BAER-INVARIANTS, ISOLOGISM, VARIETAL LAWS AND HOMOLOGY

BY

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Introduction

This paper is mainly concerned with the classification of groups and the independence of laws in varieties of groups. However, the basic ideas go over to other varieties in the sense of universal algebra, especially varieties of associative and Lie algebras. We also deal with various problems outlined below whose proper context is the theory of triple homology. In view of the somewhat unusual mixture of disciplines, we have been at pains to make as few demands on the reader as possible; in particular, for most of the paper, we do not assume any knowledge of homology.

Central to the whole paper are the groups $\mathfrak{V}M(G)$ and $\mathfrak{V}P(G)$, defined in § I.1 for any variety of groups $\mathfrak{V}$ and any group G. These are the Baer-invariants; the first modern treatment is due to Fröhlich [10], who considered associative algebras, and named the invariants after Baer's group-theoretical papers [2]. Further work on the Baer-invariants of associative algebras appears in Lue [26] and [27]. For a recent discussion that reverts to group theory and is more in the spirit of Baer's paper, see J. L. MacDonald [28]. The group $\mathfrak{V}M(G)$ is always abelian, and is the Schur multiplier of G if $\mathfrak{V}$ is the variety of abelian groups; $\mathfrak{V}P(G)$ is a central extension of $\mathfrak{V}M(G)$ by the verbal subgroup of G, and so coincides with $\mathfrak{V}M(G)$ if $G \in \mathfrak{V}$. In § I.2 we consider the classification of groups into $\mathfrak{V}$-isologism classes after P. Hall. The larger the variety $\mathfrak{V}$ the cruder the classification, all groups in $\mathfrak{V}$ falling into the same class. The problem of constructing the groups in an isologism class is postponed to § II.3, and the reader who wishes to get to this quickly should skip §§ I.3–4. In § I.3 we show how a slightly stronger property than independence of the laws of a variety $\mathfrak{V}$ can be dealt with in terms of $\mathfrak{V}P(G)$ for suitable G, and we calculate $\mathfrak{V}P(G)$ in certain cases. In § I.4 the non-finitely based variety of Vaughan-Lee's in [38] is used to construct non-finitely generated groups $\mathfrak{V}P(G)$. The calculations are rather involved, and the results are not used except to construct a counter-example in § II.4.

If G is a group in the variety $\mathfrak{V}$, and A and B are left and right $\mathfrak{V}G$-modules respectively, where $\mathfrak{V}G$ is a certain quotient ring of $\mathbf{Z}G$, various well known theories, which here coincide, define homology and cohomology groups $\mathfrak{V}_n(G, A)$ and $\mathfrak{V}^n(G, B)$ for all $n \geqslant 0$. § II.1 gives a summary of the results on varietal homology that are needed in the sequel. In § II.2 an interpretation of $\mathfrak{V}^2(G, B)$ is given after Gerstenhaber [11], and in § II.3 this is used to construct an exact sequence, c.f. Lue [27]. If $\mathfrak{V}$ is the variety of abelian groups, the minimal (or stem) groups in a $\mathfrak{V}$-isologism (or isoclinism) class may be constructed by using Schur's theory of covering groups. § II.3 ends by using the above exact sequence to give a quantative account of the failure of the theory of covering groups for an arbitrary variety, and gives a theoretical procedure for constructing the groups in a $\mathfrak{V}$-isologism class. The recursive procedure given in Evens [40] for constructing all finite p-groups is obtained by a suitable choice of $\mathfrak{V}$. The reader who is not interested in homology may take the interpretation of $\mathfrak{V}^1(G, B)$ in § II.1 (ix), and of $\mathfrak{V}^2(G, B)$ in § II.2, as definitions, and read § II.3 without further reference to §§ II.1–2.

In § II.4, various results of purely homological interest are given. A simple formula for $\mathfrak{V}_2(G, A)$ and $\mathfrak{V}^2(G, B)$ in terms of $\mathfrak{V}M(G)$ is given which is valid if $\mathfrak{V}$ is 'big enough'. Thus a variety constructed in § I.4 gives rise to a finite abelian group whose second integral homology and cohomology groups are not finitely generated. On a more theoretical level we call a variety $\mathfrak{V}$ *balanced* if $\mathfrak{V}_n(G, A) = 0$ whenever A is a projective $\mathfrak{V}G$-module and $n > 0$. This is equivalent to the condition that the groups $\mathfrak{V}_n(G, A)$ and $\mathfrak{V}^n(G, B)$, which are defined by fixing the module and varying the group, agree with the appropriate Tor and Ext defined by fixing the group and varying the module. The variety of all groups is known to be balanced, and here the Tor and Ext give rise, after a dimension shift, to the classical (co-)homology of groups. General theory also shows that abelian varieties, and varieties whose finitely generated groups are splitting groups (i.e. projectives relative to the surjections in the variety), are balanced. We conjecture that this exhausts the class of balanced varieties of groups, and produce evidence to support the conjecture. Finally, a characterization of Schur-Baer varieties, as defined by P. Hall, (see the end of § I.1), is given in terms of the exponents of (co-)homology groups.

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Preliminaries

Our notation is based on that of Hanna Neumann [30]. A *variety* is a class of groups closed under the formation of subgroups, quotient groups and Cartesian products. X_∞ denotes the group freely generated by $\{x_1, x_2, \ldots\}$. If $\mathfrak{V}$ is a variety, $V(X_\infty)$ denotes the intersection of the kernels of all homomorphisms of X_∞ into all groups in $\mathfrak{V}$. It turns out that $G \in \mathfrak{V}$ if and only if $V(X_\infty)$ is in the kernel of every homomorphism of X_∞ into G. Also, a subgroup of X_∞ is of the form $V(X_\infty)$ for some variety $\mathfrak{V}$ if and only if it is a fully invariant subgroup. Thus $\mathfrak{V} \leftrightarrow V(X_\infty)$ is an order reversing bijection between the set of varieties and the set of fully invariant subgroups of X_∞. If G is any group, $V(G)$ denotes the union of all images of $V(X_\infty)$ under all homomorphisms of X_∞ into G. It is clear that $V(G)$, the *$\mathfrak{V}$-verbal subgroup of G*, is a fully invariant subgroup, and that $G \in \mathfrak{V}$ if and only if $V(G)=1$. If $n \geqslant 1$, the subgroup of X_∞ generated by $\{x_1, \ldots, x_n\}$ is denoted by X_n; any element of X_∞ is a *word*, and an element of X_n is an *n-letter word*. Any element of $V(X_\infty)$ is a *law* of $\mathfrak{V}$, and a law which is also an n-letter word is an *n-letter law* of $\mathfrak{V}$.

If v is an n-letter word, and $G^{(n)}$ denotes the Cartesian product of n copies of G, v defines, in a natural way, a map, also denoted by v, from $G^{(n)}$ to $V(G)$. The largest normal subgroup N of G such that v factors through the natural map $G^{(n)} \to (G/N)^{(n)}$ is denoted by $v^*(G)$. Other descriptions of $v^*(G)$ appear at the beginning of § I.1. The intersection of the subgroups $v^*(G)$ for all laws v of $\mathfrak{V}$ is the *$\mathfrak{V}$-marginal subgroup* $V^*(G)$ of G, and $G/V^*(G)$ is the *marginal factor of G*.

If $v \in X_\infty$ and $\mathfrak{v} \subseteq X_\infty$, v is a *consequence* of $\mathfrak{v}$, or $\mathfrak{v}$ *implies* v, if v is a law of every variety for which $\mathfrak{v}$ is a set of laws. This is equivalent to the condition that v be in the least fully invariant subgroup of X_∞ containing $\mathfrak{v}$. If this is not the case there is a group G such that $\mathfrak{v}$ is a set of laws of G (that is, if $w \in \mathfrak{v}$ then w is in the kernel of every homomorphism of X_∞ into G), but v is not a law of G. Then G is said to *differentiate v from* $\mathfrak{v}$. If $\mathfrak{v} \subseteq V(X_\infty)$, and every law of $\mathfrak{V}$ is a consequence of $\mathfrak{v}$, $\mathfrak{v}$ is a *defining set of laws of* $\mathfrak{V}$, or $\mathfrak{v}$ *defines* $\mathfrak{V}$. A set of groups *generates* the least variety containing it (such a variety always exists); the variety generated by the group G is denoted by $\operatorname{var}(G)$.

A *$\mathfrak{V}$-splitting group* is a group $G \in \mathfrak{V}$ that is projective relative to the class of surjections in $\mathfrak{V}$. Equivalently, every extension $1 \to K \to E \to G \to 1$ with E in $\mathfrak{V}$ splits. For example, a $\mathfrak{V}$-free group of rank α (that is a group isomorphic to $F/V(F)$ where F is free of rank α) is a $\mathfrak{V}$-splitting group for any cardinal α.

The variety whose groups are all trivial will be denoted by $\mathfrak{E}$, the variety of all groups by $\mathfrak{O}$, the variety of all abelian groups by $\mathfrak{A}$, the variety of all abelian groups of exponent n by $\mathfrak{A}_n$, the variety of all nilpotent groups of class at most c by $\mathfrak{N}_c$, and the variety of all soluble groups of length at most l by $\mathfrak{S}_l$. The set of all varieties forms a complete lattice under $\vee$ and $\wedge$, *join* and (set theoretic) *intersection* respectively. Also, if $\mathfrak{U}$ and $\mathfrak{V}$ are varieties one may form the *product* $\mathfrak{U}\,\mathfrak{V}$, namely the class of all groups G such that $V(G) \in \mathfrak{U}$. Multiplication of varieties is associative, and so the usual convention for exponentiation may be used, and $\mathfrak{S}_l = \mathfrak{A}^l$. For $g_1, g_2 \in G$, $[g_1, g_2]$ denotes $g_1^{-1} g_2^{-1} g_1 g_2$; and for subgroups

H, K of G, $[H, K]$ denotes the subgroup generated by $\{[h, k]: h \in H, k \in K\}$. Commutators will be written with the left-normed convention, that is $[x_1, x_2, \ldots, x_n]$ denotes $[[\ldots[[[x_1, x_2], x_3], x_4], \ldots], x_n]$. If $\mathfrak{U}$ and $\mathfrak{V}$ are varieties, $[\mathfrak{U}, \mathfrak{V}]$ denotes the variety of all groups G such that $[U(G), V(G)] = 1$. Thus, $[\mathfrak{V}, \mathfrak{E}]$ is the variety 'centre-by-$\mathfrak{V}$' of all groups G in which $V(G)$ is central, and, in particular, $\mathfrak{N}_c$ can be defined inductively by $\mathfrak{N}_0 = \mathfrak{E}$ and $\mathfrak{N}_c = [\mathfrak{N}_{c-1}, \mathfrak{E}]$. If G is any group $\gamma_c(G)$ denotes the c-th term of the lower central series of G and $\zeta_c(G)$ the c-th term of the upper central series of G, so that $\gamma_1(G) = G$, $\gamma_2(G) = G'$ and $\zeta_1(G) = \zeta(G)$, the centre of G. Thus if $\mathfrak{V} = \mathfrak{N}_c$, $V(G) = \gamma_{c+1}(G)$ and, as is easy to prove, $V^*(G) = \zeta_c(G)$.

If G acts on a group K as a group of automorphisms in a given way K is a *G-group* (or a *G-module* if K is abelian) and $G[K$ denotes the split extension of K by G, given by $G[K = \{g[k: g \in G, k \in K\}$ with multiplication $(g_1[k_1)(g_2[k_2) = g_1 g_2[k_1^{g_2} k_2$. Usually G and K will be regarded as subgroups of $G[K$ in the obvious way, so that $g[k$ is written as gk; however, in the case when $K \lhd G$, and so K is a G-group under the action of G by conjugation (that is, $k^g = g^{-1} kg$), every element of K is also an element of G, and the more precise notation will be needed.

The cyclic group of order n generated by a is denoted by $C_n(a)$ in multiplicative notation and by $\mathbf{Z}_n(a)$ in additive notation. If M is a monoid, $\mathbf{Z}M$ denotes the corresponding integral monoid ring, and $\mathbf{Z}_n M$ the monoid ring with coefficients in the ring of integers mod n. The augmentation ideal IM is the kernel of the ring homomorphism from $\mathbf{Z}M$ to $\mathbf{Z}$ (or $\mathbf{Z}_n M$ to $\mathbf{Z}_n$) that maps each element of M to 1.

If α_i: $E_i \to G$, $i = 1, 2$ are surjections, the *fibre product*, $E_1 \underset{G}{\times} E_2$ is the subgroup $\{(e_1, e_2): e_1\alpha_1 = e_2\alpha_2\}$ of $E_1 \times E_2$. Similarly if ι_i: $B \to \zeta(E_i)$, $i = 1, 2$, are injections, $E_1 \overset{B}{\times} E_2$ denotes the quotient group of $E_1 \times E_2$ obtained by amalgamating the images of B; that is, $(E_1 \times E_2)/N$ where N is the normal subgroup $\{(b\iota_1, -b\iota_2): b \in B\}$. (Abelian groups will often be written additively in a context in which multiplicative notation is more common to avoid violating homological conventions.) Let E_i be an extension of K_i by G giving rise to an exact sequence $1 \to K_i \to E_i \to G \to 1$, and let ι_i: $B \to \zeta(K_i)$ be an injection whose image is a normal subgroup of K_i, for $i = 1, 2$. The B becomes an E_i-module by conjugation, centralized by K_i, and hence a G-module. Suppose that the same G-module structure is induced on B in either case. Then $\{(b\iota_1, -b\iota_2): b \in B\}$ is a normal subgroup of $E_1 \underset{G}{\times} E_2$, and the resulting quotient group is denoted by $E_1 \overset{B}{\underset{G}{\times}} E_2$. Note that $E_1 \underset{G}{\times} E_2$ and $E_1 \overset{B}{\underset{G}{\times}} E_2$ have natural surjections onto G, and that B has a natural injection into $E_1 \overset{B}{\times} E_2$ and $E_1 \overset{B}{\underset{G}{\times}} E_2$. If $K_i = B$ and ι_i is the identity for $i = 1, 2$, this gives rise to an extension $1 \to B \to E_1 \overset{B}{\underset{G}{\times}} E_2 \to G \to 1$, the classical Baer sum of the given extensions.

If $\mathfrak{V}$ is a variety and $G_1, G_2 \in \mathfrak{V}$, the *verbal product* of G_1 and G_2 is the group G obtained by taking the free product $G_1 * G_2$ and dividing out $V(G_1 * G_2)$. There are natural embeddings of G_1 and G_2 in G; and G has the property that given homomorphisms $\alpha_i: G_i \to H$, $H \in \mathfrak{V}$, $i=1, 2$, there exists a unique homomorphism $\alpha: G \to H$ which agrees with α_i on G_i, $i=1, 2$.

CHAPTER I
Group theory

§1. Basic Concepts

We start with a discussion of the basic properties of marginal subgroups, as introduced by P. Hall in [15]. This leads to a new construction of a variety from given varieties $\mathfrak{U}$ and $\mathfrak{V}$, namely the variety of all groups G such that $U(G) \subseteq V^*(G)$. The Baer-invariants are then defined, as in Fröhlich [10], and their basic properties are discussed. We have regretfully abandoned Fröhlich's notation as it clashes occasionally with group-theoretic practice.

LEMMA 1.1. *If G is a group, v is an s-letter word, and $\mathfrak{V}$ is a variety, then*

(i) $v^*(G) = \{g \in G: v(a_1, \ldots, a_{i-1}, a_i g, a_{i+1}, \ldots, a_s) = v(a_1, \ldots a_s)$ *for all i and all* $a_1, \ldots, a_s \in G\}$

$= \{g \in G: v(a_1, \ldots, a_{i-1}, g a_i, a_{i+1}, \ldots, a_s) = v(a_1, \ldots, a_s)$ *for all i and all* $a_1, \ldots, a_s \in G\}$.

(ii) *If $\theta: G \to H$ is a surjection, then $v^*(G)\theta \subseteq v^*(H)$. It follows that $V^*(G)$ is a characteristic subgroup of G.*

(iii) *If v is a consequence of the set of words $\mathfrak{w}$, then*

$\bigcap_{w \in \mathfrak{w}} w^*(G) \subseteq v^*(G)$; *and, in particular, if $\mathfrak{w}$ defines $\mathfrak{V}$ then*

$\bigcap_{w \in \mathfrak{w}} w^*(G) = V^*(G)$.

(iv) *The following are equivalent: $G \in \mathfrak{V}$; $V(G)=1$; $V^*(G)=G$.*

Proof. Straightforward.

Examples. 1. If $\mathfrak{V} = \mathfrak{A}$, $V^*(G) = \zeta(G)$; more generally, if $\mathfrak{V} = \mathfrak{N}_c$, $V^*(G) = \zeta_c(G)$. (See P. Hall [15].)

2. If $\mathfrak{V} = \mathfrak{A}_m$, $V^*(G)$ is the set of elements in $\zeta(G)$ of order dividing m.

If $N \lhd G$ and $\mathfrak{V}$ is a variety, define $[NV^*G]$ to be the subgroup of G generated by

$$\{v(g_1, \ldots, g_{i-1}, g_i n, g_{i+1}, \ldots, g_s)(v(g_1, \ldots, g_s))^{-1}: 1 \leqslant i \leqslant s < \infty, v \in V(X_s), g_1, \ldots, g_s \in G, n \in N\}.$$

If $E \equiv 1 \to N \to G \to G/N \to 1$, $[NV^*G]$ corresponds to $V_1(E)$ in the notation of [10]. Our notation is motivated by example 1 below.

PROPOSITION 1.2. *If $N \lhd G$ and $\mathfrak{V}$ is a variety, then*

(i) *$[NV^*G]$ is the least normal subgroup T of G, contained in N, such that $N/T \subseteq V^*(G/T)$;*

(ii) *$[NV^*G]$ is the largest normal subgroup T of G such that for every homomorphism $\theta\colon G \to H$ with $N\theta \subseteq V^*(H)$, $T \subseteq \ker\theta$;*

(iii) *If N is fully invariant subgroup of G, then so is $[NV^*G]$.*

Proof. Easy.

PROPOSITION 1.3. *If $N \lhd G$ and $\mathfrak{V}$, $\mathfrak{W}$ are varieties such that $\mathfrak{V} \subseteq [\mathfrak{W}, \mathfrak{E}]$,* then

(i) *$N \cap V(G) \supseteq [NV^*G] \supseteq [N, W(G)]$;*

(ii) *$[V^*(G), W(G)] = 1$.*

Proof. (See P. Hall [15]).

(i) Clearly $N \cap V(G) \supseteq [NV^*G]$. If $w \in W(X_t)$ is a law of $\mathfrak{W}$, then $[w, x_{t+1}]$ is a law of $\mathfrak{V}$, and hence $[NV^*G]$ contains $[w(g_1, \ldots, g_t), n]$ for all $g_1, \ldots, g_t \in G$, $n \in N$.

(ii) This follows from (i), since if $N = V^*(G)$, $[NV^*G] = 1$.

Examples. 1. If $\mathfrak{V} = \mathfrak{A}$, $[NV^*G] = [N, G]$. More generally, if $\mathfrak{V} = \mathfrak{N}_c$, $[NV^*G] = [N, G, \ldots, G]$, with c repetitions of G.

2. If $\mathfrak{V} = \mathfrak{A}_m$, $[NV^*G]$ is generated by $[N, G] \cup \{n^m\colon n \in N\}$.

COROLLARY 1.4. *If $\mathfrak{V}$ is any variety other than $\mathfrak{O}$, and F is a free group of rank >1 then $V^*(F) = 1$.*

Proof. A subgroup of a free group has a non-trivial centralizer only if it is cyclic. A free group has a non-trivial normal cyclic subgroup only if it is itself cyclic. If F is a free group of rank >1, and $\mathfrak{V} \neq \mathfrak{O}$, then $\mathfrak{V}(F) \neq 1$. These well known facts, with Proposition 1.3 (ii), establish the result. Note that the corollary generalises the fact that a free group of rank >1 has trivial centre.

Define $\mathfrak{U}\mathfrak{V}^*$ to be the class of all groups G such that $U(G) \subseteq V^*(G)$. Note that this is unrelated to the product of two classes.

PROPOSITION 1.5. (i) *$\mathfrak{U}\mathfrak{V}^*$ is a variety. If, for any group G, $UV^*(G)$ denotes the corresponding verbal subgroup, then $(UV^*)(G) = [U(G)V^*G]$.*

(ii) *If $u(x_1, \ldots, x_r)$, $v(x_1, \ldots, x_s)$ are words, then define, for $1 \leqslant i \leqslant s$,*

$$v_{(i)}u = v(x_1, \ldots, x_{i-1}, x_i u(x_{s+1}, \ldots, x_{r+s}), x_{i+1}, \ldots, x_s)\,(v(x_1, \ldots, x_s))^{-1},$$

$$v^{(i)}u = v(x_1, \ldots, x_{i-1}, u(x_{s+1}, \ldots, x_{r+s}), x_{i+1}, \ldots, x_s)\,(v(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_s))^{-1}.$$

Let $\mathfrak{u}$ and $\mathfrak{v}$ be defining sets of laws for $\mathfrak{U}$ and $\mathfrak{V}$ respectively. Then $\mathfrak{U}\mathfrak{V}^*$ is defined by each of the following sets:

$$\mathfrak{w} = \{v_{(i)}u\colon v(x_1, ..., x_s) \in \mathfrak{v},\ u \in \mathfrak{u},\ 1 \leqslant i \leqslant s < \infty\};$$

$$\mathfrak{w}' = \{v^{(i)}u\colon v(x_1, ..., x_s) \in V(X_\infty),\ u \in \mathfrak{u},\ 1 \leqslant i \leqslant s < \infty\}.$$

Proof. (i) By Proposition 1.2 (iii), $[U(X_\infty)\, V^* X_\infty]$ is fully invariant in X_∞, and by (ii) of that proposition it is in the kernel of every homomorphism of X_∞ into a group in $\mathfrak{U}\mathfrak{V}^*$. To complete the proof it is enough to check that every homomorphic image of $X_\infty/[U(X_\infty)\, V^* X_\infty]$ is in $\mathfrak{U}\mathfrak{V}^*$, and this follows from (i) of the same proposition.

(ii) $\mathfrak{U}\mathfrak{V}^*$ is defined by $\{v_{(i)}u\colon v(x_1, ..., x_s) \in V(X_\infty),\ u \in U(X_\infty),\ 1 \leqslant i \leqslant s < \infty\}$, by definition of $(UV^*)(X_\infty)$. It is easy to check that, for $\bar{u} \in U(X_\infty)$, $v_{(i)}\bar{u}$ is a consequence of $\{v_{(i)}u : u \in \mathfrak{u}\}$. The proof that it is sufficient to take $v \in \mathfrak{v}$ only is in three parts.

(A) $(v^{-1})_{(i)}u = ((v_{(i)}u)^{-1})^v$, and hence $v_{(i)}u$ implies $(v^{-1})_{(i)}u$.

(B) $(v\bar{v})_{(i)}u = (v_{(i)}u)\,(\bar{v}_{(i)}u)^{v^{-1}}$, and hence $\{v_{(i)}u, \bar{v}_{(i)}u\}$ implies $(v\bar{v})_{(i)}u$.

(C) Let $v \in V(X_s)$ and let $w_1, ..., w_s$ be any words; for some t, they are all t-letter words. Pick some fixed i, $1 \leqslant i \leqslant t$. If $u(x_1, ..., x_r) \in \mathfrak{u}$ and $1 \leqslant k \leqslant s$, then

$$w_k^{-1} w_k(x_1, ..., x_{i-1}, x_i u(x_{t+1}, ..., x_{t+r}), x_{i+1}, ..., x_t) = u_k,$$

say, is clearly a law of $\mathfrak{U}$. Then $v(w_1, ..., w_s)$ is a t-letter word, and

$$\begin{aligned} v(w_1, ..., w_s)_{(i)}u &= v(w_1u_1, ..., w_su_s)\,(v(w_1, ..., w_s))^{-1} \\ &= v(w_1u_1, ..., w_su_s)\,(v(w_1, w_2u_2, ..., w_su_s))^{-1}\, v(w_1, w_2u_2, ..., w_su_s) \ldots (v(w_1, ..., w_s))^{-1}. \end{aligned}$$

Hence $v(w_1, ..., w_s)_{(i)}u$ is a consequence of $\{v_{(1)}u_1, ..., v_{(s)}u_s\}$.

It follows that $\mathfrak{w}$ defines $\mathfrak{U}\mathfrak{V}^*$. To check that $\mathfrak{w}'$ defines $\mathfrak{U}\mathfrak{V}^*$, note that

$$v_{(i)}u = (v(x_1, ..., x_{i-1}, x_i x_{s+1}, x_{i+1}, ..., x_s)^{(s+1)}u.$$

Sometimes $\mathfrak{w}'$ can be reduced by replacing $V(X_\infty)$ by $\mathfrak{v}$, and even this set can be redundant, as is shown by the following

Examples. Let $\mathfrak{u}$ be a defining set of laws for $\mathfrak{U}$.

1. If $v = [x_1, ..., x_{c+1}]$, so that v defines $\mathfrak{N}_c$, then $\mathfrak{U}\mathfrak{N}^*{}_c$ is defined by $\{[u, x_2, ..., x_{c+1}]\colon u \in \mathfrak{u}\} = \{v^{(1)}u\colon u \in \mathfrak{u}\}$. This follows by induction on c, recalling that $N_c^*(G) = \zeta_c(G)$. In particular, $\mathfrak{N}_c\mathfrak{N}_d^* = \mathfrak{N}_{c+d}$.

2. Let $v = [x_1, x_2]$ and $w = x_1^m$, so that $\mathfrak{A}_m$ is defined by $\{v, w\}$. Then $\mathfrak{U}\mathfrak{A}_m^*$ is defined by $\{v^{(1)}u\colon u \in \mathfrak{u}\} \cup \{w^{(1)}u\colon u \in \mathfrak{u}\}$, as is easily seen.

3. Let $v = [[x_1, x_2], [x_3, x_4]]$, so that v defines $\mathfrak{S}_2$; then $\mathfrak{U}\mathfrak{S}_2^*$ is defined by $\{v^{(1)}u\colon u \in \mathfrak{u}\}$. This can be proved by an easy commutator calculation.

4. On the other hand, in example 2, if $m=2$, so that $\mathfrak{A}_2$ is defined by w and $\mathfrak{A}$ is defined by v, then $\mathfrak{A}\mathfrak{A}_2^*$ is not defined by $w^{(1)}v=[x_1, x_2]^2$. For the (restricted) wreath product $C_2 \operatorname{wr} C_\infty$ satisfies this law, but is not nilpotent. However $[x_1, x_2, x_3]$ is clearly a law in $\mathfrak{A}\mathfrak{A}_2^*$.

