FUNDAMENTAL SOLUTIONS FOR DEGENERATE PARABOLIC EQUATIONS

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Introduction

Consider a system of n stochastic differential equations

$$d\xi(t) = b(\xi(t))\,dt + \sigma(\xi(t))\,dw(t) \tag{0.1}$$

where $b = (b_1, ..., b_n)$, $\sigma = (\sigma_{ij})$ is an $n \times n$ matrix and $w = (w^1, ..., w^n)$ is n-dimensional Brownian motion. Under standard smoothness and growth conditions on b and σ, the process $\xi(t)$ is a diffusion process (see [7], [8], [11]) with the differential generator

$$L = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i},$$

where $a_{ij} = \frac{1}{2}\sum_k \sigma_{ik}\sigma_{jk}$. Denote by $q(x, t, A)$ the transition probabilities of the diffusion process. If L is elliptic then a fundamental solution for the Cauchy problem associated with the parabolic equation

$$Lu - \frac{\partial u}{\partial t} = 0 \tag{0.2}$$

can be constructed, under suitable smoothness and growth conditions on the coefficients (see [3], [1]); denote it by $K(x, t, \xi)$. It is also known (see [7], [8]) that this fundamental solution is the density function for the transition probabilities of (0.1), i.e.,

$$q(t, x, A) = \int_A K(x, t, \zeta)\, d\zeta \tag{0.3}$$

for any $t > 0$, $x \in R^n$, and for any Borel set A in R^n.

The present work is concerned with the case where L is degenerate elliptic, i.e., the

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matrix $(a_{ij}(x))$ is degenerate on some set S. The purpose of the paper is to construct a fundamental solution (or a "generalized" fundamental solution) under some conditions on the nature of S and on the coefficients of L.

In section 1 we consider the parabolic equation

$$\varepsilon \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + Lu - \frac{\partial u}{\partial t} = 0 \quad (\varepsilon > 0)$$

and show that its fundamental solution converges (as $\varepsilon \to 0$ through some sequence) to a function $K(x, t, \xi)$, provided $x \notin S$, $\xi \notin S$.

In section 2 we obtain some bounds on $K(x, t, \xi)$ away from the set S.

In sections 3, 6–10 we specialize to the case where S is an "obstacle" in the following sense: S consists of a finite disjoint union of hypersurfaces and of isolated points; the "normal diffusion" of (0.1) vanishes on S, and the "normal drift" is either identically zero ("two-sided obstacle") or it is of one sign ("one-sided obstacle").

In section 3 we construct a function $G(x, t, \xi)$ and obtain estimates on it near the set S. [In section 6 it is shown that $G(x, t, \xi)$ coincides with $K(x, t, \xi)$ if x is on that side of S with respect to which S is an obstacle.] The estimates derived in section 3 show that $G(x, t, \xi)$ decreases "almost" exponentially fast as x or ξ (depending on the sign of the normal drift at S) tends to S. This behavior is strikingly different from the behavior of Green's function in the non-degenerate case; for in the latter case G decreases to zero at a linear rate only.

In section 4 we obtain estimates on K and G near ∞. These estimates seem to be new even in the non-degenerate case (i.e., in case S is the empty set).

In section 5 it is shown that the function $K(x, t, \xi)$ constructed in section 1 satisfies the relation (0.3) provided $x \notin S$, $A \cap S = \emptyset$.

In section 6 it is shown that if S is a two-sided obstacle then

$$P_x\{\xi(t) \in S \quad \text{for all} \quad t > 0\} = 1 \quad \text{if} \quad x \in S.$$

On the other hand, if S is "strictly" one-sided obstacle, say from the exterior of S, then

$$P_x\{\xi(t) \in [S \cup (\text{int } S)]\} = 0 \quad \text{if } t > 0, \; x \in S.$$

Finally, it is proved that if S is an obstacle with respect to the exterior of S then $K(x, t, \xi) = G(x, t, \xi)$ if x is in the exterior of S.

In section 7 we construct a "generalized" fundamental solution in the case of two-sided obstacles; for $x \notin S$, it coincides with the function $K(x, t, \xi)$ (and, therefore, with $G(x, t, \xi)$ for x in the exterior of S), and, for $x \in S$, it is some measure supported on S.

In section 8 we show that if S is a strictly one-sided obstacle then the function $K(x, t, \xi)$ is well defined for all $x \in R^n$, $t > 0$, $\xi \in R^n \backslash S$, and it is a fundamental solution.

In proving the results of sections 7, 8 we make a crucial use of the probabilistic results of section 6.

In section 9 we derive lower bounds on $K(x, t, \xi)$ both near S and near ∞. These results show that the upper bounds derived in sections 3, 4 are sharp.

In section 10 we consider the Cauchy problem

$$Lu - \frac{\partial u}{\partial t} = 0 \quad \text{if } t > 0, \quad u(x, 0) = f(x).$$

It is assumed that S is either two-sided obstacle or strictly one-sided obstacle. It is proved that the solution $u(x, t) = E_x f(\xi(t))$ is continuous for $t > 0$ if $f(x)$ is measurable and, say, bounded. (When S is a two-sided obstacle, an additional condition on f is required.)

We conclude this introduction with a simple example is case $n = 1$. The equation

$$u_t = x^2 u_{xx} + b(x) u_x$$

is a special case of the equations treated in section 7, if $b(0) = 0$, and in section 8, if $b(0) \neq 0$.

1. Construction of the would-be fundamental solution

We shall denote the boundary of a set Ω by $\partial\Omega$. Let

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} \quad (a_{ij} = a_{ji}),$$

and assume:

(A). The functions

$$a_{ij}(x), \frac{\partial}{\partial x_\lambda} a_{ij}(x), \frac{\partial^2}{\partial x_\lambda \partial x_\mu} a_{ij}(x), b_i(x), \frac{\partial}{\partial x_\lambda} b_i(x)$$

are uniformly Hölder continuous in compact subsets of R^n.

Let S be a closed subset of R^n, and assume:

(B_S). The matrix $(a_{ij}(x))$ is positive definite for any $x \notin S$, and positive semi-definite for any $x \in S$.

When S is the empty set $\emptyset$, we denote the condition (B_S) by ($B_\emptyset$). When (A) and ($B_\emptyset$) hold, a fundamental solution for the parabolic equation

$$Lu - \frac{\partial u}{\partial t} = 0 \quad \text{in the strip } 0 < t < \infty, \ x \in R^n \tag{1.1}$$

is known to exist [10]. If $(a_{ij}(x))$ is uniformly positive definite and if some global bounds are assumed on the functions in (A), then a fundamental solution can be constructed having certain global bounds (see [1], [3]).

The present work is concerned with the case $S \neq \emptyset$. (The bounds derived in section 4, though, seem to be new also in case $S = \emptyset$.)

In the present section we shall construct a function $K(x, t, \xi)$ as a limit of fundamental solution $K_\varepsilon(x, t, \xi)$ for the parabolic equations

$$L_\varepsilon u - \frac{\partial u}{\partial t} = 0, \quad \text{where } L_\varepsilon = Lu + \varepsilon \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} \quad (\varepsilon > 0). \tag{1.2}$$

In the following sections we shall show, under some conditions on S and on the coefficients of L, that a fundamental solution of (1.1) coincides with $K(x, t, \xi)$, at least away from S.

Let

$$B_m = \{x; \ |x| < m\}, \quad m = 1, 2, \ldots.$$

Denote by $G_{m,\varepsilon}(x, t, \xi)$ the Green function for (1.2) in the cylinder $Q_m = B_m \times (0, \infty)$. Thus $G_{m,\varepsilon}(x, t, \xi)$, its first t-derivative and its second x-derivatives are continuous in (x, t, ξ) for $x \in \bar{B}_m$, $t > 0$, $\xi \in \bar{B}_m$, and as a function of (x, t),

$$L_\varepsilon G_{m,\varepsilon}(x, t, \xi) - \frac{\partial}{\partial t} G_{m,\varepsilon}(x, t, \xi) = 0 \quad \text{if } (x, t) \in Q_m \quad (\xi \text{ fixed in } B_m),$$

$$G_{m,\varepsilon}(x, t, \xi) \to 0 \quad \text{if } t \to 0, x \neq \xi, x \in B_m,$$

$$G_{m,\varepsilon}(x, t, \xi) = 0 \quad \text{if } t > 0, x \in \partial B_m.$$

Finally, for any continuous function $f(\xi)$ with support in B_m, the function

$$u(x, t) = \int_{B_m} G_{m,\varepsilon}(x, t, \xi) f(\xi) d\xi$$

satisfies:

$$L_\varepsilon u(x, t) = 0 \quad \text{in } Q_m,$$

$$u(x, t) \to f(x) \quad \text{if } t \to 0, x \in B_m,$$

$$u(x, t) = 0 \qquad \text{if } t > 0, x \in \partial B_m.$$

It is well known [3; p. 82] that such a function $G_{m,\varepsilon}(x, t, \xi)$ exists and is uniquely determined by the above properties.

Denote by L^*, L_ε^* the adjoint operators of L, L_ε respectively. Denote by $G^*_{m,\varepsilon}(x, t, \xi)$ the Green function for the equation

$$L_\varepsilon^* u - \frac{\partial u}{\partial t} = 0$$

in Q_m. Again, its existence and uniqueness follow from [3; p. 82]. As proved in [3; p. 84],

$$G_{m,\varepsilon}(x, t, \xi) = G^*_{m,\varepsilon}(\xi, t, x).$$

It follows that as a function of (ξ, t),

$$L_\varepsilon^* G_{m,\varepsilon}(x,t,\xi) - \frac{\partial}{\partial t} G_{m,\varepsilon}(x,t,\xi) = 0 \quad \text{if } (\xi,t) \in Q_m \quad (x \text{ fixed in } B_m).$$

LEMMA 1.1. *Let (A) hold. Then,*

(i)

$$0 \leqslant G_{m,\varepsilon}(x,t,\xi) \leqslant G_{m+1,\varepsilon}(x,t,\xi) \quad \text{if } (x,t) \in Q_m, \ \xi \in B_m, \tag{1.3}$$

$$\lim_{m\to\infty} G_{m,\varepsilon}(x,t,\xi) \equiv K_\varepsilon(x,t,\xi) \textit{ is finite for all } x \in R^n, \ t>0, \ \xi \in R^n. \tag{1.4}$$

(ii) *The functions*

$$K_\varepsilon(x,t,\xi), \ \frac{\partial}{\partial x_\lambda} K_\varepsilon(x,t,\xi), \ \frac{\partial^2}{\partial x_\lambda \partial x_\mu} K_\varepsilon(x,t,\xi), \ \frac{\partial}{\partial t} K_\varepsilon(x,t,\xi)$$

are continuous in (x,t,ξ) for $x \in R^n$, $t>0$, $\xi \in R^n$; for any continuous function $f(\xi)$ with compact support, the function

$$u(x,t) = \int_{R^n} K_\varepsilon(x,t,\xi) f(\xi) d\xi \tag{1.5}$$

satisfies

$$L_\varepsilon u - \frac{\partial u}{\partial t} = 0 \quad \textit{if } x \in R^n, \ t>0, \tag{1.6}$$

$$u(x,t) \to f(x) \quad \textit{if } t \to 0.$$

(iii) *The functions*

$$\frac{\partial}{\partial \xi_\lambda} K_\varepsilon(x,t,\xi) \ \frac{\partial^2}{\partial \xi_\lambda \partial \xi_\mu} K_\varepsilon(x,t,\xi)$$

are continuous in (x,t,ξ) for $x \in R^n$, $t>0$, $\xi \in R^n$; for any continuous function $g(x)$ with compact support, the function

$$v(\xi,t) = \int_{R^n} K_\varepsilon(x,t,\xi) g(x) dx \tag{1.7}$$

satisfies:

$$L_\varepsilon^* v - \frac{\partial v}{\partial t} = 0 \quad \textit{if } \xi \in R^n, \ t>0, \tag{1.8}$$

Proof. The proof given below exploits some ideas of S. Ito [10]. The inequalities in (1.3) are an easy consequence of the maximum principle (cf. [1], [3], [10]). In fact, for any continuous and nonnegative function $f_k(\xi)$ with support in B_m,

$$0 \leqslant \int_{B_m} G_{m,\varepsilon}(x,t,\xi) f_k(\xi) d\xi \leqslant \int_{B_{m+1}} G_{m+1,\varepsilon}(x,t,\xi) f_k(\xi) d\xi$$

by the maximum principle. Taking a sequence $\{f_k\}$ converging to the Dirac measure at ξ^0, the inequalities in (1.3), at $\xi=\xi^0$, follow.

Again, by the maximum principle,

$$\int_{B_m} G_{m,\varepsilon}(x,t,\xi)\,d\xi \leqslant 1. \tag{1.9}$$

Similarly

$$\int_{B_m} G_{m,\varepsilon}(x,t,\xi)\,dx \leqslant 1. \tag{1.10}$$

Now fix a positive integer m. Denote by $\partial/\partial T_\zeta$ the inward conormal derivative to ∂B_m at ζ. By Green's formula: for any positive integer k, $k>m$,

$$\begin{aligned} G_{k,\varepsilon}(x,t,\xi) &= \int_{B_m} G_{m,\varepsilon}(x,s,\zeta)\,G_{k,\varepsilon}(\zeta,t-s,\varepsilon)\,d\zeta. \\ &\quad + \int_0^s \int_{\partial B_m} \frac{\partial}{\partial T_\zeta} G_{m\,\varepsilon}(x,\sigma,\zeta)\,G_{k,\varepsilon}(\zeta,t-s+\sigma,\xi)\,dS_\zeta\,d\sigma \end{aligned} \tag{1.11}$$

for any $0<s<t$, $x\in B_m$, $\xi\in B_m$. Taking $s=t/2$ and using the estimates (see [3])

$$G_{m,\varepsilon}\left(x,\frac{t}{2},\zeta\right) \leqslant C_m \quad (x\in B_m,\ \zeta\in B_m) \tag{1.12}$$

$$\left|\frac{\partial}{\partial T_\zeta} G_{m,\varepsilon}(x,\sigma,\zeta)\right| \leqslant C_m \quad (\zeta\in\partial B_m,\ x\in K,\ 0<\sigma<s) \tag{1.13}$$

where K is a compact subset of B_m (C_m depends on m, ε, t, K), we get

$$\begin{aligned} G_{k,\varepsilon}(x,t,\xi) &\leqslant C_m \int_{B_m} G_{k,\varepsilon}\left(\zeta,\frac{t}{2},\xi\right) d\zeta + C_m \int_{t\,2}^t \int_{\partial B_m} G_{k,\varepsilon}(\zeta,\sigma,\xi)\,dS_\zeta\,d\sigma \\ &\leqslant C_m + C_m \int_{t/2}^t \int_{\partial B_m} G_{k,\varepsilon}(\zeta,\sigma,\xi)\,dS_\zeta\,d\sigma, \end{aligned} \tag{1.14}$$

where (1.10) has been used. If we replace the ball B_m by a ball $B_{m+\lambda}$ $(0<\lambda<1)$ with center 0 and radius $m+\lambda$, and Green's function $G_{m,\varepsilon}$ by the corresponding Green function $G_{m+\lambda,\varepsilon}$, then the constants $C_{m+\lambda}$ will remain bounded, independently of λ. In fact, this can be verified as follows: If $|x-\zeta|\geqslant c>0$, $0<s\leqslant T$, or if $0<c_0\leqslant s\leqslant T$, the inequality

$$G_{m+\lambda,\varepsilon}(x,s,\zeta)\leqslant C \quad (C \text{ depends on } c,\ c_0,\ \varepsilon,\ T \text{ but not on } \lambda) \tag{1.15}$$

follows from [3; p. 82]. For fixed x, the function

$$v(\zeta,s) = G_{m+\lambda,\varepsilon}(x,s,\zeta)$$

satisfies:

$$\begin{aligned} L_\varepsilon^* v - (\partial v/\partial s) &= 0 \quad \text{in } B_{m+\lambda}\times(0,\infty), \\ v(\zeta,s) &= 0 \quad \text{if } \zeta\in\partial B_{m+\lambda},\ s>0. \end{aligned} \tag{1.16}$$

By (1.15), if x varies in a compact set K, $K\subset B_m$, if $0<s<T$ and if ζ varies in a $B_{m+\lambda}$-neighborhood V of $\partial B_{m+\lambda}$ such that $K\cap\bar{V}=\emptyset$, then $v\leqslant C$. Using this fact and (1.16), and applying standard estimates (for instance, the Schauder-type boundary estimates [3]), we deduce that

$$\left|\frac{\partial}{\partial\zeta_i}G_{m+\lambda,\varepsilon}(x,s,\zeta)\right|\leqslant C \tag{1.17}$$

if $x\in K$, $0<s<T$, $\zeta\in V$. From this inequality and (1.15) we see that, analogously to (1.14), we have

$$G_{k,\varepsilon}(x,t,\xi)\leqslant C_{m+\lambda}+C_{m+\lambda}\int_{t/2}^{t}\int_{\partial B_{m+\lambda}}G_{k,\varepsilon}(\zeta,\sigma,\xi)\,dS_\zeta\,d\sigma,\quad C_{m+\lambda}\leqslant C_m^* \tag{1.18}$$

where the constant C_m^* is independent of λ, provided $x\in K$, $\xi\in B_m$, $t>0$. The constant C_m^* may depend on t. However, as the proof of (1.18) shows, if $t_0\leqslant t\leqslant T_0$ where $t_0>0$, $T_0>0$, then C_m^* can be taken to depend on t_0, T_0, but not on t.

Integrating both sides of (1.18) with respect to λ, $0<\lambda<1$, we get

$$G_{k,\varepsilon}(x,t,\xi)\leqslant C_m^*+C_m^*\int_{t/2}^{t}\int_{D_m}G_{k,\varepsilon}(\zeta,\sigma,\xi)\,d\zeta\,d\sigma,$$

where D_m is the shell $\{x;\,m<|x|<m+1\}$. Using (1.10) we conclude that

$$G_{k,\varepsilon}(x,t,\xi)\leqslant C_m^{**}\quad\text{if } x\in K,\ \xi\in B_m,\ t_0\leqslant t\leqslant T_0 \tag{1.19}$$

where C_m^{**} is a constant independent of k. Combining this with (1.3), the assertion (1.4) follows.

