ON THE TRACE FORMULA FOR HECKE OPERATORS

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The formula to be proved in this paper has roughly the following form:

$$\operatorname{tr}(\Gamma\alpha\Gamma \mid A_m) - \operatorname{tr}(\Gamma\alpha^{-1}\Gamma \mid B_{2-m}) = \textstyle\sum_C J(C).$$

Here Γ is a discrete subgroup of $SL_2(\mathbf{R})$ such that $SL_2(\mathbf{R})/\Gamma$ is of finite measure, m an arbitrary rational number, A_m the space of cusp forms of weight m with respect to Γ on the upper half complex plane $\mathfrak{H}$, B_{2-m} the space of integral forms of weight $2-m$ with respect to Γ, α an element of $SL_2(\mathbf{R})$ such that Γ and $\alpha^{-1}\Gamma\alpha$ are commensurable, and $J(C)$ a complex number defined for each class C of elements of $\Gamma\alpha\Gamma$ under a certain equivalence. The double cosets $\Gamma\alpha\Gamma$ and $\Gamma\alpha^{-1}\Gamma$ act on A_m and B_{2-m} respectively, under some conditions. An *integral form* of weight m is a holomorphic function $f(z)$ on $\mathfrak{H}$ which satisfies $f(\gamma(z))/f(z) = t(\gamma)(d\gamma(z)/dz)^{-m/2}$ for every $\gamma \in \Gamma$ with a certain constant factor $t(\gamma)$, and which is holomorphic at every cusp; an integral form is called a *cusp form* if it vanishes at every cusp.

If m is an integer >2, then $B_{2-m} = \{0\}$. The formula in this case was obtained by Selberg [5] and Eichler [2]. If $m=2$, B_{2-m} consists of the constants, and therefore $\operatorname{tr}(\Gamma\alpha^{-1}\Gamma \mid B_{2-m})$ is simply the number of right or left cosets in $\Gamma\alpha^{-1}\Gamma$. This case is also included in [2]. It should also be mentioned that the generalized Riemann-Roch theorem of Weil [8] is closely related to the above formula when α belongs to the normalizer of Γ.

Although our formula is given for an arbitrary rational m, the cases of integral and half integral weight with respect to an arithmetic Γ seem most significant. If m is a half integer >2, we have again $B_{2-m} = \{0\}$, and the formula is of the same nature as in the case of integral $m>2$. However, if $m=3/2$, both A_m and B_{2-m} can be non-trivial. Especially if Γ is a congruence subgroup of $SL_2(\mathbf{Z})$, it is conjecturable that $B_{\frac{1}{2}}$ is spanned by theta series of the type

$$\textstyle\sum_n \psi(n) \exp(2\pi i n^2 r z)$$

with a rational number r and a character ψ modulo a positive integer. This is at least true for the groups of low level. In such a case, tr $(\Gamma\alpha\Gamma|A_{3/2})$ is effectively computable.

The non-triviality of both A_m and B_{2-m} occurs also when $m=1$. In this case, however, the formula does not seem to bring forth any new information about the modular forms of weight 1. We can only compute the trace on the space of Eisenstein series, and also rediscover the forms whose Mellin transforms are L-functions of an imaginary quadratic field. (See 5.8 for a more detailed discussion.) Thus our formula for $m=1$ is not so effective in this sense, but it tells at least of what the trace formula should be in the extreme case $m=1$.

To prove the formula, we adapt to our setting the methods of Kappus [4] and Eichler [3], in which the forms of even positive weight were treated. In § 1, we consider automorphic forms in an axiomatic way, and then construct, in § 2, an algebraic analogue of kernel function. We work on the product of two copies of an algebraic curve, while the authors of [3] and [4] considered the composite of two copies of an algebraic function field. Although the theory of §§ 1, 2 as well as a part of later sections seems developable for the curves defined over a field of positive characteristic, our discussion is restricted to the case of characteristic 0, mainly for the sake of simplicity. In § 3, we prove the first formulation of the trace formula, which is algebro-geometric in the sense that the right hand side is expressed in terms of the fixed points of the algebraic correspondence attached to $\Gamma\alpha\Gamma$. A more group-theoretical formulation will be given in § 4. In the final § 5, we make a few remarks and discuss some features peculiar to the cases $m=3/2$ and $m=1$.

One remark, though obvious, may be added: A formula of the same type will undoubtedly be proved for higher dimensional manifolds instead of a curve. For example, we note that if Γ has neither parabolic nor elliptic elements and $\alpha\Gamma\alpha^{-1}=\Gamma$, then the above formula follows directly from the fixed point formula of Atiyah and Bott. Although such a special case is not important from an arithmetical viewpoint, the Atiyah-Bott formula will suggest a plausible form in a more general case. It should also be mentioned that we do not put any emphasis on our choice of method. The framework of the present paper has a natural limitation, while it enables us to obtain a fairly general and practical formula in the one-dimensional case with a relatively small amount of complexity. (At least we have dispensed with any discussion of convergence or limit process.) Therefore any method, either analytic, geometric or group-theoretical, may be adopted on its own merit in proving the higher dimensional generalization.

1. Axioms of automorphic forms

1.1. We fix a "universal domain" Ω of characteristic 0 in the sense of Weil's Foundations [9], and consider algebro-geometric objects rational over subfields of Ω, denoted by k, k', k_0, etc. We take these fields so that Ω has an infinite transcendence degree over them.

Let V be a complete non-singular curve, which will be fixed throughout the first two sections. If V is defined over a field k, we denote by $k(V)$ the field of k-rational functions on V, by $\Phi(k)$ the module of k-rational differential 1-forms on V, and by $D(k)$ the module of all k-rational divisors of V. The unions of $k(V)$, $\Phi(k)$, $D(k)$ for all fields k of rationality for V will be denoted by $\Omega(V)$, Φ, and D, respectively. It is necessary for our purpose to consider divisors with fractional coefficients. Therefore we put $D_{\mathbf{Q}}=D\otimes_{\mathbf{Z}}\mathbf{Q}$, $D_{\mathbf{Q}}(k)=D(k)\otimes_{\mathbf{Z}}\mathbf{Q}$, and $\deg(\Sigma_i c_i x_i)=\Sigma_i c_i$ for $\Sigma_i c_i x_i\in D_{\mathbf{Q}}$ with $c_i\in\mathbf{Q}$, $x_i\in V$. An element of $D_{\mathbf{Q}}$ is called *k-rational* if it belongs to $D_{\mathbf{Q}}(k)$. For $0\neq\alpha\in k(V)$ and $0\neq\omega\in\Phi(k)$, we can define their divisors (which are of course elements of $D(k)$) as usual, and denote them by $\operatorname{div}(\alpha)$ and $\operatorname{div}(\omega)$. Let $P(k)$ denote the set of all k-rational prime divisors of V. For each $p\in P(k)$, we can define a discrete valuation ν_p of $k(V)$ in a natural manner so that $\operatorname{div}(\alpha)=\Sigma_p\nu_p(\alpha)p$. We use the symbol ν_p also for the map $D_{\mathbf{Q}}(k)\to\mathbf{Q}$ defined by $\mathfrak{a}=\Sigma_p\nu_p(\mathfrak{a})p$ for $\mathfrak{a}\in D_{\mathbf{Q}}$, and put $\nu_p(\omega)=\nu_p(\operatorname{div}(\omega))$ for $0\neq\omega\in\Phi(k)$. For $\mathfrak{a}$, $\mathfrak{b}\in D_{\mathbf{Q}}(k)$, we write $\mathfrak{a}\geqslant\mathfrak{b}$ if $\nu_p(\mathfrak{a}-\mathfrak{b})\geqslant 0$ for all $p\in P(k)$.

1.2. To discuss automorphic forms in an axiomatic way, we consider a system $\mathfrak{F}=\{F, F', Z, \mathfrak{z}\}$ formed by the objects satisfying the following axioms (A_{1-4}).

(A_1) *F and F' are one-dimensional vector spaces over $\Omega(V)$.*

(A_2) *To each non-zero element f of F or F', one can assign an element of $D_{\mathbf{Q}}$ denoted by $\operatorname{div}(f)$ satisfying*

$$\operatorname{div}(hf)=\operatorname{div}(h)+\operatorname{div}(f)\quad \textit{for } h\in\Omega(V),\ f\in F \textit{ or } F'.$$

(A_3) *Z is a non-degenerate $\Omega(V)$-bilinear map $F\times F'\to\Phi$.*

(A_4) *$\mathfrak{z}$ is an element of $D_{\mathbf{Q}}$ such that*

$$\operatorname{div}(Z(f,g))=\mathfrak{z}+\operatorname{div}(f)+\operatorname{div}(g)\quad (0\neq f\in F,\ 0\neq g\in F').$$

For $0\neq f\in F$ and $u\in F$, we can define $h=f^{-1}u=uf^{-1}=u/f$ to be the element of $\Omega(V)$ such that $hf=u$. This applies also to the elements of F'.

To make our notation more suggestive, we use a symbol dz instead of Z, and write

$$fg\,dz=Z(f,g)\qquad (f\in F,\ g\in F'),$$
$$\operatorname{div}(dz)=\mathfrak{z}.$$

Then we have $$\operatorname{div}(fg\,dz) = \operatorname{div}(f) + \operatorname{div}(g) + \operatorname{div}(dz).$$

For the moment, dz is merely a symbol replacing Z; it has no meaning as the differential of z until § 3, where we take F and F' to be the modules of automorphic forms of weight m and $2-m$, respectively, and dz as the differential of the variable on the upper half plane.