Proposition 1.6. *If $\mathfrak{U}\subseteq\mathfrak{U}_1$, and $\mathfrak{V}\subseteq\mathfrak{V}_1$ are varieties then:*

(i) $\mathfrak{U}\mathfrak{V}^*\subseteq\mathfrak{U}_1\mathfrak{V}_1^*$;

(ii) *if* $\mathfrak{U}\subseteq[\mathfrak{W},\mathfrak{E}]$, then $\mathfrak{U}\mathfrak{V}^*\subseteq[\mathfrak{W},\mathfrak{V}]$, *and* $\mathfrak{V}\mathfrak{U}^*\subseteq[\mathfrak{W},\mathfrak{V}]$. *In particular*, $\mathfrak{U}\mathfrak{V}^*\subseteq[\mathfrak{U},\mathfrak{V}]$, *and* $\mathfrak{A}\mathfrak{V}^*\subseteq\mathfrak{V}\mathfrak{A}^*=[\mathfrak{V},\mathfrak{E}]$.

Proof. (i) If $G\in\mathfrak{U}\mathfrak{V}^*$ then $U_1(G)\subseteq U(G)\subseteq V^*(G)\subseteq V_1^*(G)$, so $G\in\mathfrak{U}_1\mathfrak{V}_1^*$.

(ii) From Proposition 1.3 (i) it follows that $[V(G)U^*G]\geqslant[W(G), V(G)]$ and hence $\mathfrak{V}\mathfrak{U}^*\subseteq[\mathfrak{W},\mathfrak{V}]$. The other result follows from the fact that $v(x_1, \ldots, x_s)^w=v(x_1[x_1, w], \ldots, x_s[x_s, w])$ and hence $[v(x_1, \ldots, x_s), w]$ is a law in $\mathfrak{U}\mathfrak{V}^*$ whenever $v(x_1, \ldots, x_s)$ and w are laws in $\mathfrak{V}$ and $\mathfrak{W}$ respectively, and $\mathfrak{U}\subseteq[\mathfrak{W},\mathfrak{E}]$.

Examples. 1. The inequality $\mathfrak{A}\mathfrak{V}^*\subseteq\mathfrak{V}\mathfrak{A}^*$ cannot in general be replaced by equality. For example, $\mathfrak{A}\mathfrak{S}_2^*$ is defined by the law $[[x_1, x_2, x_3], [x_4, x_5]]$ and hence lies in the variety $\mathfrak{A}\mathfrak{N}_2$ which contains only countably many non-isomorphic finitely generated groups. However $\mathfrak{S}_2\mathfrak{A}^*$ is the variety of all centre-by-metabelian groups which contains uncountably many non-isomorphic finitely generated groups, and so is much larger than $\mathfrak{A}\mathfrak{S}_2^*$. (See P. Hall [16]).

2. It is not true in general, even if $\mathfrak{U}\subseteq\mathfrak{V}$, that $\mathfrak{U}\mathfrak{V}^*\subseteq\mathfrak{V}\mathfrak{U}^*$. For example, $\mathfrak{N}_2\mathfrak{S}_2^*\nsubseteq\mathfrak{S}_2\mathfrak{N}_2^*$. In this case $\mathfrak{N}_2\mathfrak{S}_2^*$ is defined by the law $[[x_1, x_2, x_3, x_4], [x_5, x_6]]$, and $\mathfrak{S}_2\mathfrak{N}_2^*$ is defined by the law $[[x_1, x_2], [x_3, x_4], x_5, x_6]$. Now working inside $\gamma_6(X_\infty)/\gamma_7(X_\infty)$, and using basic commutators, it is straightforward to check that the verbal subgroup corresponding to $\mathfrak{S}_2\mathfrak{N}_2^*$ contains elements which are not in the verbal subgroup corresponding to $\mathfrak{N}_2\mathfrak{S}_2^* \bmod \gamma_7(X_\infty)$.

Proposition 1.7. *If $\mathfrak{U}$ and $\mathfrak{V}$ are varieties, then $\mathfrak{V}\subseteq\mathfrak{V}\mathfrak{U}^*$ and $\mathfrak{V}\subseteq\mathfrak{U}\mathfrak{V}^*$; and the inclusions are strict if and only if $\mathfrak{U}\neq\mathfrak{E}$ and $\mathfrak{V}\neq\mathfrak{O}$.*

Proof. The inclusions, and equality if $\mathfrak{U}=\mathfrak{E}$ or $\mathfrak{V}=\mathfrak{O}$, are clear. Now assume that $\mathfrak{U}\neq\mathfrak{E}$, and so $\mathfrak{U}\supseteq\mathfrak{A}_p$ for some prime p. Define varieties $\mathfrak{V}M_n\mathfrak{U}$ and $\mathfrak{U}M^n\mathfrak{V}$ inductively as follows: $\mathfrak{V}M_0\mathfrak{U}=\mathfrak{U}M^0\mathfrak{V}=\mathfrak{V}$; and for $n\geqslant 1$, $\mathfrak{V}M_n\mathfrak{U}=(\mathfrak{V}M_{n-1}\mathfrak{U})\mathfrak{U}^*$, $\mathfrak{U}M^n\mathfrak{V}=\mathfrak{U}(\mathfrak{U}M^{n-1}\mathfrak{V})^*$. If $\mathfrak{V}=\mathfrak{V}\mathfrak{U}^*$, then $\mathfrak{V}=\mathfrak{V}M_n\mathfrak{U}\supseteq\mathfrak{E}M_n\mathfrak{A}_p$ for all $n\geqslant 0$; and if $\mathfrak{V}=\mathfrak{U}\mathfrak{V}^*$, then $\mathfrak{V}=\mathfrak{U}M^n\mathfrak{V}\supseteq\mathfrak{A}_pM^n\mathfrak{E}$ for all $n\geqslant 0$. Now if $\bigcup_0^\infty\mathfrak{E}M_n\mathfrak{A}_p$ and $\bigcup_0^\infty\mathfrak{A}_pM^n\mathfrak{E}$ each generate $\mathfrak{O}$, it follows that $\mathfrak{V}=\mathfrak{O}$ if either $\mathfrak{V}=\mathfrak{U}\mathfrak{V}^*$ or $\mathfrak{V}=\mathfrak{V}\mathfrak{U}^*$, as required.

First consider $\mathfrak{E}M_n\mathfrak{A}_p=\mathfrak{W}_n$, say. From Example 2 after Proposition 1.5, $W_n(X_\infty)=X_\infty$ for $n=0$, and $W_n(X_\infty)=\langle[g, h], h^p: g\in X_\infty, h\in W_{n-1}(X_\infty)\rangle$ for $n>0$. But now Theorem

6.3 of Stallings [35] states that $\bigcap_0^\infty W_n(X_\infty)$ is trivial, and this gives that $\bigcap_0^\infty \mathfrak{W}_n$ generates $\mathfrak{O}$ as required.

We complete the proof by showing that $\mathfrak{W}_n = \mathfrak{A}_p M^n \mathfrak{E}$ for all $n \geqslant 0$. This is clearly true for $n=0, 1$. Assume that the result is true for $n-2$ and $n-1$, where $n>1$; we need to prove that $\mathfrak{A}_p \mathfrak{W}^*_{n-1} = \mathfrak{W}_{n-1}\mathfrak{A}^*_p$. For each $u(x_1, \ldots, x_r) \in W_{n-2}(X_r)$, $r>0$, let $\mathfrak{u}'(p) = \{u(x_1, \ldots, x_{i-1}, x_i[x_{r+1}, x_{r+2}], x_{i+1}, \ldots, x_r), u(x_1, \ldots, x_{i-1}, x_i x^p_{r+1}, x_{i+1}, \ldots, x_r)\colon 1 \leqslant i \leqslant r\}$. Then using Proposition 1.5. and the subsequent examples:

(i) $\mathfrak{W}_{n-1}$ is defined by $\mathfrak{v} = \{[u, x_{r+1}], u^p\colon r>0, u \in W_{n-2}(X_r)\}$;

(ii) $\mathfrak{W}_{n-1}$ is defined by $\mathfrak{v}' = \{u'u^{-1}\colon r>0, u \in W_{n-2}(X_r), u' \in \mathfrak{u}'(p)\}$;

(iii) it follows from (i) that $\mathfrak{A}_p\mathfrak{W}^*_{n-1}$ is defined by $\mathfrak{w} = \{[u', x_{r+1}]\,[u, x_{r+1}]^{-1}, [u, x_{r+1}z]\,[u, x_{r+1}]^{-1}, (u')^p u^{-p}\colon r>0, u \in W_{n-2}(X_r), u' \in \mathfrak{u}'(p), z = [x_{r+2}, x_{r+3}] \text{ or } x^p_{r+2}\}$;

(iv) it follows from (ii) that $\mathfrak{W}_{n-1}\mathfrak{A}^*_p$ is defined by $\mathfrak{w}' = \{[u'u^{-1}, x_{r+1}], (u'u^{-1})^p\colon r>0, u \in W_{n-2}(X_r), u' \in \mathfrak{u}'(p)\}$.

Now $\mathfrak{A}_p\mathfrak{W}^*_{n-1}$ and $\mathfrak{W}_{n-1}\mathfrak{A}^*_p$ are contained in $[\mathfrak{E}, \mathfrak{W}_{n-1}]$ by Proposition 1.6, so using (i) and (ii), each variety satisfies the laws

$$\mathfrak{w}'' = \{[u, x_{r+1}, x_{r+2}], [u^p, x_r], [u', u]\colon r>0, u \in W_{n-2}(X_r), u' \in \mathfrak{u}'(p)\}.$$

Hence, in proving that each of $\mathfrak{w}$ and $\mathfrak{w}'$ implies the other, the laws $\mathfrak{w}''$ may be used, and the result is then immediate.

We now introduce the Baer-invariants $\mathfrak{V}M(G)$ and $\mathfrak{V}P(G)$. Let $1 \to R \to F \to G \to 1$ be a free presentation of G and $\mathfrak{V}$ be a variety. Consider the group $(R \cap V(F))/[RV^*F]$; it is abelian since $[RV^*F] \supseteq [R, V(F)]$ by Proposition 1.3 (i). If $\alpha\colon G \to H$ is a homomorphism, and H has the presentation $1 \to \bar{R} \to \bar{F} \to H \to 1$, then there exists a homomorphism $\beta\colon F \to \bar{F}$ (not unique) which induces α. Clearly β also gives rise to a homomorphism $\beta^*\colon (R \cap V(F))/[RV^*F] \to (\bar{R} \cap V(\bar{F}))/[\bar{R}V^*\bar{F}]$.

Lemma 1.8. (i) *β^* is independent of the choice of β; denote it by α^*.*

(ii) *If $\gamma\colon H \to K$, and K is supplied with a presentation, then $(\alpha\gamma)^* = \alpha^*\gamma^*$.*

(iii) *$(R \cap V(F))/[RV^*F]$ is independent of the presentation G.*

Proof. (i) Let β' be a second homomorphism inducing α. Then for any $g \in F$, $g\beta \equiv g\beta'$ modulo $\bar{R}$; and so, if $v \in V(X_s)$, then $v(g_1\beta, \ldots, g_s\beta) \equiv v(g_1\beta', \ldots, g_s\beta')$ modulo $[\bar{R}V^*\bar{F}]$ for all $g_1, \ldots, g_s \in F$. Hence $\beta^* = (\beta')^*$.

(ii) is clear. Hence, taking $G=H$ and α the identity, it is easy to check that α^* is an isomorphism, which proves (iii).

In view of the lemma, we denote $(R \cap V(F))/[RV^*F]$ by $\mathfrak{V}M(G)$ (M for 'multiplier'). It is clearly functorial in G, and if $\mathfrak{V} = \mathfrak{A}$ it is the Schur multiplier of G.

In exactly the same way it may be shown that $V(F)/[RV^*F]$ is independent of the presentation, and is functorial in G; we denote this group by $\mathfrak{B}P(G)$ (P for 'pandect'). Equating $V(G)$ with $V(F)/R \cap V(F)$ gives rise to a central extension $1 \to \mathfrak{B}M(G) \to \mathfrak{B}P(G) \to V(G) \to 1$, natural in G. Of course, if $G \in \mathfrak{B}$ then $\mathfrak{B}M(G) = \mathfrak{B}P(G)$.

If $\mathfrak{W}$ is a variety containing G, these definitions may be adjusted by requiring F to be a $\mathfrak{W}$-free group; this gives rise to relative Baer-invariants $\mathfrak{W}\mathfrak{B}M(G)$ and $\mathfrak{W}\mathfrak{B}P(G)$. If $G \in \mathfrak{U}$, then clearly $\mathfrak{W}\mathfrak{B}M(G) = \mathfrak{B}M(G)$ and $\mathfrak{W}\mathfrak{B}P(G) = \mathfrak{B}P(G)$ whenever $\mathfrak{W} \supseteq \mathfrak{U}\mathfrak{B}^*$. The relative Baer-invariant $\mathfrak{B}\mathfrak{A}M(G)$ is discussed in [24.I] in the case $\mathfrak{B} \supseteq \mathfrak{A}$, and in [19] in the case $\mathfrak{B} \not\supseteq \mathfrak{A}$. It turns out that $\mathfrak{B}\mathfrak{A}M(G)$ is the first homology group of G with integral coefficients, in the homology relative to $\mathfrak{B}$; whereas, provided $\mathfrak{B} \supseteq \mathfrak{A} \operatorname{var} G$, $\mathfrak{B}M(G)$ is the second $\mathfrak{B}$-homology group of G with coefficients in $\mathbf{Z}(G)$, as proved in § II.4.

If $1 \to R \to F \to G \to 1$ is a presentation of G, F acts by conjugation on $(R \cap V(F))/[RV^*F]$; and since $[RV^*F] \supseteq [R, V(F)]$, this induces an action of G. This action is clearly compatible with the isomorphism 1^* used in Lemma 1.8 to establish that $(R \cap V(F))/[RV^*F]$ is independent of the presentation. Thus $\mathfrak{B}M(G)$ is a G-module; in fact the action of $g \in G$ on $\mathfrak{B}M(G)$ corresponds under the functoriality of $\mathfrak{B}M(G)$ to the inner automorphism of G induced by g. Similarly $\mathfrak{B}P(G)$ is a G-group, and the natural embedding of $\mathfrak{B}M(G)$ in $\mathfrak{B}P(G)$ is a G-homomorphism. We now examine some simple properties of this G-action.

LEMMA 1.9. *If K is a G-group and $\mathfrak{B}$ is a variety, the following are equivalent:*

(i) $V(G[K) \subseteq G$;

(ii) $K \subseteq V^*(G[K)$.

Proof. (ii) $\Rightarrow$ (i) is immediate, so assume (i). For any $v(x_1, \dots, x_s) \in V(X_\infty)$ and any $g_1[k_1, \dots, g_s[k_s \in G[K$, $v(g_1[k_1, \dots, g_s[k_s) = v(g_1, \dots, g_s)[k$ for some $k \in K$, and (i) implies that $k = 1$. It follows immediately that $K \subseteq V^*(G[K)$.

If the conditions of Lemma 1.9 are satisfied we say that K is a $\mathfrak{B}G$-*group* (or a $\mathfrak{B}G$-*module* if K is abelian). This clearly implies that K (as a group) is in $\mathfrak{B}$ and that $V(G[K) = V(G)$. An $\mathfrak{A}G$-group, for example, is an abelian group on which G acts trivially.

LEMMA 1.10. *If $N \lhd G$, N is a $\mathfrak{B}G$-group (with G acting via conjugation) if and only if $N \subseteq V^*(G)$.*

Proof. There is an injection θ: $G[N \to G \times G$ given by $(g[n)\theta = (gn, g)$ for all $g \in G$, $n \in N$. If $v(x_1, \dots, x_s) \in V(X)$ and $g_1[n_1, \dots, g_s[n_s \in G[N$, then $v(g_1[n_1, \dots, g_s[n_s)\theta = (v(g_1n_1, \dots, g_sn_s), v(g_1, \dots, g_s))$ and this is in $G\theta$ if and only if $v(g_1n_1, \dots, g_sn_1) = v(g_1, \dots, g_s)$. The result now follow easily using the fact that θ is an injection.

LEMMA 1.11. *Let K be a G-group and θ: $H \to G$ be a homomorphism; regard K as an H-group via θ. If K is a $\mathfrak{V}G$-group then K is a $\mathfrak{V}H$-group; and conversely, provided θ is a surjection, if K is a $\mathfrak{V}H$-group then K is a $\mathfrak{V}G$-group.*

Proof. Routine.

In particular, if K is a G-group and $N \lhd G$ such that N centralizes K, then, regarding K as a G/N-group, the above result can be applied with θ: $G \to G/N$ the natural surjection.

COROLLARY 1.12. *If B is a G-module, and $1 \to B \to E \to G \to 1$ is an extension of B by G, then B is a $\mathfrak{V}G$-module if and only if $B \subseteq V^*(E)$.*

Proof. $B \subseteq V^*(E)$ if and only if B is a $\mathfrak{V}E$-module (with E acting via conjugation) by Lemma 1.10. But $G \cong E/B$ and B as normal subgroup of E centralizes the E-module B. Lemma 1.11 now gives the result.

LEMMA 1.13. *Let K be a G-group. Then K is a $\mathfrak{V}G$-group if and only if $V(G)$ centralizes K and K is a $\mathfrak{V}(G/V(G))$-group.*

Proof. If K is a $\mathfrak{V}G$-group $K \subseteq V^*(G[K)$; so by Proposition 1.3 (ii) K is centralized by $V(G[K) = V(G)$. The result and its converse now follow immediately from Lemma 1.11.

PROPOSITION 1.14. *If $\mathfrak{U}$, $\mathfrak{V}$ are varieties, the following are equivalent:*

(i) *for all $G \in \mathfrak{U}$, $\mathfrak{V}P(G)$ is a $\mathfrak{U}G$-group;*

(ii) $\mathfrak{U}\mathfrak{V}^* \subseteq \mathfrak{V}\mathfrak{U}^*$.

Proof. Assume (ii) and let $G \in \mathfrak{U}$ have the presentation $1 \to R \to F \to G \to 1$. Put $H = F/[RV^*F]$; then by (ii) $V(H) \subseteq U^*(H)$. Hence (by Lemma 1.10) $V(H)$ is a $\mathfrak{U}H$-group, and is centralized by $R/[RV^*F]$. Lemma 1.11 gives immediately that $V(H)$ is a $\mathfrak{U}G$-group, and of course $V(H) = \mathfrak{V}P(G)$.

Now assume (i), and take $H \in \mathfrak{U}\mathfrak{V}^*$. Then for any presentation $1 \to R \to F \to H \to 1$ of H, $UV^*(F) \subseteq R$. Put $G = F/U(F)$ and $E = F/(UV^*(F))$. $\mathfrak{V}P(G) = V(E)$ is a $\mathfrak{U}G$-group by assumption, and hence a $\mathfrak{U}E$-group, (by Lemma 1.11). Let θ: $E \to H$ be the natural surjection. Then $V(H) = (V(E))\theta \subseteq (U^*(E))\theta \subseteq U^*(H)$. So $H \in \mathfrak{V}\mathfrak{U}^*$ as required.

In particular, if $G \in \mathfrak{V}$, $\mathfrak{V}P(G) = \mathfrak{V}M(G)$ is a $\mathfrak{V}G$-group; and since the property of being a $\mathfrak{U}G$-group is clearly inherited by sub-G-groups, $\mathfrak{V}M(G)$ is a $\mathfrak{U}G$-group whenever $\mathfrak{V}P(G)$ is, for any G. However, for $\mathfrak{V}M(G)$ we have a result, valid for all G, which is in general false for $\mathfrak{V}P(G)$.

PROPOSITION 1.15. *For any group G and variety $\mathfrak{V}$, $\mathfrak{V}M(G)$ is a $\mathfrak{V}G$-group.*

Proof. Let $1 \to R \to F \to G \to 1$ be a presentation of G. Then $R \cap V(F)/[RV^*F]$ is $\mathfrak{B}$-marginal in $F/[RV^*F]$, and so is a $\mathfrak{B}(F/[RV^*F])$-group annihilated by $R/[RV^*F]$. Hence $\mathfrak{B}M(G)$ is a $\mathfrak{B}G$-group.

We end this paragraph with a discussion of the order of $\mathfrak{B}M(G)$ and $\mathfrak{B}P(G)$. Following P. Hall [17], if v is a word and E a group, v is a *Schur-Baer word on E* if either $v^*(E)$ is of infinite index, or $v^*(E)$ is of finite index m, and the verbal subgroup $v(E)$ is of order dividing a power of m. If v is a Schur-Baer word on every group, v is a *Schur-Baer word.* Similarly, $\mathfrak{B}$ is a *Schur-Baer variety on E*, or is a *Schur-Baer variety*, if the corresponding statements hold with $v^*(E)$ and $v(E)$ replaced by $V^*(E)$ and $V(E)$ respectively. Clearly v is a Schur-Baer word if and only if the variety defined by v is a Schur-Baer variety.

P. Hall conjectures in [17] that every word is a Schur-Baer word; this conjecture remains open. Schur proved in [34] that $[x, y]$ is a Schur-Baer word, and this was extended by Baer in [3] to outer commutators, as defined before Theorem 3.6. In fact, it is shown in [17] that if v and w are Schur-Baer words on disjoint sets of letters, then $[w, v]$ is a Schur-Baer word; also, that the intersection of 2 Schur-Baer varieties is a Schur-Baer variety. Turner-Smith proved in [37] that, if E and all its quotient groups are residually finite, then every word is a Schur-Baer word on E; and Merzljakov proved in [29] the corresponding result for E a linear group over a field, and for E almost a residually finite p-group for infinitely many primes p.

The variety $\mathfrak{B}$ is *finitely based* if it can be defined by a finite set of laws, and hence by one law. The n-letter laws of $\mathfrak{B}$ are finitely based if they are a consequence of a finite set of laws of $\mathfrak{B}$.

THEOREM 1.16. *Every locally abelian-by-nilpotent variety $\mathfrak{B}$ is a Schur-Baer variety.*

Proof. If $\mathfrak{B}$ is finitely based, then since, by P. Hall [18], a finitely generated abelian-by-nilpotent group is residually finite, the theorem follows in this case from the result of Turner-Smith's quoted above, and appears in [17]. But since, by P. Hall [16], a finitely generated abelian-by-nilpotent group has the maximum condition on normal subgroups, it follows that the n-letter laws of any locally abelian-by-nilpotent variety are finitely based for any fixed n, and this is clearly enough to reduce the problem to the case in which $\mathfrak{B}$ is finitely based.

The next results explain our interest in Schur-Baer varieties.

THEOREM 1.17. *The following conditions on the variety $\mathfrak{B}$ are equivalent:*

(i) *$\mathfrak{B}$ is a Schur-Baer variety;*

(ii) *for every finite group* G, $\mathfrak{V}P(G)$ *is of order dividing a power of* $|G|$;

(iii) *for every finite group* G, $\mathfrak{V}M(G)$ *is of order dividing a power of* $|G|$.

Proof. Let $\mathfrak{V}$ be a Schur-Baer variety, and $1 \to R \to F \to G \to 1$ be a presentation of G, where G is of finite order m. Then since $V^*(F/[RV^*F]) \supseteq R/[RV^*F]$, the marginal factor of $F/[RV^*F]$ is of order dividing m, and hence $V(F)/[RV^*F] = \mathfrak{V}P(G)$ is of order dividing a power of m. Thus (i) $\Rightarrow$ (ii). Conversely, with the same notation, if E is a group with marginal factor isomorphic to G, it is easy to see that $V(E)$ is a homomorphic image of $\mathfrak{V}P(G)$, so (ii) $\Rightarrow$ (i). Finally (ii) $\Leftrightarrow$ (iii) since $\mathfrak{V}P(G)$ is an extension of $\mathfrak{V}M(G)$ by $V(G)$.

THEOREM 1.18. *If* $\mathfrak{V}$ *is defined by a set of Schur-Baer words, then for all finite groups* G, *every element of* $\mathfrak{V}P(G)$ *and of* $\mathfrak{V}M(G)$ *is of order dividing a power of* $|G|$.

Proof. If $\mathfrak{V}$ is defined by the set of words $\{v_i\}$, and E is any group, then $V^*(E) = \bigcap_i v_i^*(E)$ and $V(E) = \prod_i v_i(E)$. The result follows at once.

§ 2. Isologism

In [15] P. Hall introduced the concept of $\mathfrak{V}$-isologism, for any variety $\mathfrak{V}$, and in [14] showed how $\mathfrak{A}$-isologisms (or isoclinisms) can be used in the classification of p-groups; see also [13]. The connection between Baer-invariants and isologism, which we establish in this paragraph, is hinted at in the works of P. Hall quoted above.

If $1 \to K \to E \to G \to 1$ is an extension, $\zeta(K)$ becomes, by conjugation, an E-module centralized by K, and hence a G-module. If $\mathfrak{V}$ is a variety, G a group, and B a G-module, define a $\mathfrak{V}-G-B$*-extension* to be an extension $1 \to K \to E \to G \to 1$, and an inclusion $\iota\colon B \to \zeta(K)$ of G-modules, such that, regarding K and B as subgroups of E, $K \subseteq V^*(E)$ and $B \supseteq V(E) \cap K$. Note that $V(E) \cap K \subseteq \zeta(K)$ since, by Proposition 1.3, $[V^*(E), V(E)] = 1$. Also, for $\mathfrak{V}-G-B$-extensions to exist it is necessary and sufficient that B be a $\mathfrak{V}G$-module. Necessity follows from Lemmas 1.10 and 1.11, and sufficiency from Corollary 1.12.