The inequality (1.19) for m replaced by $m+1$ and $K=\bar{B}_m$ shows that the family $\{G_{k,\varepsilon}(x,t,\xi)\}$ (for $k>m$) is uniformly bounded for $x\in B_m$, $\xi\in B_m$, $t_0\leqslant t\leqslant T_0$. We can employ the Schauder-type interior estimates [3], considering the $G_{k,\varepsilon}$ first as functions of (x,t) and then as functions of (ξ,t). We conclude that there is a subsequence which is uniformly convergent to a function $G_\varepsilon(x,t,\xi)$ with the corresponding derivatives

$$\frac{\partial}{\partial x_\lambda},\ \frac{\partial^2}{\partial x_\lambda\partial x_\mu},\ \frac{\partial}{\partial t},\ \frac{\partial}{\partial\xi_\lambda},\ \frac{\partial^2}{\partial\xi_\lambda\partial\xi_\mu}, \tag{1.20}$$

in compact subsets of $\{(x,t,\xi);\,x\in B_m,\ t_0<t<T_0,\ \xi\in B_m\}$. Since however the entire sequence $\{G_{k,\varepsilon}(x,t,\xi)\}$ is convergent to $K_\varepsilon(x,t,\xi)$, the same is true of the entire sequence of each of the partial derivatives of (1.20). It follows that the function $K_\varepsilon(x,t,\xi)$ and its derivatives

$$\frac{\partial}{\partial x_\lambda}K_\varepsilon,\ \frac{\partial^2}{\partial x_\lambda\partial x_\mu}K_\varepsilon,\ \frac{\partial}{\partial t}K_\varepsilon,\ \frac{\partial}{\partial\xi_\lambda}K_\varepsilon,\ \frac{\partial^2}{\partial\xi_\lambda\partial\xi_\mu}K_\varepsilon$$

are continuous in (x, t, ξ) for x, ξ in R^n and $t>0$. Further, as a function of (x, t),

$$L_\varepsilon K_\varepsilon - \frac{\partial}{\partial t} K_\varepsilon = 0 \quad (\xi \text{ fixed}),$$

and as a function of (ξ, t)

$$L_\varepsilon^* K_\varepsilon - \frac{\partial}{\partial t} K_\varepsilon = 0 \quad (x \text{ fixed}).$$

Consequently, the functions u, v defined in (1.5), (1.7) satisfy the parabolic equations of (1.6), (1.8) respectively. It remains to show that

$$u(x, t) \to f(x) \quad \text{if } t \to 0, \tag{1.21}$$

$$v(x, t) \to g(x) \quad \text{if } t \to 0. \tag{1.22}$$

Note that (1.3), (1.4), (1.9), (1.10) imply that

$$\int_{R^n} K_\varepsilon(x, t, \xi) d\xi \leqslant 1, \quad \int_{R^n} K_\varepsilon(x, t, \xi) dx \leqslant 1. \tag{1.23}$$

We proceed to prove (1.21). Let the support of f be contained in some ball B_m. Suppose first that $f \in C^3$. For $k > m$, consider the functions

$$u_k(x, t) = \int_{R^n} G_{k,\varepsilon}(x, t, \xi) f(\xi) d\xi.$$

The uniform convergence of $\{G_{k,\varepsilon}(x, t, \xi)\}$ to $K_\varepsilon(x, t, \xi)$ implies that $u_k(x, t) \to u(x, t)$ for any $x \in R^n, t>0$. Notice next that

$$|u_.(x, t)| \leqslant (\sup |f|) \int_{R^n} G_{k,\varepsilon}(x, t, \xi) d\xi \leqslant \sup |f|$$

$$u_k(x, 0) = f(x) \text{ is a } C^3 \text{ function.}$$

Hence the Schauder-type boundary estimates [3] [for the parabolic operator $L_\varepsilon - \partial/\partial t$] imply that the sequence $\{u_k(x, t)\}$ is uniformly convergent (with its second x-derivatives) for $x \in B_m, t \geqslant 0$. It follows that $u(x, t)$ $(t>0)$ has a continuous extension $u(x, 0)$ to $t=0$ and

$$u(x, 0) = \lim_{k \to 0} u_k(x, 0) = f(x).$$

If f is only assumed to be continuous, let f_i be C^3 functions such that

$$\gamma_i \equiv \sup_{x \in R^n} |f_i(x) - f(x)| \to 0 \quad \text{if } i \to \infty,$$

and such that the support of each f_i is in B_m. Then

$$\int_{B_m} |K_\varepsilon(x, t, \xi)[f_i(\xi) - f(\xi)]| d\xi \leqslant \gamma_i \int_{B_m} K_\varepsilon(x, t, \xi) d\xi \leqslant \gamma_i$$

by (1.23). Also, by what we have already proved,

$$\delta_i(t) \equiv \left| \int_{B_m} K_\varepsilon(x,t,\xi) f_i(\xi) d\xi - f_i(x) \right| \to 0 \quad \text{if } t \to 0 \quad (\text{if fixed}).$$

It follows that

$$\overline{\lim_{t\to 0}} \, | u(x,t) - f(x) | \leqslant 2\gamma_i + \lim_{t\to 0} \delta_i(t) = 2\gamma_i.$$

Since $\gamma_i \to 0$ if $i \to \infty$, the assertion (1.21) follows. The proof of (1.22) is similar. This completes the proof of Lemma 1.1.

We now recall the definition of a fundamental solution for a nondegenerate parabolic equation. For simplicity we specialize to the case of the equation (1.2).

Definition. Let $K_\varepsilon(x, t, \xi)$ be a function defined for $x \in R^n, t>0, \xi \in R^n$, and Borel measurable in ξ (for (x, t) fixed). Suppose that for every continuous function $f(\xi)$ with compact support the function $u(x, t)$ defined by (1.5) exists and satisfies (1.6). Then we say that $K_\varepsilon(x, t, \xi)$ is a *fundamental solution* of the parabolic equation

$$L_\varepsilon u - \frac{\partial u}{\partial t} = 0 \quad \text{for } x \in R^n, t>0.$$

From now on we shall designate by $K_\varepsilon(x, t, \xi)$ the fundamental solution constructed in the proof of Lemma 1.1.

Remark. There are well known uniqueness theorems for the Cauchy problem for a parabolic equation with coefficients that may grow to ∞ as $|x| \to \infty$ (see, for instance, [3] and a recent paper [4]). When such a uniqueness theorem can be applied to the solution of (1.6), then the fundamental solution (when subject to some global growth condition as $|x| \to \infty$) is uniquely determined.

THEOREM 1.2. *Let* (A), (B_S) *hold. Then there exists a sequence $\varepsilon_m \searrow 0$ such that, as $m \to \infty$,*

$$K_{\varepsilon_m}(x, t, \xi) \to K(x, t, \xi) \tag{1.24}$$

together with the first two x-derivatives, the first two ξ-derivatives and the first t-derivative uniformly for all x, ξ in E, $\delta < t < 1/\delta$, where E is any compact set in R^n such that $E \cap S = \emptyset$, and δ is any positive number, $0<\delta<1$.

Proof. Let E_0 be a compact set which does not intersect S.

Let $B_\lambda (0 \leqslant \lambda \leqslant 1)$ be a family of bounded open sets such that $\bar{B}_\lambda \subset B_{\lambda'}$ if $\lambda < \lambda'$, $E_0 \subset B_0$, $\bar{B}_1 \cap S = \emptyset$, and such that as λ varies from 0 to 1 the boundary ∂B_λ covers simply a finite disjoint union D of shells, and $dx = \varrho dS^\lambda d\lambda$, where dS^λ is the surface element of ∂B_λ and ϱ

is a positive continuous function. It is assumed that each ∂B_λ consists of a finite number of C^3 hypersurfaces.

Taking $k\to\infty$ in (1.11) and using the monotone convergence theorem, we obtain the relation (1.11) with $G_{k,\varepsilon}$ replaced by K_ε. This relation holds also with B_m replaced by B_λ and $G_{m,\varepsilon}$ replaced by Green's function $G_{\varepsilon,\lambda}$ of $L_\varepsilon-\partial/\partial t$ in the cylinder $B_\lambda\times(0,\infty)$. The estimates (cf. (1.15), (1.17))

$$G_{\varepsilon,\lambda}(x,t,\zeta)\leqslant C_\varepsilon \quad (x\in E_0,\ \zeta\in B_\lambda,\ t_0\leqslant t\leqslant T_0 \tag{1.25}$$

$$\left|\frac{\partial}{\partial T_\zeta}G_{\varepsilon,\lambda}(x,t,\zeta)\right|\leqslant C_\varepsilon \quad (x\in E_0,\ \zeta\in\partial B_\lambda,\ 0<t<T) \tag{1.26}$$

hold, where $t_0>0$, $T_0<\infty$. Since $(a_{ij}(x))$ is positive definite for $x\in\bar{B}_1$, the constants C_ε can be taken to be independent of both ε and λ; the proof is similar to the proof of (1.15), (1.17). It follows that if $x\in E_0$, $\xi\in E_0$, $t_0\leqslant t\leqslant T_0$,

$$\begin{aligned} K_\varepsilon(x,t,\xi) &\leqslant C^*\int_{B\lambda}K_\varepsilon\left(x,\frac{t}{2},\xi\right)d\xi+C^*\int_0^{t/2}\int_{\partial B_\lambda}K_\varepsilon\left(\zeta,\frac{t}{2}+\sigma,\xi\right)dS_\zeta^\lambda\,d\sigma \\ &\leqslant C^*+C^*\int_0^{t/2}\int_{\partial B_\lambda}K_\varepsilon\left(\xi,\frac{t}{2}+\sigma,\xi\right)dS_\zeta^\lambda\,d\sigma \end{aligned} \tag{1.27}$$

where C^* is a constant independent of ε,λ; (1.23) has been used here. Integrating with respect to λ and using (1.23), we find that

$$K_\varepsilon(x,t,\xi)\leqslant C \quad (C \text{ independent of } \varepsilon). \tag{1.28}$$

This bound is valid for x, ξ in E_0 and $t\in[t_0,T_0]$; the constant C depends on E_0, t_0, T_0, but not on ε.

From the Schauder-type interior estimates applied to $K_\varepsilon(x,t,\xi)$ first as a function of (x,t) and then as a function of (ξ,t) we conclude, upon using (1.28), that

$$K_\varepsilon(x,t,\xi),\ \frac{\partial}{\partial x_\lambda}K_\varepsilon(x,t,\xi),\ \frac{\partial^2}{\partial x_\lambda\partial x_\mu}K_\varepsilon(x,t,\xi),$$

$$\frac{\partial}{\partial t}K_\varepsilon(x,t,\xi),\ \frac{\partial}{\partial \xi_\lambda}K_\varepsilon(x,t,\xi),\ \frac{\partial^2}{\partial \xi_\lambda\partial \xi_\mu}K_\varepsilon(x,t,\xi)$$

satisfy a uniform Hölder condition in (x,t,ξ) when $x\in E'$, $\xi\in E'$, $t_0+\delta\leqslant t<T_0-\delta$ for any $\delta>0$, where E' is any set in the interior of E_0; the Hölder constants are independent of ε (since $(a_{ij}(x))$ is positive definite for $x\in E_0$). Since E_0, t_0, T_0 are arbitrary, we conclude, by diagonalization, that there is a sequence $\{\varepsilon_m\}$, $\varepsilon_m\to0$ if $m\to\infty$, such that

$$K(x,t,\xi)\equiv\lim_{m\to\infty}K_{\varepsilon_m}(x,t,\xi)$$

exists, and the convergence is uniform together with the convergence of the respective first two x-derivatives, first two ξ-derivatives and first t-derivative, for all x, ξ in any compact set E, $E \cap S = \emptyset$, and for all t, $\delta \leq t \leq 1/\delta$, where δ is any positive number.

COROLLARY 1.3. *The function* $K(x, t, \xi)$ *satisfies:* (i) *as a function of* (x, t), $LK(x, t, \xi) - \partial K(x, t, \xi)/\partial t = 0$, *and* (ii) *as a function of* (ξ, t), $L^*K(x, t, \xi) - \partial K(x, t, \xi)/\partial t = 0$, *for all* $x \notin S$, $\xi \notin S$, $t > 0$.

The function $K(x, t, \xi)$ seems to be a natural candidate for a fundamental solution of (1.1). It will be shown later on that, under suitable assumptions on S and on the coefficients of L, this is "essentially" the case, at least away from S.

2. Interior estimates

We denote by D_x the vector $(\partial/\partial x_1, ..., \partial/\partial x_n)$.

LEMMA 2.1. *Let* (A), (B_S) *hold. Let* B *be a bounded domain with* C^2 *boundary* ∂B, *and let* $\bar{B} \cap S = \emptyset$. *Denote by* $G_{B,\varepsilon}(x, t, \xi)$ *the Green function of* $L_\varepsilon - \partial/\partial t$ *in the cylinder* $B \times (0, \infty)$. *Then, for any compact subset* B_0 *of* B *and for any* $\varepsilon_0 > 0$, $T > 0$,

$$G_{B,\varepsilon}(x, t, \xi) \leq (C/t^{n/2}) \quad \text{if } (x, \xi) \in (B \times B_0) \cup (B_0 \times B), \quad 0 < t \leq T, \tag{2.1}$$

$$G_{B,\varepsilon}(x, t, \xi) \leq Ce^{-c/t} \quad \text{if } (x, \xi) \in (B \times B_0) \cup (B_0 \times B), \quad |x-\xi| \geq \varepsilon_0, 0 < t \leq T, \tag{2.2}$$

$$|D_x G_{B,\varepsilon}(x, t, \xi)| \leq Ce^{-c/t} \quad \text{if } (x, \xi) \in B \times B_0, \quad |x-\xi| \geq \varepsilon_0, 0 < t < T, \tag{2.3}$$

$$|D_\xi G_{B,\varepsilon}(x, t, \xi)| \leq Ce^{-c/t} \quad \text{if } (x, \xi) \in B_0 \times B, \quad |x-\xi| \geq \varepsilon_0, 0 < t < T, \tag{2.4}$$

where C, c *are positive constants depending on* B, B_0, ε_0, T *but independent of* ε.

Proof. We write (cf. [3; p. 82])

$$G_{B,\varepsilon}(x, t, \xi) = \Gamma_\varepsilon(x, t, \xi) + V_\varepsilon(x, t, \xi \tag{2.5}$$

where $\Gamma_\varepsilon(x, t, \xi)$ is a fundamental solution for $L_\varepsilon - \partial/\partial t$ in a cylinder $Q = B' \times (0, \infty)$ and B' is an open neighborhood of $\bar{B}$ such that its closure does not intersect S. Since L is non-degenerate outside S, the construction of Γ can be carried out as in [3], and (see [3; p. 24])

$$|\Gamma_\varepsilon(x, t, \xi)| + |D_x\Gamma_\varepsilon(x, t, \xi)| \leq Ce^{-c/t} \quad \text{if } |x-\xi| \geq \varepsilon_0 > 0, 0 < t < T; \tag{2.6}$$

the positive constants C, c can be taken to be independent of ε. Notice also that

$$|\Gamma_\varepsilon(x, t, \xi)| \leq \frac{C}{t^{n/2}} \quad \text{if } 0 < t < T. \tag{2.7}$$

By the methods of [3] one can actually also prove that

$$|D_x^2\Gamma_\varepsilon(x,t,\xi)|+|D_t\Gamma_\varepsilon(x,t,\xi)|\leqslant Ce^{-c/t} \quad \text{if } |x-\xi|\geqslant\varepsilon_0>0,\ 0<t<T. \tag{2.8}$$

The points (x, ξ) in (2.6)–(2.8) vary in B'.

The function $V_\varepsilon(x, t, \xi)$, for fixed ξ in B, satisfies

$$L_\varepsilon V_\varepsilon - \frac{\partial}{\partial t} V_\varepsilon = 0 \quad \text{if } x\in B,\ 0<t<T,$$

$$V_\varepsilon(x,t,\xi) = -\Gamma_\varepsilon(x,t,\xi) \quad \text{if } x\in\partial B,\ 0<t<T,$$

$$V_\varepsilon(x,0,\xi)=0 \quad \text{if } x\in B.$$

If ξ remains in a compact subset E of B then, by (2.6) and the maximum principle,

$$|V_\varepsilon(x,t,\xi)|\leqslant Ce^{-c/t} \quad (x\in B,\ \xi\in E,\ 0<t<T). \tag{2.9}$$

This inequality together with (2.5)–(2.7) imply (2.1), (2.2) for $(x, \xi)\in B\times B_0$. Since similar inequalities hold for Green's function $G^*_{B,\varepsilon}(x, t, \xi)$ of $L^*_\varepsilon - \partial/\partial t$, and since $G_{B,\varepsilon}(x, t, \xi) = G^*_{B,\varepsilon}(\xi, t, x)$, the inequalities (2.1), (2.2) follow also when $(x, \xi)\in B_0\times B$.

