ for all $t \in [0, 1]$ and $s \in (0, 1)$. Now define $H: [0, 1] \times M_2 \times (0, +\infty) \to M_2 \times (0, +\infty)$ by

$$H(t, q, s) = (q, h(t, s)) \in M_2 \times (0, +\infty).$$

And define $H_1: [0, 1] \times M \to M$ by

$$\begin{aligned} H_1(t, q) &= q & &\text{if } \tau(q) \leqslant \lambda, \\ H_1(t, q) &= F^{-1}(H(t, F(q))) & &\text{if } \tau(q) > \lambda. \end{aligned}$$

Since, for all $t \in [0, 1]$ $H_1(t, q') = q'$ if $\tau(q') < \lambda + 1$, H_1 is C^∞ in a neighborhood of $[0, 1] \times q$ if $\tau(q) = \lambda$: for $H_1(t, q') = q'$ for all $t \in [0, 1]$ and all q' sufficiently near q. H_1 is clearly C^∞ elsewhere on $[0, 1] \times M$. H_1 is the isotopy required in statement (a) of the theorem. □

The terminology introduced by the following definition will be used in stating some of the results on the asymptotic behavior of geodesics to be given presently.

Definition. A semi-infinite geodesic $C: [0, +\infty) \to M$ on a Riemannian manifold M is *unbounded* if for every compact subset K of M there is a $t \in [0, +\infty)$ such that $C(t) \in M - K$. The geodesic C *converges to infinity* if for every compact subset K of M there is a $t \in [0, +\infty)$ such that if $s > t$ then $C(s) \in M - K$.

A semi-infinite geodesic $C: [0, +\infty) \to M$ converges to infinity if and only if C is a proper mapping (in the usual sense that if K is any compact subset of M then $C^{-1}(K)$ is a compact subset of $[0, +\infty)$). Thus if the exponential map $TM_p \to M$ is a proper mapping for a point $p \in M$ then any semi-infinite geodesic $C: [0, +\infty) \to M$ with $C(0) = p$ converges to infinity, because the composition of proper mappings is a proper mapping and C = the mapping of $[0, +\infty)$ into TM_p as a straight line (parametrized proportional to arc length) through the origin composed with the exponential map at p.

THEOREM 5. *If M is a complete Riemannian manifold on which a strictly convex exhaustion function exists, then the exponential map $\exp_p: TM_p \to M$ is for each $p \in M$ a proper* *mapping; and in particular all semi-infinite geodesics converge to infinity. More precisely, if* $\varphi: M \to R$ *is a strictly convex exhaustion function on* M *and if* p *is a point of* M *then there are constants* t_0 *and* $B>0$ *such that if* $C: [0, +\infty) \to M$ *is any arc-length parametrized semi-infinite geodesic with* $C(0)=p$ *then* $\varphi(C(t)) \geq Bt$ *for any* $t \geq t_0$.

COROLLARY. (a) *If* M *is a (necessarily complete) noncompact Riemannian manifold on which a Lipschitz continuous strictly convex exhaustion function exists, then the exponential map at each point of* M *is a proper mapping. And in fact if* p *is a point of* M *there are constants* t_0 *and* $B>0$ *such that, for any arc-length parametrized geodesic* $C: [0, +\infty) \to M$ *with* $C(0)=p$, $\operatorname{dis}(p, C(t)) > Bt$ *for all* $t \geq t_0$. (b) *If* M *is a complete noncompact Riemannian manifold whose curvature is everywhere positive then the exponential map at each point of* M *is a proper mapping. And in fact if* p *is a point of* M *then there are constants* t_0 *and* $B>0$ *such that, for any arc-length parametrized geodesic* $C: [0, +\infty) \to M$ *with* $C(0)=p$, $\operatorname{dis}(p, C(t)) \geq Bt$ *for all* $t \geq t_0$.

Proof of the corollary, Theorem 5 *being assumed.* Statement (b) of the corollary follows immediately from statement (a) and the fact that there is a Lipschitz continuous strictly convex exhaustion function on any complete Riemannian manifold whose curvature is positive everywhere ([12; p. 292] and Theorem 2 of the present paper). To prove statement (a), note that if $\varphi(C(t)) \geq Bt$ for all $t \geq t_0$ then, for all $t \geq \max\{t_0, 2B^{-1}\varphi(C(0))\}$, $\varphi(C(t)) - \varphi(C(0)) \geq \frac{1}{2}Bt$. Then if B_1 is a Lipschitz constant for φ,

$$\operatorname{dis}(C(t), C(0)) \geq (\tfrac{1}{2}BB_1^{-1})t$$

for all $t \geq \max\{t_0 2B^{-1}\varphi(C(0))\}$. □

Proof of Theorem 5. Properness of the exponential map at a point implies convergence to infinity of all semi-infinite geodesics emanating from that point. Furthermore, the existence of constants t_0 and $B>0$ with the properties indicated implies properness of the exponential map; for then, if K is a compact subset of M, $(\exp_p)^{-1}K$ is a closed subset of the compact set

$$\{v \in TM_p \mid \|v\| \leq \max(t_0, B^{-1}(\sup_K \varphi))\}.$$

Thus to prove the theorem, it is now necessary to establish only the last statement of the theorem. Furthermore, it is enough to establish the last statement in the case of C^∞ strictly convex exhaustion functions. For, if $\varphi: M \to R$ is any strictly convex exhaustion function then according to Theorem 1 (a) there is a C^∞ strictly convex exhaustion function $\tau: M \to R$

such that $|\varphi-\tau|<1$ everywhere on M. Suppose $B>0$ and t_0 are such that for any geodesic C: $[0,+\infty]\to M$ parametrized by arc length and having $C(0)=p$

$$\tau(C(t)) > Bt \quad \text{for } t \geqslant t_0.$$

Then if $t>\max(t_0, 2B^{-1})$,

$$\varphi(C(t)) > \tau(C(t)) - 1 > \tfrac{1}{2}Bt \quad \text{if } t \geqslant t_0.$$

So suppose now that φ: $M\to R$ is a C^∞ strictly convex exhaustion function and p is a point of M. Let $\beta=\inf_C(d^2/dt^2)\varphi(C(t))|_{t=0}$ with C ranging over all arc-length parameter geodesic segments having $C(0)\in\{q\in M\,|\,\varphi(q)\leqslant\varphi(p)+1\}$. Since this last set is compact and φ is strictly convex, β is a positive number. Let $\alpha=\inf_X(X\varphi)$ with X ranging over the unit vectors in TM_p; then $-\infty<\alpha\leqslant 0$. Let $\lambda=$ the unique positive number such that $\alpha\lambda+\frac{1}{2}\beta\lambda^2=1$. Suppose that C: $[0,t]\to M$ is an arc-length parameter geodesic segment with $C(0)=p$ and $C\ [0,t])\subset\{q\in M\,|\,\varphi(q)\leqslant\varphi(p)+1\}$. Then

$$\varphi(C(t)) = \varphi(p) + \int_0^t \frac{d}{ds}\varphi(C(s))\,ds \geqslant \varphi(p) + \int_0^t\left(\alpha + \int_0^s \frac{d^2}{d\xi^2}\varphi(C(\xi))\,d\xi\right)ds$$

$$\geqslant \varphi(p) + \int_0^t\left(\alpha + \int_0^s \beta d\xi\right)ds \geqslant \psi(p) + \alpha t + \tfrac{1}{2}\beta t^2.$$

Since $\varphi(p)+\alpha t+\frac{1}{2}\beta t^2>\varphi(p)+1$ if $t>\lambda$, t must be $\leqslant\lambda$. Thus if C: $[0,+\infty)\to M$ is a semi-infinite geodesic with $C(0)=p$ then there exists a positive number t_C with $t_C\leqslant\lambda$ such that $\varphi(C(t_C))=\varphi(p)+1$. Since $t\to\varphi(C(t))$ is (strictly) convex on $[0,+\infty)$, $\varphi(C't))\geqslant\varphi(p)+(t/t_C)$ if $t\geqslant t_C$. Hence

$$\varphi(C(t)) \geqslant \varphi(p) + \frac{t}{\lambda} \quad \text{if } t \geqslant \lambda.$$

Finally, if $t\geqslant\max(\lambda, 2\lambda|\varphi(p)|)$, $\varphi(C(t))\geqslant t/(2\lambda)$, so that the last statement of the theorem holds with $t_0=\max(\lambda, 2\lambda\varphi(p))$ and $B=(2\lambda)^{-1}$. □

The existence on a manifold of a convex exhaustion function which is not necessarily strictly convex does imply special properties of the geodesics on the manifold even though it is not necessarily true for such a manifold that the exponential map at a given point is a proper map. For instance, the two-dimensional cylinder $R^2/\{(x,y)\sim(x',y')$ if $x'-x\in\mathbf{Z}\}$ is an example of a complete Riemannian manifold on which a C^∞ convex exhaustion function (the coordinate y) is defined but which does not have a proper exponential map at any point.