If $\mathcal{E}_i \equiv 1 \to K_i \to E_i \to G_i \to 1$, $\iota_i\colon B_i \to \zeta(K_i)$ is a $\mathfrak{V}-G_i-B_i$-extension, $i=1, 2$, then a *weak* $\mathfrak{V}$*-homologism* $(\theta, \varphi)\colon \mathcal{E}_1 \to \mathcal{E}_2$ is a homomorphism $\theta\colon G_1 \to G_2$, and a homomorphism $\varphi\colon B_1 \to B_2$ compatible with θ (i.e. such that $(b^g)\varphi) = (b\varphi)^{g\theta}$ for all $b \in B_1$ and $g \in G_1$), satisfying the following condition. Let $\psi\colon E_1 \to E_2$ be any map (not necessarily a homomorphism) lifting θ; then $\forall r > 0$, $\forall v \in V(X_r)$, and $\forall t_1, \ldots, t_r \in E_1$ such that, $v(t_1, \ldots, t_r) \in B_1$, $v(t_1, \ldots, t_r)\varphi = v(t_1\psi, \ldots, t_r\psi)$. Since $K_2 \subseteq V^*(E_2)$, $v(t_1\psi, \ldots, t_r\psi)$ is independent of ψ, and φ is determined by θ if $B_1 = K_1 \cap V(E_1)$; however, it is easy to see that, in general, a homomorphism $\theta\colon G_1 \to G_2$ will not induce a weak homologism; since different expressions for an element b of B_1 as the value of a law of $\mathfrak{V}$ may lead to different values for $b\varphi$. If (θ, φ) is a weak $\mathfrak{V}$-homologism, and θ and φ are isomorphisms, then (θ, φ) is a *weak* $\mathfrak{V}$*-isologism*. For fixed G and B, a *weak* $\mathfrak{V}-G-B$-*isologism* is a weak $\mathfrak{V}$-isologism (θ, φ) such that $\theta = 1_G$ and $\varphi = 1_B$. $I(\mathfrak{V}, G, B)$ denotes the set of weak $\mathfrak{V}-G-B$-isologism classes and $[\mathcal{E}]$ denotes the class containing the $\mathfrak{V}-G-B$-extension $\mathcal{E}$.

THEOREM 2.1. *For any $\mathfrak{V}G$-module B, there is a natural bijection between $Hom_G(\mathfrak{V}M(G), B)$ and $I(\mathfrak{V}, G, B)$.*

Proof. Let α: $\mathfrak{V}M(G) \to B$ be a G-homomorphism and $1 \to R \to F \xrightarrow{\pi} G \to 1$ be a presentation of G. Denote $F/[RV^*F]$ and $R/[RV^*F]$ by $\bar{F}$ and $\bar{R}$ respectively. Then α can be regarded as a G-homomorphism from $\bar{R} \cap V(\bar{F})$ to B, and $\bar{F}$ can be given an action on B via that of $G \cong \bar{F}/\bar{R}$. Now let E denote the group obtained from $\bar{F}[B$ by amalgamating $\bar{R} \cap V(\bar{F})$ with its image under α in B; it is easy to check that this does not lead to collapse. Define a map θ: $E \to G$ by $(\overline{f[b})\theta = f\pi$ where $f \in F$, $b \in B$ and $\overline{f[b}$ is the corresponding element of E. Then it is easy to check that θ is a surjection with kernel $K = \{\overline{f[b} \mid f \in R, b \in B\}$. Also the obvious embedding ι: $B \to K$ maps B into $\zeta(K)$ since $\bar{R}$ centralizes B. The extension $\mathcal{E}_\alpha \equiv 1 \to K \to E \to G \to 1$, ι: $B \to \zeta(K)$, is a $\mathfrak{V}-G-B$-extension; for, using the facts that $\bar{R} \subseteq V^*(\bar{F})$ and B is a $\mathfrak{V}G$-module, if follows that $K \subseteq V^*(E)$. It is also straightforward to check that $[\mathcal{E}_\alpha]$ is independent of the presentation of G. Define Φ: $\mathrm{Hom}_G(\mathfrak{V}M(G), B) \to I(\mathfrak{V}, G, B)$ by $\alpha\Phi = [\mathcal{E}_\alpha]$. Then Φ is well-defined from above. It remains to construct an inverse Θ for Φ. Let $\mathcal{E} \equiv 1 \to K \to E \to G \to 1$, ι: $B \to \zeta(K)$, be a $\mathfrak{V}-G-B$-extension, and $1 \to R \to F \to G \to 1$ be a presentation of G. This gives rise to commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & R/[RV^*F] \arrow[r] \arrow[d, "\delta"] & F/[RV^*F] \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, "="] & 1 \\ 1 \arrow[r] & K \arrow[r] & E \arrow[r] & G \arrow[r] & 1 \end{tikzcd}$$

with exact rows, and δ induces a G-homomorphism α, say, of $R \cap V(F)/[RV^*F] = \mathfrak{V}M(G)$ into B. It is easy to verify that α depends only on $[\mathcal{E}]$, so that Θ: $I(\mathfrak{V}, G, B) \to \mathrm{Hom}_G(\mathfrak{V}M(G), B)$ defined by $[\mathcal{E}]\,\Theta = \alpha$, is well defined. It is straightforward to check that Φ and Θ are inverse maps.

It remains to connect weak isologisms with isologisms in the sense of P. Hall. Let E_i be a group, $X_i = V(E_i)$, $K_i = V^*(E_i)$, and $G_i = E_i/K_i$ for $i = 1, 2$. Then a $\mathfrak{V}$-*homologism*, (θ, ω): $E_1 \to E_2$ is a homomorphism θ: $G_1 \to G_2$ and a homomorphism ω: $X_1 \to X_2$ satisfying the following condition. Let ψ: $E_1 \to E_2$ be any map lifting θ, then for all $r > 0$, and all $v \in V(X_r)$, and all $t_1, \ldots, t_r \in E_1$, $v(t_1, \ldots, t_r)\,\omega = v(t_1\psi, \ldots, t_r\psi)$. If θ, ω are isomorphisms, (θ, ω) is a $\mathfrak{V}$-*isologism*. Thus two groups are $\mathfrak{V}$-isologic whenever evaluating the laws of $\mathfrak{V}$ in either group gives rise to essentially the same maps. If (θ, ω) is a $\mathfrak{V}$-homologism, ω is determined by θ and ω is a G-homomorphism.

THEOREM 2.2. *Let* $\mathcal{E}_i \equiv 1 \to K_i \to E_i \to G_i \to 1$, ι_i: $B_i \to \zeta(K_i)$, *be* $\mathfrak{V} - G_i - B_i$*-extensions such that* $K_i = V^*(E_i)$ *and* $B_i \iota_i = V(E_i) \cap K_i$, *for* $i = 1, 2$. *Then* $\mathcal{E}_1$ *and* $\mathcal{E}_2$ *are weakly* $\mathfrak{V}$*-isologic if and only if* E_1 *and* E_2 *are* $\mathfrak{V}$*-isologic.*

Proof. If E_1 and E_2 are $\mathfrak{V}$-isologic, it follows a fortiori that $\mathcal{E}_1$ and $\mathcal{E}_2$ are weakly $\mathfrak{V}$-isologic. For the converse, let $\mathcal{E}_1$ and $\mathcal{E}_2$ be weakly $\mathfrak{V}$-isologic. It is enough to assume that $G_1 = G_2 = G$ say, $B_1 = B_2 = B$ say, and that $\mathcal{E}_1$ and $\mathcal{E}_2$ are weakly $\mathfrak{V} - G - B$-isologic. Then with the notation of the proof of Theorem 2.1, $[\mathcal{E}_1]\Theta = [\mathcal{E}_2]\Theta = \alpha$ say, and α is a surjection. If $1 \to R \to F \to G \to 1$ is a presentation of G, there are commutative diagrams

$$\begin{array}{ccccccccc} 1 & \to & R/[RV^*F] & \to & F/[RV^*F] & \to & G & \to & 1 \\ & & \downarrow \delta_i & & \downarrow \gamma_i & & \downarrow = & & \\ 1 & \longrightarrow & K_i & \longrightarrow & E_i & \longrightarrow & G & \to & 1 \end{array}$$

with exact rows, such that δ_i induces α, for $i = 1, 2$. Let γ_i induce β_i: $V(F)/[RV^*F] \to V(E_i)$; then there are commutative diagrams

$$\begin{array}{ccccccccc} 1 & \to & \mathfrak{V}M(G) & \to & \mathfrak{V}P(G) & \to & V(G) & \to & 1 \\ & & \downarrow \alpha & & \downarrow \beta_i & & \downarrow = & & \\ 1 & \to & B & \longrightarrow & V(E_i) & \longrightarrow & V(G) & = & 1 \end{array}$$

with exact rows which are central extensions. The bottom rows of the diagrams are determined by the top row and α, up to an automorphism of $V(E_i)$ that fixes B and $V(G)$ elementwise. It follows that there is an isomorphism ω: $V(E_1) \to V(E_2)$ such that $\beta_2 = \beta_1 \omega$, fixing B and the factor group $V(G)$. Then $(1_G, \omega)$ is a $\mathfrak{V}$-isologism: $E_1 \to E_2$ as required.

Theorem 2.2 gives a correspondence between $\mathfrak{V}$-isologism classes with $\mathfrak{V}$-marginal factor G and certain weak $\mathfrak{V}$-isologism classes, and hence, by Theorem 2.1, with certain G-homomorphisms α: $\mathfrak{V}M(G) \to B$. We now consider the problems of ascertaining which groups G can appear as $\mathfrak{V}$-marginal factors and which $\alpha \in \mathrm{Hom}_G(\mathfrak{V}M(G), B)$ then give rise to $\mathfrak{V}$-isologism classes with this $\mathfrak{V}$-marginal factor.

For any $g_1, ..., g_r \in G$ and $v \in V(X_r)$ such that $v(g_1, ..., g_r) = 1$ let $\bar{v}(g_1, .., g_r)$ denote the corresponding element of $\mathfrak{V}M(G)$, and let $w_i(x_1, ..., x_{r+1})$ denote $v_{(i)}u$ where $u = x_1$, see Proposition 1.5. For any $g \neq 1$, $g \in V^*(G)$, let

$$S(g) = \{\bar{w}_i(g_1, ..., g_r, g) \colon r \geqslant i \geqslant 1, v \in V(X_r), g_1, ..., g_r \in G\}.$$

THEOREM 2.3. $\alpha \in \mathrm{Hom}_G(\mathfrak{V}M(G), B)$ *defines a* $\mathfrak{V}$*-isologism class, with* G *as* $\mathfrak{V}$*-marginal factor, if and only if* α *is onto, and for each* $g \in V^*(G)$, $g \neq 1$, $S(g) \nsubseteq \ker \alpha$. *If these conditions are satisfied, and* $1 \to R \to F \to G \to 1$ *is a presentation of* G *with* $\bar{F} = F/[RV^*F]$, *then*

$F_1 = \bar{F}/\ker\alpha$ is a representative of this $\mathfrak{V}$-isologism class, and so is F_1/T for any $T \lhd F_1$ such that $T \cap V(F_1) = 1$. Moreover, every representative of this class arises in this way from some presentation of G and some T.

Proof. If α is onto, and $1 \to R \to F \to G \to 1$ is a presentation of G, then the corresponding weak $\mathfrak{V}$-isologism class is represented by $\mathcal{E}_\alpha$, as constructed in Theorem 2.1, and $\mathcal{E}_\alpha$ is $1 \to R_1 \to F_1 \to G \to 1$ where F_1 is as above, and $R_1 = \bar{R}/\ker\alpha$, $\bar{R} = R/[RV^*F]$. Then B is embedded in R_1 with image $R_1 \cap V(F_1)$, and if $f \in V^*(F_1)$ and f maps to g in G, $g \in V^*(G)$; and the condition that if $g \neq 1$, $S(g) \nsubseteq \ker\alpha$ ensures that $g = 1$. Hence $R_1 = V^*(F_1)$, and $F_1/V^*(F_1) \cong G$. If $T \lhd F_1$, and $T \cap V(F_1) = 1$, then for any $t \in T$, $f_1, \ldots, f_r \in F_1$, $v \in V(X_r)$, $r \geqslant i \geqslant 1$, $w_i(f_1, \ldots, f_r, t) \in T \cap V(F_1) = 1$. Hence $T \subseteq V^*(F_1) = R_1$. It follows that F_1/T is $\mathfrak{V}$-isologic to F_1.

Conversely, if E satisfies $E/V^*(E) \cong G$ and $V^*(E) \cap V(E) \cong B$, then let F be a free group with E as an epimorphic image, and R be the kernel of the composite $F \to E \to G$. It is straightforward to check that E is an epimorphic image of the corresponding group F_1, that the corresponding $\alpha \in \mathrm{Hom}_G(\mathfrak{V}M(G), B)$ is onto, and that if $g \neq 1$, $g \in V^*(G)$, then since any $e \in E$ which maps to g is not in $V^*(E)$, $S(g) \nsubseteq \ker\alpha$.

Theorem 2.3 provides a procedure for constructing all $\mathfrak{V}$-isologism classes with G as marginal factor. Consider first the identity map $1 = 1_{\mathfrak{V}M(G)}$; this will give rise to such a $\mathfrak{V}$-isologism class if and only if $S(g) \neq 1$ for each $g \in V^*(G)$, $g \neq 1$. Also, if 1 does not give rise to such an isologism class, then neither will any $\alpha \in \mathrm{Hom}_G(\mathfrak{V}M(G), B)$ for any B. For example, if $\mathfrak{V} = \mathfrak{A}$ and G is a finite abelian group, then G can be the $\mathfrak{V}$-marginal (or central) factor of a group if and only if the two largest invariants of G coincide (see [14]). If 1 does give such a $\mathfrak{V}$-isologism class, one then investigates for each surjection $\alpha \in \mathrm{Hom}_G(\mathfrak{V}M(G), B)$ whether $S(g) \nsubseteq \ker\alpha$. There are some short cuts which can be used in testing the properties of $S(g)$; for example it is sufficient to consider just those $v \in V(X_\infty)$ which lie in some defining set for $\mathfrak{V}$. Notice too that

$$S(g) = \{\bar{v}(g, g_2, \ldots, g_r) - \bar{v}(1, g_2, \ldots, g_r) : r \geqslant 1,\ g_2, \ldots, g_r \in G,\ v \in V(X_r) \quad \text{and} \quad v(1, g_2, \ldots, g_r) = 1\}.$$

Finally, if α_1 and α_2 are surjections satisfying this condition, they determine the same $\mathfrak{V}$-isologism class if and only if there is an automorphism of G, such that the corresponding automorphism of $\mathfrak{V}M(G)$ induces a bijection between $\ker\alpha_1$ and $\ker\alpha_2$.

In the case $\mathfrak{V} = \mathfrak{A}$, this procedure was used in [13] in the calculation of all $\mathfrak{V}$-isologism classes (or families) represented by groups of order dividing 64. Again, if $\mathfrak{V} = \mathfrak{A}$, every weak $\mathfrak{V} - G - B$-isologism class contains an element $1 \to K \to E \to G \to 1$, $\iota: B \to \zeta(K)$, in which $K = B$ and $\iota = 1_B$. (If the class corresponds to the identity map on $\mathfrak{V}M(G)$—the Schur multiplier of G—then E is a 'covering group' of G). In particular, every $\mathfrak{V}$-isologism class contains a group E such that $V^*(E) \subseteq V(E)$. The classical proof depends on the fact that, when $\mathfrak{V} = \mathfrak{A}$, subgroups of $\mathfrak{V}$-free groups are $\mathfrak{V}$-splitting groups, and subgroups of $\mathfrak{V}$-marginal subgroups are normal. Clearly the only varieties with these properties are $\mathfrak{A}$, $\mathfrak{A}_m$ for m a positive square free integer, and $\mathfrak{E}$. We shall return to this point in § II.3.

Examples 1. Let $\mathfrak{V} = \mathfrak{N}_2$, E be nilpotent of class exactly 3, $K = V^*(E) = \zeta_2(E)$, and $B = V(E) = \gamma_2(E)$. Then $V^*(E) = \zeta_2(E) \supseteq \gamma_2(E) = V(E)$. Moreover, any group D which is $\mathfrak{V}$-isologic to E is of class exactly 3, and so $V^*(D) \supseteq V(D)$.

2. Let $\mathfrak{V} = \mathfrak{N}_2$ and $G = C_2(\alpha_1) \times C_2(\alpha_2)$. Take a presentation $1 \to R \to F \to G \to 1$, where F is freely generated by $\{y_1, y_2\}$, and $y_i \mapsto \alpha_i$, $i = 1, 2$. It is easy to see that $\mathfrak{V}M(G) = V(F)/[RV^*F] = \mathbf{Z}_2(z_1) \oplus \mathbf{Z}_2(z_2)$, where z_i is represented by $[y_2, y_1, y_i]$, $i = 1, 2$. Clearly, $\forall\, g \in G \backslash \{1\}$, $S(g) = \mathfrak{V}M(G)$; so if $\alpha\colon \mathfrak{V}M(G) \to B$ is any surjection with $B \neq 0$, $S(g) \nsubseteq \ker \alpha$. Such an α is either an isomorphism or has kernel of order 2. (This latter is a possibility since G, being abelian, acts trivially on $\mathfrak{V}M(G)$.) But the group of automorphisms of G acts transitively on the set of subgroups of $\mathfrak{V}M(G)$ of order 2, so there are exactly 2 $\mathfrak{V}$-isologism classes with marginal factor isomorphic to G. The first of these is represented by $F/[RV^*F] = F/\gamma_4\gamma_3^2$, where $\gamma_4 = \gamma_4(F)$, and γ_3^2 is the subgroup of F generated by the squares of the elements of $\gamma_3(F)$. To find a smaller representative of the class, a normal subgroup T of F is needed such that $R \supseteq T \supseteq \gamma_4\gamma_3^2$, and $T \cap \gamma_3 \subseteq \gamma_4\gamma_3^2$; then F/T is $\mathfrak{V}$-isologic to $F/\gamma_4\gamma_3^2$. It is easy to see that T must be of index at least 64 in F. The $\mathfrak{A}$-isologism class (or family) Γ_{17} in the Hall-Senior tables [13] is the only family of rank 6 (i.e. whose minimal representatives are of order 2^6) to lie in this $\mathfrak{N}_2$-isologism class. The other $\mathfrak{N}_2$-isologism class with marginal factor isomorphic to G clearly contains every group of order 16 and class 3; these all lie in Γ_3. The families Γ_6 and Γ_7 of rank 5, and Γ_{15}, Γ_{16} and Γ_{18} of rank 6, also lie in this class.

3. Let $\mathfrak{V} = \mathfrak{N}_2$, and $G = D_8 = \langle a_1, a_2 \colon a_1^2 = a_2^2 = [a_1, a_2]^2 = 1 \rangle$. Take a presentation with $y_i \mapsto a_i$, $i = 1, 2$ as in Example 2. In this case $\mathfrak{V}M(G) = (\mathbf{Z}_4(z_1) \oplus \mathbf{Z}_4(z_2))/\langle 2(z_1 + z_2) \rangle$, where z_i is represented by $[y_2, y_1, y_i]$, $i = 1, 2$. In this case there are two new features. Firstly, G acts non-trivially on $\mathfrak{V}M(G)$, a_1 and a_2 each sending every element to minus itself; and secondly, $S([a_1, a_2]) = 2\mathfrak{V}M(G)$, so only surjections α whose kernel does not contain this subgroup give rise to $\mathfrak{V}$-isologism classes.

If G is a finite group of order m, does it follow that every isologism class with marginal factor isomorphic to G has a representative of order dividing a power of m or at least of finite order? For an arbitrary variety these questions remain open. Clearly a necessary condition for an affirmative answer to the first question (assuming such isologism classes exist) is that $\mathfrak{V}M(G)$ or equivalently $\mathfrak{V}P(G)$ be of finite order dividing a power of m; for example if $\mathfrak{V}$ is a Schur-Baer variety (see Theorem 1.17). If $\mathfrak{V}MG$ has this property then it remains to decide whether F has a normal subgroup T of index a power of m such that $T \cap V(F) \subseteq [RV^*F]$ and $T \subseteq R$. Now suppose that $\bar{R}$, or equivalently $\bar{F}$, is residually finite. Then, since $V(\bar{F})$ is finite, $\bar{F}$ has a normal subgroup $\bar{T}$ of finite index such that $\bar{T} \cap V(\bar{F}) = 1$, and if $\mathfrak{V}$ is locally nilpotent, $\bar{T}$ can clearly be chosen to have index a power of m. Using Theorem 1.16, we have proved

THEOREM 2.4. *If $\mathfrak{V}$ is a locally abelian-by-nilpotent variety, and G is a finite group of order m, then every $\mathfrak{V}$-isologism class with marginal factor isomorphic to G has a finite representative, and, if $\mathfrak{V}$ is locally nilpotent, has a representative of order dividing a power of m.*

We now investigate the abelian group structure on $I(\mathfrak{V}, G, B)$ induced by the natural bijection between $\mathrm{Hom}_G(\mathfrak{V}MG, B)$ and $I(\mathfrak{V}, G, B)$ in Theorem 2.1. Let $E(\mathfrak{V}, G, B)$ denote the class of $\mathfrak{V}-G-B$-extensions, and let $\mathcal{E}_i \equiv 1 \to K_i \to E_i \to G \to 1$, $\iota_i \colon B \to \zeta(K_i)$, $\mathcal{E}_i \in E(\mathfrak{V}, G, B)$, $i=1, 2$. Put $\mathcal{E}_1 + \mathcal{E}_2 \equiv 1 \to K_1 \overset{B}{\times} K_2 \to E_1 \overset{B}{\underset{G}{\times}} E_2 \to G \to 1$, $\iota \colon B \to \zeta(K_1 \times K_2)$; see the preliminaries for the notation. It is easy to check that $\mathcal{E}_1 + \mathcal{E}_2 \in E(\mathfrak{V}, G, B)$. If $V(E_i) \cap K_i = 1$, call $\mathcal{E}_i$ a *null* extension. Clearly the sum of two null extensions is again null. Now the map Θ constructed in the proof of Theorem 2.1 is defined in terms of a map Ψ say of $E(\mathfrak{V}, G, B)$ into $\mathrm{Hom}_G(\mathfrak{V}M(G), B)$, and it is easy to see that $\mathcal{E}\Psi = 0$ if and only if $\mathcal{E}$ is a null extension. The proof of the following theorem is now straightforward, and we omit it.

THEOREM 2.5. *The above addition on $E(\mathfrak{V}, G, B)$ induces the structure of an abelian group on $I(\mathfrak{V}, G, B)$ in such a way that the natural bijection between $I(\mathfrak{V}, G, B)$ and $\mathrm{Hom}_G(\mathfrak{V}MG, B)$ of Theorem 2.1 becomes an isomorphism. Two elements of $E(\mathfrak{V}, G, B)$ belong to the same element of $I(\mathfrak{V}, G, B)$ if and only if they become isomorphic in the obvious sense on the addition to each of a (possibly different) null extension. If $1 \to K \to E \to G \to 1$, $\iota \colon B \to \zeta(K)$, represents an element of $I(\mathfrak{V}, G, B)$, then $1 \to K \to E \to G \to 1$, $-\iota \colon B \to \zeta(K)$ represents the inverse element.*

§3. Pandects

In this paragraph and the next we consider the 'pandects' of the variety $\mathfrak{V}$, namely the groups $\mathfrak{V}P(G)$, where G is a free abelian group, or more generally any relatively free group. If $G = X_\infty / X'_\infty$ and v is a law of $\mathfrak{V}$, then v determines an element $\bar{v}$ of $\mathfrak{V}P(G)$; and we consider the connection between the condition that a set of laws $\mathfrak{v}$ should define $\mathfrak{V}$, and the condition that $\mathfrak{V}P(G)$ should be generated as End(G)-group, by $\{\bar{v}: v \in \mathfrak{v}\}$. The paragraph ends with the explicit calculation of the pandects of a class of varieties.

Throughout this paragraph, α denotes a positive integer or ∞.

If $\mathfrak{U}$ and $\mathfrak{V}$ are varieties, then the $\mathfrak{U}$-α-pandect of $\mathfrak{V}$ is the group $\mathfrak{V}P(X_\alpha/U(X_\alpha)) = V(X_\alpha)/UV^*(X_\alpha)$, see Proposition 1.5; it will be denoted by $V/U-\alpha$. In particular, if $\mathfrak{U}=\mathfrak{A}$, this will be abbreviated to the *α-pandect* of $\mathfrak{V}$; this is a central factor of X_α by Proposition 1.6.

Examples. 1. If $\mathfrak{V}=\mathfrak{N}_c$, then $V/A-\alpha$ is the Schur multiplier of the $\mathfrak{V}$-free group of rank α; and this a free-abelian group, freely generated by the basic commutators of weight $c+1$ on at most α letters.

2. It will be proved in Chapter II that $V/A-\alpha$ is the group denoted in [24. III] by $\mathfrak{B}_0(\prod, H_2(-, \mathbf{Z}))$, where $\prod$ is a free abelian group of rank α. This was calculated for finite α in [25] for the cases $\mathfrak{V}=\mathfrak{N}_c \cap \mathfrak{S}_2$, $\mathfrak{S}_2$, and $\mathfrak{S}_2\mathfrak{A}^*=[\mathfrak{E}, \mathfrak{S}_2]$. Taking direct limits gives $V/A-\infty$.

Clearly the $\mathfrak{U}-\alpha$-pandect of $\mathfrak{V}$ depends functorially on the $\mathfrak{U}$-free groups $F_\alpha(\mathfrak{U})$ of rank α. In particular, if $E(\alpha, \mathfrak{U})$ denotes the semigroup of endomorphisms of $F_\alpha(\mathfrak{U})$, then $V/U-\alpha$ is an $E(\alpha, \mathfrak{U})$-group. Notice, however, that $E(\alpha, \mathfrak{A})$ has a natural ring structure but $V/A-\alpha$ is not an $E(\alpha, \mathfrak{A})$-module. For example, if $\mathfrak{V}=\mathfrak{W}=\mathfrak{A}$, then $V/W-2$ is generated freely by (an element represented by) $[x_2, x_1]$, and twice the identity of $E(\alpha, \mathfrak{A})$ induces four times the identity on $V/W-2$ (as $[x_2, x_1] \mapsto [x_2^2, x_1^2]=[x_2, x_1]^4$ in $V/W-2$).

If $\alpha \leqslant \beta \leqslant \infty$, the natural injection of X_α in X_β has a right inverse, and gives rise to an embedding of $V/U-\alpha$ in $V/U-\beta$ as a direct summand. We identify $V/U-\alpha$ with this subgroup of $V/U-\beta$, and for any $v \in V(X_\alpha)$, $\bar{v}$ will denote the element $(UV^*(X_\beta))v$ of $V/U-\beta$, for all $\beta \geqslant \alpha$, with a similar convention for subsets; the value of β, when significant, will be clear from the context.

The next two results are easy.