From (2.6), (2.8) we see that for any ξ in a compact subset E of B there is a function $f(x, t)$ which coincides with $-\Gamma_\varepsilon(x, t, \xi)$ for $x\in\partial B$, $0<t<T$, and which satisfies

$$|f(x,t)|+|D_tf(x,t)|+|D_xf(x,t)|+|D_x^2f(x,t)|\leqslant C^*e^{-c/t} \quad (x\in B,\ 0<t<T)$$

where C^* is a constant independent of ξ, ε. We use here the fact that ∂B is in C^2. Notice that

$$L_\varepsilon(V_\varepsilon - f) - \frac{\partial}{\partial t}(v_\varepsilon - f) = -L_\varepsilon f + \frac{\partial f}{\partial t} \equiv \tilde f,$$

$$|\tilde f(x,t)|\leqslant C^{**}e^{-c/t} \qquad (x\in B,\ 0<t<T),$$

$$V_\varepsilon - f = 0 \quad \text{if } x\in\partial B,\quad 0<t<T \quad \text{or if } x\in B,\ t=0;$$

the constant C^{**} is independent of ε. By the proof of the $(1+\delta)$-estimate in [3; Chap. 7] we conclude that

$$|D_x[V_\varepsilon(x,t,\xi)-f(x,t)]|\leqslant C_1C^{**}e^{-c/t} \quad \text{if } x\in B,\ 0<t<T,$$

where C_1 is a constant independent of ε. Recalling (2.5), (2.6), the assertion (2.3) follows. A similar inequality holds for Green's function $G^*_{B,\varepsilon}$; since $G_{B,\varepsilon}(x, t, \xi) = G^*_{B,\varepsilon}(\xi, t, x)$, this inequality gives (2.4).

THEOREM 2.2. *Let* (A), (B_S) *hold. Let E be any compact subset in R^n such that $E\cap S=\varnothing$ and let ε_0, T be any positive numbers. Then*

$$K(x,t,\xi)\leqslant\frac{C}{t^{n/2}} \quad \textit{if } x\in E,\ \xi\in E,\ 0<t<T, \tag{2.10}$$

$$K(x,t,\xi) \leqslant Ce^{-c/t} \quad \text{if } x\in E,\ \xi\in E,\ |x-\xi| \geqslant \varepsilon_0,\ 0<t<T, \tag{2.11}$$

where C, c are positive constants.

Proof. Let B_λ $(0 \leqslant \lambda \leqslant 1)$ be an increasing family of bounded open sets with C^2 boundary, as in the proof of Theorem 1.2. Let F be a compact subset of B_0. Recall that $\bar{B}_1 \cap S = \emptyset$. We proceed as in the proof of Theorem 1.2 to employ the relation (1.11) with B_m replaced by B_λ and with G_m replaced by $G_{B_\lambda,\varepsilon}$:

$$G_{k,\varepsilon}(x,t,\xi) = \int_{B_\lambda} G_{B_\lambda,\varepsilon}(x,s,\zeta)\, G_{k,\varepsilon}(\zeta,t-s,\xi)\, d\zeta + \int_0^s \int_{\partial B_\lambda} \frac{\partial}{\partial T_\zeta} G_{B_\lambda,\varepsilon}(x,\sigma,\zeta)\, G_{k,\varepsilon}(\zeta,t-s+\sigma,\xi)\, dS_\zeta^\lambda\, d\nu. \tag{2.12}$$

From the proof of Lemma 2.1 we see that the estimates (2.1)–(2.4) hold for $G_{B_\lambda,\varepsilon}$ with constants C, c independent of λ. Using (2.1), (2.4) for $B = B_\lambda$ in (2.12), we obtain, after applying the inequality (1.10) for $m = k$, integrating with respect to λ $(0<\lambda<1)$ and applying once more (1.10) with $m = k$,

$$G_{k,\varepsilon}(x,t,\xi) \leqslant \frac{C}{t^{n/2}} \quad \text{provided } x\in F,\ \xi\in F,\ 0<t<T.$$

Taking $k \to \infty$, we get

$$K_\varepsilon(x,t,\xi) \leqslant \frac{C}{t^{n/2}} \quad \text{if } x\in F,\ \xi\in F,\ 0<t<T. \tag{2.13}$$

Taking $\varepsilon = \varepsilon_m \to \infty$, the inequality (2.10) follows.

To prove (2.11), let A, F be disjoint compact domains, $(A \cup F) \cap S = \emptyset$, and let ∂F be in C^2. Consider the function

$$v_\varepsilon(x,t) = K_\varepsilon(x,t,\xi) \quad \text{for } x\in F,\ 0<t<T \quad (\xi \text{ fixed in } A).$$

Denote by $G_{F,\varepsilon}(x,t,\xi)$ the Green function of $L_\varepsilon - \partial/\partial t$ in $F \times (0,\infty)$. By Lemma 2.1,

$$|D_\zeta G_{F,\varepsilon}(x,t,\zeta)| \leqslant Ce^{-c/t} \quad \text{if } \zeta\in\partial F,\ x\in F_0,\ 0<t<T, \tag{2.14}$$

where F_0 is any compact subset in the interior of F.

We have the following representation for $v_\varepsilon(x,t)$:

$$v_\varepsilon(x,t) = \int_0^t \int_{\partial F} \frac{\partial}{\partial T_\zeta} G_{F,\varepsilon}(x,s,\zeta)\, v_\varepsilon(\zeta,s)\, dS_\zeta\, ds \quad (x\in \operatorname{int} F,\ 0<t<T) \tag{2.15}$$

Indeed, this formula is valid for $v_{k,\varepsilon}(x,t) \equiv G_{k,\varepsilon}(x,t,\xi)$ since $v_{k,\varepsilon}(x,0) = 0$. Taking $k \to \infty$ and using the monotone convergence theorem, (2.15) follows.

Substituting the estimates (2.13), (2.14) into the right-hand side of (2.15), we obtain

$$v_\varepsilon(x,t) \leqslant \frac{C'}{t^{n/2}} e^{-c'/t} \leqslant Ce^{-c/t}$$

where C', c', C, c are positive constants independent of ε. Taking $\varepsilon=\varepsilon_m\to 0$, the assertion (2.11) follows.

3. Boundary estimates

We shall need the condition:

(C) There is a finite number of disjoint sets $G_1, \ldots, G_{k_0}, G_{k_0+1}, \ldots, G_k$ such that each $G_i (1\leqslant i\leqslant k_0)$ consists of one point z_i and each $G_j(k_0+1\leqslant j\leqslant k)$ is a bounded closed domain with C^3 connected boundary ∂G_j. Further,

$$a_{ij}(z_l)=0,\ b_i(z_l)=0 \quad \text{if } 1\leqslant l\leqslant k_0;\ 1\leqslant i,j\leqslant n, \tag{3.1}$$

$$\sum_{i,j=1}^{n} a_{ij}(x)\nu_i\nu_j=0 \quad \text{for } x\in\partial G_j \quad (k_0+1\leqslant j\leqslant k), \tag{3.2}$$

$$\sum_{i=1}^{n}\left(b_i(x)-\sum_{j=1}^{n}\frac{\partial a_{ij}(x)}{\partial x_j}\right)\nu_i\leqslant 0 \quad \text{for } x\in\partial G_j \quad (k_0+1\leqslant j\leqslant k) \tag{3.3}$$

where $\nu=(\nu_1, \ldots, \nu_n)$ is the outward normal to ∂G_j at x.

Let $\Omega=\bigcup_{j=1}^{k} G_j$, $\hat{\Omega}=R^n\backslash\Omega$, $\partial G_j=G_j=\{z_j\}$ if $1\leqslant j\leqslant k_0$, $\partial\Omega=\bigcup_{j=1}^{k}\partial G_j$.

In this section, and in sections 6–10, we shall assume that

$$S=\partial\Omega. \tag{3.4}$$

Let $\{N_m\}$ be a sequence of domains with C^3 boundary ∂N_m, such that $\bar{N}_m\subset N_{m+1}\subset\hat{\Omega}$, $\bigcup_m N_m=\hat{\Omega}$. We take N_m such that ∂N_m consists of two disjoint parts: $\partial_1 N_m$ which lies in $(1/m)$-neighborhood of $\partial\Omega$ and $\partial_2 N_m$ which is the sphere $|x|=m$.

Denote by $G_m(x,t,\xi)$ the Green function for $L-\partial/\partial t$ in $N_m\times(0,\infty)$. By arguments similar to those used in the proofs of Lemma 1.1 and Theorem 2.2, we have:

$$0\leqslant G_m(x,t,\xi)\leqslant G_{m+1}(x,t,\xi), \tag{3.5}$$

$$G(x,t,\xi)=\lim_{m\to\infty}G_m(x,t,\xi) \quad \text{is finite} \tag{3.6}$$

for all x, ξ in $\hat{\Omega}$, $t>0$. Further

$$G_m(x,t,\xi)\leqslant\frac{C}{t^{n/2}} \quad \text{if } x\in E,\ \partial\in E,\ 0<t<T, \tag{3.7}$$

$$G_m(x,t,\xi)\leqslant Ce^{-c/t} \quad \text{if } x\in E,\ \xi\in E,\ |x-\xi|\geqslant\varepsilon_0,\ 0<t<T, \tag{3.8}$$

$$G(x,t,\xi)\leqslant\frac{C}{t^{n/2}} \quad \text{if } x\in E,\ \xi\in E,\ 0<t<T, \tag{3.9}$$

$$G(x,t,\xi)\leqslant Ce^{-c/t} \quad \text{if } x\in E,\ \xi\in E,\ |x-\xi|\geqslant\varepsilon_0,\ 0<t<T, \tag{3.10}$$

where E is any compact set such that $E \subset \hat{\Omega}$, T and ε_0 are any positive numbers, and C, c are positive constants depending on E, ε_0, T but independent of m. We also have, by the strong maximum principle [3], that $G(x, t, \xi) > 0$ if $x \in \hat{\Omega}$, $t > 0$, $\xi \in \hat{\Omega}$. Finally,

$$LG(x,t,\xi) - \frac{\partial}{\partial t} G(x,t,\xi) = 0 \quad \text{if } x \in \hat{\Omega},\ t > 0 \quad (\xi \text{ fixed in } \hat{\Omega}), \tag{3.11}$$

$$L^*G(x,t,\xi) - \frac{\partial}{\partial t} G(x,t,\xi) = 0 \quad \text{if } \xi \in \hat{\Omega},\ t > 0 \quad (x \text{ fixed in } \hat{\Omega}). \tag{3.12}$$

Notice that in proving (3.5)–(3.12) we do not use the conditions (3.1)–(3.3).

Denote by $R(x)$ the distance from $x \in \Omega$ to the set $\hat{\Omega}$. This function is in C^2 in some Ω-neighborhood of $\partial\Omega$ and also up to the boundary $\bigcup_{j=k_0+1}^{k} \partial G_j$.

THEOREM 3.1. *Let* (A), (B_S), (C) *and* (3.4) *hold. Let* E *be any compact subset of* $\hat{\Omega}$. *Then for any* $T > 0$ *and for any* $\varrho > 0$ *sufficiently small, there are positive constants* C, γ *such that*

$$G(x,t,\xi) \leq C \exp\left\{-\frac{\gamma}{t}(\log R(x))^2\right\} \tag{3.13}$$

if $\xi \in E$, $x \in \hat{\Omega}$, $R(x) < \varrho$, $0 < t < T$.

COROLLARY 3.2. *If in Theorem* 3.1, *the condition* (3.3) *is replaced by the condition*

$$\sum_{i=1}^{n}\left(b_i(x) - \sum_{j=1}^{n}\frac{\partial a_{ij}(x)}{\partial x_j}\right)\nu_i \geq 0 \quad \textit{for } x \in \partial G_j (k_0 + 1 \leq j \leq k), \tag{3.14}$$

then

$$G(x,t,\xi) \leq C \exp\left\{-\frac{\gamma}{t}(\log R(\xi))^2\right\} \tag{3.15}$$

if $x \in E$, $\xi \in \hat{\Omega}$, $R(\xi) < \varrho$, $0 < t < T$.

The point of these results will become obvious when, in section 6, we shall prove that

$$K(x,t,\xi) = G(x,t,\xi) \quad \text{if } x \in \hat{\Omega},\ \xi \in \hat{\Omega},\ t > 0.$$

Proof of Theorem 3.1. For any $\varepsilon > 0$, denote by M_ε the set of all points $x \in \hat{\Omega}$ for which $R(x) < \varepsilon$, and by Γ_ε the set of all points $x \in \hat{\Omega}$ with $R(x) = \varepsilon$. The number ε is such that $E \cap \bar{M}_\varepsilon = \emptyset$ and $R(x)$ is in $C^2(M_\varepsilon)$; later on we shall impose another restriction on the size of ε (depending only on the coefficients of L).

Let $M_{\varepsilon,m} = M_\varepsilon \cap N_m$. Its boundary $\partial M_{\varepsilon,m}$ consists of Γ_ε and of $\partial_1 N_m$ (the "inner" boundary of N_m), provided m is sufficiently large, say $m \geq m_0(\varepsilon)$.

For $m \geq m_0(\varepsilon)$, consider the function

$$v(x,t) = G_m(x,t,\xi) \quad \text{for } x \in M_{\varepsilon,m},\ 0 < t < T \quad (\xi \text{ fixed in } E).$$

If $x \in \partial_1 N_m$, $v(x, t) = 0$. If $x \in \Gamma_\varepsilon$, $0 < t < T$, then, by (3.8),

$$0 \leqslant v(x, t) \leqslant Ce^{-c/t}.$$

Finally, $v(x, 0) = 0$ if $x \in M_{\varepsilon, m}$. We shall compare $v(x, t)$ with a function of the form

$$w(x\,t) = C \exp\left\{-\frac{\gamma}{t}(\log R(x))^2\right\} \quad (\gamma(\log \varepsilon) \leqslant c) \tag{3.16}$$

where γ is a sufficiently small positive constant independent of m. Notice that $w(x, 0) = 0$ if $x \in M_{\varepsilon, m}$, $w(x, t) \geqslant 0$ if $x \in \partial_1 N_m$, and $w(x, t) \geqslant Ce^{-c/t}$ if $x \in \Gamma_\varepsilon$, $0 \leqslant t \leqslant T$. Hence, if we can show that

$$Lw - w_t < 0 \quad \text{for } x \in M_{\varepsilon, m},\ 0 < t < T, \tag{3.17}$$

then, by the maximum principle,

$$G_m(x, t, \xi) \equiv v(x, t) \leqslant w(x, t).$$

Taking $m \to \infty$, the assertion (3.13) follows.

To prove (3.17), set $\Phi = 1/w$. Then

$$w_{x_i} = -\frac{1}{\Phi}\frac{2\gamma}{t}\frac{\log R}{R} R_{x_i},$$

$$w_{x_i x_j} = \frac{1}{\Phi}\left\{\frac{4\gamma^2}{t^2}\frac{(\log R)^2}{R^2} R_{x_i} R_{x_j} - \frac{2\gamma}{t}\frac{1}{R^2} R_{x_i} R_{x_j} + \frac{2\gamma}{t}\frac{\log R}{R^2} R_{x_i} R_{x_j} - \frac{2\gamma}{t}\frac{\log R}{R} R_{x_i x_j}\right\},$$

$$-w_t = -\frac{1}{\Phi}\frac{\gamma}{t^2}(\log R)^2.$$

Hence

$$[Lw - w_t]\Phi = \frac{4\gamma^2}{t^2}\frac{(\log R)^2}{R^2}\sum a_{ij} R_{x_i} R_{x_j} - \frac{2\gamma}{t}\frac{1}{R^2}\left(1 + \log\frac{1}{R}\right)\sum a_{ij} R_{x_i} R_{x_j}$$

$$+ \frac{2\gamma}{t}\frac{1}{R}\left(\log\frac{1}{R}\right)\sum a_{ij} R_{x_i x_j} + \frac{2\gamma}{t}\frac{1}{R}\left(\log\frac{1}{R}\right)\sum b_i R_{x_i} - \frac{\gamma}{t^2}(\log R)^2.$$

Setting

$$\mathcal{A} = \sum a_{ij} R_{x_i} R_{x_j},$$

$$\mathcal{B} = \sum b_i R_{x_i} + \sum a_{ij} R_{x_i x_j},$$

we find that

$$(Lw - w_t)\Phi = \frac{4\gamma^2}{t^2}\frac{(\log R)^2}{R^2}\mathcal{A} - \frac{2\gamma}{t}\frac{1 + \log(1/R)}{R^2}\mathcal{A} + \frac{2\gamma}{t}\frac{\log(1/R)}{R}\mathcal{B} - \frac{\gamma}{t^2}(\log R)^2. \tag{3.19}$$

By (3.1), (3.2), $\mathcal{A} = 0$ on $\partial\Omega$. Since $\mathcal{A} \geqslant 0$ everywhere, we conclude that

$$\mathcal{A} \leqslant C_0 R^2 \quad \text{if} \quad 0 \leqslant R(x) \leqslant 1 \quad (C_0 \text{ positive constant}). \tag{3.20}$$

When $\mathcal{A} = 0$ we have (by [6])

$$\sum a_{ij} R_{x_i x_j} = -\sum \frac{\partial a_{ij}}{\partial x_j} \nu_i \quad \text{on } \partial\Omega.$$

Recalling (3.1)–(3.3) we deduce that $\mathcal{B} \leq 0$ on $\partial\Omega$, so that

$$\mathcal{B} \leq C_0 R \quad \text{if } 0 \leq R(x) \leq 1 \quad (C_0 \text{ positive constant}). \tag{3.21}$$

Now, if γ is sufficiently small then, by (3.20),

$$\frac{4\gamma^2}{t^2} \frac{(\log R)^2}{R^2} \mathcal{A} \leq \frac{1}{2} \frac{\gamma}{t^2} (\log R)^2.$$

Since also
$$-\frac{2\gamma}{t} \frac{1 + \log(1/R)}{R^2} \mathcal{A} < 0 \quad \text{if } R(x) < \varepsilon,\ \varepsilon < 1,$$

we conclude from (3.19) that

$$(Lw - w_t)\Phi < \frac{2\gamma}{t} \frac{\log(1/R)}{R} \mathcal{B} - \frac{1}{2} \frac{\gamma}{t^2} (\log R)^2.$$

Using (3.21) we see shat if ε is sufficiently small then (3.17) holds.

