ON THE CLASSIFICATION OF KLEINIAN GROUPS: I—KOEBE GROUPS

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In this paper we exhibit a class of Kleinian groups called Koebe groups, and prove that every uniformization of a closed Riemann surface can be realized by a unique Koebe group.

The first examples of these groups were due to Klein [6] who constructed them using his combination theorem. Koebe [7, 8] proved a general uniformization theorem for Kleinian groups constructed in this fashion.

Our existence theorem is based on the generalized combination theorems [11, 12], and uses Bers' technique of variation of parameters using quasiconformal mappings [3]. The uniqueness theorem is a generalization of Koebe's original proof [7], but with weaker hypotheses.

Detailed statements of theorems, and outlines of proofs appear in section 2. The theorems are all formulated in terms of Kleinian groups; equivalent formulations in terms of uniformizations of Riemann surfaces appear in [13], where our main result was first announced.

1. Definitions

1.1. We denote the group of all fractional linear transformations by SL'. A Kleinian group G is a subgroup of SL' which acts discontinuously at some point of $\hat{C}=C\cup\{\infty\}$. The set of points at which G acts discontinuously is denoted by $\Omega=\Omega(G)$, and its complement $\Lambda=\Lambda(G)$ is the limit set. A component of $\Omega(G)$ is called a *component* of G; a component Δ of G is *invariant* if $g\Delta=\Delta$, for all $g\in G$.

1.2. We will be primarily concerned with Kleinian groups which have an invariant component. For any $g\in SL'$, we will not distinguish between the Kleinian group G with invariant component Δ, and the group $g\circ G\circ g^{-1}$ with invariant component $g(\Delta)$.

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1.3. An isomorphism $\psi: G \to G^*$ between Kleinian groups is called *type-preserving* if ψ preserves the square of the trace of every elliptic element; and if both ψ and ψ^{-1} preserve parabolic elements.

1.4. Two Kleinian groups G and G^*, with invariant components Δ and Δ^* respectively, are called *weakly similar* if there is an orientation preserving homeomorphism $\varphi: \Delta \to \Delta^*$, where $\varphi \circ g \circ \varphi^{-1}$ defines an isomorphism from G onto G^*. In this case, φ is called a *weak similarity*.

If the isomorphism $g \to \varphi \circ g \circ \varphi^{-1}$ is type-preserving, then φ is a *similarity*, and G and G^* are called *similar*; if in addition, φ is conformal, then G and G^* are called *conformally similar*.

1.5. A parabolic element g of a Kleinian group G is called *accidental* if there is a weak similarity φ, so that $\varphi \circ g \circ \varphi^{-1}$ is loxodromic (including hyperbolic).

1.6. If G is a Kleinian group with invariant component Δ, and H is a subgroup of G, then H has a *distinguished invariant component* $\Delta(H) \supset \Delta$.

1.7. A subgroup H of a Kleinian group G with invariant component Δ is called a *factor subgroup* if H is a maximal subgroup of G satisfying

(i) $\Delta(H)$ is simply-connected,

(ii) H with invariant component $\Delta(H)$ contains no accidental parabolic elements, and

(iii) every parabolic element of G, whose fixed point lies in $\Lambda(H)$, is an element of H.

1.8. A Kleinian group G is *elementary* if $\Lambda(G)$ is finite; it is *Fuchsian* if $\Lambda(G)$ is a circle (or line).

1.9. A finitely generated Kleinian group G with an invariant component is called a *Koebe group* if every factor subgroup of G is either elementary or Fuchsian.

2. Statements of results

2.1. Our main result is that every finitely generated Kleinian group with an invariant component is conformally similar to a unique Koebe group. We, in fact, prove a stronger result.

THEOREM 1. *Let G be a finitely generated Kleinian group with an invariant component. Then there is a unique Koebe group G^*, and there is a unique (up to elements of SL') conformal similarity between G and G^*.*

2.2. We prove this theorem in several steps. In section 5 we use the combination theorems [11, 12] (see section 4) and the decomposition theorem [14] (see section 3) to prove a topological existence theorem.

THEOREM 2. *Every finitely generated Kleinian group with an invariant component is similar to a Koebe group.*

In section 7 we combine Theorem 2 with Bers' technique of variation of parameters by quasiconformal mappings [3] to prove the existence part of Theorem 1.

THEOREM 3. *Every finitely generated Kleinian group with an invariant component is conformally similar to a Koebe group.*

In section 10 we prove the uniqueness of Koebe groups.

THEOREM 4. *If φ is a conformal similarity between Koebe groups, then $\varphi \in SL'$.*

In broad outline, the proof of uniqueness follows Koebe's original proof [7]. The estimates in section 9 for loops invariant under parabolic transformations are new. Also, we do not assume any knowledge of how the groups are constructed, and so we need to know that similarities preserve factor subgroups.

THEOREM 5. *Let G and G^* be Kleinian groups with invariant components, and let ψ: $G \to G^*$ be a type-preserving isomorphism. Then H is a factor subgroup of G if and only if $\psi(H)$ is a factor subgroup of $\psi(G)$.*

The proof of Theorem 5 is based on the following generalization of a result in [15].

THEOREM 6. *If there is a type-preserving isomorphism from a Kleinian group G onto a finitely generated Fuchsian group of the first kind, then G has a simply-connected invariant component, and G contains no accidental parabolic elements.*

Theorems 5 and 6 are proven in section 8.

3. Decomposition

3.1. Let H be a subgroup of the Kleinian group G. A set $A \subset C$ is said to be *precisely invariant under H in G* if

(i) $g(A) = A$, for all $g \in H$, and
(ii) $g(A) \cap A = \emptyset$, for all $g \in G - H$.

If Y is a connected subset of Ω/G and A is a connected component of $p^{-1}(Y)$, then A is precisely invariant under $H = \{g \in G \mid gA = A\}$ in G. In this case, we say that A *covers* Y, and that H is a *covering subgroup* of Y.

In general, for a Kleinian group G and subset $A \subset \hat{C}$, $H = \{g \in G \mid gA = A\}$ is called the *stabilizer of A in G.*

3.2. For the remainder of this section we assume that G is a given finitely generated Kleinian group with an invariant component Δ.

Ahlfors finiteness theorem [1] asserts that $S=\Delta/G$ is a finite Riemann surface (i.e., a closed surface from which a finite number of points have been deleted), and that the projection $p: \Delta \to S$ is branched over at most a finite number of points of S.

3.3. It was shown in [14] that there is a set of simple disjoint loops $w_1, ..., w_k$ on S, which divide S into sub-surfaces $Y_1, ..., Y_s$, so that the following hold.

(i) Each covering subgroup of each Y_i is a factor subgroup of G; every factor subgroup of G is a covering subgroup of some Y_i; two factors subgroups are conjugate in G if and only if they are covering subgroups of the same Y_i.

(ii) If H and H' are factor subgroups of G, then $H \cap H'$ is either trivial, elliptic cyclic or parabolic cyclic.

(iii) If A covers some Y_i with covering subgroup H, and A' covers some Y_i' with covering subgroup H', and $\bar{A} \cap \bar{A}' \neq \emptyset$, then $\bar{A} \cap \bar{A}'$ is a simple loop which, except perhaps for a parabolic fixed point, covers some w_j with covering subgroup $H \cap H'$.

(iv) If W covers some w_j then W lies on the boundary of two regions A and A', where A covers some Y_i with covering subgroup H, and A' covers some Y_i' with covering subgroup H'. Then $H \cap H'$ is the stabilizer of W.

In particular, every covering subgroup of each w_j is either trivial, or elliptic cyclic or parabolic cyclic.

(v) Every elliptic or parabolic element of G is contained in some factor subgroup.

