CURVATURE ESTIMATES FOR MINIMAL HYPERSURFACES

BY

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In [12] J. Simons initiated a study of minimal cones from a more differential geometric point of view than had previously been attempted. One of Simons' main results was an identity for the Laplacian of the second fundamental form of minimal hyper-surfaces. Coupling this identity with an analysis of the first eigenvalue of a certain differential operator, he was able to prove that no non-trivial n-dimensional stable minimal cones exist in $\mathbf{R}^{n+1}$ for $n \leqslant 6$. He was thus able to demonstrate that any boundary of least area in $\mathbf{R}^{n+1}$, $n \leqslant 6$, must in fact be a hyperplane, because Fleming [7] had demonstrated that the non-existence of non-trivial stable minimal cones in $\mathbf{R}^n$ implies the result that the only boundaries of least area in $\mathbf{R}^n$ are the hyperplanes.

Simons was in fact able to deduce that, for $n \leqslant 7$, the only entire solutions of the minimal surface equation

$$(1+|\nabla u|^2)\sum_{j=1}^{n}\frac{\partial^2 u}{\partial x_j^2} - \sum_{i,j=1}^{n}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j} = 0 \tag{*}$$

are linear functions, because De Giorgi [6] had improved Fleming's result in the non-parametric case, by showing that the non-existence of non-trivial stable minimal cones in $\mathbf{R}^n$ implies that the only *non-parametric* boundaries of least area in $\mathbf{R}^{n+1}$ are the hyper-planes. The conjecture that the only entire solution of (*) are linear functions was known as the Bernstein conjecture, after Bernstein [2]. Prior to Simons' paper, it had been settled in the case $n=2$ by Bernstein [2], $n=3$ by De Giorgi [6] and $n=4$ by Almgren [1]. Sub-sequent to Simons' paper the conjecture was finally completely settled; it was shown to be false for $n>7$ by Bombieri, De Giorgi and Giusti [3].

In the case $n=2$ Heinz [8] considered solutions of (*) which were defined over a disc $\{x \in \mathbf{R}^2 : |x-x_0| < R\}$. He proved there is an absolute constant β such that

$$(\varkappa_1^2+\varkappa_2^2)(x_0) \leqslant \beta/R^2, \tag{**}$$

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where $\varkappa_1$, $\varkappa_2$ are principal curvatures of the graph of the solution u of (*). In the case when u was an entire solution of (*) Heinz let $R\to\infty$ in (**) and hence proved $\varkappa_1=\varkappa_2\equiv 0$; i.e., (**) implies Bernstein's theorem in the case $n=2$. The result (**) and its proof have been refined by various authors, and a parametric version was obtained by Osserman (See [11]). However, the methods used were all strictly 2-dimensional.

In this paper we will use Simons' identity for the Laplacian of the second fundamental form for minimal hypersurfaces to obtain a number of new estimates for the curvatures of stable minimal hypersurfaces M which are immersed in a Riemannian manifold N. Under suitable restrictions on N, we will in fact obtain (see Theorem 3) a pointwise bound for the principal curvatures of M, provided dim $(M)\leqslant 5$. In the special case when $N=\mathbf{R}^{n+1}$, when M is an area minimizing hypersurface with boundary outside the ball $\{x\in\mathbf{R}^{n+1}: |x-x_0|<R\}$ and when $n\leqslant 5$, Theorem 3 gives the inequality (cf. (**) above)

$$\sum_{i=1}^{n}\varkappa_i^2(x_0)\leqslant\beta/R^2,$$

where $\varkappa_1, ..., \varkappa_n$ are the principal curvatures of M and β is an absolute constant. When $\partial M=\phi$, we can let $R\to\infty$ and deduce $\varkappa_i\equiv 0$, $i=1, ..., n$; i.e., we obtain a proof of Bernstein's Theorem for $n\leqslant 5$. A Bernstein-type result which is valid in a more general setting is given in Theorem 2.

In the final section of the present paper we give a simplified proof of Simons' result that there are no non-trivial 6-dimensional stable minimal cones in $\mathbf{R}^7$.

§ 1. Notation and preliminary results

In this section, we set up our terminology and record Chern's [4] computation o Simons' inequality for minimal hypersurfaces. (See inequality (1.20) below.) We then demonstrate that this inequality gives a better lower bound (inequality (1.34)) for the Laplacian of $|A|$ (A = second fundamental form of M) than had previously been realized.

Let M be an oriented n-dimensional manifold immersed in an oriented $(n+1)$-dimensional Riemannian manifold N. We choose a local field of orthonormal frames $e_1, ..., e_{n+1}$ in N such that, restricted to M, the vectors $e_1, ..., e_n$ are tangent to M. With respect to this frame field of N, let $\omega_1, ..., \omega_{n+1}$ be the field of dual frames. Then the structure equations of N are given by

$$d\omega_i=-\sum_{j=1}^{n+1}\omega_{ij}\wedge\omega_j,\quad \omega_{ij}+\omega_{ji}=0 \tag{1.1}$$

$$d\omega_{ij}=-\sum_{k=1}^{n+1}\omega_{ik}\wedge\omega_{kj}+\Omega_{ij}, \tag{1.2}$$

where

$$\Omega_{ij} = \frac{1}{2}\sum_{k,l=1}^{n+1} K_{ijkl}\,\omega_k \wedge \omega_l$$

and

$$K_{ijkl} + K_{ijlk} = 0.$$

We restrict these forms to M. Then

$$\omega_{n+1} = 0. \tag{1.3}$$

Since $0 = d\omega_{n+1} = -\sum_{i=1}^{n} \omega_{n+1,i} \wedge \omega_i$, by Cartan's lemma we can write

$$\omega_{n+1,i} = \sum_j h_{ij}\omega_j, \quad h_{ij} = h_{ji}. \tag{1.4}$$

Here and in what follows, the range of summation is from 1 to n.