Proof of Corollary 3.2. The formal adjoint of Lu is

$$L^* u = \sum a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum \tilde{b}_i \frac{\partial u}{\partial x_i} + \tilde{c} u$$

where
$$\tilde{b}_i = -b_i + 2 \sum \frac{\partial a_{ij}}{\partial x_j},$$

$$\tilde{c} = \sum \frac{\partial^2 a_{ij}}{\partial x_i \partial x_j} - \sum \frac{\partial b_i}{\partial x_i}. \tag{3.22}$$

Since
$$\tilde{b}_i - \sum \frac{\partial a_{ij}}{\partial x_j} = -\left(b_i - \sum \frac{\partial a_{ij}}{\partial x_j}\right),$$

the condition (3.14) implies the condition (3.3) for L^*. The proof of (3.17) remains valid for L^* (with a trivial change due to the term $\tilde{c}w$). We conclude that Green's function $G_m^*(x, t, \xi)$ corresponding to $L^* - \partial/\partial t$ in $N_m \times (0, \infty)$ satisfies:

$$G_m^*(x, t, \xi) \leq w(x, t) \quad (x \in M_{\varepsilon, m},\ 0 < t < T,\ \xi \in E).$$

Recalling that $G_m(x, t, \xi) = G_m^*(\xi, t, x)$ and taking $m \to \infty$, the assertion (3.15) follows.

We shall now assume that

$$\mathcal{A}(x) = O(R^{p+1}), \quad \text{as } R(x) \to 0, \tag{3.23}$$

where p is a positive number, $p > 1$.

THEOREM 3.3. *Let* (A), (B_S), (C), (3.4) *and* (3.23) *hold. Let* E *be any compact subset of* $\hat{\Omega}$. *Then, for any* $T>0$ *and for any* $\varrho>0$ *sufficiently small, there are positive constants* C, γ *such that*

$$G(x,t,\xi) \leqslant C \exp\left\{-\frac{\gamma}{t}(R(x))^{1-p}\right\} \tag{3.24}$$

if $\xi \in E$, $x \in \hat{\Omega}$, $R(x)<\varrho$, $0<t<T$.

COROLLARY 3.4. *If in Theorem* 3.3, *the condition* (3.3) *is replaced by the condition* (31.4), *then*

$$G(x,t) \leqslant C \exp\left\{-\frac{\gamma}{t}(R(\xi))^{1-p}\right\} \tag{3.25}$$

if $x \in E$, $\xi \in \hat{\Omega}$, $R(\xi)<\varrho$, $0<t<T$.

Proof of Theorem 3.3. We proceed as in the proof of Theorem 3.1, but change the definition of $w(x,t)$. First we consider the interval $0<t<\delta$ (δ is small and will be determined later on), and take

$$w(x,t) = C \exp\left\{-\frac{\gamma}{t}(R(x))^{1-p}\right\}. \tag{3.26}$$

If we prove that, for any $\gamma>0$ sufficiently small and independent of m, (3.17) holds for $x \in M_{\varepsilon,m}$, $0<t<\delta$, then the inequality (3.24), for $0<t<\delta$, follows as in the proof of Theorem 3.1. To prove (3.17), set $\Phi = 1/w$. Then

$$w_{x_i} = \frac{1}{\Phi}\frac{\gamma}{t}\frac{p-1}{R^p}R_{x_i},$$

$$w_{x_i x_j} = \frac{1}{\Phi}\left\{\frac{\gamma^2(p-1)^2}{t^2 R^{2p}}R_{x_i}R_{x_j} - \frac{\gamma p(-1)}{tR^{p+1}}R_{x_i}R_{x_j} + \frac{\gamma(p-1)}{tR^p}R_{x_i x_j}\right\},$$

$$-w_t = \frac{1}{\Phi}\left\{-\frac{\gamma}{t^2}\frac{1}{R^{p-1}}\right\}.$$

Hence
$$(Lw-w_t)\Phi = \frac{\gamma^2(p-1)^2}{t^2}\frac{\mathcal{A}}{R^{2p}} - \frac{\gamma p(p-1)}{t}\frac{\mathcal{A}}{R^{p+1}} + \frac{\gamma(p-1)}{t}\frac{\mathcal{B}}{R^p} - \frac{\gamma}{t^2 R^{p-1}}. \tag{3.27}$$

If γ is sufficiently small, then, by (3.23),

$$\frac{\gamma^2(p-1)^2}{t^2}\frac{\mathcal{A}}{R^{2p}} \leqslant \frac{1}{3}\frac{\gamma}{t^2}\frac{1}{R^{p-1}}.$$

By (3.21)
$$\frac{\gamma(p-1)}{t}\frac{\mathcal{B}}{R^p} \leqslant \frac{1}{3}\frac{\gamma}{t^2}\frac{1}{R^{p-1}}$$

if $0<t<\delta$ and δ is sufficiently small. From (3.27) we then conclude that (3.17) holds if $0<t<\delta$. As mentioned above, this implies (3.24) for $0<t<\delta$. In order to prove (3.24) for $\delta<t<T$ we introduce another comparison function, namely,

$$w^0(x,t)=\hat{C}\exp\left\{-\frac{\hat{\gamma}}{(t+1)^{\lambda}}(R(x))^{1-p}\right\}$$

where $\hat{C},\hat{\gamma},\lambda)$ are positive numbers. With $\Phi=1/w^0$, we have

$$(Lw^0-w_t^0)\Phi=\frac{\hat{\gamma}^2(p-1)^2}{(t+1)^{2\lambda}}\frac{\mathcal{A}}{R^{2p}}-\frac{\hat{\gamma}p(p-1)}{(t+1)_\lambda}\frac{\mathcal{A}}{R^{p+1}}+\frac{\hat{\gamma}(p-1)}{(t+1)^{\lambda}}\frac{\mathcal{B}}{R^p}-\frac{\hat{\gamma}\lambda}{(t+1)^{\lambda+1}}\frac{1}{R^{p-1}}. \tag{3.28}$$

We choose λ (independently of $\hat{\gamma}$) so large that $\lambda>1$ and

$$(p-1)\frac{\mathcal{B}}{R}<\frac{1}{3}\frac{\lambda}{T+1};$$

this is possible by (3.21). With λ fixed we net choose $\hat{\gamma}$ so small that

$$\frac{\hat{\gamma}(p-1)^2}{(\delta+1)^{\lambda-1}}\frac{\mathcal{A}}{R^{p+1}}\leqslant\frac{1}{3}\lambda.$$

It then follows from (3.28) that $Lw^0-w_t^0<0$ if $x\in M_{\varepsilon,m}$, $\delta<t<T$. Notice that if $\hat{\gamma}$ is sufficienty small and $\hat{C}$ is sufficiently large (both independent of m), then, by (3.8),

$$G_m(x,t,\xi)\leqslant w^0(x,\ )\quad(\xi\text{ fixed in }E) \tag{3.30}$$

if $x\in\Gamma_\varepsilon$, $0<t<T$. The same inequality clearly holds also if $x\in\partial_1N_m$, $t>0$ and, by what we have already proved above, for $x\in M_{\varepsilon,m}$, $t=\delta$. Hence, we can apply the maximum principle and conclude that (3.30) holds for $x\in M_{\varepsilon,m}$, $\delta<t<T$. Taking $m\to\infty$, the proof of (3.24), for $\delta<t<T$, follows.

The proof of Corollary 3.4 is obtained by applying the proof of Theorem 3.3 to the equation $L^*u-\partial u/\partial t=0$; the proof of Corollary 3.2. The details may be omitted.

Remark 1. Suppose Ω consists of a finite disjoint union of closed domains G_j, i.e., $k_0=0$. The estimates of Theorems 3.1, 3.3 show that $G(x,t,\xi)$ is actually Green's function for $L-\partial/\partial t$ in $\hat{\Omega}\times(0,\infty)$. When L is nondegenerate, Green's function vanishes for $x\in\partial\Omega$ at a linear rate, i.e., $\partial G(x,t,\xi)/\partial\nu\neq 0$ (ν is the normal to $\partial\Omega$ at x); in fact this is a consequence of the maximum principle (see [3]). In the present case where L degenerates on $\partial\Omega$, Green's function vanishes on $\partial\Omega$ at a rate faster than any power of $R(x)$.

Remark 2. Set $\Omega_0=\text{int}\,\Omega$. In Theorems 3.1, 3.3 and their corollaries we were concerned with Green's function $G(x,t,\xi)$ for x,ξ in $\hat{\Omega}$. Similarly one can construct a Green function

$G_0(x, t, \xi)$ for x, ξ in Ω_0. If (A), (B_S), (C) and (3.4) hold with ν (in (C)) being the inward normal to ∂G_j at $x(k_0+1 \leq j \leq k)$ then (3.13) holds with $G(x, t, \xi)$ replaced by $G_0(x, t, \xi)$; $\xi \in E$, $x \in \Omega_0$, $0<t<T$, $\operatorname{dist}(x, \partial\Omega)<\varrho$, where E is any compact subset of Ω_0. Similarly, if (3.3) is replaced by (3.14) (ν the inward normal) then (3.15) holds with $G(x, t, \xi)$ replaced by $G_0(x, t, \xi)$; $x \in E$, $\xi \in \Omega_2$, $0<t<T$, $\operatorname{dist}(\xi, \partial\Omega)<\varrho$. The assertions of Theorem 3.3 and Corollary 3.4 also extend to $G_0(x, t, \xi)$. Note that $G_0(x, t, \xi)=0$ if $x \in G_j^0$, $\xi \in G_h^0$ and $j \neq h$; $G_i^0 = \operatorname{int} G_i$.

4. Estimates near infinity

In this section we replace the conditions (C), (3.4) by the much weaker condition:

$$S \text{ is a compact set.} \tag{4.1}$$

Let $\hat{S} = R^n \setminus S$.

THEOREM 4.1. *Let* (A), (B_S) *and* (4.1) *hold. Assume also that*

$$\sum_{i,j=1}^{n} a_{ij}(x) x_i x_j \leq C_0(1+|x|^4), \tag{4.2}$$

$$-\left[\sum_{i=1}^{n} x_i b_i(x) + \sum_{i=1}^{n} a_{ii}(x)\right] \leq C_0(1+|x|^2) \tag{4.3}$$

where C_0 is a positive constant. Let E be any bounded subset of $\hat{S}$. Then, for any $T>0$ and for any ϱ sufficiently large, there are positive constants C, γ such that

$$K(x, t, \xi) \leq C \exp\left\{-\frac{\gamma}{t}(\log|x|)^2\right\} \tag{4.4}$$

if $\xi \in E$, $|x|>\varrho$, $0<t<T$.

Notice that the closure of E may intersect S

COROLLARY 4.2. *If, in Theorem* 4.1, *the condition* (4.3) *is replaced by the conditions*

$$\sum_{i=1}^{n} x_i b_i(x) + \sum_{i=1}^{n} a_{ii}(x) \leq C_0(1+|x|^2), \tag{4.5}$$

$$\sum_{i,j=1}^{n} \frac{\partial^2 a_{ij}(x)}{\partial x_i \partial x_j} - \sum_{i=1}^{n} \frac{\partial b_i(x)}{\partial x_i} \leq [\log(2+|x|)]^2 \eta(|x|) \quad (\eta(r) \to 0 \quad \text{if } r \to \infty), \tag{4.6}$$

then

$$K(x, t, \xi) \leq C \exp\left\{-\frac{\gamma}{t}(\log|\xi|)^2\right\} \tag{4.7}$$

if $x \in E$, $|\xi|>\varrho$, $0<t<T$.

Proof of Theorem 4.1. Consider first the case where $\bar{E} \cap S = \emptyset$. For any $\varrho > 0$, m positive integer, let

$$N_{m,\varrho} = \{x; \varrho < |x| < m\}, \quad \Delta_\varrho = \{x; |x| = \varrho\}, \quad \Delta_m = \{x; |x| = m\}.$$

The number ϱ is sufficiently small (to be determined later on), whereas $m > \varrho$. The boundary of $N_{m,\varrho}$ then consists of the spheres Δ_ϱ, Δ_m. Proceeding similarly to the proof of Theorem 3.1, we shall compare the function $v(x, t) = G_{m,\varepsilon}(x, t, \xi)$ (ξ fixed in E) with a function $w(x, t)$ in the cylinder $N_{m,\varrho} \times (0, T)$. We take

$$w(x,t) = C \exp\left\{-\frac{\gamma}{t}(\log|x|)^2\right\} \tag{4.8}$$

where C, γ are positive constants. It is clear that (3.19) holds with $R(x) = |x|$, L replaced by L_ε, a_{ij} replaced by $a_{ij}^\varepsilon = a_{ij} + \varepsilon\delta_{ij}$, where

$$\mathcal{A} = \frac{1}{|x|^2}\sum a_{ij}^\varepsilon(x) x_i x_j$$

$$\mathcal{B} = \frac{1}{|x|}[\sum x_i b_i(x) + \sum a_{ii}^\varepsilon(x)] - \frac{1}{|x|^3}\sum a_{ij}^\varepsilon(x) x_i x_j.$$

By (4.2), (4.3) we have, for all $R(x) = |x|$ sufficiently large,

$$\mathcal{A} \leqslant C_0 R^2, \quad -\mathcal{B} \leqslant C_0 R \quad (C_0 \text{ positive constant}).$$

Now choose γ so small that

$$\frac{4\gamma^2}{t^2}\frac{(\log R)^2}{R^2}\mathcal{A} \leqslant \frac{1}{3}\frac{\alpha}{t^2}(\log R)^2 \tag{4.9}$$

Next choose ϱ such that if $R(x) = |x| \geqslant \varrho$,

$$-\frac{2\gamma}{t}\frac{1+\log(1/R)}{R^2}\mathcal{A} = \frac{2\gamma}{t}\frac{\log R - 1}{R^2}\mathcal{A} < \frac{1}{3}\frac{\gamma}{t^2}(\log R)^2, \tag{4.10}$$

$$\frac{2\gamma}{t}\frac{\log(1/R)}{R}\mathcal{B} = -\frac{2\gamma}{t}\frac{\log R}{R}\mathcal{B} < \frac{1}{3}\frac{\gamma}{t^2}(\log R)^2 \tag{4.11}$$

for all $0 < t < T$. It follows that $L_\varepsilon w - w_t < 0$ if $x \in N_{m,\varrho}$, $0 < t < T$.

Notice that ϱ was chosen independently of γ. With ϱ now fixed, we further decrease γ (if necessary) so that

$$v(x, t) \leqslant w(x, t) \quad \text{if } x \in \Delta_\varrho, \quad 0 < t \leqslant T$$

for some positive constant C (in (4.8)). The last inequality evidently holds also if $x \in \Delta_m$, $0 < t < T$ or if $x \in N_{m,\varrho}$, $t = 0$. Applying the maximum principle, we get

$$G_{m,\varepsilon}(x, t, \xi) = v(x, t) \leqslant w(x, t) \quad \text{if } x \in N_{m,\varrho}, 0 < t < T.$$

From this the assertion (4.4) follows by taking first $m \to \infty$ and then $\varepsilon = \varepsilon_m \to 0$.

So far we have proved (4.4) only in case $\bar{E} \cap S = \emptyset$. Now let E be any bounded set disjoint to S. Let Σ be a sphere containing both E and S in its interior Δ. From what we have proved so far we know that if $x \in N_{m,\varrho}$ then

$$G_{m,\varepsilon}(x, t, \xi) \leqslant w(x, t) \tag{4.12}$$

if $\xi \in \Sigma$, $0 < t < T$. Now, as a function of (ξ, t) the function $w(x, t)$ satisfies:

$$\left(L_\varepsilon^* - \frac{\partial}{\partial t}\right) w = \left[\tilde{c}(\xi) - \frac{\gamma}{t^2}(\log|x|)^2\right] w(x, t) < 0$$

if ϱ is sufficiently large and $\xi \in \Delta$, $0 < t < T$. Hence, by the maximum principle, (4.12) holds also for $\xi \in \Delta$, $0 < t < T$. Taking $m \to \infty$ and then $\varepsilon = \varepsilon_m \to 0$, the inequality (4.4) follows.

Proof of Corollary 4.2. We apply the proof of Theorem 4.1 to the adjoint L^* of L (cf. the proof of Corollary 3.2). Since (4.9)–(4.11) remain valid (with $\mathcal{B}$ replaced by $-\mathcal{B}$) with the factor 1/3 on the right-hand sides replaced by 1/4, it remains to show that

$$\tilde{c}(x) < \frac{1}{4} \frac{\gamma}{t^2} (\log R)^2,$$

where $\tilde{c}$ is defined in (3.22). In view of (4.6), this inequality holds if $0 < t < T$, provided ϱ is sufficiently small and $R(x) = |x| > \varrho$.

Suppose next that (4.2) is replaced by

$$\sum_{i,j=1}^{n} a_{ij}(x) x_i x_j \leqslant C_0(1 + |x|^{4-p}) \quad (0 < p \leqslant 2). \tag{4.13}$$

Then we can use, for $0 < t < \delta$, the comparison function

$$w(x, t) = C \exp\left\{-\frac{\gamma}{t}|x|^p\right\}. \tag{4.14}$$

In fact one easily verifies that $L_\varepsilon w - w_t < 0$ for $x \in N_{m,\varrho}$, $0 < t < \delta$, provided γ and δ are sufficiently small. For $\delta < t < T$ we use the comparison function

$$w^0(x, t) = C \exp\left\{-\frac{\hat{\gamma}}{(t+1)^\lambda}|x|^p\right\}. \tag{4.15}$$

Choosing first λ sufficiently large, and then $\hat{\gamma}$ sufficiently small, we find that $L_\varepsilon w^0 - \partial w^0/\partial t < 0$ if $x \in N_{m,\varrho}$, $\delta < t < T$.

With the aid of these comparison functions we obtain:

THEOREM 4.3. *Let* (A), (B_S), (4.1), (4.13) *and* (4.3) *hold. Let E be any bounded subset of $\hat{S}$. Then, for any $T > 0$ and for any ϱ sufficiently large, there are positive constants C, γ such that*

$$K(x, t, \xi) \leqslant C \exp\left\{-\frac{\gamma}{t}|x|^p\right\} \tag{4.16}$$

if $\xi \in E$, $|x| > \varrho$, $0 < t < T$.

