INTEGRAL MEANS, UNIVALENT FUNCTIONS AND CIRCULAR SYMMETRIZATION

BY

ALBERT BAERNSTEIN II

Washington University, St. Louis, Mo. USA(1)

1. Introduction

We begin by considering the class S of all functions $f(z)$ holomorphic and univalent in the unit disk $|z|<1$ with $f(0)=0$, $f'(0)=1$, and denote by $k(z)$ the Koebe function,

$$k(z)=\frac{z}{(1-z)^2},$$

which maps the unit disk conformally onto the complex plane slit along the negative real axis from $-\frac{1}{4}$ to $-\infty$. The Koebe function is known to be extremal for many problems involving S. The first result in this paper asserts this is the case for a large class of problems about integral means. Specifically, I will prove the following theorem.

THEOREM 1. *Let Φ be a convex non-decreasing function on $(-\infty, \infty)$. Then for $f\in S$ and $0<r<1$,*

$$\int_{-\pi}^{\pi}\Phi(\log|f(re^{i\theta})|)\,d\theta\leqslant\int_{-\pi}^{\pi}\Phi(\log|k(re^{i\theta})|)\,d\theta. \tag{1}$$

If equality holds for some $r\in(0,1)$ and some strictly convex Φ, then $f(z)=e^{-i\alpha}k(ze^{i\alpha})$ for some real α.

In particular, we have for $0<r<1$,

$$\int_{-\pi}^{\pi}|f(re^{i\theta})|^p\,d\theta\leqslant\int_{-\pi}^{\pi}|k(re^{i\theta})|^p\,d\theta \quad (0<p<\infty),$$

$$\int_{-\pi}^{\pi}\log^+|f(re^{i\theta})|\,d\theta\leqslant\int_{-\pi}^{\pi}\log^+|k(re^{i\theta})|\,d\theta. \tag{2}$$

(1) This research was supported by NSF Grant GP-38959

The best previously known result in the direction of (2) is apparently that of Bazilevic [4], who showed that (2) holds for $p=1, 2$ if a universal constant is added to the right hand side. MacGregor [19] and Wilken [26] have proved (2) for f close-to-convex and $p \geqslant 1$.

It remains an interesting open question whether (2) holds for any values of p if f and k are replaced by their derivatives. The inequality is sometimes false for $0<p<\frac{1}{3}$, since k' belongs to the Hardy class H^p for these p, whereas Lohwater, Piranian, and Rudin [18] have constructed a function in S whose derivative does not belong to any H^p. For f close-to-convex and $p \geqslant 1$ MacGregor [19] has proved that (2) holds with f and k replaced by $f^{(n)}$ and $k^{(n)}$ for all positive integers n.

The most famous unsolved problem about S is Bieberbach's conjecture, which asserts that if $f(z)=\Sigma a_n z^n \in S$ then $|a_n| \leqslant n$. In other words, the Koebe function is extremal for the problem of maximizing the absolute value of the n'the coefficient. By taking $p=1$ in (2) and using Cauchy's formula, one can easily obtain the bound $|a_n| \leqslant \frac{1}{2}\, en$. The best known bound at this time is $|a_n| \leqslant 1.081\, n$, due to Fitzgerald [7]. Since Theorem 1 provides considerably more information than just the inequality between the L^1 norms, it is conceivable that there might be some way of using its full strength to obtain further results about the coefficients.

Consider now a not necessarily univalent function $f(z)=\Sigma a_n z^n$ which is holomorphic in $|z|<1$. Rogosinski made the conjecture, more general than Bieberbach's, that if f is subordinate to some function $g \in S$, then $|a_n| \leqslant n$. It was proved (see, e.g. [14, p. 422]) by Littlewood that

$$\int_{-\pi}^{\pi} |f(re^{i\theta})|\, d\theta \leqslant \int_{-\pi}^{\pi} |g(re^{i\theta})|\, d\theta.$$

Using this inequality together with Theorem 1, we once again obtain the coefficient estimate $|a_n| \leqslant \frac{1}{2}\, en$, which is apparently the best one known in this context, a fact pointed out to me by Ch. Pommerenke.

The proof of Theorem 1, and of the other results in this paper, is based on considerations involving a certain auxiliary function. Let $u(z)$ be an extended real valued function defined in an annulus $r_1<|z|<r_2$. We suppose $u(re^{i\theta})$ is, for each $r \in (r_1, r_2)$, a Lebesgue integrable function of θ, and define a new function u^* in the semi-annulus $\{re^{i\theta}: r_1<r<r_2, 0 \leqslant \theta \leqslant \pi\}$ by

$$u^*(re^{i\theta}) = \sup_E \int_E u(re^{i\omega})\, d\omega \tag{3}$$

where the sup is taken over all measurable sets $E \subset [-\pi, \pi]$ with $|E|=2\theta$. Here, and throughout the paper,

$$|E| = \text{Lebesgue measure of } E.$$

The reader may recognize u^* as the integral from $-\theta$ to θ of the symmetric non-increasing equimeasurable rearrangement of u. (See Proposition 2 of § 3). The usefulness of u^* for our purposes stems from the following result.

THEOREM A. *Suppose u is subharmonic in the annulus $r_1 < |z| < r_2$. Then u^* is subharmonic in the semi-annulus $\{re^{i\theta}: r_1 < r < r_2, 0 < \theta < \pi\}$.*

Theorem A was proved by the author in [1] for functions of the form $u = \log |g|$, where g is an entire function. It is not difficult to adapt that proof to the more general situation considered here. Different, simpler, proofs have recently been discovered by M. Essén and P. Sjögren. In § 2 we present Sjögren's proof of (an extended version of) Theorem A.

The original application of Theorem A, in [1], was to obtain a precise estimate of the size of the set where certain functions meromorphic in the plane are large. Refinements of this result appear in [3]. In [2], two variants of u^* were used to obtain a result about entire functions that generalizes the Wiman-Valiron $\cos \pi\varrho$ theorem.

In § 3 of this paper we prove some simple real variable results, and in § 4 some results about Green's functions. In § 5 we prove Theorem 1. Here is the idea. Let u and v be the Green's functions, with pole at $\zeta = 0$, of the ranges of f and k respectively. Extend u and v to the whole plane by setting them equal to zero outside their original domains of definition. Then u and v are subharmonic in the plane, except for a logarithmic singularity at the origin. From Proposition 3 in § 3 and Cartan's formula it will follow that the conclusion of Theorem 1 holds if and only if $u^* \leqslant v^*$ everywhere in the upper half plane. The key to proving this inequality is the fact that the proposed extremal function v^* is harmonic in the upper half plane, whereas, by Theorem A, u^* is subharmonic there.

In § 6 we present some complements and extensions of Theorem 1. The proofs require only very slight modifications of the proof of Theorem 1. Theorem 2 asserts that the conclusion of Theorem 1 holds if f and k are replaced by their reciprocals. Theorem 3 is an analog of Theorem 1 for univalent functions in an annulus, and Theorem 4 asserts that Theorem 1 remains true for normalized weakly univalent functions in the sense of Hayman [11].

In §§ 7 and 8 we consider integral means inequalities associated with circular symmetrization. Let D be a domain in the extended plane. The circular symmetrization of D is the domain D^* defined as follows: For each $t \in (0, \infty)$ let $D(t) = \{\theta \in [0, 2\pi]: te^{i\theta} \in D\}$. If $D(t) = [0, 2\pi]$ then the intersection of D^* with the circle $|z| = t$ is the full circle, and if

$D(t)$ is empty then so is the intersection of D^* with $|z|=t$. If $D(t)$ is a proper subset of $[0, 2\pi]$ and $|D(t)|=\alpha$ then the intersection of D^* with $|z|=t$ is the single arc $\{te^{i\theta}: |\theta|<\alpha/2\}$. Moreover, D^* contains the origin, or the point at ∞, if and only if D does.

Accounts of the theory of symmetrization may be found in [22] and [12]. We point out that our use of the notation u^* for functions u differs from that in [12]. Our results all concern circular symmetrization, but the corresponding results for Steiner symmetrization are also true and can be proved by straightforward modification of the proofs for the circular case.

Assume now that D possesses a Green's function. Fix $z_0 \in D$ and let $u(z)$ be the Green's function of D with pole at z_0. Let v be the Green's function of D^* with pole at $|z_0|$. We set u and v equal to 0 outside D and D^* respectively.

THEOREM 5. *Let Φ be as in Theorem* 1. *Then*

$$\int_{-\pi}^{\pi} \Phi(u(re^{i\theta}))\,d\theta \leqslant \int_{-\pi}^{\pi} \Phi(v(re^{i\theta}))\,d\theta \qquad (0<r<\infty). \tag{4}$$

It follows from (4) that

$$\sup_\theta u(re^{i\theta}) \leqslant \sup_\theta v(re^{i\theta}),$$

and from this follows very easily the inequality

$$\lim_{z\to z_0} u(z) + \log|z-z_0| \leqslant \lim_{z\to z_0} v(z) + \log|z-|z_0||,$$

that is, symmetrization increases the inner radius, a well-known result of Pólya and Szegö [22].

It seems probable that if equality holds in (4) for some r and some strictly convex Φ then D must be a rotation of D^*, but this does not follow from our proof.

For $\lambda>0$ let

$$D_\lambda = \{z \in D: u(z) > \lambda\}.$$

The question is raised in [13] whether $(D_\lambda)^* \subset (D^*)_\lambda$. From considerations in § 3 it is easy to see that this holds for all λ if and only if

$$\frac{\partial u^*}{\partial\theta} \leqslant \frac{\partial v^*}{\partial\theta} \tag{5}$$

throughout the upper half plane. Theorem 5 is equivalent to the inequality $u^* \leqslant v^*$, but we have not been able to prove the more precise inequality (5).

Theorem 5 and a result of Lehto's [17] lead to a strong symmetrization principle for functions holomorphic in a disk. Let f be holomorphic in $|z|<1$ and D be the set of all values taken on by f. Let D_0 be a simply connected domain containing D^*. We assume D_0 is not the whole plane, and let F be a conformal map of $|z|<1$ onto D_0 with $F(0)=|f(0)|$.

THEOREM 6. *If f is holomorphic in the unit disk and F is as just described, then, for* $0<r<1$,

$$\int_{-\pi}^{\pi} \Phi(\log|f(re^{i\theta})|)\,d\theta \leqslant \int_{-\pi}^{\pi} \Phi(\log|F(re^{i\theta})|)\,d\theta, \tag{6}$$

where Φ is as in Theorem 1.

By considering, for example, the L^2 norms of $|f|$ and $|F|$ in (6) we obtain

$$|f'(0)| \leqslant |F'(0)|, \tag{7}$$

a result due to Hayman [12].

We note an important special case of Theorem 6. If f is univalent, then D and D^* are simply connected and we may take for F a conformal map onto D^* (with $F(0)=|f(0)|$). For Steiner symmetrization with respect to the real axis, the appropriate analogue of Theorem 6 involves integrals of $\operatorname{Re} f$ and $\operatorname{Re} F$. When D^* is multiply connected, it is conceivable that (6) could be sharpened by replacing the F there by the projection of the conformal map onto the universal covering surface of D^*. In this context, it is not even known if the inequality corresponding to (7) is true.