To define "k-rational elements" of F and F', fix any non-zero element w of F. Let k_0 be a field of rationality for V, div (w), and $\mathfrak{z}$. Take a non-zero element η of $\Phi(k_0)$. There is a uniquely determined element v of F' such that $wv\,dz = \eta$. For any field k of rationality for V containing k_0, put
$$F(k) = k(V)w, \quad F'(k) = k(V)v.$$
Then we see that

(A_5) *Z maps $F(k) \times F'(k)$ onto $\Phi(k)$; div (f) is k-rational if $f \in F(k)$ or $F'(k)$; $F(k_1) = F(k) \otimes_k k_1$, $F'(k_1) = F'(k) \otimes_k k_1$ if $k \subset k_1$.*

Hereafter we fix k_0, and consider only the fields k containing k_0 as basic fields. Such a field k will be called a field of rationality for $\mathfrak{F}$. For each $p \in P(k)$ and $f \in F$ or F', we define a rational number $\nu_p(f)$ by div $(f) = \sum_p \nu_p(f)p$.

A simple example of $\mathfrak{F}$ is obtained by taking $F = \Phi$, $F' = \Omega(X)$, $Z(f, g) = fg$, and $\mathfrak{z} = 0$.

Remark. The modules $F(k)$ and $F'(k)$ depend on the choice of w. Instead of taking w, one could start with (A_5) as an additional axiom.

1.3. Let us now introduce a module $R(k)$, which may be called the module of "$F(k)$-valued adeles" in a weak sense. To be precise, we consider a map $b: P(k) \to F(k)$ which assigns to each $p \in P(k)$ an element b_p of $F(k)$, such that $\nu_p(b_p) \geqslant 0$ for all except a finite number of p's. We denote by $R(k)$ the module of all such b, addition being defined by $(b+c)_p = b_p + c_p$. We write $b = (b_p)$, and call b_p *the p-component of* b. For $a \in k(V)$ and $b \in R(k)$, we can define an element ab of $R(k)$ by $(ab)_p = a \cdot b_p$. Each $c \in F(k)$ defines an element of $R(k)$ whose p-component equals c for every $p \in P(k)$. In this way $F(k)$ can be identified with a submodule of $R(k)$.

Now for $\mathfrak{a} \in D_{\mathbf{Q}}(k)$, put
$$R(\mathfrak{a}, k) = \{b \in R(k) \mid \nu_p(b_p) \geqslant -\nu_p(\mathfrak{a}) \text{ for all } p \in P(k)\},$$
$$F(\mathfrak{a}, k) = \{f \in F(k) \mid \operatorname{div}(f) \geqslant -\mathfrak{a}\}$$
$$= R(\mathfrak{a}, k) \cap F(k),$$
$$F(\mathfrak{a}) = \{f \in F \mid \operatorname{div}(f) \geqslant -\mathfrak{a}\}.$$

Taking F' in place of F, we define $R'(k)$, $R'(\mathfrak{a}, k)$, $F'(\mathfrak{a}, k)$ and $F'(\mathfrak{a})$ in the same manner. Also we put

$$L(\mathfrak{a}, k) = \{f \in k(V) \mid \operatorname{div}(f) \geqslant -\mathfrak{a}\},$$
$$L(\mathfrak{a}) = \{f \in \Omega(V) \mid \operatorname{div}(f) \geqslant -\mathfrak{a}\},$$
$$l(\mathfrak{a}) = \dim L(\mathfrak{a}).$$

Here dim stands for the dimension of a vector space over Ω. If $0 \neq u \in F(k)$, then

$$F(\mathfrak{a}, k)u^{-1} = L(\mathfrak{a} + \operatorname{div}(u), k), \quad F(\mathfrak{a})u^{-1} = L(\mathfrak{a} + \operatorname{div}(u)), \tag{1.3.1}$$

hence $F(\mathfrak{a}, k)$ is finite dimensional over k, and

$$\dim F(\mathfrak{a}) = l(\mathfrak{a} + \operatorname{div}(u)),$$
$$F(\mathfrak{a}) = F(\mathfrak{a}, k) \otimes_k \Omega.$$

1.4. For $p \in P(k)$ and $\omega \in \Phi(k)$, we define the residue of ω at p, denoted by $\operatorname{Res}_p(\omega)$ as follows. Put $p = p_1 + \ldots + p_s$ with points p_i on V. Let $\bar{k}$ be the algebraic closure of k, and t an element of $\bar{k}(V)$ such that $v_{p_i}(t) = 1$. Define $\operatorname{Res}_{p_i}(\omega)$ as usual to be the coefficient of t^{-1} in the power-series expansion of ω/dt in t with coefficients in $\bar{k}$. Then we put

$$\operatorname{Res}_p(\omega) = \textstyle\sum_{i=1}^{s} \operatorname{Res}_{p_i}(\omega).$$

We see easily that $\operatorname{Res}_p(\omega) = \operatorname{Tr}_{k(p_1)/k}(\operatorname{Res}_{p_1}(\omega))$, and as is well known,

$$\textstyle\sum_{p \in P(k)} \operatorname{Res}_p(\omega) = 0 \quad \text{for all } \omega \in \Phi(k).$$

1.5. PROPOSITION. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two elements of $D_{\mathbf{Q}}$ such that*

$$\mathfrak{a} + \mathfrak{b} = \mathfrak{z};\ \mathfrak{a} + \operatorname{div}(f) \in D \text{ for every non-zero } f \in F. \tag{1.5.1}$$

Then $\mathfrak{b} + \operatorname{div}(g) \in D$ for every non-zero $g \in F'$. Moreover, if $\mathfrak{a}$ and $\mathfrak{b}$ are k-rational, the vector space $F(\mathfrak{a}, k)$ is dual to $R'(k)/[R'(\mathfrak{b}, k) + F'(k)]$ by the k-bilinear pairing

$$(f, v) \to \langle f, v \rangle = \textstyle\sum_{p \in P(k)} \operatorname{Res}_p(f v_p dz)$$

for $f \in F(\mathfrak{a}, k)$, $v = (v_p) \in R'(k)$.

Proof. First note that $\mathfrak{a} + \operatorname{div}(f) \in D$ holds for all non-zero $f \in F$ if it holds for at least one f. Now let $0 \neq g \in F'$. Then $fg\,dz \in \Phi$, so that $\operatorname{div}(fg\,dz) \in D$. Subtracting $\mathfrak{a} + \operatorname{div}(f)$ from $\operatorname{div}(fg\,dz)$, one finds $\operatorname{div}(g) + \mathfrak{b} \in D$. Now the duality in the case $F = \Phi$, $F' = \Omega(V)$, $\mathfrak{z} = 0$ is well known. In fact, let $R_0(k)$ (resp. $R_0(\mathfrak{b}, k)$) denote the module $R'(k)$ (resp. $R'(\mathfrak{b}, k)$) defined with $F' = \Omega(V)$. In this special case, we see that $\mathfrak{b} \in D$, and

$$F(\mathfrak{a}, k) = \{\omega \in \Phi(k) \mid \operatorname{div}(\omega) \geqslant \mathfrak{b}\},$$

and this vector space is dual to $R_0(k)/[R_0(\mathfrak{b}, k) + k(V)]$. (See e.g. Chevalley [1], especially

p. 30, Th. 2. See also Weil [8], pp. 58–59, and Eichler [3], p. 177.) In the general case, take any non-zero $w \in F'(k)$. Then

$$F(\mathfrak{a}, k) w\, dz = \{\eta \in \Phi(k) \,|\, \operatorname{div}(\eta) \geqslant \mathfrak{b} + \operatorname{div}(w)\},$$

$$w^{-1} R'(\mathfrak{b}, k) = R_0(\mathfrak{b} + \operatorname{div}(w), k).$$

Further for $f \in F(k)$ and $v \in R'(k)$, we have $\operatorname{Res}_p(f v_p dz) = \operatorname{Res}_p(f w w^{-1} v_p dz)$. Therefore our assertion for F and R' reduces to the above special case.

1.6. Let $\mathfrak{a}$, $\mathfrak{b}$ be as in Proposition 1.5 under the assumption (1.5.1). Let $0 \neq f \in F$, $0 \neq g \in F'$, $\omega = fg\,dz$. Then

$$\mathfrak{a} + \mathfrak{b} + \operatorname{div}(f) + \operatorname{div}(g) = \operatorname{div}(\omega),$$

$$\dim F(\mathfrak{a}) = l(\mathfrak{a} + \operatorname{div}(f)),$$

$$\dim F'(\mathfrak{b}) = l(\mathfrak{b} + \operatorname{div}(g)) = l(\operatorname{div}(\omega) - \mathfrak{a} - \operatorname{div}(f)),$$

hence by the Riemann-Roch theorem, we obtain

$$\dim F(\mathfrak{a}) - \dim F'(\mathfrak{b}) = \deg(\mathfrak{a} + \operatorname{div}(f)) - \mathfrak{g} + 1, \tag{1.6.1}$$

where $\mathfrak{g}$ is the genus of V.