COROLLARY 4.4. *If in Theorem* 4.3, *the condition* (4.3) *is replaced by the conditions* (4.5) *and*

$$\sum_{i,j=1}^{n} \frac{\partial^2 a_{ij}(x)}{\partial x_i \partial x_j} - \sum_{i=1}^{n} \frac{\partial b_i(x)}{\partial x_i} \leqslant (1 + |x|^p)\eta(|x|) \quad (\eta(r) \to 0 \quad \textit{if } r \to \infty), \tag{4.17}$$

then

$$K(x, t, \xi) \leqslant C \exp\left\{-\frac{\gamma}{t}|\xi|^p\right\} \tag{418}$$

if $x \in E$, $|\xi| > \varrho$, $0 < t < T$.

The proof of the corollary is obtained by applying the proof of Theorem 4.3 (with the same comparison functions w, w^0 as in (4.14), (4.15)) to L^*.

Remark 1. Denote by $\tilde{S}$ the unbounded component of $R^n \backslash S$. One can construct the function $G(x, t, \xi)$, for x, ξ in $\tilde{S}$ and $t > 0$, in the same way that we have constructed $G(x, t, \xi)$ for x, ξ in $\hat{\Omega}$, $t > 0$, as a limit of Green's functions $G_m(x, t, \xi)$ (cf. the remark following (3.12)). Using the same comparison functions as in Theorems 4.1, 4.3 and Corollaries 4.2, 4.4, we can estimate the functions $G_m(x, t, \xi)$ and, consequently, also $G(x, t, \xi)$. The estimates on G are the same as for K, except that now $\bar{E} \cap S$ is required to be empty.

Remark 2. Let $\mathcal{M}$ be an affine matrix. If we change the definition of $w(x, t)$ in (4.8), replacing $|x|$ by $|\mathcal{M}x|$, then we can establish the estimate (4.4) when (4.2), (4.3) are replaced by the more general conditions

$$\sum_{i,j=1}^{n} a_{ij}(x) \tilde{R}_{x_j} \tilde{R}_{x_j} \leqslant C_0(1 + |x|^2),$$

$$-\left[\sum_{i=1}^{n} b_i(x) \tilde{R}_{x_i} + \sum_{i,j=1}^{n} a_{ij}(x) \tilde{R}_{x_i x_j}\right] \leqslant C_0(1 + |x|),$$

where $\tilde{R}(x) = |\mathcal{M}x|$. Similar remarks apply also to the other results of this section.

5. Relation between K and a diffusion process

If the symmetric matrix $(a_{ij}(x))$ is positive semi-definite and the a_{ij} belong to $C^2(R^n)$, then (by [2] or [12]) there exists an $n \times n$ matrix $\sigma(x) = (\sigma_{ij}(x))$ which is Lipschitz continuous, uniformly in compact subsets of R^n, such that

$$\sigma(x)\sigma^*(x) = 2(a_{ij}(x)) \quad [\sigma^* = \text{transpose of } \sigma],$$

i.e., $\Sigma\, \sigma_{ik}(x)\sigma_{jk}(x) = 2a_{ij}(x)$. If

$$\sum_{i=1}^{n} a_{ii}(x) \leqslant C(1+|x|^2), \tag{5.1}$$

then, clearly,

$$|\sigma(x)| \leqslant C(1+|x|) \tag{5.2}$$

with a different constant C. Conversely, (5.2) implies (5.1) and, in fact, implies

$$\sum_{i,j=1}^{n} |a_{ij}(x)| \leqslant C(1+|x|^2).$$

We shall now assume that (5.1) holds and, in addition,

$$\sum_{i=1}^{n} |b_i(x)| \leqslant C(1+|x|). \tag{5.3}$$

Set $b=(b_1, \ldots, b_n)$. Since we always assume that (A) holds, the functions $\sigma(x)$, $b(x)$ are uniformly Lipschitz continuous in compact subsets of R^n.

Consider the system of n stochastic differential equations

$$d\xi(t) = \sigma(\xi(t))\,dw(t) + b(\xi(t))\,dt \tag{5.4}$$

where $w(t)$ is n-dimensional Brownian motion. It is well known (see, for instance, [7], [8], [11]) that this system has a unique solution $\xi(t)$ (for $t>0$) for any prescribed initial condition $\xi(0)=x$. The process $\xi(t)$ defines a time-homogeneous diffusion process, and the transition probabilities are given by

$$P(t, x, A) = E_x(\xi(t) \in A) \tag{5.5}$$

for any Borel set A in R_n.

Definition. If there is a function $\Gamma(x, t, \xi)$ defined for all x, ξ in R_n and $t>0$ and Borel measurable in ξ for fixed (x, t), such that

$$P(t, x, A) = \int_A \Gamma(x, t, \xi)\,d\xi \tag{5.6}$$

for any Borel set A in R^n and for any $x \in R^n$, $t>0$, then we call $\Gamma(x, t, \xi)$ *the fundamental solution* of the parabolic equation (1.1).

Note that $\Gamma(x, t, \xi)$, if existing, is uniquely determined, for each (x, t) almost everywhere in ξ. Note also that for any continuous function $f(\xi)$ with compact support.

$$E_x[f(\xi(t))] = \int_{R^n} \Gamma(x, t, \xi) f(\xi)\,d\xi. \tag{5.7}$$

Suppose now that $f(\xi)$ is also in C^2. If the matrix $(a_{ij}(x))$ is positive definite, the $\sigma_{ij}(x)$ are in C^2 (by [2]). But then, by [7], [8], the left-hand side of (5.7), $u(x, t)$, is a classical solution of the Cauchy problem

$$Lu - u_t = 0 \quad \text{if } t > 0, x \in R^n, \tag{5.8}$$

$$u(x, 0) = f(x) \quad \text{if } x \in R^n. \tag{5.9}$$

If f is just assumed to be continuous, let $f_m(x)$ be a C^2 function with uniformly bounded supports such that $f_m \to f$ uniformly in R^n, as $m \to \infty$. Let $u_m(x, t) = E_x(f_m(\xi(t)))$. Then

$$Lu_m - \frac{\partial u_m}{\partial t} = 0 \quad \text{if } t > 0, x \in R^n,$$

$$u_m(x, 0) = f_m(x) \quad \text{if } x \in R^n.$$

Noting that $u_m(x, t) \to u(x, t)$ as $m \to \infty$, uniformly in (x, t) in bounded sets of $R^n \times [0, \infty)$, (5.9) follows. Applying to u_m the Schauder-type interior estimates [3] we also find that $\{u_m\}$ converges to u together with the first two x-derivatives and the first t-derivative. Consequently, u is a solution of (5.8). We have thus proved that for any continuous function f with compact support, the right-hand side of (5.7) is a classical solution of (5.8), (5.9). Thus, when the matrix $(a_{ij}(x))$ is positive definite $\Gamma(x, t, \xi)$ is a fundamental solution in the usual sense (see Section 1). When $(a_{ij}(x))$ is uniformly positive definite and the a_{ij}, b_i satisfy some boundedness conditions at ∞, this fundamental solution Γ can be constructed by the parametrix method [3]. Under milder growth conditions it was constructed in [4].

THEOREM 5.1. *Let* (A), (B_S) *and* (5.1), (5.3) *hold. Then*

$$\lim_{\varepsilon \to 0} K_\varepsilon(x, t, \xi) \text{ exists for all } x \notin S, \xi \notin S, t > 0, \tag{5.10}$$

and the function $K(x, t, \xi) = \lim_{\varepsilon \to 0} K_\varepsilon(x, t, \xi)$ *satisfies:*

$$P_x(\xi(t) \in A) = \int_A K(x, t, \xi) d\xi \tag{5.11}$$

for any Borel set A *with* $A \cap S = \emptyset$.

Proof. In section 1 we have proved that there is a sequence $\{\varepsilon_m\}$ converging to zero such that

$$K_{\varepsilon_m}(x, t, \xi) \to K(x, t, \xi) \quad \text{as } m \to \infty \tag{5.12}$$

for all $x \notin S$, $\xi \notin S$, $t > 0$; the convergence is uniform when x, ξ vary in any compact set E, $E \cap S = \emptyset$, and t varies in any interval $(\delta, 1/\delta)$, $\delta > 0$. The same proof shows that anv sequence $\{\varepsilon'_m\}$ converging to zero has a subsequence $\{\varepsilon''_m\}$ such that

$$K_{\varepsilon''_m}(x, t, \xi) \to M(x, t, \xi) \quad \text{as } m \to \infty$$

for some function M, and the convergence is uniform in the same sense as before. If we can show that $M(x, t, \xi) = K(x, t, \xi)$ then the assertion (5.10) follows.

If we show that

$$P_x(\xi(t) \in A) = \int_A M(x, t, \xi)\, d\xi \tag{5.13}$$

for any bounded Borel set A, $\bar{A} \cap S = \emptyset$, then, by applying this to the particular sequence $\{\varepsilon_m\}$ we derive (5.13) with M replaced by K. This will show both that $M = K$ (so that (5.10) is true) and that (5.11) holds. Thus, in order to complete the proof of the theorem it remains to verify (5.13).

For any $\varepsilon > 0$, consider the stochastic differential system

$$d\xi^\varepsilon(t) = \sigma^\varepsilon(\xi^\varepsilon(t))\, dw(t) + b(\xi^\varepsilon(t))\, dt \tag{5.14}$$

where σ^ε is such that $\sigma^\varepsilon(\sigma^\varepsilon)^* = 2(a_{ij} + \varepsilon^2 \delta_{ij})$; here $(\sigma^\varepsilon)^* =$ transpose of σ^ε. We then have

$$P_x(\xi^\varepsilon(t) \in A) = \int_A K_\varepsilon(x, t, \xi)\, d\xi. \tag{5.15}$$

Indeed, by the argument following (5.7), for any continuous function f with compact support, the function $Ef(\xi^\varepsilon(t))$ is a solution of (5.8), (5.9). The function

$$\int_{R^n} K_\varepsilon(x, t, \xi) f(\xi)\, d\xi$$

is also a solution of (5.8), (5.9). Since both solutions are bounded (the boundedness of the second solution follows from the proof of Theorem 4.1) they must coincide (by [3; p. 56, Problem 2]). Taking a sequence of f's which converges to the characteristic function of A, (5.15) follows.

Since (by [2]) $\sigma^\varepsilon(x, t) \to \sigma(x, t)$ uniformly on compact sets, as $\varepsilon \to 0$, a standard argument shows (cf. [6]) that

$$E_x |\xi(t) - \xi(t)|^2 \to 0 \quad \text{if } \varepsilon \to 0. \tag{5.16}$$

Suppose now that A is a ball of radius R and denote by B_ϱ ($\varrho > 0$) the ball of radius ϱ concentric with A. From (5.16) it follows that if $\varrho < R < \varrho'$ then

$$\overline{\lim_{\varepsilon \to 0}}\, P_x(\xi^\varepsilon(t) \in B_\varrho) \leqslant P_x(\xi(t) \in A),$$

$$\underline{\lim_{\varepsilon \to 0}}\, P_x(\xi^\varepsilon(t) \in B_{\varrho'}) \geqslant P_x(\xi(t) \in A).$$

By (5.15) and Theorem 1.2 we also have

$$P_x(\xi^\varepsilon(t) \in B_{\varrho'} / B_\varrho) = \int_{B_{\varrho'} / B_\varrho} K_\varepsilon(x, t, \xi)\, d\xi \leqslant C(\varrho' - \varrho)$$

provided ϱ' is sufficiently close to R (so that $\bar{B}_{\varrho'} \cap S = \emptyset$), where C is a constant independent of ε. From the last three relations we deduce that

$$P_x(\xi^\varepsilon(t) \in A) \to P_x(\xi(t) \in A) \quad \text{if } \varepsilon \to 0. \tag{5.17}$$

Taking $\varepsilon = \varepsilon''_m \to 0$, the right-hand side of (5.15) converges to the right-hand side of (5.13). If A is a ball then, by (5.17), the left-hand side of (5.15) converges to the left-hand side of (5.13). We have thus established (5.13) in case A is a ball with $\bar{A} \cap S = \emptyset$. But then (5.13) follows also for any Borel set A with $A \cap S = \emptyset$.

THEOREM 5.2. *Let* (A), (B_S), (4.1) *and* (5.1), (5.3) *hold. Then, for any* $x \in S$,

$$K(x, t, \xi) \equiv \lim_{\varepsilon \to 0} K_\varepsilon(x, t, \xi) \tag{5.18}$$

exists for all $\xi \notin S$, $t > 0$; *the convergence is uniform with respect to* (ξ, t) *in compact subsets of* $(R^n \backslash S) \times [0, \infty)$, *and* (5.11) *holds for any Borel set* A *with* $A \cap S = \emptyset$. *Finally, for any disjoint compact sets* M, E *in* R^n *with* $S \subset M$, *and for any* $T > 0$,

$$K(x, t, \xi) \leqslant Ce^{-c/t} \quad \textit{for all } x \in M, \xi \in E, 0 < t < T \tag{5.19}$$

where C, c *are positive constants depending on* M, E, T.

Proof. Let E be a compact set, $E \cap S = \emptyset$, and let M be a bounded neighborhood of S such that $\bar{M} \cap E = \emptyset$. For fixed ξ in E, consider the function

$$v_\varepsilon(x, t) = K_\varepsilon(x, t, \xi) \quad \text{for } x \in M, 0 < t < T.$$

If $x \in \partial M$, $0 < t < T$ then, by the results of sections 1, 2,

$$0 \leqslant v_\varepsilon(x, t) \leqslant Ce^{-c/t}$$

where C, c are positive constants independent of ξ, ε. Further,

$$v_\varepsilon(x, 0) = 0 \quad \text{if } x \in M,$$

$$L_\varepsilon v_\varepsilon - \frac{\partial v_\varepsilon}{\partial t} = 0 \quad \text{if } x \in M, t > 0.$$

Hence, by the maximum principle,

$$0 \leqslant v_\varepsilon(x, t) \leqslant Ce^{-c/t} \quad \text{if } x \in M, 0 \leqslant t \leqslant T,$$

i.e.,

$$0 \leqslant K_\varepsilon(x, t, \xi) \leqslant Ce^{-c/t} \quad \text{if } x \in M, 0 \leqslant t \leqslant T, \xi \in E. \tag{5.20}$$

Fix x in S and consider the function

$$\phi_\varepsilon(\xi, t) = K_\varepsilon(x, t, \xi) \quad \text{for } \xi \in E, 0 \leqslant t \leqslant T.$$

By (5.20) this function is bounded. Since $\phi_\varepsilon(\xi, 0)=0$ if $\xi\in E$, and

$$L_\varepsilon^*\phi_\varepsilon-\frac{\partial}{\partial t}\phi_\varepsilon=0 \quad \text{if } \xi\in E, 0<t\leqslant T,$$

and since L^* is nondegenerate for $\xi\in E$, we can apply the Schauder-type estimates [3] in order to conclude the following:

For any sequence $\{\varepsilon_m'\}$ converging to 0 there is a subsequence $\{\varepsilon_m^*\}$ such that $\{\phi_{\varepsilon_m^*}\}$ is convergent to some function $\phi(\xi, t)=\hat{K}(x, t, \xi)$, together with the first t-derivative and the first two ξ-derivatives, uniformly for ξ in any set interior to E and t in $[0, T]$. By diagonalization, there is a subsequence $\{\varepsilon_m''\}$ of $\{\varepsilon_m^*\}$ for which

$$K_{\varepsilon_m''}(x, t, \xi)\to\hat{K}(x, t, \xi) \quad \text{for all } \xi\in R^n\backslash S, t>0;$$

the first t-derivatives and the first two ξ-derivatives also converge, and the convergence is uniform for (ξ, t) in compact subsets of $(R^n\backslash S)\times[0, \infty)$.

Notice that the sequence $\{\varepsilon_m''\}$ may depend on the parameter x. Now let A be a Borel set such that $\bar{A}\cap S=\emptyset$. Taking, in (5.15), $x\in S$ and $\varepsilon=\varepsilon_m''\to 0$, and noting (upon using (5.20)) that the proof of (5.17) remains valid for $x\in S$, we conclude that

$$P_x(\xi(t)\in A)=\int_A \hat{K}(x, t, \xi)d\xi.$$

Thus, $K(x, t, \xi)$ is independent of the particular sequence $\{\varepsilon_m'\}$ that we have started with. It follows that (5.18) holds. The other assertions of the lemma now follow immediately; in particular, (5.19) follows from (5.20).

From the above proof we see that, for fixed x in S.

$$L^*K(x, t, \xi)-\frac{\partial}{\partial t}K(x, t, \xi)=0 \quad \text{if } \xi\notin S,\ t>0.$$

Theorem 5.3. *Let* (A), (B_S), (4.1) *and* (5.1), (5.3) *hold. Then for any disjoint compact sets M, E in R^n with $S\subset M$, and for any $T>0$*

$$K_\varepsilon(x, t, \xi)\leqslant Ce^{-c/t} \quad \text{for all } x\in E, \xi\in M, 0<t<T, \tag{5.21}$$

$$K(x, t, \xi)\leqslant Ce^{-c/t} \quad \text{for all } x\in E, \xi\in M\backslash S, 0<t<T, \tag{5.22}$$

where C, c are positive constants depending on M, E, T.

Indeed, we apply the argument which led to (5.20) to L^*, $K_\varepsilon^*(x, t, \xi)$ instead of L, $K_\varepsilon(x, t, \xi)$. We then get

$$K_\varepsilon^*(x, t, \xi)\leqslant Ce^{-c/t}$$

if $x\in M$, $\xi\in E$, $0<t<T$. Since $K^*(x, t, \xi)=K_\varepsilon(\xi, t, x)$, (5.21) follows. Since $K_\varepsilon(\xi, t, x)\to K(\xi, t, x)$ as $\varepsilon\to 0$, provided $\xi\notin S$, $x\notin S$, (5.22) also follows.