Assume again that f is univalent and F is the conformal map onto D^*. Write

$$f(z)=\sum_{n=0}^{\infty} a_n z^n, \quad F(z)=\sum_{n=0}^{\infty} A_n z^n.$$

Then $|a_0|=|A_0|$ by hypothesis and $|a_1|\leqslant|A_1|$ by (7). Is it true that $|a_n|\leqslant|A_n|$? Even if this is not true in general, the situation for subordinate functions suggests that perhaps the weaker result

$$\sum^{n} |a_k|^2 \leqslant \sum_{0}^{n} |A_k|^2 \tag{8}$$

is true. A proof of (8) when F is a conformal map onto the right half plane would lead to a proof of Littlewood's conjecture, $|a_n|\leqslant 4n|a_0|$ for non-zero univalent f.

In § 8 we consider domains D contained in the unit disk. Theorem 7 is an analogue of Theorem 5 in which u and v are harmonic measures associated with D and D^* instead of Green's functions. This theorem generalizes a result of Haliste's [9], and provides a solution of the Carleman-Milloux problem much more refined than the one commonly known.

I am grateful to P. Sjögren for communicating to me the proof of Theorem A′ that appears in § 2 and to Matts Essén for many helpful comments. The simple proof of Proposition 5 is due to him. John Lewis called by attention to the classes $S(d)$ discussed in

§ 6. W. H. J. Fuchs pointed out to me the existence of Lehto's result used in §§ 4 and 6. Especially, it is a very great pleasure for me to acknowledge the many helpful conversations I have had during the course of this research with my colleague, Professor Richard Rochberg.

2. Proof of theorem A.

We recall that if u is subharmonic in an annulus $r_1 < |z| < r_2$ then $u(re^{i\omega})$ is a Lebesgue integrable function of ω for $r_1 < r < r_2$ ([24, p. 4]). Moreover, the mean value $N(r, u)$, defined by

$$N(r, u) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(re^{i\omega})\, d\omega, \tag{9}$$

is a continuous function of r on (r_1, r_2) ([24, p. 5]).

We are going to prove a more general version of Theorem A, about differences of subharmonic functions, which will be needed for the proof of Theorem 5.

THEOREM A′. *Suppose $u = u_1 - u_2$, where u_1 and u_2 are subharmonic in $r_1 < |z| < r_2$. Define*

$$u^{\#}(re^{i\theta}) = u^*(re^{i\theta}) + 2\pi N(r, u_2)$$

where u^* is defined by (3) *and N by* (9). *Then $u^{\#}$ is subharmonic in $\{re^{i\theta}: r_1 < r < r_2, 0 < \theta < \pi\}$ and is continuous on $\{re^{i\theta}: r_1 < r < r_2, 0 \leqslant \theta \leqslant \pi\}$.*

Proof. First we note that for each $re^{i\theta}$, $r_1 < r < r_2$, $0 \leqslant \theta \leqslant \pi$, there exists a set $E \subset [-\pi, \pi]$ with $|E| = 2\theta$ for which the sup in (3) is attained, i.e.

$$u^*(re^{i\theta}) = \int_E u(re^{i\omega})\, d\omega.$$

This follows from Proposition 1 in § 3. Moreover, letting E^c denote the complement of E in $[-\pi, \pi]$, we have

$$u^{\#}(re^{i\theta}) = \int_E u(re^{i\omega})\, d\omega + \int_{-\pi}^{\pi} u_2(re^{i\omega})\, d\omega = \int_E u_1(re^{i\omega})\, d\omega + \int_{E^c} u_2(re^{i\omega})\, d\omega. \tag{9 a}$$

Observe that, for any E with $|E| = 2\theta$ the right hand side is $\leqslant u^{\#}(re^{i\theta})$. Now we prove the continuity statement. If u_1 and u_2 are continuous in the annulus then the proof is quite straightforward and we omit it. For the general case we use a regularization argument. Take R_1, R_2 with $r_1 < R_1 < R_2 < r_2$. For sufficiently small $\delta > 0$ and $R_1 < |z| < R_2$ define

$$(A_\delta u)(z) = \frac{1}{\pi\delta^2}\int_0^\delta\int_{-\pi}^{\pi} u(z+te^{i\varphi})\,t\,d\varphi\,dt.$$

Define $A_\delta u_1$ and $A_\delta u_2$ similarly. The subharmonicity of u_1 and u_2 implies $A_\delta u_i(z) \geqslant u_i(z)$ for $R_1 < |z| < R_2$ and $i=1, 2$. This, together with (9a) shows that

$$u^{\#}(z) \leqslant (A_\delta u)^{\#}(z) \quad (R_1 < |z| < R_2, \ \mathrm{Im}\, z \geqslant 0).$$

(We are computing $(A_\delta u)^{\#}$ relative to the decomposition $A_\delta u = A_\delta u_1 - A_\delta u_2$.) Take $re^{i\theta}$ with $R_1 < r < R_2$, $0 \leqslant \theta \leqslant \pi$. Choose E with $|E| = 2\theta$ so that (9a) holds with $A_\delta u$ in place of u. Then

$$\begin{aligned} 0 \leqslant (A_\delta u)^{\#}(re^{i\theta}) - u^{\#}(re^{i\theta}) &\leqslant \int_E (A_\delta u_1 - u_1)(re^{i\omega})\,d\omega + \int_{E^c} (A_\delta u_2 - u_2)(re^{i\omega})\,d\omega \\ &\leqslant \int_{-\pi}^{\pi} (A_\delta u_1 - u_1)(re^{i\omega})\,d\omega + \int_{-\pi}^{\pi} (A_\delta u_2 - u_2)(re^{i\omega})\,d\omega \\ &= N(r, A_\delta u_1) - N(r, u_1) + N(r, A_\delta u_2) - N(r, u_2) \end{aligned} \tag{10}$$

I claim that

$$\lim_{\delta\to 0} N(r, A_\delta u_i) = N(r, u_i) \quad (i=1,2) \tag{11}$$

uniformly for $r \in (R_1, R_2)$. This, together with (10), shows that $u^{\#}$ is the uniform limit as $\delta \to 0$ of $(A_\delta u)^{\#}$ in the upper half of $R_1 < |z| < R_2$. Since $A_\delta u_1$ and $A_\delta u_2$ are continuous for each δ, the same is true of $(A_\delta u)^{\#}$ and hence of $u^{\#}$.

Let v stand for u_1 or u_2. Then

$$N(r, A_\delta v) = \frac{1}{\pi\delta^2}\int_0^\delta t\,dt \int_{-\pi}^{\pi} d\theta \int_{-\pi}^{\pi} v(re^{i\theta} + te^{i\varphi})\,d\varphi \tag{12}$$

Now

$$\int_{-\pi}^{\pi} v(re^{i\theta}+te^{i\varphi})\,d\varphi = \int_{-\pi}^{\pi} v(re^{i\theta}+te^{i(\varphi+\theta)})\,d\varphi = \int_0^{\pi} v(re^{i\theta}+te^{i(\theta+\omega)}) + v(re^{i\theta}+te^{i(\theta-\varphi)})\,d\varphi$$

So, for $0 < t \leqslant \delta$,

$$\begin{aligned} \int_{-\pi}^{\pi} d\theta \int_{-\pi}^{\pi} v(re^{i\theta}+te^{i\varphi})\,d\varphi &= \int_0^{\pi} d\varphi \int_{-\pi}^{\pi} [v((r+te^{i\varphi})\,e^{i\theta}) + v((r+te^{-i\varphi})\,e^{i\theta})]\,d\theta \\ &= \int_0^{\pi} N(|r+te^{i\varphi}|, v) + N(|r+te^{-i\varphi}|, v)\,d\varphi \leqslant 2\pi \sup\{N(s,v),\ |s-r| \leqslant t\} \\ &\leqslant 2\pi \sup\{N(s,v) \colon |s-r| \leqslant \delta\}. \end{aligned}$$

Substituting in (12), we find

$$N(r, A_\delta v) \leqslant \sup\{N(s, v): |s-r| \leqslant \delta\}. \tag{13}$$

Since $N(r, v) \leqslant N(r, A_\delta v)$ and $N(r, v)$ is uniformly continuous on $[R_1, R_2]$, the desired statement (11) follows from (13). This completes the proof of continuity.

For the proof of subharmonicity we need a result about subsets of the unit circle T. As usual, we identify T with the quotient of the real line modulo the subgroup $2\pi Z$. For $E \subset T$ let E_ε denote the translate of E by ε.

LEMMA. *Let E be a measurable subset of T, with $0 < |E| < 2\pi$. Then there exists $\delta > 0$ such that*

$$|E_\varepsilon \cap E_{-\varepsilon}| \leqslant |E| - 2\varepsilon \tag{14}$$

for $0 < \varepsilon < \delta$.

Proof. If E is an interval, or differs from an interval by a null set, then equality holds in (14) for all sufficiently small ε. In the contrary case, there are points $a_1 < b_1 < a_2 < b_2 < a_1 + 2\pi$ such that a_1 and a_2 are density points of E^c and b_1 and b_2 are density points of E. We may assume $0 < a_1$ and $b_2 < 2\pi$. Choose c with $b_1 < c < a_2$ and let χ denote the characteristic function of E. Then, for $\varepsilon > 0$,

$$\begin{aligned}\int_0^c \chi(t+\varepsilon)\chi(t-\varepsilon)\,dt &\leqslant \int_0^{a_1} \chi(t+\varepsilon)\,dt + \int_{a_1}^{b_1} \chi(t-\varepsilon)\,dt + \int_{b_1}^{c} \chi(t+\varepsilon)\,dt \\ &= \int_0^c \chi(t+\varepsilon)\,dt - \int_{a_1}^{b_1} \chi(t+\varepsilon)\,dt + \int_{a_1}^{b_1} \chi(t-\varepsilon)\,dt \\ &= \int_0^c \chi(t+\varepsilon)\,dt + \int_{a_1-\varepsilon}^{a_1+\varepsilon} \chi(t)\,dt - \int_{b_1+\varepsilon}^{b_1+\varepsilon} \chi(t)\,dt.\end{aligned}$$

Choose $\delta > 0$ such that $0 < \varepsilon < \delta$ implies

$$\int_{a_1-\varepsilon}^{a_1+\varepsilon} \chi(t)\,dt \leqslant \frac{\varepsilon}{2}, \quad \int_{b_1-\varepsilon}^{b_1+\varepsilon} \chi(t)\,dt \geqslant \frac{3\varepsilon}{2}.$$

Then

$$\int_0^c \chi(t+\varepsilon)\chi(t-\varepsilon)\,dt \leqslant \int_0^c \chi(t+\varepsilon)\,dt - \varepsilon \quad (0 < \varepsilon < \delta). \tag{15}$$

Similarly, using

$$\int_c^{2\pi} \chi(t+\varepsilon)\chi(t-\varepsilon)\,dt \leqslant \int_c^{a_2} \chi(t+\varepsilon)\,dt + \int_{a_2}^{b_2} \chi(t-\varepsilon)\,dt + \int_{b_2}^{2\pi} \chi(t+\varepsilon)\,dt$$

we obtain (with perhaps a smaller δ)

$$\int_c^{2\pi} \chi(t+\varepsilon)\,\chi(t-\varepsilon)\,dt \leqslant \int_c^{2\pi} \chi(t+\varepsilon)\,dt - \varepsilon \quad (0<\varepsilon<\delta).$$

This, combined with (15), yields, for $0<\varepsilon<\delta$,

$$|E_\varepsilon \cap E_{-\varepsilon}| = \int_0^{2\pi} \chi(t+\varepsilon)\,\chi(t-\varepsilon)\,dt \leqslant \int_0^{2\pi} \chi(t+\varepsilon)\,dt - 2\varepsilon = |E_\varepsilon| - 2\varepsilon,$$

and the lemma is proved.