By using (1.1)–(1.4), we obtain

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{1.5}$$

$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} + \frac{1}{2}\sum_{k,l} R_{ijkl}\,\omega_k \wedge \omega_l, \tag{1.6}$$

where

$$R_{ijkl} = K_{ijkl} + h_{ik}h_{jl} - h_{il}h_{jk}. \tag{1.7}$$

The form $\sum_{i,j} h_{ij}\omega_i\omega_j$ and the scalar $(1/n)\sum_i h_{ii} = H$ are called respectively the second fundamental form and the mean curvature of the immersed manifold M. If H is identically zero, M is said to be minimal.

Now exterior differentiate (1.4) and define h_{ijk} by

$$\sum_k h_{ijk}\omega_k = dh_{ij} - \sum_k h_{ik}\,\omega_{kj} - \sum_k h_{kj}\,\omega_{ki}. \tag{1.8}$$

Then

$$\sum_{j\ k} (h_{ijk} + \tfrac{1}{2} K_{n+1,ijk})\,\omega_j \wedge \omega_k = 0, \tag{1.9}$$

$$h_{ijk} - h_{ikj} = K_{n+1,ikj} = -K_{n+1,ijk}. \tag{1.10}$$

Next, we exterior differentiate (1.8) and define h_{ijkl} by

$$\sum_l h_{ijkl}\,\omega_l = dh_{ijk} - \sum_l h_{ljk}\,\omega_{li} - \sum_l h_{ijl}\,\omega_{lk} - \sum_l h_{ilk}\,\omega_{lj}. \tag{1.11}$$

Then

$$\sum_{k,l} (h_{ijkl} - \tfrac{1}{2}\sum_m h_{im}R_{m,kl} - \tfrac{1}{2}\sum_m h_{mj}R_{mikl})\,\omega_k \wedge \omega_l = 0, \tag{1.12}$$

and

$$h_{ijkl} - h_{ijlk} = \sum_m h_{im}R_{mjkl} + \sum_m h_{mj}R_{mikl}. \tag{1.13}$$

Let us now denote the covariant derivative of K_{ijkl}, as a curvature tensor of N, by $K_{ijkl;m}$. Then restricting to M, we obtain

$$K_{n+1,ijk;l} = K_{n+1,ijkl} - K_{n+1,i,n+1,k}h_{jl} - K_{n+1,ij,n+1}h_{kl} + \sum_m h_{ml}K_{mijk}, \tag{1.14}$$

where

$$\sum_l K_{n+1,ijkl}\omega_l = dK_{n+1,ijk} - \sum_m K_{n+1,mjk}\omega_{mi} - \sum_m K_{n+1,imk}\omega_{mj} - \sum_m K_{n+1,ijm}\omega_{mk}. \tag{1.15}$$

The Laplacian Δh_{ij} of the second fundamental form h_{ij} is defined by

$$\Delta h_{ij} = \sum_k h_{ijkk}. \tag{1.16}$$

From (1.10), we obtain

$$\Delta h_{ij} = \sum_k h_{ikjk} - \sum_k K_{n+1,ijkk} = \sum_k h_{kijk} - \sum_k K_{n+1,ijkk}. \tag{1.17}$$

Also, from (1.13) we obtain

$$h_{kijk} = h_{kikj} + \sum_m h_{km}R_{mijk} + \sum_m h_{mi}R_{mkjk}. \tag{1.18}$$

Then if we replace h_{kikj} in (1.18) by $h_{kkij} - K_{n+1,kikj}$ (by (1.10)) and if we substitute the right hand side of (1.18) into h_{kijk} of (1.17), we obtain

$$\Delta h_{ij} = \sum_k (h_{kkij} - K_{n+1,kikj} - K_{n+1,ijkk}) + \sum_k (\sum_m h_{km}R_{mijk} + \sum_m h_{mi}R_{mkjk}). \tag{1.19}$$

From (1.7), (1.14) and (1.19) we then obtain

$$\begin{aligned}\Delta h_{ij} = &\sum_k h_{kkij} - \sum_k K_{n+1,kik;j} - \sum_k K_{n+1,ijk;k} + \sum_k (-h_{kk}K_{n+1,ij,n+1} - h_{ij}K_{n+1,k,n+1,k}) \\ &+ \sum_{m,k} (h_{mj}K_{mkik} + h_{mi}K_{mkjk} + 2h_{mk}K_{mijk}) \\ &+ \sum_{m,k} (h_{mi}h_{mj}h_{kk} + h_{km}h_{ki}h_{mj} - h_{km}h_{km}h_{ij} - h_{mi}h_{mk}h_{kj}).\end{aligned} \tag{1.20}$$