3.4. It was also shown in [14] that if we appropriately choose a complete set of non-conjugate factor subgroups, then G can be constructed from these subgroups using weak versions of the combination theorems. We essentially reprove this in section 5.

3.5. The properties listed in 3.3 are obviously invariant under small deformations of the loops w_j; in particular, we can assume that each w_j is smooth.

4. Combination theorems

4.1. We state the combination theorems here in the form that we will use them. Except for conclusion (iv), these are special cases of the results in [11] and [12]. Under more stringent hypotheses, conclusion (iv) is proven in [16]; that proof is easily adapted to this case.

4.2. For any Kleinian group G, we let $°\Omega = °\Omega(G)$ be $\Omega(G)$ with all fixed points of elliptic elements deleted, so that G acts freely on $°\Omega$. A *fundamental set* D for G is a subset of $°\Omega$ satisfying

(i) $g(D) \cap D = \emptyset$, for all $g \in G - 1$,

(ii) $\cup_{g \in G} g(D) = {}^{\circ}\Omega$, and

(iii) ∂D has zero 2-dimensional measure.

4.3. Combination Theorem I. *Let H be a finite or parabolic cyclic subgroup of both the Kleinian groups G_1 and G_2. Assume that there is a simple closed curve W which divides $\hat{C}$ into two topological discs B_1 and B_2, where $\bar{B}_i \cap \Omega(H)$ is precisely invariant under H in G_i, and $W \cap \Omega(H) \subset \Omega(G_i)$, $i = 1, 2$. Assume that there are fundamental sets D_1, D_2, E, for G_1, G_2, H, respectively, where $D_i \subset E$, $i = 1, 2$. We also assume that if H is parabolic, then H is its own normalizer in either G_1 or G_2. Then*

(i) *G, the group generated by G_1 and G_2, is Kleinian;*

(ii) *G is the free product of G_1 and G_2 with amalgamated subgroup H;*

(iii) *$D = D_1 \cap D_2$ is a fundamental set for G;*

(iv) *every elliptic or parabolic element of G is conjugate in G to some element of either G_1 or G_2; and*

(v) *If $z \in \Lambda(G)$, then either z is a translate of some point of $\Lambda(G_1)$, or z is a translate of some point of $\Lambda(G_2)$, or there is a sequence $\{g_n\}$ of distinct elements of G so that $g_{n+1}(W)$ separates z from $g_n(W)$, and $z = \lim g_n(W)$.*

4.4. Combination Theorem II. *Let H_1 and H_2 be subgroups of the Kleinian group G_1, where H_1 and H_2 are finite or parabolic cyclic. Suppose there are open topological discs B_1, B_2, bounded by simple closed curves W_1, W_2, respectively, where B_i is precisely invariant under H_i in G_1, $i = 1, 2$, either $\bar{B}_1$ is precisely invariant under H_1 in G_1, or $\bar{B}_2$ is precisely invariant under H_2 in G_1, and $g(\bar{B}_1) \cap \bar{B}_2 = \emptyset$, for all $g \in G_1$. Suppose further that there are fundamental sets D_1, E_1, E_2, for G_1, H_1, H_2, respectively so that $D_1 \subset E_i$, and $W_i \cap E_i \subset W_i \cap D_i$, $i = 1, 2$. Assume also that there is an element $f \in SL'$ so that $f(W_1) = W_2$, $f \circ H_1 \circ f^{-1} = H_2$, and $f(B_1) \cap B_2 = \emptyset$. Then*

(i) *G, the group generated by G_1 and f is Kleinian;*

(ii) *every relation in G is a consequence of the relations in G_1, together with $f \circ H_1 \circ f^{-1} = H_2$;*

(iii) *$D = D_1 - \{(D_1 \cap B_1) \cup (D_2 \cap \bar{B}_2)\}$ is a fundamental set for G;*

(iv) *every elliptic or parabolic element of G is conjugate in G to some element of G_1; and*

(v) *if $z \in \Lambda(G)$, then either z is a translate of some point of $\Lambda(G_1)$, or z is a translate of a fixed point of f, or there is a sequence $\{g_n\}$ of distinct elements of G, so that $g_{n+1}(W_1)$ separates z from $g_n(W_1)$, and $z = \lim g_n(W_1)$.*

5. Topological construction

5.1. As in section 3, we assume throughout this section that G is a given finitely generated Kleinian group with an invariant component Δ. We set $S=\Delta/G$, and let $w_1, \ldots, w_k$ be the loops on S which divide it into the subsurfaces $Y_1, \ldots, Y_s$.

5.2. If the number of loops $k=0$, then G is a factor subgroup of itself; hence Δ is simply connected, and G contains no accidental parabolic elements. If Δ is hyperbolic, then one easily sees that G is conformally similar to a Fuchsian group; if Δ is not hyperbolic, then of course G is elementary.

5.3. Proceeding inductively with the proof of Theorem 2, we first take up the case that some w_i, we now call it w, is dividing.

Let W cover w; if necessary we adjoin a parabolic fixed point to W, so that it becomes a simple loop. Let B_1 and B_2 be the topological discs bounded by W.

We choose base points $\tilde{o}$ on W, and $o=p(\tilde{o})$, and after deleting the points of ramification from S to get a subsurface S', we define the subgroup τ_i of $\pi_1(S', o)$ to be generated by those loops on S' at o, which do not cross w, and whose liftings starting or ending at $\tilde{o}$ do not enter B_i.

Having chosen base points, there is a natural homomorphism from $\pi_1(S', o)$ onto G. We let G_i be the image of τ_i under this homomorphism.

Let H be the stabilizer of W in G. One sees at once that B_i is precisely invariant under H in G_i; in fact, except perhaps for a parabolic fixed point, $\bar{B}_i$ is precisely invariant under H in G_i.

The loop w divides S into two subsurfaces X_1 and X_2, where near W, $X_1=p(B_2)$ and $X_2=p(B_1)$. Near W, there is a connected component A_i of $p^{-1}(X_i)$. One sees at once that G_i is the stabilizer of A_i in G. Since $A_i/G_i \cup (\bar{B}_i \cap \Omega(H))/H$ is already a finite Riemann surface, $\Delta(G_i)/G_i=\hat{X}_i$ is X_i with a disc or punctured disc sewn in along the boundary loop w.

We originally had S cut up into subsurfaces $Y_1, \ldots, Y_s$, by the loops $w=w_1, \ldots, w_k$. Those of the loops $w_2, \ldots, w_k$ which lie in $\hat{X}_1$ cut it up into subsurfaces $\hat{Y}_1, \ldots, \hat{Y}_{\hat{s}}$. Since each covering subgroup in G_1 of each $\hat{Y}_i$ is a factor subgroup of G, one easily sees that it is also a factor subgroup of G_1.

From the way we have constructed G_1, it is clear that if $g \in G_1$ doesn't lie in a covering subgroup of some $\hat{Y}_i$, then g is loxodromic, and there is some lifting W_j of some w_j which separates the fixed points of g. We conclude that g does not lie in any factor subgroup of G_1, and so the covering subgroups of the $\hat{Y}_i$ are precisely the factor subgroups of G_1.

It now follows from our induction hypothesis that G_1 is similar to a Koebe group G_1^*;

we denote the similarity by φ_1: $\Delta(G_1)\to\Delta(G_1^*)=\Delta_1^*$. The same reasoning shows that there is Koebe group G_2^*, and a similarity φ_2: $\Delta(G_2)\to\Delta(G_2^*)=\Delta_2^*$.

Since B_i is precisely invariant under the cyclic subgroup H in G_i, $\varphi(B_i)$ is precisely invariant under $H_i^*=\varphi_i\circ H\circ\varphi_i^{-1}$ in G_i^*, and so there is a circular disc contained in $\varphi_i(B_i)$ which is also precisely invariant under H_i^* in G_i^*. After an obvious deformation, we can assume that $\varphi_i(B_i)$ is a circular disc.