The behavior of the geodesics of this example is a special case of the behavior described in the following theorem:

THEOREM 6. *If M is a Riemannian manifold on which an exhaustion function exists which is convex outside some compact subset of M, then any geodesic which is unbounded converges to infinity. Specifically, if $\varphi: M \to R$ is an exhaustion function which is convex outside some compact set and if $C: [0, +\infty) \to M$ is a semi-infinite geodesic which is unbounded, then there are constants t_0 and $B > 0$ such that*

$$\varphi(C(t)) \geqslant Bt \quad \text{if } t > t_0.$$

COROLLARY. (a) *If M is a Riemannian manifold on which there is a Lipschitz continuous exhaustion function which is convex outside a compact subset of M and if $C: [0, +\infty) \to M$ is a semi-infinite arc-length parameter geodesic on M which is unbounded, then there exist constants t_0 and $B > 0$ such that*

$$\operatorname{dis}(C(0), C(t)) \geqslant Bt \quad \text{if } t \geqslant t_0.$$

(b). *If M is a complete Riemannian manifold whose curvature is nonnegative outside some compact subset of M and if $C: [0, +\infty) \to M$ is a semi-infinite unbounded arc-length parameter geodesic then there are constants t_0 and $B > 0$ such that*

$$\operatorname{dis}(C(0), C(t)) \geqslant Bt \quad \text{if } t \geqslant t_0.$$

Proof of the corollary, Theorem 6 *being assumed.* Since on a complete Riemannian manifold whose curvature is nonnegative outside a compact set there is a Lipschitz continuous exhaustion function which is convex outside some compact set ([12; p. 292]), statement (b) of the corollary is implies by the statement (a) of the corollary. To verify statement (a) let $\varphi: M \to R$ be an exhaustion function of the sort indicated. Then by part (b) of the theorem for some t_0 and $B > 0$

$$C(\varphi(t)) \geqslant Bt \quad \text{if } t \geqslant t_0.$$

Then

$$C(\varphi(t)) - C(\varphi(0)) \geqslant \tfrac{1}{2}Bt,$$

if

$$t > \max t_0 \frac{2C(\varphi(0))}{B}.$$

For such t

$$\operatorname{dis}(C(0), C(t)) \geqslant (\tfrac{1}{2}BB_1^{-1})t$$

where B_1 = a Lipschitz constant for φ. □

Proof of Theorem 6. Let $\varphi: M \to R$ be an exhaustion function which is convex on $M - K$, K being a compact subset of M, and $C: [0, +\infty) \to M$ be an unbounded semi-infinite geodesic. Set $\lambda_0 = 1 + \sup_K \varphi$ and $\lambda_1 = \max\{\lambda_0, 1 + \varphi(C(0))\}$. Then since φ is an exhaustion function and C is unbounded, the set $\{t \in [0, +\infty) \,|\, \varphi(C(t)) \geq \lambda\}$ is not empty. Set t_1 = infinum of this set. Since $\varphi(C(0)) < \lambda_1$, $t_1 > 0$. Now in a neighborhood of t_1, $t \to \varphi(C(t))$ is a convex function; and since $\varphi(C(t)) < \varphi(C(t_1)) = \lambda_1$ if $t < t_1$ the convex function $t \to \varphi(C(t))$ near t_1 is a strictly monotone increasing function near t_1 and in particular $\varphi(c(t)) > \lambda_1$ for $t > t_1$ but t sufficiently close to t_1.

Suppose that for some $t_2 > t_1$, $\varphi(C(t_2)) = \lambda_1$. Then, if t_2' = the infimum of the set of t_2 such that $\varphi(C(t_2)) = \lambda_1$ and $t_2 > t_1$, $\varphi(C(t_2')) = \lambda_1$ but $\varphi(C(t)) > \lambda_1$ for all $t \in (t_1, t_2)$. In particular $t \to \varphi(C(t))$ is convex and nonconstant on (t_1, t_2) but attains a maximum value at some point of (t_1, t_2), contradicting the maximum principle for convex functions. Thus $\varphi(C(t))$ must be $> \lambda_1$ for all $t > t_1$, so that $\varphi(C(t)) \in M - K$ for all $t > t_1$ and $t \to \varphi(C(t))$ is a convex function on $[t_1, +\infty)$.

Put $t_0' = t_1 + 1$. Then because $t \to \varphi(C(t))$ is convex on $[t_1, +\infty)$,

$$\varphi(C(t)) \geq (t - t_0') \cdot \{\varphi(C(t_0')) - \varphi(C(t_1))\} + \varphi(C(t_1))$$

if $t \geq t_0'$. Hence if

$$t \geq t_0' + \frac{2|\varphi(C(t_1))|}{\varphi(C(t_0')) - \varphi(C(t_1))},$$

then

$$\varphi(C(t)) \geq \tfrac{1}{2}(t - t_0)\{\varphi(C(t_0')) - \varphi(C(t_1))\},$$

so that Theorem 6 holds for

$$t_0 = t_0' + \frac{2|\varphi(C(t_1))|}{\varphi(C(t_0')) - \varphi(C(t))},$$

and

$$B = \varphi(C(t_0')) - \varphi(C(t_1)). \qquad \square$$

§ 3. Integrals of nonnegative subharmonic functions on manifolds of positive curvature

If S is a closed convex hypersurface in Euclidean space R^n and S_ε is the hypersurface obtained by displacing each point of S distance ε along the exterion unit normal to S at that point, then the (n-dimensional) volume enclosed between S and S_ε is greater than $\varepsilon \times$ the $(n-1)$-dimensional volume of S. This fact is in agreement with the intuitive notion that the exterion normals of a convex surface diverge. The following theorem is a result of similar nature in a more general setting:

THEOREM 7(a). *If M is a Riemannian manifold and $\varphi\colon M \to \mathbf{R}$ is an exhaustion function which is C^∞ on M and strictly convex outside some compact subset of M and which is*

Lipschitz continuous on M and if $f: M \to \mathbf{R}$ is a C^∞ nonnegative subharmonic function which is not identically zero, then there is a positive constant A_f such that

$$\int_{\{q\in M \mid \varphi(q)\leqslant \lambda\}} f \geqslant A_f \lambda$$

for all sufficiently large $\lambda \in \mathbf{R}$. Here the integral is taken relative to the measure induced on M by its Riemannian metric.

(b) *If M is a complete noncompact Riemannian manifold whose sectional curvature is positive outside some compact subset, then the integral over M of any nonnegative C^∞ subharmonic function which is not equal to zero is (positively) infinite and in particular the volume of M is infinite.*

Proof. By virtue of Theorem 1, 7 (a) implies 7 (b). To prove 7 (a), first note that it is enough to consider the case of oriented manifolds M, since the result for a nonorientable manifold follows from that case by consideration of the orientable double covering. So suppose from now to the conclusion of the proof that M is oriented, and let Ω be the Riemannian volume form on M. Let $\dim M = n$.

