THE VOLUME OCCUPIED BY NORMALLY DISTRIBUTED SPHERES

BY

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1. Introduction

In this paper we consider N spheres of radius R whose centres lie at points $\mathbf{z}_1, \ldots, \mathbf{z}_N$ in three dimensional Euclidean space, and the points $\mathbf{z}_i$ are independently distributed in a three-dimensional spherical normal (i.e. Gaussian) distribution with zero means, unit standard deviations, and zero correlations. The spheres can therefore overlap. Let V be the total volume covered by the spheres. We estimate the mean and variance of V, and prove that, when normalised by scale and location, its distribution tends asymptotically to normality if N has a Poisson distribution with mean λ, and λ tends to infinity. We also prove that if N is a fixed number, the same result holds when N tends to infinity.

2. The mean value of V

We denote points in the space by vectors $\mathbf{z}$ or $\mathbf{x}$. Let $I(\mathbf{z})$ be a random indicator function equal to unity if z is covered by at least one sphere, and equal to zero otherwise. Then the volume covered is

$$V = \int I(\mathbf{z})\, d\mathbf{z}, \tag{1}$$

where the integral is taken over the whole of space. Let $F(\mathbf{z})$ be the integral of the normal distribution over a sphere of radius R and centre $\mathbf{z}$. Then the mean value of V is the expectation

$$E(V) = \int EI(\mathbf{z})\, d\mathbf{z} = \int (1 - e^{-\lambda F(\mathbf{z})})\, d\mathbf{z} = 4\pi \int_0^\infty z^2 (1 - e^{-\lambda F(z)})\, dz, \tag{2}$$

where $z = |\mathbf{z}|$, and we have written $F(z)$ for $F(\mathbf{z})$. We also write $F_1(x) = F(z)$ where $x = z - R$. We first show that $E(V)$ is asymptotically equal to

$$\tfrac{4}{3}\pi(2 \log \lambda - 2 \log (2 \log \lambda))^{3/2}. \tag{3}$$

To do this we must first obtain bounds for $F(z)$ and $F_1(x)$. Suppose $z>R$, so that $x>0$. Write

$$D=(2\log\lambda-2\log(2\log\lambda))^{\frac{1}{2}}, \tag{4}$$

and

$$x=D+u.$$

Then we prove

$$e^{-\lambda F_1(x)}<K_1e^{-\frac{1}{2}K_3D|u|},\quad \text{for } -D\leqslant u\leqslant 0, \tag{5}$$

$$>K_1'>0,\qquad \text{for } u\geqslant -2D^{-1}, \tag{6}$$

$$1-e^{-\lambda F_1(x)}>K_2'>0,\qquad \text{for } -D\leqslant u\leqslant 2D^{-1} \tag{7}$$

$$<K_2e^{-Du},\qquad \text{for } u\geqslant 0, \tag{8}$$

where K_1, K_1', K_2, K_2', K_3, are fixed positive constants depending only on R. To do this we must estimate $F_1(x)=F(z-R)$. $F(z)$ is the integral from zero to R of the non-central χ^2-distribution with non-centrality parameter z^2. We do not have to calculate it exactly but only to obtain upper and lower bounds. From the properties of the normal distribution we can write for $x>0$,

$$F_1(x)=\int_0^{2R}(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}(x+t)^2}J(t)\,dt, \tag{9}$$

where $J(t)$ is the integral of a bivariate circular normal distribution with zero means and unit standard deviations, over a circle of radius $\{t(2R-t)\}^{\frac{1}{2}}$, with centre at the origin. This integral is clearly less than

$$\tfrac{1}{2}t(2R-t)\leqslant Rt,\quad (0\leqslant t\leqslant 2R)$$

and greater than

$$\tfrac{1}{2}t(2R-t)e^{-\frac{1}{2}R^2}>\tfrac{1}{2}tRe^{-\frac{1}{2}R^2},\quad \text{for } 0\leqslant t\leqslant R.$$

We then have

$$F_1(x)<(2\pi)^{-\frac{1}{2}}Re^{-\frac{1}{2}x^2}\int_0^{2R}e^{-tx-\frac{1}{2}t^2}t\,dt<(2\pi)^{-\frac{1}{2}}Re^{-\frac{1}{2}x^2}x^{-2}. \tag{10}$$

Similarly

$$F_1(x)>\tfrac{1}{2}(2\pi)^{-\frac{1}{2}}Re^{-R^2-\frac{1}{2}x^2}\int_0^R e^{-tx}t\,dt$$

$$>\tfrac{1}{2}(2\pi)^{-\frac{1}{2}}Re^{-R^2}(1-(1+R^2)e^{-R^2})e^{-\frac{1}{2}x^2}x^{-2},\quad \text{for } x\geqslant R, \tag{11}$$

and

$$>\tfrac{1}{4}(2\pi)^{-\frac{1}{2}}R^3e^{-2\frac{1}{2}R^2},\quad \text{for } x<R. \tag{12}$$