Now assuming M is minimal in N, so that $\Sigma_k h_{kk} = 0$, we obtain

$$\begin{aligned}\sum_{i,j} h_{ij}\Delta h_{ij} = &- \sum_{i,j,k} h_{ij}K_{n+1,kik;j} - \sum_{i,j,k} h_{ij}K_{n+1,ijk;k} - \sum_{i,j,k} h_{ij}^2 K_{n+1,k,n+1,k} \\ &+ \sum_{m,i,j,k} (2h_{mj}h_{ij}K_{mkik} + 2h_{mk}h_{ij}K_{mijk}) - (\sum_{i,j} h_{ij}^2)^2.\end{aligned} \tag{1.21}$$

Up to now, we have followed the exposition in [5]. In order to proceed, we assume that the sectional curvatures of N are bounded between K_1 and K_2 and

$$|\nabla K|^2 = \sum_{i,j,k,l,m} K_{ijkl;m}^2 \leq c^2. \tag{1.22}$$

For any point $p \in M$, we can choose our frame $\{e_1, \dots, e_n\}$ at that point so that

$$h_{ij} = \lambda_i \delta_{ij}. \tag{1.23}$$

At such a point we have

$$\sum_{m,i,j,k}(2h_{mj}h_{ij}K_{mkik}+2h_{mk}h_{ij}K_{mijk})=\sum_{i,k}(2\lambda_i^2K_{ikik}+2\lambda_k\lambda_iK_{kiik})=\sum_{i,k}(\lambda_i^2-2\lambda_i\lambda_k+\lambda_k^2)K_{ikik}$$
$$=\sum_{i,k}(\lambda_i-\lambda_k)^2K_{ikik}\geqslant K_2\sum_{i,k}(\lambda_i-\lambda_k)^2=2nK_2\sum_i\lambda_i^2-2K_2(\sum_i\lambda_i)^2=2nK_2\sum_i\lambda_i^2. \quad (1.24)$$

It then follows from (1.21), (1.22), (1.23) and (1.24) that

$$\sum_{i,j}h_{ij}\Delta h_{ij}\geqslant -2\sqrt{\sum_{i,j}h_{ij}^2}\sqrt{\sum_{i,j,k,l,m}K_{ijkl;m}^2}-nK_1(\sum_{i,j}h_{ij}^2)+2nK_2(\sum_{i,j}h_{ij}^2)-(\sum_{i,j}h_{ij}^2)^2$$
$$\geqslant -2c\sqrt{\sum_{i,j}h_{ij}^2}+n(2K_2-K_1)(\sum_{i,j}h_{ij}^2)-(\sum_{i,j}h_{ij}^2)^2 \quad (1.25)$$

Now let

$$|A|^2=\sum_{i,j}h_{ij}^2. \quad (1.26)$$

Then (1.25) shows that, at all points where $|A|\neq 0$,

$$2|A|\Delta|A|+2(\nabla|A|)^2=\Delta|A|^2=2\sum_{i,j,k}h_{ijk}^2+2\sum_{i,j}h_{ij}\Delta h_{ij}$$
$$\geqslant 2\sum_{i,j,k}h_{ijk}^2-4c|A|+2n(2K_2-K_1)|A|^2-2|A|^4. \quad (1.27)$$

The crucial point now is to give a lower bound for $\Sigma_{i,j,k}h_{ijk}^2$ in terms of $|\nabla|A||^2$. First, by using (1.10) together with the inequality

$$2|K_{n+1,iji}|\leqslant K_1-K_2,$$

we obtain

$$\sqrt{\sum_{i\neq j}h_{iji}^2}\geqslant\sqrt{\sum_{i\neq j}h_{iij}^2}-\sqrt{\sum_{i,j}(h_{iji}-h_{iij})^2}\geqslant\sqrt{\sum_{i\neq j}h_{iij}^2}-\sqrt{\sum_{i,j}K_{n+1,iji}^2}$$
$$\geqslant\sqrt{\sum_{i\neq j}h_{iij}^2}-\tfrac{1}{2}\sqrt{n(n-1)}\,(K_1-K_2). \quad (1.28)$$

Also,

$$\sum_{i,j,k}h_{ijk}^2-|\nabla|A||^2=[(\sum_{i,j}h_{ij}^2)(\sum_{i,j,k}h_{ijk}^2)-\sum_k(\sum_{i,j}h_{ij}h_{ijk})^2](\sum_{i,j}h_{ij}^2)^{-1}$$
$$=\tfrac{1}{2}\sum_{i,j}\sum_{s,t,k}(h_{ij}h_{stk}-h_{st}h_{ijk})^2(\sum_{i,j}h_{ij}^2)^{-1}, \quad (1.29)$$

and, using (1.23),

$$\sum_{i,j,s,t,k}(h_{ij}h_{stk}-h_{st}h_{ijk})^2=\sum_{i,s,t,k}(h_{ii}h_{stk}-h_{st}h_{iik})^2+(\sum_{s,t}h_{st}^2)(\sum_{\substack{i\neq j\\ k}}h_{ijk}^2)$$
$$\geqslant(\sum_i h_{ii}^2)\sum_{\substack{s\neq t\\ k}}h_{stk}^2+\sum_{s,t}h_{st}^2(\sum_{\substack{i\neq j\\ k}}h_{ijk}^2)=2(\sum_{s,t}h_{st}^2)(\sum_{\substack{i\neq j\\ k}}h_{ijk}^2). \quad (1.30)$$