Now we can prove that $u^{\#}$ is subharmonic. For fixed r and ϱ, $0<\varrho<r$, define $r(\psi)$ and $\alpha(\psi)$ for ψ real by

$$r(\psi) = |r+\varrho e^{i\psi}|, \quad \alpha(\psi) = \arg(r+\varrho e^{i\psi}), \quad \left(|\arg| < \frac{\pi}{2}\right).$$

Then
$$r+\varrho e^{i\psi} = r(\psi)\, e^{i\alpha(\psi)}.$$

Note that $r(\psi)=r(-\psi)$ and $\alpha(\psi) = -\alpha(-\psi)$. For any function u we have

$$\begin{aligned}\int_{-\pi}^{\pi} u(re^{i\omega}+\varrho e^{i\psi})\,d\psi &= \int_0^{\pi} u(re^{i\omega}+\varrho e^{i(\omega+\psi)}) + u(re^{i\omega}+\varrho e^{i(\omega-\psi)})\,d\psi \\ &= \int_0^{\pi} u(r(\psi)\,e^{i(\omega+\alpha(\psi))}) + u(r(\psi)\,e^{i(\omega-\alpha(\psi))})\,d\psi. \qquad (16)\end{aligned}$$

Let u be as in the statement of the theorem. Fix $re^{i\theta}$ with $r_1<r<r_2$, $0<\theta<\pi$. As in (9a) there is a set E with $|E|=2\theta$ such that

$$u^{\#}(re^{i\theta}) = \int_E u_1(re^{i\omega})\,d\omega + \int_F u_2(re^{i\omega})\,d\omega \qquad (17)$$

where $F=E^c$. Since u_1 and u_2 are subharmonic we have, for sufficiently small ϱ and $j=1,2$,

$$u_j(re^{i\omega}) \leqslant \frac{1}{2\pi}\int_0^{\pi} u_j(r(\psi)\,e^{i(\omega+\alpha(\psi))}) + u_j(r(\psi)\,e^{i(\omega-\alpha(\psi))})\,d\psi.$$

We put these inequalities in (17) and reverse the order of integration. The result may be written

$$\begin{aligned}u^{\#}(re^{i\theta}) \leqslant{}& \frac{1}{2\pi}\int_0^{\pi}\left[\left[\int_{E_{\alpha(\psi)}} + \int_{E_{-\alpha(\psi)}} u_1(r(\psi)\,e^{i\omega})\,d\omega\right] d\psi\right. \\ &+ \frac{1}{2\pi}\int_0^{\pi}\left[\left[\int_{F_{\alpha(\psi)}} + \int_{F_{-\alpha(\psi)}} u_2(r(\psi)\,e^{i\omega})\,d\omega\right] d\psi. \right. \qquad (18)\end{aligned}$$

Let δ be associated with E as in the lemma. We assume that ϱ is small enough so that $0 \leqslant \alpha(\psi) < \delta$ and $\theta + \alpha(\psi) \leqslant \pi$ for $0 \leqslant \psi \leqslant \pi$. Fix such a ψ. By the lemma, we can choose a set C with

$$C \subset (E_{\alpha(\psi)} \cup E_{-\alpha(\psi)}) - (E_{\alpha(\psi)} \cap E_{-\alpha(\psi)})$$

such that if $A = (E_{\alpha(\psi)} \cap E_{-\alpha(\psi)}) \cup C$ then

$$|A| = |E| - 2\alpha(\psi) = 2(\theta - \alpha(\psi)). \tag{19}$$

Let $B = (E_{\alpha(\psi)} \cup E_{-\alpha(\psi)}) - C$. Then

$$A \cup B = E_{\alpha(\psi)} \cup E_{-\alpha(\psi)}, \quad A \cap B = E_{\alpha(\psi)} \cap E_{-\alpha(\psi)}. \tag{20}$$

Since $$|A| + |B| = |A \cap B| + |A \cup B| = |E_{\alpha(\psi)}| + |E_{-\alpha(\psi)}| = 4\theta$$

it follows that

$$|B| = 2(\theta + \alpha(\psi)). \tag{21}$$

By (20), for every integrable function g we have

$$\int_A g + \int_B g = \int_{E_{\alpha(\psi)}} g + \int_{E_{-\alpha(\psi)}} g.$$

A similar identity holds with A and B replaced by their complements and $E_{\alpha(\psi)}$, $E_{-\alpha(\psi)}$ replaced by $F_{\alpha(\psi)}$, $F_{-\alpha(\psi)}$. Thus

$$\begin{aligned}\int_{E_{\alpha(\psi)}} + \int_{E_{-\alpha(\psi)}} u_1(r(\psi)\, e^{i\omega})\, d\omega + \int_{F_{\alpha(\psi)}} + \int_{F_{-\alpha(\psi)}} u_2(r(\psi)\, e^{i\omega})\, d\omega \\ = \int_A u_1(r(\psi)\, e^{i\omega})\, d\omega + \int_{A^c} u_2(r(\psi)\, e^{i\omega})\, d\omega + \int_B u_1(r(\psi)\, e^{i\omega})\, d\omega + \int_{B^c} u_2(r(\psi)\, e^{i\omega})\, d\omega \\ \leqslant u^*(r(\psi)\, e^{i(\theta - \alpha(\psi))}) + u^*(r(\psi)\, e^{i(\theta + \alpha(\psi))}).\end{aligned} \tag{22}$$

The inequality follows from (19) and (21). Substituting (22) in (18) and recalling (16), we obtain

$$u^*(re^{i\theta}) \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} u^*(re^{i\theta} + \varrho\, e^{i\psi})\, d\psi,$$

which proves the subharmonicity of u^*.

3. Some real variable results

Let $g(x)$ be a real valued integrable function on $[-\pi, \pi]$. The *distribution function* of g is the function $\lambda(t)$ defined by

$$\lambda(t) = |\{x \colon g(x) > t\}|.$$

The definition of Lebesgue integral leads to the well known formula

$$\int_{-\pi}^{\pi} g = -\int_{-\infty}^{\infty} t\,d\lambda(t).$$

In particular, two functions with the same distribution function have the same integral over $[-\pi, \pi]$.

We define now

$$g^*(\theta) = \sup_{|E|=2\theta} \int_E g \quad (0 \leqslant \theta \leqslant \pi). \tag{23}$$

PROPOSITION 1. *For each* $\theta \in [0, \pi]$ *there exists a set* E *with* $|E| = 2\theta$ *for which the supremum in* (23) *is attained.*

Proof. This is obviously true for $\theta = 0$ and $\theta = \pi$. Take $\theta \in (0, \pi)$. There exists $t \in (-\infty, \infty)$ such that $\lambda(t+) = \lambda(t) \leqslant 2\theta \leqslant \lambda(t-)$. Let $A = \{x: g(x) > t\}$, $B = \{x: g(x) \geqslant t\}$. Then $|A| = \lambda(t)$, $|B| = \lambda(t-)$. Take E such that $|E| = 2\theta$ and $A \subset E \subset B$. Let F be any set with $|F| = 2\theta$. Then

$$\int_F g = \int_F [g(x) - t]\,dx + 2\theta t \leqslant \int_{-\pi}^{\pi} [g(x) - t]^+\,dx + 2\theta t = \int_E [g(x) - t]\,dx + 2\theta t = \int_E g.$$

This proves the proposition.

The *symmetric non-increasing rearrangement* of g is the extended real valued function $G(x)$, defined on $[-\pi, \pi]$ as follows. If $\lambda(t)$ is continuous and strictly decreasing then G on $[0, \pi]$ is the inverse function of $\frac{1}{2}\lambda$. In general,

$$G(x) = \inf\{t: \lambda(t) \leqslant 2x\} \quad (0 \leqslant x < \pi).$$

$$G(\pi) = \lim_{x \to \pi-} G(x) = \text{ess. inf}\, g.$$

We set $G(x) = G(-x)$ for $-\pi \leqslant x \leqslant 0$. Then G is non-increasing on $[0, \pi]$ and it is easily verified that G has the same distribution function as g. (The reader should be aware of the fact that in the literature the function we are calling G is often called g^*.)

The relation between our functions G and g^* is given by the following formula.

PROPOSITION 2. *For* $g \in L^1[-\pi, \pi]$,

$$g^*(\theta) = \int_{-\theta}^{\theta} G(x)\,dx \quad (0 \leqslant \theta \leqslant \pi). \tag{24}$$

Proof. For $\theta = 0$ both sides are zero and for $\theta = \pi$ both sides equal the integral of g

over $[-\pi, \pi]$. Take $\theta \in (0, \pi)$ and let E and t be as in the proof of Proposition 1. Then

$$g^*(\theta) = \int_E g = \int_{-\pi}^{\pi} [g(x) - t]^+ \, dx + 2\theta t = \int_{-\pi}^{\pi} [G(x) - t]^+ \, dx + 2\theta t. \tag{25}$$

(The last equation follows from the fact that $[g(x) - t]^+$ and $[G(x) - t]^+$ have the same distribution function.) Sets of the form $\{x \colon G(x) > s\}$ are intervals symmetric around the origin, and G and g are equi-distributed. These facts imply $G(x) \geqslant t$ for $|x| \leqslant \theta$ and $G(x) \leqslant t$ for $|x| \geqslant \theta$. Thus

$$\int_{-\pi}^{\pi} [G(x) - t]^+ \, dx + 2\theta t = \int_{-\theta}^{\theta} [G(x) - t] \, dx + 2\theta t = \int_{-\theta}^{\theta} G(x) \, dx,$$

which, together with (25), proves (24).

Our next result asserts that inequalities between the convex integral means of two functions are equivalent to inequalities between their * functions.

PROPOSITION 3. *For $g, h \in L^1[-\pi, \pi]$ the following statements are equivalent.*

(a) *For every convex non-decreasing function Φ on $(-\infty, \infty)$,*

$$\int_{-\pi}^{\pi} \Phi(g(x)) \, dx \leqslant \int_{-\pi}^{\pi} \Phi(h(x)) \, dx.$$

(b) *For every $t \in (-\infty, \infty)$,*

$$\int_{-\pi}^{\pi} [g(x) - t]^+ \, dx \leqslant \int_{-\pi}^{\pi} [h(x) - t]^+ \, dx.$$

(c) $$g^*(\theta) \leqslant h^*(\theta) \quad (0 \leqslant \theta \leqslant \pi).$$

This proposition is a variant on Exercises 249 and 250 of [10].