According to part of the proof of Theorem 4 (a), there is a λ_1 such that φ has no critical points such that $\varphi(p) \geqslant \lambda_1$. Consider from here on only λ which are $\geqslant \lambda$. Then (by a standard argument: see [19]; also cf. the proof of Theorem 4 (a) of the present paper): the set $\{q\in M \mid \varphi(q)=\lambda\}$ is a compact (embedded) submanifold of M of dimension $(n-1)$ and this submanifold is the topological boundary of $\{q\in M \mid \varphi(q)\leqslant\lambda\}$. The set $\{q\in M \mid \varphi(q)\leqslant\lambda\}$ will hereafter be denoted by M^φ_λ and the set $\{q\in M \mid \varphi(q)=\lambda\}$ by $\partial M^\varphi_\lambda$.

The submanifold $\partial M^\varphi_\lambda$ has smoothly varying distinguished unit normal at each point, namely $\operatorname{grad}\varphi/\|\operatorname{grad}\varphi\|$. Thus $\partial M^\varphi_\lambda$ inherits an orientation from the orientation of M. Let ω_λ be the volume $(n-1)$-form on $\partial M^\varphi_\lambda$ determined by this orientation and by the induced Riemannian metric on $\partial M^\varphi_\lambda$. Then

$$\Omega_p = (\omega_\lambda)_p \wedge (d\varphi/\|\operatorname{grad}\varphi\|)_p.$$

This equality is verified by evaluating both sides on an n-tuple of orthonormal vectors in TM_p of the type $(e_1, \ldots, e_{n-1}, \operatorname{grad}\varphi/\|\operatorname{grad}\varphi\|)$, the e_i's being then necessarily tangent to $\partial M^\varphi_\lambda$.

The integral curves of the vector field $\operatorname{grad}\varphi/\|\operatorname{grad}\varphi\|^2$ emanating from the points of $\partial M^\varphi_{\lambda_1}$ generate a diffeomorphism $H: \partial M^\varphi_{\lambda_1} \times \mathbf{R}^+ \to \{q\in M \mid \varphi(q)\geqslant\lambda_1\}$ (here $\mathbf{R}^+ = \{t\in\mathbf{R}(t\geqslant 0\})$; specifically, $H(p,t)=$ the point with parameter value t along the integral curve of $\operatorname{grad}\varphi/\|\operatorname{grad}\varphi\|^2$ which has parameter value 0 at p (see [19, p. 13] and the proof of Theorem

4(a) of the present paper). Note that $\varphi(H(p,t))=t$ so that from the formula for Ω_p in the previous paragraph and Fubini's theorem it follows that for any $\lambda \geqslant \lambda_1$

$$\int_{M_\lambda^\varphi - M_{\lambda_1}^\varphi} f\Omega = \int_{\lambda_1}^{\lambda} \left(\int_{\partial M_t^\varphi} \frac{f}{\|\text{grad } \varphi\|} \omega_t \right) dt.$$

Suppose φ is Lipschitz continuous on M with Lipschitz constant $B(>0)$ on M. Then $\|\text{grad } \varphi\| \leqslant B$ everywhere so

$$\int_{M_\lambda^\varphi - M_{\lambda_1}^\varphi} f\Omega \geqslant \frac{1}{B^2} \int_{\lambda_1}^{\lambda} \left(\int_{\partial M_t^\varphi} f \|\text{grad } \varphi\| \omega_t \right) dt.$$

It will now be shown that the integral $\int_{\partial M_t^\varphi} f\|\text{grad } \varphi\|\omega_t$ is nonzero for some value of $t > \lambda_1$.

Because f is subharmonic, there is no point $p \in M$ such that $f(p) = \sup_{q\in M} f(q)$ unless f is constant on M. Hence if K is any compact subset of M there is a point $q \in M - K$ such that $f(q) > 0$; for if $f|(M-K)$ were $\equiv 0$ then there would be a point $p \in K$ with $f(p) = \sup_{q\in M} f(q)$ so that f would be constant and so $\equiv 0$ on M. Since $M_{\lambda_1}^\varphi$ is compact and $M - M_{\lambda_1}^\varphi = \bigcup_{t>\lambda_1} \partial M_t^\varphi$, there is a point $p \in \partial M_t^\varphi$ with $f(p) > 0$. Since $\|\text{grad } \varphi\|$ is nowhere zero on $M - M_{\lambda_1}^\varphi$, for some λ_2

$$\int_{\partial M_{\lambda_2}^\varphi} f \|\text{grad } \varphi\| \omega_{\lambda_2} > 0.$$

If, for all $t \geqslant \lambda_2$,

$$\int_{\partial M_t^\varphi} f \|\text{grad } \psi\| \omega_t \geqslant \int_{\partial M_{\lambda_2}^\varphi} f \|\text{grad } \varphi\| \omega_{\lambda_2},$$

then for $\lambda \geqslant \lambda_2$

$$\int_{M_\lambda^\varphi} f\Omega \geqslant \int_{M_\lambda^\varphi - M_{\lambda_1}^\varphi} f\Omega \geqslant \frac{1}{B^2} \int_{\lambda_1}^{\lambda} \left(\int_{\partial M_t^\varphi}^{\lambda} f \|\text{grad } \varphi\| \omega_t \right) dt \geqslant (\lambda - \lambda_1) \left(\frac{1}{B^2} \int_{\partial M_{\lambda_2}^\varphi} f \|\text{grad}\| \omega_{\lambda_2} \right),$$

from which the conclusion of (part (a) of) the theorem follows. Thus to complete the proof of the theorem, it is enough to show that

$$\int_{\partial M_t^\varphi} f \|\text{grad } \varphi\| \omega_t \geqslant \int_{\partial M_{\lambda_2}^\varphi} f \|\text{grad } \varphi\| \omega_{\lambda_2}.$$

For notational convenience, let $I(t)$ denote the integral $\int_{\partial M_t^\varphi} f\|\text{grad } \varphi\|\omega_t$. Also, let H_t: $\partial M_{\lambda_1}^\varphi \to \partial M_t^\varphi$, $t \geqslant \lambda_1$, be defined by $H_t(p) = H(p,t)$ where H: $\partial M_{\lambda_1}^\varphi \times \mathbf{R}^+ \to \{q \in M \mid \varphi(q) \geqslant \lambda_1\}$ is the diffeomorphism defined previously. Then

$$I(t) = \int_{p \in \partial M^{\varphi}_{\lambda_1}} (f(H_t(p)) \|(\operatorname{grad} \varphi)_{H_t(p)}\| H_t^* \omega_t(p).$$

From this formula it is clear by differentiation under the integral sign that $I(t)$ is a C^∞ function of t (for $t \geq \lambda_1$). Since the one parameter family H_t is generated by $\operatorname{grad} \varphi / \|\operatorname{grad} \varphi\|^2$, it follows that for any $t > \lambda_1$

$$I'(t) = \int_{\partial M^{\varphi}_t} \frac{(\operatorname{grad} \varphi) f}{\|\operatorname{grad} \varphi\|^2} \|\operatorname{grad} \varphi\| \omega_t + \int_{\partial M^{\varphi}_t} f \frac{1}{\|\operatorname{grad} \varphi\|^2} D_{\operatorname{grad} \varphi} \|\operatorname{grad} \varphi\| \omega_t + \int_{\partial M^{\varphi}_t} f \|\operatorname{grad} \varphi\| \alpha_t$$

where α_t = the variational derivative of the $(n-1)$-form ω_t relative to the variation vector field $\operatorname{grad} \varphi / \|\operatorname{grad} \varphi\|^2$ and $D_{\operatorname{grad} \varphi} \|\operatorname{grad} \varphi\|$ = the result of applying the vector field $\operatorname{grad} \varphi$ to the function $\|\operatorname{grad} \varphi\|$. The standard variation of area formula (see, for instance, [27]) gives that

$$\alpha_t = -\left\langle \frac{\operatorname{grad} \varphi}{\|\operatorname{grad} \varphi\|^2}, K \right\rangle \omega_{t_0} = -\frac{1}{\|\operatorname{grad} \varphi\|^2} (K\varphi)\, \omega_t$$

where K = the mean curvature vector of ∂M^{φ}_t at the point at which α_t is being evaluated. That is, $K = \Sigma_{i=1}^{n-1} (D_{e_i} e_i)^N$ where: $e_1, \ldots, e_{n-1}$ is an orthonormal $(n-1)$-frame defined in a neighborhood in M of the point of ∂M^{φ}_t at issue and each e_i is tangent to ∂M^{φ}_t at every point of ∂M^{φ}_t at which it is defined; D is the covariant operator on M; and $(\ )^N$ denotes the component normal to ∂M^{φ}_t.