But by using (1.28) we obtain

$$\sum_{\substack{i\neq j\\ k}}h_{ijk}^2\geqslant\sum_{i\neq j}h_{iji}^2+\sum_{i\neq j}h_{ijj}^2=2\sum_{i\neq j}h_{iji}^2\geqslant\frac{2}{1+\varepsilon}(\sum_{i\neq j}h_{iij}^2)-\frac{n(n-1)}{2\varepsilon}(K_1-K_2)^2 \quad (1.31)$$

for all $\varepsilon>0$. Here we have used the fact that

$$\sqrt{A}\geqslant\sqrt{B}-\sqrt{C}\text{ implies }A\geqslant\frac{B}{1+\varepsilon}-\frac{C}{\varepsilon}$$

for any non-negative A, B, C and any $\varepsilon>0$.

Then since

$$|\nabla|A||^2=\sum_k(\sum_{i,j}h_{ij}h_{ijk})^2(\sum_{i,j}h_{ij}^2)^{-1}=\sum_\kappa(\sum_i h_{ii}h_{iik})^2(\sum_i h_{ii}^2)^{-1}\leqslant\sum_{i,\kappa}h_{iik}^2=\sum_{i\neq k}h_{iik}^2+\sum_i h_{iii}^2$$
$$=\sum_{i\neq k}h_{iik}^2+\sum_i(\sum_{j\neq i}h_{jji})^2\leqslant\sum_{i\neq k}h_{iik}^2+(n-1)\sum_{i\neq j}h_{jji}^2=n\sum_{i\neq j}h_{jji}^2, \tag{1.32}$$

we can conclude from (1.29), (1.30) and (1.31) that

$$\sum_{i,j,k}h_{ijk}^2-|\nabla|A||^2\geqslant\sum_{\substack{i\neq j\\ k}}h_{ijk}^2\geqslant\frac{2}{1+\varepsilon}(\sum_{i\neq j}h_{iij}^2)-\frac{n(n-1)}{2\varepsilon}(K_1-K_2)^2$$
$$\geqslant\frac{2}{(1+\varepsilon)n}|\nabla|A||^2-\frac{n(n-1)}{2\varepsilon}(K_1-K_2)^2. \tag{1.33}$$

Combining (1.27) and (1.33), we then have

$$|A|\Delta|A|+|A|^4\geqslant\frac{2}{(1+\varepsilon)n}|\nabla|A||^2-\frac{n(n-1)}{2\varepsilon}(K_1-K_2)^2-2c|A|+n(2K_2-K_1)|A|^2. \tag{1.34}$$

at all points where $|A|\neq 0$. Actually since $|A|\Delta|A|=\frac{1}{2}\Delta|A|^2-|\nabla|A||^2$ we can in fact see that this inequality must be globally true in the distribution sense, even if $|A|$ vanishes at various points.

The next important inequality is the stability inequality. Recall that a minimal hypersurface is called stable if and only if the second variation of the area functional is non-negative for all compactly supported deformations. Of course it is true that area minimizing hypersurfaces are stable in this sense. A direct computation (cf. [4]) shows that, if M is stable, then

$$\int_M[f\Delta f+(\sum_i K_{n+1,i,n+1,i}+|A|^2)f^2]\leqslant 0$$

for any smooth function f with compact support in M. Therefore

$$\int_M[f\Delta f+(nK_2+|A|^2)f^2]\leqslant 0;$$

that is

$$\int_M(nK_2+|A|^2)f^2\leqslant\int_M|\nabla f|^2. \tag{1.35}$$

Replacing f by $|A|^{1+q}f \quad (q \geq 0)$ this gives

$$\int_M (nK_2|A|^{2+2q}+|A|^{4+2q})f^2 \leq \int_M (1+q)^2|A|^{2q}|\nabla|A||^2 f^2+|A|^{2+2q}|\nabla f|^2$$
$$+(2+2q)\int_M |A|^{1+2q}f(\nabla|A|)\cdot(\nabla f). \tag{1.36}$$

On the other hand if we multiply the inequality (1.34) by $f^2|A|^{2q}$ and integrate over M we obtain (after integrating by parts in $\int_M |A|^{1+2q}(\Delta|A|)f^2$)

$$\frac{2}{(1+\varepsilon)n}\int_M |A|^{2q}|\nabla|A||^2 f^2 \leq -(1+2q)\int_M |A|^{2q}|\nabla|A||^2 f^2$$
$$+\int_M \left\{n(K_1-2K_2)|A|^2+|A|^4+|\nabla|A||^2+2c|A|+\frac{n(n-1)}{2\varepsilon}(K_1-K_2)^2\right\}|A|^{2q}f^2$$
$$-2\int_M |A|^{1+2q}f(\nabla|A|)\cdot(\nabla f). \tag{1.37}$$