We normalize G_1^* and G_2^* so that $B_1^*=\varphi_2(B_1)$ is $|z|>1$, and so that $B_2^*=\varphi_1(B_2)$ is $|z|<1$. We further normalize and deform φ_1 and φ_2 near W so that $\varphi_1|W=\varphi_2|W$. Now $H^*=\varphi_1\circ H_1\circ\varphi_1^{-1}=\varphi_2\circ H_2\circ\varphi_2^{-1}$ is a common subgroup of G_1^* and G_2^*.

The only non-trivial hypothesis of 4.3 that we need to verify is that H^* is its own normalizer is either G_1^* or G_2^*. In order to see that H is its own normalizer is either G_1 or G_2, we look at the covering regions which border on W, and the factor subgroups J_1, J_2 which stabilize these regions. We easily observe that if H is parabolic and H is not its own normalizer in G_i, then H is not its own normalizer in J_i, and so J_i must be elementary. If J_1 and J_2 were both elementary, then $J_1\cup J_2$ would also be elementary, contradicting the maximality of J_i. We conclude that the hypothesis of 4.3 are satisfied both for the groups G_1 and G_2 with common subgroup H and for G_1^* and G_2^* with common subgroup H^*.

One sees at once that the factor subgroups of G_1^* and of G_2^* are factor subgroups of G^*, the group generated by G_1^* and G_2^* and as above one easily sees that every factor subgroup of G^* is a conjugate of one of these. Hence G^* is a Koebe group.

We can combine φ_1 and φ_2 to obtain a similarity φ: $\Delta\to\Delta^*$, the invariant component of G^*. We define $\varphi|A_i=\varphi_i$, $i=1, 2$, and observe that φ is continuous across W. We then use conclusion (ii) of 4.3. to define φ on Δ so that $\varphi(\Delta)=\Delta^*$ and $\varphi\circ G\circ\varphi^{-1}=G^*$. It then follows from conclusion (iv) of 4.3 that $g\in G$ is parabolic if and only if $\varphi\circ g\circ\varphi^{-1}$ is. Hence φ is a similarity between G and the Koebe group G^*.

5.4. Proceeding with our induction, we now take up the case that none of the loops $w_1, \ldots, w_k$ is dividing. Each loop w_i when raised to some least power α_i lifts to a loop; if $\alpha_i=\infty$, then the corresponding element of G is parabolic. We assume that $\alpha_1\leqslant\alpha_2\leqslant\ldots\leqslant\alpha_k$. We let $w=w_1$, let W_1 be some connected component of $p^{-1}(w)$, and we let H_1 be the stabilizer of W_1 in G; if H_1 is parabolic, we adjoin the fixed point of H_1 to W_1, so that in any case W_1 is a simple closed curve. W_1 divides $\hat{C}$ into two topological discs; we will call one of them B_1.

We choose base points o on w, and $\tilde{o}$ on W_1, let S' be as in 5.3, and let τ_1 be the subgroup of $\pi_1(S', o)$ generated by loops on S' at o, which do not cross w, and whose liftings, starting and ending at $\tilde{o}$, do not enter B_1. Let G_1 be the image of τ_1 in G under the natural homomorphism.

Since w is non-dividing, $G_1 \neq G$. The translates of W_1 under G cut up Δ into regions; let A be that region which has W_1 on its boundary and which does not intersect B_1. Let f be some element of $G - G_1$, where $W_2 = f(W_1)$ lies on the boundary of A. Let $H_2 = f \circ H_1 \circ f^{-1}$, and let B_2 be the topological disc bounded by W_2, where $B_2 \cap A = \emptyset$.

One easily sees that A is invariant under G_1, that B_i is precisely invariant under H_i in G_1, $i = 1, 2$, that $f(B_1) \cap B_2 = \emptyset$, and that $g(B_1) \cap B_2 = \emptyset$ for all $g \in G_1$.

We let $X = S - \{w\}$, so that X has two boundary loops. We observe that $\hat{X} = \Delta(G_1)/G_1$ is X with two discs or punctured discs sewn in along the boundary loops. The loops $w_2, ..., w_k$ divide $\hat{X}$ into regions $\hat{Y}_1, ..., \hat{Y}_{\hat{s}}$; as in the previous case, one easily sees that the covering subgroups of the $\hat{Y}_i$ are precisely the factor subgroups of G_1. Hence there is a Koebe group G_1^* with invariant component Δ_1^*, and there is a similarity $\varphi_1: \Delta(G_1) \to \Delta_1^*$.

For $i = 1, 2$, $B_i^* = \varphi_1(B_i)$ is a topological disc which is precisely invariant under $H_i^* = \varphi_1 \circ H_i \circ \varphi^{-1}$ in G_1^*. B_i^* contains a circular disc with the same properties; hence after a minor deformation, we can assume that B_i^* is a circular disc. Next one easily constructs an element $f^* \in SL'$ so that f^* maps the boundary of B_1^* onto the boundary of B_2^*, $f(B_1^*) \cap B_2^* = \emptyset$ and f^* conjugates H_1^* into H_2^* inducing the same isomorphism as $\varphi_1 \circ f \circ \varphi_1^{-1}$. We next deform φ_1 near W_2 so that $\varphi_1 \circ f \,|\, W_1 = f^* \circ \varphi_1 \,|\, W_1$.

We remark next that the choice of w_1 to minimize the order α_1 guarantees that if H_1 is parabolic, then no factor subgroup of G can be elementary, and so H_1 is its own normalizer in G_1. Hence $\bar{B}_i$ is precisely invariant under H_i in G_1, $\bar{B}_i^*$ is precisely invariant under H_i^* in G_1^*, $g(\bar{B}_1) \cap \bar{B}_2 = \emptyset$ for all $g \in G_1$, and $g^*(\bar{B}_1^*) \cap \bar{B}_2^* = \emptyset$ for all $g^* \in G_1^*$.

We conclude that the hypotheses of 4.4 hold for both G_1 with subgroups H_1 and H_2, and for G_1^* with subgroups H_1^* and H_2^*.

Let G^* be the group generated by G_1^* and f^*. As in the preceding case, we observe that G^* is a Koebe group.

We have φ_1 defined on A. We define $\varphi: \Delta \to \Delta^*$, the invariant component of G^*, by $\varphi \,|A = \varphi_1$ and then we use conclusion (ii) of 4.4 together with the fact that $\varphi_1 \circ f \,|\, W_1 = f^* \circ \varphi_1 \,|\, W_1$ to define φ as a homeomorphism of Δ onto Δ^*; it follows from conclusion (iv) of 4.4 that $g \in G$ is parabolic if and only if $\varphi \circ G \circ \varphi^{-1}$ is. This concludes the proof of Theorem 2.

6. Extended Kleinian groups

6.1. In what follows we will be dealing with *extended Kleinian groups;* that is, discontinuous groups of possibly orientation-reversing conformal self-maps of $\hat{C}$.

6.2. We collect here some of the basic facts about Kleinian groups which also hold for extended Kleinian groups. The proofs, which are straightforward generalizations of those in the classical case, are omitted.

Let $\tilde{G}$ be an extended Kleinian group. We define the set of discontinuity Ω and the limit set Λ exactly as in the classical case.

We normalize $\tilde{G}$ so that $\infty \in \Omega$ and so that ∞ is not a fixed point of any element of $\tilde{G}$. Then every $g \in \tilde{G}$ can be represented by a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1,$$

where either $g(z) = (az+b)(cz+d)^{-1}$ or $g(z) = (a\bar{z}+b)(c\bar{z}+d)^{-1}$.