Now for any vector field defined on an open set on which φ is C^∞ and convex $V(V\varphi) - (D_V V)\varphi \geq 0$. (This standard fact is established by checking that the value of the left-hand side at a point p depends only on the value V_p of V at p and then choosing a vector field W with $W_p = V_p$ and $D_W W = 0$. Then $W(W\varphi) - (D_W W)\varphi = W(W\varphi)$ = the second derivative at p of φ along the geodesic through p whose tangent at $p = W_p$. This second derivative is nonnegative if φ is convex). Hence

$$(D_{e_i} e_i)\varphi \leq e_i(e_i \varphi) = 0:$$

here $e_i(e_i\varphi) = 0$ because $e_i \varphi \equiv 0$ along ∂M^{φ}_t. Thus

$$K\varphi = \sum_i (D_{e_i} e_i)^N \varphi = \sum_i (D_{e_i} e_i)\, \varphi \leq 0$$

and α_t is therefore a nonnegative $(n-1)$-form on ∂M^{φ}_t. So

$$\int_{\partial M^{\varphi}_t} f \|\operatorname{grad} \varphi\| \alpha_t \geq 0.$$

Put $N = \operatorname{grad}\varphi/\|\operatorname{grad}\varphi\|$. Then since $N\langle N, N\rangle = 0$, $\langle D_N N, N\rangle = 0$ and hence $\langle D_N N, \operatorname{grad}\varphi\rangle = 0$ or $(D_N N)\varphi = 0$. Thus that $N(N\varphi) - (D_N N)\varphi \geq 0$ implies that $N(N\varphi) \geq 0$. Since $D_{\operatorname{grad}\varphi}\|\operatorname{grad}\varphi\| = \|\operatorname{grad}\varphi\| D_N \|\operatorname{grad}\varphi\| = \|\operatorname{grad}\varphi\| N(\langle N, \operatorname{grad}\varphi\rangle) = \|\operatorname{grad}\varphi\| N(N)$, $D_{\operatorname{grad}\varphi}\|\operatorname{grad}\varphi\| \geq 0$. Consequently

$$\int_{\partial M_t^\varphi} f \frac{1}{\|\operatorname{grad}\varphi\|^2} D_{\operatorname{grad}\varphi}\|\operatorname{grad}\varphi\|\,\omega_t \geq 0.$$

Finally by Stokes' theorem

$$\int_{\partial M_t^\varphi} \frac{(\operatorname{grad}\varphi) f}{\|\operatorname{grad}\varphi\|^2}\|\operatorname{grad}\varphi\|\,\omega_t = \int_{M_t^\varphi} \Delta f \Omega \geq 0$$

since $\Delta f \geq 0$. Thus $I'(t)$ is the sum of three nonnegative terms and so $I(t)$ is a nondecreasing function of t for $t \geq \lambda_1$. □

It is more natural to estimate the growth of an integral as a function of its domain of integration in terms of its value on Riemannian balls than on the sublevel sets of more or less arbitrary exhaustion functions. The following theorem gives an estimate of this more natural sort. In this theorem, $B(p; r)$ denotes the open Riemannian ball about p of radius r.

THEOREM 8(a). *If M is a noncompact Riemannian manifold on which there exists an exhaustion function $\varphi: M \to R$ which is C^∞ and Lipschitz continuous and which is strictly convex outside some compact subset of M and if $f: M \to \mathbf{R}$ is C^∞ nonnegative subharmonic function which is not identically zero on M, then there exists a positive constant B_f such that for every $p \in M$*

$$\int_{B(p,\lambda)} f \geq B_f \lambda$$

for all sufficiently large $\lambda \in \mathbf{R}$.

(b) *If M is a complete noncompact Riemannian manifold whose sectional curvature is positive outside some compact subset of M and if $f: M \to \mathbf{R}$ is an C^∞ nonnegative subharmonic function which is not identically zero on M, then there exists a positive constant B_f such that for every $p \in M$,*

$$\int_{B(p;\lambda)} f \geq B_f \lambda$$

for all sufficiently large $\lambda \in \mathbf{R}$.

Proof. As before, part (a) implies part (b) by virtue of Theorem 1 (b). As in the proof of Theorem 7 (a), let λ_1 be a real number such that for no critical point of p is $\varphi(p) \geqslant \lambda_1$. If C is an integral curve of $\operatorname{grad} \varphi / \|\operatorname{grad} \varphi\|^2$ with $C(0) \in \partial M^\varphi_{\lambda_1}$ (the notations of the proof of Theorem 7 (a) are continuing to be used here) then C is defined on all of $\mathbf{R}^+$ and $\|(\operatorname{grad} \varphi)(C(t))\|$, $t \in \mathbf{R}^+$, is a nondecreasing function on $\mathbf{R}^+$: the first of these assertions was established in the proof of Theorem 4 (a) and the last assertion follows immediately from the fact established in the proof of Theorem 7 (a) that

$$D_{\operatorname{grad} \varphi} \|\operatorname{grad} \varphi\| \geqslant 0.$$

Since $\{q \in M \mid \varphi(q) \geqslant \lambda_1\}$ is a union of integral curves of $\operatorname{grad} \varphi / \|\operatorname{grad} \varphi\|^2$ emanating from $\partial M^\varphi_\lambda$,

$$\inf_{q \ni \varphi(q) \geqslant \lambda_1} \|(\operatorname{grad} \varphi)_q\| \geqslant \inf_{q \ni \varphi(q) = \lambda_1} \|(\operatorname{grad} \varphi)_q\| > 0.$$

(In fact, the two infima are equal since the reverse inequality between them holds a priori). Let $\varepsilon = \inf_{q \ni \varphi(q) \geqslant \lambda_1} \|\operatorname{grad} \varphi)_q\|$. For any integral curve $C\colon \mathbf{R}^+ \to M$ of $\operatorname{grad} \varphi / \|\operatorname{grad} \varphi\|^2$ with $C(0) \in \partial M^\varphi_{\lambda_1}$, the length of $C \mid [0, t]$ is

$$\int_0^t \|C'(\tau)\| d\tau = \int_0^t \frac{1}{\|(\operatorname{grad} \varphi)(C(\tau))\|} d\tau,$$

which is $\leqslant \varepsilon^{-1} t$. So for any $q \in M$ with $\varphi(q) \geqslant \lambda_1$, $\operatorname{dis}(q, \partial M^\varphi_{\lambda_1}) \leqslant \varepsilon^{-1}(\varphi(q) - \lambda_1)$.