By adding (1.36), (1.37) and using the inequality

$$2q|A|^{1+2q}f(\nabla|A|)\cdot(\nabla f)\leq \varepsilon q^2 f^2|A|^{2q}|\nabla|A||^2+\varepsilon^{-1}|A|^{2+2q}|\nabla f|^2,$$

we then have

$$\left(\frac{2}{(1+\varepsilon)n}-(1+\varepsilon)q^2\right)\int_M |A|^{2q}|\nabla|A||^2 f^2$$
$$\leq \int_M (1+\varepsilon^{-1})|A|^{2q}\left\{|A|^2|\nabla f|^2+n(K_1-3K_2)|A|^2 f^2+2c|A|f^2+\frac{n(n-1)}{2\varepsilon}(K_1-K_2)^2 f^2\right\}. \tag{1.38}$$

Inequality (1.38) will be of central importance in what follows.

§ 2. Main results

Throughout this section we will assume M is a stable immersion, so that all the inequalities of § 1 are valid.

First of all, we obtain an L_p estimate for A by using (1.38) together with (1.36).

THEOREM 1. *For each $p\in[4,4+\sqrt{8/n})$ and for each non-negative smooth function f with compact support in M, we have*

$$\int_M |A|^p f^p \leq \beta\int_M [|\nabla f|^p+(c^{2/3}+K_1-K_2+\max\{-K_2,0\})^{p/2}f^p],$$

where β is a constant depending only on n and p.

Proof. Let $q=(p-4)/2$, so that $q>0$ and $q^2<2/n$. By using (1.38) with ε chosen small enough to ensure $2/[(1+\varepsilon)n]-(1+\varepsilon)q^2>0$, we have

$$\int_M |A|^q\,|\nabla|A|\,|^2 f^2 \leqslant \beta_1 \int_M (|A|^{p-2}|\nabla f|^2+(K_1-3K_2)|A|^{p-2}f^2+c|A|^{p-3}f^2+(K_1-K_2)^2|A|^{p-4}f^2), \tag{2.1}$$

where β_1 depends only on n, p.

On the other hand, (1.36) says

$$\int_M |A|^p f^2 \leqslant \int_M ((1+q)^2|A|^{2q}\,|\nabla|A|\,|^2 f^2+2(1+q)(|A|^q f\nabla|A|)\cdot(|A|^{p/2-1}\nabla f)) + \int_M (|A|^{p-2}\,|\nabla f|^2-nK_2|A|^{p-2}f^2). \tag{2.2}$$

By the Cauchy inequality we have

$$(|A|^q f\nabla|A|)\cdot(|A|^{p/2-1}\nabla f) \leqslant \tfrac{1}{2}|A|^{2q}f^2|\nabla|A||^2+\tfrac{1}{2}|A|^{p-2}|\nabla f|^2, \tag{2.3}$$

and by using Young's inequality we have the following for each $\varepsilon>0$:

$$c|A|^{p-3} \leqslant \varepsilon|A|^p+\beta_2 c^{p/3}, \tag{2.4}$$

$$|A|^{p-2}|\nabla f|^2 = f^2(|A|^{p-2}(|\nabla f|^2/f^2)) \leqslant \varepsilon|A|^p f^2+\beta_3(|\nabla f|^p/f^{p-2}), \tag{2.5}$$

$$\max\{(K_1-3K_2)|A|^{p-2},\ -nK_2|A|^{p-2}\} \leqslant \varepsilon|A|^p+\beta_4\,(\max\{K_1-3K_2,\ -nK_2,\ 0\})^{p/2} \tag{2.6}$$

$$(K_1-K_2)^2|A|^{p-4} \leqslant \varepsilon|A|^p+\beta_5(K_1-K_2)^{p/2}, \tag{2.7}$$

where $\beta_2, \ldots, \beta_5$ are determined by ε, p.

Now let M_+ be defined by

$$M_+ = \{x\in M\colon f\neq 0\}.$$

Then using (2.1), (2.3)–(2.7) in the inequality (2.2) we obtain

$$(1-\beta_6\varepsilon)\int_M |A|^p f^2 \leqslant \beta_7 \int_{M_+} \{|\nabla f|^p/f^{p-2}+(c^{p/3}+(K_1-K_2)^{p/2} + (\max\{K_1-3K_2,\ -nK_2, 0\})^{p/2})f^2\} \tag{2.8}$$

where β_6 depends on n, p and β_7 depends on ε, n, p.

If we now take $\varepsilon=1/(2\beta_6)$ and replace f by $f^{p/2}$ the required inequality easily follows.

Suppose now that we have a constant R_0 with $0<R_0\leqslant\infty$ and a family of subsets $\{B_R\}_{R\in(0,R_0)}$ defined by

$$B_R = \{x\in M\colon r(x)\leqslant R\},$$

where r is a given Lipschitz function on M with

$$|\nabla r| \leqslant 1 \quad \text{a.e. on } M.$$

Suppose also that each B_R is compact and

$$M = \bigcup_{R \in (0, R_0)} B_R.$$

We have in mind the particular cases where the B_R are either geodesic balls of radius R in M or the intersection with M of geodesic balls of radius R in N. (We note that in the former case, the immersion of M into N need not even be proper.)