6. The behavior of $\xi(t)$ near S

In section 3 we have introduced the condition (C). In this section we shall need also other similar conditions:

(C_0) The condition (C) holds with one exception, namely, the condition (3.3) is omitted.

(C′) The condition (C) holds with one exception, namely, the inequality (3.3) is replaced by the inequality (3.14).

(C*) The condition (C) holds with one exception, namely, the inequality (3.3) is replaced by equality, i.e.,

$$\sum_{i=1}^{n}\left(b_i(x)-\sum_{j=1}^{n}\frac{\partial a_{ij}(x)}{\partial x_j}\right)\nu_i=0 \quad \text{for } x\in\partial G_j \quad (k_0+1\leqslant j\leqslant k). \tag{6.1}$$

(C**). There is a finite number of disjoint closed bounded domains $G_j (1\leqslant j\leqslant k)$ with C^3 connected boundary ∂G_j, such that

$$\sum a_{ij}(x)\nu_i\nu_j=0 \quad \text{for } x\in\partial G_j \quad (1\leqslant j\leqslant k), \tag{6.2}$$

$$\sum_{i=1}^{n}\left(b_i(x)-\sum_{j=1}^{n}\frac{\partial a_{ij}(x)}{\partial x_j}\right)\nu_i>0 \quad \text{for } x\in\partial G_j \quad (1\leqslant j\leqslant k) \tag{6.3}$$

where $\nu=(\nu_1, \ldots, \nu_n)$ is the outward normal to ∂G_j at x.

We shall also need the following condition:

(A_0) The inequalities (5.2), (5.3) hold, and $\sigma(x)$, $b(x)$ are uniformly Lipschitz continuous in compact subsets of R^n. Finally, the matrix $a=\frac{1}{2}\sigma\sigma^*$ is continuously differentiable in R^n.

Notice that if (A), (B_S) and (5.1), (5.3) hold, then the condition (A_0) is satisfied.

If (A_0) and (C′) hold then, by [6],

$$P_x\{\exists\, t>0 \quad \text{such that } \xi(t)\in\Omega\}=0 \quad \text{if } x\in\Omega, \tag{6.4}$$

i.e., if $\xi(0)=x\in\hat{\Omega}$ then with probability one $\xi(t)$ remains in $\hat{\Omega}$ for all $t>0$. Thus we may consider $\partial\Omega$ as an *obstacle* for $\xi(t)$ *from the side* $\hat{\Omega}$, or briefly, as a *one-sided obstacle*.

If (A_0) and (C) hold then, by [6],

$$P_x\{\exists\, t>0 \quad \text{such that } \xi(t)\in(\partial\Omega\cup\hat{\Omega})\}=0 \quad \text{if } x\in\Omega_0 \tag{6.5}$$

where $\Omega_0=\text{int}\,\Omega$. Thus $\partial\Omega$ is an obstacle for $\xi(t)$ from the side Ω_0. If, in particular, (C^*) holds, then $\partial\Omega$ is an obstacle from both sides $\hat{\Omega}$ and Ω_0; we then say that $\partial\Omega$ is a *two-sided obstacle*.

THEOREM 6.1. *Let* (A_0), (C*) *hold. Then, for any* $1\leqslant j\leqslant k$,

$$P_x\{\xi(t)\in\partial G_j \quad \textit{for all } t>0\}=1 \quad \textit{if } x\in\partial G_j. \tag{6.6}$$

Proof. Since (6.6) is obvious if $x=z_j$, $1\leqslant j\leqslant k_0$, it remains to consider the case where $k_0+1\leqslant j\leqslant k$.

Let $R(x)$ be a function such that $R(x) = \text{dist.}\ (x, \partial G_j)$ if x is in a small Ω-neighborhood of ∂G_j; $R(x) = -\text{dist.}\ (x, \partial G_j)$ if x is in a small Ω_0-neighborhood of ∂G_j; $R(x) \neq 0$ if $x \notin \partial G_j$; $R(x) = \text{const.}$ if $|x|$ is sufficiently large, and $R(x)$ is in $C^2(R^n)$. Then

$$\begin{aligned} LR^2(x) &= 2\sum a_{ij} R_{x_i} R_{x_j} + 2R\{\sum a_{ij} R_{x_i x_j} + \sum b_i R_{x_i}\} \\ &\equiv 2\mathcal{A} + 2R\mathcal{B} \leqslant CR^2, \end{aligned}$$

since $\mathcal{A} = O(R^2)$, $|\mathcal{B}| = O(R)$ if R is small, and $\mathcal{A} = \mathcal{B} = 0$ if $|x|$ is large. By Ito's formula,

$$E_x R^2(\xi(t)) - R^2(x) = E_x \int_0^t LR^2(\xi(s))ds \leqslant CE_x \int_0^t R^2(s))ds.$$

If $x \in \partial G_j$ then $R(x) = 0$. Setting $\phi(t) = E_x R^2(\xi(t))$, we then have

$$\phi(t) \leqslant C \int_0^t \phi(s)ds, \ \phi(0) = 0.$$

Hence $\phi(t) = 0$ for all t, i.e., $R^2(\xi(t)) = 0$ a.s. This proves (6.6).

Theorem 6.2. *Let* (A_0), (C^{**}) *hold. Then, for any* $t > 0$,

$$P_x(\xi(t) \in G_j) = 0 \quad \text{if } x \in \partial G_j \quad (1 \leqslant j \leqslant k). \tag{6.7}$$

In view of (6.4) and (6.7), if $x \in \hat{\Omega} \cup \partial\Omega$, then with probability one, $\xi(t) \in \hat{\Omega}$. This motivates us to call $\partial\Omega$ a *strictly one-sided obstacle, from the side* $\hat{\Omega}$, when the condition (C^{**}) holds.

Set
$$\varrho(x) = \text{dist.}\ (x, \partial\Omega).$$

We shall first establish the following lemma.

Lemma 6.3. *Let* (A_0), (C_0) *hold. Then*

$$E_x \varrho^2(\xi(t)) \leqslant Ct^2 \quad \text{if } x \in \partial\Omega,\ 0 < t < 1 \quad (C \text{ constant}). \tag{6.8}$$

Proof. Since $\varrho(\xi(t)) \equiv 0$ if $x = z_j$ $(1 \leqslant j \leqslant k_0)$, it remains to prove (6.8)in case $x \in \partial_0\Omega$, where

$$\partial_0\Omega \bigcup_{j=k_0+1}^{k} \partial G_j.$$

Set
$$\varrho_0(x) = \text{dist.}\ (x, \partial_0\Omega).$$

Let $M(x)$ be a C^2 function in R^n such that

$$M(x) = \begin{cases} \varrho_0(x), \text{ if } x \text{ is in a small } \hat{\Omega}\text{-neighborhood of } \partial_0\Omega, \\ -\varrho_0(x), \text{ if } x \text{ is in a small } \Omega\text{-neighborhood of } \partial_0\Omega, \\ |x|, \text{ if } |x| \text{ is sufficiently large}, \end{cases}$$

and $M(x) \neq 0$ if $x \notin \partial_0 \Omega$. If $x \in \partial_0 \Omega$ then, by Ito's formula,

$$M(\xi(t)) = \int_0^t M_x \sigma dw + \int_0^t LM ds.$$

Squaring both sides and taking the expectation, we obtain

$$E_x M^2(\xi(t)) \leqslant C E_x \int_0^t |M_x \sigma|^2 ds + C E_x \left(\int_0^t |LM| ds \right)^2. \tag{6.9}$$

Near $\partial_0 \Omega$,

$$|M_x \sigma|^2 = \sum_i \left(\sum_j \sigma_{ij} \frac{\partial \varrho_0}{\partial x_j} \right)^2 = O(\varrho^2) = O(M^2),$$

by (3.2), and near ∞,

$$|M_x \sigma|^2 = O(|x|^2) = O(M^2)$$

by (5.2). Next, for $|x|$ large

$$|LM| \leqslant C|x| = CM$$

by (5.2), (5.3), and for $|x|$ in a bounded set,

$$|LM| \leqslant C.$$

Using all these estimates in (6.9), and using Schwarz's inequality, we get

$$E_x M^2(\xi(t)) \leqslant C \int_0^t E_x M^2(\xi(s)) ds + Ct \int_0^t E_x M^2(\xi(s)) ds + Ct^2.$$

By iteration we then obtain

$$E_x M^2(\xi(t)) \leqslant Ct^2,$$

and this implies (6.8).

Proof of Theorem 6.2. For any $\varepsilon > 0$, let $G_{i,\varepsilon}$ be the set of points $x \in G_i$ with $\varrho(x) < \varepsilon$. The boundary $\partial G_{i,\varepsilon}$ of $G_{i,\varepsilon}$ consists of ∂G_i and $\partial' G_{i,\varepsilon}$; the latter is the set of all points x in G_i with $\varrho(x) = \varepsilon$. Denote by τ_ε the hitting time of $\partial' G_{i,\varepsilon}$.

Let ε_0 be a small positive number, so that $\varrho \in C^2$ in G_{i,ε_0}. Let

$$\Psi(x) = \begin{cases} -\varrho(x) & \text{if } x \in G_{i,\varepsilon_0}, \\ 0 & \text{if } x \notin G_i. \end{cases} \tag{6.10}$$

Then $D_x \Psi$ is continuous, and $D_x^2 \Psi$ is piecewise continuous, with discontinuity of the first kind across ∂G_i.

Define

$$\mathcal{A} = \sum a_{ij} \varrho_{x_i} \varrho_{x_j}$$

$$\mathcal{B} = \sum a_{ij} \varrho_{x_i x_j} + \sum b_i \varrho_{x_i}$$

for $x \in G_{t,\varepsilon_0}$. Then

$$L\Psi(x) = -(2\mathcal{A} + 2\varrho\mathcal{B}) \quad \text{if } x \in G_{t,\varepsilon_0}.$$

Hence, by (6.2), (6.3), if ε_0 is sufficiently small then

$$L\Psi(x) \geq \begin{cases} \beta\varrho(x) & \text{if } x \in G_{t,\varepsilon_0} \quad (\beta \text{ positive constant}), \\ 0 & \text{if } x \notin G_t. \end{cases}$$

By an approximation argument (see [5]) one can justify the use of Ito's formula for $\Psi(\xi(t))$. Recalling (6.10), (6.11) and taking $0 < \varepsilon < \varepsilon_0$, we then get

$$0 \geq E_x\Psi(\xi(t \wedge \tau_\varepsilon)) = E_x\int_0^{t\wedge\tau_\varepsilon} L\Psi(\xi(s))\,ds \geq 0 \quad (x \in \partial G_t).$$

Hence $E_x\Psi(\xi(t \wedge \tau_\varepsilon)) = 0$; by (6.10) this implies

$$P_x(\xi(t \wedge \tau_\varepsilon) \in \partial' G_{t,\varepsilon}) = 0$$

i.e.,

$$P_x(\tau_\varepsilon > t) = 1.$$

Since this is true for any $t > 0$, $P_x(\tau_\varepsilon = \infty) = 1$, i.e.,

$$P_x(\xi(t) \in G_t \backslash G_{t,\varepsilon}) = 0.$$

Since this is true for any $0 < \varepsilon < \varepsilon_0$,

$$P_x(\xi(t) \in \text{int } G_t) = 0 \quad (x \in \partial G_t). \tag{6.12}$$

Thus, in order to complete the proof of Theorem 6.2 it remains to show that

$$P_x(\xi(t) \in \partial\Omega) = 0 \quad \text{if } x \in \partial\Omega,\ t > 0 \tag{6.13}$$

Let $\Psi(x)$ be a C^2 function in $\hat{\Omega} \cup \partial\Omega$ such that

$$\Psi(x) = \begin{cases} \varrho(x) & \text{if } 0 \leq \varrho(x) < r_1, \\ 1 & \text{if } \varrho(x) > 1 \end{cases}$$

where $0 < r_1 < 1$, and $\Psi(x) > 0$ if $\varrho(x) > 0$. If r_1 is sufficiently small then, by (6.2), (6.3), $L\Psi(x) \geq \alpha_0 > 0$ if $\varrho(x) < r_1$. Hence, for all $x \in \hat{\Omega} \cup \partial\Omega$,

$$L\Psi(x) \geq \alpha_0 - C_1\Psi(x) \quad (C_1 \text{ positive constant}). \tag{6.14}$$

Notice also that for all $x \in \hat{\Omega} \cup \partial\Omega$,

$$L\Psi(x) \leq \alpha_1 \quad (\alpha_1 \text{ positive constant}). \tag{6.15}$$

By (6.12) and (6.4),

$$P_x\{\exists t > 0 \text{ such that } \xi(t) \notin \hat{\Omega} \cup \partial\Omega\} = 0 \quad \text{if } x \in \partial\Omega.$$

Hence, if $x \in \partial\Omega$, we can apply Ito's formula to get

$$E_x \Psi(\xi(t)) = \int_0^t E_x[L\Psi(\xi(s))]\,ds. \tag{6.16}$$

Using (6.14)–(6.16) we find that

$$E_x \Psi(\xi(t)) \geq \alpha_1 t,$$

$$E_x \Psi(\xi(t)) \geq \alpha_0 t - C_1 E_x \int_0^t \Psi(\xi(s))\,ds.$$

Hence,
$$\alpha_0 t \leq E_x \Psi(\xi(t)) + \tfrac{1}{2}\alpha_1 C_1 t^2.$$

Consequently
$$\frac{\alpha}{2} t \leq E_x \varrho(\xi(t)), \quad \text{if } 0 < t < t^* \quad (x \in \partial\Omega) \tag{6.17}$$

provided t^* is sufficiently small and α is any positive constant such that $\alpha\Psi(x) \leq \alpha_0 \varrho(x)$ for all $x \in \hat{\Omega}$.

Set
$$\delta_x(t) = P_x(\xi(t) \in \partial\Omega).$$

Then, by (6.17) and Lemma 6.2,

$$\frac{\alpha}{2} t \leq E_x\{\chi_{\xi(t)\in\hat{\Omega}}\, \varrho(\xi(t))\} \leq \{E_x \chi_{\xi(t)\in\hat{\Omega}}\}^{1/2} \{E_x \varrho^2(\xi(t))\}^{1/2} \leq C\{1-\delta_x(t)\}^{1/2} t.$$

It follows that
$$\frac{\alpha}{2C} \leq (1-\delta_x(t))^{1/2},$$

i.e.,
$$\delta_x(t) \leq \delta = 1 - \frac{\alpha^2}{4C^2} < 1 \quad \text{if } 0 < t < t^*. \tag{6.18}$$

By the Markov property, if $t = s + r$ where s, r are positive numbers smaller than t^*,

$$P_x(\xi(t) \in \partial\Omega) = E_x\{\chi_{\xi(s)\in\partial\Omega} P_{\xi(s)}(\xi(r) \in \partial\Omega)\} + E_x\{\chi_{\xi(s)\in\hat{\Omega}} P_{\xi(s)}(\xi(r) \in \partial\Omega)\}.$$

The second term vanishes, by (6.4). Applying (6.18) to use the first term, we get

$$P_x(\xi(t) \in \partial\Omega) \leq \delta E_x\{\chi_{\xi(s)\in\partial\Omega}\} = \delta P_x(\xi(s) \in \partial\Omega) \leq \delta^2.$$

Similarly,
$$P_x(\xi(t) \in \partial\Omega) \leq \delta^m$$

for any m, if $t < t^* m$. Taking $m \to \infty$, the assertion (6.13) follows.

We shall now establish a relation between the functions $K(x, t, \xi)$ and $G(x, t, \xi)$, $G_0(x, t, \xi)$

THEOREM 6.4. *If* (A), (B_S), (C'), (3.4) *and* (5.1), (5.3) *hold, then*

$$K(x, t, \xi) = G(x, t, \xi) \quad if\ x \in \hat{\Omega},\ \xi \in \hat{\Omega},\ t > 0. \tag{6.19}$$

If (A), (B_S), (C), (3.4) *and* (5.1), (5.3) *hold, then*

$$K(x, t, \xi) = G_0(x, t, \xi) \quad if\ x \in \Omega_0,\ \xi \in \Omega_0,\ t > 0. \tag{6.20}$$

The function G was constructed in section 2, and the function G_0 was defined at the end of section 2.

Proof. Let $f(x)$ be a continuous nonnegative function with support in a compact Borel set A, $A \subset \hat{\Omega}$. Choose m so large that $A \subset N_m$, and consider the function

$$u_m(x, t) = \int_A G_m(x, t, \xi) f(\xi) d\xi \tag{6.21}$$

It satisfies:
$$Lu_m - \frac{\partial u_m}{\partial t} = 0 \qquad \text{if } x \in N_m,\ t > 0,$$

$$u_m(x, 0) = f(x) \quad \text{if } x \in N_m,$$

$$u_m(x, t) = 0 \qquad \text{if } x \in \partial N_m,\ t > 0.$$

Using Ito's formula, we get

$$u_m(x, t) = E_x\{u(\xi(\tau_m), t - \tau_m)\} = E_x\{f(\xi(\tau_m))\chi_{\tau_m = t}\}$$

where τ_m is the first time the process $(s, \xi(s))$ hits the set $\{\partial N_m \times (0, t)\} \cup \{N_m \times \{t\}\}$. If (C') holds then (6.4) holds, so that $\tau_m \to t$ a.s. as $m \to \infty$. Hence

$$\lim_{m\to\infty} u_m(x, t) = E_x f(\xi(t)) = \int_A K(x, t, \xi)) f(\xi) d\xi,$$

by Lemma 5.1. Since on the other hand, by (6.21),

$$\lim_{m\to\infty} u_m(x, t) = \int_A G(x, t, \xi) f(\xi) d\xi,$$

the assertion (6.19) holds. The proof of (6.20) is similar.