1.7. Let k_1 be an extension (either algebraic or transcendental) of k. Then we can embed $R'(k)$ into $R'(k_1)$ as follows. To each $b = (b_p) \in R'(k)$, we assign $b^* = (b_q^*) \in R'(k_1)$ by

$$b_q^* = \begin{cases} b_p & \text{if } q \leqslant p, \\ 0 & \text{if } q \leqslant p \quad \text{for no } p \in P(k). \end{cases}$$

This embedding maps $R'(\mathfrak{b}, k) + F'(k)$ into $R'(\mathfrak{b}, k_1) + F'(k_1)$. (Note that $F'(k)$ is not necessarily mapped into $F'(k_1)$.) Further it maps $R'(k)/[R'(\mathfrak{b}, k) + F'(k)]$ injectively into $R'(k_1)/[R'(\mathfrak{b}, k_1) + F'(k_1)]$, and the latter can be identified with the tensor product of the former with k_1 over k. This is also compatible with the duality with $F(\mathfrak{a}, k_1) = F(\mathfrak{a}, k) \otimes_k k_1$ explained in Proposition 1.5.

1.8. Let $\mathfrak{a}$ and $\mathfrak{b}$ be elements of $D_{\mathbf{Q}}$ satisfying (1.5.1). Take a field k of rationality for $\mathfrak{F}$ and $\mathfrak{a}$. Take also a non-zero element v of $F'(k)$ and a prime divisor $q \in P(k)$ of degree one (i.e., a k-rational point of V) that is disjoint with div (v), $\mathfrak{a}$, and $\mathfrak{z}$. Let t_q be an element of $k(V)$ such that $\nu_q(t_q) = 1$. (We call such a t_q a *k-rational local parameter at q*.) Then we can find a basis $\{w_1, \ldots, w_n\}$ of $F(\mathfrak{a}, k)$ such that

$$\begin{aligned} & w_i v\,dz = (t_q^{\alpha_i} + c_{i1} t_q^{\alpha_i+1} + \ldots)\, dt_q, \\ & 0 \leqslant \alpha_1 < \ldots < \alpha_n \end{aligned} \tag{1.8.1}$$

with $c_{ij} \in k$. Subtracting a suitable linear combination $\Sigma_{j>i} b_j w_j$ from w_i, we may assume that

(1.8.2) *The coefficient of* $t_q^{\alpha_j} dt_q$ *in* $w_i v dz$ *is* 0 *if* $j > i$.

We call $\{w_1, \ldots, w_n\}$ *a q-basis of* $F(\mathfrak{a}, k)$ *relative to* v *and* t_q, if (1.8.1) and (1.8.2) are satisfied.

Now define an element u_i of $R'(k)$ so that

$$u_{ip} = 0 \quad \text{for} \quad p \neq q,$$
$$u_{iq} = t_q^{-\alpha_i - 1} v.$$

Then $\langle w_i, u_j \rangle = \mathrm{Res}_q (w_i u_{jq} v^{-1} v dz) = \delta_{ij}$ by virtue of (1.8.2). Therefore $u_1, \ldots, u_n$ form a basis of $R'(k)/[R'(\mathfrak{b}, k) + F'(k)]$, hence every element of $R'(k)$ is congruent to a linear combination of the form $\Sigma_i c_i u_i$ with c_i in k modulo $R'(\mathfrak{b}, k) + F'(k)$. We state this fact as

1.9. Proposition. *Let* q, t_q, *and* v *be as above. Then, for every* $r \in R'(k)$, *there exists an element* s *of* $R'(k)$ *such that*

$$r - s \in R'(\mathfrak{b}, k) + F'(k),$$
$$s_p = 0 \quad \text{for } p \neq q,$$
$$s_q = \left(\textstyle\sum_{i=1}^{n} c_i t_q^{-\alpha_i - 1}\right) v$$

with $c_i \in k$.

2. An algebraic kernel function

2.1. Let V and $\mathfrak{F}$ be the same as before, and k a field of rationality for $\mathfrak{F}$. The purpose of this section is to construct an "algebraic kernel function" which will play an essential role in the computation of the trace of a Hecke operator in the next section. In the construction we shall be considering "generic points" of V over k in the sense of [9]. If x is a generic point of V over k, then $k(x)$ is a subfield of Ω, isomorphic to $k(V)$ over k by the map $g \mapsto g(x)$ for $g \in k(V)$. Here $g(x)$ is the value of g at the point x. For our purpose, it is absolutely necessary to distinguish $k(V)$ from $k(x)$. (Note that $k(V)$ is linearly disjoint with Ω over k.) It is also necessary to consider the functions and the divisors on the product $V \times V$, which is a non-singular surface rational over k. There are three types of k-rational prime divisors of $V \times V$:

(i) $p \times V$ with $p \in P(k)$,

(ii) $V \times p$ with $p \in P(k)$,

(iii) a k-rational prime divisor of $V \times V$ which has a non-trivial intersection with any divisor of the above two types.

A prime divisor of the type $p \times V$ or $V \times p$ is called *left constant* or *right constant*, respectively. A divisor of the third type is called *non-constant*. Let $k(V \times V)$ denote the field of k-rational functions on the surface $V\times V$. For each prime divisor $\mathfrak{P}$ of the above three types, we can define a discrete valuation $\nu_{\mathfrak{P}}$ of $k(V\times V)$.

Now we identify $k(V)\otimes_k k(V)$ with a subring of $k(V\times V)$ in a natural manner. Namely, for $\alpha\in k(V)$, $\beta\in k(V)$, we view $\alpha\otimes\beta$ as the element of $k(V\times V)$ defined by $(\alpha\otimes\beta)(x, y)=\alpha(x)\beta(y)$ with a generic point (x, y) of $V\times V$ over k. Then we define a module $E(k)$ by

$$E(k)=k(V\times V)\otimes_K(F(k)\otimes_k F'(k)) \quad (K=k(V)\otimes_k k(V)).$$

To be more explicit, $E(k)$ is a one-dimensional vector space over $k(V\times V)$ formed by all the expressions of the form

$$W=H\otimes f\otimes g$$

with $H\in k(V\times V)$, $f\in F(k)$, $g\in F'(k)$, under the rule

$$H\otimes\alpha f\otimes\beta g=(\alpha\otimes\beta)H\otimes f\otimes g$$

for $\alpha\in k(V)$, $\beta\in k(V)$. For a k-rational prime divisor $\mathfrak{P}$ of $V\times V$, we define $\nu_{\mathfrak{P}}(W)$ as follows:

$$\nu_{p\times V}(W)=\nu_{p\times V}(H)+\nu_p(f),$$
$$\nu_{V\times p}(W)=\nu_{V\times p}(H)+\nu_p(g),$$
$$\nu_{\mathfrak{P}}(W)=\nu_{\mathfrak{P}}(H) \quad \text{if } \mathfrak{P} \text{ is non-constant.}$$

To express W, it is often convenient to use the notation

$$W(x, y)=H(x, y)f(x)g(y)$$

with a generic point (x, y) of $V\times V$ over k. For example, given $f\in F(k)$, $g\in F'(k)$ and an element η of $k(x, y)$, we shall be speaking of the element W of $E(k)$ defined by

$$W(x, y)=\eta f(x)g(y).$$

This means $W=H\otimes f\otimes g$ with the element H of $k(V\times V)$ defined by $H(x, y)=\eta$. Here the symbols $f(x)$, $g(y)$ are meaningless only by themselves; x and y are merely to indicate "the left and right variables".

2.2. Let (x, y) be a generic point of $V\times V$ over k, and let $k_1=k(x)$. For every $H\in k(V\times V)$, define an element H_1 of $k_1(V)$ by $H_1(y)=H(x, y)$. (Note that y is generic on V over k_1.) Then $H\mapsto H_1$ gives an isomorphism of $k(V\times V)$ onto $k_1(V)$. Take a non-constant prime divisor $\mathfrak{P}$ of $k(V\times V)$. As a k-rational algebraic cycle, $\mathfrak{P}$ has a generic point of the form (x, y') over k, where $y'\in V$ and $k(x, y')$ is algebraic over $k(x)=k_1$. Let $\mathfrak{y}'$ be the k_1-rational prime divisor of V, that is the sum of all conjugates of y' over k_1. Then we see that the isomorphism $H \mapsto H_1$ sends the "place" $\mathfrak{P}$ of $k(V \times V)$ to the "place" $\mathfrak{y}'$ of $k_1'(V)$, since $H_1(y') = H(x, y')$. More symbolically, one has

$$H_1 \bmod \mathfrak{y}' = H \bmod \mathfrak{P}.$$

Especially we have

$$\nu_{\mathfrak{P}}(H) = \nu_{\mathfrak{y}'}(H_1). \tag{2.2.1}$$

As a special case, take as $\mathfrak{P}$ the locus Δ of (x, x) on $V \times V$ over k. We call Δ *the diagonal* on $V \times V$. In this case, we consider the k_1-rational prime divisor consisting of the point x, which we denote by the same letter x. Then we have

$$\nu_\Delta(H) = \nu_x(H_1). \tag{2.2.2}$$

2.3. Now we consider two elements $\mathfrak{a}$ and $\mathfrak{b}$ of $D_{\mathbf{Q}}$ under a set of conditions

$$\mathfrak{a} + \mathfrak{b} = \mathfrak{z}; \quad \mathfrak{a} + \operatorname{div}(f) \in D \text{ for } 0 \neq f \in F; \quad \mathfrak{b} + \operatorname{div}(g) \in D \quad \text{for } 0 \neq g \in F'. \tag{2.3.0}$$

As seen in Proposition 1.5, the last two conditions are equivalent. These $\mathfrak{a}$, $\mathfrak{b}$ will be always the same throughout this section.