Thus there exist positive constants K_4, K_5, such that, for $x>R$, $0<K_4<F_1(x)(e^{-\frac{1}{2}x^2}x^{-2})^{-1}<K_5$. Now put $x=D+u$. If $R-D\leqslant u\leqslant 0$, we have

$$\exp-\{\lambda F_1(x)\}<\exp-\{\lambda K_4x^{-2}e^{-\frac{1}{2}(D+u)^2}\}$$

$$<\exp-\left\{K_4\left(\frac{2\log\lambda}{x^2}\right)e^{-Du-\frac{1}{2}u^2}\right\}<\exp-K_3\{\tfrac{1}{2}+\tfrac{1}{2}D|u|\},$$

where $K_3>0$, and λ is sufficiently large,

$$< K_1 e^{-K_3 D|u|}, \tag{13}$$

where K_3 can be defined so that this holds also for $-D \leqslant u \leqslant R-D$. (6) is obvious by using the reverse inequality with K_5. Now consider (8). We have $u>0$, and

$$\begin{aligned} 1-e^{-\lambda F_1(x)} &< \lambda F_1(x) < \lambda K_5 (D+u)^{-2} e^{-\frac{1}{2}(D+u)^2} \\ &< \lambda K_5 (D+u)^{-2} \exp -\tfrac{1}{2}\{(2\log\lambda - 2\log(2\log\lambda))^{\frac{1}{2}} + u\}^2 \\ &< K_2 \exp -\tfrac{1}{2}\{u^2 + 2uD\} < K_2 \exp -uD, \end{aligned}$$

for λ sufficiently large. (7) is also obvious by using the reverse inequality with K_4. Finally for $z \leqslant R$, it is sufficient in what follows to use $F(R) \leqslant F(z) \leqslant 1$.

We can now estimate the mean. Write $z = Dy$. Then from (2) we have

$$(4\pi D^3)^{-1} E(V) = \int_0^\infty y^2 (1 - e^{-\lambda F(Dy)})\, dy.$$

Using (5) and (8), and uniform convergence under the integral sign in any pair of intervals $(0,\ 1-\delta)$, $(1+\delta,\ \infty)$ where $\delta>0$, the integral tends to $\frac{1}{3}$, and (3) is proved.

3. A lower bound for the variance of V

We now obtain an expression for the variance which cannot be explicitly calculated, but is such that we can obtain upper and lower bounds for its asymptotic behaviour. We shall show that there exist positive constants K_6, K_7 such that, for all sufficiently large λ,

$$K_6 D^{-1} < \mathrm{Var}\,(\mathrm{V}) < K_7 D^{-1}. \tag{14}$$

Thus the variance decreases as λ increases.

We first recall a lemma of Bernstein [4] which will be needed in the proof of the convergence of the distribution to normality. Suppose that $X = Y + Z$, where X, Y, Z are random variables with finite variances, and whose distribution depends on a parameter λ. Then whether or not Y and Z are independent, if $\mathrm{Var}(Z)\{\mathrm{Var}(Y)\}^{-1}$ tends to zero as λ increases, then $\mathrm{Var}(X)\{\mathrm{Var}(Y)\}^{-1}$ tends to unity. Furthermore, under the same assumption, if the distribution of Y after possible rescaling and relocation by its mean and standard deviation, tends to the normal distribution with zero mean and unit standard deviation, then so also does that of X.

If $\mathbf{z}_1$, $\mathbf{z}_2$ are the vectors from the origin to two points, the variance of V is given by

$$\mathrm{Var}\,(V) = \iint \{E(I(\mathbf{z}_1) I(\mathbf{z}_2)) - E(I(\mathbf{z}_1))\, E(I(\mathbf{z}_2))\}\, d\mathbf{z}_1\, d\mathbf{z}_2, \tag{15}$$

where the integrals are taken over the whole of space.

Let S_1, S_2, be spheres of radius R around the points $\mathbf{z}_1$, $\mathbf{z}_2$, and define J_1, J_2, J_3 to be the integrals of the normal distribution over the regions defined by the part of S_1 outside S_2, the part of S_1 inside S_2, and the part of S_2 outside S_1. Then we can rewrite the above expression as

$$\operatorname{Var}(V) = 4\pi \int_0^\infty z_1^2 \, dz_1 \int d\mathbf{z}_2 \, e^{-\lambda(J_1+J_2+J_3)}(1-e^{-\lambda J_2}), \tag{16}$$

where $z_1 = |\mathbf{z}_1|$ is integrated from zero to infinity, and $\mathbf{z}_2$ is integrated over the whole of space. In fact, however, if $\mathbf{z}_1$ is given, $\mathbf{z}_2$ has only to be integrated over a sphere of centre $\mathbf{z}_1$ and radius $2R$, since outside this sphere the integrand is zero. As before, we write $x_1 = z_1 - R = |z_1| - R$, $x_2 = z_2 - R$.