The isometric circle of g is the circle centered at $g^{-1}(\infty)$ with radius $|c|^{-1}$. Every element $g \in \tilde{G}$ can be written as $g = e \circ r$, where r is inversion in the isometric circle of g, and e is a Euclidean motion.

If we choose R so large that $U = |z| > R$ is precisely invariant under the identity in $\tilde{G}$, then

$$\text{dia}^2\, g(U) \geqslant k\,|c|^{-4},$$

and so

$$\Sigma\,|c|^{-4} < \infty,$$

where the sum is taken over all non-trivial elements of $\tilde{G}$.

6.3. If G is a Koebe group, then there is a natural *extended Koebe group* $\tilde{G} \supset G$, where $\tilde{G}$ is generated by G together with the reflections in the limit circles of the Fuchsian factor subgroups.

LEMMA 1. *If G is a Koebe group with invariant component Δ, and $\tilde{G}$ is the extended Koebe group, then Δ is precisely invariant under G in $\tilde{G}$.*

Proof. One easily sees that we can write a general element of $\tilde{G}$ as $r_n \circ r_{n-1} \circ \ldots \circ r_1 \circ g$ where each r_i is an inversion in a limit circle C_i of some Fuchsian factor subgroup H_i, $g \in G$ and $r_i \neq r_{i+1}$.

For each i, we let Δ_i be the component of H_i, which does not contain Δ.

Observe that $g(\Delta) = \Delta$; $r_1 \circ g_1(\Delta) \subset \Delta_1$, which is disjoint from Δ. Then since $r_2 \neq r_1$, $r_2 \circ r_1 \circ g_1(\Delta) \subset \Delta_2$. Continuing in this manner, we see that if $g \in \tilde{G} - G$, then $g(\Delta) \cap \Delta = \varnothing$.

The proof above actually shows more.

LEMMA 2. *Let $H_1, \ldots, H_p$ be a complete list of non-conjugate Fuchsian factor subgroups of G_j, let $\{g_{ij}\}$ be a collection of generators for H_i, and let r_i be inversion in the limit circle of H_i. Then the relations in G, together with the relations $\{g_{ij} \circ r_i = r_i \circ g_{ij},\ r_i^2 = 1\}$ form a complete set of relations for $\tilde{G}$.*

6.4. Lemma 3. $\Omega(\tilde{G})/\tilde{G} = \Delta/G$.

Proof. Since Δ is precisely invariant under G in $\tilde{G}$, we need to show that the translates of Δ cover $\Omega(\tilde{G})$.

If z is not in $\Lambda(\tilde{G})$, and not in Δ, then $z \in \Omega(G)$, and so [14] there is a limit circle C_1 of some Fuchsian factor subgroup which separates z from Δ. Let r_1 be inversion in C_1; if $z \notin r_1(\Delta)$, then there is a limit circle C_2 on the boundary of $r_1(\Delta)$, where C_2 separates z from C_1. Continuing in this manner, we get a sequence of limit circles $C_1, C_2, \ldots$, where C_n separates z from C_{n-1}.

It follows from Lemma 1 that if we look at all translates under $\tilde{G}$ of all the limit circles of Fuchsian factor subgroups then any two of them are either disjoint or tangent. Since there are only finitely many inequivalent Fuchsian factor subgroups, and each of them is finitely generated and of the first kind [14], the spherical diameter of $C_n \to 0$, and so $z \in \Lambda(\tilde{G})$.

7. The existence theorem

7.1. In this section we prove Theorem 3. Let G be a finitely generated Kleinian group with invariant component Δ. Let G^* be a similar Koebe group, with invariant component Δ^*, and let φ: $\Delta^* \to \Delta$ be the similarity. It was observed by Bers [3] that we can assume that φ is quasiconformal; i.e., φ has locally square integrable derivatives satisfying:

$$\varphi_{\bar{z}} = \mu(z)\varphi_z, \quad \text{where ess sup } |\mu| = k < 1. \tag{1}$$

7.2. Since φ is a similarity, the dilatation $\mu(z)$, defined in (1), satisfies

$$\mu \circ g(z)\overline{g'(z)}/g'(z) = \mu(z), \tag{2}$$

for every $g \in G^*$.

7.3. We let $\tilde{G}^*$ be the extended Koebe group and we define μ for z in $\Omega(\tilde{G}^*)$ as follows: If $z \in \Delta$, then μ is defined above.

If $z \in \Omega(\tilde{G}^*)$, then by Lemma 3, there is a $g \in \tilde{G}^*$ with $g(z) \in \Delta$. If g preserves orientation, we define $\mu(z)$ by (2).

If g reverses orientation, we define $\mu(z)$ by

$$\mu \circ g(z)\overline{g'(z)}/g'(z) = \overline{\mu(z)}, \tag{3}$$

where $g'(z) = g_{\bar{z}}$.

One easily sees that these definitions are consistent, so that μ is well defined.

If $z \in \Lambda(\tilde{G}^*)$, we set $\mu(z) = 0$.

The function μ is measurable and ess sup $|\mu| < 1$.

7.4. It was shown by Ahlfors and Bers [2] that given $\mu(z)$ measurable with ess sup $|\mu|<1$, then there is a quasiconformal homeomorphism w of $\hat{C}$, satisfying (1), and that if v is any other solution of (1), then $v=f\circ w$, where $f\in SL'$.

7.5. Let w be some solution of (1) as above. If g is an orientation-preserving element of G^*, then it follows from (2) that

$$(w\circ g)_{\bar{z}}=\mu(z)(w\circ g)_z,$$

hence there is an element $\psi(g)\in SL'$, with $w\circ g\circ w^{-1}=\psi(g)$.

Similarly if $g\in\tilde{G}^*$ is orientation-reversing, then it follows from (3) that

$$\overline{(w\circ g)}_{\bar{z}}=\mu(z)\overline{(w\circ g)}_z;$$

hence there is an orientation-reversing conformal transformation $\psi(g)=w\circ g\circ w^{-1}$.

7.6. The group $G'=\psi(G^*)$ is a Kleinian group with invariant component $\Delta'=w(\Delta)$, and since w is a homeomorphism, ψ is type-preserving; hence $w\,|\Delta^*$ is a similarity.

Elementary factor subgroups are obviously preserved under isomorphisms, and each non-elementary factor subgroup is the stabilizer of its limit set, hence both ψ and ψ^{-1} preserve factor subgroups.

For each Fuchsian factor subgroup H^* of G^*, there is an inversion $r^*\in\tilde{G}^*$, whose fixed point set is $\Lambda(H^*)$. Hence the fixed point set of $\psi(r^*)$ is $\Lambda(\psi(H))$. Since $\psi(g^*)$ is an orientation-reversing fractional linear transformation, its fixed point set is either finite or a circle. We conclude that G' is a Koebe group.

7.7. Since φ and w are both similarities, $\varphi\circ w^{-1}\colon\Delta'\to\Delta$ is a similarity between G' and G. One computes that for $z\in\Delta'$, $(\varphi\circ w^{-1})_{\bar{z}}=0$, and so $\varphi\circ w^{-1}$ is a conformal similarity between G' and G. This completes the proof of Theorem 3.

8. Isomorphisms

8.1. In this section we prove Theorems 5 and 6. We start with some observations about Fuchsian groups.

8.2. LEMMA 4. *Let $\psi\colon G\to G'$ be a type-preserving isomorphism between finitely generated Fuchsian groups. Then G and G' are of the same kind.*

Proof. It is well known that if G is of the second kind then G can be written as a free product of cyclic groups where every elliptic or parabolic element is a conjugate of some element in one of these cyclic groups. If G is of the first kind and purely hyperbolic, then it cannot be decomposed as a non-trivial free product; if it is not purely hyperbolic, then it contains elliptic or parabolic elements $g_1, \ldots, g_n$, belonging to distinct conjugacy classes of maximal cyclic subgroups of G, where $g_n \circ \ldots \circ g_1$ lies in the commutator subgroup.