PROPOSITION 3.1. *If $\mathfrak{U}$ and $\mathfrak{V}$ are varieties, $\mathfrak{v} \subseteq V(X_\alpha)$, and v is a consequence of $\mathfrak{v}$, then $\bar{v}$ is in the $E(\alpha, \mathfrak{U})$-closure of $\bar{\mathfrak{v}}$.*

COROLLARY 3.2. *In particular, if $\mathfrak{V}$ is defined by a set of n-letter laws, and if $n \leqslant \alpha \leqslant \infty$, then $V/U-\alpha$ is the $E(\alpha, \mathfrak{U})$-closure of $V/U-n$. Thus n defines a 'stable range' for the groups $V/U-\alpha$.*

Our next aim is to consider a weak converse to Proposition 3.1; for example, if $\mathfrak{v} \subseteq V(X_\alpha)$ and $\bar{\mathfrak{v}}$ generates $V/U-\alpha$ as $E(\alpha, \mathfrak{U})$-group, in what sense does $\mathfrak{v}$ define the α-letter laws of $\mathfrak{V}$?

Examples 1. Clearly, the larger $\mathfrak{U}$ is, the more information $V/U-\alpha$ contains about the α-letter laws of $\mathfrak{V}$. In particular if $\mathfrak{U}=\mathfrak{O}$, $V/U-\alpha$ is $V(X_\alpha)$; whereas if $\mathfrak{U}=\mathfrak{E}$, $V/U-\alpha$ is trivial for all α and all $\mathfrak{V}$.

2. If $\mathfrak{V}=\mathfrak{S}_2$, the 2-pandect is trivial, but $V(X_2)$ is not trivial.

3. The $\mathfrak{U}$-pandect of $\mathfrak{V}\neq\mathfrak{O}$, for large enough (finite) α and $\mathfrak{U}\neq\mathfrak{E}$, is powerful enough to distinguish $\mathfrak{V}$ from $\mathfrak{O}$; for if $\mathfrak{U}\neq\mathfrak{E}$ and $\mathfrak{V}\neq\mathfrak{O}$, then $\mathfrak{U}\mathfrak{V}^*\neq\mathfrak{V}$ (by Proposition 1.7), and so, for some finite α, $V/U-\alpha$ is not trivial.

4. If $\mathfrak{W}$ and $\mathfrak{V}$ are varieties such that $\mathfrak{W}\cap\mathfrak{U}\mathfrak{V}^*=\mathfrak{V}$, then $V/U-\alpha$ is generated by the image of $W(X_\alpha)$. If $\mathfrak{W}=\mathfrak{O}$ this can only happen if $\mathfrak{V}=\mathfrak{W}$ or $\mathfrak{U}=\mathfrak{E}$ (by Proposition 1.7). Suppose that $\mathfrak{W}$ is an non-nilpotent variety which contains a maximal nilpotent proper subvariety. Then if $\mathfrak{U}$ is any nilpotent variety $\mathfrak{U}\mathfrak{V}^*$ is also nilpotent and hence $\mathfrak{W}\cap\mathfrak{U}\mathfrak{V}^*=\mathfrak{V}$. The same remarks remain valid if "nilpotent" is replaced by "soluble" throughout. Clearly any non-nilpotent (non-soluble) variety which satisfies the ascending chain condition on nilpotent (soluble) subvarieties has a maximal nilpotent (soluble) subvariety. Examples are given by $\mathrm{var}(S_3)$ in the nilpotent case, and $\mathrm{var}(A_5)$ in the soluble case; for in both these cases the variety in question is a Cross variety, and has only a finite number of subvarieties (see [30] ch. 5).

The following definitions strengthen the concept of independence of laws, so that Proposition 3.1 can be stated in a form in which the converse is also true. If $\mathfrak{U}$ and $\mathfrak{V}$ are varieties, $\mathfrak{v}$ is a set of laws of $\mathfrak{V}$, and v is a law of $\mathfrak{V}$, a group G *$\mathfrak{U}\mathfrak{V}^*$-differentiates v from* $\mathfrak{v}$ if $G\in\mathfrak{U}\mathfrak{V}^*$ and G differentiates v from $\mathfrak{v}$. If no such group G exists, then v is a *$\mathfrak{U}\mathfrak{V}^*$-consequence of* $\mathfrak{v}$. Finally, $\mathfrak{V}$ is *$\mathfrak{U}\mathfrak{V}^*$-defined by* $\mathfrak{v}$, and $\mathfrak{v}$ is a *$\mathfrak{U}\mathfrak{V}^*$-basis for* $\mathfrak{V}$, if every law of $\mathfrak{V}$ is a $\mathfrak{U}\mathfrak{V}^*$ consequence of $\mathfrak{v}$.

THEOREM 3.3. *With the above notation, the following are equivalent:*

(i) *v is a $\mathfrak{U}\mathfrak{V}^*$-consequence of $\mathfrak{v}$;*

(ii) *v is a consequence of $\mathfrak{v}$ together with the laws of $\mathfrak{U}\mathfrak{V}^*$;*

(iii) *for any α such that v and $\mathfrak{v}$ are contained in X_α, $\bar{v}$ is in the $E(\alpha,\mathfrak{U})$-closure of $\bar{\mathfrak{v}}$ in $V/U-\alpha$;*

(iv) *there exists an α such that v and $\mathfrak{v}$ are contained in X_α and $\bar{v}$ is in the $E(\alpha,\mathfrak{U})$-closure of $\bar{\mathfrak{v}}$ in $V/U-\alpha$.*

Proof. Clearly (i) is equivalent to (ii).

Now assume that (ii) holds, and that v and $\mathfrak{v}$ lie in X_α for some α. Then, by Proposition

3.1, $\bar{v}$ is in the $E(\alpha, \mathfrak{U})$-closure of $\bar{\mathfrak{v}} \cup UV^*(X_\alpha)$, which is just the $E(\alpha, \mathfrak{U})$-closure of $\bar{\mathfrak{v}}$ as required for (iii).

Clearly (iii) implies (iv). Now assume that (iv) holds, where $v = v(x_1, \ldots, x_m)$ for $m \leqslant \alpha$, and that there exists a group G such that $G \in \mathfrak{U}\mathfrak{B}^*$, $\mathfrak{v}$ is a set of laws for G, and there exist $g_1, \ldots, g_m \in G$ such that $v(g_1, \ldots, g_m) \neq 1$. Now choose a homomorphism θ: $X_\alpha \to G$ such that $x_i\theta = g_i$ for $i = 1, \ldots, m$. Then θ induces a homomorphism θ': $V/U - \alpha \to G$ since $UV^*(G) = 1$. Clearly the $E(\alpha, \mathfrak{U})$-closure of $\mathfrak{v}$ in $V/U - \alpha$ is in the kernel of θ', whereas $v(x_1, \ldots, x_m)\theta' = v(g_1, \ldots, g_m) \neq 1$. Hence no such group G exists, and so (iv) implies (i).

COROLLARY 3.4. *The α-letter laws of $\mathfrak{B}$ have a finite $\mathfrak{U}\mathfrak{B}^*$-basis if and only if $V/U - \alpha$ is finitely generated as $E(\alpha, \mathfrak{U})$-group.*

COROLLARY 3.5. *$\mathfrak{B}$ is $\mathfrak{U}\mathfrak{B}^*$-defined by n-letter laws if and only if $V/U - \infty$ is the $E(\infty, \mathfrak{U})$-closure of $V/U - n$.*

We look now at the structure of the pandects of $\mathfrak{B}$ when $\mathfrak{B}$ is defined by outer commutator words. We know of no finitely based variety whose α-pandects, for finite α, are not finitely generated as abelian groups. The next result shows that no such variety can be defined by outer commutator words. Finally we calculate precisely the pandects of of the variety of all polynilpotent groups of fixed class row.

If $1 \leqslant \alpha, \beta \leqslant \infty$, let $E(\alpha, \beta)$ denote the set of maps from $\{x_1, \ldots, x_\alpha\}$ into $\{x_1, \ldots, x_\beta\}$ (with the obvious interpretation for α or β infinite). Let $E(\alpha)$ denote $E(\alpha, \alpha)$ regarded as a monoid. There is an obvious embedding of $E(\alpha)$ in $E(\alpha, \mathfrak{A})$, and hence of $\mathbf{Z}E(\alpha)$ in $\mathbf{Z}E(\alpha, \mathfrak{A})$; thus $V/A - \alpha$ may be regarded as a $\mathbf{Z}E(\alpha)$-module. Now, if α is finite, $\mathbf{Z}E(\alpha)$ is finitely generated as an abelian group; and so is any finitely generated $\mathbf{Z}E(\alpha)$-module. Recall that *outer commutators* are defined inductively as follows: x_1 is an outer commutator; and if $u(x_1, \ldots, x_m)$ and $v(x_1, \ldots, x_n)$ are outer commutators, then so is $[u(x_1, \ldots, x_m), v(x_{m+1}, \ldots, x_{m+n})]$.

THEOREM 3.6. *Let $\mathfrak{v}$ be a set of outer commutators on at most α letters defining the variety $\mathfrak{B}$. Then the β-pandect of $\mathfrak{B}$ is generated as abelian group by the image of $\bar{\mathfrak{v}}$ under $E(\alpha, \beta)$.*

Proof. Let $v(x_1, \ldots, x_r) \in \mathfrak{v}$; then in $V/A - \beta$, $\bar{v}(g_1^{h_1}, \ldots, g_r^{h_r}) = \bar{v}(g_1[g_1, h_1], \ldots, g_r[g_r, h_r]) = \bar{v}(g_1, \ldots, g_r)$, for all $g_i, h_i \in X_\beta$, $i = 1, \ldots, r$. From the construction of outer commutators and the commutator relations it is now straightfoward to check that, in $V/A - \beta$, for all $g_1, \ldots, g_r$, g in X_β, and for all i, $\bar{v}(g_1, \ldots, g_{i-1}, g_i g, g_{i+1}, \ldots, g_r) = \bar{v}(g_1, \ldots, g_r) + \bar{v}(g_1, \ldots, g_{i-1}, g, g_{i+1}, \ldots, g_r)$ and $\bar{v}(g_1, \ldots, g_{i-1}, g_i^{-1}, g_{i+1}, \ldots, g_r) = -\bar{v}(g_1, \ldots, g_r)$. It follows immediately that the β-pandect of $\mathfrak{B}$ is generated by $\{v(x_{i_1}, \ldots, x_{i_r})\colon 1 \leqslant i_j \leqslant \beta,\ v(x_1, \ldots, x_r) \in \mathfrak{v}\}$.

COROLLARY 3.7. *In the above situation, $V/A-\alpha$ is generated as $\mathbf{Z}(E_\alpha)$-module by $\bar{\mathfrak{v}}$.*

Under more restrictive hypotheses, the pandects of $\mathfrak{V}$ may be calculated precisely. We shall take the basic commutators in $\{x_1, x_2, \dots\}$ to be defined and ordered as follows. The basic commutators of weight 1 are $x_1, x_2, \dots$ ordered by $x_1 < x_2 < \dots$. If basic commutators of all weights less than n have been defined and ordered, then the basic commutators of weight n are all the commutators of the form $[c_1, c_2]$ where c_1 and c_2 are basic commutators, the sum of whose weights is n, such that $c_1 > c_2$ and, if $c_1 = [c_3, c_4]$ where c_3 and c_4 are basic commutators, then $c_4 \leqslant c_2$. For the ordering, basic commutators of weight n are greater than those of smaller weight and ordered in some arbitrary fixed way amongst themselves.

If $\mathfrak{P}_{c_1 \dots c_r}$ denotes the polynilpotent variety of class row $(c_1, \dots, c_r)$ (that is $\mathfrak{P}_{c_1 \dots c_r} = \mathfrak{N}_{c_r}\mathfrak{N}_{c_{r-1}} \dots \mathfrak{N}_{c_1}$) then $\mathfrak{P}_{c_1 \dots c_r}$ is defined by the law $w_r = w_r(x_1, \dots, x_{c_1 \dots c_r})$ defined inductively as follows. Let $w_1 = w_1(x_1, \dots, x_{c_1}) = [x_1, \dots, x_{c_1}]$. If $w_1, \dots, w_{i-1}$ have been defined, such that $w_j = w_j(x_1, \dots, x_{c_1 \dots c_j})$ then $w_i = [w_{i-1}^{(1)}, w_{i-1}^{(2)}, \dots, w_{i-1}^{(ci)}]$, where $w_{i-1}^{(k)} = w_{i-1}(x_{(k-1)(c_1 \dots c_{i-1})+1}, \dots, x_{kc_1 \dots c_{i-1}})$.

THEOREM 3.8. (T. C. Hurley). *The β-pandect of the polynilpotent variety $\mathfrak{P}_{c_1 \dots c_r}$ of class row $(c_1, \dots, c_r)$ is freely generated as abelian group by the set of elements represented by basic commutators of weight $c_1 \dots c_r$ in $P_{c_1 \dots c_r}(X_\beta)$.*

Proof. Let $\mathfrak{P}_{(i)}$ denoted $\mathfrak{P}_{c_1 \dots c_i}$. Since $P_{(r)}(X_\beta) \subseteq \gamma_{c_1 \dots c_r}(X_\beta)$, and $(AP^*_{(r)})(X_\beta) \subseteq \gamma_{(c_1 \dots c_r)+1}(X_\beta)$, it is clear that the elements represented by the basic commutators of weight $c_1 \dots c_r$ in $P_{(r)}(X_\beta)$ freely generate (as abelian group) a subgroup of the β-pandect of $\mathfrak{P}_{(r)}$. It is therefore sufficient to show that these elements generate the β-pandect, that is, by Theorem 3.7, that each image of w_r under $E(c_1 \dots c_r, \beta)$ can be written as a product of such elements and their inverses $\operatorname{mod}(AP^*_{(r)})(X_\beta)$. The argument will be by induction on r. The case $r=1$ gives the well-known result that $\gamma_{c_1}/\gamma_{c_1+1}$ is freely generated, as abelian group, by the basic commutators of weight c_1. Assume now that the result is true for $\mathfrak{P}_{r-1}$. If $\gamma \in E(c_1 \dots c_r, \beta)$ then $w_r\gamma = [w_{r-1}^{(1)}\gamma, \dots, w_{r-1}^{(c_1)}\gamma]$. Now each $w_{r-1}^{(k)}$ is the image of $w_{r-1}^{(1)}$ under an element of $E(c_1 \dots c_{r-1}, \beta)$, and hence, by the inductive hypothesis, can be expressed $\operatorname{mod}(AP^*_{(r-1)})(X_\beta)$ as a product of basic commutators of weight $c_1 \dots c_{r-1}$ in $P_{(r-1)}(X_\beta)$ and their inverses. Hence $w_r\gamma$ can be expressed $\operatorname{mod}(AP^*_{(r)})(X_\beta)$ as a product of commutators of the form $[u_1, \dots, u_{c_r}]$ and their inverses, where each u_i is a basic commutator of weight $c_1 \dots c_{r-1}$ in $P_{(r-1)}(X_\beta)$. Now $[u_1, \dots, u_{c_r}]$ is not necessarily a basic commutator, but it can be expressed as a product of basic commutators of weight c_r in $\{u_1, \dots, u_{c_r}\}$. Now, since u_j and u_k have the same weights, if $u_j = [v_1, v_2]$ then v_2 has smaller weight than

u_k. It follows immediately that a basic commutator in $\{u_1, ..., u_{c_r}\}$ is also a basic commutator in $\{x_1, x_2, ...\}$. Since $u_1, ..., u_{c_r}$ are in $P_{(r-1)}(X_\beta)$, any commutator of weight c_r in $u_1, ..., u_{c_r}$ is in $\gamma_{c_r}(P_{(r-1)}(X_\beta)) = P_{(r)}(X_\beta)$. This completes the inductive step.

§ 4. Infinitely generated examples

The problem of deciding which varieties are not finitely based has recently received much attention. In this paragraph, which is based on the work of Vaughan-Lee, we exhibit various varieties $\mathfrak{B}$ which have the stronger property of having (in the notation of § 3) no finite $\mathfrak{A}_2\mathfrak{B}^*$-basis. This is equivalent to a non-finite-generation property of the pandects of $\mathfrak{B}$ (see § 3); a homological interpretation will be given in Chapter II. In [38], Vaughan-Lee finds a variety $\mathfrak{B}$ with an infinite set of independent laws, and we shall show that these laws are in fact $\mathfrak{A}_2\mathfrak{B}^*$-independent. This set of laws requires infinitely many letters, and by a further refinement of Vaughan-Lee's argument we shall find a variety $\mathfrak{Y}$ with infinitely many $\mathfrak{A}_2\mathfrak{Y}^*$-independent 3-letter laws. Bounding the number of letters will give rise to a striking counter-example in homology.

THEOREM 4.1. (Vaughan-Lee [38]). *For each $n \geq 1$, let v_n denote the word $[[x_1, x_2, x_3], [x_4, x_5], ..., [x_{2n+2}, x_{2n+3}], [x_1, x_2, x_3]]$. Let $\mathfrak{B}$ be the variety defined by*

$$\{v_1, v_2, v_3, ...\} \cup \{x_1^{16}, [[x_1, x_2, x_3], [x_4, x_5, x_6], [x_7, x_8]]\},$$

and let $\mathfrak{B}_n$ be the variety defined by leaving out the law v_n from this set. Then $\mathfrak{B}$ is a proper subvariety of $\mathfrak{B}_n$ for each $n = 1, 2, \ldots$.

The proof of this result is given in [38], and we shall reproduce just enough of the details to enable us to prove our stronger result.

Let A_n be the group generated by elements $a_1, ..., a_{2n}$, subject to the relations $a_i^2 = [a_i, a_j, a_k] = 1$, $1 \leq i, j, k \leq 2n$. Thus A_n/A_n' and A_n' are elementary abelian 2-groups. Denote the coset $A_n'g$ by $\bar{g}$ for $g \in A_n$, with a similar convention for B_n as defined below.

Let B_n be generated by $\{b_g\colon g \in A_n\} \cup \{c_{\bar{g}}\colon \bar{g} \in A_n/A_n'\}$ subject to the relations $b_g^2 = [b_g, b_h, b_k] = 1$, $g, h, k \in A_n$, and

(i) $[b_g, b_h] = 1$ if $\bar{g} \neq \bar{h}$;

(ii) $[b_g, b_h] = c_{\bar{g}}$ or 1 if $\bar{g} = \bar{h}$.

In case (ii) the choice of $c_{\bar{g}}$ or 1 depends on a subtle rule which will not be given here; instead, the consequences of the rule that will be needed are given.

(α) $[b_g, b_{([a_1, a_2] \ldots [a_{2n-1}, a_{2n}] g)}] = c_{\bar{g}}$,

(hence B_n is generated by $\{b_g\colon g \in A_n\}$), and B_n', as a vector space over $\mathbf{Z}_2$, has a basis $\{c_{\bar{g}}\colon \bar{g} \in A_n/A_n'\}$.

(β) Every permutation of the generating set $\{b_g: g \in A_n\}$ given by right multiplication by any fixed element of A_n induces an automorphism of B_n, so that B_n is an A_n-group. Hence there is an A_n-isomorphism between B_n/B'_n and $\mathbf{Z}_2 A_n$ in which, $\forall g \in A_n$, $\bar{b}_g \leftrightarrow g$ and an A_n/A'_n-isomorphism between B'_n and $\mathbf{Z}_2(A_n/A'_n)$ in which, $\forall \bar{g} \in A_n/A'_n$, $c_{\bar{g}} \leftrightarrow \bar{g}$. It will be convenient to denote b_1 and $c_{\bar{1}}$ by b and c respectively, so that, $\forall g \in A_n$, $b_g = b^g$ and $c_{\bar{g}} = c^g$.

(γ) Every permutation of the generating set $\{b_g: g \in A_n\}$ given by an automorphism of A_n obtained from a permutation of $\{a_i\}$ induces an automorphism of B_n.

Using (β), put $D_n = A_n [B_n$; then

(δ) $\forall d \in B_n$, $\forall m \neq n$, and $\forall g_1, \ldots, g_{2m} \in D_n$, $[d, [g_1, g_2], \ldots, [g_{2m-1}, g_{2m}], d] = 1$

(ε) The map θ: $B_n \times D_n^{(2n)} \to B'_n$ defined by $(d, g_1, \ldots, g_{2n})\theta = [d, [g_1, g_2], \ldots, [g_{2n-1}, g_{2n}], d]$, $\forall d \in B_n$ and $\forall g_1, \ldots, g_{2n} \in D_n$, factors as $B_n \times D_n^{(2n)} \to (B_n/B'_n) \otimes \Lambda^{2n}(A_n/A'_n) \to B'_n$, where the first map is the natural surjection, and the second is the homomorphism induced by $\bar{b}_g \otimes \bar{g}_1 \wedge \ldots \wedge \bar{g}_{2n} \to c_{\bar{g}}^{\Delta}$, where $\Delta = 0$ if $\bar{g}_1, \ldots, \bar{g}_{2n}$ are linearly dependent, i.e. if $\bar{g}_1 \wedge \ldots \wedge \bar{g}_{2n} = 0$, and is 1 otherwise. Here $\bar{g}_i$ denotes $g_i \bmod B_n A'_n$.

It is clear from these results, in particular from (δ) and (ε), that D_n satisfies all the laws used in the definition of $\mathfrak{V}$, with the exception of v_n, and so the theorem follows.

It is also clear from (ε) that B'_n is $\mathfrak{V}$-marginal in D_n, and our next aim is to replace D_n by a quotient group whose derived group will be $\mathfrak{V}$-marginal.

LEMMA 4.2. *Let $r > 2$, and let $h_1, \ldots, h_r \in D_n$, $h_i = f_i d_i$, $f_i \in B_n$, $d_i \in A_n$. Then, equating B_n/B'_n with $\mathbf{Z}_2 A_n$ as in (β),* $[h_1, \ldots, h_r]B'_n = \bar{f}_1 d_1([d_1, d_2] - d_2)(1 - d_3) \ldots (1 - d_r) + \bar{f}_2(1 - d_1) \times d_2(1 - d_3) \ldots (1 - d_r) + \bar{f}_3(1 - [d_1, d_2]) d_3(1 - d_4) \ldots (1 - d_r)$.

Proof. A straightforward commutator calculation.

Using (β) to equate B'_n with $\mathbf{Z}_2(A_n/A'_n)$, let $I \subset B'_n$ be the augmentation ideal. Since A_n/A'_n is elementary abelian of rank $2n$, it is easy to see that $I \supset I^2 \supset \ldots \supset I^{2n} \supset I^{2n+1} = 0$ is a properly descending sequence of subgroups, with I^{2n} of order 2, the non-zero element being $(1 - a_1) \ldots (1 - a_{2n})$.

LEMMA 4.3. *With θ as in (ε), $(\gamma_r(D_n) \times D_n^{(2n)})\theta = I^{r-1}$, for $r > 2$.*

Proof. This follows at once from Lemma 4.2. Applying the lemma to the law v_n gives $v_n(D_n) = V(D_n) = I^2$, so that D_n/I^3 satisfies all the laws used to define $\mathfrak{V}$ with the exception of v_n. Now consider $v_n(d_1, d_2, d_3, g_1, \ldots, g_{2n})$, $d_i, g_i \in D_n$. It follows from (ε) that multiplying g_i, $i = 1, \ldots, 2n$, by any element of D'_n does not affect the value of v_n. Also, multiplying d_i, $i = 1, 2, 3$, by any element of D'_n multiplies $[d_1, d_2, d_3]$ by an element of $\gamma_4(D_n)$, and so, by (ε) and Lemma 4.3, changes the value of v_n by an element of I^3. Hence D_n/I^3 $\mathfrak{A}\mathfrak{V}^*$-differentiates v_n from the other laws used in the definition of $\mathfrak{V}$. In fact since $D_n/D'_n \in \mathfrak{A}_2$, $D_n/I^3\mathfrak{A}_2\mathfrak{V}^*$-differentiates the laws. This proves:

THEOREM 4.4. *With the above notation, v_n is $\mathfrak{A}_2\mathfrak{V}^*$-independent of $\{v_i: i \neq n\} \cup \{x_1^{16}, [[x_1, x_2, x_3], [x_4, x_5, x_6], [x_7, x_8]]\}$.*

COROLLARY 4.5. *The A_2-∞-pandect of $\mathfrak{V}$ is not finitely generated as $E(\infty, \mathfrak{A}_2)$-group.*

In [8], Bryant proves that the variety $\mathfrak{U} = \mathfrak{B}_4\mathfrak{B}_2$ is not finitely based, where $\mathfrak{B}_n$ is the variety of all groups of exponent dividing n. He constructs groups which differentiate between laws of the form $(x_1^2 \dots x_n^2)^4$, and it is straightforward to check that these groups do in fact $\mathfrak{A}\mathfrak{U}^*$-differentiate between the laws. This gives rise to analogues for Theorem 4.4 and Corollary 4.5 for the variety $\mathfrak{B}_4\mathfrak{B}_2$ and its ∞-pandect.

We now construct a variety $\mathfrak{Y}$ whose 3-pandect is not finitely generated as $E(3, \mathfrak{A}_2)$-group and, hence, as abelian group. It is easy to construct infinitely many independent 2-letter laws by replacing the letters x_j in v_i by independent generators of F'_2, but it is not easy to decide whether these give a non-finitely generated 2-pandect. As a preliminary step we revert to the case of infinitely many generators.

LEMMA 4.6. *Let $v'_n = v'_n(x_1, \dots, x_{2n+1}; x_{2n+2}, \dots, x_{4n+1})$ be the law $[[x_1, \dots, x_{2n+1}], [x_{2n+2}, x_{2n+3}], \dots, [x_{4n}, x_{4n+1}], [x_1, \dots, x_{2n+1}]]$ and $\mathfrak{V}'$ be the variety defined by $\{v'_n: n = 1, 2, \dots\}$. Then with the above notation, $V'(D_n) = v'_n(D_n) = I^{2n}$, and D_n $\mathfrak{A}_2\mathfrak{V}'^*$-differentiates v'_n from $\{v'_i: i \neq n\}$.*

Proof. The proof of Theorem 4.4 needs only the slightest adjustment to give this result.

We use Lemma 4.6 and the laws v'_n as the basis for our construction of the variety $\mathfrak{Y}$, rather than Theorem 4.4 and the laws v_n, only because our attempts to use the latter (and simpler) situation failed. To use the groups D_n to differentiate 3-letter laws they must each be embedded in a 3-generator group. In view of (γ), D_n is a $C_{2n}(s)$-group, where s acts on A_n by $a_i^s = a_{i+1}$, $1 \leqslant i \leqslant 2n-1$, and $a_{2n}^s = a_1$; on B_n by $b_g \mapsto b_{g^s}$; and hence on D_n. Then $C_{2n}(s)[D_n$ is generated by $\{s, a_1, b\}$. Unfortunately, to get the construction to work, a slightly more complicated group is needed, namely $(C_{2n}(s) \times C_2(t))\,[(D_n \times D_n)$, where s acts on each copy of D_n as above, and t interchanges the copies of D_n. Call this group G_n; G_n is generated by $\{s, t, (a_1, 1), (b, 1)\}$, and only the subgroup generated by $\{s, t, (ba_1, 1)\}$ is needed. It remains to define $\mathfrak{Y}$. Put

$$\begin{aligned} w_n &= w_n(x_1, x_2, x_3) \\ &= v'_n([x_1, x_2], [x_1, x_2]^{x_3}, \dots, [x_1, x_2]^{x_3^{2n}};\ [x_1, x_2], [x_1, x_2]^{x_3}, \dots, [x_1, x_2]^{x_3^{2n-1}}), \end{aligned}$$

and let $\mathfrak{Y}$ be the variety defined by $\{w_n: n = 2, 4, 8, \dots\}$.