The variance is certainly greater than the integral (16) taken only over values of z_1, $\mathbf{z}_2$ such that

$$D + D^{-1} \leqslant x_1 \leqslant D + 2D^{-1}, \qquad D - 2D^{-1} \leqslant x_2 \leqslant D - D^{-1}. \tag{17}$$

As we are concerned with an asymptotic bound, we can suppose λ sufficiently large for D to be large compared with R. For any $\mathbf{z}_1$ with z_1 of the order of D or larger, the range of integration of $\mathbf{z}_2$ will be over a bounded region in which the surface of the sphere $|z_1| = \text{constant}$, and the surface $|z_2| = \text{constant}$, will be practically planes. In what follows we shall describe the situation as if they were in fact planes. This introduces a small error which we take care of by choosing the various constants involved to be larger or smaller than would be required if the surface really were a plane so that for large enough λ, the resulting inequalities will be true.

We integrate $\mathbf{z}_2$ in the region defined by (17), and under the further condition that the perpendicular distance from $\mathbf{z}_2$ on to the line of the vector $\mathbf{z}_1$ is not greater than $\frac{1}{2}(RD^{-1})^{\frac{1}{2}}$ (the factor $\frac{1}{2}$ is introduced to make sure the curvature does not affect the result). Then the sphere S_2 will cover the point $(x_1 z_1^{-1})\mathbf{z}_1$, and will in fact cover an octant of S_1 defined by the region below a plane through the point $\mathbf{z}_1$ perpendicular to the vector $\mathbf{z}_1$, and two perpendicular planes containing the vector $\mathbf{z}_1$. The integral of the density over this octant will therefore be greater than $\frac{1}{8}F(z_1 - R) = \frac{1}{8}F_1(x_1)$. We have

$$1 - e^{-\lambda J_2} > (\text{constant}) > 0,$$

from (7). We also have

$$\exp -\lambda(J_1+J_2+J_3) > \exp -2\lambda(J_2+J_3) > \exp -2\lambda F(z_2) > K_8 > 0,$$

on using (6). Inserting these bounds in (16), and integrating z_1, z_2 subject to the prescribed restrictions, we get, as in (14),

$$\operatorname{Var}(V) > K_6 D^{-1}, \quad K_6 > 0,$$

for all λ greater than some constant depending only on R.

4. An upper bound for the variance of V

We now obtain an upper bound for the variance. This is much more complicated. We write $V = V_1 + V_2 + V_3$ where V_1 is the covered set inside the sphere $|\mathbf{z}| - R < D - D^{-\frac{1}{2}}$, V_2 is the covered set lying in the region $D - D^{-\frac{1}{2}} \leqslant |z| - R \leqslant D + D^{-\frac{1}{2}}$, and V_3 is the covered set in $|z| - R > D + D^{-\frac{1}{2}}$. These three quantities are correlated, but we shall show that Var (V_1) and Var (V_3) are asymptotically negligible compared with Var (V_2), and using Bernstein's lemma we need consider only the latter. The Var (V_i) are given by (16) with z_1, z_2 both confined to the three above regions.

Consider first Var (V_1). This will be given by the integral (16) with $|z_1|$, $|z_2| \leqslant D - D^{-\frac{1}{2}} + R$, i.e. with x_1, $x_2 \leqslant D - D^{-\frac{1}{2}}$. The integral will be twice the corresponding integral with the additional restriction that $x_2 \leqslant x_1$. Var (V_1) is thus less than

$$8\pi \int_R^{D-D^{-\frac{1}{2}}+R} z_1^2 \, dz_1 \int d\mathbf{z}_2 e^{-\lambda(J_1+J_2)}),$$

together with an integral over $|z_1| \leqslant R$ which is easily shown to be negligible. $J_1 + J_2$ depends only on z_1, and the integral over $\mathbf{z}_2$ is bounded by $(4/3)\pi(2R)^3$, whilst from (5),

$$e^{-\lambda(J_1+J_2)} < K_1 e^{-\frac{1}{2} D K_3 |u|},$$

where $x_1 = D + u$, and $-D \leqslant u \leqslant D^{-\frac{1}{2}}$. Thus the integral is less than

$$\frac{32}{3}\pi^2 (2R)^3 K_1 \int_{D^{-\frac{1}{2}}}^{D} (D + R - |u|)^2 e^{-\frac{1}{2}K_3 D|u|} d|u| < \frac{64}{3}\pi^2 K_1 K_3^{-1} (2R)^3 D^2 e^{-\frac{1}{2}K_3 D^{\frac{1}{2}}}, \quad (18)$$

which tends to zero much faster than $D^{-\frac{1}{2}}$ as $D \to \infty$.

Now consider Var (V_3). Then the integrand is less than ($|z_1| = R + D + u$)

$$(D + R + u)^2 (1 - e^{-\lambda(J_1+J_2)}) < (D + R + u)^2 K_2 e^{-uD}.$$

The integral over the region outside $|\mathbf{z}| = D + D^{-\frac{1}{2}} + R$, is therefore not greater than

$$\frac{32}{3}\pi(2R)^3 K_2 \int_{D^{-\frac{1}{2}}}^{\infty} (D + R + u)^2 e^{-uD} du < (\text{constant})\, D e^{-D^{\frac{1}{2}}}, \text{ for } D \text{ sufficiently large.} \quad (19)$$