Now let $f = \gamma \circ r$, where γ is the Lipschitz function on R with $\gamma(t) \equiv 1$ for $t \leq \theta R$, ($\theta \in (0,1)$ a given constant) $\gamma(t) \equiv 0$ for $t > R$ and with $\gamma(t)$ decreasing linearly for $t \in (\theta R, R)$. It is then not difficult to see that Theorem 1 implies

$$\int_{B_{\theta R}} |A|^p \leq \tilde{\beta} R^{-p} |B_R|, \quad R \in (0, R_0), \quad \theta \in (0,1), \quad p \in (0, 4 + \sqrt{8/n}), \tag{2.9}$$

where $\tilde{\beta} = \beta\{(1-\theta)^{-p} + [R^2(c^{2/3} + K_1 - K_2 + \max\{-K_2, 0\})]^{p/2}\}$ (β as in Lemma 1) and $|B_R|$ is the n-dimensional volume of B_R.

Note that if $c = 0$ and $K_1 = K_2 \geq 0$, then (2.9) gives

$$\int_{B_{\theta R}} |A|^p \leq \frac{\beta}{(1-\theta)^p} R^{-p} |B_R|.$$

Thus if $\lim_{R \to \infty} R^{-p} |B_R| = 0$ for some $p \in (0, 4 + \sqrt{8/n})$, we must then have $|A| = 0$; that is, we have the Bernstein type result stated in the following theorem.

THEOREM 2. *Suppose* $K_1 = K_2 \geq 0$, $c = 0$ *and* $\lim_{R \to \infty} R^{-p} |B_R| = 0$ *for some* $p \in (0, 4 + \sqrt{8/n})$. *Then M is totally geodesic.*

Remarks. 1. Suppose N is complete and M is a boundary of least area in N in the sense that $M = \partial U = \partial \bar{U}$ for some open $U \subset N$ and $\operatorname{Vol}(\partial U \cap A) \leq \operatorname{Vol}(\partial A \cap U)$ for each open $A \subset N$ with compact closure. Then we can take r to be geodesic distance in N and prove that

$$|B_R| \leq \tfrac{1}{2} \operatorname{Vol}(S_R),$$

where S_R is the geodesic sphere of radius R in N.

In particular, if N is flat we deduce that $|B_R|$ has order at most R^n and hence there is a p satisfying the conditions of Theorem 2 provided $n < 4 + \sqrt{8/n}$, that is, $n \leq 5$. Thus we deduce that any boundary of least area in N is totally geodesic if $n \leq 5$. In particular, we deduce Bernstein's theorem for minimal graphs in $\mathbf{R}^{n+1}$ when $n \leq 5$.

The dimensional restriction $n \leq 5$ can be relaxed if the volume growth of N is small. For example if N is the product of the $(n-4)$-dimensional torus and the 5-dimensional Euclidean space, then all boundaries of least area in N are totally geodesic.

In the case $n \leqslant 5$ we now show that one can actually obtain a pointwise bound for $|A|$, provided appropriate restrictions are imposed on N.

We here continue to use the family of subsets $\{B_R\}_{R\in(0,R_0)}$ introduced above.

THEOREM 3. *Suppose N is simply connected, complete and has non-positive curvature $(K_1 \leqslant 0)$. Then if $n \leqslant 5$ and*

$$R^2(c^{2/3}+|K_2|)+R^{-n}|B_R| \leqslant \beta_0,$$

we have

$$\sup_{B_{\theta R}} |A| \leqslant \beta R^{-1}$$

for each $\theta \in (0,1)$, where β is a constant depending only on θ, n and β_0.

Remarks. We note that in the special case when $N=\mathbf{R}^{n+1}$, if M is a boundary of least area in $\{x\in\mathbf{R}^{n+1}: |x-x_0|<R_0\}$ and if B_R is the intersection with M of the ball $\{x\in\mathbf{R}^{n+1}: |x-x_0|\leqslant R\}$, then (because $R^{-n}|B_R| \leqslant (n+1)\omega_{n+1}/2$—see Remark 1 after Theorem 2) the inequality of the theorem implies

$$|A|(x_0) \leqslant \beta_1/R, \quad R<R_0, \tag{2.10}$$

where β_1 is an absolute constant. (Note that β_1 can be computed explicitly.) If $R_0=\infty$ we can let $R\to\infty$ in (2.10) and obtain another proof of Bernstein's theorem for $n\leqslant 5$.

It is an open question whether or not an inequality like (2.10) is true in the case $n=6$. In the case $n=2$ an inequality of the form (2.10) was first established for non-parametric surfaces in [8]. An analogous result, also in the case $n=2$, was established by Osserman (see [11]). Osserman's result was proved subject to the assumption that the Gauss map omits a neighborhood of S^2; no stability condition was assumed. However, since it is not clear whether or not a 2-dimensional boundary of least area must have a Gauss map which omits a neighborhood of S^2 (at least when we restrict the Gauss map to B_R, $R<R_0$), our inequality seems to be of some interest even in the case $n=2$.