THEOREM 4.7. *With the above notation, the laws* w_n, $n=2, 4, 8, \ldots$ *are* $\mathfrak{A}_2\mathfrak{Y})^*$*-independent.*

COROLLARY 4.8. The $\mathfrak{A}_2-3$*-pandect of* $\mathfrak{Y}$ *is not finitely generated as abelian group. In particular,* $\mathfrak{Y}$ *is not a Schur-Baer variety.*

Proof of Theorem 4.7. It will be shown that G_n $\mathfrak{A}_2\mathfrak{Y})^*$-differentiates w_n from $\{w_i: i \neq n, i=2, 4, 8, \ldots\}$. Since $[x_1, x_2]$ and its conjugates take values in $D_n \times D_n$, and D_n satisfies the laws v_i' for $i \neq n$, it follows that G_n satisfies the laws w_i for $i \neq n$. By the same token, if e is the non-zero element of I^{2n}, $Y(G_n)=w_n(G_n) \subseteq \langle (e, 1), (1, e)\rangle$. In fact w_n is not a law in G_n, for $w_n((ba_1, 1), t, s)$ has each component in $D_n \times D_n$ equal to $v_n'(ba_1, ba_2, \ldots, ba_{2n}, ba_1; ba_2, \ldots, ba_{2n})=e$, by a simple application of Lemma 4.2 and (ε). It remains to check that G_n' and s^2 are w_n-marginal. This will imply that $Y(G_n)$ is central in G_n, so $Y(G_n)=\langle (e, e)\rangle$ is of order 2.

Say that $g \in G_n$ is *marginal in the first place* if $w(g_1 g, g_2, g_3)=w(g_1, g_2, g_3), \forall g_1, g_2, g_3 \in G_n$, with similar conventions for the second and third places. Say that $S \subseteq G_n$ is marginal in the ith place if all its elements are.

Since D_n' is v_n'-marginal in D_n, $D_n' \times D_n'$ is marginal in G_n, and $D_n \times D_n$ (and a fortiori G_n') is marginal in the third place. Let $g_1, g_2, g_3 \in G_n$ and put $[g_1, g_2]^{g_3^i}=(d_{1,i+1}, d_{2,i+1})$ for $i=0, 1, \ldots, 2n$. The image of d_{ji} in A_n/A_n' will be denoted by $\bar{d}_{ji}$; observe that $\bar{d}_{j,2n+1}=\bar{d}_{j1}$. Let $d_{j1}=f_j d_j$, $j=1, 2$, $f_j \in B_n$, $d_j \in A_n$. Then $\bar{f}_j \in B_n/B_n' \cong \mathbf{Z}_2 A_n$ as in (β). Let $\bar{\bar{f}}_j$ be the image of $\bar{f}_j$ in $\mathbf{Z}_2(A_n/A_n')$ and $\alpha_j=\alpha(f_j)$ be the image of $\bar{\bar{f}}_j$ under the augmentation $\mathbf{Z}_2(A_n/A_n') \to \mathbf{Z}_2$. Then

$$w_n(g_1, g_2, g_3) = (e\alpha_1\Delta_1, e\alpha_2\Delta_2) \tag{1}$$

where $\Delta_j=0$ if $\bar{d}_{j1}, \ldots, \bar{d}_{j,2n}$ are linearly independent and is 1 otherwise.
For

$$w_n(g_1, g_2, g_3) = (([d_{11}, d_{12}, \ldots, d_{1,2n}, d_{11}], d_{11}, \ldots, d_{1,2n})\theta, ([d_{21}, d_{22}, \ldots, d_{2,2n}, d_{21}], d_{21}, \ldots, d_{1,2n})\theta),$$

and hence this formula is certainly correct in the j-th place if $\Delta_j=0$, so suppose $\bar{d}_{j1}, \ldots, \bar{d}_{j,2n}$ are linearly independent. Then Lemma 4.2, together with (ε), gives the jth component of $w_n(g_1, g_2, g_3)$ as the sum of 3 terms. The third term vanishes trivially; the second vanishes since $(1-\bar{d}_{j1})(1-\bar{d}_{j,2n+1})=(1-\bar{d}_{j1})^2=0$ in $B_n'=\mathbf{Z}_2(A_n/A_n')$; and the first gives $e\alpha_j$, for

$$(1-\bar{d}_{j2})\ldots(1-\bar{d}_{j,2n})(1-\bar{d}_{j1}) = e \quad \text{as} \quad \{\bar{d}_{j2}, \ldots, \bar{d}_{j,2n}, \bar{d}_{j1}\}$$

is a basis for A_n/A_n'.

The next step is to prove that s^2 is marginal in the third place. Since it has already been shown that $D_n \times D_n$ is marginal in the third place it is enough to prove that $w_n(g_1, g_2, g_3) = w_n(g_1, g_2, g_3 s^2)$ whenever $g_3=s^r t^s$. Consider first the case when r is even. Then

$$d_{j,n+1}=d_{j1}^{g_3^n}=d_{j1}^{s^{rn}}=d_{j1}; \quad \text{so} \quad \Delta_j=0,$$

and $w_n(g_1, g_2, g_3) = w_n(g_1, g_2, g_3 s^2) = 0$. Assume now that r is odd. Since α_j is independent of g_3, the linear independence of $\{\bar{d}_{j1}, \ldots, \bar{d}_{j,2n}\}$ is the only problem. Clearly

$$\begin{aligned} d_{j,i+1} &= d_{j1}^{s^{ri}}, \quad \text{if } \varepsilon = 0 \quad \text{or} \quad i \text{ is even} \\ &= d_{3-j,1}^{s^{ri}}, \quad \text{if } \varepsilon = 1 \quad \text{and} \quad i \text{ is odd.} \end{aligned} \tag{2}$$

If $\varepsilon = 0$, letting i go from 0 to $2n-1$, $s^{ri} = (s^r)^i$ goes through all elements of $C_{2n}(s)$. If $\varepsilon = 1$, letting i go from 0 to $2n-2$ through even values, s^{ri} goes through all non-generators of $C_{2n}(s)$, and letting i go from 1 to $2n-1$ through odd values, s^{ri} goes through all generators of $C_{2n}(s)$. Thus in either case replacing r by another odd integer permutes the elements $\{d_{j1}, \ldots, d_{j,2n}\}$ among themselves, and does not affect the linear independence of $\{\bar{d}_{ji}\}$. Hence s^2 is marginal in the third place.

The final step is to check that

$$\forall g_1, g_2, g_3 \in G_n \quad \text{and} \quad \forall g \in G'_n, \quad \text{and for} \quad g = s^2,$$

$w_n(g_1 g, g_2, g_3) = w_n(g_1, g_2 g, g_3) = w_n(g_1, g_2, g_3)$. By the above calculation it is enough to consider $g_3 = st^\varepsilon$, $\varepsilon = 0$ or 1; also it is enough to prove that $w_n(g_1 g, g_2, g_3) = w_n(g_1, g_2, g_3)$. An easy calculation shows that

$$\begin{aligned} G'_n = \{&(d, d)(1, d')(f, f)(f', 1) \colon d, d' \in A_n, f, f' \in B_n, \\ &\bar{d}' = \sum_1^{2n} k_i \bar{a}_i \quad \text{where } \sum_i k_i \quad \text{is even, } \alpha(f') = 0\}\,(D'_n \times D'_n). \end{aligned}$$

It follows that multiplying g_1 by an element of G'_n multiplies $[g_1, g_2]$ by an element of the form $(d', d')(1, d'')(f', f'') \bmod D'_n \times D'_n$, where, if

$$\bar{d}' = \sum_i k'_i \bar{a}_i \quad \text{and} \quad \bar{d}'' = \sum_i k''_i \bar{a}_i, \sum_1^{2n} k'_i \equiv \sum_1^n k''_{2i} \equiv \sum_1^n k'_{2i-1} \equiv 0 \bmod 2$$

and $\alpha(f') = \alpha(f'') = 0$. It follows from (1) that the term (f', f'') can be ignored. With d_{ji} as above, let

$$\bar{d}_{11} = \sum_i l_i \bar{a}_i, \quad \text{and} \quad \bar{d}_{12} = \sum_i m_i \bar{a}_i.$$

(It is clear from (2) that Δ_1 depends only on $\bar{d}_{11}$ and $\bar{d}_{12}$.) Then it follows from (2) that either $\varepsilon = 0$ and $m_i = l_{i-1}$ for all i (suffices mod $2n$) or $\varepsilon = 1$ and $\bar{d}_{12} = \bar{d}_{21}$. Thus, multiplying g_1 by $(d', d'd'')$ replaces l_i by $l_i + k'_i$, and replaces m_i by $m_i + k'_{i-1}$ if $\varepsilon = 0$, or by $m_i + k'_{i-1} + k''_{i-1}$ if $\varepsilon = 1$. It will be shown in Proposition 4.11 that

$$\Delta_1 = \Delta(\bar{d}_{11}, \ldots, \bar{d}_{1,2n}) \equiv \left(\sum_1^n l_{2i}\right)\left(\sum_1^n m_{2i}\right) + \left(\sum_2^n l_{2i-1}\right)\left(\sum_1^n m_{2i-1}\right) \bmod 2.$$

It is clear from the conditions on $\{k'_i\}$ and $\{k''_i\}$ that of the four factors in this expression, either all or none is changed mod 2, so that Δ_1 remains unchanged. Similarly for Δ_2.

The effect of multiplying g_1 by s^2 will be to replace $[g_1, g_2]$ by $[g_1, g_2]^{s^2}[s^2, g_2]$. Let

$$[s^2, g_2] \equiv \left(\sum_1^{2n} l_i' a_i, \sum_1^{2n} m_i' a_i\right)(h, h') \bmod D_n' \times D_n', h, h' \in B_n.$$

Then

$$\sum_1^n l_{2i}' \equiv \sum_1^n m_{2i}' \equiv \sum_1^n l_{2i-1}' \equiv \sum_1^n m_{2i-1}' \equiv 0 \bmod 2$$

and $\alpha(h) = \alpha(h') = 0$. It is clear that in (1) α_1 and α_2 are unaltered. To evaluate the effect on Δ_1, notice that l_i is replaced by $l_{i-2}+l_i'$ and m_i by $m_{i-2}+l_{i-1}'$ if $\varepsilon=0$ and $m_{i-2}+m_{i-1}'$ if $\varepsilon=1$ for each i. Hence none of

$$\sum_1^n l_{2i-1}, \sum_1^n l_{2i}, \sum_1^n m_{2i}, \sum_1^n m_{2i-1}$$

is altered mod 2, and so Δ_1 is unchanged. Similarly for Δ_2, and, subject to Proposition 4.11 the theorem is proved.

The problem has been reduced to a question about determinants. Put $r=2n$ and let $x_i, y_i, i=1, \ldots, r$ be indeterminates over the field of $\mathbf{C}$ complex numbers. The assumption that n is a power of 2 will be held in abeyance. Let $\Delta(x, y)$ denote the determinant of

$$\begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_r \\ x_{r-1} & x_r & x_1 & \cdots & x_{r-2} \\ x_{r-3} & x_{r-2} & x_{r-1} & \cdots & x_{r-4} \\ \vdots & \vdots & \vdots & & \vdots \\ x_3 & x_4 & x_5 & \cdots & x_2 \\ y_r & y_1 & y_2 & \cdots & y_{r-1} \\ y_{r-2} & y_{r-1} & y_r & \cdots & y_{r-3} \\ y_{r-4} & y_{r-3} & y_{r-2} & \cdots & y_{r-5} \\ \vdots & \vdots & \vdots & & \vdots \\ y_2 & y_3 & y_4 & \cdots & y_1 \end{bmatrix}$$

Rearranging the rows so that x and y rows alternate it can be seen that this is equal (up to sign) to the block circulant

$$\begin{bmatrix} X_1 & X_2 & \cdots & X_n \\ X_n & X_1 & \cdots & X_{n-1} \\ \vdots & \vdots & & \vdots \\ X_2 & X_3 & \cdots & X_1 \end{bmatrix}$$

where

$$X_1 = \begin{pmatrix} x_1 & x_2 \\ y_r & y_1 \end{pmatrix}, \ldots, X_n = \begin{pmatrix} x_{r-1} & x_r \\ y_{r-2} & y_{r-1} \end{pmatrix}$$

Let α be a primitive n-th root of unity, then it follows from T. Muir [41] Chapter 12, § 514 that this can be expressed as

$$(-1)^{(n-1)(n-2)/2}\prod_{k=0}^{n-1}|X_1+\alpha^k X_2+\ldots+\alpha^{(n-1)k}X_n|.$$

Let ω be an r-th root of unity, and let

$$f_\omega(\underline{x},\underline{y})=\left(\sum_1^n x_{2i}\,\omega^{2i}\right)\left(\sum_1^n y_{2i}\,\omega^{2i}\right)-\left(\sum_1^n x_{2i-1}\,\omega^{2i-1}\right)\left(\sum_1^n y_{2i-1}\,\omega^{2i-1}\right).$$

LEMMA 4.9. *Let ξ be a primitive r-th root of unity, then*

$$\Delta(\underline{x},\underline{y})=\prod_0^{n-1} f_{\xi^k}(\underline{x},\underline{y}).$$

Proof. It is clear that the product has the same coefficient of $x_1^n y_1^n$ as $\Delta(\underline{x},\underline{y})$, so it remains to check that this product is equal to

$$\pm\prod_{k=0}^{n-1}|X_1+\alpha^k X_2+\ldots+\alpha^{(n-1)k}X_n|\quad\text{where } \alpha=\xi^2.$$

Now

$$\begin{aligned}
&\prod_0^{n-1}|X_1+\xi^{2k}X_2+\ldots+\xi^{2(n-1)k}X_n|\\
&\quad=\prod_0^{n-1}\begin{vmatrix} x_1+\xi^{2k}x_3+\ldots+\xi^{2(n-1)k}x_{r-1}, & x_2+\xi^{2k}x_4+\ldots+\xi^{2(n-1)k}x_r\\ y_r+\xi^{2k}y_2+\ldots+\xi^{2(n-1)k}y_{r-2}, & y_1+\xi^{2k}y_3+\ldots+\xi^{2(n-1)k}y_{r-1}\end{vmatrix}\\
&\quad=-\prod_0^{n-1}\frac{1}{\xi^2}\left[\left(\sum_1^n x_{2i}\,\xi^{2ik}\right)\left(\sum_1^n y_{2i}\,\xi^{2ik}\right)-\left(\sum_1^n x_{2i-1}\,\xi^{(2i-1)k}\right)\left(\sum_1^n y_{2i-1}\,\xi^{(2i-1)k}\right)\right]\\
&\quad=-\prod_0^{n-1} f_{\xi^k}(\underline{x},\underline{y}).
\end{aligned}$$

PROPOSITION 4.10. *If n is a power of 2,*

$$\Delta(\underline{x},\underline{y})\equiv\left(\sum_1^n x_{2i-1}\right)\left(\sum_1^n y_{2i-1}\right)+\left(\sum_1^n x_{2i}\right)\left(\sum_1^n y_{2i}\right)\bmod 2.$$

Proof. In the cyclotomic field $\mathbf{Q}(\xi)$, the prime ideal (2) ramifies and is the n-th power of the prime ideal $(1-\xi)$. Hence Δ is even if and only if $1-\xi$ divides $f_{\xi^i}(\underline{x},\underline{y})$ for some i, and this is true if and only if $f_{\xi^0}(\underline{x},\underline{y})$ is even. But

$$f_{\xi^0}(\underline{x},\underline{y})=\left(\sum_1^n x_{2i}\right)\left(\sum_1^n y_{2i}\right)-\left(\sum_1^n x_{2i-1}\right)\left(\sum_1^n y_{2i-1}\right)$$

and so the result follows.

CHAPTER II

§1. Homological machinery

The homology we shall employ is a special case of a number of theories described by various authors. Theories of Barr and Beck [6], and of Rinehart [33], are most suited to our needs; for a brief account of how these and other theories are to be adapted to the varietal situation, see [24, I] § 2.

If $\mathfrak{B}$ is a variety, and $\Pi \in \mathfrak{B}$, $(\mathfrak{B}, \Pi)$ denotes a category whose objects are the groups in $\mathfrak{B}$ supplied with a fixed surjection (usually suppressed) onto Π. In particular Π, as an object of $(\mathfrak{B}, \Pi)$, will be assumed to be supplied with the identity map. The morphisms of $(\mathfrak{B}, \Pi)$ are the group homomorphisms for which the obvious triangle commutes; two of these homomorphisms are already required to be surjections, if the third is as well, the morphism is a *surjection in* $(\mathfrak{B}, \Pi)$. A reason for using $(\mathfrak{B}, \Pi)$ rather than $\mathfrak{B}$ is that if B is a Π-module we wish to have a functor that takes G in $\mathfrak{B}$ to $\mathrm{Der}(G, B)$; but this is possible only if there is a fixed homomorphism of G into Π, so that B becomes a G-module. It is possible, and usual, to take the objects of $(\mathfrak{B}, \Pi)$ to be groups in $\mathfrak{B}$ with any fixed homomorphism into Π. The restriction to surjections produces occasional slight simplifications. Throughout this section, G will denote an object in $(\mathfrak{B}, \Pi)$.

(i) If $\mathcal{A}$ is an abelian category with enough projectives, and T: $(\mathfrak{B}, \Pi) \to \mathcal{A}$ is any functor, then for every integer $n \geq 0$ there is a functor $\mathfrak{B}_n(-, T)$ from $(\mathfrak{B}, \Pi)$ to $\mathcal{A}$, the n-th *derived functor* of T. The image of a morphism α under $\mathfrak{B}_n(-, T)$ will be denoted by α_n, and *not* by $\mathfrak{B}_n(\alpha, T)$; see (iii).

(ii) If G is a $\mathfrak{B}$-free group (or more generally a $\mathfrak{B}$-splitting group) then $\mathfrak{B}_n(G, T) = 0$ for $n > 0$, and $\mathfrak{B}_0(G, T) = T(G)$. (see [6], 4.4 and § 5 or [33], Proposition 2.7, Definition 2.6 and Proposition 2.4).

(iii) If α: $E \to G$ is a surjection in $(\mathfrak{B}, \Pi)$ there are objects $\mathfrak{B}_n(\alpha, T)$, and morphisms, in $\mathcal{A}$ that make a long exact sequence

$$\to \mathfrak{B}_n(\alpha, T) \to \mathfrak{B}_n(E, T) \xrightarrow{\alpha_n} \mathfrak{B}_n(G,T) \to \mathfrak{B}_{n-1}(\alpha, T) \to \dots \xrightarrow{\alpha_0} \mathfrak{B}_0(G, T) \to 0.$$

A commutative square

$$\begin{tikzcd} E_1 \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & G_1 \arrow[d, "\delta"] \\ E_2 \arrow[r, "\beta"] & G_2 \end{tikzcd}$$

in $(\mathfrak{V}, \Pi)$, with α and β surjections, gives rise to a commutative diagram

$$\begin{tikzcd}
\dots \arrow[r] & \mathfrak{V}_{n+1}(G_1, T) \arrow[r] \arrow[d, "\gamma_{n+1}"'] & \mathfrak{V}_n(\alpha, T) \arrow[r] \arrow[d] & \mathfrak{V}_n(E_1, T) \arrow[r] \arrow[d, "\delta_n"'] & \dots \\
\dots \arrow[r] & \mathfrak{V}_{n+1}(G_2, T) \arrow[r] & \mathfrak{V}_n(\beta, T) \arrow[r] & \mathfrak{V}_n(E_2, T) \arrow[r] & \dots .
\end{tikzcd}$$

In particular, the commutative square being a morphism of α into β, $\mathfrak{V}_n(-, T)$ defines a functor on the appropriate morphism category (see [6], Proposition 2.2, in which α is not required to be a surjection, thus introducing an extra term on the right; or [33], Theorem 2.18. That the exact sequences in these theories agree when α is a surjection is proved in [24, II] Remark 1.3).

(iv) The right Π-modules B for which $\Pi[B$ lies in $\mathfrak{V}$ are precisely the right $\mathfrak{V}\Pi$-modules, where $\mathfrak{V}\,\Pi$ is a certain quotient ring of $\mathbf{Z}\Pi$. (See Knopfmacher [22], or [24, I] § 1.) This is consistent with the terminology of § I.1. If P in $(\mathfrak{V}, \Pi)$ is $\mathfrak{V}$-freely generated by $\mathbf{y}$, then $IP \otimes_P \mathfrak{V}\Pi$ is freely generated, as right $\mathfrak{V}\Pi$-module, by $\{(1-y)\otimes 1;\ y \in \mathbf{y}\}$, ([24, I], Lemma 1.2). If $\mathfrak{V}$ contains $\mathfrak{A}$ var Π, then clearly $\mathfrak{V}\Pi = \mathbf{Z}\Pi$. Similar results hold for left modules; the same quotient ring $\mathfrak{V}\Pi$ occurs.

If A is a left $\mathfrak{V}\Pi$-module, and T: $(\mathfrak{V}, \Pi) \to \mathcal{Ab}$ is defined by $T(G) = IG \otimes_G A$, then $\mathfrak{V}_n(G, T)$ and $\mathfrak{V}_n(\alpha, T)$ will be denoted by $\mathfrak{V}_n(G, A)$ and $\mathfrak{V}_n(\alpha, A)$ respectively. If $A = \mathfrak{V}\Pi$, regarded as a left $\mathfrak{V}\Pi$-module, then T and its derived functors will be taken to have values in the category of right $\mathfrak{V}\Pi$-modules. If B is a right $\mathfrak{V}\Pi$-module, and S: $(\mathfrak{V}, \Pi) \to \mathcal{Ab}^{op}$ is defined by $S(G) = \mathrm{Der}(G, B)$, then $\mathfrak{V}_n(G, S)$ and $\mathfrak{V}_n(\alpha, S)$ will be denoted by $\mathfrak{V}^n(G, B)$ and $\mathfrak{V}^n(\alpha, B)$ respectively. These functors are all additive in the module.

The group Π is suppressed in the notation as it plays only a minor role. If $\Gamma \to \Pi$ is a fixed surjection, $G \in (\mathfrak{V}, \Gamma)$, and T: $(\mathfrak{V}, \Pi) \to \mathcal{A}$, then G defines an object G^* of $(\mathfrak{V}, \Pi)$, T defines a functor T^*: $(\mathfrak{V}, \Gamma) \to \mathcal{A}$, and $\mathfrak{V}_n(G^*, T)$ is naturally isomorphic to $\mathfrak{V}_n(G, T^*)$ for all $n \geqslant 0$. For example, writing G for G^*, $\mathfrak{V}_n(G, A)$ takes the same values if G is regarded as an object of $(\mathfrak{V}, \Pi)$ or of $(\mathfrak{V}, \Gamma)$. In particular, if G is fixed, we may take $\Pi = G$.

(v) A short exact sequence of left (right) $\mathfrak{V}\Pi$-modules gives rise to a long exact sequence in homology or cohomology as in the classical theory; however, since varietal homology does not in general vanish in positive dimensions on projective modules, and similarly for cohomology and injective modules, these exact sequences are of little use, beyond the simple fact that the (co-)homology groups are functorial in the module.

(vi) $\mathfrak{V}_0(G, A) = IG \otimes_G A$ and $\mathfrak{V}^0(G, B) = \mathrm{Der}(G, B)$. If α: $E \to G$ is a surjection in $(\mathfrak{V}, \Pi)$, then $\mathfrak{V}_0(\alpha, A) = (R/[R, S]) \otimes_\Pi A$ and $\mathfrak{V}^0(\alpha, B) = \mathrm{Hom}_\Pi(R/[R, S], B)$, where $R = \ker \alpha$, and $S = \ker(E \to \Pi)$; here E acts on the right on $R/[R, S]$ by conjugation, and this induces an action of Π via $E\to\Pi$. (The evaluation of $\mathfrak{B}_0(G, A)$ and $\mathfrak{B}^0(G, B)$ is easy in any appropriate theory; for a discussion of $\mathfrak{B}_0(\alpha, A)$ and $\mathfrak{B}^0(\alpha, B)$, see the remarks before [24, I] Theorem 2.1).

(vii) If $\mathfrak{B}=\mathfrak{O}$, the universal variety, then for all $n>0$, $\mathfrak{O}_n(G, A)=H_{n+1}(G, A)$ and $\mathfrak{O}^n(G, B)=H^{n+1}(G, B)$; see [5] or [33]. $H_n(G, A)$ is defined as $\mathrm{Tor}_n^{\mathbf{Z}G}(\mathbf{Z}, A)$, and so, using the exact sequence $0\to IG\to \mathbf{Z}G\to\mathbf{Z}\to 0$, $\mathrm{Tor}_n^{\mathbf{Z}G}(IG, A)=H_{n+1}(G, A)$ for $n>0$; thus $\mathfrak{O}_n(G, A)=\mathrm{Tor}_n^{\mathbf{Z}G}(IG, A)$ for $n\geqslant 0$, by (vi). Similarly $\mathfrak{O}^n(G, B)=\mathrm{Ext}^n_{\mathbf{Z}G}(IG, B)$ for $n\geqslant 0$.

It is a basic fact of the homology theories in question that they give rise to the usual Tor and Ext if $\mathfrak{B}$ is an abelian variety. For example, if $\mathfrak{B}=\mathfrak{A}$, $\mathfrak{B}_n(G, A)=\mathrm{Tor}_n(G, A)$ and $\mathfrak{B}^n(G, B)=\mathrm{Ext}^n(G, B)$; these vanish for $n>1$. If $\mathfrak{B}=\mathfrak{A}_m$, then G, A and B must be of exponent dividing m, and $\mathfrak{B}_n(G, A)=\mathrm{Tor}_n^{\mathbf{Z}_m}(G, A)$, $\mathfrak{B}^n(G, B)=\mathrm{Ext}^n_{\mathbf{Z}_m}(G, B)$. These vanish if $n>0$ and m is square free.