Proof. (a) $\Rightarrow$ (b) is trivial. For (b) $\Rightarrow$ (c) we may assume $0 < \theta < \pi$ and let $t = H(\theta)$, where H is the symmetric non-increasing rearrangement of h. Then for any set E with $|E| = 2\theta$,

$$\begin{aligned} \int_E g = \int_E [g(x) - t] \, dx + 2\theta t &\leqslant \int_{-\pi}^{\pi} [g(x) - t]^+ \, dx + 2\theta t \\ &\leqslant \int_{-\pi}^{\pi} [h(x) - t]^+ \, dx + 2\theta t \\ &= \int_{-\pi}^{\pi} [H(x) - t]^+ \, dx + 2\theta t = \int_{-\theta}^{\theta} H(x) \, dx = h^*(\theta), \end{aligned}$$

which proves (c). For (c)$\Rightarrow$(b) we may assume $t <$ ess. sup$g$ and choose θ so that $G(\theta-)\geqslant t\geqslant G(\theta+)$ (For $t\leqslant$ ess. inf$g$ take $\theta=\pi$). Then

$$\int_{-\pi}^{\pi}[g(x)-t]^+\,dx=\int_{-\pi}^{\pi}[G(x)-t]^+\,dx=\int_{-\theta}^{\theta}[G(x)-t]\,dx=g^*(\theta)-2\theta t\leqslant h^*(\theta)-2\theta t$$
$$=\int_{-\theta}^{\theta}[H(x)-t]\,dx\leqslant\int_{-\pi}^{\pi}[H(x)-t]^+\,dx=\int_{-\pi}^{\pi}[h(x)-t]^+\,dx,$$

which proves (b).

Now we turn to (b)$\Rightarrow$(a). A simple limiting argument involving truncations of Φ shows that it is sufficient to prove the inequality in (a) for convex non-decreasing functions Φ which are equal to a constant α on some interval $(-\infty, -M)$. Consideration of $\Phi=(\Phi-\alpha)+\alpha$ shows that there is no loss of generality in assuming $\alpha=0$. Since Φ' is non-decreasing there is a positive measure μ on $(-\infty,\infty)$ defined by $\mu(-\infty,s)=\Phi'(s-)$. Then

$$\Phi(s)=\int_{-\infty}^{s}\Phi'(t)\,dt=-\int_{-\infty}^{s}\Phi'(t)\,d(s-t)=\int_{-\infty}^{s}(s-t)\,d\mu(t).$$

Thus

$$\Phi(s)=\int_{-\infty}^{\infty}(s-t)^+\,d\mu(t)\qquad(-\infty<s<\infty).$$

The inequality in (a) follows at once from this representation together with hypothesis (b) and Fubini's theorem. This completes the proof of Proposition 3.

Using (a)$\Leftrightarrow$(c) we note that Theorem 1 may be succinctly restated as

$$(\log|f|)^*\leqslant(\log|k|)^*\quad(f\in S)$$

The other theorems may be restated similarly.

4. Some properties of Green's functions

A domain D in the extended plane is said to possess a Green's function, or be of hyperbolic type, if there exists a positive function on D which is harmonic in D except at a point $z_0\in D$ where it has a logarithmic pole. The smallest such function is called the Green's function of D, with pole at z_0. If there is a Green's function of D with pole at z_0 then there is a Green's function of D with a pole at any other point of D.

We will say that D has a *classical Green's function* if each Green's function $u(z)$ tends to zero as z approaches the boundary of D. A sufficient condition for D to posses a classical Green's function is that every boundary point of D be contained in a continuum which is contained in the boundary.

In this paper whenever a Green's function exists it will be extended to the whole Riemann sphere by setting it equal to zero in the complement of D. Thus, if u is a classical Green's function with pole at z_0 then u is continuous on the sphere, except at z_0. At a point in the complement of D u is obviously $\leqslant$ its mean value on circles centered at that point, hence u is subharmonic on the sphere, except at z_0.

The proofs of our theorems about integral means of analytic functions $f(z)$ involve transferring the integrals from the z-plane into the image plane. The transfer is accomplished via the following result.

PROPOSITION 4. *Let f be holomorphic in $|z|<1$ and D be the range of f. Assume D has a Green's function and let u be the Green's function of D with pole at $f(0)$. Then for* $0<\varrho<\infty$ *and* $0<r<1$,

$$\int_{-\pi}^{\pi} \log^+ \frac{|f(re^{i\theta})|}{\varrho}\, d\theta \leqslant \int_{-\pi}^{\pi} [u(\varrho\, e^{i\varphi}) + \log r]^+\, d\varphi + 2\pi \log^+ \frac{|f(0)|}{\varrho}. \tag{27}$$

If f is univalent then equality holds.

Proof. For $0<r<1$ and ζ complex consider the function

$$N(r,\zeta) = N(r,\zeta,f) = \sum \log^+ \frac{r}{|z_i|} \tag{28}$$

where the sum is taken over the roots z_i of $f(z)=\zeta$ in $|z|<r$, counted according to multiplicity. Cartan's formula ([14, p. 214]) asserts that

$$\int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})|\, d\theta = \int_{-\pi}^{\pi} N(r, e^{i\varphi})\, d\varphi + 2\pi \log^+ |f(0)|. \tag{29}$$

The function $u(f(z)) + \log|z|$ is superharmonic in $|z|<1$ and has non-negative boundary values. Therefore

$$u(f(z)) \geqslant -\log|z| \quad (|z|<1). \tag{30}$$

For $0<r<1$ let $D_r = \{\zeta \in D : u(\zeta) > -\log r\}$. By (30) we have $f(|z|<r) \subset D_r$, thus $N(r,\zeta)=0$ for $\zeta \notin D_r$. Lehto [17] has shown that $N(r,\zeta)$ is a subharmonic function of ζ, except for a logarithmic pole at $\zeta = f(0)$. Thus $u(\zeta)+\log r - N(r,\zeta)$ is superharmonic in D_r and has non-negative boundary values. It follows that $N(r,\zeta) \leqslant u(\zeta) + \log r$ in D_r, so that

$$N(r,\zeta) \leqslant [u(\zeta) + \log r]^+ \tag{31}$$

for all $\zeta \in \mathbf{C}$. Putting this inequality in (29) we obtain the desired integral inequality (27).

If f is univalent then the Green's function u is given by $u(\zeta) = -\log|f^{-1}(\zeta)|$. Comparing with the definition (28) of $N(r, \zeta)$ we see that equality holds in (31), and hence in (27). This completes the proof of Proposition 4.

The extremal functions in our theorems about integral means all have the property that their associated domains D coincide with their circular symmetrization, that is, $D = D^*$. The next result provides some explanation as to why this is so. Let D be a domain with a classical Green's function and let u be the Green's function of D with pole at z_0. We assume z_0 is finite, the case $z_0 = \infty$ requiring only obvious modifications. Then we can write

$$u(z) = u_1(z) - \log |z - z_0|$$

where u_1 is subharmonic in the whole finite plane. If $z_0 \neq 0$ then, by Theorem A′,

$$u^{\#}(re^{i\theta}) = u^*(re^{i\theta}) + \int_{-\pi}^{\pi} \log |re^{i\varphi} - z_0| \, d\varphi \tag{32}$$

is subharmonic in the upper half plane. If $z_0 = 0$ then u is subharmonic in $0 < |z| < \infty$ so that u^* is subharmonic in the upper half plane.

PROPOSITION 5. *Let D and u be as just described. Suppose that $D = D^*$ and that $z_0 = r_0 \geq 0$. Let $D^+ = D \cap (\operatorname{Im} z > 0)$. Then*

(a) $u^*(z)$ *is harmonic in* D^+, *when* $r_0 = 0$,

(b) $u^*(z) + 2\pi \log^+ \dfrac{|z|}{r_0}$ *is harmonic in* D^+, *when* $r_0 > 0$.

Proof. Consider case (a). Define h by

$$h(re^{i\theta}) = \int_{-\theta}^{\theta} u(re^{i\varphi}) \, d\varphi \quad (0 \leq \theta \leq \pi, r > 0).$$

Since u is harmonic in $D - \{0\}$ and D contains a neighborhood of the arc $\{re^{i\varphi}: |\varphi| \leq \theta\}$ whenever it contains $re^{i\theta}$ we have, for $re^{i\theta} \in D^+$,

$$\frac{\partial^2 h}{\partial (\log r)^2}(re^{i\theta}) = \int_{-\theta}^{\theta} \frac{\partial^2 u}{\partial (\log r)^2}(re^{i\varphi}) \, d\varphi = \int_{-\theta}^{\theta} -\frac{\partial^2 u}{\partial \varphi^2}(re^{i\varphi}) \, d\varphi$$

$$= -\frac{\partial u}{\partial \theta}(re^{i\theta}) + \frac{\partial u}{\partial \theta}(re^{-i\theta}) = -\frac{\partial^2 h}{\partial \theta^2}(re^{i\theta}).$$

Thus h is harmonic in D^+, and is continuous in the closed upper half plane, except at the orgin. If $re^{i\theta} \notin D^+$, with $r > 0$, $0 \leq \theta \leq \pi$, then, since $u \geq 0$ everywhere and $u = 0$ outside D, it follows from the the definition of u^* that $u^*(re^{i\theta}) = h(re^{i\theta})$. From $u(z) = u_1(z) - \log |z|$ with u_1 harmonic in a neighborhood of 0 it follows that

$$\lim_{z \to 0} [u^*(z) - h(z)] = 0.$$

The same relation holds as $z\to\infty$, since u is continuous at ∞.

We have shown that u^*-h is subharmonic in D^+, continuous and bounded on the closure, and equal to zero on the boundary. Thus $u^*\leqslant h$ in D^+. But the definition of u^* shows that $u^*\geqslant h$. We conclude that $u^*=h$, and hence that u^* is harmonic in D^+.

Turning to the case $r_0>0$ we observe that u^*-h is still continuous in the closed extended upper half plane and equal to zero outside D^+. (The simple proof that $u^*(z)-h(z)\to 0$ as $z\to r_0$ is left to the reader.)

Since $u(z)+\log|z-r_0|$ is harmonic in D it follows that

$$h_1(re^{i\theta})=\int_{-\theta}^{\theta}[u(re^{i\varphi})+\log|re^{i\varphi}-r_0|]\,d\varphi$$

is harmonic in D^+. Define

$$u_2(re^{i\theta})=u^*(re^{i\theta})+\int_{-\theta}^{\theta}\log|re^{i\varphi}-r_0|\,d\varphi.$$

Comparing with (32), we find

$$u^{\#}(re^{i\theta})-u_2(re^{i\theta})=\int_{-\pi+\theta}^{\pi-\theta}\log|re^{i(\pi-\varphi)}-r_0|\,d\varphi. \tag{33}$$

The function on the right is a harmonic function of $re^{i\theta}$ in the upper half plane. Since $u^{\#}$ is subharmonic in the upper half plane the same is true of u_2. Thus $u_2-h_1=u^*-h$ is subharmonic in D^+. As in case (a) it follows that $u^*=h$ throughout the upper half plane, and hence that $u_2=h_1$ there. The harmonicity of $h_1=u_2$ in D^+ implies, in view of (33), that of $u^{\#}$ there. Evaluating the mean value in (32), we find

$$u^{\#}(re^{i\theta})=u^*(re^{i\theta})+2\pi\log^+\frac{r}{r_0}+\log r_0,$$

so that $u^*+2\pi\log^+(r/r_0)$ is harmonic in D^+, as claimed.

We point out a consequence of the equation $u^*=h$ established above.

COROLLARY. *Under the hypotheses of Proposition* 5, $u(\bar z)=u(z)$ *for all* z *and, for every* $r>0$, $u(re^{i\theta})$ *is a decreasing function of* θ *on* $[0,\pi]$.

Proof. If this were not the case we could find an r and θ such that

$$u^*(re^{i\theta})>\int_{-\theta}^{\theta}u(re^{i\varphi})\,d\varphi,$$

which violates $u^*=h$.