Thus we can confine the integral to the region where $D - D^{-\frac{1}{2}} \leqslant |z| - R = x \leqslant D + D^{-\frac{1}{2}}$. Furthermore this integral is twice the corresponding integral with $x_2 \leqslant x_1$. We write $x_1 = D + u_1$, $x_2 = D + u_2$, and consider the three separate cases;

$$0 \leqslant u_2 \leqslant u_1 \leqslant D^{-\frac{1}{2}}, \quad -D^{-\frac{1}{2}} \leqslant u_2 \leqslant 0 \leqslant u_1 \leqslant D^{-\frac{1}{2}}, \quad -D^{-\frac{1}{2}} \leqslant u_2 \leqslant u_1 \leqslant 0. \quad (20)$$

In the first case put $v_1 = u_1 - u_2$, and let v_2 be the perpendicular distance of the point $\mathbf{x}_2$ from the line of the vector $\mathbf{x}_1$. In what follows we write $\mathbf{x}_3$ for the vector from the origin to the nearest point of the lens shaped region common to the two spheres S_1 and S_2. Then $x_3 \geqslant x_1$, $x_3 \geqslant x_2$. Ignoring the curvature of the spheres $|x_1| = \text{constant}$, $|x_2| = \text{constant}$, S_2 would contain the point $\mathbf{x}_1$ if $v_2 < (2Rv_1)^{\frac{1}{2}}$. However for λ and D sufficiently large it is sufficient to replace $|\mathbf{x}_3|$ by $|\mathbf{x}_1|$ whenever $|\mathbf{x}_3| > |\mathbf{x}_1|$ and integrate over the range $v_2 < 2(2Rv_1)^{\frac{1}{2}}$. We therefore first consider the region of integration defined by $0 \leqslant u_2 \leqslant u_1 \leqslant D^{-\frac{1}{2}}$, and $v_2 < 2(2Rv_1)^{\frac{1}{2}}$. Integrating over v_2 first and using the upper bound (8), we get that the integral is not greater than

$$32\pi^2 R \int_0^{D^{-\frac{1}{2}}} du_1 \int_0^{u_1} du_2 (u_1 - u_2)(D + R + u_1)^2 K_2 e^{-u_1 D}$$

$$< 16\pi^2 RK_2 (D + D^{-\frac{1}{2}})^2 \int_0^{D^{-\frac{1}{2}}} u_1^2 e^{-u_1 D} du_1 < 16\pi^2 RK_2 (D + D^{-\frac{1}{2}})^2 D^{-3} < (\text{const})\, D^{-1}. \quad (21)$$

Consider also the value of this integral when u_1 is taken over the smaller region $\alpha D^{-1} \leqslant u_1 \leqslant D^{-\frac{1}{2}}$, where α is a suitably chosen large fixed number. Then the integral will be less than

$$16\pi^2 RK_2 (D + D^{-\frac{1}{2}})^2 \int_{\alpha D^{-1}}^{\infty} u_1^2 e^{-u_1 D} du_1 \quad (22)$$

which can be verified to be less than a constant times

$$(2 + 2\alpha + \alpha^2) D^{-1} e^{-\alpha}. \quad (23)$$

Thus given any small positive number $\varepsilon > 0$, it is possible to choose α large and fixed so that the contribution to the integral for the variance of V_2, of the region outside $u_1 < \alpha D^{-1}$, is less than $\varepsilon \operatorname{Var}(V)$.

Now suppose that $0 \leqslant u_2 \leqslant u_1 \leqslant D^{-\frac{1}{2}}$ as before but $v_2 \geqslant 2(2Rv_1)^{\frac{1}{2}}$. We also must have $v_2 < 2R$. We now need to estimate $u_3 = x_3 - D$. The following theory is described as if $\mathbf{z}_1$ and $\mathbf{z}_2$ were parallel but the resulting small error for D large is taken care of by the fact that we take $v_2 \geqslant 2(2Rv_1)^{\frac{1}{2}}$ instead of $v_2 \geqslant (2Rv_1)^{\frac{1}{2}}$. Then by using straightforward geometry we can verify that

$$u_3 - u_1 > \frac{3}{32R}(v_2^2 - 2Rv_1) > \frac{3}{32R}(v_2 - (2Rv_1)^{\frac{1}{2}})^2. \quad (24)$$

The region of space common to S_1 and S_2 can be enclosed in a sphere whose nearest point to the origin is $x_3 = D + u_3$. Then the last term in the integrand in (16) is majorised by $1 - e^{-\lambda F_1(D+u_3)} < K_2 e^{-u_3 D}$. The contribution to the integral from this region is therefore not greater than

$$4\pi\int_0^{D^{-\frac{1}{2}}}(D+R+u_1)^2du_1\int_0^{u_1}du_2\int_{2(2Rv_1)^{\frac{1}{2}}}^{\infty}2\pi v_2K_2e^{-u_1D}dv_2$$
$$< 8\pi^2K_2\int_0^{D^{-\frac{1}{2}}}(D+R+u_1)^2du_1\int_0^{u_1}du_2\int_{2(2Rv_1)^{\frac{1}{2}}}^{\infty}v_2e^{-u_1D-\frac{3}{32R}(v_2-(2Rv_1)^{\frac{1}{2}})^2}dv_2,$$