(viii) If $\mathfrak{W}$ is a variety containing $\mathfrak{B}$ there are "change of variety" homomorphisms $\varphi_n(\mathfrak{B}, \mathfrak{W}, G, A)$: $\mathfrak{W}_n(G, A)\to\mathfrak{B}_n(G, A)$ and $\varphi^n(\mathfrak{B}, \mathfrak{W}, G, B)$: $\mathfrak{B}^n(G, B)\to\mathfrak{W}^n(G, B)$. They are isomorphisms for $n=0$, and if $n=1$ they are surjections in homology and injections in cohomology; see [24, III], § 1 and Corollary 2.2. They are isomorphisms if $n=1$ and $\mathfrak{B}$ contains $\mathfrak{A}$ var G; see [25]. Similar results hold if G is replaced by a surjection, except that there is no analogue for the isomorphism obtained if $n=1$ and $\mathfrak{B}$ contains $\mathfrak{A}$ var G.

(ix) $\mathfrak{B}^1(G, B)$ classifies extensions $1\to B\to E\to G\to 1$ such that conjugation in E induces the given G-module structure on B, as in the classical theory, with the additional condition that E lies in $\mathfrak{B}$. If $1\to B\to E_i\to G\to 1$, $i=1, 2$, are extensions as above, so is their Baer sum $1\to B\to E_1 \overset{B}{\underset{G}{\times}} E_2\to G\to 1$. With this addition, the isomorphism classes of extensions form a group, with identity represented by the extension $1\to B\to G[B\to G\to 1$; and the correspondence between the set of isomorphism classes of extensions and $\mathfrak{B}^1(G, B)$ is a group isomorphism; see [7] or [33], § 3. This interpretation is easily seen to be respected by change of variety homomorphisms in the obvious sense; which explains why $\varphi^1(\mathfrak{B}, \mathfrak{W}, G, B)$ is an injection, and is an isomorphism if $\mathfrak{B}\supseteq\mathfrak{A}$ var G. The interpretation is also natural in G and B.

(x) If α: $D\to G$ is a surjection in $(\mathfrak{B}, \Pi)$ with kernel R, $\mathfrak{B}^1(\alpha, B)$ classifies extensions $1\to B\to K\to R\to 1$ where K is a $\mathfrak{B}D$-group as in § I.1, $B\to K$ and $K\to R$ are D-homomorphisms, and the action of K on itself by conjugation agrees with the action defined via $K\to R\to D$. Here D acts on B via $D\to G$, and on R by conjugation; note that B and R are both $\mathfrak{B}D$-groups, and that B is mapped into $\zeta(K)$. If $1\to B\to L\to R\to 1$ is another such extension it is equivalent to $1\to B\to K\to R\to 1$ if there is a D-isomorphism between K and L inducing the identity on B and R. The set of equivalence classes forms a group under Baer sum (with the obvious action of D on the middle term) isomorphic to $\mathfrak{B}^1(\alpha, B)$; the isomorphism is natural in $\mathfrak{B}$, α and B.

The homomorphism $\mathfrak{V}^1(D, B)\to\mathfrak{V}^1(\alpha, B)$ of (iii) corresponds to sending the extension $1\to B\to E\to D\to 1$ to the extension $1\to B\to K\to R\to 1$, where K is the kernel of the composite $E\to D\to G$. The action of D on K is induced by the action of E on K by conjugation; this is well defined since B centralizes K. See [33] § 4.

(xi) If $\mathfrak{W}\supseteq\mathfrak{V}$ there is an exact sequence

$$\mathfrak{W}_2(G, A)\xrightarrow{\varphi_2}\mathfrak{V}_2(G, A)\to\mathfrak{V}_0(G, \mathfrak{O}_1(-, A))\to\mathfrak{W}_1(G, A)\xrightarrow{\varphi_1}\mathfrak{V}_1(G, A)\to 0$$

where φ_1 and φ_2 are the change of variety homomorphisms of (viii); see [24, III]. If $\mathfrak{V}\supseteq\mathfrak{A}$. var G and $\mathfrak{W}=\mathfrak{O}$, φ_1 is an isomorphism, φ_2 is an injection, and the resulting short exact sequence $0\to\mathfrak{O}_2(G, A)\to\mathfrak{V}_2(G, A)\to\mathfrak{V}_0(G, \mathfrak{O}_1(-, A))\to 0$ splits; see [25]. Similar results, with the arrows reversed, hold in cohomology. In § II.3 we shall obtain a generalization of the cohomology exact sequence in which G is not required to lie in $\mathfrak{V}$.

The next result is needed to calculate a term in these exact sequences.

(xii) With T as in (i), $\mathfrak{V}_0(G, T)$ may be calculated as follows. Take a $\mathfrak{V}$-free (or $\mathfrak{V}$-splitting) group P_0, and a surjection $P_0\to G$ with kernel R. This supplies P_0 with a surjection onto Π. Take another $\mathfrak{V}$-free (or $\mathfrak{V}$-splitting) group Q, with a homomorphism into R whose image generates R as normal subgroup of P_0. Let P_1 be the verbal product of P_0 and Q. There are two surjections, d_0 and d_1, of P_1 onto P_0, defined as follows. Each maps P_0, as a verbal factor of P_1, identically onto P_0; d_0 maps Q into R by the given homomorphism and d_1 maps Q to the identity. The composites of d_0 and of d_1 with $P_0\to G$ give coincident surjections of P_1 onto G, and so, after a further composition, of P_0 onto Π; thus P_1 may be regarded as an element of $(\mathfrak{V}, \Pi)$. Then $\mathfrak{V}_0(G, T)=\operatorname{coker}(d_0T-d_1T)$. This is the beginning of the 'construction pas à pas' of André [1]; a slightly more general construction is given in the proof of [33], Proposition 2.4, see [ibid], Definition 2.6., which also arises from the construction of Tierney and Vogel [36].

(xiii) In conclusion, there is a number of obvious commutative diagrams, too large to enumerate. For example let $(\mathcal{V}, \mathfrak{U})$ be the category of varieties containing $\mathfrak{U}$, with morphisms the inclusions, and let $\mathcal{M}$ be the category of left $\mathfrak{U}\Pi$-modules. Then for every integer $n\geqslant 0$, $\mathfrak{V}_n(G, A)$ defines a functor from $(\mathcal{V}, \mathfrak{U})^{op}\times(\mathfrak{U}, \Pi)\times\mathcal{M}$ to $\mathcal{Ab}$. The homomorphism corresponding to a change of variety appears in (viii).

§ 2. An interpretation of $\mathfrak{V}^2(G, B)$

Let α: $D\to G$ be a surjection in $\mathfrak{V}$, and let B be a right $\mathfrak{V}G$-module. By splicing two short exact sequences, one sees from (x) that $\mathfrak{V}^1(\alpha, B)$ classifies exact sequences $1\to B\to K\to D\xrightarrow{\alpha}G\to 1$ of $\mathfrak{V}D$-groups, as defined after Lemma I. 1.9, where D acts on itself and on G via conjugation, the homomorphisms are all homomorphisms of D-groups, the action of

D on B is induced by the action of G on B via α, and the action of K on itself by conjugation agrees with the action defined via $K \to D$. This implies that B is embedded in $\zeta(K)$. If $1 \to B \to K_i \to D \xrightarrow{\alpha} G \to 1$, $i = 1, 2$, are two such sequences they are *equivalent*, and correspond to the same element of $\mathfrak{V}^1(\alpha, B)$, if and only if there is a D-isomorphism of K_1 onto K_2 giving rise to a commutative diagram,

$$\begin{tikzcd} 1 \arrow[r] & B \arrow[r] \arrow[d, "="] & K_1 \arrow[r] \arrow[d] & D \arrow[r, "\alpha"] \arrow[d, "="] & G \arrow[r] \arrow[d, "="] & 1 \\ 1 \arrow[r] & B \arrow[r] & K_2 \arrow[r] & D \arrow[r, "\alpha"] & G \arrow[r] & 1. \end{tikzcd}$$

The equivalence class containing a sequence S will be denoted by $[S]$. Let $\mathrm{Ext}^1(\mathfrak{V}, \alpha, B)$ be the set of equivalence classes of such sequences, so that there is a natural bijection between $\mathrm{Ext}^1(\mathfrak{V}, \alpha, B)$ and $\mathfrak{V}^1(\alpha, B)$. This becomes an isomorphism when the obvious addition is defined in $\mathrm{Ext}^1(\mathfrak{V}, \alpha, B)$, see (x). Information on the naturality of this isomorphism in α will be needed. If $S_1 \equiv 1 \to B \to K_1 \to D_1 \xrightarrow{\alpha} G \to 1$ defines an element of $\mathrm{Ext}^1(\mathfrak{V}, \alpha, B)$, and if θ: $D_2 \to D_1$ is a homomorphism, with $D_2 \in \mathfrak{V}$, such that $\theta\alpha = \beta$, say, is a surjection, then put $S_2 \equiv 1 \to B \to K_2 \to D_2 \xrightarrow{\beta} G \to 1$, where $K_2 = K_1 \underset{D_1}{\times} D_2$. The homomorphisms are the natural ones, and with the natural action of D_2 on K_2, S_2 clearly defines an element of $\mathrm{Ext}^1(\mathfrak{V}, \beta, B)$. Thus θ defines a map θ^*: $\mathrm{Ext}^1(\mathfrak{V}, \alpha, B) \to \mathrm{Ext}^1(\mathfrak{V}, \beta, B)$ sending S_1 to S_2. Also θ defines a morphism of β to α (put $\gamma = \theta$ and $\delta = 1_G$ in (iii), the roles of α and β are reversed), and hence θ induces a homomorphism θ^1: $\mathfrak{V}^1(\alpha, B) \to \mathfrak{V}^1(\beta, B)$; and if $s_1 \in \mathfrak{V}^1(\alpha, B)$, $s_2 \in \mathfrak{V}^1(\beta, B)$ correspond to S_1 and S_2 respectively, then $s_1\theta^1 = s_2$. The more general case in which G is allowed to vary is similar, but will not be needed.

Put $C^2(\mathfrak{V}, G, B) = \bigcup_\alpha \mathrm{Ext}^1(\mathfrak{V}, \alpha, B)$, the union being taken over all surjections α in $\mathfrak{V}$ with image G. Working within a Grothendieck universe, $C^2(\mathfrak{V}, G, B)$ becomes a set. If

$$S_i \equiv 1 \to B \to K_i \to D_i \to G \to 1,\ [S_i] \in C^2(\mathfrak{V}, G, B),\ i = 1, 2,$$

let $[S_1] + [S_2] = [S]$, where $S \equiv 1 \to B \to K_1 \overset{B}{\times} K_2 \to D_1 \underset{G}{\times} D_2 \to G \to 1$; the action of $D_1 \underset{G}{\times} D_2$ on $K_1 \underset{B}{\times} K_2$ is the natural one. It is easy to see that addition is well defined, and makes $C^2(\mathfrak{V}, G, B)$ a commutative monoid, with identity the sequence $1 \to B \xrightarrow{=} B \to G \xrightarrow{=} G \to 1$, where $B \to G$ is the trivial map. This addition will become compatible with the addition already defined in $\mathrm{Ext}^1(\mathfrak{V}, \alpha, B)$ on the introduction of the equivalence relation of similarity below. (The proof of this statement is contained in the proof of Theorem 2.1.).

If $S \equiv 1 \to B \to K \to D \to G \to 1$ defines an element of $C^2(\mathfrak{V}, G, B)$, say that S and $[S]$ are *null* if the commutative diagram

$$\begin{tikzcd} & 1 \arrow[r] & K \arrow[r] \arrow[d, "="] & E \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, "="] & 1 \\ 1 \arrow[r] & B \arrow[r] & K \arrow[r] & D \arrow[r] & G \arrow[r] & 1 \end{tikzcd}$$

with exact rows can be completed, with $E \in \mathfrak{V}$, in such a way that the action of E on K by conjugation agrees with the action via $E \to D$. It is easy to see that the null elements of $C^2(\mathfrak{V}, G, B)$ form a submonoid. Say that two elements of $C^2(\mathfrak{V}, G, B)$ are *similar* if they become equal on adding to each a (possibly different) null element. This is clearly an equivalence relation that respects addition. Let $\mathrm{Ext}^2(\mathfrak{V}, G, B)$ denote the resulting quotient monoid.

THEOREM 2.1. $\mathrm{Ext}^2(\mathfrak{V}, G, B)$ *is an abelian group, naturally isomorphic to* $\mathfrak{V}^2(G, B)$.

Proof. Define $\Theta: C^2(\mathfrak{V}, G, B) \to \mathfrak{V}^2(G, B)$ as follows. If $[S] \in \mathrm{Ext}^1(\mathfrak{V}, \alpha, B)$ corresponds to $s \in \mathfrak{V}^1(\alpha, B)$, let $[S]\Theta$ be the image of s under $\mathfrak{V}^1(\alpha, B) \to \mathfrak{V}^2(G, B)$ as in (iii). Then, in particular, for a homomorphism α with domain a $\mathfrak{V}$-free group P (and such an α always exists), since $\mathfrak{V}^2(P, B) = 0$, the exactness of $\mathfrak{V}^1(\alpha, B) \to \mathfrak{V}^2(G, B) \to \mathfrak{V}^2(P, B)$ shows that Θ is onto. If the domain of α is an arbitrary group D in $\mathfrak{V}$, the exactness of $\mathfrak{V}^1(D, B) \to \mathfrak{V}^1(\alpha, B) \to \mathfrak{V}^2(G, B)$, together with the last part of (x), shows that $[S]\Theta = 0$ if and only if S is null. It remains to check that Θ is additive. Let $S_i \equiv 1 \to B \to K_i \to D_i \xrightarrow{\alpha_i} G \to 1$ define an element of $\mathrm{Ext}^1(\mathfrak{V}, \alpha_i, B)$, s_i be the corresponding element of $\mathfrak{V}^1(\alpha_i, B)$, and t_i be the image of s_i in $\mathfrak{V}^1(\alpha, B)$, $i = 1, 2$; here $\alpha: D_1 \times_G D_2 \to G$ is the canonical surjection, and the morphism of α to α_i is given by the projection $\theta_i: D_1 \times_G D_2 \to D_i$. Let $t \in \mathfrak{V}^1(\alpha, B)$ correspond to $[S_1] + [S_2]$. Since the composite $\mathfrak{V}^1(\alpha_i, B) \to \mathfrak{V}^1(\alpha, B) \to \mathfrak{V}^2(G, B)$ coincides with the homomorphism $\mathfrak{V}^1(\alpha_i, B) \to \mathfrak{V}^2(G, B)$, see (iii), it is enough to check that $t_1 + t_2 = t$. Now, if $R_j = \ker \alpha_j$, t_i corresponds to $\mathcal{T}_i \equiv 1 \to B \to K_i \times R_j \to D_1 \times_G D_2 \to G \to 1$, $j = 3 - i$, where the homomorphisms and the action of $D_1 \times_G D_2$ on $K_i \times R_j$ are the obvious ones; see the discussion of the functoriality of $\mathrm{Ext}^1(\mathfrak{V}, \alpha, B)$ in α before the statement of the theorem. So $t_1 + t_2$ corresponds to $\mathcal{T}$, where $\mathcal{T}$ is obtained by adding $\mathcal{T}_1$ and $\mathcal{T}_2$ as elements of $\mathrm{Ext}^1(\mathfrak{V}, \alpha, B)$; see (x). It is easy to see that $\mathcal{T} \equiv 1 \to B \to K_1 \overset{B}{\times} K_2 \to D_1 \times_G D_2 \to G \to 1$, again with the obvious maps and action; but $[\mathcal{T}] = [S_1] + [S_2]$. This completes the proof that Θ induces an isomorphism between $\mathrm{Ext}^2(\mathfrak{V}, G, B)$ and $\mathfrak{V}^2(G, B)$. It is easy to see how $\mathrm{Ext}^2(\mathfrak{V}, G, B)$ becomes a functor of $\mathfrak{V}$, G and B (varying G requires the introduction of a 'base' group Π), and the naturality of the isomorphism is then routine.

Note. If $S_i \equiv 1 \to B \to K_i \to D_i \to G \to 1$ defines an element of $C^2(\mathfrak{V}, G, B)$, $i = 1, 2$, $[S_1]$ is *related* to $[S_2]$ if there is a commutative diagram

$$\begin{array}{ccccccccc}
1 & \to & B & \to & K_1 & \to & D_1 & \to & G & \to & 1 \\
 & & \downarrow{=} & & \downarrow & & \downarrow & & \downarrow{=} & & \\
1 & \to & B & \to & K_2 & \to & D_2 & \to & G & \to & 1,
\end{array}$$

where $K_1 \to K_2$ is a D_1-homomorphism, D_1 acting on K_2 via $D_1 \to D_2$. It is easy to see, as in Gerstenhaber [11], that similarity is the finest equivalence relation on $C^2(\mathfrak{V}, G, B)$ with the property that related elements are equivalent.

The second cohomology group can also be interpreted in terms of obstructions. This was first done for the variety of all groups by Eilenberg and MacLane in [9]; for a treatment based on the Gruenberg resolution, see [12]. A triple-theoretic technique dealing with commutative algebras was produced by Barr in [4], and generalized by Grace Orzech in [32]. Her results include the case of varieties of groups.

Let $1 \to K \to E \to G \to 1$ be exact and let $B = \zeta(K)$. Then E acts by conjugation on K, and this gives rise to a homomorphism θ: $G \to \operatorname{Out} K$, the group of outer automorphisms of K. A G-module structure is induced on B by θ. Conversely, given θ: $G \to \operatorname{Out} K$, θ makes $B = \zeta(K)$ into a G-module, and obstruction theory asks whether θ arises as above from an extension, and if so, in how many ways. The solution is obtained by associating with every such θ an element of $H^3(G, B) = \mathfrak{O}^2(G, B)$ in such a way that those arising from extensions correspond to 0. If θ does arise from such an extension, then $\mathfrak{O}^1(G, B)$ acts naturally, faithfully, and transitively on the set of all such extensions. Moreover, every element of $\mathfrak{O}^2(G, B)$ does arise from some θ.

Suppose that in the above situation $E \in \mathfrak{V}$, $\mathfrak{V}$ some variety. Let $N = \operatorname{im} \theta$, and M be the inverse image of N in Aut K. It is easy to see that K is a $\mathfrak{V}M$-group, and this makes B a $\mathfrak{V}G$-module. If θ: $G \to \operatorname{Out} K$ satisfies this condition, and $G \in \mathfrak{V}$, call θ a $\mathfrak{V}-G-B$-*core*. Orzech's obstruction theory then associates with every $\mathfrak{V}-G-B$-core an element of $\mathfrak{V}^2(G, B)$, those that arise from extensions are those that are associated with 0, and given a fixed $\mathfrak{V}-G-B$-core associated with 0, $\mathfrak{V}^1(G, B)$ acts naturally, faithfully and transitively on the set of extensions giving rise to the given core. Introducing an equivalence relation on the set of $\mathfrak{V}-G-B$-cores they can be made into a group Obs$(\mathfrak{V}, G, B)$, after the style of [9], so that the above theory gives rise to an injection of Obs$(\mathfrak{V}, G, B)$ into $\mathfrak{V}^2(G, B)$, natural in $\mathfrak{V}$ and G; this injection is an isomorphism if the $\mathfrak{V}$-free groups of large enough rank have trivial centres. This condition on the centres cannot be dispensed with; for example it is clear that Obs$(\mathfrak{V}, G, B)$ is trivial if $\mathfrak{V}$ is an abelian variety, whereas the only abelian varieties $\mathfrak{V}$ for which $\mathfrak{V}^2(G, B)$ is always trivial are those of exponent m, where m is 0 or a positive square-free integer.

The requirement that B be the whole of the centre of K has two disadvantages. Firstly, it means that Obs$(\mathfrak{V}, G, B)$ is not functorial in B, and secondly that some elements of $\mathfrak{V}^2(G, B)$ may not arise as 'obstructions'. We now indicate how a theory without these drawbacks can be constructed, see Gerstenhaber [11].

Given θ, $G \in \mathfrak{V}$, B, K, M and N as above, there is an exact sequence $S_\theta \equiv 1 \to B \to K \to D \to$

$G \to 1$, where $D = M \times_N G$, $K \to D$ is induced by the homomorphism $K \to M$ given by the action of K on itself by conjugation, and $D \to G$ is the canonical projection. If D acts on K via the projection $D \to M$, S_θ defines an element of $C^2(\mathfrak{V}, G, B)$. The $\mathfrak{V}-G-B$-core defined by θ arises from the extension $1 \to K \to E \to G \to 1$ if and only if the commutative diagram

$$\begin{tikzcd}[column sep=small] & 1 \arrow[r] & K \arrow[r] \arrow[d, "="] & E \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, "="] & 1 \\ 1 \arrow[r] & B \arrow[r] & K \arrow[r] & D \arrow[r] & G \arrow[r] & 1 \end{tikzcd} \tag{3}$$

can be completed in such a way that the action of E on K by conjugation agrees with the action via $E \to D$.

Now call a commutative diagram

$$\begin{tikzcd}[column sep=small] 1 \arrow[r] & B \arrow[r] \arrow[d, "\iota"] & K \arrow[r] \arrow[d, "="] & L \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, "="] & 1 \\ 1 \arrow[r] & \zeta(K) \arrow[r] & K \arrow[r] & D \arrow[r] & G \arrow[r] & 1, \end{tikzcd}$$

where B is now just a G-submodule of $\zeta(K)$, the rows define elements of $C^2(\mathfrak{V}, G, B)$, ι is the inclusion, and the given action of L on K agrees with the action via $L \to D$, a *relative* $\mathfrak{V}-G-B$-core. Thus a relative $\mathfrak{V}-G-B$-core may be throught of as a partially solved extension problem. Call an exact sequence $1 \to K \to E \to G \to 1$ that fits into the commutative diagram (3), with D replaced by L, and the same condition on the action of E on K, a *solution* to the above relative $\mathfrak{V}-G-B$-core.

The connection with cohomology is now immediate. A relative $\mathfrak{V}-G-B$-core as above is determined by its top row, and so there is a bijection between $C^2(\mathfrak{V}, G, B)$ and the set of (isomorphism classes of) relative $\mathfrak{V}-G-B$-cores, in which the cores with a solution correspond to null sequences. This gives rise to a map of the set of relative $\mathfrak{V}-G-B$-cores onto $\mathfrak{V}^2(G, B)$ in which the cores with a solution are those that map to 0. Moreover the map is natural in $\mathfrak{V}$, G and B. Finally it can be shown, as in [11], that if a relative $\mathfrak{V}-G-B$-core has a solution, $\mathfrak{V}^1(G, B)$ acts naturally, faithfully, and transitively on the set of solutions. If $1 \to B \to T \to G \to 1$ corresponds to an element of $\mathfrak{V}^1(G, B)$, it acts by sending $1 \to K \to E \to G \to 1$ to $1 \to K \to E \overset{B}{\underset{G}{\times}} T \to G \to 1$.

§ 3. An exact sequence

The object of this paragraph is to establish, for any variety $\mathfrak{V}$, group G, and $\mathfrak{V}G$-module B, an exact sequence

$$0 \to \mathfrak{V}^1(G/V(G), B) \to \mathfrak{O}^1(G, B) \to \mathrm{Hom}_G(\mathfrak{V}M(G), B) \to \mathfrak{V}^2(G/V(G), B) \to \mathfrak{O}^2(G, B).$$

Since B is a $\mathfrak{V}G$-module, it may also be regarded as a $\mathfrak{V}(G/V(G))$-module by Lemma I. 1.13. It will be shown in the next paragraph that this exact sequence coincides with the cohomology sequence in (xi) provided that $G \in \mathfrak{V}$. The proof will depend entirely on the interpretations of the groups in the sequence in terms of extensions. It is easy to construct a long exact sequence connecting $\mathfrak{V}^n(G/V(G), B)$, $\mathfrak{O}^n(G, B)$, and a 'mystery' term in each dimension, by mapping a simplicial resolution of G in $\mathfrak{O}$ onto a simplicial resolution of $G/V(G)$ in $\mathfrak{V}$, applying $\mathrm{Der}(-, B)$, and forming the long exact cohomology sequence of the resulting short exact sequence of complexes. However, the proof that the long exact sequence extends the above sequence would not be instructive. An interesting homological approach requires a homological interpretation of $\mathfrak{V}M(G)$; for $G \in \mathfrak{V}$ there is a simple interpretation in terms of triple homology, see § 4. For arbitrary G it might be possible, following Fröhlich [10], to regard $\mathfrak{V}M(-)$ as the first derived functor of the functor from $\mathfrak{O}$ to $\mathfrak{V}$ taking G to $G/V(G)$, and then to use, for example, the homotopical theory of Keune [21]. This certainly gives the correct answer if $\mathfrak{V}$ is an abelian variety. The paragraph ends with a discussion of the significance of the exact sequence in the classification of the groups in an isologism class. If $\mathfrak{V} = \mathfrak{A}_p$, this gives a simple algorithm for constructing all p-groups. Finally, Lue in [27] constructs, by homological means, a long exact sequence that agrees with the above as far as the term $\mathfrak{V}^2(G/V(G), B)$.

THEOREM 3.1. *If $\mathfrak{V}$ is a variety, G a group, and B a $\mathfrak{V}G$-module, there is an exact sequence*

$$0 \to \mathfrak{V}^1(G/V(G), B) \to \mathfrak{O}^1(G, B) \to \mathrm{Hom}_G(\mathfrak{V}M(G), B) \to \mathfrak{V}^2(G/V(G), B) \to \mathfrak{O}^2(G, B).$$

Here $\mathfrak{V}^i(G/V(G), B) \to \mathfrak{O}^i(G, B)$ *is the composite of the change of variety homomorphism* $\mathfrak{V}^i(G/V(G), B) \to \mathfrak{O}^i(G/V(G), B)$ *of* (viii) *with the canonical map* $\mathfrak{O}^i(G/V(G), B) \to \mathfrak{O}^i(G, B)$.

Proof. Let $1 \to B \to E_1 \to G/V(G) \to 1$ correspond to an element of $\ker(\mathfrak{V}^1(G/V(G), B) \to \mathfrak{O}^1(G, B))$. This gives rise to a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & B \arrow[r] \arrow[d, "="] & E_2 \arrow[r] \arrow[d, "\gamma"] & G \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & B \arrow[r] & E_1 \arrow[r] & G/V(G) \arrow[r] & 1 \end{tikzcd}$$

with $E_1 \in \mathfrak{V}$ whose top row splits. If $\beta\colon G \to E_2$ splits the top row, then since $E_1 \in \mathfrak{V}$, $\beta\gamma$ splits the bottom row. This gives exactness at $\mathfrak{V}^1(G/V(G), B)$.