Proof of Theorem 3. By (1.27) it is not difficult to check that the function $u=R^{-2}\beta_0^2+|A|^2$ satisfies an inequality of the form

$$\Delta u+\beta_1(R^{-2}+|A|^2)u \geqslant 0, \tag{2.11}$$

where β_1 is a constant depending only on n.

We now need to recall a well known result from the theory of elliptic equations (see for example [10], Theorem 5.3.1): Suppose ϕ is a non-negative function satisfying

$$\Delta\phi+c\phi \geqslant 0$$

on some ball K_R of radius R in $\mathbf{R}^n$. Then for each $\varepsilon > 0$ and each $\theta \in (0,1)$

$$\sup_{K_{\theta R}} \phi \leqslant c_1 \left\{ R^{-n} \int_{K_R} \phi^2 dx \right\}^{1/2}, \tag{2.12}$$

where c_1 depends only on n, ε, θ and $R^\varepsilon \int_{K_R} |c|^{(n+\varepsilon)/2} dx$.

The same argument can be used to bound functions u satisfying (2.11) on M. The only difficulty in modifying the proof from $\mathbf{R}^n$ to the present manifold setting is that one needs a suitable Sobolev inequality. Under the hypotheses stated in Theorem 3 such an inequality has been proved in [9]. In fact it is proved in [9] that if N is simply connected, complete and has non-positive curvature then

$$\left\{ \int_M f^{n/(n-1)} \right\}^{(n-1)/n} \leqslant c_2 \int_M |\nabla f|$$

for any smooth f with compact support in M, where c_2 depends only on n.

Thus we can copy the $\mathbf{R}^n$ proof of (2.12) and obtain

$$\sup_{B_{\theta R}} |A|^2 \leqslant c_3 \left\{ R^{-n} \int_{B_R} (R^{-2} \beta_0^2 + |A|^2)^2 \right\}^{1/2},$$

where c_3 depends on $R^\varepsilon \int_{B_R} (R^{-2} + |A|^2)^{(n+\varepsilon)/2}$, n, ε and θ. Choosing $\varepsilon > 0$ such that $n + \varepsilon < 4 + \sqrt{8/n}$ (which can be done for $n \leqslant 5$) and using (2.9), we then have Theorem 3.

§ 3. Minimal cones in $\mathbf{R}^{n+1}$

We conclude this paper with a simplified proof of Simons' theorem concerning the non-existence of stable 6-dimensional minimal cones in $\mathbf{R}^7$.

We let C be an n-dimensional stable minimal cone in $\mathbf{R}^{n+1}$ with vertex 0. That is, C is a union of rays emanating from 0 such that $C - \{0\}$ is a n-dimensional C^∞ stable minimal submanifold of $\mathbf{R}^{n+1}$.

Using (1.27) and (1.29) together with the fact that $c = K_1 = K_2 = 0$ (since $N = \mathbf{R}^{n+1}$ in this case), we have

$$|A| \Delta |A| + |A|^4 = \tfrac{1}{2} |A|^{-2} \sum_{i,j,r,s,k} \sigma_{ijrsk}^2$$

at all points of $C - \{0\}$ for which $|A| \neq 0$, where

$$\sigma_{ijrsk} = h_{ij} h_{rsk} - h_{rs} h_{ijk}, \quad i, j, r, s, k = 1, \ldots, n.$$

Then clearly, since $h_{ij} = h_{ji}$ and $h_{ijk} = h_{ikj}$ (by (1.10)), we then have

$$|A| \Delta |A| + |A|^4 \geqslant 2 |A|^{-2} \sum_{k=1}^{n} \sum_{j \neq n, r \neq n, s \neq n} \sigma_{njrsk}^2. \tag{3.1}$$

If we now choose a frame $e_1, \ldots, e_n$ at a given point x in such a way that h_{ij} is diagonal and such that e_n is in the radial direction $x/|x|$, then we have

$$h_{ij}=0,\ i\neq j,\ h_{nn}=0 \quad \text{and} \quad h_{ij,n}=-|x|^{-1}h_{ij},\ i,j=1,\ldots,n.$$

Then

$$\sigma_{njrsk}=-h_{rs}h_{njk}=|x|^{-1}h_{rs}h_{jk}.$$

Thus, since $h_{nn}=0$ and $h_{ij}=0$ for $i\neq j$, (3.1) gives

$$|A|\Delta|A|+|A|^4\geqslant 2|A|^{-2}|x|^{-2}|A|^4=2|x|^{-2}|A|^2 \tag{3.2}$$

As with inequality (1.34), this inequality holds globally in the distribution sense even if $|A|$ vanishes at various points.

We will also need the following formula (which is a special case of the co-area formula) for integration over C:

$$\int_C \phi\, dV_n=\int_0^\infty\int_{\partial B_R}\phi(x)\,dV_{n-1}(x)\,dR=\int_0^\infty R^{n-1}\int_{\partial B_1}\phi(R\xi)\,dV_{n-1}(\xi)\,dR. \tag{3.3}$$

Here ϕ is an arbitrary summable function on C and B_R denotes the intersection with C of the ball in $\mathbf{R}^{n+1}$ with radius R and center 0.