Take any field k of rationality for $\mathfrak{F}$, $\mathfrak{a}$, $\mathfrak{b}$. Let x be a generic point of V over k, and let $k_1 = k(x)$. Define the k_1-rational prime divisor x as above, and take a k_1-rational local parameter t_x at x of the form $t_x = \tau - \tau(x)$ with a non-constant element τ of $k(V)$. (This special form of t_x will simplify our later discussion.) Let $\{f_1, \ldots, f_n\}$ be a basis of $F(\mathfrak{a}, k)$ over k, and take non-zero elements u of $F(k)$ and v of $F'(k)$. These f_i, u, and v will be fixed throughout this section. Consider the power-series expansion of f_i/u:

$$f_i/u = \textstyle\sum_{r=0}^{\infty} \varphi_{ir} t_x^r$$

with $\varphi_{ir} \in k_1$. Since f_i/u is k-rational, we see that f_i/u is finite at x, and $\varphi_{i0} = (f_i/u)(x) \neq 0$. Now take an x-basis $\{w_1, \ldots, w_n\}$ of $F(\mathfrak{a}, k_1)$ relative to t_x and v in the sense of 1.8. Then

$$w_i v dz = (t_x^{\beta_i} + \ldots) dt_x \quad (i = 1, \ldots, n;\ 0 \leqslant \beta_1 < \ldots < \beta_n) \tag{2.3.1}$$

with non-negative integers β_i. Since $fvdz$ is finite and $\neq 0$ at x for every non-zero $f \in F(k)$, we have $\beta_1 = 0$. Put $w_i = \sum_{j=1}^n c_{ij} f_j$ with $c_{ij} \in k_1$, and $\zeta = uvdz/dt_x$. Since $dt_x = d\tau$, we see that $\zeta \in k(V)$, and

$$w_i v dz = (\textstyle\sum_{j=1}^n c_{ij} f_j/u)\, \zeta\, dt_x = (\sum_{j,r} c_{ij} \varphi_{jr} t_x^r)\, \zeta\, dt_x.$$

Therefore

$$\sum_{j=1}^n c_{ij} \varphi_{jr} = \begin{cases} 0 & \text{if } r < \beta_i, \\ \zeta(x)^{-1} & \text{if } r = \beta_i. \end{cases} \tag{2.3.2}$$

Take elements $g_1, \ldots, g_n$ of $R'(k)$ so that they form a basis of $R'(k)/[R'(\mathfrak{b}, k) + F'(k)]$ dual to $\{f_1, \ldots, f_n\}$. Consider the embedding of $R'(k)$ into $R'(k_1)$ defined in 1.7, and denote by g_i^* the image of g_i by this embedding. Now define an element G_0 of $R'(k_1)$ by

$$(2.3.3) \qquad G_0 = \sum_{i=1}^{n} (f_i/u)(x)\, g_i^*.$$

By Proposition 1.9, there exists an element s of $R'(k_1)$ such that

$$(2.3.4) \qquad \begin{cases} s_p = 0 & \text{for all } p \in P(k_1) \quad \text{other than } x, \\ s_x = \left(\sum_{i=1}^{n} a_i t_x^{-\beta_i - 1}\right) v \end{cases}$$

with $a_i \in k_1$, and

$$(2.3.5) \qquad G_0 - s \in R'(\mathfrak{b}, k_1) + F'(k_1).$$

Then $\mathrm{Res}_x\,(f_i s_x dz) = \langle f_i, s\rangle = \langle f_i, \Sigma_j (f_j/u)(x) g_j^*\rangle = (f_i/u)(x)$. Therefore $\mathrm{Res}_x\,(f s_x dz) = (f/u)(x)$ for all $f \in F(\mathfrak{a}, k_1)$. Substituting $w_i = \Sigma_j c_{ij} f_j$ for f, we obtain

$$\mathrm{Res}_x\,(w_i s_x dz) = \sum_j c_{ij} (f_j/u)(x) = \sum_j c_{ij} \varphi_{j0} = \begin{cases} 0 & \text{if } i > 1, \\ \zeta(x)^{-1} & \text{if } i = 1, \end{cases}$$

by virtue of (2.3.2). On the other hand,

$$\begin{aligned} \mathrm{Res}_x\,(w_i s_x dz) &= \mathrm{Res}_x\,(w_i v \cdot v^{-1} s_x dz) \\ &= \mathrm{Res}_x\left[(t_x^{\beta_i} + \ldots)\left(\sum_j a_j t_x^{-\beta_j - 1}\right) dt_x\right] = a_i. \end{aligned}$$

Therefore

$$a_i = \begin{cases} 0 & (i > 1), \\ \zeta(x)^{-1} & (i = 1), \end{cases}$$

hence (2.3.4) becomes

$$(2.3.6) \qquad \begin{cases} s_p = 0 & \text{for } x \neq p \in P(k_1), \\ s_x = \zeta(x)^{-1} t_x^{-1} v. \end{cases}$$

2.4. By (2.3.5), we have

$$(2.4.1) \qquad G_0 - s = A_0 + B_0$$

with $A_0 \in R'(\mathfrak{b}, k_1)$ and $B_0 \in F'(k_1)$. Define an element B of $E(k)$ by

$$B(x, y) = (B_0/c)(y)\, u(x)\, c(y)$$

with any non-zero $c \in F'(k)$. Note that $B_0/c \in k_1(V)$, and $(B_0/c)(y)$ is an element of the field $k_1(y) = k(x, y)$. Similarly, for every $p \in P(k_1)$, we can define elements G_p, A_p, and S_p of $E(k)$ by

$$\begin{aligned} G_p(x, y) &= (G_{0p}/c)(y)\, u(x)\, c(y), \\ A_p(x, y) &= (A_{0p}/c)(y)\, u(x)\, c(y), \\ S_p(x, y) &= (s_p/c)(y)\, u(x)\, c(y). \end{aligned}$$

Obviously B, G_p, A_p, S_p do not depend on the choice of c. (As for u, it has been fixed at the beginning.) From our definition of G_0, we obtain

$$(2.4.2) \qquad G_p(x, y) = \sum_{i=1}^{n} f_i(x)\, g_{ip}^*(y) \quad (p \in P(k_1)).$$

With these elements of $E(k)$, the equality (2.4.1) becomes

(2.4.3) $$G_p - S_p = A_p + B \quad (p \in P(k_1)).$$

More precisely, one has

(2.4.4) $$\sum_{i=1}^{n} f_i(x)\, g^*_{ip}(y) = A_p(x, y) + B(x, y) \quad \text{for } x \neq p \in P(k_1),$$

(2.4.5) $$-S_x(x, y) = A_x(x, y) + B(x, y).$$

Since $A_0 \in R'(\mathfrak{b}, k_1)$, we have

(2.4.6) $$\nu_{V\times p}(A_p) \geqslant -\nu_p(\mathfrak{b}) \quad (p \in P(k)),$$

hence

(2.4.7) $$\nu_{V\times p}(B) \geqslant -\nu_p(\mathfrak{b}) \quad (p \in P(k)),$$

unless $\nu_p(g_{ip}) < -\nu_p(\mathfrak{b})$ for some i. Furthermore, by virtue of (2.2.2), we have

(2.4.8) $$\nu_\Delta(A_x) = \nu_x(A_{0x}) \geqslant 0.$$

Now let $\mathfrak{P}$ be a non-constant k-rational prime divisor of $V \times V$, with a generic point (x, y') over k. Let $\mathfrak{y}'$ be the k_1-rational prime divisor of V corresponding to the point y' as defined in 2.2. Since $-s_{\mathfrak{y}'} = A_{0\mathfrak{y}'} + B_0$, we have, by (2.2.1),

$$\nu_{\mathfrak{P}}(B) = \nu_{\mathfrak{y}'}(B_0/v) = \nu_{\mathfrak{y}'}((s_{\mathfrak{y}'}/v) + (A_{0\mathfrak{y}'}/v)).$$

Since $A_0 \in R'(\mathfrak{b}, k_1)$, we have $\nu_{\mathfrak{y}'}(A_{0\mathfrak{y}'}/v) \geqslant 0$. If $\mathfrak{P} \neq \Delta$, we have $s_{\mathfrak{y}'} = 0$, while if $\mathfrak{P} = \Delta$, we have $\mathfrak{y}' = x$ and $\nu_x(s_x/v) = \nu_x(s_x) = -1$. Thus

(2.4.9) $$\nu_\Delta(B) = -1; \quad \nu_{\mathfrak{P}}(B) \geqslant 0 \quad \textit{for every non-constant } \mathfrak{P} \neq \Delta.$$

We are going to normalize B so that it has a pole only at some pre-assigned constant divisors. To do this, we have to impose the following conditions on $\mathfrak{a}$:

(2.4.10) $$F(\mathfrak{a} - p) \neq F(\mathfrak{a}) \quad \textit{for every point } p \textit{ of } V;$$

(2.4.11) $$F'(\mathfrak{b} + p) = F'(\mathfrak{b}) \quad \textit{for every point } p \textit{ of } V.$$

Note that $\dim F(\mathfrak{a} - p) \geqslant \dim F(\mathfrak{a}) - 1$. By (1.6.1), we have

$$\dim F(\mathfrak{a} - p) - \dim F'(\mathfrak{b} + p) = \dim F(\mathfrak{a}) - \dim F'(\mathfrak{b}) - 1,$$

hence (2.4.10) is equivalent to (2.4.11).