To define $\mathfrak{O}^1(G, B) \to \mathrm{Hom}_G(\mathfrak{V}M(G), B)$, identify $\mathrm{Hom}_G(\mathfrak{V}M(G), B)$ with $I(\mathfrak{V}, G, B)$, as in § I.2. If $1 \to B \to E \to G \to 1$, $1_B : B \to B$ defines an element of $I(\mathfrak{V}, G, B)$, and this gives the required map; it is clearly a homomorphism. Now it is easy to see that the image of $\mathfrak{V}^1(G/V(G), B)$ in $\mathfrak{O}^1(G, B)$ consists of those elements that are represented by extensions $1 \to B \to E \to G \to 1$ such that $V(E) \cap B = 1$, and this is the kernel of $\mathfrak{O}^1(G, B) \to \mathrm{Hom}_G(\mathfrak{V}M(G), B)$.

To define $\mathrm{Hom}_G(\mathfrak{V}M(G), B) \to \mathfrak{V}^2(G/V(G), B)$, equating $\mathrm{Hom}_G(\mathfrak{V}M(G), B)$ with $I(\mathfrak{V}, G, B)$ as above, and $\mathfrak{V}^2(G/V(G), B)$ with $\mathrm{Ext}^2(\mathfrak{V}, G/V(G), B)$ as in § 2, start by defining Ψ from the set $E(\mathfrak{V}, G, B)$ of $\mathfrak{V}-G-B$-extensions to $\mathrm{Ext}^2(\mathfrak{V}, G/V(G), B)$ as follows. If $\mathcal{E} \equiv 1 \to K \to E \to G \to 1$, ι: $B \to \zeta(K)$ is a $\mathfrak{V}-G-B$-extension, let $\mathcal{E}\Psi$ be the similarity class containing $1 \to B \to K \to D/V(D) \to G/V(G) \to 1$, where $D = E/B$. Note that, since K is a $\mathfrak{V}E$-group centralized by B, K is a $\mathfrak{V}D$-group, and hence a $\mathfrak{V}(D/V(D))$-group; see Lemmas I. 1.11 and I. 1.13. Regarding $E(\mathfrak{V}, G, B)$ as a monoid, as in § 1.2, Ψ is clearly a homomorphism, and if $V(E) \cap K = 1$, the commutative diagram

$$\begin{array}{ccccccccc}
1 \to & K & \to & E/V(E) & \to & G/V(G) & \to 1 \\
 & \downarrow{=} & & \downarrow & & \downarrow{=} & \\
1 \to B \to & K & \to & D/V(D) & \to & G/V(G) & \to 1
\end{array}$$

shows that $\mathcal{E}\Psi$ is a null sequence. Thus Ψ induces a homomorphism of $I(\mathfrak{V}, G, B)$ into $\mathrm{Ext}^2(\mathfrak{V}, G/V(G), B)$ and hence of $\mathrm{Hom}_G(\mathfrak{V}M(G), B)$ into $\mathfrak{V}^2(G/V(G), B)$.

It is clear that the composite $\mathfrak{O}^1(G, B) \to \mathrm{Hom}_G(\mathfrak{V}M(G), B) \to \mathfrak{V}^2(G/V(G), B)$ is zero, but the exactness at this point is harder. If $1 \to R \to F \to G \to 1$ is a presentation of G, and $\alpha \in \mathrm{Hom}_G(\mathfrak{V}M(G), B)$, it follows from the proof of Theorem I.2.1 that α corresponds to an element of $I(\mathfrak{V}, G, B)$ containing a $\mathfrak{V}-G-B$-extension $\mathcal{E}_\alpha \equiv 1 \to K \to E \to G \to 1$, ι: $B \to \zeta(K)$ where E is obtained from $(F/[RV^*F])$ [B by amalgamating $R \cap V(F)/[RV^*(F)]$ with its image under α in B. Then $\mathcal{E}_\alpha$ fits into the commutative diagram

$$\begin{array}{ccccccc}
 & 1 & & 1 & & 1 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
1 \longrightarrow & B & \to & K & \to & R/R \cap V(G) & \to 1 \\
 & \downarrow & & \downarrow & & \downarrow & \\
1 \longrightarrow & X & \to & E & \longrightarrow & F/V(F) & \longrightarrow 1 \\
 & \downarrow & & \downarrow & & \downarrow & \\
1 \to & V(G) & \to & G & \longrightarrow & G/V(G) & \longrightarrow 1 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 1 & & 1 & & 1 &
\end{array} \tag{4}$$

with exact rows and columns, and the image of $[\mathcal{E}_\alpha]$ in $\mathrm{Ext}^2(\mathfrak{V}, G/V(G), B)$ is $1 \to B \to K \to$

$F/V(F) \to G/V(G) \to 1$. It is clear that if the image of $[\mathcal{E}_\alpha]$ is 0, a commutative diagram

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & B \arrow[r] \arrow[d, "="] & K \arrow[r] \arrow[d] & R/R \cap V(F) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & B \arrow[r] & T \arrow[r] \arrow[d] & F/V(F) \arrow[r] \arrow[d] & 1 \\
 & & G/V(G) \arrow[r, "="'] \arrow[d] & G/V(G) \arrow[d] & \\
 & & 1 & 1 &
\end{tikzcd} \tag{5}$$

with exact rows and columns and with $T \in \mathfrak{B}$ can be constructed. (This is just a rearrangement of the diagram used to define a null sequence in $C^2(\mathfrak{B}, G, B)$.) The actions of $F/V(F)$ on K defined via $E \to F/V(F)$ and $T \to F/V(F)$ are required to agree. We now construct a $\mathfrak{B}-G-B$-extension $\mathcal{E}$ such that $\mathcal{E}$ is in the image of $\mathfrak{O}^1(G, B)$ and $[\mathcal{E}_\alpha] = [\mathcal{E}]$. Since $T \in \mathfrak{B}$ the middle row of (5) splits; and hence the top row splits, and $K = S \times B$ for some subgroup S of K, since B is central in K. Regarding S as a subgroup of E, since X centralizes K, S is normal in E. Thus there is an exact sequence $1 \to B \to E/S \to G \to 1$ which defines an element of $\mathfrak{O}^1(G, B)$. Then the $\mathfrak{B}-G-B$-extension $\mathcal{E} \equiv 1 \to B \to E/S \to G \to 1$, $1_B\ B \to B$ is its image in $I(\mathfrak{B}, G, B)$, and clearly $[\mathcal{E}]\Theta = \alpha$ (in the notation of Theorem I.2.1), and so $[\mathcal{E}] = [\mathcal{E}_\alpha]$ as required.

Let $1 \to B \to K \to D_1 \to G/V(G) \to 1$ represent an element of $\mathrm{Ext}^2(\mathfrak{B}, G/V(G), B)$, with a suitable action of D_1 on K. Its image in $\mathrm{Ext}^2(\mathfrak{O}, G, B)$ is represented by $1 \to B \to K \to D_2 \to G \to 1$, where there is a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & B \arrow[r] \arrow[d, "="] & K \arrow[r] \arrow[d, "="] & D_2 \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & B \arrow[r] & K \arrow[r] & D_1 \arrow[r] & G/V(G) \arrow[r] & 1.
\end{tikzcd} \tag{6}$$

Clearly $D_1 \cong D_2/V(D_2)$. The top row is a null sequence if and only if (6) can be extended to form a commutative diagram

$$\begin{tikzcd}
 & 1 \arrow[r] & K \arrow[r] \arrow[d, "="] & E \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & B \arrow[r] \arrow[d, "="] & K \arrow[r] \arrow[d, "="] & D_2 \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & B \arrow[r] & K \arrow[r] & D_1 \arrow[r] & G/V(G) \arrow[r] & 1
\end{tikzcd}$$

where E acts in the appropriate way on K. But this is just the condition that the given element of $\mathrm{Ext}^2(\mathfrak{B}, G/V(G), B)$ be the image of the element of $I(\mathfrak{B}, G, B)$ defined by $1\to K\to E\to G\to 1$, $B\to\zeta(K)$. This gives exactness at $\mathfrak{B}^2(G/V(G), B)$, and completes the proof of the theorem.

Remark. The exact sequence of the theorem is natural in $\mathfrak{B}$, G and B. Moreover, if $\mathfrak{W}$ is any variety containing $\mathfrak{B}$ and G, the proof of the theorem gives, without significant adjustment, an exact sequence

$$1\to\mathfrak{B}^1(G/V(G), B)\to\mathfrak{W}^1(G, B)\to\mathrm{Hom}_G(\mathfrak{W}\mathfrak{B}M(G), B)\to\mathfrak{B}^2(G/V(G), B)\to\mathfrak{W}^2(G, B)$$

where $\mathfrak{W}\mathfrak{B}M(G)$ is the relative Baer-invariant defined after Lemma I.1.8.

We return to the problem of isologism classes, as in § I.2; or more precisely, the construction of groups within an isologism class, aiming at Theorem 3.2 below.

Consider first the case $\mathfrak{B}=\mathfrak{A}$. Then, as in (vii), $\mathfrak{B}^2(G/V(G), B)=\mathrm{Ext}^2(G/G', B)=0$. Thus the exact sequence becomes, in the classical notation,

$$0\to\mathrm{Ext}(G/G', B)\to H^2(G, B)\to\mathrm{Hom}(M, B)\to 0$$

where M is the Schur multiplier of G, and G acts trivially on B (and on M). This sequence is well known. If $B=M$, and $1\to M\to E\to G\to 1$ represents an element of $H^2(G, B)$ that maps to 1_M, E is a covering group of G. The sequence gives the familiar fact that G has at most $|\mathrm{Ext}(G/G', B)|$ non-isomorphic covering groups. (At most, since non-isomorphic extensions may have isomorphic terms.)

Returning to an arbitrary variety, if $\alpha\in\mathrm{Hom}_G(\mathfrak{B}M(G), B)$, the corresponding weak $\mathfrak{B}$–G–B-isologism class has a representative of the form $1\to B\to E\to G\to 1$, 1_B, if and only if its image in $\mathfrak{B}^2(G/V(G), B)$ is zero. If this is not the case, consider a representative $1\to K\to E\to G\to 1$, ι: $B\to\zeta(K)$. Regarding B as a subgroup of E, put $D=E/B$, so that G is a quotient of D.

The kernel of the natural surjection of D onto $D/V(D)\underset{G/V(G)}{\times}G$ is $V(D)\cap K/B$, which is trivial since $B\supseteq V(E)\cap K$ by assumption; thus $D\cong D/V(D)\underset{G/V(G)}{\times}G$. Using the fact that the exact sequence is natural in the group, let β be the image of α under the natural map $\mathrm{Hom}_G(\mathfrak{B}M(G), B)\to\mathrm{Hom}_D(\mathfrak{B}M(D), B)$. It is easy to see that β is the image of the element of $\mathfrak{O}^1(D, B)$ defined by the extension $1\to B\to E\to D\to 1$, and so maps to 0 in $\mathfrak{B}^2(D/V(D), B)$.

Conversely, if $\tau\in\mathfrak{B}^2(G/V(G), B)$ is the image of α, let $S\to G/V(G)$ be a surjection, with $S\in\mathfrak{B}$ such that the image of τ in $\mathfrak{B}^2(S, B)$ is 0. Such a surjection always exists; for example S may be taken to be a $\mathfrak{B}$-free group.

Put $D=S\underset{G/V(G)}{\times}G$: then if L is the kernel of $S\to G/V(G)$, D is an extension of L by G; $L\subseteq V^*(D)$, regarding L as a subgroup of D; and $D/V(D)\cong S$. Let β be the image of α under the natural map $\mathrm{Hom}_G(\mathfrak{B}M(G), B)\to\mathrm{Hom}_D(\mathfrak{B}M(D), B)$. Then the image of β in

$\mathfrak{V}^2(D/V(D), B) \cong \mathfrak{V}^2(S, B)$ is zero. Hence β is the image of some element of $\mathfrak{O}^1(D, B)$, represented by $1 \to B \to E \to D \to 1$, say. If K is the kernel of the composite $E \to D \to G$, this gives rise to an extension $1 \to K \to E \to G \to 1$, and an injection $\iota\colon B \to \zeta(K)$. We claim that this is a $\mathfrak{V}-G-B$-extension corresponding to α. Since the image of K in D is clearly contained in $V^*(D)$, K/B is $\mathfrak{V}$-marginal in E/B. The fact that $1 \to B \to E \to D \to 1$ gives rise to β now shows, after an easy argument, that K is marginal in E. Since $V(E) \cap K$ is clearly mapped to 1 in D, it follows that $B \supseteq V(E) \cap K$. Thus $\mathcal{E} \equiv 1 \to K \to E \to G \to 1$, $\iota\colon B \to \zeta(K)$ is a $\mathfrak{V}-G-B$-extension. Let $\alpha' \in \mathrm{Hom}_G(\mathfrak{V}M(G), B)$ be the corresponding homomorphism. It remains to check that $\alpha' = \alpha$. Since α and α' are both mapped to β under the natural map $\mathrm{Hom}_G(\mathfrak{V}M(G), B) \to \mathrm{Hom}_D(\mathfrak{V}M(D), B)$, it is enough to show that this map is an injection, or that $\mathfrak{V}M(D) \to \mathfrak{V}M(G)$ is a surjection. Now by Theorem 3.2 of Fröhlich [10], if $1 \to L \to D \to G \to 1$ is an extension with $L \subseteq V^*(D)$, there is an exact sequence $\mathfrak{V}M(D) \to \mathfrak{V}M(G) \to L \to D/V(D) \to G/V(G) \to 1$, where the maps are the natural ones. These conditions are satisfied here, and $L \to D/V(D)$ is an injection, so $\mathfrak{V}M(D) \to \mathfrak{V}M(G)$ is a surjection, as required. Thus we have arrived at an element $\mathcal{E}$ of the weak $\mathfrak{V}-G-B$-isologism class corresponding to α and every element of the class arises in this way. Note that the number of isomorphism classes of E corresponding to a given S is at most $|\mathfrak{V}^1(S, B)|$. In the special case in which α corresponds to a $\mathfrak{V}$-isologism class, we have

THEOREM 3.2. *Given any $\mathfrak{V}$-isologism class with marginal factor G, let $\alpha\colon \mathfrak{V}M(G) \to B$ be the corresponding surjection. Let $\tau \in \mathfrak{V}^2(G/V(G), B)$ be the image of α under the homomorphism $\mathrm{Hom}_G(\mathfrak{V}M(G), B) \to \mathfrak{V}^2(G/V(G), B)$ of Theorem 3.1. Let $S \to G/V(G)$ be a surjection, with $S \in \mathfrak{V}$ such that the image of τ in $\mathfrak{V}^2(S, B)$ is 0. Put $D = S \times_{G/V(G)} G$, and let $\beta \in \mathrm{Hom}_D(\mathfrak{V}M(D), B)$ correspond to α. Let $1 \to B \to E \to D \to 1$ correspond to an element of $\mathfrak{O}^1(D, B)$ that maps to β under the homomorphism $\mathfrak{O}^1(D, B) \to \mathrm{Hom}_D(\mathfrak{V}M(D), B)$ of Theorem 3.1. Then E belongs to the given $\mathfrak{V}$-isologism class, every group in the $\mathfrak{V}$-isologism class arises in this way, and for a given choice of S there are at least one and at most $|\mathfrak{V}^1(G, B)|$ possible isomorphism classes for E.*

Sometimes a weak $\mathfrak{V}-G-B$-isologism class will contain extensions $1 \to K \to E \to G \to 1$, $\iota\colon B \to \zeta(K)$, in which K is abelian, even if the case $K = B$ does not arise. It is easy to construct examples when this does and when it does not happen. In this case, K is a $\mathfrak{V}G$-module, and if $\tau \in \mathfrak{V}^2(G/V(G), B)$ is the image of α, the image of τ in $\mathfrak{V}^2(G/V(G), K)$ is 0. The above discussion, in which the group was lifted to 'kill' an element of $\mathfrak{V}^2(G/V(G), B)$ has a simpler analogue in the case of embedding the module, except that K is restricted to being abelian. The fact that such a K does not always exist reflects the fact that varietal cohomology does not in general vanish on injective $\mathfrak{V}G$-modules.

The explicit calculation of the groups $\mathfrak{V}^2(G/V(G), B)$ using André's 'construction pas à pas' in [1], or the construction of Tierney and Vogel in [36], is generally a practical if tedious business if $\mathfrak{V}$ is a small variety such as $\mathfrak{N}_2$, and G is a well-behaved group; c.f. [20]. However, if $G/V(G)$ is a $\mathfrak{V}$-splitting group, then $\mathfrak{V}^1(G/V(G), B)$ and $\mathfrak{V}^2(G/V(G), B)$ are always trivial. For a discussion of varieties whose finitely generated groups have this property, see § 4. Here we consider the case $\mathfrak{V}=\mathfrak{A}_p$, p a prime, so that every group in the variety is $\mathfrak{V}$-free.

THEOREM 3.3. *Let $\mathfrak{V}=\mathfrak{A}_p$, G be a group of order p^n, and α: $\mathfrak{V}M(G)\to B$ be a surjection corresponding to a $\mathfrak{V}$-isologism class, where B is of order p^m. Then there is exactly one group of order p^{m+n} in the isologism class corresponding to α, and this is obtained as follows. If $a_1, \ldots, a_r \in G$ define a $\mathbf{Z}_p$-basis for $G/V(G)$, so that $a_1, \ldots, a_r$ generate G, and if F is freely generated by $\{y_1, \ldots, y_r\}$, let $1\to R\to F\to G\to 1$ be the presentation defined by $y_i \mapsto a_i$, $i=1, \ldots, r$. Equating $R\cap V(F)/[RV^*F]=R/[RV^*F]$ with $\mathfrak{V}M(G)$, the quotient E, say, of $F/[RV^*F]$ by* ker α *is the required group.*

Moreover, there is exactly one group of order p^k in the isologism class whenever $t\geqslant m+n$, namely $E\times C$, where C is elementary abelian of rank $t-(m+n)$.

Proof. The construction and uniqueness of E follow easily from Theorems I.2.3 and II.3.2. Note that, since $R\subseteq V(F)$, the subgroup T as in Theorem I.2.3, must be trivial.

It is clear that, if C is an elementary abelian p-group, then E and $E\times C$ are $\mathfrak{V}$-isologic, and the uniqueness of $E\times C$ again follows easily from the uniqueness statement of Theorem 3.2.

Remarks. 1. It follows from Theorem 3.3 that every genus, in the terminology of Hall and Senior [13], is a union of $\mathfrak{A}_2$-isologism classes.

2. Theorem 3.3 gives a recursive procedure for constructing all finite p-groups, which produces each p-group once and only once (c.f. Evens [40]). The amount of computation required to construct all groups of order 128 by hand using this method would be unbearable.

Perfect groups. Define a $\mathfrak{V}$-stem-extension of a group G to be an extension $1\to B\to E\to G\to 1$ such that $B\subseteq V(E)\cap V^*(E)$. The subgroup K of the group M is *small* if M has no proper subgroup L such that $KL=M$. If M, or the Frattini subgroup of M, is finitely generated this is equivalent to the condition that K lie in the Frattini subgroup of M, and is in general a stronger condition. If $1\to B\to E\to G\to 1$ is a $\mathfrak{V}$-stem-extension of G, B is embedded in E as a small subgroup. For L is a subgroup of E such that $BL=E$, then since $B\subseteq V^*(E)$ it follows that $V(L)=V(E)$; thus $B\subseteq V(E)\subseteq L$, so $L=E$.

If the group G and variety $\mathfrak{V}$ are such that the image of the identity map under the homomorphism $\mathrm{Hom}_G(\mathfrak{V}M(G), \mathfrak{V}M(G)) \to \mathfrak{V}^2(G/V(G), \mathfrak{V}M(G))$ is zero, then $\mathfrak{V}M(G)$ is characterised as a G-module by the following property: given any $\mathfrak{V}$-stem-extension $1 \to B \to D \to G \to 1$, there exists a $\mathfrak{V}$-stem-extension $1 \to \mathfrak{V}M(G) \to E \to G \to 1$ and a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \mathfrak{V}M(G) \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, "="] & 1 \\ 1 \arrow[r] & B \arrow[r] & D \arrow[r] & G \arrow[r] & 1 \end{tikzcd}$$

where the vertical arrows represent surjections. This observation follows easily from the earlier part of this section. A $\mathfrak{V}$-stem-extension $1 \to \mathfrak{V}M(G) \to E \to G \to 1$ is called a $\mathfrak{V}$-*stem-cover* of G and E a $\mathfrak{V}$-*covering group* of G. The existence of a $\mathfrak{V}$-stem-cover of G is clearly equivalent to the above assumption on the vanishing of the image of the identity. It also follows from the exact sequence of Theorem 3.1 that if one $\mathfrak{V}$-stem-cover of G exists, then there are precisely $\mathfrak{V}^1(G/V(G), \mathfrak{V}M(G))$ equivalence classes of such $\mathfrak{V}$-stem-covers under the usual equivalence relation for extensions. A sufficient condition for the existence of $\mathfrak{V}$-stem-covers of G is that $G/V(G)$ should be a $\mathfrak{V}$-splitting group, for then $\mathfrak{V}^2(G/V(G), B)$ (and $\mathfrak{V}^1(G/V(G), B)$) vanishes for any $G/V(G)$-module B, see § 1, (ii).

We wish to consider briefly what happens under the stronger assumption that $G/V(G)$ is trivial, or, as we shall say, that G is $\mathfrak{V}$-*perfect*. (If $\mathfrak{V}$ is a soluble variety of exponent zero, as is the case with the great majority of varieties mentioned in this paper, then a group G is $\mathfrak{V}$-perfect if and only if it is perfect.) If $\mathfrak{V}$ is any variety and G is $\mathfrak{V}$-perfect then $V^*(G) \subseteq \zeta(G)$, since $[V^*(G), V(G)] = 1$, with equality if $\mathfrak{V}$ has exponent zero. Also, if G is $\mathfrak{V}$-perfect and $1 \to B \to E \to G \to 1$ is an extension in which B is embedded as a small subgroup of E, then since $V(E)$ maps onto $V(G) = G$ it follows that E is also $\mathfrak{V}$-perfect. In particular, this holds if $1 \to B \to E \to G \to 1$ is a $\mathfrak{W}$-stem extension of G for any variety $\mathfrak{W}$, or if B is in the Frattini subgroup of E and B or E is finitely generated.

Now let $\mathfrak{V}$ be a variety of exponent zero, G be a $\mathfrak{V}$-perfect group and $1 \to B \to E \to G \to 1$ be an extension in which B is embedded as small subgroup of E, then $V(E) = E' = E$ and $V^*(E) = \zeta(E)$. It follows that $1 \to B \to E \to G \to 1$ is a $\mathfrak{V}$-stem extension of G if and only if it is an $\mathfrak{A}$-stem-extension of G. Thus if $\mathfrak{V}$ is a variety of exponent zero and G is a $\mathfrak{V}$-perfect group, $\mathfrak{V}M(G)$ is the Schur multiplier $\mathfrak{A}M(G)$ of G; and a slight generalisation of the above argument shows that if $\mathfrak{V}$ is of exponent n, and G is a $\mathfrak{V}$-perfect group, then $\mathfrak{V}M(G) = \mathfrak{A}M(G) \otimes \mathbf{Z}/n\mathbf{Z}$. Since the Schur multiplier of every known finite simple group has been calculated (at the moment of writing the latest is O'Nan's putative group) this gives many more situations in which $\mathfrak{V}M(G)$ is known.

Finally, let G be a perfect group, and $1 \to B \to E \to G \to 1$ be an $\mathfrak{A}$-stem-extension. Then $\zeta_2(E) = N_2^*(E) = \zeta(E)$ since E is $\mathfrak{N}_2$-perfect. It follows, as observed by Schur, that every $\mathfrak{A}$-stem-extension $1 \to C \to D \to E \to 1$ gives rise to an $\mathfrak{A}$-stem-extension $1 \to A \to D \to G \to 1$ where $D \to G$ is the composite $D \to E \to G$. Hence $\mathfrak{A}M(E)$ is isomorphic to the kernel of the surjection $\mathfrak{A}M(G) \to B$ arising from the 'universal covering property' of $\mathfrak{A}M(G)$. These results can be translated at once to the case of $\mathfrak{V}$-perfect groups if $\mathfrak{V}$ is of exponent 0.

For a somewhat different treatment of the results in the case of algebras, see Lue [42].

§ 4. The structure of varietal (co-)homology groups

An interpretation of $\mathfrak{V}M(G)$ in terms of varietal homology is given when $G \in \mathfrak{V}$. This is most successful under the stronger hypothesis that $\mathfrak{A}$ var $G \subseteq \mathfrak{V}$, in which case the second varietal (co-)homology groups can be calculated in terms of $\mathfrak{V}M(G)$. In particular, if $\mathfrak{Y}$ is as in § I.4, $\mathfrak{Y}^2(C_2 \times C_2 \times C_2, \mathbf{Z}_2)$ is of uncountable rank. This reflects the fact that uncountably many varieties may be obtained by deleting arbitrary sets of laws from those used to define $\mathfrak{Y}$.

Balanced varieties are discussed; the search for balanced varieties in universal algebra seems an important problem.

If G is a finite group of order m, the classical (co-)homology groups in positive dimension with coefficients in any module have exponent dividing m. Results in this direction are produced for other varieties, giving a characterization of Schur–Baer varieties.

LEMMA 4.1. *Let $G \in \mathfrak{V}$, A and B be left and right $\mathfrak{V}G$-modules respectively, and $F \to G$ be a surjection, with F free. Then $F \to G$ factors through $F/V(F) = P$ say, and the induced homomorphisms $\mathfrak{O}_0(F, A) \to \mathfrak{O}_0(P, A)$ and $\mathfrak{O}^0(P, B) \to \mathfrak{O}^0(F, B)$ are isomorphisms.*

Proof. By (vi) $\mathfrak{O}_0(F, A) = IF \underset{F}{\otimes} A \cong (IF \underset{F}{\otimes} \mathfrak{V}G) \underset{G}{\otimes} A$. Also,

$$\mathfrak{O}_0(P, A) = IP \underset{P}{\otimes} A \cong (IP \underset{P}{\otimes} \mathfrak{V}G) \underset{G}{\otimes} A.$$

But by (iv) the natural homomorphism of $IF \underset{F}{\otimes} \mathfrak{V}G$ onto $IP \underset{P}{\otimes} \mathfrak{V}G$ is an isomorphism; and since the identification of $\mathfrak{V}_0(G, A)$ with $IG \underset{G}{\otimes} A$ in (vi) is natural in G, the result for homology follows. The proof for cohomology is similar.