8.3. LEMMA 5. *Let G be a finitely generated Kleinian group with an invariant component. If every factor subgroup of G is cyclic, then there is a type-preserving isomorphism of G onto a Fuchsian group of the second kind.*

Proof. It was shown in [14] that the intersection of any two factor subgroups is a maximal cyclic subgroup of G, hence in this case the intersection of distinct factor subgroups is trivial. Again using [14] we conclude that we can choose a complete set $H_1, \ldots, H_s$ of non-conjugate factor subgroups, so that G is the free product, in the sense of combination theorem I, of $H_1, \ldots, H_s$. The construction of a Fuchsian group of the second kind which is a free product of elliptic and parabolic cyclic groups is classical (it is also a straightforward application of 4.3).

8.4. We now prove Theorem 6. Let $\psi: G \to G'$ be a type-preserving isomorphism from the finitely generated Kleinian group G onto the Fuchsian group of the first kind G'. Let Δ be some component of G, and let G_0 be the stabilizer of Δ; by Ahlfor's finiteness theorem [1], G_0 is finitely generated and has Δ as an invariant component.

We first show that G_0 contains a non-elementary factor subgroup. If not, then [14] Δ would be the only component of G_0, so that $G_0 = G$. Then by Lemma 5, we would have a type-preserving isomorphism of G onto a Fuchsian group of the second kind, which, by Lemma 4, cannot occur.

Now let H be a non-elementary factor subgroup of G. Then there is a type-preserving isomorphism of H onto a Fuchsian group of the first kind, and so by Lemma 4, $\psi(H)$ is of the first kind.

Since $\psi(H)$ and $\psi(G)$ are both finitely generated and of the first kind, $[G:H] < \infty$, and so $\Lambda(G) = \Lambda(H)$.

It was shown in [17] (see also Bers [4] and Kra and Maskit [9]) that H is either quasi-Fuchsian (i.e., a perhaps trivial quasiconformal deformation of a Fuchsian group) or degenerate (i.e., $\Omega(H)$ is connected and simply-connected). Hence either $G = H$, or H is quasi-Fuchsian, and $[G:H] = 2$.

It remains only to show that the latter case cannot occur; we assume it does. Using the Nielsen realization theorem [5] (for proof, see Marden [10], or Zieschang [18]), there is a quasiconformal homeomorphism

$$w: \hat{C} \to \hat{C}, \quad \text{where} \quad w \circ g \circ w^{-1} = \psi(g), \quad \text{for all} \quad g \in H.$$

We assume that G' operates on the upper half plane; we let r denote reflection in the real line, and we let j be some element of $G-H$. We observe that $\chi = r \circ w \circ j \circ w^{-1}$ maps U onto itself, and that for every $g \in \psi(H)$, $\chi \circ g \circ \chi^{-1} = \psi(j) \circ g \circ \psi(j^{-1})$, which is impossible since $\psi(j)$ preserves orientation while χ reverses orientation.

This concludes the proof of Theorem 6.

8.5. We now prove Theorem 5. We have a type-preserving isomorphism $\psi: G \to G^*$, and H is a factor subgroup of G. We need to show that $\psi(H)$ is a factor subgroup of G^*, in fact, it suffices to show that $\psi(H)$ satisfies conditions (i)–(iii) of 1.7.

If H is elementary, then one easily sees that $\psi(H)$ is also elementary with the same number of limit points.

If H is non-elementary, there is a type-preserving isomorphism of H onto a Fuchsian group of the first kind, hence by Theorem 6, $\psi(H)$ satisfies (i) and (ii) of 1.7.

If g^* is a parabolic element of G^*, and the fixed point of g^* lies in $\Lambda(\psi(H))$, then the group $\hat{H}$ generated by $\psi(H)$ and g^* has a connected limit set, and so $\Delta(\hat{H})$ is simply-connected. If the Riemann map from $\Delta(\hat{H})$ induces a type-preserving isomorphism on $\psi(H)$, then the image of $\psi(H)$ must be of the first kind and $[\hat{H}: \psi(H)] < \infty$; hence some power of g^* lies in $\psi(H)$, and so the fixed point of $\psi^{-1}(g^*)$ lies in $\Lambda(H)$.

If the Riemann map does not induce a type-preserving isomorphism on $\psi(H)$, then $\psi(H)$ contains a parabolic element g' which is accidental as an element of $\hat{H}$. As in the decomposition theorem, let W be a simple loop which is precisely invariant under the cyclic group generated by g' in $\hat{H}$. Now W separates $\Lambda(\hat{H})$ into two non-empty sets, and since g' is not accidental as an element of $\psi(H)$, $\Lambda(\psi(H))$ is contained in one of them. Hence the fixed point of g' is also the fixed point of some $h \circ g^* \circ h^{-1}$, where $h \in \hat{H}$, and $h \circ g^* \circ h^{-1} \notin H$. Since g' and $h \circ g^* \circ h^{-1}$ have the same fixed point, they commute. Hence $\psi^{-1}(g')$ and $\psi^{-1}(h \circ g^* \circ h^{-1})$ have the same fixed point, and so $\psi^{-1}(h \circ g^* \circ h^{-1}) \in H$.

9. Structure loops

9.1. Throughout this section G is a Koebe group, with invariant component Δ; $\tilde{G}$ is the extended Koebe group; $w_1, \ldots, w_k$ is the set of loops on $S = \Delta/G$; as in 3.3, they divide S into the regions $Y_1, \ldots, Y_s$.

Each connected component of $p^{-1}(w_i)$ is, after adjoining a parabolic fixed point if necessary, a simple closed curve, called a *structure loop*. We define the structure loops so that the set of structure loops is invariant under $\tilde{G}$.

The purpose of this section is to prove certain uniform estimates for structure loops. Since there are only finitely many equivalence classes of structure loops [14], it will suffice to prove each lemma for a given structure loop W and its translates under $\tilde{G}$.

9.2. We will use the following notations and normalizations throughout this section.

Let H be the stabilizer of W in $\tilde{G}$; then H is either finite or parabolic cyclic.

If H is finite, we normalize $\tilde{G}$ so that $\infty \in \Omega(\tilde{G})$, ∞ is not fixed by any element of $\tilde{G}$, and so that W and all its translates under $\tilde{G}$ are uniformly bounded.

We decompose $\tilde{G}$ into cosets $\tilde{G} = \Sigma_p g_p H$, so that the set $\{g_p(W)\}$ is precisely the set of distinct translates of W. We write

$$g_p = \begin{pmatrix} a_p & b_p \\ c_p & d_p \end{pmatrix},$$

and let δ_p be the distance from $g_p^{-1}(\infty)$ to W. Since $W \subset \Omega(\tilde{G})$, there is a $\delta > 0$, so that $\delta_p \geqslant \delta$.

If H is parabolic, then we normalize $\tilde{G}$ so that $\infty \in \Omega(\tilde{G})$, ∞ is not fixed by any element of $\tilde{G}$, W and all its translates are bounded, H has its fixed point at the origin, and H is generated by $h: z \to z(z+1)^{-1}$.

Let $\bar{H}$ be the stabilizer of the origin in $\tilde{G}$. Since H is the intersection of two factor subgroups and at most one of them can be elementary, $\bar{H}$ contains a reflection in a circle invariant under H. Hence $\bar{H}$ contains a rank 2 free abelian group $\hat{H}$, where $[\bar{H}: \hat{H}] \leqslant 4$, and $\hat{H}$ is generated by h and $\hat{h}: z \to z(i\varrho z+1)^{-1}$, $\varrho > 0$.