For certain simply connected domains D with $D=D^*$ the above result was proved by Jenkins [15].

5. Proof of Theorem 1.

Let $f \in S$ and k be the Koebe function. Using (b) $\Rightarrow$ (a) of Proposition 3 with $g(\theta) = \log|f(re^{i\theta})|$, $h(\theta) = \log|k(re^{i\theta})|$ and $t = \log \varrho$ we see that the inequality of Theorem 1 will hold provided we can show that

$$\int_{-\pi}^{\pi} \log^+ \frac{|f(re^{i\theta})|}{\varrho}\, d\theta \leqslant \int_{-\pi}^{\pi} \log^+ \frac{|k(re^{i\theta})|}{\varrho}\, d\theta \tag{34}$$

for all $r \in (0, 1)$ and all $\varrho > 0$.

Let $u(\zeta)$ and $v(\zeta)$ be the Green's functions, with pole at $\zeta = 0$, of the ranges of f and k respectively. These Green's functions are classical, in the sense of § 4. Thus, when defined to be zero outside their original domains, u and v are subharmonic in $0 < |\zeta| < \infty$. In view of Proposition 4, inequality (34) is equivalent to

$$\int_{-\pi}^{\pi} [u(\varrho e^{i\varphi}) + \log r]^+\, d\varphi \leqslant \int_{-\pi}^{\pi} [v(\varrho e^{i\varphi}) + \log r]^+ d\varphi.$$

By (c) $\Rightarrow$ (b) of Proposition 3, this inequality will follow from the inequality

$$u^*(\varrho e^{i\varphi}) \leqslant v^*(\varrho e^{i\varphi}) \quad (0 < \varrho < \infty,\ 0 \leqslant \varphi \leqslant \pi). \tag{35}$$

By Theorem A, u^* is subharmonic in the (open) upper half plane. The range of k is the slit plane $\mathbf{C} - (-\infty, -\frac{1}{4})$, so, by Proposition 5, v^* is harmonic in the upper half plane. Fix $\varepsilon > 0$ and define the function Q by

$$Q(\varrho e^{i\varphi}) = u^*(\varrho e^{i\varphi}) - v^*(\varrho e^{i\varphi}) - \varepsilon\varphi \quad (0 < \varrho < \infty,\ 0 \leqslant \varphi \leqslant \pi).$$

Then Q is subharmonic in the upper half plane, and, by the continuity statement in Theorem A′, is continuous in the closed upper half plane except at the origin.

Let d be the distance from the origin to the complement of the range of f. Then $d \geqslant \frac{1}{4}$ We can write

$$u(\zeta) = -\log|\zeta| + u_1(\zeta) \tag{36}$$

where u_1 is harmonic in $|\zeta| < d$. From $f'(0) = 1$ it follows that $u_1(0) = 0$, and from this it follows that

$$u^*(\varrho e^{i\varphi}) = -2\pi \log \varrho + o(1)$$

as $\varrho \to 0$, uniformly in φ. The same relation holds for v^*. Thus

$$\limsup_{\zeta \to 0} Q(\zeta) = 0. \tag{37}$$

Since $u(\zeta)\to 0$ and $v(\zeta)\to 0$ as $\zeta\to\infty$ the same is true of u^* and v^*. Thus

$$\limsup_{\zeta\to\infty} Q(\zeta)=0. \tag{38}$$

From the definition of the * function, we have

$$Q(\varrho)=0 \quad (\varrho>0). \tag{39}$$

From (36) we obtain

$$u^*(\varrho e^{i\pi})=\int_{-\pi}^{\pi} u(\varrho e^{i\varphi})\,d\varphi=-2\pi\log\varrho \quad (0<\varrho\leqslant d). \tag{40}$$

Define v_1 by

$$v(\zeta)=-\log|\zeta|+v_1(\zeta).$$

Then $v_1(0)=0$, v_1 is harmonic in $|\zeta|<\frac{1}{2}$, and v_1 is subharmonic in the whole plane. Thus

$$v^*(\varrho e^{i\pi})=\int_{-\pi}^{\pi} v(\varrho e^{i\varphi})\,d\varphi=-2\pi\log\varrho+\int_{-\pi}^{\pi} v_1(\varrho e^{i\varphi})\,d\varphi\geqslant-2\pi\log\varrho \quad (0<\varrho<\infty). \tag{41}$$

Comparing (40) and (41), we see that

$$Q(\varrho e^{i\pi})\leqslant 0 \quad (0<\varrho\leqslant d). \tag{42}$$

Let M be the supremum of Q in the upper half plane. We want to show that $M=0$. Let $\{\zeta_n\}$ be a sequence such that $Q(\zeta_n)\to M$. A subsequence of $\{\zeta_n\}$ converges to some point ζ_0 in the extended closed upper half plane. Since Q is subharmonic, we may, by the maximum principle, assume that ζ_0 is a boundary point. If $\zeta_0=\infty$ or if ζ_0 is a point of the real axis in $[-d,\infty]$ then, in view of (37), (38), (39) and (40), $M=0$.

The remaining possibility is $\zeta_0=\varrho_0 e^{i\pi}$, with $d<\varrho_0<\infty$. Then

$$Q(\varrho_0 e^{i\pi})-Q(\varrho_0 e^{i\varphi})\geqslant 0 \quad (0\leqslant\varphi\leqslant\pi). \tag{43}$$

Let G be the symmetric non-increasing rearrangement of the function $\varphi\to u(\varrho_0 e^{i\varphi})$. The continuity of u implies that G is continuous on $[-\pi,\pi]$. By Propositon 2,

$$\frac{\partial u^*}{\partial\varphi}(\varrho_0 e^{i\varphi})=2G(\varphi) \quad (0\leqslant\varphi\leqslant\pi).$$

The circle $|\zeta|=\varrho_0$ intersects the complement of the range of f. Thus $\inf_\varphi u(\varrho_0 e^{i\varphi})=G(\pi)=0$. Hence

$$\lim_{\varphi\to\pi-}\frac{\partial u^*}{\partial\varphi}(\varrho_0 e^{i\varphi})=0.$$

The same is true with u^* replaced by v^*. Hence

$$\lim_{\varphi\to\pi-}\frac{\partial Q}{\partial\varphi}(\varrho_0 e^{i\varphi})=-\varepsilon.$$

In view of the mean value theorem this is inconsistent with (43). We conclude that $-\infty<\zeta_0<-d$ is impossible, and thus have shown $M=0$. Hence $Q\leqslant 0$, and, since ε was arbitrary, (35) is established and the proof of the inequality statement in Theorem 1 is complete.

We prove now the statement in Theorem 1 regarding equality. The notations used are the same as above. Suppose f is not the Koebe function or one of its rotations. Then $d>\frac{1}{4}$. If $\varrho_0>\frac{1}{4}$ then v can not be harmonic in any annulus $\varrho_0<|\zeta|<\varrho_1$, since v achieves its minimum value zero at the interior points $\varrho e^{i\pi}$, $\varrho\in(\varrho_0,\varrho_1)$. It follows that $v_1(\zeta)=-\log|\zeta|+v(\zeta)$ fails to be harmonic in $|\zeta|<\varrho$ as soon as $\varrho>\frac{1}{4}$. Let h be harmonic in $|\zeta|<\varrho$ and equal to v_1 on $|\zeta|=\varrho$.
Then

$$0=v_1(0)<h(0)=\frac{1}{2\pi}\int_{-\pi}^{\pi}v_1(\varrho\, e^{i\varphi})\,d\varphi \quad (\varrho>\tfrac{1}{4}).$$

Thus (see (41))

$$v^*(\varrho e^{i\pi})>-2\pi\log\varrho \quad (\tfrac{1}{4}<\varrho<\infty). \tag{44}$$

The function u^*-v^* is subharmonic in the upper half plane. We showed above that it is continuous and bounded in the closed half plane, and is $\leqslant 0$ on the real axis. By (40) and (44) it is strictly negative on the interval $(-d,\,-\frac{1}{4})$, hence is strictly negative in the open upper half plane.

That is,

$$u^*(\varrho e^{i\varphi})<v^*(\varrho e^{i\varphi}) \quad (0<\varrho<\infty,\ 0<\varphi<\pi). \tag{45}$$

Fix $r\in(0,1)$. I claim there is an interval $J\subset(0,\infty)$ such that $\varrho\in J$ implies

$$\int_{-\pi}^{\pi}[u(\varrho\, e^{i\varphi})+\log r]^+\,d\varphi<\int_{-\pi}^{\pi}[v(\varrho\, e^{i\varphi})+\log r]^+\,d\varphi. \tag{46}$$

For $u(\zeta)=-\log|\zeta|$ we may take $J=(r,r+\varepsilon)$, where $\varepsilon>0$ is sufficiently small (direct verification, using $v=-\log|k^{-1}|$.) Any other u has the property

$$\inf_{\varphi} u(\varrho\, e^{i\varphi})<\sup_{\varphi J} u(\varrho\, e^{i\varphi})$$

for all $\varrho\leqslant d$. The inf is a continuous function of ϱ, equals zero for $\varrho=d$, and tends to ∞ as $\varrho\to 0$. We take for J an interval such that

$$\inf_{\varphi} u(\varrho\, e^{i\varphi})<-\log r<\sup_{\varphi} u(\varrho\, e^{i\varphi}) \quad (\varrho\in J).$$

Let $E(\varrho)=\{\varphi\colon u(\varrho e^{i\varphi})+\log r>0\}$. Then $0<|E(\varrho)|<2\pi$ for $\varrho\in J$.

Hence

$$\int_{-\pi}^{\pi}(u(\varrho e^{i\varphi})+\log r)^{+}\,d\varphi=\int_{E(\varrho)}(u(\varrho e^{i\varphi})+\log r)\,d\varphi$$

$$\leqslant u^{*}(\varrho e^{i|E(\varrho)|/2})+|E(\varrho)|\log r<v^{*}(\varrho e^{i|E(\varrho)|/2})+|E(\varrho)|\log r,$$

where the strict inequality is by (45). The right hand side is, by an obvious argument, $\leqslant$

$$\int_{-\pi}^{\pi}(v(\varrho e^{i\varphi})+\log r)^{+}\,d\varphi.$$

Thus we have proved (46).

In view of Proposition 4, inequality (46) may be written as

$$\int_{-\pi}^{\pi}\log^{+}\frac{|f(re^{i\theta})|}{\varrho}\,d\theta<\int_{-\pi}^{\pi}\log^{+}\frac{|k(re^{i\theta})|}{\varrho}\,d\theta\quad(\varrho\in J).\tag{47}$$

Let Φ be a non-decreasing strictly convex function on $(-\infty,\infty)$. (By strictly convex, we mean Φ is not linear on any interval). Let J' be the interval $\log J$. We may assume J' is a finite open interval. Let s_0 be a number lying to the left of J', and decompose Φ as follows,

$$\Phi=\Phi_1+\Phi_2,$$

where Φ_1 coincides with Φ on $(-\infty,s_0]$ and is linear on $[s_0,\infty)$ with slope $\Phi'(s_0-)$. Then Φ_1 and Φ_2 are non-decreasing and convex on $(-\infty,\infty)$, and Φ_2 is strictly convex on (s_0,∞), $\Phi_2(s)=0$ for $s\leqslant s_0$. We may write, as in the proof of (b) $\Rightarrow$ (a) of Proposition 3,

$$\Phi_2(s)=\int_{-\infty}^{\infty}(s-t)^{+}\,d\mu(t)\tag{48}$$

where μ is a positive measure. Since Φ_2 is not linear on J', we must have

$$\mu(J')>0.\tag{49}$$

From the inequality statement of Theorem 1 we have

$$\int_{-\pi}^{\pi}(\log|f(re^{i\theta})|-t)^{+}\,d\theta\leqslant\int_{-\pi}^{\pi}(\log|k(re^{i\theta})|-t)^{+}\,d\theta$$

for all t, and by (47), we have strict inequality for $t\in J'$. Using (48), (49), we denote

$$\int_{-\pi}^{\pi}\Phi_2(\log|f(re^{i\theta})|)\,d\theta<\int_{-\pi}^{\pi}\Phi_2(\log|k(re^{i\theta})|)\,d\theta.\tag{50}$$

Since
$$\int_{-\pi}^{\pi}\Phi_1(\log|f(re^{i\theta})|)\,d\theta\leqslant\int_{-\pi}^{\pi}\Phi_1(\log|k(re^{i\theta})|)\,d\theta,$$

and $\Phi=\Phi_1+\Phi_2$, (50) remains true when Φ_2 is replaced by Φ, and we are done.