Now let A_f be a positive constant satisfying the conclusion of Theorem 7 (a). And let p be any point of M. Choose a positive real number r_0 such that $B(p; r_0) \supseteq M^\varphi_{\lambda_1}$ ($M^\varphi_{\lambda_1}$ is compact so such a choice is possible). Then for any $\lambda \geqslant \lambda_1$, $M^\varphi_\lambda \subseteq N(p; r_0 + \varepsilon^{-1}(\lambda - \lambda_1))$ by virtue of the estimate which concludes the previous paragraph. Hence, for $r \geqslant r_0$, $B(p; r) \supset M^\varphi_{\lambda_1 + \varepsilon(r - r_0)}$ so that

$$\int_{B(p;r)} f \geqslant \int_{M^\varphi_{\lambda_1 + \varepsilon(r - r_0)}} f.$$

For r sufficiently large

$$\int_{M^\varphi_{\lambda_1 + \varepsilon(r - r_0)}} f \geqslant A_f(\lambda_1 + \varepsilon(r - r_0)) \geqslant \tfrac{1}{2} A_f \varepsilon r,$$

the first inequality holding by Theorem 7 (a) and the second by elementary considerations. Thus for r sufficiently large

$$\int_{B(p;r)} f \geqslant \tfrac{1}{2} A_f \varepsilon r$$

so that the positive constant $\frac{1}{2} A_f \varepsilon$ is an acceptable choice for the B_f required for Theorem 8 (a). □

§ 4. The total curvature of manifolds of positive curvature

It is a well-known result of Cohn-Vossen [5] that if the total curvature integral $\int_M K dA$ on a complete orientable two-dimensional Riemannian manifold M is absolutely convergent then $\int_M K dA \leqslant \chi(M)$: here $\chi(M)$ = the Euler characteristic of M, K = the Gaussian curvature, and dA = the measure induced on M by the Riemannian metric. It can be shown by examples ([24]) that a corresponding extension of the higher-dimensional analogue of the Gauss-Bonnet theorem ([1], [4]) to the case of noncompact manifolds does not hold in general (even in even-dimensions: The failure in case of odd dimension is obvious, since in that case the generalized Gauss-Bonnet integrand is zero). The purpose of this section is to state and prove such an extension for four-dimensional manifolds whose curvature is positive outside some compact set. The argument used to prove this result for four-dimensional manifolds also provides a simple proof of the Cohn-Vossen inequality in the case of the curvature's being positive outside some compact set. Related results for four-dimensional manifolds, in which however the curvature is required to be nonnegative everywhere on the manifold, are given in [23] and [28].

THEOREM 9. *If M is a complete oriented Riemannian manifold of dimension four whose sectional curvature if positive outside some compact subset of M, then the integral $\int_M \Theta$ of the generalized Gauss-Bonnet integrand Θ is (absolutely) convergent and*

$$\int_M \Theta \leqslant \chi(M).$$

The Euler characteristic $\chi(M)$ of M is necessarily defined and finite by virtue of Theorem 4(b). Since the integrand Θ is positive at any point of M at which all sectional curvatures are positive (Chern's theorem, cf. [23], [28]), the integral's absolute convergence and the required inequality follow if a uniform upper bound $\int_U \Theta \leqslant \chi(M)$ is known, where U varies over some increasing family of closure-compact open subregions of M whose union $= M$. The following lemma will be used to obtain such a bound.

LEMMA 5. *Suppose that U is an open set with compact closure and C^∞ boundary in a four-dimensional oriented Riemannian manifold M and that the sectional curvature of M is positive in a neighborhood of the boundary of U. Suppose also that there is a C^∞ function φ: $M \to \mathbf{R}$ such that*

(a) $U = \{p \in M \mid \varphi(p) < 0\}$,
(b) $\operatorname{grad} \varphi|_p \neq 0$ *if* $\varphi(p) = 0$,
(c) φ *is geodesically convex in a neighborhood of any point* $p \in M$ *having* $\varphi(p) = 0$.

Then

$$\int_U \Theta \leqslant \chi(U).$$

The proof of this lemma is obtained by modification of the classical argument in [4] to include the terms arising from the fact that U has (possibly) nonempty boundary.

Proof. Let Y be a C^∞ unit vector field defined everywhere in a neighborhood of U except perhaps at some isolated points and equal to $(\operatorname{grad}\varphi)/\|\operatorname{grad}\varphi\|$ in a neighborhood of the boundary of U. Such a vector field may be obtained for instance by first observing that there is a C^∞ function $\varphi': M\to\mathbf{R}$ with isolated nondegenerate critical points on M which agrees with φ in a neighborhood of the boundary of U ([20; pp. 12–16]) and then taking $Y=\operatorname{grad}\varphi'/\|\operatorname{grad}\varphi'\|$. The vector field Y extends in an obvious way to a vector field $\tilde{Y}$ on the double $\tilde{U}$ of U: if $\tilde{U}=U_1\cup U_2$ with $U_1=U$ then $\tilde{Y}=-Y$ on U_2. Since the antipodal map on the 3-sphere has degree $+1$, the index of a singularity of $\tilde{Y}$ on U_2 equals the index of the corresponding singularity of Y on $U(=U_1)$. Thus $\chi(\tilde{U})$, which equals the sum of the indices of the singularities of $\tilde{Y}$ on $\tilde{U}$, equals twice the sum of the indices of Y on U. On the other hand, $\chi(\tilde{U})=2\chi(U)$. Hence $\chi(U)=$the sum of the indices of the singularities of Y (on U).

Let $\omega_1, \ldots, \omega_4$ be a local orthonormal oriented coframe field on M. Define the connection forms ω_{ij} and curvature forms Ω_{ij} by

$$d\omega_i=\sum_{j=1}^{4}\omega_j\wedge\omega_{ji},$$

$$\Omega_{ij}=d\omega_{ij}-\sum_{k=1}^{4}\omega_{ik}\wedge\omega_{kj}.$$

The generalized Gauss-Bonnet integrand Θ is given by

$$\Theta=\frac{1}{32\pi^2}(\Omega_{12}\wedge\Omega_{34}-\Omega_{13}\wedge\Omega_{24}+\Omega_{23}\wedge\Omega_{14}).$$

(For this formula and all the following related results see [4]). The form Θ, which appears to depend on the choice of the local oriented orthonormal coframe $\omega_1, \ldots, \omega_4$, can be shown by computation to be independent of this choice and thus to be a well-defined C^∞ form on M. Considered as a form on the bundle of oriented orthonormal frames, Θ is exact (whereas Θ is in general not exact considered as a form on M): namely,

$$\Theta=d\Pi,$$

where

$$\prod = \frac{1}{\pi^2}\{\tfrac{1}{2}\Phi_0 - \tfrac{1}{3}\Phi_1\},$$

and

$$\Phi_0 = \omega_{14} \wedge \omega_{24} \wedge \omega_{34},$$
$$\Phi_1 = \Omega_{12} \wedge \omega_{34} - \Omega_{13} \wedge \omega_{24} + \Omega_{23} \wedge \omega_{14}.$$

The form Π may be considered to be a form on the (sphere) bundle of unit vectors in the tangent bundle of M; precisely, if $x \in M$ and $e_1, ..., e_4$ and $e_1', ..., e_4'$ are two oriented orthonormal frames at x then

$$\Pi(e_1, ..., e_4) = \Pi(e_1', ..., e_4')$$

if $e_4 = e_4'$. This fact is again demonstrated by a computation. On the bundle of unit tangent vectors it of course remains true that $\Theta = d\Pi$.

Let $x_1, ..., x_k$ be the isolated points of U at which Y is not defined. These points are necessarily finite in number since Y is defined in a neighborhood of the boundary of U and U has compact closure. The limit

$$\lim_{\varepsilon \to 0^+} \int_{\partial B(x_l;\varepsilon)} Y^*\prod$$

exists and is finite (where $Y^*\Pi$ = "pull-back" of Π under the map Y of $U - \{x_1, ..., x_k\}$ into the bundle of unit tangent vectors and $\partial B(x_l; \varepsilon)$ = the boundary of the Riemannian ball of radius ε about x_l). Denote this limit by $\int_{x_l} Y^*\Pi$. Then Stokes' theorem yields

$$\int_U \Theta = \int_{U-\{x_1,...,x_k\}} \Theta = \int_{\partial U} Y^*\prod - \sum_l \int_{x_l} Y^*\prod.$$

The minus sign in the final expression arises as usual from the fact that the exterior normal of $U - B(x_l; \varepsilon)$ is the interior normal of $B(x_l; \varepsilon)$. As in [4], the integral $\int_{x_l} - Y^*\Pi =$ the index of Y at x_l. Thus

$$\int_U \Theta = \int_{\partial U} Y^*\prod + \chi(U).$$

To establish the lemma, it thus remains to be shown only that $\int_{\partial U} Y^*\Pi \leq 0$.