Let t denote the map $z \to z^{-1}$. We choose a fundamental domain D for $\bar{H}$, where every point of $t(D)$ is closer to $t(W)$ than to any translate of $t(W)$ under $t \circ \bar{H} \circ t$. We set $\gamma = W \cap D$. We assume that D has been chosen so that γ is connected, and so that $\infty \in D$.

We decompose G into cosets $\tilde{G} = \Sigma_p g_p \bar{H}$; since the set $\{(g_p \circ \bar{h})^{-1}(\infty)\}$, $\bar{h} \in \bar{H}$, is invariant under $\bar{H}$, we can assume that $g_p^{-1}(\infty) \in D$. We again write

$$g_p = \begin{pmatrix} a_p & b_p \\ c_p & d_p \end{pmatrix}.$$

Since we are interested in uniform estimates, it will suffice to consider those translates of W of the form $\hat{g}_{pq}(W) = g_p \circ \hat{h}^q(W)$.

In any case, we assume as in 3.5 that W is smooth except perhaps at a parabolic fixed point; an easy example shows that the derivative need not be continuous at a parabolic fixed point.

We denote the Euclidean diameter of any set A by dia (A).

In each inequality below, k denotes some positive constant.

9.3. LEMMA 6. *If $\{W_j\}$ is an enumeration of the structure loops of $\tilde{G}$, then*

$$\Sigma_j \operatorname{dia}^2 (W_j) < \infty.$$

Proof. We first consider the case that H is finite. Inverting in the isometric circle of g_p yields

$$\operatorname{dia} g_p(W) \leqslant 2\,|c_p|^{-2}\delta_p^{-1} \leqslant k\,|c_p|^{-2}.$$

Hence

$$\Sigma_p \operatorname{dia}^2 g_p(W) \leqslant k\Sigma_p\,|c_p|^{-4} < \infty.$$

In the parabolic case, we let δ_{pq} be the distance from $t\circ g_p^{-1}(\infty)$ to $t\circ \hat{h}^q(W)$, and we observe that

$$\delta_{p0} \geqslant k, \quad \text{and} \quad \delta_{pq} \geqslant kq \quad q \neq 0. \tag{4}$$

We write $\hat{g}_{pq}(W) = (g_p\circ t)\circ(t\circ\hat{h}^q\circ t)(t(W))$, and observe that $|d_p|^{-1}$ is the radius of the isometric circle of $(g_p\circ t)$, and so

$$\operatorname{dia} \hat{g}_{pq}(W) \leqslant 2\,|d_p|^{-2}\delta_{pq}^{-1}. \tag{5}$$

Since D is relatively compact in $C-\{0\}$,

$$k^{-1} \leqslant |d_p|\,|c_p|^{-1} \leqslant k. \tag{6}$$

Combining (4), (5), and (6), we obtain

$$\Sigma_{pq} \operatorname{dia}^2 \hat{g}_{pq}(W) \leqslant k\,\Sigma_{pq}|d_p|^{-4}\,\delta_{pq}^{-2} \leqslant k\,\Sigma_p|c_p|^{-4}\,\Sigma_q|q|^{-2} \leqslant k\,\Sigma_p|c_p|^{-4} < \infty.$$

9.4. LEMMA 7. *Every structure loop has finite length.*

Proof. It suffices to consider the case that H is parabolic, and W is normalized as in 9.2. Then near 0 the length of W is given by

$$\Sigma \int_\gamma |h'(z)|\,|dz| = \Sigma \int_\gamma |rz+1|^{-2}\,|dz| \leqslant \Sigma|r|^{-2}\int_\gamma |z+r^{-1}|^{-2}\,|dz| \leqslant k\,\Sigma|r|^{-2},$$

where the sum is taken for $|r|$ sufficiently large.

9.5. For any structure loop W, let $L(W)$ be its length.

LEMMA 8. *There is a constant k so that for any structure loop W, $L(W) \leqslant k$* dia (W).

Proof. We first take up the case that H is finite. Then

$$L(g_p(W)) = \int_W |g_p'(z)|\,|dz| = |c_p|^{-2}\int_W |z-g_p^{-1}(\infty)|^{-2}\,|dz|$$
$$\leqslant |c_p|^{-2}\,\delta_p^{-2}\,L(W) \leqslant k|c_p|^{-2}\,L(W). \tag{7}$$

We also observe that if dia $W = |x-y|$, then

$$\operatorname{dia}(g_p(W)) \geqslant |g_p(x)-g_p(y)| \geqslant |x-y|\,|c_p|^{-2}\,|x-g_p^{-1}(\infty)|^{-1}\,|y-g_p^{-1}(\infty)|^{-1} \geqslant k\,|c_p|^{-2} \operatorname{dia}(W). \tag{8}$$

Combining (7) and (8), we obtain

$$L(g_p(W))\,\mathrm{dia}^{-1}(g_p(W)) \leqslant kL(W)\,\mathrm{dia}^{-1}(W).$$

For the case that H is parabolic, we again let δ_{pq} be the distance from $t\circ g_p^{-1}(\infty)$ to $t\circ \hat{h}^q(W)$, and we invert in the isometric circle of $g_p\circ t$ to obtain

$$\mathrm{dia}\,\hat{g}_{pq}(W) \geqslant \delta_{pq}^{-1}\,|d_p|^{-2}. \tag{9}$$

We also compute

$$\begin{aligned} L(\hat{g}_{pq}(W)) &= \Sigma_{r=-\infty}^{\infty}\int_{t(\gamma)} |(\hat{g}_{pq}\circ h^r\circ t)'(z)|\,|dz| \\ &= |d_p|^{-2}\,\Sigma_{r=-\infty}^{\infty}\int_{t(\gamma)} |z - t\circ g_p^{-1}(\infty) + r + i\varrho q|^{-2}\,|dz|. \end{aligned} \tag{10}$$

One easily sees that for $z\in t(\gamma)$,

$$|z - t\circ g_p^{-1}(\infty)| \geqslant k, \quad \text{and} \quad |z - t\circ g_p^{-1}(\infty) + r + i\varrho q|^2 \geqslant k(r^2 + \varrho^2 q^2). \tag{11}$$

Combining (10) and (11), we obtain

$$L(\hat{g}_{po}(W)) \leqslant k|d_p^{-2}|, \quad \text{and} \quad L(\hat{g}_{pq}(W)) \leqslant k\,|d_p|^{-2}\,|q|^{-1}. \tag{12}$$

We also easily observe that

$$\delta_{po} \leqslant k, \quad \text{and} \quad \delta_{pq} \leqslant k\,|q|. \tag{13}$$

Combining (9), (12) and (13), we obtain

$$L(\hat{g}_{pq}(W))\,\mathrm{dia}^{-1}\,\hat{g}_{pq}(W) \leqslant k.$$

9.6. If x and y are distinct points on a structure loop W, then we denote by $E(x, y)$ the length of the shorter arc of W connecting x to y.