Let us now prove a few lemmas which are necessary for our process of normalization.

2.5. Lemma. *Let $0 \neq \xi \in k(V \times V)$. Suppose that the pole of ξ consists of the diagonal Δ with multiplicity one, and possibly right or left constant divisors. Let p be a k-rational point of V such that $p \times V$ is not contained in the pole of ξ, and let η be the element of $k(V)$ defined* *by* $\eta(y)=\xi(p, y)$ *with any generic point* y *of* V *over* k. *Then* $\nu_q(\eta)\geqslant\nu_{V\times q}(\xi)$ *for* $p\neq q\in P(k)$, *and* $\nu_p(\eta)\geqslant\nu_{V\times p}(\xi)-1$.

Proof. Take an element π of $k(V)$ so that $\nu_q(\pi)=\nu_{V\times q}(\xi)$. Define an element ξ' of $k(V\times V)$ by $\xi'(x, y)=\pi(y)^{-1}\xi(x, y)$ with a generic point (x, y) of $V\times V$ over k. Then $\nu_{V\times q}(\xi')=0$. If $q\neq p$, we see that ξ' is finite at (p, q) and $\xi'(p, y)=\pi(y)^{-1}\eta(y)$. Therefore $\nu_q(\pi^{-1}\eta)\geqslant 0$, hence $\nu_q(\eta)\geqslant\nu_q(\pi)=\nu_{V\times q}(\xi)$. Next take an element γ of $k(V)$ so that $\nu_p(\gamma)=1$. Define two elements α and β of $k(V\times V)$ by $\alpha(x, y)=\gamma(x)$, $\beta(x, y)=\gamma(y)$. Put $e=\nu_{V\times p}(\xi)$. Since $\nu_\Delta(\alpha-\beta)=1$, we see that $\beta^{-e}(\beta-\alpha)\xi$ is finite at (p, p). (In fact, $\nu_Q(\beta^{-e}(\beta-\alpha)\xi)\geqslant 0$ for all k-rational prime divisors Q of $V\times V$ passing through (p, p).) Therefore, specializing $\beta^{-e}(\beta-\alpha)\xi$ to $p\times V$, we find $\nu_p(\gamma^{1-e}\eta)\geqslant 0$, so that $\nu_p(\eta)\geqslant e-1$, q.e.d.

2.6. LEMMA *Put* $r=\dim F'(\mathfrak{b})$. *Let* $q_1, \ldots, q_s$ *be independent generic points of* V *over a field of rationality for* $\mathfrak{F}$ *and* $\mathfrak{a}$. *Then*

$$\dim F(\mathfrak{a}+\textstyle\sum_{i=1}^{s}q_i)=\begin{cases}\dim F(\mathfrak{a}) & \text{if } s\leqslant r,\\ \dim F(\mathfrak{a})+s-r & \text{if } s>r.\end{cases}$$

Proof. Let $0\neq f\in F(k)$, $0\neq g\in F'(k)$ with a field k of rationality for $\mathfrak{F}$ and $\mathfrak{a}$. Put $\omega=fg\,dz$ and $\mathfrak{b}=\mathfrak{a}+\operatorname{div}(f)$. Then

$$\dim F(\mathfrak{a}+\textstyle\sum_{i=1}^{s}q_i)=l(\mathfrak{b}+\sum_{i=1}^{s}q_i),$$
$$r=l(\operatorname{div}(\omega)-\mathfrak{b}).$$

Therefore our assertion can be written as

$$l(\mathfrak{b}+\textstyle\sum_{i=1}^{s}q_i)=\begin{cases}l(\mathfrak{b}) & \text{if } s\leqslant r,\\ l(\mathfrak{b})+s-r & \text{if } s>r,\end{cases}$$

which is nothing else than Weil [10, p. 11, Prop. 8].

2.7. LEMMA. *Let* k *be a field of rationality for* $\mathfrak{F}$; p *a* k*-rational point of* V; t_p *a* k*-rational local parameter at* p; w *an element of* $F(k)$ *such that* $0\leqslant\nu_p(w)<1$; $\mathfrak{b}$ *an element of* $D_{\mathbf{Q}}(k)$ *such that* $\operatorname{div}(w)+\mathfrak{b}\in D$. *Further let* c *be an integer such that*

$$(*)\qquad \deg(\operatorname{div}(w)+\mathfrak{b})+c-1>2\mathfrak{g}-2,$$

where $\mathfrak{g}$ *is the genus of* V, *and let* $e=\nu_p(\mathfrak{b})-\nu_p(w)$. *Then there exists an element* f *of* $F(k)$ *such that* $\nu_p(f-t_p^{e-c}w)\geqslant\nu_p(w)+e-c+1$, *and* $\nu_q(f)\geqslant-\nu_q(\mathfrak{b})$ *for* $p\neq q\in P(k)$.

Proof. By (1.3.1), $a\mapsto aw$ gives a k-linear isomorphism of $L(\operatorname{div}(w)+c'p+\mathfrak{b}, k)$ onto $F(c'p+\mathfrak{b}, k)$ for any integer c'. By (*), we have

$$l(\operatorname{div}(w)+cp+\mathfrak{b})=\deg(\operatorname{div}(w)+\mathfrak{b})+c-\mathfrak{g}+1,$$

and this holds also when c is replaced by $c-1$. Therefore

$$F(cp+\mathfrak{d},\, k)/F((c-1)p+\mathfrak{d},\, k)$$

is one-dimensional, hence our assertion.

2.8. Let us now fix a field k_0 of rationality for $\mathfrak{F}$ and $\mathfrak{a}$, and take an extension k of k_0, which is algebraically closed, and which has an infinite transcendence degree over k_0. Hereafter we take this k as our basic field. Since k is algebraically closed, $P(k)$ can be identified with the set of all k-rational points of V.

Now, with this k, we fix u, v, f_i, g_i, x, t_x, and define G_p, S_p, A_p, and B as in 2.4. We shall now show that under the condition (2.4.10), A_p and B can be chosen so that

$$\nu_{p\times V}(B) \geqslant -\nu_p(\mathfrak{a}) \quad \text{for every } p\in P(k)-\{q_1, \dots, q_r\}, \tag{2.8.1}$$

$$\nu_{q_i\times V}(B) \geqslant -1 \quad \text{for } i=1, \dots, r, \tag{2.8.2}$$

with r independent generic points $q_1, \dots, q_r$ of V over k_0 rational over k, where $r=\dim F'(\mathfrak{b})$.

To show this, we start from any choice of A_p and B as in 2.4, and observe that $\nu_{q\times V}(B)+\nu_q(\mathfrak{a})\in \mathbf{Z}$ for all $q\in P(k)$ by virtue of (2.3.0). Let us fix one $p\in P(k)$, and put $\nu_p(\mathfrak{a})=-e-e'$ with $e\in\mathbf{Z}$ and $0\leqslant e'<1$, $\nu_{p\times V}(B)-e-e'=-c$. Then $c\in\mathbf{Z}$. Assume that (2.8.1) is not satisfied for this p, i.e., $c>0$. (The number of such points p is of course finite.) Take a non-zero element w of $F(k)$ so that $\nu_p(w)=e'$, and take a k-rational local parameter t_p at p. Put, for each $q\in P(k)$,

$$A_q(x, y) = t_p(x)^{e-c}a_q(x, y)\,w(x)\,v(y),$$
$$B(x, y) = t_p(x)^{e-c}b(x, y)\,w(x)\,v(y)$$

with elements a_q and b of $k(x, y)$. Then

$$t_p(x)^{c-e}\,\textstyle\sum_{i=1}^{n}(f_i/w)(x)\,(g_{iq}/v)(y) = a_q(x, y)+b(x, y), \tag{2.8.3}$$

$$\nu_{p\times V}(b) = \nu_{p\times V}(B)-e-e'+c = 0. \tag{2.8.4}$$

Consider (2.8.3) as an equality about the elements of $k(V\times V)$, and take it modulo $p\times V$. Since $f_i\in F(\mathfrak{a})$, we have $\nu_p(f_i/w)\geqslant e$, hence the left hand side is 0 modulo $p\times V$. This together with (2.8.4) shows

$$\nu_{p\times V}(a_q) = 0. \tag{2.8.5}$$

By (2.4.6), we have $\qquad \nu_{V\times q}(a_q) \geqslant -\nu_q(\mathfrak{b})-\nu_q(v).$

Let β be the element of $k(V)$ defined by $\beta(y)=b(p, y)$. Then β is exactly b mobulo $p\times V$, which is the same as $-a_q$ modulo $p\times V$. By (2.4.9), Δ is the only non-constant pole of b, so that (2.8.3) shows that a_q has the same property. By (2.8.4) and (2.8.5), we can apply Lemma 2.5 to b and a_q modulo $p \times V$. Then we find

$$\nu_q(\beta) \geqslant \begin{cases} \nu_{V\times q}(a_q) \geqslant -\nu_q(\mathfrak{b}) - \nu_q(v) & \text{if } q \neq p, \\ \nu_{V\times p}(a_p) - 1 \geqslant -\nu_p(\mathfrak{b}) - \nu_p(v) - 1 & \text{if } q = p. \end{cases}$$