Let x be a point of the boundary ∂U of U. Choose unit tangent vectors e_1, e_2, e_3 at x in such a way that $e_1, e_2, e_3, Y(x)$ is an oriented orthonormal frame at x and that the second fundamental S_Y form of ∂U (relative to the normal Y to ∂U) is diagonal, i.e.

$$S_Y(e_i, e_j) = \lambda_i \delta_{ij} \quad \text{for some } \lambda_i \in \mathbf{R}.$$

The second covariant differential $D^2\varphi(e_i, e_i)$ equals by definition $\bar{e}_i(\bar{e}_i\varphi)-(D_{\bar{e}_i}\bar{e}_i)\varphi$ where $\bar{e}_i$ is any extension of e_i to a vector field near x; it is nonnegative because φ is convex near x. On the other hand, if the vector field $\bar{e}_i$ is taken to be tangent near x to ∂U then $\bar{e}_i(\bar{e}_i\varphi)=0$ at x so that

$$0 \leqslant D^2\varphi(e_i, e_i) = -(D_{\bar{e}_i}\bar{e}_i)\varphi.$$

But

$$-(D_{\bar{e}_i}\bar{e}_i)\varphi = -\langle D_{\bar{e}_i}\bar{e}_i, \operatorname{grad}\varphi\rangle = -\|\operatorname{grad}\varphi\|\langle D_{\bar{e}_i}\bar{e}_i, Y\rangle = -\|\operatorname{grad}\ \|S_Y(e_i, e_i).$$

Hence

$$0 \leqslant -\|\operatorname{grad}\varphi\|S_Y(e_i, e_i),$$

and so

$$\lambda_i \leqslant 0 \quad \text{for each } i=1, 2, 3.$$

Also

$$\lambda_i\delta_{ij} = S_Y(e_i, e_j) = \langle D_{\bar{e}_j}\bar{e}_i, Y\rangle = \omega_{i4}(e_j).$$

Now Φ_0 and Φ_1, being 3-forms, are multiplies of the volume form ∂U when restricted to ∂U. Specifically, if $\psi=$ the volume for m of ∂U then

$$\Phi_0 = \Phi_0(e_1, e_2, e_3)\psi \quad \text{and} \quad \Phi_1 = \Phi_1(e_1, e_2, e_3)\psi$$

because

$$\psi(e_1, e_2, e_3) = 1,$$

and

$$\Phi_0(e_1, e_2, e_3) = (\omega_{14}\wedge\omega_{24}\wedge\omega_{34})(e_1, e_2, e_3) = \lambda_1\lambda_2\lambda_3$$

since $\omega_{i4}(e_j)=\lambda_i\delta_{ij}$. Since each $\lambda_i \leqslant 0$, $\Phi_0(e_1, e_2, e_3)\leqslant 0$. Next,

$$\begin{aligned}\Phi_1(e_1, e_2, e_3) &= (+\Omega_{12}\wedge\omega_{34}-\Omega_{13}\wedge\omega_{24}+\Omega_{23}\wedge\omega_{14})(e_1, e_2, e_3)\\ &= \lambda_3\Omega_{12}(e_1, e_2)=\lambda_2\Omega_{13}(e_1, e_3)+\lambda_1\Omega_{23}(e_2, e_3).\end{aligned}$$

Since $\Omega_{ij}(e_i, e_j)=$ the negative of the sectional curvature of the 2-plane spanned by e_i and $e_j (i\neq j)$, $\Omega_{ij}(e_i, e_j)\leqslant 0$ at x by hypothesis. Again since each $\lambda_i\leqslant 0$, $\Phi_1(e_1, e_2, e_3)\geqslant 0$. Finally

$$(Y^*\prod)(e_1, e_2, e_3) = \frac{1}{\pi^2}\{\tfrac{1}{2}\Phi_0-\tfrac{1}{3}\Phi_1\}(e_1, e_2, e_3)\leqslant 0.$$

Hence

$$\int_{\partial U} Y^*\prod \leqslant 0. \qquad \square$$

Proof of Theorem 9. Let $\varphi: M \to \mathbf{R}$ be a C^{∞} exhaustion function on M which is strictly convex outside some compact subset of M: the existence of such a function is guaranteed by Theorem 1(b). According to (the proof of) Theorem 4(a) there is a $\lambda_0 \in \mathbf{R}$ such that

(a) φ is strictly convex on $\varphi^{-1}((\lambda_0, +\infty))$

(b) φ has no critical points in $\varphi^{-1}((\lambda_0, +\infty))$ and (hence) for any $\lambda \in (\lambda_0, +\infty)$, $\varphi^{-1}((-\infty, \lambda])$ has the homotopy type of M.

For each $\lambda \in (\lambda_0, +\infty)$, the interior $\overset{0}{M}_{\lambda}$ of $M_{\lambda} = \varphi^{-1}((-\infty, \lambda])$ satisfies the hypotheses (for U) of Lemma 5 with φ of the lemma = the present $\varphi - \lambda$. But also $\chi(M_{\lambda}) = \chi(M)$ since M_{λ} has the homotopy type of M. Thus for any $\lambda \in (\lambda_0, +\infty)$

$$\int_{M\lambda} \Theta \leqslant \chi(M_{\lambda}) = \chi(M).$$

Since $\bigcup_{\lambda} \overset{0}{M}_{\lambda} = M$, the conclusions of the theorem now follow from the remarks made immediately after the statement of theorem. □

To prove the analogue of Theorem 9 for two-dimensional manifolds, one need only establish the analogue for the case of two dimensions of Lemma 5 by reasoning similar to but simpler than that needed in the case of four dimensions. The proof of the theorem as given then applies to the case of two dimensions.

§ 5. Function-theoretic properties of noncompact Kähler manifolds of positive curvature

The purpose of this section is to state and prove some results on the function theory of Kähler manifolds which have an exhaustion function which is strictly convex (or strictly convex outside some compact set); these results then apply to Kähler manifolds whose curvature is everywhere positive (or positive outside some compact set). The basic source of the relationship between strictly convex functions and function theory is the following lemma:

LEMMA 6. *Let f: $M \to \mathbf{R}$ be a C^2 function on a Kähler manifold M; if f is convex then f is plurisubharmonic and if f is strictly convex then f is strictly plurisubharmonic.*

This lemma in fact holds without the assumption that f is C^2, as shown in [10].

Proof of Lemma 6. Let L_f denote the Levi form of f defined by

$$L_f = 4 \sum_{i,j} \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} dz_i \otimes d\bar{z}_j$$

where $(z_1, ..., z_n)$ is a local holomorphic coordinate system (it is easily verified by a computation that L_f thus defined locally is independent of coordinate choice so that L_f is in fact globally defined). The form L_f is nonnegative definite (on the holomorphic tangent spaces of U) if and only if f is plurisubharmonic and positive definite if and only if f is strictly plurisubharmonic.