LEMMA 9. *There is a constant k so that for all pairs of points x, y lying on a structure loop,*

$$E(x, y) \leqslant k\,|x - y|.$$

Proof. We first take up the case that H is finite, and observe that for x and y lying on a fixed loop W, which is smooth,

$$E(x, y) \leqslant k\,|x - y|.$$

Then

$$E(g_p(x), g_p(y)) = \int |g_p'(z)|\,|dz| = |c_p|^{-2}\int |z - g_p^{-1}(\infty)|^{-2}\,|dz|.$$

Since $|z-g_p^{-1}(\infty)|$ is bounded from both above and below,

$$E(g_p(x), g_p(y)) \leqslant k|c_p|^{-2}E(x, y) \leqslant k|c_p|^{-2}|x-y|$$
$$\leqslant k|g_p(x)-g_p(y)|\,|x-g_p^{-1}(\infty)|\,|y-g_p^{-1}(\infty)| \leqslant k|g_p(x)-g_p(y)|.$$

We turn next to the parabolic case. We assume first that $q \neq 0$, and that x and y both lie on the same side of $t(\gamma)$. We assume for simplicity that x and y both lie to the right of $t(\gamma)$, that y lies to the right of x, and we first take up the case that $|y| \leqslant |q|$. Then

$$\begin{aligned} &E(\hat{g}_{pq}\circ t(x), \hat{g}_{pq}\circ t(y))\,|\hat{g}_{pq}\circ t(x)-\hat{g}_{pq}\circ t(y)|^{-1} \\ &\quad\leqslant |x-t\circ g_p^{-1}(\infty)+i\varrho q|\,|y-t\circ g_p^{-1}(\infty)+i\varrho q|\,|x-y|^{-1}\int_x^y |z-t\circ g_p^{-1}(\infty)+i\varrho q|^{-2}\,|dz| \\ &\quad\leqslant k|q|^2\,|x-y|^{-1}\int_x^y |q|^{-2}\,|dz| \leqslant k. \end{aligned} \tag{14}$$

If $|y|>|q|$ and $|y| \leqslant 2|x|$, then

$$\begin{aligned} &E(\hat{g}_{pq}\circ t(x), \hat{g}_{pq}\circ t(y))\,|\hat{g}_{pq}\circ t(x)-\hat{g}_{pq}\circ t(y)|^{-1} \\ &\quad\leqslant |x-t\circ g_p^{-1}(\infty)+i\varrho q|\,|y-t\circ g_p^{-1}(\infty)+i\varrho q|\,|x-y|^{-1}\int_x^y |z-t\circ g_p^{-1}(\infty)+i\varrho q|^{-2}\,|dz| \\ &\quad\leqslant k|x|\,|y|\,|x-y|^{-1}\int_x^y |z|^{-2}\,|dz| \leqslant k|x|^2\,|x-y|^{-1}\int_x^y |x|^{-2}\,|dz| \leqslant k. \end{aligned} \tag{15}$$

Continuing the case that x and y both lie on the same side of $t(\gamma)$, $|y|>|q|$, we assume $|y|>2|x|$ and we choose an appropriate fundamental domain γ' for the action of H on W, where one endpoint of $t(\gamma')$ is at $t(x)$. Then

$$\begin{aligned} &E(\hat{g}_{pq}\circ t(x), \hat{g}_{pq}\circ t(y))\,|\hat{g}_{pq}\circ t(x)-\hat{g}_{pq}\circ t(y)|^{-1} \\ &\quad\leqslant \int_x^y |\hat{g}_{pq}\circ t(x)-\hat{g}_{pq}\circ t(y)|^{-1}\,|(\hat{g}_{pq}\circ t)'(z)|\,|dz| \\ &\quad\leqslant k|x+i\varrho q|\,|y+i\varrho q|\,|x-y|^{-1}\,\Sigma_{r=1}^{N}\int_{t(\gamma')} |z+i\varrho q+r|^{-2}\,|dz| \\ &\quad\leqslant k|x+i\varrho q|\,|y+i\varrho q|\,|x-y|^{-1}\,\Sigma_{r=1}^{N}|x+i\varrho q+r|^{-2} \\ &\quad\leqslant k|x+i\varrho q|\,|y+i\varrho q|\,|x-y|^{-1}\,\Sigma_{r=1}^{N}(|x|^2+\varrho^2q^2+r^2)^{-1} \\ &\quad\leqslant k|x+i\varrho q|\,\Sigma_{r=1}^{N}(|x|^2+\varrho^2q^2+r^2)^{-1} \leqslant k|x+i\varrho q|\,(|x|^2+\varrho^2q^2)^{-1/2} \leqslant k. \end{aligned} \tag{16}$$

We next consider the case that x and y lie on opposite sides of $t(\gamma)$, or in $t(\gamma)$, and $|x-y| \leqslant q$. Then

$$\begin{aligned} E(\hat{g}_{pq}\circ t(x), \hat{g}_{pq}\circ t(y))\,|\hat{g}_{pq}\circ t(x)-\hat{g}_{pq}\circ t(y)|^{-1} &\leqslant k|x-y|^{-1}\,|x+i\varrho q|\,|y+i\varrho q|\int_x^y |z+i\varrho q|^{-2}\,|dz| \\ &\leqslant k|x-y|^{-1}\,|q|^2\int_x^y |q|^{-2}\,|dz| \leqslant k. \end{aligned} \tag{17}$$

We next take up the case that x and y lie on opposite sides of $t(\gamma)$ and $|x-y|\geqslant|q|$. Then

$$E(\hat{g}_{pq}\circ t(x), \hat{g}_{pq}\circ t(y))\,|\hat{g}_{pq}\circ t(x)-\hat{g}_{pq}\circ t(y)|^{-1}$$
$$\leqslant\left(\int_{-\infty}^{x}+\int_{y}^{\infty}\right)|\hat{g}_{pq}\circ t(x)-\hat{g}_{pq}\circ t(y)|^{-1}\,|(\hat{g}_{pq}\circ t)'(z)|\,|dz|. \tag{18}$$

We bound these two integrals separately. For the integral on the right, we choose a fundamental domain γ' for the action of H on W, so that $t(y)$ lies on one endpoint of γ'. Then

$$\int_{y}^{\infty}|\hat{g}_{pq}\circ t(x)-\hat{g}_{pq}\circ t(y)|^{-1}\,|(\hat{g}_{pq}\circ t)'(z)|\,|dz|$$
$$\leqslant k|x-y|^{-1}\,|x+i\varrho q|\,|y+i\varrho q|\,\Sigma_{r=1}^{\infty}\int_{t(\gamma')}|z+i\varrho q+r|^{-2}\,|dz|$$
$$\leqslant k|y+i\varrho q|\,\Sigma_{r=1}^{\infty}\int_{t(\gamma')}|z+i\varrho q+r|^{-2}\,|dz|$$
$$\leqslant k|y+i\varrho q|\,\Sigma_{r=1}^{\infty}(|y|^2+r^2+\varrho^2q^2)^{-1}\leqslant k|y+i\varrho q|\,(|y|^2+\varrho^2q^2)^{-1/2}\leqslant k. \tag{19}$$

The integral on the left in (18) is bounded similarly.

For $q=0$, and x and y lying on the same side of $t(\gamma)$, then we can simply set $q=0$ in (15) and (16). If x and y lie on opposite sides of $t(\gamma)$, or if one or both of them lies in $t(\gamma)$, and $|x-y|\leqslant 2$, then we integrate along the finite arc of $t(W)$, and we obtain the desired result as in (17). Similarly, if $|x-y|\geqslant 2$, then we integrate along the infinite arc of $t(W)$, and the desired bound is obtained as in (19).

Finally, the case $y=\infty$, or $x=\infty$, can be treated by taking the appropriate limits in (16), and (19).

10. Uniqueness

10.1. In this section we prove Theorem 4. We assume that G is a Koebe group normalized so that $\infty\in\Delta$, so that $|z|>1$ is precisely invariant under the identity in G and so that all structure loops lie in $|z|<1$. We also assume that we are given a second Koebe group G^*, with invariant component Δ^*, and that we are given a conformal similarity $\varphi\colon \Delta\to\Delta^*$, between G and G^*. We normalize φ so that near ∞

$$\varphi(z)=z+O(|z|^{-1}). \tag{20}$$

10.2. Let $\tilde{G}$ and $\tilde{G}^*$ be the respective extended Koebe groups. Using Lemmas 1, 2 and 3 together with Theorem 5, we see that we can extend φ to be a conformal homeomorphism $\varphi\colon \Omega(\tilde{G})\to\Omega(\tilde{G}^*)$, where $\varphi\circ g\circ\varphi^{-1}$ defines a type-preserving isomorphism of $\tilde{G}$ onto $\tilde{G}^*$.