By (2.4.11), we have $\operatorname{div}(\beta v) \geqslant -\mathfrak{b}$. By Lemma 2.7, we can find an element f of $F(k)$ such that

$$\nu_p(f - t_p^{e-c} w) \geqslant e + e' - c + 1,$$

$$\nu_q(f) \geqslant -\nu_q(\mathfrak{a}) \quad \text{for } q \neq p, p_1, \ldots, p_\lambda,$$

$$\nu_{p_i}(f) \geqslant -1 \quad \text{for } i = 1, \ldots, \lambda,$$

if we choose sufficiently many k-rational points $p_1, \ldots, p_\lambda$ of V not involved in $\mathfrak{a}$. (We take $\mathfrak{a} + p_1 + \ldots + p_\lambda$ to be the divisor $\mathfrak{d}$ of Lemma 2.7.) Then we define an element H_0 of $F'(k_1)$ and an element H of $E(k)$ by

$$H_0 = (f/u)(x)\beta v,$$

$$H(x, y) = f(x)\beta(y)v(y) = (H_0/v)(y)u(x)v(y).$$

Then $H_0 \in F'(\mathfrak{b}, k_1)$, and $\nu_{V\times q}(H) = \nu_q(\beta v) \geqslant -\nu_q(\mathfrak{b})$ for every $q \in P(k)$. Further we have

$$B(x, y) - H(x, y) = t_p(x)^{e-c}[b(x, y) - (t_p^{c-e} f/w)(x)\beta(y)]w(x)v(y),$$

hence

$$\nu_{p\times V}(B - H) \geqslant e + e' - c + 1.$$

We see also that

$$\nu_{q\times V}(B - H) \geqslant \operatorname{Min}(\nu_{q\times V}(B), \; -\nu_q(\mathfrak{a})) \quad \text{for } q \in P(k) - \{p, p_1, \ldots, p_\lambda\}.$$

The points p_i can be chosen so that $\nu_{p_i\times V}(B) \geqslant 0$. Then we have

$$\nu_{p_i\times V}(B - H) \geqslant -1.$$

Now replace B_0 by $B_0 - H_0$. Then B and $A_{\mathfrak{z}}$ are replaced by $B - H$ and $A_{\mathfrak{z}} + H$ for every $\mathfrak{z} \in P(k_1)$. Apply the same procedure to the new B with the same point p if still $\nu_{p\times V}(B) - e - e' < 0$, or with other p for which (2.8.1) does not hold. After a finite number of steps, we can now assume that B has the following properties:

$$\nu_{p\times V}(B) \geqslant -\nu_p(\mathfrak{a}) \quad \text{if } p \in P(k) - M,$$

$$\nu_{p\times V}(B) = -1 \quad \text{if } p \in M,$$

with a finite set M of independent generic points over k_0, all rational over k.

Take r independent generic points $q_1, \ldots, q_r$ of V over k_0, independent of the points of M. Apply again the procedure of taking (2.8.3) modulo $p \times V$ for each $p \in M$. Let w_p, b_p, and β_p denote the functions w, b, and β defined above for this p (with respect to the new B). Since $e=e'=0$ and $c=1$ in the present case, we have $B(x, y)=t_p(x)^{-1}b_p(x, y)\,w_p(x)\,v(y)$; β_p is b_p modulo $p\times V$, and div $(\beta_p v)\geqslant -\mathfrak{b}$. By Lemma 2.6,

$$F(\mathfrak{a}+p+q_1+\ldots+q_r)/F(\mathfrak{a}+q_1+\ldots+q_r)$$

is one-dimensional, hence there exists, for each $p\in M$, an element f_p of $F(k)$ such that

$$\begin{aligned} &\nu_p(f_p-t_p^{-1}w_p)\geqslant 0 \\ &\nu_q(f_p)\geqslant -\nu_q(\mathfrak{a}) \quad \text{for } q\in P(k)-\{p, q_1, \ldots, q_r\}, \\ &\nu_{q_i}(f_p)\geqslant -1 \qquad \text{for } i=1, \ldots, r. \end{aligned}$$

Define an element J of $E(k)$ by

$$J(x, y)=\Sigma_{p\in M}f_p(x)\beta_p(y)v(y).$$

Then $\nu_{V\times q}(J)\geqslant -\nu_q(\mathfrak{b})$ for all $q\in P(k)$, and

$$\begin{aligned} &\nu_{q\times V}(B-J)\geqslant 0 \qquad \text{for } q\in M, \\ &\nu_{q_i\times V}(B-J)\geqslant -1 \qquad \text{for } i=1, \ldots, r, \\ &\nu_{q\times V}(B-J)\geqslant -\nu_q(\mathfrak{a}) \quad \text{for } q\in P(k)-M\cup\{q_1, \ldots, q_r\}. \end{aligned}$$

Therefore, replacing B and $A_{\mathfrak{z}}$ by $B-J$ and $A_{\mathfrak{z}}+J$, we obtain the desired properties (2.8.1, 2), retaining the properties (2.4.2–9). It is this B which was to be constructed and which may be called an algebraic kernel function. We conclude this section by proving two propositions concerning the behavior of B at its pole.

2.9. Proposition. *Let (x, y) be a generic point of $V\times V$ over k with the same x as in 2.4, and let $k_2=k(y)$, $0\neq w\in F'(k)$. Define an element B_w of $k_2(V)$ by $B(x, y)=B_w(x)u(x)w(y)$. Then $\nu_y(B_w uw\,dz)=-1$, and* $\mathrm{Res}_y\,(B_w uw\,dz)=1$.

Proof. Let us first consider the case where w is the element v with which we constructed B. Take the local parameter $t_x=\tau-\tau(x)$ with $\tau\in k(V)$ as in 2.3. Let s_x and S_x be as in 2.3 and 2.4, and put $\zeta=uv\,dz/dt_x$. Then $\zeta\in k(V)$, $S_x(x, y)=(s_x/v)(y)u(x)v(y)$, and $s_x=\zeta(x)^{-1}t_x^{-1}v$ by (2.3.6). Define an element S^* of $k_2(V)$ by $S^*(x)=(s_x/v)(y)$. Define also a k_2-rational local parameter t_y at y by $t_y=\tau-\tau(y)$. Then

$$S^*(x)=\zeta(x)^{-1}t_x(y)^{-1}=\zeta(x)^{-1}[\tau(y)-\tau(x)]^{-1}=-\zeta(x)^{-1}t_y(x)^{-1},$$

hence $S^*=-1/\zeta t_y$, and $S^*uv\,dz=-t_y^{-1}dt_y$. Therefore $\nu_y(S^*)=-1$ and $\mathrm{Res}_y(S^*uv\,dz)=-1$. By (2.4.5), we have $-S_x=A_x+B$. Define an element A^* of $k_2(V)$ by $A_x(x, y)=A^*(x)u(x)v(y)$. Then $-B_v=A^*+S^*$. By (2.2.2) and (2.4.8), we have $\nu_y(A^*)=\nu_\Delta(A_x)\geqslant 0$. Therefore

$\nu_y(B_v) = \nu_y(S^*) = -1$, and $B_v uvdz$ has the same residue as $-S^* uvdz$ at y. This settles the case $w = v$. In the general case, put $w = \alpha v$ with $\alpha \in k(V)$. Then $B_w = \alpha(y)^{-1} B_v$, so that $B_w uwdz = \alpha(y)^{-1} \alpha \cdot B_v uvdz$, hence our assertion.

2.10. PROPOSITION. *Let the notation be the same as in Proposition 2.9. Suppose $r > 0$, $w \in F'(\mathfrak{b}, k)$, and $\nu_{q_i}(w) = 0$ for $i = 1, \ldots, r$. Then $\nu_{q_i}(B_w uwdz) = -1$ for $i = 1, \ldots, r$, and $B_w uwdz$ has a pole only at $y, q_1, \ldots, q_r$. Moreover, if c_i is the element of $k(V)$ defined by*

$$c_i(y) = \mathrm{Res}_{q_i}(B_w uwdz) \quad (i = 1, \ldots, r),$$

then $c_1 w, \ldots, c_r w$ form a basis of $F'(\mathfrak{b}, k)$ over k, and for every $g \in F'(\mathfrak{b})$, one has

$$(*) \qquad g = -\sum_{i=1}^{r} (g/w)(q_i) \cdot c_i w.$$

Note that, since $r = \dim F'(\mathfrak{b})$, the assumption $r > 0$ implies the existence of a non-zero element of $F'(\mathfrak{b}, k)$. Moreover, since the q_i are generic over a field k_0 of rationality for $\mathfrak{b}$, we have $\nu_{q_i}(w) = 0$ for every non-zero $w \in F'(\mathfrak{b}, k_0)$.