6. Complements and extensions of Theorem I.

THEOREM 2. *For $f \in S$ and Φ as in Theorem* 1

$$\int_{-\pi}^{\pi} \Phi\left(\log \frac{1}{|f(re^{i\theta})|}\right) d\theta \leqslant \int_{-\pi}^{\pi} \Phi\left(\log \frac{1}{|k(re^{i\theta})|}\right) d\theta \quad (0 < r < 1).$$

If equality holds for some $r \in (0, 1)$ and some strictly convex Φ, then $f(z) = e^{-i\alpha}k(ze^{i\alpha})$ for some real α.

Proof. Assume f is not the Koebe function or one of its rotations. As in the proof of Theorem 1, it is sufficient to prove, for all $\varrho > 0$,

$$\int_{-\pi}^{\pi} \log^+ \frac{1}{\varrho |f(re^{i\theta})|} d\theta \leqslant \int_{-\pi}^{\pi} \log^+ \frac{1}{\varrho |k(re^{i\theta})|} d\theta \tag{51}$$

with strict inequality for ϱ in some open interval. By Jensen's formula and the fact that $f \in S$,

$$\int_{-\pi}^{\pi} \log (\varrho |f(re^{i\theta})|) \, d\theta = 2\pi(\log \varrho - \log r).$$

Thus

$$\int_{-\pi}^{\pi} \log^+(\varrho |f(re^{i\theta})|) \, d\theta = 2\pi(\log \varrho - \log r) + \int_{-\pi}^{\pi} \log^+ \frac{1}{\varrho |f(re^{i\theta})|} d\theta, \tag{52}$$

and the same is true when f is replaced by k. By Theorem 1, the left hand side of (52) does not decrease when f is replaced by k, and is strictly increased for ϱ in the reciprocal of the interval J of (47). Thus (51) holds, and the theorem is proved.

We turn our attention now to an analogue of class S for functions univalent in an annulus. For $R > 1$ let $A(R)$ denote the annulus $1 < |z| < R$, and let $S(R)$ denote the class of all holomorphic univalent functions f in $A(R)$ which satisfy

$$|f(z)| > 1 \quad \text{for} \quad z \in A(R), \quad |f(z)| = 1 \quad \text{for} \quad |z| = 1.$$

Thus f maps $A(R)$ onto a ring domain in the ζ-plane whose bounded complementary component is the disk $|\zeta| \leqslant 1$. There is a unique function $k_R \in S(R)$ which maps $A(R)$ onto the exterior of the unit disk with a slit along the negative real axis from some point $-d(R)$ to $-\infty$, and which satisfies $k_R(1) = 1$.

THEOREM 3. *For $f \in S(R)$, Φ as in Theorem* 1, *and* $1 < r < R$,

$$\int_{-\pi}^{\pi} \Phi(\log |f(re^{i\theta})|) \, d\theta \leqslant \int_{-\pi}^{\pi} \Phi(\log |k_R(re^{i\theta})|) \, d\theta, \tag{a}$$

$$\int_{-\pi}^{\pi} \Phi\left(\log \frac{1}{|f(re^{i\theta})|}\right) d\theta \leqslant \int_{-\pi}^{\pi} \Phi\left(\log \frac{1}{|k_R(re^{i\theta})|}\right) d\theta. \tag{b}$$

If equality holds for some $r \in (1, R)$ and some strictly convex Φ, then $f(z) = e^{i\beta} k_R(ze^{i\alpha})$ for some real α and β.

From the theorem it follows that for each fixed $z \in A(R)$ the quantity $|f(z)|$ is maximized and minimized over $S(R)$ by rotations of k_R. This distortion result was proved first by Groetzsch [8], who used the method of extremal length, and has also been proved by Duren and Schiffer [6] by means of a variational argument.

To prove part (a) of Theorem 3 it suffices, as in the proof of Theorem 1, to prove

$$\int_{-\pi}^{\pi} \log^+ \frac{|f(re^{i\theta})|}{\varrho}\, d\theta \leqslant \int_{-\pi}^{\pi} \log^+ \frac{|k_R(re^{i\theta})|}{\varrho}\, d\theta \tag{53}$$

for $1 < r < R$, $\varrho > 0$ and $f \in S(R)$. As before, the first step is to transform the proposed inequality into an equivalent one involving integrals of the inverse functions. Suppose g is holomorphic in $A(R)$ and has a continuous extension to $|z| = 1$. If g has no zeros, then, for $1 < r < R$

$$\int_{-\pi}^{\pi} \log|g(re^{i\theta})|\, d\theta = \int_{-\pi}^{\pi} \log|g(e^{i\theta})|\, d\theta + 2\pi B \log r \tag{54}$$

where B is the winding number of the curve $g(e^{i\theta})$ around the point $\zeta = 0$. This is easily deduced from the fact that the mean value on the left in (54) is a linear function of $\log r$. If g has exactly one zero $z_0 \in A(R)$ and no zeros on $|z| = 1$ then (54) holds provided we add $2\pi \log^+(r/|z_0|)$ to the right hand side.

Let $f \in S(R)$ and $D = f(A(R))$. Define u by

$$\begin{aligned} u(\zeta) &= 0 \quad \text{for} \quad |\zeta| \leqslant 1 \\ &= -\log |f^{-1}(\zeta)| \quad \text{for} \quad \zeta \in D \\ &= -\log R \quad \text{for} \quad \zeta \notin D,\ |\zeta| > 1. \end{aligned}$$

The argument in the proof of Cartan's formula [14, p. 214] together with the above facts about mean values leads to the formula

$$\begin{aligned} \int_{-\pi}^{\pi} \log^+ \frac{|f(re^{i\theta})|}{\varrho}\, d\theta &= \int_{-\pi}^{\pi} [u(\varrho e^{i\varphi}) + \log r]^+\, d\varphi \quad (\varrho \geqslant 1) \\ &= 2\pi \log \frac{r}{\varrho} \qquad (0 < \varrho \leqslant 1) \end{aligned}$$

Thus (53) is equivalent to the inequality

$$\int_{-\pi}^{\pi} [u(\varrho\, e^{i\theta}) + \log r]^+ \, d\varphi \leqslant \int_{-\pi}^{\pi} [v(\varrho\, e^{i\theta}) + \log r]^+ \, d\varphi \tag{55}$$

where v is obtained from k_R in the same way that u is obtained from f. By Proposition 3, (55) is implied by

$$u^*(\varrho e^{i\varphi}) \leqslant v^*(\varrho e^{i\varphi}) \quad (0 < \varrho < \infty,\ 0 \leqslant \varphi \leqslant \pi). \tag{56}$$

For $0 \leqslant \varrho \leqslant 1$ both sides of (56) are zero, so it suffices or prove (56) for $\varrho e^{i\varphi} \in H$, where $H = \{\zeta\colon |\zeta| > 1,\ \operatorname{Im} \zeta > 0\}$. u is subharmonic in $|\zeta| > 1$. Therefore u^* is subharmonic in H. The argument in the proof of Proposition 5 shows that v^* is harmonic in H.

Let ζ_0 be a point of smallest absolute value in the unbounded complementary component of $D = f(A(R))$, and put $d = |\zeta_0|$. Then each circle $|\zeta| = \varrho$ with $\varrho \geqslant d$ intersects the complement of D. From well-known symmetrization results [16, Theorems 8.3 and 8.4] it follows that the modulus of the ring domain $D_1 = \{\zeta\colon |\zeta| > 1,\ \zeta \notin (-\infty, -d]\}$ is $\geqslant$ the modulus of D, with strict inequality unless D_1 is a rotation of D. The modulus of D is the same as that of

$$k_R(A(R)) = \{\zeta\colon |\zeta| > 1,\ \zeta \notin (-\infty,\ -d(R))\}.$$

Hence $d \geqslant d(R)$, and strict inequality holds unless f is a rotation of k_R.

For $\varrho \geqslant d$ we have, as in the proof of Theorem 1,

$$u_\theta^*(\varrho\, e^{i\pi}) = v_\theta^*(\varrho\, e^{i\pi}) = -\log R.$$

Also, u is harmonic in $1 < |\zeta| < d$, so

$$u^*(\varrho\, e^{i\pi}) = \int_{-\pi}^{\pi} u(\varrho\, e^{i\varphi})\, d\varphi = A \log \varrho + B \quad (1 \leqslant \varrho \leqslant d).$$

$B = 0$, since $u = 0$ on $|\zeta| = 1$, while, by the Cauchy-Riemann equations

$$A = \int_{-\pi}^{\pi} \frac{\partial u}{\partial \varrho}(e^{i\varphi})\, d\varphi = -\int_{-\pi}^{\pi} \frac{\partial}{\partial \varphi} \arg(f^{-1}(e^{i\varphi}))\, d\varphi = -2\pi.$$

Similarly,

$$\begin{aligned} v^*(\varrho e^{i\pi}) &= -2\pi \log \varrho \quad (1 \leqslant \varrho \leqslant d(R)) \\ &> -2\pi \log \varrho \quad (d(R) < \varrho < \infty). \end{aligned}$$

It follows that (56) holds for $\varrho e^{i\pi}$ when $1 \leqslant \varrho \leqslant d$ and, if f is not a rotation of k_R, strict inequality holds for $d(R) < \varrho \leqslant d$. The proof of (a) of Theorem 3 and the accompanying equality statement can now be accomplished by an argument entirely analogous to that in the proof of Theorem 1, except now the full upper half plane is replaced by H.

Part (b) is obtained from (a) just as Theorem 2 was obtained from Theorem 1. This time we use

$$\int_{-\pi}^{\pi} \log(\varrho|f(re^{i\theta})|)\,d\theta = 2\pi(\log\varrho + \log r)$$

for all $f \in S(R)$ and $1 \leqslant r \leqslant R$.

Kirwan and Schober [27] have used the L^1 case of Theorem 3 to obtain bounds for the coefficients in the Laurent expansion of $f \in S(R)$ which are better than any previously known.

The convex integral means of $\log|k_R|$ are in fact extremal for a class more extensive than $S(R)$. We state this as a corollary.