Put $w=v_2^2$, and use the fact that $(v_2-(2Rv_1)^{\frac{1}{2}})^2>\frac{1}{4}v_2^2$. Then the above is less than

$$4\pi^2K^2\int_0^{D^{-\frac{1}{2}}}(D+R+u_1)^2du_1\int_0^{u_1}du_2\int_{8Rv_1}^{\infty}e^{-u_1D-\frac{3D}{128R}w}dw$$
$$< 4\pi^2K_2\frac{128R}{3D}\int_0^{D^{-\frac{1}{2}}}(D+R+u_1)^2e^{-u_1D}u_1du_1<(\text{constant})\,(D+R+D^{-\frac{1}{2}})^2D^{-3}$$
$$<(\text{constant})\,D^{-1}. \tag{25}$$

Consider also the similar integral over the region

$$\alpha D^{-1}\leqslant u_1\leqslant D^{-\frac{1}{2}},$$

where α is chosen sufficiently large as before. Then given ε small and fixed, we can choose α sufficiently large and fixed such that for all D greater than some constant we have that the contribution to the integral of the part where $u_1>\alpha D^{-1}$ is less than

$$(\text{constant})\,D^{-1}\int_{\alpha D^{-1}}^{D^{-\frac{1}{2}}}(D+u_1)^2e^{-u_1D}u_1\,du_1<(\text{constant})\,D\int_{\alpha}^{D^{\frac{1}{2}}}e^{-y}y\,dy. \tag{26}$$

Choosing α sufficiently large, greater than unity, and dependent only on ε for all large D, this is less than (for $D>1$ say),

$$(\text{constant})\,D^{-1}\alpha e^{-\alpha}. \tag{27}$$

We now consider the second case, i.e. where

$$-D^{-\frac{1}{2}}\leqslant u_2\leqslant 0\leqslant u_1\leqslant D^{-\frac{1}{2}}.$$

Define $v_1=u_1-u_2$, and v_2 as before. First suppose that $v_2\leqslant 2(2Rv_1)^{\frac{1}{2}}=2(2R(u_1+|u_2|))^{\frac{1}{2}}$. The contribution to the integral is not greater than

$$32\pi^2R\int_0^{D^{-\frac{1}{2}}}(D+R+u_1)^2du_1\int_0^{D^{-\frac{1}{2}}}d|u_2|\,(u_1+|u_2|)\,K_1K_2e^{-\frac{1}{2}K_3D|u_2|-Du_1}$$
$$<(\text{constant})\,(D+R+D^{-\frac{1}{2}})^2D^{-3}<(\text{constant})\,D^{-1}. \tag{28}$$

Following the same argument as above we see that given $\varepsilon>0$, there exists α sufficiently large so that if we integrate u_1 over the range $(\alpha D^{-1},\ D^{-\frac{1}{2}})$, the contribution to the integral is less than

$$\text{(constant) } D^{-1}\alpha e^{-\alpha}, \tag{29}$$

which can be made less than εD^{-1}.

Now suppose $v_2 > 2(2R(u_1 + |u_2|))^{\frac{1}{2}}$. Then the contribution to the integral is less than

$$8\pi^2 \int_0^{D^{-\frac{1}{2}}} (D+R+u_1)^2 \, du_1 \int_0^{D^{-\frac{1}{2}}} d|u_2| \int_{2(2R(u_1+|u_2|))^{\frac{1}{2}}}^{\infty} K_1 K_2 e^{-\frac{1}{2}K_3 D|u_2| - Du_3} v_2 \, dv_2. \tag{30}$$

From (24) we have

$$u_3 - u_2 \geqslant \frac{3}{32R}(v_2 - (2Rv_1)^{\frac{1}{2}})^2 > \frac{3}{128R} v_2^2.$$

Put $w = v_2^2$. Then the innermost integral in (30) is not greater than

$$\tfrac{1}{2} K_1 K_2 e^{-\frac{1}{2}K_3 D|u_2|} \int_{8R(u_1+|u_2|)}^{\infty} e^{\frac{3D}{128R}w - Du_1} \, dw < \frac{64R}{3} D^{-1} K_1 K_2 e^{-\frac{1}{2}K_3 D|u_2| - Du_1}.$$

Inserting this in (30) and integrating with respect to u_1 and $|u_2|$, we obtain an upper bound

$$\text{(constant) } D^{-1}. \tag{31}$$

As before if we restrict u_1 to the range $(\alpha D^{-1}, D^{-\frac{1}{2}})$ we obtain an upper bound

$$KD^{-1}\alpha e^{-\alpha}, \tag{32}$$

which can be made arbitrarily small compared with Var (V), by choosing α large.

Finally consider the case where

$$-D^{-\frac{1}{2}} \leqslant u_2 \leqslant u_1 \leqslant 0.$$

Defining $u_3 = x_3 - D$ as before we consider separately the cases $u_3 \leqslant 0$, $u_3 \geqslant 0$. Write $u_3^+ = \max(0, u_3)$.