PROPOSITION 4.2. *With the assumptions of Lemma 4.1, let $R = \ker F \to G$. Then $\mathfrak{O}_1(P, A) \cong (V(F)/[V(F), R]) \underset{G}{\otimes} A$, and $\mathfrak{O}^1(P, B) \cong \mathrm{Hom}_G(V(F)/[V(F), R], B)$.*

Proof. By (iii) and Lemma 4.1, there is an exact sequence $\mathfrak{O}_1(F, A) \to \mathfrak{O}_1(P, A) \to \mathfrak{O}_0(F \to P, A) \to 0$. But $\mathfrak{O}_1(F, A) = 0$ by (ii), and $\mathfrak{O}_0(F \to P, A) = (V(F)/[V(F), R]) \underset{G}{\otimes} A$ by (vi). The proof for cohomology is similar.

Note. The isomorphisms in Lemma 4.1 and Proposition 4.2 are natural in the group and module. Varying the group requires that F and P be considered as elements of $(\mathfrak{O}, G)$. Also, the results remain true if $\mathfrak{O}$ is replaced by any variety $\mathfrak{W}$ containing $\mathfrak{V}$, and F by $F/W(F)$.

THEOREM 4.3. *If $\mathfrak{V}$ is a variety containing G, and A and B are left and right $\mathfrak{V}G$-modules respectively, then $\mathfrak{V}_0(G, \mathfrak{O}_1(-, A)) \cong \mathfrak{V}M(G) \otimes_G A$, and $\mathfrak{V}_0(G, \mathfrak{O}^1(-, B)) \cong \mathrm{Hom}_G(\mathfrak{V}M(G), B)$. In particular, $\mathfrak{V}_0(G, \mathfrak{O}_1(-, \mathfrak{V}G)) \cong \mathfrak{V}M(G)$.*

Proof. We first show that $\mathfrak{V}_0(G, \mathfrak{O}_1(-, \mathfrak{V}G)) \cong \mathfrak{V}M(G)$ when $\mathfrak{V} \supseteq \mathfrak{A}\,\mathrm{var}\,G$, so that $\mathfrak{V}G = \mathbf{Z}G$, as in (iv). Take a presentation $1 \to R \to F \to G \to 1$, and let $P_0 = F/V(F)$, so that P_0 is mapped onto G with kernel $R/V(F)$. Let Q be a $\mathfrak{V}$-free group, and $Q \to R/V(F)$ be a surjection. If P_1 is the verbal product of P_0 and Q, homomorphisms d_0 and d_1 of P_1 onto P_0 are defined as in (xii). Let e_i: $\mathfrak{O}_1(P_1, \mathfrak{V}G) \to \mathfrak{O}_1(P_0, \mathfrak{V}G)$ be the homomorphism induced by d_i, $i = 0,1$. Then $\mathfrak{V}_0(G, \mathfrak{O}_1(-, \mathfrak{V}G)) = \mathrm{coker}(e_0 - e_1)$, as in (xii). By Proposition 4.2, $\mathfrak{O}_1(P_0, \mathfrak{V}G) \cong V(F)/[V(F), R]$, and by Proposition I.1.3 (i), $[RV^*F] \supseteq [V(F), R]$. But clearly im $(e_0 - e_1)$ consists of all elements represented by terms of the form

$$v(g_1h_1, \ldots, g_rh_r)v(g_1, \ldots, g_r)^{-1}\colon v \in V(X_r),\ g_1, \ldots, g_r \in P_0,\ h_1, \ldots, h_r \in R/V(F).$$

Thus coker $(e_0 - e_1) = V(F)/[RV^*F] = \mathfrak{V}P(G) = \mathfrak{V}M(G)$, since $G \in \mathfrak{V}$.

Now consider the case when $\mathfrak{V}$ is an arbitrary variety containing G. Since $\mathfrak{O}_1(P_0, \mathfrak{V}G) = (V(F)/[V(F), R]) \otimes_G \mathfrak{V}G$, and tensoring is right exact (i.e. cokernel preserving), it follows from the first part of the proof that $\mathfrak{V}_0(G, \mathfrak{O}_1(-, \mathfrak{V}G)) \cong \mathfrak{V}M(G) \otimes_G \mathfrak{V}G$. But $\mathfrak{V}M(G)$ is already a $\mathfrak{V}G$-module by Proposition I.1.15, so $\mathfrak{V}M(G) \otimes_G \mathfrak{V}G \cong \mathfrak{V}M(G)$. A repetition of this argument gives $\mathfrak{V}_0(G, \mathfrak{O}_1(-, A)) \cong \mathfrak{V}M(G) \otimes_G A$ for an arbitrary left $\mathfrak{V}G$-module A, and a similar argument gives the corresponding result for cohomology.

Note. The isomorphisms of the theorem are natural in the variety, group and module. Varying the variety and the group requires the introduction of a base variety $\mathfrak{U}$ and a base group Π as in (xiii). Also, the theorem may be generalized by replacing $\mathfrak{O}$ by any variety $\mathfrak{W}$ containing $\mathfrak{V}$, provided that $\mathfrak{V}M(G)$ is replaced by the relative Baer-invariant $\mathfrak{W}\mathfrak{V}M(G)$ defined after Lemma I.1.8.

As a rule, the smaller the variety $\mathfrak{V}$ the easier it is to deal with the ring $\mathfrak{V}G$; thus the following result is sometimes of use.

THEOREM 4.4. *If $\mathfrak{U} \subseteq \mathfrak{V}$ the following are equivalent:*

(a) *for all $G \in \mathfrak{U}$, $\mathfrak{V}M(G)$ is naturally isomorphic to $\mathfrak{V}_0(G, \mathfrak{O}_1(-, \mathfrak{U}G))$;*

(b) $\mathfrak{U}\mathfrak{V}^* \subseteq \mathfrak{V}\mathfrak{U}^*$.

Proof. By Theorem 4.3, (a) is equivalent to the statement that $\mathfrak{V}M(G)$ is naturally isomorphic to $\mathfrak{V}M(G)\otimes_G \mathfrak{U}G$, i.e. that $\mathfrak{V}M(G)$ is a $\mathfrak{U}G$-module. This is equivalent to (b) by Proposition I.1.14.

COROLLARY 4.5. *If $\mathfrak{V}$ is a variety of exponent* 0, *and G is an abelian group in $\mathfrak{V}$, then* $\mathfrak{V}M(G)\cong\mathfrak{V}_0(G, \mathfrak{O}_1(-, \mathbf{Z}))$.

Proof. Put $\mathfrak{U}=\mathfrak{A}$, so that $\mathfrak{U}\Pi=\mathbf{Z}$. The result then follows by Proposition I.1.6 (ii).

Note. In [25], $\mathfrak{V}_0(G, \mathfrak{O}_1(-, \mathbf{Z}))$, which is now seen to be $\mathfrak{V}M(G)$, is calculated for G a finitely generated abelian group, and $\mathfrak{V}$ the varieties $\mathfrak{N}_c\cap\mathfrak{S}_2$, $\mathfrak{S}_2$, and $[\mathfrak{E}, \mathfrak{S}_2]$; the case of $\mathfrak{S}_2$ is covered in § I.3 in the more general setting of polynilpotent varieties, at least if G is a free abelian group. The generalization to an arbitrary finitely generated abelian group is easy.

The next result gives a more striking connection between Baer-invariants and homology for 'large' varieties.

THEOREM 4.6. *If $\mathfrak{V}\supseteq\mathfrak{A}$ var G, and A and B are left and right G-modules respectively, then* $\mathfrak{V}_2(G, A)\cong\mathfrak{V}M(G)\otimes_G A\oplus\mathfrak{O}_2(G, A)$, *and* $\mathfrak{V}^2(G, B)\cong\mathrm{Hom}_G(\mathfrak{V}M(G), B)\oplus\mathfrak{O}^2(G, B)$. *In particular*, $\mathfrak{V}_2(G, \mathbf{Z}G)\cong\mathfrak{V}M(G)$.

Proof. Immediate from Theorem 4.3 and the splitting of the short exact sequence in (xi).

Note. The isomorphisms of the theorem are natural in the variety, group and module, as in the note to Theorem 4.3. However, the variety $\mathfrak{O}$ plays a crucial role here and cannot be replaced by another variety $\mathfrak{W}$ as in the note to Theorem 4.3. The reason is that the short exact sequences in (xi) depend on the fact that $\mathfrak{O}$ is balanced, see below.

COROLLARY 4.7. *If the variety $\mathfrak{Y}$ is as in § I.4, and $E=C_2(p_1)\times C_2(p_2)\times C_2(p_3)$ is an elementary abelian group of order* 8, *then $\mathfrak{Y}_2(E, \mathbf{Z}_2)$ and $\mathfrak{Y}^2(E, \mathbf{Z}_2)$ are vector spaces over $\mathbf{Z}_2$ of countably infinite dimension and dimension the continuum respectively.*

Proof. Since $\mathfrak{O}_2(E, \mathbf{Z}_2)=H_3(E, \mathbf{Z}_2)$ and $\mathfrak{O}^2(E, \mathbf{Z}_2)=H^3(E, \mathbf{Z}_2)$ are finite, and $\mathfrak{Y}M(E)$ is clearly at most countable, both statements of the theorem will follow easily from the fact that $\mathrm{Hom}_E(\mathfrak{Y}M(E), \mathbf{Z}_2)$ is infinite. For n a power of 2 greater than 1, the group G_n of § I.4 has the property that the subgroup H_n generated by G_n' and the squares in G_n is $\mathfrak{Y}$-marginal in G_n. Take the presentation $1\to R\to F\to E\to 1$, where F is freely generated by $\{y_1, y_2, y_3\}$, and $y_i\mapsto p_i$ for all i. Define β_n: $F\to G_n$ by $y_1\beta_n=(ba_1, 1)$, $y_2\beta_n=t$, $y_3\beta_n=s$.

This gives rise to a commutative diagram

$$\begin{array}{ccccccccc} 1 \to & R & \to & F & \longrightarrow & E & \longrightarrow 1 \\ & \downarrow{\scriptstyle \gamma_n} & & \downarrow{\scriptstyle \beta_n} & & \downarrow{\scriptstyle \alpha_n} & \\ 1 \to & H_n & \to & G_n & \to & G_n/H_n & \to 1. \end{array}$$

Since $H_n \subseteq Y^*(G_n)$, by Proposition I.1.2 (ii) γ_n induces a homomorphism of $R/[RY^*F]$ into H_n, and hence a homomorphism γ_n^* of $Y(F)/[RY^*F] = \mathfrak{Y}M(E)$ into $Y(G_n)$. Now w_m, as in the definition of $\mathfrak{Y}$, defines an element of $\ker \gamma_n^*$ if and only if $w_m((ba_1, 1), t, s) = 1$, and this happens if and only if $m \neq n$. It follows that the homomorphisms γ_n^* are distinct, and so $\mathrm{Hom}_E(\mathfrak{Y}M(E), \mathbf{Z}_2)$ is infinite as required.

Note. 1. It is clear that $\mathfrak{Y}_2(E, \mathbf{Z})$ is not finitely generated, and that $\mathfrak{Y}^2(E, \mathbf{Z})$ is of cardinal the continuum. Also, the choice of E is not critical in that it may be replaced by any group T having E as a homomorphic image, provided that $\mathfrak{A}$ var $T \subseteq \mathfrak{Y}$.

Note 2. If S is any subset of $\{w_m\}$ it follows that there is a homomorphism of $\mathfrak{Y}M(E)$ into $\mathbf{Z}_2$ whose kernel contains the element defined by w_m if and only if $w_m \in S$. Thus uncountably many homomorphisms of $\mathfrak{Y}M(E)$ into $\mathbf{Z}_2$ arise from the uncountably many varieties that may be obtained from $\mathfrak{Y}$ by omitting an arbitrary set of laws from $\{w_m\}$. An alternative way of looking at the situation is as follows. The homomorphism γ_n^* gives rise to an element of $\mathfrak{Y}^2(E, \mathbf{Z}_2)$, and hence, in the terminology of § II.2, to a class of relative $\mathfrak{Y} - E - \mathbf{Z}_2$-cores. Let $\bar{G}_n$ be the subgroup of G_n generated by $(ba_1, 1)$, s and t, and $\bar{H}_n$ be the subgroup of $\bar{G}_n$ generated by $\bar{G}_n'$ and the squares in $\bar{G}_n$, so that $\bar{G}_n/\bar{H}_n \cong E$. Then $\bar{G}_n$, or more strictly the exact sequence $1 \to \bar{H}_n \to \bar{G}_n \to E \to 1$, is a solution to one of the above cores, but a solution that does not lie in $\mathfrak{Y}$. Thus, loosely speaking, a variety is not finitely based if there are infinitely many groups lying 'just outside' it. These groups differentiate the laws of the variety, but, subject to suitable conditions on the marginal factors, they may also be thought of as solving varietal obstruction problems that cannot be solved within the variety.

A variety $\mathfrak{V}$ for which $\mathfrak{V}_n(G, A) = \mathfrak{V}^n(G, B) = 0$ for all $G \in \mathfrak{V}$, all $n > 0$, and all projective left $\mathfrak{V}G$-modules A and injective right $\mathfrak{V}G$-modules B will be called *balanced.* If $\mathfrak{V}$ is balanced then

$$\mathfrak{V}_n(G, A) = \mathrm{Tor}_n^{\mathfrak{V}G}(IG \underset{G}{\otimes} \mathfrak{V}G, A) \quad \text{and} \quad \mathfrak{V}^n(G, B) = \mathrm{Ext}_{\mathfrak{V}G}^n(IG \underset{G}{\otimes} \mathfrak{V}G, B)$$

for all left $\mathfrak{V}G$-modules A, right $\mathfrak{V}G$-modules B, and $n \geqslant 0$; for it is easy to see, using (vi), that this is true for $n = 0$, and this together with the exact sequence in (v) and the property of being balanced, characterises the above Tor and Ext. It is easy to see that a (necessary and) sufficient condition for $\mathfrak{V}$ to be balanced is that $\mathfrak{V}_n(G, A) = 0$ for all $G \in \mathfrak{V}$ and all

$n > 0$, where A is a free $\mathfrak{V}G$-module of rank 1. The case of an arbitrary free module follows by taking direct limits, and of an arbitrary projective module by additivity. The result for cohomology then follows from the spectral sequence $\mathrm{Ext}^p_{\mathfrak{V}G}(\mathfrak{V}_q(G, \mathfrak{V}G), B) \underset{p}{\Rightarrow} \mathfrak{V}^n(G, B)$, c.f. [24.II], or more simply from any construction of varietal (co-)homology. The definition of 'balanced' readily extends to any variety in the sense of universal algebra; see [24, I] § 3 for a general discussion of the problem. Known examples of balanced varieties include all abelian varieties; the varieties of all associative algebras, Lie algebras, and restricted Lie algebras, over a field; and, of course, the variety of all groups. See, for example, [33]. We now consider the problem of determining which varieties of groups are balanced.

THEOREM 4.7. *If* $\mathfrak{E} \neq \mathfrak{U}$, *and* $\mathfrak{A}\mathfrak{U} \subseteq \mathfrak{V} \subset \mathfrak{O}$, *there is a* $\mathfrak{U}$-*free group* G *of finite rank, a projective left* G-*module* A, *and an injective right* G-*module* B, *such that* $\mathfrak{V}_2(G, A) \neq 0$ *and* $\mathfrak{V}^2(G, B) \neq 0$. *In particular,* $\mathfrak{V}$ *is not balanced.*

Proof. Note that, since $\mathfrak{V} \supseteq \mathfrak{A}\mathfrak{U}$, for any $G \in \mathfrak{U}$, all G-modules are $\mathfrak{V}G$-modules. Dealing first with homology, let A be a free G-module of rank 1. Then by Theorem 4.6, $\mathfrak{V}_2(G, A) \cong BM(G) = \mathrm{V}/U - \alpha$, the $\mathfrak{U} - \alpha$-pandect of $\mathfrak{V}$, where α is the rank of G, c.f. § I.3. But if α is a large enough integer $V/U - \alpha$ is non-trivial, see example 3 after Corollary I.3.2. Turning to cohomology, Theorem 4.6 reduces the problem to finding a $\mathfrak{U}$-free group G and an injective G-module B such that $\mathrm{Hom}_G(\mathfrak{V}M(G), B) \neq 0$. With the same choice of G as before such a B exists; for example $\mathfrak{V}M(G)$ can be embedded in an injective G-module.

We now consider varieties $\mathfrak{V}$ whose finitely generated groups are $\mathfrak{V}$-splitting groups. Peter M. Neumann shows in [31] that every locally finite variety, of square-free exponent, whose nilpotent subvarieties are abelian, has this property; and that the last two conditions are necessary. The set of varieties satisfying all three conditions is closed under products, provided the factors have co-prime exponents, under finite joins and under the formation of subvarieties. Also, it includes the varieties $\mathfrak{A}_p$ for all primes p, and, as is shown in [31], $\mathrm{var}(A_5)$, where A_5 is the alternating group on 5 symbols.

THEOREM 4.8. *If* $\mathfrak{V}$ *is a variety whose finitely generated groups are* $\mathfrak{V}$-*splitting groups, then* $\mathfrak{V}$ *is balanced.*

Proof. It will be shown that $\mathfrak{V}_n(G, A) = 0$ for $n > 0$, $G \in \mathfrak{V}$, and any $\mathfrak{V}G$-module A; this gives the result by the remarks before Theorem 4.7. If G is finitely generated, then $\mathfrak{V}_n(G, A) = 0$ for $n > 0$ by (ii); the result for arbitrary G follows by taking direct limits. In more detail, if H is a subgroup of G, so that A is a $\mathfrak{V}$H-module, let $C(H)$ denote the complex of abelian groups used to calculate $\mathfrak{V}_*(H, A)$ using the Barr-Beck resolution. It is easy to see that $C(G)$ is the set-theoretic union of $\{C(H)$: H is a finitely generated subgroup of $G\}$. (This is false in general for cohomology.) Since $C(H)$ is acyclic for all finitely generated H, so is $C(G)$, and the result follows.

Examples. 1. If p and q are distinct primes, it follows from Theorem 4.8 and the preceding remarks that $\mathfrak{A}_p\mathfrak{A}_q$ is balanced, whereas, by Theorem 4.7, $\mathfrak{A}\mathfrak{A}_q$ is not.

2. In [23] a group in $\mathfrak{A}_3\mathfrak{A}_2$ is produced which does not split over its Sylow 3-subgroup. It follows that, in this variety, the one-dimensional cohomology groups are not all trivial.

If $\mathfrak{B}$ is a variety, G a finite group in $\mathfrak{B}$ of order m, and A and B are left and right $\mathfrak{B}G$-modules respectively, does it follow that $\mathfrak{B}_n(G, A)$ and $\mathfrak{B}^n(G, B)$ are of exponent dividing m for all $n > 0$? The answer is almost certainly 'no', though we know of no counter-example. On the other hand, we have very little positive information; see, however, K. W. Johnson [20]. The result is well known for the variety of all groups, and since $\mathfrak{B}_1(G, A)$ is a quotient group of $\mathfrak{O}_1(G, A)$ and $\mathfrak{B}^1(G, B)$ is a subgroup of $\mathfrak{O}^1(G, B)$, it follows in dimension one for an arbitrary variety. If $\mathfrak{B}$ is a balanced variety, it follows by the usual dimension-shift argument for any positive dimension; though this gives no new information in the known balanced varieties.

THEOREM 4.9. *The following conditions on the variety $\mathfrak{B}$ are equivalent.*

(a) *$\mathfrak{B}$ is a Schur-Baer variety (see the end of* § I.1)*;*

(b) *for every finite group G in $\mathfrak{B}$ and every left $\mathfrak{B}G$-module A, $\mathfrak{B}_2(G\,A)$ is of exponent dividing a power of $|G|$; and if A is finitely generated, $\mathfrak{B}_2(G, A)$ is finite;*

(c) *is obtained from* (b) *by replacing 'left' by 'right', 'A' by 'B', and '$\mathfrak{B}_2(G, A)$' by '$\mathfrak{B}^2(G, B)$'.*

Proof. By Theorem I.1.17, (a) is equivalent to the statement that $\mathfrak{B}M(G)$ is of finite order dividing a power of $|G|$ whenever G is finite. Since $\mathfrak{B}M(G)$ is a $\mathfrak{B}G$-module by Proposition I.1.15, it follows at once from the exact sequence of Theorem 3.1 that (a) is equivalent to (c). The corresponding exact sequence in homology has only been established for $G \in \mathfrak{B}$, see (xi) and Theorem 4.3; but this is clearly enough to establish the equivalence between (b) and (c).

References

[1]. ANDRÉ, M., *Méthode simpliciale en algèbre homologique et algèbre commutative.* Lecture Notes in Math. 32, Springer-Verlag, Berlin 1967.

[2]. BAER, R., Representations of groups as quotient groups, I–III. *Trans. Amer. Math. Soc.*, 58 (1945), 295–419.

[3]. —— Endlichkeitskriterien für Kommutatorgruppen. *Math. Ann.*, 124 (1952), 161–177.

[4]. BARR, M., Cohomology and obstructions: commutative algebras. *Seminar on triples and categorical homology theory.* Lecture Notes in Math. 80, 357–375, Springer-Verlag, Berlin 1969.

[5]. BARR, M. & BECK, J., Acyclic models and triples. *Proc. Conference on Categorical Algebra, La Jolla,* Springer, New York, 1966, 336–343.

[6]. —— Homology and standard constructions. *Seminar on triples and categorical homology theory.* Lecture Notes in Math. 80, 245–335, Springer, Berlin 1969.

[7]. BECK, J., Triples and cohomology. Thesis, Columbia University, New York 1965.

[8]. BRYANT, R. M., Some infinitely based varieties of groups. *J. Austral. Math. Soc.*, 16 (1973), 29–32.

[9]. EILENBERG, S. & MACLANE, S., Cohomology theory in abstract groups, II, Group extensions with a non-abelian kernel. *Ann. of Math.*, 48 (1947), 326–341.

[10]. FRÖHLICH, A., Baer-invariants of algebras. *Trans. Amer. Math. Soc.*, 109 (1963), 221–244.

[11]. GERSTENHABER, M., On the deformation of rings and algebras, II. *Ann. of Math.*, 84 (1966), 1–19.

[12]. GRUENBERG, K. W., *Cohomological topics in group theory.* Lectures notes in Math. 143, Springer, New York 1970.

[13]. HALL, M. & SENIOR, J. K., *The Groups of Order* $2^n (n \leqslant 6)$. Macmillan, New York, 1964.

[14]. HALL, P., The classification of prime power groups. *J. Reine Angew. Math.*, 182 (1940), 130–141.

[15]. —— Verbal and marginal subgroups. *J. Reine Angew. Math.*, 182 (1940), 156–157.

[16]. —— Finiteness conditions for soluble groups. *Proc. London Math. Soc.*, (3) 4 (1954), 419–436.

[17]. —— *Nilpotent Groups.* Canad. Math. Congress, Univ. of Alberta, 1957. Queen Mary College Math. Notes, London, 1970.

[18]. —— On the finiteness of certain soluble groups. *Proc. London Math. Soc.*, (3) 9 (1959), 595–622.

[19]. JOHNSON, K. W., Varietal generalisations of Schur multipliers, stem extensions and stem covers. *J. Reine Angew. Math.*, 270 (1974), 169–183.

[20]. —— A computer calculation of homology in varieties of groups. *J. London Math. Soc.*, (2) 8 (1974), 247–252.

[21]. KEUNE, F. J., Homotopical algebra and algebraic K-theory. Thesis, University of Amsterdam, 1972.

[22]. KNOPFMACHER, J., Extensions in varieties of groups and algebras. *Acta Math.*, 115 (1966), 17–50.

[23]. KOVACS, L. G., NEUMANN, B. H. & DE VRIES, H., Some Sylow subgroups. *Proc. Roy. Soc. Ser.* A, 260 (1961), 304–316.

[24]. LEEDHAM-GREEN, C. R., Homology in varieties of groups, I–III. *Trans. Amer. Math. Soc.*, 162 (1971), 1–33.

[25]. LEEDHAM-GREEN, C. R. & HURLEY, T. C., Homology in varieties of groups, IV. *Trans. Amer. Math. Soc.*, 170 (1972), 293–303.

[26]. LUE, A. S.-T., Baer-invariants and extensions relative to a variety. *Proc. Cambridge Philos. Soc.*, 63 (1967), 569–578.

[27]. —— Cohomology of algebras relative to a variety. *Math. Z.*, 121, (1971), 220–232.

[28]. MACDONALD, J. L., Group derived functors. *J. Algebra*, 10 (1968), 448–477.

[29]. MERZLJAKOV, JU. I., Verbal and marginal subgroups of linear groups. *Dokl. Akad. Nauk SSSR*, 177 (1967), 1008–1011. (English translation: *Soviet Math. Dokl.*, 8 (1967), 1538–1541.)

[30]. NEUMANN, H., *Varieties of Groups.* Springer-Verlag, Berlin, 1967.

[31]. NEUMANN, Peter M., Splitting groups and projectives in varieties of groups. *Quart. J. Math. Oxford*, (2) 18 (1967), 325–332.

[32]. ORZECH, Grace, Obstruction theory in algebraic categories, I–II. *J. Pure Appl. Algebra*, 2 (1972), 287–340.

[33]. RINEHART, G. S., Satellites and cohomology. *J. Algebra*, 12 (1969), 295–329, Errata, *J. Algebra*, 14 (1970), 125–126.

[34]. SCHUR, I., Über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen. *J. Reine Angew. Math.*, 127 (1904), 20–50.

[35]. STALLINGS, J., Homology and central series of groups. *J. Algebra*, 2 (1965), 170–181.

[36]. TIERNEY, M. & VOGEL, W., Simplicial resolutions and derived functors. *Math. Z.*, 111 (1969), 1–14.

[37]. TURNER-SMITH, R. F., Finiteness conditions for verbal subgroups. *J. London Math. Soc.*, 41 (1966), 166–176.

[38]. VAUGHAN-LEE, M. R., Uncountably many varieties of groups. *Bull. London Math. Soc.*, 2 (1970), 280–286.

[39]. WILSON, J. C. R., Topics in verbal subgroups. Thesis, London University, 1972.

Additional references

[40]. EVENS, L., Terminal p-groups. *Illinois J. Math.*, 12 (1968), 682–699.

[41]. MUIR, T., *A Treatise on the Theory of Determinants*. Dover Publications, 1960.

[42]. LUE, A. S.-T., Connected Algebras and Universal Covering. *Comment. Math. Helv.*, 48 (1973) 370–375.

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