THEOREM 6.5. *If* (A), (B_S), (C'), (3.4) *and* (5.1), (5.3) *hold, then*

$$K(x, t, \xi) = 0 \quad if\ x \in \hat{\Omega},\ \xi \in \Omega_0. \tag{6.22}$$

If (A), (B_S), (C), (3.4) *and* (5.1), (5.3) *hold, then*

$$K(x, t, \xi) = 0 \quad if\ x \in \Omega_0,\ \xi \in \hat{\Omega}. \tag{6.23}$$

Indeed, this follows from Lemma 5.1 and (6.4) (when (C′) holds), (6.5) (when (C) holds).

7. Construction of generalized fundamental solution in case of two-sided obstacle

We consider in this section the case where $\partial\Omega$ is a two-sided obstacle, i.e., (C*) holds. We shall also assume:

(D) Denote by L_i the restriction of the elliptic operator L of the manifold ∂G_i, $k_0+1\leqslant i\leqslant k$. Then, each L_i is elliptic on ∂G_i.

Thus, in local coordinates $\theta_1, .., \theta_{\lambda-1}$ of ∂G_i,

$$L_i=\sum_{\lambda,\mu=1}^{n-1}\alpha^i_{\lambda\mu}(\theta)\frac{\partial^2}{\partial\theta_\lambda\partial\theta_\mu}+\sum_{\lambda=1}^{n-1}\beta^i_\lambda\frac{\partial}{\partial\theta_\lambda}$$

and the $(n-1)\times(n-1)$ matrix $(\alpha^i_{\lambda\mu}(\theta))$ is positive definite for each θ.

Denote by $\hat{K}_i(x, t, \xi)$ the fundamental solution of L_i for the cylinder $\partial G_i\times(0, \infty)$. Its existence is well known (see, for instance, [9]). For $x\in\partial G_i$, denote by $K_i(x, t, d\xi)$ $(k_0+1\leqslant i\leqslant k)$ the measure supported on ∂G_i with density $\hat{K}_i(x, t, \xi)\,dS^i_\xi$, where dS^i_ξ is the surface element on ∂G_i. For $1\leqslant i\leqslant k_0$, let

$$K_i(z_i, t, d\xi)=\text{the Dirac measure concentrated at } \xi=z_i.$$

Now define $K(x, t, \xi)=0$ if $x\notin\partial\Omega$, $\xi\in\partial\Omega$, $t>0$, and set

$$\Gamma(x,t,d\xi)\begin{cases}K(x,t,\xi)\,d\xi & \text{if } x\notin\partial\Omega,\ t>0,\\ K_i(x,t,d\xi) & \text{if } x\in\partial G_i,\ t>0\quad (k_0+1\leqslant i\leqslant k),\\ K_i(z_i,t,d\xi) & \text{if } t>0 \qquad\qquad (1\leqslant i\leqslant k_0).\end{cases}\tag{7.1}$$

In view of Theorems 6.4 and 6.5,

$$\Gamma(x,t,d\xi)=\begin{cases}G(x,t,\xi)\,d\xi & \text{if } x\in\hat{\Omega},\ \xi\in\hat{\Omega},\ t>0,\\ G_0(x,t,\xi)\,d\xi & \text{if } x\in\Omega_0,\ \xi\in\Omega_0,\ t>0,\\ 0 & \text{if } x\in\hat{\Omega},\ \xi\in\Omega_0,\ t>0 \text{ or } x\in\Omega_0,\ \xi\in\hat{\Omega},\ t>0.\end{cases}$$

THEOREM 7.1. *Let* (A), (B_S), (C*), (3.4), (D) *and* (5.1), (5.3) *hold. Then, for any Borel set A in R^n,*

$$E_x(\xi(t)\in A)=\int_A\Gamma(x,t,d\xi).\tag{7.2}$$

Definition. $\Gamma(x, t, d\xi)$ is called *the generalized fundamental solution* for (1.1).

For $x\notin\partial\Omega$, it is given by $K(x, t, \xi)\,d\xi$, and for $x\in\partial\Omega$ it is a certain measure supported on $\partial\Omega$.

Proof of Theorem 7.1. Consider first the case where $x \notin \partial\Omega$. If $A \cap (\partial\Omega) = \emptyset$ then (7.2) is a consequence of Theorem 5.1. If $A \subset \partial\Omega$ then both sides of (7.2) vanish. The truth of (7.2) for any Borel set A follows from the preceding special cases, upon writing $A = (A \cap \partial\Omega) \cup (A \setminus \partial\Omega)$.

Consider next the case where $x \in \partial\Omega$. If $x \in \partial G_j$ and $1 \leqslant j \leqslant k_0$, then $x = z_j$ and, by the definition of Γ,

$$\int_A \Gamma(z_j, t, d\xi) = \begin{cases} 1 & \text{if } z_j \in A, \\ 0 & \text{if } z_j \notin A. \end{cases}$$

On the other hand, by Lemma 6.1,

$$E_{z_j}(\xi(t) \in (A) = \begin{cases} 1 & \text{if } z_j \in A, \\ 0 & \text{if } z_j \notin A. \end{cases}$$

Thus (7.2) follows. If $x \in \partial G_j$ and $k_0 + 1 \leqslant j \leqslant k$, then by Lemma 6.1 $\xi(t)$ remains on ∂G_j for all $t > 0$. Let

$$\hat{u}(x, t) = \int_{\partial G_j} \hat{K}_j(x, t, y) f(y) dS_y^j, \ f \text{ continuous } (x \in \partial G_j),$$

and extend $\hat{u}$ into a neighborhood of ∂G_j by defining it as constant along normals. Applying Ito's formula to $\hat{u}(\xi(s), t-s)$ and taking E_x, where $x \in \partial G_j$, we find that

$$E_x f(\xi(t)) = \int_{\partial G_j} \hat{K}_j(x, t, y) f(y)\, dS_y^j.$$

Hence,

$$P_x(\xi(t) \in B) = \int_B \hat{K}_j(x, t, \xi) dS_\xi^j \tag{7.3}$$

for any Borel set B in ∂G_j.

Again, by Lemma 5.1,

$$P_x(\xi(t) \in A) = P_x[\xi(t) \in (A \cap \partial G_j)]$$

for any Borel set A in R^n. Using (7.3) with $B = A \cap \partial G_j$, we get

$$P_x(\xi(t) \in A) \int_{A \cap \partial G_j} \hat{K}_j(x, t, \xi) dS_\xi^j = \int_A \Gamma(x, t, d\xi)$$

where the definition of Γ has been used in the last step. We have thus completed the proof of the theorem.

Remark 1. The estimates derived in section 2 for the functions G, G_0 are, by Theorem 6.4, estimates on Γ.

Remark 2. We have assumed in Theorem 7.1 that the L_i $(k_0 + 1 \leqslant i \leqslant k)$ are non-degenerate elliptic operators on ∂G_i. Suppose now that a particular L_i degenerates along a C^3

$(n-2)$-dimensional manifold Δ, $\Delta \subset G_i$, and that Δ is a two-sided obstacle. Then we can analyze the generalized fundamental solution $\hat{K}_i$ on ∂G_i by the same procedure as in Theorem 7.1. Thus, if the restriction of L_i to Δ is non-degenerate, then $\hat{K}_i(x, t, d\xi)$ will be (on ∂G_i) of the form $K_i(x, t, \xi)\, dS^i_\xi$ if $x \notin \Delta$; for $x \in \Delta$ it is given by some measure supported on Δ. (If Δ consists of one point z then this measure is the Dirac measure concentrated at z.) If the restriction of L_i to Δ is degenerate on an $(n-2)$-dimensional manifold then we can further explore the situation by the method of Theorem 7.1. Thus, in general, the measure $\hat{K}_i$ may consist of densities distributed on submanifolds of ∂G_i of any dimension l, $0 \leq l \leq n-2$.

Remark 3. For any $\delta > 0$, denote by V^δ the δ-neighborhood of $\partial\Omega$. If $x \in \partial\Omega$,

$$\lim_{\varepsilon\to 0} \int_{V_\delta} K_\zeta(x, t, \xi)\, d\xi = \lim_{\varepsilon\to 0} P_x(\xi^\varepsilon(t) \in V_\delta) = P_x(\xi(t) \in \partial\Omega) = 1, \tag{7.4}$$

where (5.15) and Lemma 6.1 have been used. This implies that, for any $\alpha > 0$,

$$\sup_{\xi\in V_\delta} \{K_\varepsilon(x, t, \xi)\, [\mathrm{dist.}(\xi, \partial\Omega)]^{1-\alpha}\} \to \infty \quad \text{if } \varepsilon \to 0; \tag{7.5}$$

for, otherwise, the left-hand side of (7.4) would converge to 0 as $\varepsilon \to 0$.

8. Existence of fundamental solution in case of strictly one-sided obstacle

We shall now replace the condition (C*) by the condition (C**). We define

$$\Gamma(x, t, \xi) = K(x, t, \xi) \quad \text{if } x \in R^n, t > 0, \xi \notin \partial\Omega. \tag{8.1}$$

For definiteness we also set $\Gamma(x, t, \xi) = 0$ if $x \in R^n, t > 0, \xi \in \partial\Omega$. Notice, by Theorem 6.5, that

$$\Gamma(x, t, \xi) = 0 \quad \text{if } x \in \hat{\Omega}, t > 0, \xi \in \Omega_0.$$

by Theorem 6.4,

$$\Gamma(x, t, \xi) = G(x, t, \xi) \quad \text{if } x \in \hat{\Omega}, t > 0, \xi \in \hat{\Omega}.$$

Thus, the boundary estimates derived in section 3 apply to Γ.

THEOREM 8.1. *Let* (A), (B_S), (C**), (3.4) *and* (5.1), (5.3) *hold. Then* $\Gamma(x, t, \xi)$ *is the fundamental solution of the parabolic equation* (1.1).

Proof. We have to verify the relation

$$P_x(\xi(t) \in A) = \int_A K(x, t, \xi)\, d\xi \tag{8.2}$$

for any Borel set A. Consider first the case where $x \notin \partial\Omega$. For any $\delta > 0$, let V_δ be the δ-neighborhood of $\partial\Omega$.

If δ is sufficiently small, then $x \notin V_\delta$. Using Theorem 5.3, we get

$$\int_{A \cap V_\delta} K_\varepsilon(x, t, \xi)\, d\xi \leqslant C \int_{A \cap V_\delta} d\xi \leqslant C\delta,$$

$$\int_{A \cap V_\delta} K(x, t, \xi) d\xi \leqslant C\delta.$$

Recalling that for each δ fixed,

$$\int_{A \setminus V_\delta} K_\varepsilon(x, t, \xi)\, d\xi \to \int_{A \setminus V_\delta} K(x, t, \xi)\, d\xi \quad \text{if } \varepsilon \to 0,$$

we conclude that

$$\int_A K_\varepsilon(x, t, \xi)\, d\xi \to \int_A K(x, t, \xi)\, d\xi \quad \text{if } \varepsilon \to 0. \tag{8.3}$$

Using the estimate (5.21) of Theorem 5.3 and the estimate (2.13), we can argue as in the proof of (5.17) to deduce the relation

$$P_x(\xi^\varepsilon(t) \in A) \to P_x(\xi(t) \in A) \quad \text{if } \varepsilon \to 0 \tag{8.4}$$

provided A is a ball. Taking $\varepsilon \to 0$ in (5.15) and using (8.3), (8.4), the relation (8.2) follows in case A is a ball. This relation is therefore valid also for any Borel set A.

Consider next the case where $x \in \partial\Omega$. By Theorem 5.2,

$$\int_{A \setminus V_\delta} K_\varepsilon(x, t, \xi)\, d\xi \to \int_{A \setminus V_\delta} K(x, t, \xi)\, d\xi \quad \text{if } \varepsilon \to 0. \tag{8.5}$$

Suppose A is a ball. By Theorem 5.2, $K_\varepsilon(x, t, \xi) \leqslant C$ if ξ belongs to a small neighborhood of $A \setminus V_\delta$. Hence, the argument used to prove (5.17) can be applied also here to deduce that

$$P_x(\xi^\varepsilon(t) \in A \setminus V_\delta) \to P_x(\xi(t) \in A \setminus V_\delta) \quad \text{if } \varepsilon \to 0 \tag{8.6}$$

Taking $\varepsilon \to 0$ in (5.15) (with A replaced by $A \setminus V_\delta$) and using (8.5), (8.6), we get

$$P_x(\xi(t) \in A \setminus V_\delta) = \int_{A \setminus V_\delta} K(x, t, \xi) d\xi \tag{8.7}$$

for any $\delta > 0$. Since $K(x, t, \xi) \geqslant 0$ for all ξ, the monotone convergence theorem yields

$$\lim_{\delta \to 0} \int_{A \setminus V_\delta} K(x, t, \xi)\, d\xi = \int_A K(x, t, \xi)\, d\xi. \tag{8.8}$$

Using Theorem 6.2 we also have

$$\lim_{\delta \to 0} P_x(\xi(t) \in A \setminus V_\delta) = P_x(\xi(t) \in A \setminus \partial\Omega) = P_x(\xi(t) \in A). \tag{8.9}$$

Taking $\delta \to 0$ in (8.7) and using (8.8), (8.9), the assertion (8.2) follows in case A is a ball. But then (8.2) clearly holds also for any Borel set A.

Remark 1. From Theorem 6.2 and (8.2) it follows that

$$K(x, t, \xi) = 0 \quad \text{if } x \in \partial\Omega,\ t > 0,\ \xi \in \Omega. \tag{8.10}$$

From Theorem 6.2, $P_x(\xi(t) \in \hat{\Omega}) = 1$ if $x \in \partial\Omega$. Hence, by the strong maximum principle [3], $K(x, t, \xi) > 0$ if $x \in \partial\Omega$, $t > 0$, $\xi \in \hat{\Omega}$. If A is a closed ball in $\hat{\Omega}$, and A' is a closed ball in the interior of A, then (cf. the proof of Lemma 10.2)

$$\varliminf_{x \in \Omega,\, x \to y} P_x(\xi(t) \in A) \geqslant P_y(\xi(t) \in A') = \int_{A'} K(y, t, \xi)\, d\xi > 0$$

if $y \in \partial\Omega$. It follows that

$$P_x(\xi(t) \in A) > 0 \quad \text{if } x \in \Omega,\ \text{dist.}\ (x, \partial\Omega) < \varepsilon_0$$

for some ε_0 small. Applying the strong maximum principle to $\int_A K(x, t, \xi)\, d\xi$, as a function of (x, t), we conclude that

$$\int_A K(x, t, \xi)\, d\xi > 0 \quad \text{if } x \in \Omega,\ t > 0.$$

Applying once more the maximum principle, to $K(x, t, \xi)$ as a function of (ξ, t), we conclude that

$$K(x, t, \xi) > 0 \quad \text{if } x \in \Omega,\ t > 0,\ \xi \in \hat{\Omega}. \tag{8.11}$$

Remark 2. Theorem 8.1 extends without difficulty to the case where the condition (C^{**}) is replaced by the more general condition where the inequality (6.3) holds for $j = 1, \ldots, l$ and the reverse inequality holds for $j = l+1, \ldots, k$. In case $n = 1$ we can just assume that each G_i consists of one point z_i and either $a(z_i) = 0$, $b(z_i) > 0$ or $a(z_i) = 0$, $b(z_i) < 0$.

Remark 3. One can easily combine cases of strictly one-sided obstacles with two-sided obstacles. Thus, if ∂G_i is a strictly one-sided obstacle with respect to either G_i or $R^n \setminus G_i$, for $i = 1, \ldots, h$, and if $G_{h+1}, \ldots, G_k$ are two-sided obstacles, then (7.2) holds with Γ define das follows:

$$\Gamma(x, t, d\xi) = \begin{cases} K(x, t, \varepsilon)\, d\xi & \text{if } x \notin \bigcup_{i=h+1}^{k} \partial G_i, \\ \Gamma_i(x, t, d\xi) & \text{if } x \in \partial G_i \quad (h+1 \leqslant i \leqslant k) \end{cases}$$

where Γ_i is a measure defined as in Theorem 7.1.

Remark 4. Remark 2 following the proof of Theorem 7.1 extends to the case that L_t degenerates on Δ and Δ is strictly one-sided obstacle for L_t.

Remark 5. Theorem 8.1 extends to the case where S is any compact subset of R^n such that

$$P_x\{\xi(t)\in S\}=0 \quad \text{for all } x\in R^n,\ t>0. \tag{8.12}$$

Let S be a C^1 manifold of dimension k $(0\leqslant k\leqslant n-1)$, and denote by $d(x)(x\in S)$ the rank of the linear operator $(a_{ij}(x))$ restricted to the linear space normal to S at x. By Theorem 3.1 of [5], if

$$d(x)\geqslant 3 \quad \text{for all } x\in S \tag{8.13}$$

then (8.12) holds for all $x\notin S$. We claim that (8.12) holds also for $x\in S$. To prove it note, by Theorem 3.1 of [5], that

$$P_x\{\xi(t)\in S\setminus V_\delta\}=0 \quad \text{if } t>0,$$

for any δ-neighborhood V_δ of x. Hence $P_x(\xi(t)\in S\setminus\{x\})=0$. Thus, it remains to prove that

$$P_x\{\xi(t)=x\}=0 \quad \text{if } t>0 \quad (x\in S). \tag{8.14}$$

Suppose for simplicity that $x=0$. Let $\varrho(x)$ be a function in $C^2(R^n)$ such that

$$\varrho(x)=\begin{cases}|x|^2 & \text{if } |x| \text{ is small,}\\ 1 & \text{if } |x| \text{ is large,}\end{cases}$$

and $\varrho(x)>0$ if $x\neq 0$. Since $\Sigma a_{ii}(0)>0$,

$$\gamma_0-C_0\varrho(x)\leqslant L\varrho(x)\leqslant\gamma_1 \quad (x\in R^n) \tag{8.15}$$

where γ_0, C_0, γ_1 are positive constants. By Ito's formula,

$$E_0\varrho(\xi(t))=E_0\int_0^t L\varrho(\xi(s))\,ds\leqslant\gamma_1 t,$$

$$E_0\varrho(\xi(t))=E_0\int_0^t L\varrho(\xi(s))\,ds\geqslant\gamma_0 t-C_0E_0\int_0^t\varrho(\xi(s))\,ds.$$

Hence
$$\gamma_0 t\leqslant E_0\varrho(\xi(t))+C_0\int_0^t\gamma_1 s\,ds=E_0\varrho(\xi(t))+\tfrac{1}{2}C_0\gamma_1t^2.$$

It follows that
$$\gamma' t\leqslant E_0\varrho(\xi(t)) \quad (\gamma' \text{ positive constant})$$

if t is sufficiently small, say $t\leqslant t^*$. Hence

$$\gamma t\leqslant E_0|\xi(t)|^2 \quad \text{if } t\leqslant t^* \quad (\gamma \text{ positive constant}). \tag{8.16}$$

Setting $\delta_x(t)=P_x(\xi(t)=0)$, we then have

$$\gamma t\leqslant E_0\{\chi_{\xi(t)\neq 0}|\xi(t)|^2\}\leqslant\{E_0\chi_{\xi(t)\neq 0}\}^{1/2}\{E_0|\xi(t)|^4\}^{1/2}\leqslant C\{1-\delta_0(t)\}^{1/2}\,t,$$

since $E_0|\xi(t)|^4 \leqslant Ct^2$. Hence

$$\delta_0(t) \leqslant \delta < 1 \quad \text{if } 0 < t < t^* \quad (\delta \text{ constant}).$$

We can now proceed to establish (8.14) by the argument following (6.18).