Let p be a point of M and V be a unit holomorphic tangent vector at p. There exists a holomorphic normal coordinate system $(z_1, ..., z_n)$ centered at p such that $\partial/\partial z_1 = V$. Let the corresponding real coordinate system be $(x_1, y_1, ..., x_n, y_n)$ where $z_i = x_i + \sqrt{-1}y_i$. Then $D_{\partial/\partial x}\partial/\partial x|_p = 0$ and $D_{\partial/\partial x}\partial/\partial y|_p = 0$ as well as $D_{\partial/\partial x}(\partial/\partial y)|_p = 0|$. In such a coordinate system

$$\begin{aligned} L_f(V, V) &= L_f\left(\frac{\partial}{\partial z_1}\bigg|_p, \frac{\partial}{\partial z_1}\bigg|_p\right) = 4\frac{\partial^2 f}{\partial z_1 \partial \bar{z}_1}\bigg|_p = \frac{\partial^2 f}{\partial x_1^2}\bigg|_p + \frac{\partial^2 f}{\partial y_1^2}\bigg|_p \\ &= \frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_1}\right)\bigg|_p - \left(D_{\partial/(\partial x_1)}\frac{\partial}{\partial x_1}\right)\bigg|_p f + \frac{\partial}{\partial y_1}\left(\frac{\partial f}{\partial y_1}\right)\bigg|_p - \left(D_{\partial/(\partial y_1)}\frac{\partial}{\partial y_1}\right)\bigg|_p f \\ &= D_f^2\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}\right)\bigg|_p + D_f^2\left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_1}\right)\bigg|_p, \end{aligned}$$

where D_f^2 is the second covariant derivative of f. D_f^2 is nonnegative definite when f is convex and positive definite when f is strictly convex. Thus L_f is nonnegative definite when f is convex and positive definite when f is strictly convex. □

THEOREM 10(a). *If M is a Kähler manifold which has a strictly convex exhaustion function then M is a Stein manifold.* (b) *If M is a noncompact complete Kähler manifold with everywhere positive sectional curvature, then M is a Stein manifold.*

Proof. In view of Theorem 1(b), part (a) implies part (b). In part (a), it is sufficient by Theorem 2(a) to consider the case in which the strictly convex exhaustion function is C^∞. Lemma 6 then implies that the exhaustion function is strictly plurisubharmonic. Since any complex manifold with a strictly plurisubharmonic exhaustion function is necessarily Stein manifold (see [8], [17]) part (a) follows. □

If a Kähler manifold has an exhaustion function which is strictly convex outside some compact set, but not everywhere on the manifold, then the manifold need not be a Stein manifold. However, such a manifold is "a Stein manifold outside some compact set" in a sense made precise by the following theorem.

THEOREM 11(a). *If M is a noncompact Kähler manifold which has an exhaustion function which is strictly convex outside some compact set, then M can be obtained from a Stein space by blowing up a finite number of points to compact subvarieties.* (b). *If M is a complete* *noncompact Kähler manifold which has positive curvature outside some compact set then M can be obtained from a Stein space by blowing up a finite number of points to compact subvarieties.*

In both part (a) and part (b), it need not be assumed that the metric on M is a Kähler metric everywhere on M. It is sufficient in part (a) to assume that M has a Kähler metric defined outside some compact subset of M such that the exhaustion function is strictly convex outside some compact set relative to this metric. Similarly in part (b) it is sufficient to assume that M has a complete Riemannian metric which has positive curvature outside some compact set and which is a Kähler metric outside some compact set. The proof to be given now of the theorem as stated also applies to these more general hypotheses.

Proof of Theorem 11. As in previous cases part (a) implies part (b) by virtue of a previous theorem (Theorem 1b) in this case). In part (a), Theorem 2(a) implies that there is a C^∞ exhaustion function which is strictly convex outside some compact set. By Lemma 6, this exhaustion function is strictly plurisubharmonic outside some compact set. Part (a) now follows from the following theorem of Narasimhan.

THEOREM. *If a noncompact complex manifold M has a C^∞ exhaustion function which is strictly plurisubharmonic outside a compact set then M can be obtained from a Stein space by blowing up a finite number of points to compact subvarieties.*

In this theorem, the hypothesis that the exhaustion function be C^∞ can be weakened: (see [22] and [25]), but the version given is sufficient for the purpose at hand. For the convenience of the reader, a brief sketch of the proof of Narasimhan's theorem will be given. The proof is given in detail in [22] and some related results, which would also suffice to complete the proof of Theorem 11 are given in [26].

Let $\varphi: M \to \mathbf{R}$ be an exhaustion function which is strictly plurisubharmonic outside some compact set K. Set $\lambda = 1 + \sup_K \varphi$. The set $M_\lambda = \{p \in M \mid \varphi(p) \leqslant \lambda\}$ is compact, since φ is an exhaustion function; and $K \subset M_\lambda$ so that φ is strictly plurisubharmonic on $M - M_\lambda$. If V is a compact connected subvariety of positive dimension of M then $V \cap (M - M_\lambda) = \varnothing$. For if $V \cap (M - M_\lambda) \neq \varnothing$ then the maximum of φ on V is attained at a point of $V \cap (M - M_\lambda)$ at which point φ is strictly plurisubharmonic. Such an occurrence would violate the maximum principle for strictly plurisubharmonic functions (see [16], for example).

Since all the compact connected positive-dimensional subvarieties of M are contained in the compact set M_λ, it follows that the set of maximal connected positive-dimensional subvarieties is finite. Let $\tilde{M}$ = the analytic space obtained by "blowing down" these subvarieties to points and let $\tilde{M}_\eta$, $\eta > \lambda$, be the space obtained by the same process from $M_\eta (= \{p \in M \mid \varphi(p) < \eta\})$.

Each of the spaces M_η, $\eta > \lambda$, is holomorphically convex by a theorem of H. Cartan [2] on holomorphic equivalence relations. Since each such M_η clearly has no compact positive-dimensional subvarieties, each is a Stein space. Thus $\tilde{M} = \bigcup_{\eta > \eta} \tilde{M}_\eta$ is an increasing union of a one-parameter family of Stein spaces and is hence a Stein space ([6]). □

It follows from (the proof of) Narasimhan's theorem that a noncompact complex manifold which has an exhaustion function which is strictly plurisubharmonic outside some compact set and which has no compact subvarieties of positive dimension is a Stein manifold (in that case, the Stein space $\tilde{M}$ = the manifold M, which is thus necessarily a Stein manifold). In particular, if such a manifold is a Kähler manifold which is diffeomorphic to Euclidean space, then it has no compact positive-dimensional subvarieties and so is a Stein manifold: for any such subvariety would be homologous to zero and a compact subvariety of a Kähler manifold cannot be homologous to zero. These observations lead to the following theorem.

THEOREM 12. *If M is a complete noncompact Kähler manifold whose curvature is everywhere nonnegative and whose curvature is positive outside some compact set then M is a Stein manifold.*

Proof. As shown in the proof of Theorem 11, M has a C^∞ exhaustion function which is strictly plurisubharmonic outside some compact set. Furthermore, it has been shown by Cheeger and Gromoll ([3, § 3]) that a Riemannian manifold of everywhere nonnegative sectional curvature and positive sectional curvature outside a compact set is diffeomorphic to euclidean space. That M is a Stein manifold now follows from the remarks preceding the statement of the theorem. □

§ 6. Some generalizations of the results of the previous sections

Many of the preceding results were stated and proved in less than maximum generality so that the central concepts of the proofs would not be obscured by excessive technical detail. In this section, some generalizations will be discussed which can be proved by essentially the same methods; the modifications of the previous arguments necessary to prove these generalizations will be briefly outlined. Each generalization and modified argument will be listed according to the section in which the original theorem appeared.

§ 1. The approximation result for strictly convex functions, Theorem 2, can be generalized to the larger class of functions described by the following definition. For this definition, let M be a Riemannian manifold and for each $p \in M$ let d_p: $M \to \mathbf{R}$ be the function whose value at $q \in M$ is the square of the Riemannian distance from q to p.

Definition: Let $\delta: M \to \mathbf{R}$ be a continuous function. A function $\varphi: M \to \mathbf{R}$ is *δ-convex* if for every $p \in M$ the function $\varphi - \frac{1}{2}\delta(p)d_p$ is strictly convex in a neighborhood of p.

If $\delta \equiv 0$ then δ-convexity is equivalent to strict convexity. The function d_p is C^∞ and strictly convex in a neighborhood of p: its second derivative at p along any arc-length parameter geodesic through p is 2. Thus the δ-convexity condition is, for C^2 functions φ, the condition that the second derivative at p along such geodesics be greater than $\delta(p)$. The argument used to prove Lemma 3 can be used to show that a δ-convex function can be approximated in a neighborhoods of compact sets by C^∞ δ-convex functions (with Lipschitz constants being approximated). Then the proof of Theorem 2 can be easily modified to prove that Theorem 2 still holds if "strict convexity" is replaced by "δ-convexity" (for any fixed continuous function $\delta: M \to \mathbf{R}$) throughout. The original statement of Theorem 2 corresponds then of course to $\delta \equiv 0$.