Returning to (16) we have

$$e^{-\lambda(J_1+J_2+J_3)} \leqslant e^{-\frac{1}{2}\lambda(J_1+2J_2+J_3)} = e^{-\frac{1}{2}\lambda F_1(x_1) - \frac{1}{2}\lambda F_1(x_2)}, \tag{33}$$

and consequently $$e^{-\lambda(J_1+J_2+J_3)} \leqslant K_1^2 e^{-\frac{1}{4}K_3 D|u_1| - \frac{1}{4}K_3 D|u_2|}.$$

Put $v_1 = |u_2| - |u_1|$ and define v_2 as before. It is easy to see that, ignoring curvature, $u_3^+ = 0$ if

$$v_2 < (2R|u_1|)^{\frac{1}{2}} + (2R|u_2|)^{\frac{1}{2}},$$

and therefore, for D reasonably large, we can ignore the curvature if we take, say

$$v_2 < 4(2R|u_1|)^{\frac{1}{2}} + 4(2R|u_2|)^{\frac{1}{2}}, \tag{34}$$

and ignore u_3^+ whenever it is greater than zero. The contribution to the integral is not greater than twice

$$64\pi^2 K_1^2 \int_0^{D^{-\frac{1}{2}}} D^2 d|u_1| \int_{|u_1|}^{D^{-\frac{1}{2}}} d|u_2| \{(2R|u_1|)^{\frac{1}{2}} + (2R|u_2|)^{\frac{1}{2}}\}^2 e^{-\frac{1}{4}K_3 D(|u_1|+|u_2|)}.$$

Since $\{(2R|u_1|)^{\frac{1}{2}} + (2R|u_2|)^{\frac{1}{2}}\}^2 \leqslant 8R|u_2|$, the integral is less than

$$128\pi^2 K_1^2 \int_0^{D^{-\frac{1}{2}}} D^2 e^{-\frac{1}{4}K_3 D|u_1|} d|u_1| \int_{|u_1|}^{D^{-\frac{1}{2}}} 8R|u_2| e^{-\frac{1}{4}K_4 D|u_2|} d|u_2| < (\text{constant})\, D^{-1}.$$

Furthermore if $|u_1| > \alpha D^{-1}$ and α is large and fixed, the contribution is less than

$$(\text{constant})\, D^{-1} \alpha e^{-\frac{1}{4}K_3 \alpha}.$$

Now consider the case where $v_2 \geqslant 4(2R|u_1|)^{\frac{1}{2}} + 4(2R|u_2|)^{\frac{1}{2}}$. Then, from the previous calculations and (24), we have, putting

$$v_3 = v_2 - 4(2R|u_1|)^{\frac{1}{2}} - 4(2R|u_2|)^{\frac{1}{2}} > 0,$$

$$u_3^+ > -|u_1| + \frac{3}{32R}(v_2 - (2Rv_1)^{\frac{1}{2}})^2 > -|u_1| + \frac{3}{32R}(v_2 - (2R|u_2| - 2R|u_1|)^{\frac{1}{2}})^2$$

$$> -|u_1| + \frac{3}{32R}(v_3 + 3(2R|u_2|)^{\frac{1}{2}} + 3(2R|u_1|)^{\frac{1}{2}})^2 > \frac{3}{12R} v_3^2.$$

We can now write an upper bound to this contribution to the integral as

$$8\pi^2 \int_0^{D^{-\frac{1}{2}}} D^2 d|u_1| \int_{|u_1|}^{D^{-\frac{1}{2}}} d|u_2| \int_0^{2R} K_1^2 K_2 e^{-K_3 D(|u_1|+|u_2|) - \frac{1}{12R} D v_3^2} v_3\, dv_3.$$

Putting $w = v_3^2$ and integrating first with respect to w, the above is less than

$$(\text{constant})\ D^{-1}, \tag{35}$$

and as before, if $|u_1| > \alpha D^{-1}$, we can choose α sufficiently large so that the above integral is less than

$$(\text{constant})\ D^{-1} e^{-\alpha} < \varepsilon. \tag{36}$$

We have therefore shown that

$$\text{Var}\,(V_1) = o(\text{Var}\,(V_2)), \tag{37}$$

$$\text{Var}\,(V_3) = o(\text{Var}\,(V_2)), \tag{38}$$

$$\text{Var}\,(V_2) < (\text{constant})\ D^{-1}, \tag{39}$$

so that

$$\text{Var}\,(V) < (\text{constant})\ D^{-1}, \tag{40}$$

as required.

5. A central limit theorem

Now define V_4 to be the volume of the set covered by spheres and lying in the range

$$D-\alpha D^{-1} \leqslant x_1,\ x_2 \leqslant D+\alpha D^{-1},$$

where α and D are chosen so large after choosing ε arbitrarily small, that

$$\operatorname{Var}(V_4) = \operatorname{Var}(V)(1+\theta\varepsilon),\ (|\theta|<1). \tag{41}$$

Then by Bernstein's lemma it is sufficient to show that the normalised value of V_4 has a distribution which tends to normality.