COROLLARY. *Suppose f is holomorphic and univalent in $A(R)$ and that the bounded component C of the complement of $f(A(R))$ is contained in $|\zeta| \leqslant 1$. Then* (a) *of Theorem* 3 *holds for f.*

Proof. Let g be a conformal map of the complement of C onto $|w| > 1$ with $g(\infty) = \infty$. Then $g \circ f \in A(R)$, and Schwarz's lemma applied to $1/g^{-1}(1/w)$ leads to the pointwise estimate $|f(z)| \leqslant |g \circ f(z)|$. Thus the corollary follows from Theorem 3.

We return now to functions holomorphic in the unit disk and will show that Theorems 1 and 2 extend to a class of functions which are only "approximately" univalent. Let p be a positive integer. Following Hayman [11] we say that a function f holomorphic in $|z| < 1$ is *weakly p-valent* if for each $\varrho > 0$ either every point on the circle $|\zeta| = \varrho$ is covered exactly p times by f (counting multiplicity) or else there exists a point on $|\zeta| = \varrho$ which is covered less than p times. This class contains the class of circumferentially mean p-valent functions, but neither contains nor is contained in the class of areally mean p-valent functions. These latter classes are studied in [12].

I do not know whether or not Theorem 4 below remains true for normalized areally mean p-valent functions. Spencer [25] has shown that, for $p = 1$, $|a_2| \leqslant 2$ holds in this class, and also (unpublished) that there exist functions in this class with $|a_3| > 3$.

THEOREM 4. *Suppose f is weakly p-valent in $|z| < 1$ and its power series expansion has the form*

$$f(z) = z^p + \textit{higher powers of } z.$$

Then, if Φ is as in Theorem 1 *and* $0 < r < 1$

$$\int_{-\pi}^{\pi} \Phi(\log|f(re^{i\theta})|)\,d\theta \leqslant \int_{-\pi}^{\pi} \Phi(\log|k(re^{i\theta})|^p)\,d\theta, \tag{a}$$

$$\int_{-\pi}^{\pi} \Phi\left(\log \frac{1}{|f(re^{i\theta})|}\right) d\theta \leqslant \int_{-\pi}^{\pi} \Phi\left(\log \frac{1}{|k(re^{i\theta})|^p}\right) d\theta. \tag{b}$$

If equality holds for some $r \in (0, 1)$ *and some strictly convex* Φ *then* $f(z) = (e^{-i\alpha}k(ze^{i\alpha}))^p$ *for some real* α.

Proof. By Lemma 3 of [11], $f^{1/p}$ is weakly 1-valent. Thus, it suffices to prove the theorem when $p = 1$. Let D denote the range of f. By Lemma 4 and Theorem III of [11] there exists a number $d_f \geqslant \frac{1}{4}$ such that D contains the disk $|\zeta| < d_f$ but does not contain any circle $|\zeta| = \varrho$ for $\varrho \geqslant d_f$. If f is not the Koebe function or one of its rotations then $d_f > \frac{1}{4}$.

Assume for now that f has an analytic extension to $|z| \leqslant 1$. Then, by Lehto's Theorem (cf. proof of Proposition 4),

$$u(\zeta) = N(1, \zeta, f)$$

is subharmonic in $0 < |\zeta| < \infty$. For this u, the inequality

$$u^*(\varrho e^{i\varphi}) \leqslant v^*(\varrho e^{i\varphi}) \quad (0 < \varrho < \infty, 0 \leqslant \varphi \leqslant \pi) \tag{57}$$

with v as in the proof of Theorem 1 can be proved by exactly the same argument as in that proof. That the u here has all of the necessary properties for the argument to go through follows from the above discussion together with some simple considerations which we leave to the reader.

For $0 < r < 1$ the inequality $N(r, \zeta, f) \leqslant [u(\zeta) + \log r]^+$ follows easily from the definitions. So, using Cartan's formula (29),

$$\int_{-\pi}^{\pi} \log^+ \frac{|f(re^{i\theta})|}{\varrho} d\theta \leqslant \int_{-\pi}^{\pi} [u(\varrho e^{i\varphi}) + \log r]^+ d\varphi.$$

This, together with (57) and the arguments in Theorems 1 and 2, proves the inequalities in Theorem 4 for f analytic on the closed disk. For a general f the inequalities may be obtained by considering $R^{-1}f(Rz)$ and letting $R \to 1-$.

Because of the limiting process the argument just given is not good enough to establish the uniqueness part of Theorem 4. Here is one way to obtain the uniqueness. For $d \in (\frac{1}{4}, 1)$ there exists a unique function $g_d \in S$ which maps $|z| < 1$ onto the plane minus a circularly forked slit of the form

$$\{\zeta: -\infty < \zeta \leqslant -d\} \cup \{\zeta: |\zeta| = d, \alpha \leqslant |\arg \zeta| \leqslant \pi\}.$$

If $d \leqslant d_f$, where d_f is as at the beginning of this proof, then the argument in Theorem 1 shows that in fact the convex integral means of $\log |f|$ are dominated by those of $\log |g_d|$. For $d > \frac{1}{4}$ the strictly convex integral means of $\log |g_d|$ are, by Theorem 1, strictlv less than those of $\log |k|$, and the uniqueness in Theorem 4 readily follows.

By considering L^1 norms in Theorem 4 we obtain the coefficient estimate $|a_n| < (e/2)n$ for weakly univalent functions. The best previously known estimate in this case was $a_n = O(n^2)$.

Netanyahu [20] studied the class $S(d)$ of all functions $f \in S$ whose image domain contains the disk $|\zeta| < d$. He proved, using the Schiffer and Julia variations, that $|a_2|$ is maximized in this class by the function g_d. As just pointed out, all the convex integral means of log $|f|$ for $f \in S(d)$ are dominated by those of log $|g_d|$, and consideration of L^2 norms leads at once to Netanyahu's result. It is conjectured, but has not been proved, that $|a_3|$ is also maximized in this class by g_d.

7. Proof of Theorems 5 and 6.

In Theorem 5 we assume from the outset that the Green's functions u and v of D and D^* are classical, i.e. have continuous, hence subharmonic, extensions to the respective complements when set equal to zero there. The general case may be obtained from this special case by an obvious approximation process.

Consider first the case when the pole z_0 is at 0. Then, by Theorem A, u^* is subharmonic in the upper halfplane. Let $D^*_+ = D^* \cap (\text{Im } z > 0)$. By Proposition 5, v^* is harmonic in D^*_+.

The argument that follows is in many respects similar to the one used in proving Theorem 1. For $\varepsilon > 0$ let

$$Q(z) = u^*(z) - v^*(z) - \varepsilon\theta \quad (z = re^{i\theta}).$$

Then Q is bounded in the closed upper half plane, and is continuous, except at 0 and ∞. Let $M = \sup\{Q(z): \text{Im } z > 0\}$. As in the proof of Theorem 1 we want to show that $M \leqslant 0$.

Let $\{z_n\}$ be a sequence in the upper half plane such that $Q(z_n) \to M$ and $z_n \to z'$, where z' is some point of the closed extended upper half plane. Since Q is subharmonic in D^*_+ we may assume $z' \notin D^*_+$. If z' lies on the positive real axis then $M = 0$, as required.

Suppose next that $z' \notin D^{*!}_+$ and $z' = Re^{i\varphi}$ with $0 < R < \infty$, $0 < \varphi \leqslant \pi$. If the circle $|z| = R$ does not intersect D then $u^*(z') = v^*(z') = 0$, so that $M < 0$, which is impossible. If the circle intersects D in a set of measure $2m$, but is not entirely contained in D, then $m \leqslant \varphi \leqslant \pi$. On this circle u and v are zero off a set of measure $2m$, so, by Proposition 2 and the continuity of the non-increasing rearrangements,

$$\lim_{\theta \to \varphi -} \frac{\partial u^*}{\partial \theta}(Re^{i\theta}) = \lim_{\theta \to \varphi -} \frac{\partial v^*}{\partial \theta}(Re^{i\theta}) = 0.$$

Hence
$$\lim_{\theta \to \varphi -} \frac{\partial Q}{\partial \theta}(Re^{i\theta}) = -\varepsilon < 0.$$

By the mean value theorem, it follows that the maximum M of Q cannot be achieved at a point z' of this sort.

Consider now the possibility that $z'=-R$ with $0<R<\infty$, and the circle $|z|=R$ lies entirely in D. Let $R_1<|z|<R_2$ be the largest annulus containing $-R$ which is contained entirely in D. Then u and v are harmonic in $R_1<|z|<R_2$, hence

$$Q(-r)=\int_0^{2\pi} u(re^{i\theta})\,d\theta-\int_0^{2\pi} v(re^{i\theta})\,d\theta-\varepsilon\pi$$

is a linear function of $\log r$ for $r\in I=(R_1, R_2)$. If Q achieves its maximum at $-R$ then $Q(-r)$ must be constant for $r\in I$. Either $R_1>0$ or $R_2<\infty$ (perhaps both). Say $R_1>0$. Then $Q(-R_1)=Q(-R)=M$. But the circle $|z|=R_1$ is not contained entirely in D, so, by the preceeding paragraph, $M=Q(-R)$ is impossible. The case $R_2<\infty$ is disposed of in the same way.

The only remaining possibilities are $z'=0$ or $z'=\infty$. Consider $z'=0$. Let $|z|<R_1$ be the largest disk centered at 0 which is contained in D. Then

$$u(z)=-\log|z|+u_1(z),\quad v(z)=-\log|z|+v_1(z)$$

where u_1 and v_1 are harmonic in $|z|<R_1$. We have

$$Q(re^{i\theta})=(2u_1(0)-2v_1(0)-\varepsilon)\theta+o(1)=A\theta+o(1)$$

uniformly in θ, as $r\to 0$. If $A\leqslant 0$ then $M=0$. If $A>0$ then, since $Q(-r)$ is constant on $(0, R_1)$, $M=A\pi=Q(-R_1)$. Since the circle $|z|=R_1$ is not contained entirely in D, this last possibility does not occur, as noted above.

The case $z'=\infty$ is split into the sub-cases $\infty\in D$ or $\infty\notin D$. If $\infty\notin D$ then $u(z)$ and $v(z)$ tend to 0 as $z\to\infty$, thus the same is true of u^* and v^*, and we deduce $M=0$. If $\infty\in D$ then u and v are harmonic in some neighborhood $|z|>R_2$ of ∞, and we deduce as in the case $z'=0$ that $M=0$.

We have shown that $Q\leqslant 0$ in the upper half plane. This implies

$$u^*(re^{i\theta})\leqslant v^*(re^{i\theta})\quad (0<r<\infty,\ 0\leqslant\theta\leqslant\pi),$$

and the integral inequality in Theorem 5 follows from Proposition 3.

In case the pole z_0 is at ∞ the desired result follows by inversion from the case $z_0=0$. For $0<|z_0|<\infty$ we write

$$u(z)=u_1(z)-\log|z-z_0|. \tag{58}$$

Then u_1 is harmonic in D and subharmonic in the whole plane. By Theorem A′ and the discussion in the proof of Proposition 5

$$u^*(re^{i\theta})+2\pi\log^+\frac{r}{|z_0|}$$

is subharmonic in the upper half plane. By Proposition 5,

$$v^*(re^{i\theta}) + 2\pi \log^+ \frac{r}{|z_0|}$$

is harmonic in D^*_+. Thus, once again,

$$Q(z) = u^*(z) - v^*(z) - \varepsilon\theta$$

is subharmonic in D^*_+. The proof that $Q \leqslant 0$ is accomplished just like it was for $z_0 = 0$, except for trivial modifications concerning the possibility $z' = 0$. (Use the argument appearing above for $z' = \infty$). We also point out that $Q(-r)$, which involves the difference of the mean values of u and v, is still a linear function of $\log r$ in the appropriate intervals. This may be seen by considering the decomposition (58) of u and the corresponding decomposition of v. This completes the proof of Theorem 5.