10.3. The structure loops for $\tilde{G}$ divide $\hat{C}$ into sets called *structure regions*. If a structure region A intersects Δ, then $A \cap \Delta$ is a covering region of some $Y_i \subset \Delta/G$; hence $A \cap \Delta$ is precisely invariant under a factor subgroup H of G.

If H is elementary, then of course $A = (A \cap \Delta) \cup \Lambda(H)$.

If H is Fuchsian, then denoting reflection in $\Lambda(H)$ by r, we see that $A = (A \cap \Delta) \cup r(A \cap \Delta) \cup \Lambda(H)$.

If H is elementary, we set $\tilde{H} = H$. If H is Fuchsian, we let $\tilde{H}$ be the group generated by H and r. In either case A is precisely invariant under $\tilde{H}$ in $\tilde{G}$.

10.4. Every structure region A is equivalent under $\tilde{G}$ to one with non-trivial intersection with Δ. Hence for every structure region A, the stabilizer $\tilde{H}$ of A in $\tilde{G}$ is either elementary or a Fuchsian group extended by a reflection.

10.5. One sees at once that if W is a structure loop for $\tilde{G}$, then $\varphi(W)$ is a structure loop for $\tilde{G}^*$. Using (20) one sees that the structure loops $\{\varphi(W)\}$ are also uniformly bounded.

10.6. We enumerate the structure loops for $\tilde{G}$ as $\{W_p\}$.

LEMMA 10. *For any* ζ *with* $|\zeta|>2$,

$$\sum_p \left| \int_{W_p} \frac{\varphi(z)\,dz}{z-\zeta} \right| < \infty.$$

Proof. For each W_p, pick some point z_p on W_p, and observing that $|z-\zeta|>1$, we have

$$\left| \int_{W_p} \frac{\varphi(z)-\varphi(z_p)}{z-\zeta}\,dz \right| \leqslant L(W_p)\,\mathrm{dia}\,\varphi(W_p).$$

Hence, using Lemmas 6 and 8, we obtain

$$\sum_p \left| \int_{W_p} \frac{\varphi(z)\,dz}{z-\zeta} \right| \leqslant k\,\Sigma_p\,\mathrm{dia}\,W_p\,\mathrm{dia}\,\varphi(W_p) \leqslant k(\Sigma_p\,\mathrm{dia}^2\,W_p)^{1/2}\,(\Sigma_p\,\mathrm{dia}^2\,\varphi(W_p))^{1/2} < \infty.$$

10.7. The circle $|z|=1$ is also considered to be a structure loop; it is contained in some structure region A, for which it is the *outer structure loop*. The other structure loops on the boundary of A are called the *inner structure loops*.

For any other structure region A, the *outer structure loop* is that structure loop on the boundary of A which separates A from ∞; the other structure loops on the boundary of A are called the *inner structure loops*.

We orient all structure loops, including $|z|=1$ so as to have positive orientation as loops in C.

10.8. Lemma 11. *For any structure region A with outer structure loop W, and inner structure loops $\{V_p\}$, and for any ζ with $|\zeta|>2$,*

$$\int_W \frac{\varphi(z)\,dz}{z-\zeta}=\sum_p \int_{V_p} \frac{\varphi(z)\,dz}{z-\zeta}.$$

Proof. If $\tilde{H}$, the stabilizer of A is finite, then the sum is finite, and $\varphi(z)(z-\zeta)^{-1}$ is holomorphic in A.

If $\tilde{H}$ is elementary but not finite, it has a single limit point a. For $\varrho>0$ sufficiently small, the circle $|z-a|=\varrho$ lies inside W, and intersects only finitely many of the V_p. Let U_ϱ be $|z-a|=\varrho$ where each arc of this circle which does not lie in A has been replaced by the shorter arc of the appropriate inner structure loop, so that U_ϱ lies in $\bar{A}$. We reorder the loops $\{V_p\}$ so that the first n of them lie outside U_ϱ. Then

$$\int_W \frac{\varphi(z)\,dz}{z-\zeta}=\sum_{p=1}^{n} \int_{V_p} \frac{\varphi(z)\,dz}{z-\zeta}+\int_{U_\varrho} \frac{\varphi(z)\,dz}{z-\zeta}.$$

As $\varrho\to 0$, $L(U_\varrho)\to 0$ by Lemma 9, and of course the integrand is bounded. Hence letting $\varrho\to 0$ and $n\to\infty$, we obtain the desired result.

We next consider the case that $\tilde{H}$ is extended Fuchsian with limit circle $|z-a|=\sigma$. For $\varrho>\sigma$ and ϱ sufficiently small, the circle $|z-a|=\varrho$ lies inside the outer structure loop of A, and intersects only finitely many inner structure loops. As in the preceding case, we replace circular chords of $|z-a|=\varrho$ by shorter arcs of structure loops to obtain a new loop U_ϱ lying in the closure of A.

We denote reflection in $\Lambda(\tilde{H})$ by r. We order the inner structure loops so that the first n lie either outside U_ϱ or inside $r(U_\varrho)$, and we observe that

$$\int_W \frac{\varphi(z)\,dz}{z-\zeta}=\sum_{p=1}^{n} \int_{V_p} \frac{\varphi(z)\,dz}{z-\zeta}+\int_{U_\varrho} \frac{\varphi(z)\,dz}{z-\zeta}-\int_{r(U_\varrho)} \frac{\varphi(z)\,dz}{z-\zeta}.$$

As $\varrho\to\sigma$, the sum on the right converges to the sum over all inner structure loops of A.

We remark that φ is continuous across $\Lambda(\tilde{H})$. For we can extend φ to be a homeomorphism on $\Omega(\tilde{H})$ which conjugates $\tilde{H}$ into $\varphi\circ\tilde{H}\circ\varphi^{-1}$. Using the fact that every point on $\Lambda(\tilde{H})$ either is a parabolic fixed point, or it can be realized as a nested sequence of translates of some axis in H, we see that φ is continuous across $\Lambda(H)$.

Using Lemma 9 again, the lengths of the loops U_ϱ are bounded, and so

$$\int_{U_\varrho} \frac{\varphi(z)\,dz}{z-\zeta}-\int_{r(U_\varrho)} \frac{\varphi(z)\,dz}{z-\zeta}\to 0.$$

10.9. We now prove Theorem 4. Using (20), we see that for $|\zeta|>2$,

$$\varphi(\zeta)=\zeta-(2\pi i)^{-1}\int_{|z|=1}\frac{\varphi(z)\,dz}{z-\zeta}. \tag{21}$$

The circle $|z|=1$ is the outer structure loop for some structure region A_1. By Lemma 11, we can replace the integral in (21) by the sum of integrals over the inner structure loops of A_1. Each of these inner structure loops is in turn an outer structure loop for another structure region. Hence, if we enumerate the structure loops as $\{W_{pq}\}$, where each W_{pq} is separated from ∞ by p other structure loops, then as a consequence of Lemma 11, we obtain

$$\Sigma_q\int_{W_{pq}}\frac{\varphi(z)\,dz}{z-\zeta}=\Sigma_q\int_{W_{p+1,q}}\frac{\varphi(z)\,dz}{z-\zeta}.$$

Hence, for every p,

$$\varphi(\zeta)=\zeta-(2\pi i)^{-1}\Sigma_q\int_{W_{pq}}\frac{\varphi(z)\,dz}{z-\zeta}.$$

It follows from Lemma 10 that as $p\to\infty$,

$$\Sigma_q\int_{W_{pq}}\frac{\varphi(z)\,dz}{z-\zeta}\to 0.$$

Hence $\varphi(\zeta)=\zeta$.

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