The assertion (8.12) can be proved also in cases where $d(y) \geqslant 2$ for all $y \in S$. For $x \notin S$, one applies Theorems 4.1, 4.2 of [5]. If $x \in S$, we cannot reduce the proof of (8.12) to that of proving (8.14) as before; instead, we proceed directly to prove (8.12) by the argument used to prove (8.14), employing the function

$$\tilde{\varrho}(\xi) = \varrho(\text{dist.}\ (\xi, S))$$

instead of $\varrho(\xi)$. Note that also ϱ satisfies the differential inequalities of (8.15).

9. Lower bounds on the fundamental solution

In Theorem 3.1 we have derived the bound

$$G(x, t, \xi) \leqslant C \exp\left\{-\frac{c}{t}(\log R(x))^2\right\} \quad (C>0, c>0) \tag{9.1}$$

if ξ varies in a compact set E of $\hat{\Omega}$, $0<t<T$, $x \in \hat{\Omega}$, and $R(x)$ is sufficiently small. Recall that the condition (C) was assumed in that theorem.

We shall now assume that the condition (C') holds and that

$$\sum_{i,j=1}^{n} a_{ij}(x) R_{x_i} R_{x_j} \geqslant \alpha R^2 \quad (\alpha \text{ positive constant}) \tag{9.2}$$

for all x in some $\hat{\Omega}$-neighborhood of $\partial\Omega$, where $R(x) = \text{dist.}\ (x, \partial\Omega)$. We shall then derive the estimate

$$G(x, t, \xi) \geqslant N \exp\left\{-\frac{\nu}{t}(\log R(x))^2\right\} \quad (N>0, \nu>0) \tag{9.3}$$

for some positive constants N, ν, for all $\xi \in E$, $0<t<T$, $x \in \hat{\Omega}$, provided $R(x)$ is sufficiently small.

To do this, we compare (for fixed $\xi \in E$) the function

$$v(x,t) = G(x, t, \xi) \quad (x \in \hat{\Omega},\ 0 < R(x) < \varepsilon,\ 0<t<T)$$

with a function $w(x, t)$ of the form

$$w(x,t) = N \exp\left\{-\frac{\nu}{t}(\log R(x))^2\right\},$$

where ε is sufficiently small, N is sufficiently small, and ν is sufficiently large. We fix ε such that $\varepsilon < 1$, dist. $(x, \xi) \geqslant c_0 > 0$ if $\xi \in E$, $x \in \hat{\Omega}$ and $R(x) > \varepsilon$, and such that $R(x)$ is in C^2

if $x\in\hat{\Omega}$, $R(x)<\varepsilon$. Fix m so large that N_m (defined in section 3) contains the set where $x\in\hat{\Omega}$, $R(x)=\varepsilon$. By [0],

$$G_m(x, t, \xi)>w(x, t) \quad \text{if } x\in\hat{\Omega},\ R(x)=\varepsilon,\ 0<t<T$$

provided N is sufficiently small and ν is sufficiently large.

Since $G(x, t, \xi)\geqslant G_m(x, t, \xi)$, we have

$$v(x, t)>w(x, t) \quad \text{if } x\in\hat{\Omega},\ R(x)=\varepsilon,\ 0<t<T.$$

Notice also that

$$v(x, 0)=w(x, 0)=0 \quad \text{if } x\in\hat{\Omega},\ 0<R(x)<\varepsilon,$$

$$\varliminf_{R(x)\to 0}\,[v(x, t)-w(x, t)]=\varliminf_{R(x)\to 0} v(x, t)\geqslant 0 \quad \text{if } 0<t<T.$$

Hence, if

$$Lw-w_t>0 \quad \text{for } x\in\hat{\Omega},\ 0<R(x)<\varepsilon,\ 0<t<T, \tag{9.4}$$

then the maximum principle can be applied; it yields the assertion (9.3). Now, the left-hand side of (9.4) can be expressed by (3.19) with $\gamma=\nu$. Since, by (C′), $\mathcal{B}/R>-C$, it is clear that if ν is sufficiently large, then the first term on the right-hand side (with $\gamma=\nu$) dominates the negative contribution of each of the remaining terms. Thus (9.4) holds.

Similarly one can prove that, when (9.2) and the condition (C) hold,

$$G(x, t, \xi)\geqslant N\exp\left\{-\frac{\nu}{t}(\log R(\xi))^2\right\} \quad (N>0, \nu>0) \tag{9.5}$$

provided $x\in E$, $0<t<T$, $\xi\in\hat{\Omega}$, $R(\xi)<\varepsilon$. We can thus state:

THEOREM 9.1. *Let* (A), (B_S), (C′), (3.4) *and* (9.2) *hold. Let E be any compact subset of $\hat{\Omega}$. Then, for any $T>0$ and for any $\varrho>0$ sufficiently small, there are positive constants N, ν such that* (9.3) *holds if $\xi\in E$, $x\in\hat{\Omega}$, $R(x)<\varrho$, $0<t<T$. If the condition* (C′) *is replaced by the condition* (C), *then* (9.5) *holds for $x\in E$, $\xi\in\hat{\Omega}$, $R(\xi)<\varrho$, $0<t<T$.*

If the condition (9.2) is replaced by the weaker condition

$$\sum a_{ij}(x)R_{x_i}R_{x_j}\geqslant \alpha R^{p+1} \quad (\alpha>0, p>1) \tag{9.6}$$

for all x in some $\hat{\Omega}$-neighborhood of $\partial\Omega$, then we can establish, instead of (9.3), (9.5), the inequalities

$$G(x, t, \xi)\geqslant N\exp\left\{-\frac{\nu}{t}(R(x))^{1-p}\right\},$$

$$G(x, t, \xi)\geqslant N\exp\left\{-\frac{\nu}{t}(R(\xi))^{1-p}\right\}$$

respectively (for x, t, ξ in the same sets as before).

Finally, lower bounds at ∞, supplementary to the upper bounds derived in section 4, can also be obtained using the above comparison function $w(x)$ with $R(x)=|x|$, or, more generally, with $R(x)=|\mathcal{M}x|$ where $\mathcal{M}$ is an affine matrix.

10. The Cauchy problem

Consider the Cauchy problem

$$Lu-u_t=0 \quad \text{if } x\in R^n, t>0, \tag{10.1}$$

$$u(x, 0)=f(x) \quad \text{if } x\in R^n,$$

where $f(x)$ is a bounded Borel measurable function. We define the solution of this problem to be the function

$$u(x, t)=E_x f(\xi(t)). \tag{10.2}$$

When the matrix $(a_{ij}(x))$ is positive definite and $f(x)$ is continuous, the function $u(x, t)$ is a classical solution of the Cauchy problem (see section 5).

The purpose of this section is to investigate the continuity of $u(x, t)$ when $(a_{ij}(x))$ is degenerate and f is continuous or just measurable.

THEOREM 10.1. *Let σ_{ij}, b_i be uniformly Lipschitz continuous in compact subsets of R^n and let* (5.2), (5.3) *hold. If $f(x)$ is bounded continuous function, then $u(x, t)$ is continuous in $(x, t)\in R^n\times[0, \infty)$, and $u(x, 0)=f(x)$.*

Proof. It is well known [8] that

$$E|\xi_y(t)-\xi_x(s)|^2\leqslant\eta(|x-y|^2+|t-s|) \quad (\eta(r)\to 0 \quad \text{if } r\to 0) \tag{10.3}$$

where $\xi_z(t)$ is the solution $\xi(t)$ of (5.4) with $\xi_z(0)=z$. Hence, by the Lebesgue bounded convergence theorem,

$$Ef(\xi_y(t))\to Ef(\xi_x(s)) \quad \text{if } x\to y, t\to s.$$

This proves the continuity of $u(y, t)$ at (x, s); $x\in R^n$, $s\geqslant 0$. Notice that $u(x, 0)=E_x f(\xi(0))=f(x)$.

We now consider the more general case where $f(x)$ is Borel measurable. When (a_{ij}) is uniformly positive definite and a fundamental solution $\Gamma(x, t, \xi)$ can be constructed by the parametrix method [3], the solution of the Cauchy problem can be written in the form

$$\int\Gamma(x, t, \xi)f(\xi)\,d\xi;$$

one can then show (using continuity properties of Γ) that this solution is continuous in (x, t) in $R^n \times (0, \infty)$. We shall prove here a similar result in case (a_{ij}) is degenerate.

LEMMA 10.2. *Let σ_{ij}, b_i be uniformly Lipschitz continuous in compact subsets of R^n and let* (5.2), (5.3) *hold. Let A be a bounded domain with C^1 boundary and suppose that $P_x(\xi(s) \in \partial A) = 0$ for some $x \in R^n$, $s > 0$. Then the function*

$$(y, t) \to P_y(\xi(t) \in A)$$

is continuous at the point $(y, t) = (x, s)$.

Proof. From (10.3) it follows that

$$\overline{\lim_{y \to x,\, t \to s}} P_y(\xi(t(\in A) \leqslant P_x(\xi(s) \in A_\delta) \quad \text{for any } \delta > 0,$$

where A_δ is a δ-neighborhood of A. Taking $\delta \to 0$, we get

$$\overline{\lim_{y \to x,\, t \to s}} P_y(\xi(t) \in A) \geqslant P_x(\xi(s) \in A \cup \partial A) = P_x(\xi(s) \in A).$$

Similarly,

$$\underline{\lim_{y \to x,\, t \to s}} P_y(\xi(t \in A) \geqslant P_x(\xi(s) \in A)$$

and the proof is complete.

THEOREM 10.3. *Let $f(x)$ be a bounded Borel measurable function in R^n, and let* (4.6) *and the assumptions of Theorem* 8.1 *hold. Then the solution $u(x, t)$ is continuous in $(x, t) \in R^n \times (0, \infty)$.*

Proof. If A is as in Lemma 10.2 then, by Theorem 8.1,

$$P_x(\xi(t) \in \partial A) = \int_{\partial A} K(x, t, \xi)\, d\xi = 0 \quad (t > 0).$$

Thus, by Lemma 10.2, the function

$$(x, t) \to P_x(\xi(t) \in A) \quad \text{is continuous in } R^n \times (0, \infty). \tag{10.4}$$

Consider now the special case where f has compact support. For any $\varepsilon > 0$, let $g(x)$ be a simple function such that

$$\sup |g| \leqslant 1 + \sup |f|,$$

$g(x) \equiv \alpha_1$ (α_i constant) if $x \in A_i$ $(1 \leqslant i \leqslant l)$, $A_i \cap A_j = \varnothing$ if $i \neq j$, $\bigcup_{j=1}^{l} A_i$ contains the support of f, each A_i is bounded, $g(x) = 0$ if $x \notin \bigcup_{i=1}^{l} A_i$ and $|f(x) - g(x)| < \varepsilon$ almost everywhere. Let B_i be bounded domains with C^1 boundary such that $B_i \supset A_i$ and the Lebesgue measure of $\bigcup_{i=1}^{l} (B_i \backslash A_i)$ is less than ε.

Then, for all (x, t), (x', t'),

$$\left| \int_{R^n} K(x,t,\xi) g(\xi)\, d\xi \int_{R^n} K(x,t,\xi) f(\xi)\, d\xi \right| < \varepsilon,$$

$$\left| \int_{R^n} K(x',t',\xi) g(\xi)\, d\xi - \int_{R^n} K(x',t',\xi) f(\xi)\, d\xi \right| < \varepsilon.$$

Further, if $(x',t') \to (x,t)$, $t>0$,

$$\overline{\lim} \left| \int_{R^n} K(x',t',\xi) g(\xi)\, d\xi - \int_{R^n} K(x,t,\xi) g(\xi)\, d\xi \right|$$
$$\leqslant (1+\sup|f|) \left\{ \overline{\lim} \int_E K(x',t',\xi)\, d\xi + \int_E K(x,t,\xi)\, d\xi \right.$$

by (10,4), where $E = \bigcup_{i=1}^{l} (B_i \backslash A_i)$. From the proof of Lemma 10.2,

$$\overline{\lim} \int_E K(x',t',\xi)\, d\xi \leqslant \int_{E\delta} K(x,t,\xi)\, d\xi$$

where E_δ is any δ-neighborhood of E.

Putting these estimates together, we conclude that if $(x',t') \to (x,t)$, $t>0$, then

$$\overline{\lim} \,|u(x',t') - u(u,t)| \leqslant 2\varepsilon + 2(1+\sup|f|) \int_{E\delta} K(x,t,\xi)\, d\xi.$$

Since ε and δ are arbitrary, the left-hand side can be made arbitrarily small. Consequently u is continuous at (x,t).

Consider now the general case where f_m is a bounded measurable function. Let

$$f_m(x) = \begin{cases} f(x) & \text{if } |x| < m, \\ 0 & \text{if } |x| > m. \end{cases}$$

Denote the solution of the Cauchy problem corresponding to f_m by u_m. By what we have already proved, each u_m is continuous. By Corollary 4.2, $u_m \to u$ uniformly on compact subsets. Consequently, u is continuous.

Consider next the case of two-sided obstacle, where only a generalized fundamental solution exists. We first take

$$f(x) = \chi_A(x), \tag{10.5}$$

the characteristic function of a set A. We assume:

(E). A is a bounded domain with C^1 boundary, and it intersects precisely one of the sets ∂G_i; further, $k_0+1 \leqslant i \leqslant k$ and the intersection $\partial A \cap \partial G_i$ is a C^1 $(n-2)$-dimensional hypersurface.

THEOREM 10.4. *Let the assumptions of Theorem* 7.1 *and* (10.5), (E) *hold. Then the solution* $u(x,t)$ *is continuous in* $(x,t) \in R^n \times (0,\infty)$.

Proof. It is enough to prove the continuity of $u(y, t)$ at $y \in \partial\Omega$. In view of Lemma 10.2, it suffices to prove that

$$P_y(\xi(t) \in \partial A) = 0 \quad \text{if } y \in \partial G_j, \quad t > 0. \tag{10.6}$$

In view of Theorem 6.1, the left-hand side of (10.6) vanishes if $j \neq i$. If $j = i$, then, by Theorems 6.1, 7.1,

$$P_y(\xi(t) \in \partial A) = P_y\{\xi(t) \in (\partial A \cap G_i)\}$$

$$= \int_{\partial A \cap G_i} \hat{K}_i(x, t, \xi)\, dS_\xi^i = 0.$$

Thus the proof is complete.

Remark 1. If A contains in its interior the point z_i and does not intersect the other sets G_j, $j \neq i$, then the assertion of Theorem 10.4 is again valid.

Remark 2. Theorem 10.4 extends to any measurable function $f(x)$ which can be approximated uniformly on compact subsets of R^n by simple functions of the form $\Sigma\, c_j \chi_{A_j}$, provided each set A_j is a bounded closed domain, and either $A_j \cap \partial\Omega = \emptyset$, or A_j satisfied the condition (E), or A_j contains in its interior one point z_i but does not intersect the remaining sets G_l, $l \neq i$. In particular, Theorem 10.4 remains valid for any bounded Borel measurable function $f(x)$ which is continuous at all the points of $\partial\Omega$.

Remark 3. The assertion of Theorem 10.4 is clearly false if $\partial A \cap \partial G_i$ contains a set of positive surface area, or if A consists of one point z_i, $1 \leq i \leq k_0$.

Remark 4. If f is a bounded continuous function in R^n, then $u(x, t)$ is continuous (by Theorem 10.1). Let

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \neq z_i, \\ f_i & \text{if } x = z_i \quad (f_i \neq f(z_i)) \end{cases}$$

for some i, $1 \leq i \leq k_0$. Denote by $\tilde{u}$ the solution corresponding to $\tilde{f}$. Then $\tilde{u}(x, t) = u(x, t)$ if $x \neq z_i$, but

$$\tilde{u}(z_i, t) = f_i \neq f(z_i) = u(z_i, t).$$

Consequently, $u(x, t)$ is discontinuous at the points (z_i, t), $t \geq 0$.

Remark 5. It is easily seen that Theorems 10.1, 10.3 and remark 2 extend to the case where $f(x)$ is assumed to have a polynomial growth.

Remark 6. If S is as in remark 5 at the end of section 8, so that (8.12) holds, then Theorem 10.1 remains valid even if one changes the definition of $f(x)$, in an arbitrary manner, on the set S. Further, the solution $u(x, t)(t > 0)$ does not change when one changes the definition of f on S.

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