Suppose that C is a "cone" with vertex at the origin of coordinates and a finite number of smooth sides. Let it subtend a solid angular region such that the parts of its intersection with the unit sphere which are nearer to the sides than $2\cos^{-1}RD^{-1}$, tend to zero in area relative to the area on the unit sphere within the cone, as D tends to infinity. Then uniformly in this condition, the variance of the part of V_4 in the cone will tend to $\omega(4\pi)^{-1}\operatorname{Var}(V_4)$ as D tends to infinity. Here ω is the solid angle of the cone, and the convergence is uniform in the shape and size of the cone.

To prove the tendency to the normal distribution we shall use Liapounov's theorem in a form in which the distributions of the individual terms are identical and independent, but vary with the number of terms. We therefore need a uniform bound on the fourth (say) moment. Consider the fourth moment of the volume V_4. This is

$$E(V_4-E(V_4))^4 = E\iiiint \prod_{i=1}^{4}\{I(\mathbf{z}_i)-EI(\mathbf{z}_i)\}\,d\mathbf{z}_1\,d\mathbf{z}_2\,d\mathbf{z}_3\,d\mathbf{z}_4, \tag{42}$$

where the integral is taken over the range

$$R+D-\alpha D^{-1} \leqslant z_1,\ z_2,\ z_3,\ z_4 \leqslant R+D+\alpha D^{-1}.$$

If any sphere S_i does not intersect any of the other spheres, the expectation of the integrand is zero. We therefore have two possible cases. In the first case the four spheres intersect in two pairs which are mutually disjoint. In the second case the four spheres intersect in such a way that they form a single connected set.

In the first case the intersection can occur in three ways, and integrating over the whole of the above range and using the previous results we obtain a value which is slightly less than $3(\operatorname{Var}(V_4))^2$. The deficit is due to the need to avoid cases where one pair overlaps part of the other.

In the second case we have

$$E\{(I(\mathbf{z}_1)-E(I(\mathbf{z}_1))\dots(I(\mathbf{z}_4)-E(I(\mathbf{z}_4))\}\leqslant \tfrac{1}{4}\sum_i E(I(\mathbf{z}_i)-E(I(\mathbf{z}_i))^4$$
$$\leqslant \tfrac{1}{4}\sum_i E(I(\mathbf{z}_i)-E(I(\mathbf{z}_i)))^2\leqslant \tfrac{1}{4}\sum_i e^{-\lambda F(z_i)}(1-e^{-\lambda F(z_i)}). \tag{43}$$

The integral with respect to $\mathbf{z}_1$ of $e^{-\lambda F(\mathbf{z}_1)}(1-e^{-\lambda F(\mathbf{z}_1)})$ over the region is less than

$$4\pi\int_0^{\alpha D^{-1}}(D+R+u)^2K_2e^{-Du}du+4\pi\int_{-\alpha D^{-1}}^{0}(D+R-u)^2K_1e^{-K_3D|u|}du, < (\text{constant})\, D.$$

Now integrating with respect to $\mathbf{z}_2$, $\mathbf{z}_3$, $\mathbf{z}_4$, and using the fact that at very worst $|\mathbf{z}_1-\mathbf{z}_i|<6R$, the total integral (42) is not greater than

$$(\text{constant})\ \alpha D(\pi R^3)^3(2\alpha D^{-1})^3<(\text{constant})\,\alpha^4D^{-2}<(\text{constant})\,\alpha^4(\operatorname{Var}(V_2))^2 \tag{45}$$

The constant depends only on R and ε.

Now consider any cone C of the form considered above. Provided the conditions on this cone are satisfied uniformly, we have, uniformly in all such cones,

$$E(V_4C-E(V_4C))^4<(\text{constant})\{E(V_4C)^2\}^2, \tag{46}$$

where the constant depends only on R and α.

In order to prove (46) we apply the same argument as above to the region V_4C. Provided the cone C subtends a solid angular region which is such that the parts of its intersection with the unit sphere which are nearer to its sides than $2\cos^{-1}RD^{-1}$ have an area which tends to zero relative to the area on this unit sphere subtended by the cone, (46) will hold uniformly in the shape of the cone so long as the rate at which this relative area tends to zero is bounded above independently of the shape of the cones considered. We now define these cones which are almost, but not quite, sectors of the sphere and which certainly satisfy this condition. Then the argument given above to obtain an upper bound to $E(V_4-E(V_4))^4$ applies without change to obtain (46).

Now choose any fixed axis OZ in space, passing through the origin O. We divide the whole of space into $2n$ regions by n planes through the axis OZ, each of which makes an angle $2\pi n^{-1}$ with its nearest neighbours. We replace each of the n planes by a double cone of vertex O whose half angle is $\cos^{-1}4RD^{-1}$. Let R_0 be the set sum of all the regions outside these double cones and inside the region $D-\alpha D^{-1}\leqslant x\leqslant D+\alpha D^{-1}$. The angular measure of R_0 is thus less than $8n\pi RD^{-1}$. Removing R_0 from this region containing V_4, we have $2n$ regions which are almost those formed by the slices of the sphere but are such that the distance between any pair is greater than $2R$. Write $V_4=V_{4,1}+\dots+V_{4,2n}$. Then the random quantities $V_{4,i}$ are all independent and have the same distribution with variances which are equal to $(1-\eta)\operatorname{Var}(V_4)(2n)^{-1}$, where η is a small number which tends to zero as D increases, provided n increases sufficiently slowly with D. This follows from the fact that the relative contribution to the variance of the set covered by the spheres and lying in the region R_0 tends to zero if we take n as, say, the integral part of $D^{\frac{1}{2}}$.