For the proof of Theorem 6, we let $D = f(|z| < 1)$. Our hypothesis that D^* is contained in a simply connected domain D_0 which is not the whole plane insures that D^* has a Green's function, which in turn implies that D has a Green's function. (One way of veryfying the last statement is to consider an exhaustion of D by regular domains and use Theorem 5). Let u,,v, and w be the Green's functions of D, D^*, and D_0, respectively. The pole of u is to be at z_0 and the pole of v and w at $|z_0|$. Since $D^* \subset D_0$ we have $v \leqslant w$ throughout the plane. This, together with Theorem 5, implies,

$$\int_{-\pi}^{\pi} [u(\varrho e^{i\varphi}) + \log r]^+ \, d\varphi \leqslant \int_{-\theta}^{\pi} [w(\varrho e^{i\varphi}) + \log r]^+ \, d\varphi$$

for all positive ϱ and r. From Proposition 4 we thus obtain

$$\int_{-\pi}^{\pi} \log^+ \frac{|f(re^{i\theta})|}{\varrho} \, d\theta \leqslant \int_{-\pi}^{\pi} \log^+ \frac{|F(re^{i\theta})|}{\varrho} \, d\theta.$$

So, by Proposition 3, all of the convex integral means of $\log |f(re^{i\theta})|$ are dominated by those of $\log |F(re^{i\theta})|$.

8. Circular symmetrization and harmonic measures

Let D be a connected open subset of $|z| < 1$, and set

$$\alpha = \partial D \cap (|z| = 1), \quad \beta = \partial D \cap (|z| < 1).$$

We assume that both α and β are non-empty. Let $u(z)$ be the harmonic measure of α with respect to D. Precisely, u is the harmonic function in D, constructed by Perron's method, corresponding to the boundary function χ_α (χ = characteristic function). An account of

Perron's method and of other notions and results from potential theory used in the sequel may be found in [23].

Let v denote the harmonic measure of $\alpha^* = \partial D^* \cap (|z| = 1)$ with respect to D^*. Extend u and v to the whole open disk $|z| < 1$ by setting them equal to zero outside D and D^*, respectively.

THEOREM 7. *Let Φ be as in Theorem 1 and u and v be the harmonic measures just described. Then*

$$\int_{-\pi}^{\pi} \Phi(u(re^{i\theta}))\, d\theta \leqslant \int_{-\pi}^{\pi} \Phi(v(re^{i\theta}))\, d\theta \quad (0 < r < 1).$$

In particular,
$$\sup_\theta u(re^{i\theta}) \leqslant \sup_\theta v(re^{i\theta}) = v(r) \tag{59}$$

Haliste [9] proved the analogue of (59) for certain domains under Steiner symmetrization. She gave two proofs, one based on Ahlfors' distortion theorem and the other on the theory of Brownian motion. The Brownian motion proof may be adapted to prove (59) for subdomains of the unit disk. On the other hand, Haliste's Steiner symmetrization result follows from (59) by means of a logarithmic transformation.

Haliste also proved inequalities of the type (59) for Steiner symmetrization in n-space. C. Borell [5] has recently extended these results by proving the full analogue of Theorem 7 for this situation.

With D as in Theorem 7, let D^{**} be the unit disk with the circular projection of β onto the negative real axis removed. Then $D^* \subset D^{**}$ and, letting w denote the harmonic measure of $|z| = 1$ with respect to D^{**}, we have $v \leqslant w$ everywhere by D^*. So by (59),

$$\sup_\theta u(re^{i\theta}) \leqslant w(r).$$

This inequality is the solution of the Carleman-Milloux problem found in [21, Theorem 1, p. 107].

For the proof of Theorem 7 we impose at first some restrictions on D. These are that all points of β be regular points for the Dirichlet problem in D, all points of $\beta^* = \partial D^* \cap (|z| < 1)$ be regular for the Dirichlet problem in D^*, and that $\beta \subset (|z| \leqslant R)$ for some $R < 1$. Then α is the whole unit circle. The harmonic measures u and v are subharmonic in $|z| < 1$, continuous on $|z| < 1$, equal to 1 on $|z| = 1$, and equal to 0 at points of $|z| < 1$ outside D and D^*, respectively. The argument in the proof of Proposition 5 shows that v^* is harmonic in $D^*_+ = \{z \in D^*: \operatorname{Im} z > 0\}$. Thus

$$Q(z) = u^*(z) - v^*(z) - \varepsilon\theta$$

is subharmonic in D_+^* and continuous on the closed upper half disk, except at 0. Since

$$u^*(e^{i\theta}) = v^*(e^{i\theta}) = 2\theta \quad (0 \leqslant \theta \leqslant \pi),$$

we have $Q \leqslant 0$ on $(\partial D_+^*) \cap (|z| = 1)$. A repetition of the argument used in the proof of Theorem 5 now shows that $Q \leqslant 0$, hence $u^* \leqslant v^*$, throughout the upper half disk, and thus the convex integral means of u are dominated by those of v.

Next, we continue to assume that β is bounded away from $|z| = 1$, but drop the assumption about boundary regularity. (Of course, every point of $|z| = 1$ is still regular). Let $\{D_n\}$ be an exhaustion of D by domains which satisfy the hypotheses of the case already proved. Then, letting u_n and v_n be the harmonic measures of $|z| = 1$ with respect to D_n and D_n^*, it follows by a routine argument that $u_n \nearrow u$ and $v_n \nearrow v$ in D. The convex integral means of each u_n are dominated by those of v_n, hence the same is true for u and v.

Finally, we consider the general case in which the inner boundary β may have limit points on $|z| = 1$. For $0 < R < 1$ let $D_R = D \cup (R < |z| < 1)$ and let u_R and v_R be the harmonic measures of $|z| = 1$ with respect to D_R and D_R^*. The convex integral means of u_R are dominated by those v_R, so, to finish the proof, it will suffice to show that

$$u_R(z) \searrow u(z), \; v_R(z) \searrow v(z) \quad (z \in D, \; R \to 1).$$

The family of functions u_R is clearly decreasing as R increases. Thus the u_R converge to a function u_1 which is harmonic in D. Since $u_R \geqslant u$, we have $u_1 \geqslant u$ in D.

Let h be an upper function for the problem of determining u. Thus h is superharmonic and bounded below in D, and, for each $z_0 \in \alpha$, $z_1 \in \beta$

$$\liminf_{z \to z_0} h(z) \geqslant 1, \quad \liminf_{z \to z_1} h(z) \geqslant 0.$$

The function $h - u_1$ is superharmonic in D. Since $0 \leqslant u_1 \leqslant 1$, we have, for $z_0 \in \alpha$,

$$\liminf_{z \to z_0} (h(z) - u_1(z)) \geqslant 0. \tag{60}$$

Suppose z_0 is a point of β which is regular for the Dirichlet problem in D. Regularity is a local property, so z_0 is also regular for the Dirichlet problem is D_R, as soon as $R > |z_0|$. Thus, for such R, $u_R(z) \to 0$ as $z \to z_0$, and hence $u_1(z) \to 0$ as $z \to z_0$. Thus (60) holds for regular $z_0 \in \beta$. The set of irregular points is a countable union of compact sets of capacity zero, and hence is a set of inner harmonic measure zero. We have shown that $h(z) - u_1(z)$ has non-negative boundary values, except perhaps on such a set. By the extended minimum principle, $h - u_1 \geqslant 0$ in D. Since u is the lower envelope of the set of all upper functions, it follows that $u \geqslant u_1$, and hence $u = u_1$, in D. This proves that $u_R \searrow u$. The proof that $v_R \searrow v$ is the same.

References

[1]. BAERNSTEIN, A., Proof of Edrei's spread conjecture. *Proc. London Math. Soc.*, (3) 26 (1973), 418–434.
[2]. —— A generalization of the cos $\pi\varrho$ Theorem. To appear in *Trans. Amer. Math. Soc.*
[3]. —— Functions extremal for the spread relation. To appear.
[4]. BAZILEVIC, I. E., On distortion theorems and coefficients of univalent functions (Russian), *Mat. Sb. N.S.*, 28 (70) (1951), 147–164.
[5]. BORELL, C., An inequality for a class of harmonic functions in n-space. Preprint.
[6]. DUREN, P. & SCHIFFER, M., A variational method for functions schlicht in an annulus. *Arch. Rational Mech. Anal.*, 9 (1962), 260–272.
[7]. FITZGERALD, C. H., Quadratic inequalities and coefficient estimates for schlicht functions. *Arch. Rational Mech. Anal.*, 46 (1972), 356–368.
[8]. GROETZSCH, H., Über einige Extremalprobleme der konformen Abbildung, *Leipzig, Berichte*, 80 (1928), 367–376.
[9]. HALISTE, K., Estimates of harmonic measures. *Ark. Mat.*, 6 (1965), 1–31.
[10]. HARDY, G. H., LITTLEWOOD, J. & PÓLYA, G., *Inequalities*, Cambridge, 1959.
[11]. HAYMAN, W. K., Some applications of the transfinite diameter to the theory of functions. *J. Anal. Math.*, 1 (1951), 155–179.
[12]. —— *Multivalent Functions*. Cambridge, 1967.
[13]. —— *Research Problems in Function Theory*. London, 1967.
[14]. HILLE, E., *Analytic Function Theory*, Vol. II. Boston, 1962.
[15]. JENKINS, J. A., On circularly symmetric functions. *Proc. Amer. Math. Soc.*, 6 (1955), 620–624.
[16]. —— *Univalent Functions and Conformal Mapping*. Berlin-Göttingen-Heidelberg, 1958.
[17]. LEHTO, O., A majorant principle in the theory of functions. *Math. Scand.*, 1 (1953).
[18]. LOHWATER, A. J., PIRANIAN, G. & RUDIN, W., The Derivative of a schlicht function. *Math. Scand.*, 3 (1955), 103–106.
[19]. MACGREGOR, T. H., Applications of extreme-point theory to univalent functions. *Michigan Math. J.*, 19 (1972), 361–376.
[20]. NETANYAHU, E., On univalent functions in the unit disk whose image contains a given disk. *J. Anal. Math.*, 23 (1970), 305–322.
[21]. NEVANLINNA, R., *Analytic Functions*. Berlin-Heidelberg-New York, 1970.
[22]. PÓLYA, G. & SZEGÖ, G., *Isoperimetric Inequalities in Mathematical Physics*. Princeton, 1951.
[23]. OHTSUKA, M., *Dirichlet Problem, Extremal Length, and Prime Ends*. New York, 1970.
[24]. RADO, T., *Subharmonic Functions*. Berlin, 1937.
[25]. SPENCER, D. C., On mean one-valent functions. *Ann. of Math.*, (2) 42 (1941).
[26]. WILKEN, D. R., The integral means of close-to-convex functions. *Michigan Math. J.*, 19 (1972), 377–379.
[27]. KIRWAN, B. & SCHOBER, G., Integral means for univalent functions in an annulus. University of Maryland Technical Report, 1974.

Received October 2, 1973