We now take f to be a C^1 function on $C-\{0\}$ with compact support in $C-\{0\}$. Then, multiplying by f^2 in (3.2) and integrating by parts, we have

$$2\int_C |A|^2 f^2 r^{-2}\leqslant \int_C (|A|^4-|\nabla|A||^2)f^2-2\int_C |A| f(\nabla f)\cdot(\nabla|A|), \tag{3.4}$$

where r is defined on C by

$$r(x)=|x|.$$

On the other hand if we use (1.36) with $f|A|$ in place of f (and with $K_2=0$), we have

$$\int_C |A|^4 f^2\leqslant \int_C |\nabla(f|A|)|^2=\int|\nabla|A||^2 f^2+\int|A|^2|\nabla f|^2+2\int|A|f(\nabla f)\cdot(\nabla|A|). \tag{3.5}$$

Combining (3.4) and (3.5) we then have

$$2\int_C |A|^2 f^2 r^{-2}\leqslant \int_C |A|^2|\nabla f|^2. \tag{3.6}$$

We now assert that (3.6) is valid even if f does not have compact support in $C-\{0\}$, provided that

$$\int_C |A|^2 f^2 r^{-2}<\infty. \tag{3.7}$$

This is proved by applying (3.6) to the function $\gamma_\varepsilon f$, where γ_ε is any smooth function on

$C-\{0\}$ with

$$\gamma_\varepsilon(x)\equiv 1 \quad \text{for} \quad \varepsilon \leqslant |x| \leqslant \varepsilon^{-1},\ |\nabla\gamma_\varepsilon(x)| \leqslant 2/|x| \quad \text{for all } x,$$
$$\text{and } \gamma_\varepsilon(x)\equiv 0 \quad \text{for} \quad |x| \leqslant \varepsilon/2 \quad \text{or} \quad |x| \geqslant 2\varepsilon^{-1},$$

and then letting $\varepsilon \to 0$.

We now show that, if $n \leqslant 6$, (3.6) cannot possibly hold for all f satisfying (3.7) unless $|A|\equiv 0$. To prove this we take $\varepsilon \in (0, \frac{1}{2})$ and take

$$f = r^{1+\varepsilon} r_1^{1-\frac{n}{2}-2\varepsilon},$$

where r_1 is defined by

$$r_1 = \max\{1, r\}.$$

This choice of f is valid because, using (3.3) together with the fact that $|A(x)| = |x|^{-1}$ $|A(x/|x|)|$, one can easily check that (3.7) holds. Then (3.6) gives

$$2\int_C |A|^2 r^{2\varepsilon} r_1^{2-n-4\varepsilon} \leqslant \left(\frac{n}{2}-2+\varepsilon\right)^2 \int_{C\{x:|x|>1\}} |A|^2 r^{2-n-2\varepsilon} + (1+\varepsilon)^2 \int_{C\{x:|x|<1\}} |A|^2 r^{2\varepsilon}. \quad (3.3)$$

Now for $n \leqslant 6$ we can choose ε such that $(\frac{1}{2}n-2+\varepsilon)^2 < 2$ and $(1+\varepsilon)^2 < 2$. (3.3) then gives

$$\int_C |A|^2 r^{2\varepsilon} r_1^{2-n-4\varepsilon} = 0;$$

that is $|A|\equiv 0$ as required.

References

[1]. Almgren, F. J., Jr., Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem. *Ann. of Math.*, 84 (1966), 277–292.

[2]. Bernstein, S., Sur un theoreme de geometrie et ses applications aux equations aux derivees partielles du type elliptique. *Comm. de la Soc. Math. de Kharkov* (2éme Ser.), 15 (1915–1917), 38–45.

[3]. Bombieri, E., De Giorgi, E. & Giusti, E., Minimal cones and the Bernstein problem. *Invent. Math.*, 7 (1969), 243–268.

[4]. Chern, S. S., *Minimal submanifolds in a Riemannian manifold.* Mimeographed Lecture Notes, Univ. of Kansas, 1968.

[5]. Chern, S. S., DoCarmo, M. & Kobayashi, S., Minimal submanifolds of a sphere with second fundamental form of constant length. In *Functional Analysis and Related Fields*, edited by F. Browder, Springer-Verlag, Berlin, 1970.

[6]. De Giorgi, E., Una estensione del teorema di Bernstein, *Ann. Scuola Norm. Sup. Pisa*, III, 19, (1965), 79–85.

[7]. Fleming, W. H., On the oriented Plateau problem, Rend. Circ. Mat., Palermo, 2 (1962), 1–22.

[8]. Heinz, E., Über die Lösungen der Minimalflächengleichung, *Nachr. Akad. Wiss. Göttingen Math., Phys. Kl II*, (1952), 51–56.

[9] HOFFMAN, D. & SPRUCK, J , Sobolev inequalities on Riemannian manifolds. To appear in *Comm. Pure Appl. Math.*

[10]. MORREY, C. B., *Multiple integrals in the calculus of variations*. New York, Springer-Verlag, 1966.

[11]. OSSERMAN, R., *A survey of minimal surfaces*. Van Nostrand Reinhold Math. Studies, 1969.

[12]. SIMONS, J., Minimal varieties in Riemannian manifolds, *Ann. of Math.*, 88 (1968), 62–105.

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