Now using Liapounov's version of the central limit theorem, which holds for cases where the common distribution of the variates may change with n provided the conditions are satisfied uniformly, it follows that the distribution of $(V_4 - E(V_4))(\text{Var}(V_4))^{-\frac{1}{2}}$ tends to a normal distribution with zero mean and unit standard deviation. Using Bernstein's lemma and the inequalities on the variances of the covered volume in all the regions omitted, it follows that a similar result holds for the distribution of V, i.e. that

$$(V - E(V))(\text{Var}(V))^{-\frac{1}{2}}$$

converges in distribution to a normal distribution with zero mean and unit standard deviation.

6. The case where N is fixed

We now consider the case where N, the number of spheres, is no longer a Poisson variate, but a fixed number which increases indefinitely. Let the volume covered by the N spheres be V'. Then following a similar argument to that above we find, analogously to (2),

$$E(V') = 4\pi \int_0^\infty z^2\{1 - (1 - F(z))^N\}\, dz, \tag{47}$$

and, analogously to (16),

$$\text{Var}(V') = 4\pi \int_0^\infty z_1^2\, dz_1 \int d\mathbf{z}_2[\{1 - J_1 - J_2 - J_3\}^N - (1 - J_1 - J_2)^N(1 - J_2 - J_3)^N]. \tag{48}$$

To evaluate these integrals we proceed as follows. When $0 \leqslant x \leqslant 1$ we have

$$e^{-\frac{1}{2}x} \geqslant (1 + x)^{-1} \geqslant e^{-x} \geqslant (1 - x). \tag{49}$$

From this it follows that for $0 \leqslant x \leqslant 1$,

$$\begin{aligned} |e^{-Nx} - (1 - x)^N| &\leqslant (1 + x)^{-N} - (1 - x)^N \leqslant Nx^2(1 + x)^{-N} \\ &\leqslant Nx^2 e^{-\frac{1}{2}Nx} \leqslant N^{-1}(N^2x^2 e^{-\frac{1}{2}Nx}) \leqslant 16N^{-1}e^{-4} < N^{-1}. \end{aligned} \quad (50$$

Define $D_1 = (2 \log N - 2 \log (2 \log N))^{\frac{1}{2}}$ and consider the integrals (47) and (48) over the range $0 \leqslant z_1,\ z_2 \leqslant 2D_1$. Then the error due to replacing the integrands in (47) and (48) by those in (2) and (16) taken over the same range is less than

$$\tfrac{4}{3}\pi(2D_1)^3 N^{-1},$$

and

$$4\pi(2D_1)^3 N^{-1}(\tfrac{4}{3}\pi(2D_1)^3),$$

respectively. For the region where $z_1 > 2D_1$, the error is easily verified to be $o(N^{-1})$, when integrated over the whole region. Similarly for $|z_2| > 2D_1$. We therefore have, if $N = \lambda$,

$$E(V') = E(V) + o((\text{Var}\,(V))^{\frac{1}{2}}), \tag{51}$$

$$\text{Var}\,(V') = \text{Var}\,(V) + o((\text{Var}\,(V)). \tag{52}$$

Now write $G(V)$ for the cumulative distribution of V when N is a Poisson variate with mean λ, and $G_N(V')$ for the cumulative distribution when N is a fixed number. Then

$$G(V) = e^{-\lambda} \sum_{0}^{\infty} (n\,!)^{-1} \lambda^n G_n(V). \tag{53}$$

Furthermore for any fixed value V, $G_{n+1}(V) \leqslant G_n(V)$. Then using (51) and (52), it follows from simple inequalities that the distribution of

$$\{V' - E(V')\}\{\text{Var}(V')\}^{-\frac{1}{2}}$$

also converges to a normal distribution with zero mean and unit standard deviation.

This is in marked contrast with the problem considered in [2], [3], of determining the distribution of the volume occupied by random intersecting spheres whose centres are uniformly distributed over a cube. If the expected number of spheres divided by the volume of the cube is defined as the density, and the volume is allowed to increase indefinitely whilst the density remains constant, the volume covered also has a distribution which tends to normality, but in the two cases where N is a Poisson variate with mean $\lambda = N_0$, and where $N = N_0$ is fixed, the variances and distributions are asymptotically unequal.

7. Conclusion

Notice that if A is the set covered by the spheres, we can write $A = A_1 + A_2 - A_3$ where A_1 is a sphere of radius D, and A_2, A_3 are random sets such that their measures, divided by $(4/3)\pi D^3$, converge in probability to zero.

The results of the present paper can also be compared with those of Efron [1] who obtained the expected volume of the smallest convex cover of a set of N points distributed in a spherical normal distribution, but did not obtain the variance. It seems probable that the variance in his problem is asymptotically the same as in the present one, and that a central limit theorem holds, but this has not yet been proved.

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