A convex function is δ-convex for any everywhere negative δ. Thus the following generalization (Theorem 1′) of Theorem 1 follows from the generalized version of Theorem 2 together with the facts: on any complete (noncompact) Riemannian manifold of nonnegative curvature there is a convex exhaustion function ([3]) and on any complete (noncompact) Riemannian manifold whose curvature is nonnegative outside a compact set there is an exhaustion function which is convex outside some compact set ([12]).

THEOREM 1′. (a) *If M is a complete noncompact Riemannian manifold of nonnegative curvature and if $\delta: M \to \mathbf{R}$ is any everywhere negqtive continuous function, then there exists a Lipschitz continuous C^∞ exhaustion function $\varphi: M \to \mathbf{R}$ which is δ-convex.*

(b) *If M is a complete noncompact Riemannian manifold whose sectional curvature is nonnegative outside a compact subset K_1 of M and if $\delta: M \to \mathbf{R}$ is any everywhere negative continuous function, then there exists a Lipschitz continuous C^∞ exhaustion function $\varphi: M \to \mathbf{R}$ and a compact subset K_2 of M such that φ is δ-convex on $M - K_2$.*

The method used in § 1 of obtaining C^∞ approximations on all of the manifold from C^∞ approximations in neighborhoods of compact sets applies not only to strictly convex (or more generally δ-convex) functions but also to other classes of functions: in particular, it applies to strictly subharmonic and (on a complex manifold) plurisubharmonic functions. Some of the theorem resulting from this method are given in [13], and a discussion of the method in a general setting is given in [14].

§ 2. The usual product metric on the manifold $S^1 \times \mathbf{R}$ is a complete metric of nonnegative (in fact, identically zero) curvature; also, the projection map $S^1 \times \mathbf{R} \to \mathbf{R}$ is a C^∞ function which is convex relative to this metric, and the square of this function is a C^∞ convex exhaustion function. Thus, in Theorem 1(b) the positive curvature hypothesis cannot be replaced by the hypothesis of nonnegative curvature, and in Theorem 1(a) the strict convexity hypothesis cannot be replaced by the hypothesis of (not necessarily strict) convexity. However, a result giving a strong topological restriction on complete noncompact manifolds of nonnegative curvature does hold (see [3] and [23]).

THEOREM (Cheeger-Gromoll). *If M is a complete noncompact Riemannian manifold of nonnegative curvature, then there exists a compact totally geodesic submanifold S of M with the property that M is diffeomorphic to the total space of the normal bundle of S in M.*

The method of proof of this theorem given in [23], which uses the convex exhaustion function constructed in [3], can be modified to apply to the exhaustion function constructed in [13] on a complete noncompact manifold whose curvature is nonnegative outside a compact set: this exhaustion function is convex outside some compact set. This modified argument yields the extension up to homotopy type of Theorem 4(b) to the case of curvature nonnegative (rather than positive, as originally assumed) outside a compact set.

§ 3. In Theorem 7, the restriction of the statements to the integrals of C^∞ nonnegative subharmonic functions f can be weakened: the theorem remains true for continuous nonnegative subharmonic functions. The proof follows from application of an appropriate method of approximating continuous subharmonic functions by C^∞ ones; this method is discussed in [13]. It is also shown in [13] that it is enough in Theorem 7(b) to assume that M has nonnegative, not necessarily positive, curvature outside some compact set. A related result of S. T. Yau [29] is that the volume of a complete noncompact manifold of nonnegative Ricci curvature is infinite.

The role played in Theorem 7 by convexity of the exhaustion function can in fact be played nearly as well by a subharmonic exhaustion function as the following theorem ([12; p. 288]) shows:

THEOREM (Greene and Wu). *Let M be a noncompact oriented C^∞ Riemannian manifold on which there exists a continuous exhaustion function $\varphi: M \to \mathbf{R}$ and a compact set $K_\varphi \subset M$ such that*

(a) $\varphi \mid (M - K_\varphi)$ *is* C^2,

(b) $\varphi \mid (M - K_\varphi)$ *is (uniformly) Lipschitz continuous,*

(c) $\varphi \mid (M - K_\varphi)$ *is subharmonic.*

Then, if f is a continuous nonnegative subharmonic function such that $\{p \in M \mid f(p) > 0,\ \varphi(p) > \max K_\varphi,\ \operatorname{grad} \varphi(p) \neq 0\} \neq \emptyset$, *there exist constants $A_f > 0$ and t_0 such that*

$$\int_{M_t^\varphi} f \geqslant A_f (t - t_0),$$

and in particular $\int_M f = +\infty$.

Finally, S.-T. Yau has demonstrated a number of results on integrals of nonconstant subharmonic functions on arbitrary complete Riemannian manifolds without any curvature assumptions [29].

§ 4. In Theorem 9, the hypothesis that M have positive sectional curvature outside some compact set can be replaced by the hypothesis that M have nonnegative sectional curvature outside some compact set. The method of proof of this stronger result differs from the proof given for Theorem 9 in technical detail only. Specifically, the exhaustion function $\varphi: M \to \mathbf{R}$ constructed in [12] on any complete noncompact manifold M whose curvature is nonnegative outside some compact set has the property that for all sufficiently large λ the sublevel set $\varphi^{-1}((-\infty, \lambda))$ has the homotopy type of M. This fact can be shown using the method of [3] or [23] developed to prove the same fact in the case that M has everywhere nonnegative curvature. Thus to complete the proof of the generalization of Theorem 9 it need only be shown that

$$\int_{\varphi^{-1}((-\infty, \lambda))} \Theta \leqslant \chi(\varphi^{-1}(-\infty, \lambda))$$

for all sufficiently large λ. The convex set $\varphi^{-1}((-\infty, \lambda))$ need not have C^{∞} boundary so that Lemma 4 cannot be applied directly. However, by approximating φ by the convolution smoothings φ_{ε} discussed previously, approximations of $\varphi^{-1}((-\infty, \lambda))$ are obtained in the sense that the measure of the symmetric difference of $\varphi^{-1}((-\infty, \lambda))$ and $\varphi_{\varepsilon}^{-1}((-\infty, \lambda))$ goes to 0 as $\varepsilon \to 0^{+}$. Moreover, the domains $\varphi_{\varepsilon}^{-1}((-\infty, \lambda))$ approach being convex in an appropriate sense. Then the required inequality on $\int_{\varphi^{-1}((-\infty, \lambda))} \Theta$ follows from Lemma 4 by a limit argument. A detailed discussion of the reasoning to be used is given in [23], where the case of everywhere nonnegative curvature is discussed.

The restriction of Theorem 9 to manifolds of dimension (two and) four is a consequence of the fact that the algebraic Hopf conjecture fails in general for manifolds of dimension six or greater ([7]; see also [18]). But in dimension six a partial result is available: the boundary terms have in this case the sign required to make the inequality of Lemma 3 hold (cf. [23]). But, since the integrand Θ need not be nonnegative, only the following result is implied: If M is a complete noncompact manifold of dimension six whose sectional curvature is nonnegative outside some compact set and if $\varphi: M \to \mathbf{R}$ is the exhaustion function on M constructed in [12] (which is convex outside some compact set), then

$$\limsup_{\lambda \to +\infty} \int_{\varphi^{-1}((-\infty, \lambda))} \Theta \leqslant \chi(M).$$

§ 5. Theorem 10(b) was originally demonstrated in [11] by much more complicated methods than those employed here. Theorem 11 was also discussed in [11]. Some additional results closely related to Theorems 10, 11 and 12 are given in [9], [13], and [25].

Added in proof. (*November* 3, 1976); R. Walter has pointed out to us that Lemma 5 of the present paper also follows from Theorem 4.2.2. of his paper [28].

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