Proof. Define an element B' of $k(V \times V)$ by $B(x, y) = B'(x, y) u(x) w(y)$. Then $\nu_{p \times V}(B') = \nu_{p \times V}(B) - \nu_p(u) \geqslant -\nu_p(\mathfrak{a}) - \nu_p(u)$ for $p \in P(k) - \{q_1, \ldots, q_r\}$, and $\nu_{q_i \times V}(B') \geqslant -1$. By Lemma 2.5, we have $\nu_{q_i}(B_w) \geqslant -1$, $\nu_y(B_w) \geqslant -1$, and $\nu_q(B_w) \geqslant \nu_{q \times V}(B')$ for all $q \in P(k_2) - \{y, q_1, \ldots, q_r\}$. Therefore, for every $g \in F'(\mathfrak{b})$, we have

$$\begin{aligned} \mathrm{div}\,(B_w ug\,dz) &\geqslant -y - \textstyle\sum_{i=1}^{r} q_i + \mathrm{div}\,(u) + \mathrm{div}\,(g) + \mathfrak{z} - \mathfrak{a} - \mathrm{div}\,(u) \\ &\geqslant -y - \textstyle\sum_{i=1}^{r} q_i. \end{aligned}$$

This shows that the differential form $B_w ug dz$ has no pole except at $y, q_1, \ldots, q_r$. Especially this applies to the case $g = w$. By Proposition 2.9, we have

$$\mathrm{Res}_y\,(B_w ug dz) = \mathrm{Res}_y\,((g/w) \cdot B_w uwdz) = (g/w)(y).$$

By our definition of c_i,

$$\mathrm{Res}_{q_i}(B_w ug dz) = \mathrm{Res}_{q_i}((g/w) \cdot B_w uwdz) = (g/w)(q_i) \cdot c_i(y).$$

Since the sum of all residues is 0, we obtain

$$(g/w)(y) + \textstyle\sum_{i=1}^{r} (g/w)(q_i) c_i(y) = 0,$$

which proves the equality (*). This shows also that $F'(\mathfrak{b})$ is contained in the Ω-linear span of $c_1 w, \ldots, c_r w$. Since $r = \dim F'(\mathfrak{b})$, the $c_i w$ must form a basis of $F'(\mathfrak{b})$ over Ω. It follows that $c_i \neq 0$, hence $\nu_{q_i}(B_w uwdz) = -1$. This completes the proof.

3. The trace formula: first formulation

3.1. For $\alpha=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbf{C})$ and $z \in \mathbf{C} \cup \{\infty\}$, we put, as usual,

$$\alpha(z)=(az+b)/(cz+d),$$

and denote by $\mathfrak{H}$ the complex upper half plane:

$$\mathfrak{H}=\{z\in\mathbf{C} \mid \mathrm{Im}\,(z)>0\}.$$

We are going to discuss automorphic forms of an arbitrary rational weight. Fix a "weight" m which is a rational number, and consider the set $\mathfrak{G}_m$ consisting of all couples $(\alpha, h(z))$ formed by an element $\alpha=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of $\mathrm{SL}_2(\mathbf{R})$ and a holomorphic function $h(z)$ on $\mathfrak{H}$ of the form $h(z)=t\cdot(cz+d)^m$ with $t\in\mathbf{C}$, $|t|=1$. We make $\mathfrak{G}_m$ a group by defining the law of multiplication by

$$(\alpha, h(z))(\beta, j(z))=(\alpha\beta, h(\beta(z))j(z)).$$

Let $\tau=(\alpha, h(z))\in\mathfrak{G}_m$. For a meromorphic function f on $\mathfrak{H}$, we define a function $f|\tau$ by

$$(f|\tau)(z)=f(\alpha(z))h(z)^{-1}.$$

Let Γ be a discrete subgroup of $\mathrm{SL}_2(\mathbf{R})$ such that $\mathfrak{H}/\Gamma$ is of finite measure with respect to $y^{-2}dxdy$. (We denote the quotient space by $\mathfrak{H}/\Gamma$ although we let Γ act on the left of $\mathfrak{H}$.) By a *proper lifting of* Γ *of weight* m, we understand a map $L\colon \Gamma\to\mathfrak{G}_m$ satisfying the following conditions (3.1.1–3):

(3.1.1) *L is an injective homomorphism of Γ into $\mathfrak{G}_m$ such that $P\circ L$ is the identity map of Γ, where P is the natural projection map of $\mathfrak{G}_m$ onto $SL_2(\mathbf{R})$.*

(3.1.2) $L(-1)=(-1, 1)$ *if* $-1\in\Gamma$.

(3.1.3) *If* $\gamma=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\in\Gamma$, $L(\gamma)=(\gamma, t\cdot(cz+d)^m)$, *then* $t^n=1$ *for some positive integer* n.

Since Γ is finitely generated, we can take n to be independent of γ.

Let us fix such a lifting $L\colon \Gamma\to\mathfrak{G}_m$, and put

$$L(\gamma)=(\gamma, j(\gamma, z)) \quad (\gamma\in\Gamma).$$

Then $j(\gamma\delta, z)=j(\gamma, \delta(z))j(\delta, z)$. Now we can define a proper lifting L' of Γ of weight $2-m$ by

$$\begin{aligned} L'(\gamma) &= (\gamma, j'(\gamma, z)), \\ j'(\gamma, z) &= j(\gamma, z)^{-1}(cz+d)^2 \\ &= j(\gamma, z)^{-1}(dz/d\gamma(z)) \qquad \text{for } \gamma=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\in\Gamma. \end{aligned}$$

3.2. Let $\mathfrak{H}^*$ denote the union of $\mathfrak{H}$ and the cusps of Γ. Then $\mathfrak{H}^*/\Gamma$ has a natural structure of a compact Riemann surface. We take a projective non-singular curve V which is complex analytically isomorphic to $\mathfrak{H}^*/\Gamma$, and fix a Γ-invariant holomorphic map φ: $\mathfrak{H}^* \to V$ through which $\mathfrak{H}^*/\Gamma$ is isomorphic to V. (We call such (V, φ) a *model of* $\mathfrak{H}^*/\Gamma$.) Then $\mathbf{C}(V)\circ\varphi$ is the field of all Γ-automorphic functions: we identify $\mathbf{C}(V)\circ\varphi$ with $\mathbf{C}(V)$ if there is no fear of confusion.

For fixed L and L' as above, let F (resp. F') denote the module of all meromorphic functions f on $\mathfrak{H}$ which satisfy $f|L(\gamma)=f$ (resp. $f|L'(\gamma)=f$) for all $\gamma \in \Gamma$, and which are meromorphic at every cusp of Γ in the sense explained below. We see that F is either $\{0\}$ or one-dimensional over $\mathbf{C}(V)$. In the following treatment, let us simply assume that F is not $\{0\}$, without discussing the condition for the non-triviality of F. Then F' is also non-trivial, and we obtain a system $\mathfrak{F}=\{F, F', Z, \mathfrak{z}\}$ satisfying the axioms (A_{1-4}) of 1.2 as follows. For $(f, g) \in F \times F'$, define $Z(f, g)$ to be the differential form on V which can be identified with $fg\,dz$. The divisor $\operatorname{div}(f) = \sum_p \nu_p(f)p$ for $f \in F$ or $f \in F'$ can be defined in the following way. Let $p=\varphi(z_0) \in V$ with $z_0 \in \mathfrak{H}^*$. If $z_0 \in \mathfrak{H}$, consider the expansion of f at z_0:

$$f(z) = c_k(z-z_0)^k + c_{k+1}(z-z_0)^{k+1} + \dots, \quad c_k \neq 0. \tag{3.2.1}$$

Then we put $\nu_p(f) = k/e_p$, where e_p is the order of the group

$$\{\gamma \in \Gamma \mid \gamma(z_0) = z_0\}/(\Gamma \cap \{\pm 1\}). \tag{3.2.2}$$

If z_0 is a cusp of Γ, the last group is free cyclic. Take an element δ of Γ that generates this free cyclic group, and take an element $\varrho^* = (\varrho, \xi(z))$ of $\mathfrak{G}_m$ so that $\varrho(\infty) = z_0$. Then

$$\varrho^{*-1}L(\delta)\varrho^* = \left(\varepsilon\begin{bmatrix}1 & h\\ 0 & 1\end{bmatrix}, e^{2\pi i r}\right) \tag{3.2.3}$$

with $\varepsilon = \pm 1$, $h \in \mathbf{R}$, and $0 \leqslant r < 1$. By (3.1.3), r must be rational. Changing δ for δ^{-1} if necessary, we may assume $h>0$. Now we say that f is *meromorphic at* z_0 if

$$f|\varrho^* = \textstyle\sum_{n \in \mathbf{Z}} c_n \exp[2\pi i(n+r)z/h]$$

with only finitely many non-zero c_n for $n<0$. Then we put $\nu_p(f) = n+r$ with the smallest n for which $c_n \neq 0$. Finally we put

$$\mathfrak{z} = \operatorname{div}(dz) = -\textstyle\sum_{p \in R}(1-e_p^{-1})\,p,$$

where R is the set of all the points of V corresponding to the elliptic points and the cusps of Γ; e_p denotes the order of the group (3.2.2) for each point $p=\varphi(z_0) \in V$; especially $e_p = \infty$ if p corresponds to a cusp. It is now easy to verify that $\mathfrak{F}$ actually satisfies (A_{1-4}) of 1.2. (As for (A_4), see for example [6, § 2.4].)

3.3. For each $p \in V$, there is a unique rational number μ'_p such that

$$(3.3.1) \qquad 0 \leqslant \mu'_p < 1, \quad \nu_p(g) \equiv \mu'_p \bmod \mathbf{Z} \quad \text{for } 0 \neq g \in F'.$$

This is because F' is one-dimensional over $\mathbf{C}(V)$. We see that $e_p \mu'_p \in \mathbf{Z}$ (if $e_p < \infty$), hence $\mu'_p \leqslant 1 - e_p^{-1}$. Put $\mu_p = 1 - e_p^{-1} - \mu'_p$ for each $p \in V$, and define two elements $\mathfrak{a}$ and $\mathfrak{b}$ of $D_{\mathbf{Q}}$ by

$$(3.3.2) \qquad \mathfrak{a} = -\Sigma_{p \in R} \mu_p p, \quad \mathfrak{b} = -\Sigma_{p \in R} \mu'_p p.$$

Then we see that $\mathfrak{a}$ and $\mathfrak{b}$ satisfy the condition (2.3.0), and we have

$$(3.3.3) \qquad 0 \leqslant \mu_p \leqslant 1, \quad \nu_p(f) \equiv \mu_p \bmod \mathbf{Z} \quad \text{for } 0 \neq f \in F.$$

Obviously $\mu_p = \mu'_p = 0$ if $p \notin R$. Moreover one has

$$(3.3.4) \qquad F'(\mathfrak{b}) = F'(0),$$

so that $F'(\mathfrak{b})$ consists of all the elements of F' which are holomorphic on $\mathfrak{H}$ and also holomorphic at every cusp. We have $0 \leqslant \mu_p \leqslant 1 - e_p^{-1}$ if p corresponds to an elliptic point. If p corresponds to a cusp, define r by (3.2.3) under the condition $0 \leqslant r < 1$. Then $\mu_p = r$ or 1 according as $r > 0$ or $= 0$. Therefore $F(\mathfrak{a})$ consists of all the elements of F which are holomorphic on $\mathfrak{H}$ and vanish at every cusp. Thus $F(\mathfrak{a})$ is the vector space of all cusp forms with respect to the automorphic factor $j(\gamma, z)$.

Let $\mathfrak{g}$ denote the genus of V. Define $v(\mathfrak{H}/\Gamma)$ by

$$v(\mathfrak{H}/\Gamma) = (2\pi)^{-1} \int_{\mathfrak{H}/\Gamma} y^{-2}\, dx\, dy.$$

It is well known that $v(\mathfrak{H}/\Gamma) = 2\mathfrak{g} - 2 + \Sigma_{p \in R}(1 - e_p^{-1})$. Moreover one has

$$(3.3.5) \qquad \deg(\operatorname{div}(f)) = (m/2) \cdot v(\mathfrak{H}/\Gamma) \quad \text{for } 0 \neq f \in F.$$

To see this, take a positive integer n so that $mn \in \mathbf{Z}$ and $j(\gamma, z)^{2n} = (d\gamma(z)/dz)^{-mn}$ for all $\gamma \in \Gamma$. Such an integer n always exists by virtue of (3.1.3). Then $\operatorname{div}(f) = (2n)^{-1} \operatorname{div}(f^{2n})$, hence (3.3.5) follows from [6, Prop. 2.16].

If $0 \neq f \in F$ and $0 \neq g \in F'$, we have

$$\deg(\operatorname{div}(f) + \mathfrak{a}) = (m-1) v(\mathfrak{H}/\Gamma)/2 + \Sigma_{p \in R}\{(1 - e_p^{-1})/2 - \mu_p\} + \mathfrak{g} - 1,$$
$$\deg(\operatorname{div}(g) + \mathfrak{b}) = (1 - m/2) v(\mathfrak{H}/\Gamma) - \Sigma_{p \in R} \mu'_p.$$

By (1.6.1), we obtain

$$(3.3.6) \qquad \dim F(\mathfrak{a}) - \dim F'(\mathfrak{b}) = (m-1) v(\mathfrak{H}/\Gamma)/2 + \Sigma_{p \in R}\{(1 - e_p^{-1})/2 - \mu_p\}.$$

Further it can easily be verified that

(3.3.7) *If* $(m/2-1)v(\mathfrak{H}/\Gamma)+\Sigma_{p\in R}\mu'_p>1$, *then* $F'(\mathfrak{b})=\{0\}$, *and the condition* (2.4.10) *is satisfied.*

3.4. Lemma. *Let* $(\beta, h)\in\mathfrak{G}_m$ *and* $\beta(z_0)=z_0$ *with* $z_0\in\mathfrak{H}$. *Put* $\sigma=\begin{bmatrix}\bar{z}_0 & z_0\\ 1 & 1\end{bmatrix}$. *Then* $\sigma^{-1}\beta\sigma=\begin{bmatrix}\bar{\zeta} & 0\\ 0 & \zeta\end{bmatrix}$, $h(z_0)=\eta$ *with complex numbers* ζ *and* η *such that* $|\zeta|=|\eta|=1$. *If* $\beta=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ *and* $h(z)=t\cdot(cz+d)^m$, *then* $\zeta=cz_0+d$, $\eta=t\zeta^m$. *Especially if* $\beta\in\Gamma$ *and* $(\beta,h)=L(\beta)$, *then* $\eta=j(\beta,z_0)=\zeta^{-2e\mu}$, *where* $e=e_p$ *and* $\mu=\mu_p$ *with* $p=\varphi(z_0)$.

Proof. Almost all assertions can be verified in a straightforward manner. The relation $\eta=\zeta^{-2e\mu}$ can be obtained by considering the expansion (3.2.1) for $0\neq f\in F$ and the equality $f|L(\beta)=f$. (This is a consequence of our assumption $F\neq\{0\}$.)

3.5. Hereafter till the end of this section, we fix an element $\tau=(\alpha,h)$ of $\mathfrak{G}_m$ satisfying the following two conditions:

(3.5.1) Γ *and* $\alpha^{-1}\Gamma\alpha$ *are commensurable*;

(3.5.2) $$L(\alpha\gamma\alpha^{-1})=\tau\cdot L(\gamma)\tau^{-1}\quad \textit{for all } \gamma\in\Gamma\cap\alpha^{-1}\Gamma\alpha.$$

Put $\Gamma^*=L(\Gamma)$. Then, by [7, Prop. 1.1], the projection map $\Gamma^*\tau\Gamma^*\to\Gamma\alpha\Gamma$ is one-to-one. Let β^* denote the element of $\Gamma^*\tau\Gamma^*$ corresponding to an element β of $\Gamma\alpha\Gamma$, and put

(3.5.3) $$\beta^*=(\beta, h(\beta,z))\quad(\beta\in\Gamma\alpha\Gamma).$$

Especially $\tau=\alpha^*=(\alpha,h(\alpha,z))$. By [7, Prop. 1.0, Prop. 1.1], Γ^* is commensurable with $\tau\Gamma^*\tau^{-1}$. Moreover, if $\Gamma\alpha\Gamma=\bigcup_\nu\Gamma\alpha_\nu$ is a disjoint union, then $\Gamma^*\alpha^*\Gamma^*=\bigcup_\nu\Gamma^*\alpha^*_\nu$ is a disjoint union.

Now we define a linear transformation $[\Gamma\alpha\Gamma]^*$ on F by

$$f|[\Gamma\alpha\Gamma]^*=\Sigma_\nu f|\alpha_\nu^*=\Sigma_\nu f(\alpha_\nu(z))h(\alpha_\nu,z)^{-1}\quad(f\in F).$$

It can easily be verified that $[\Gamma\alpha\Gamma]^*$ maps $F(\mathfrak{a})$ into itself.

Furthermore, put $\Gamma_*=L'(\Gamma)$, and

(3.5.4) $$\beta_*=(\beta, h'(\beta,z)),$$

$$h'(\beta,z)=h(\beta^{-1},\beta(z))(d\beta(z)/dz)^{-1}\quad\text{for } \beta\in\Gamma\alpha^{-1}\Gamma.$$

Then it can easily be seen that α_*^{-1} satisfies (3.5.1, 2) with respect to L'. Therefore, with a disjoint union $\Gamma\alpha^{-1}\Gamma=\bigcup_\nu\Gamma\beta_\nu$, we define a linear transformation $[\Gamma\alpha^{-1}\Gamma]_*$ on F' by

$$f|[\Gamma\alpha^{-1}\Gamma]_*=\Sigma_\nu f|\beta_{\nu*}=\Sigma_\nu f(\beta_\nu(z))h'(\beta_\nu,z)^{-1}\quad(f\in F').$$

This maps $F'(\mathfrak{b})$ into itself. Our main purpose is to prove a trace-formula of the form

$$\operatorname{tr}([\Gamma\alpha\Gamma]^* \mid F(\mathfrak{a})) - \operatorname{tr}([\Gamma\alpha^{-1}\Gamma]_* \mid F'(\mathfrak{b})) = \Sigma_{\xi\in\Xi} I(\xi),$$

where $I(\xi)$ is a certain complex number defined for each "fixed point" ξ of $\Gamma\alpha\Gamma$ on V, whose precise description will be given in the next paragraph.

3.6. Define an algebraic curve $T = T(\Gamma\alpha\Gamma)$ on $V \times V$ by

$$T = \{\varphi(z) \times \varphi(\alpha(z)) \mid z \in \mathfrak{H}^*\},$$

where (V, φ) is the model of $\mathfrak{H}^*/\Gamma$ fixed in 3.2. Let us now assume the following condition on $\Gamma\alpha\Gamma$:

(3.6.1) *If π denotes the natural map of $SL_2(\mathbf{R})$ onto $SL_2(\mathbf{R})/\{\pm 1\}$, one has*

$$\pi(\alpha^{-1}\Gamma\alpha \cap \Gamma) = \pi(\alpha^{-1}\Gamma\alpha) \cap \pi(\Gamma).$$

This is satisfied whenever $-1 \in \Gamma$. Let $\Gamma\alpha\Gamma = \bigcup_{\nu=1}^{d} \Gamma\alpha_\nu$ be a disjoint union. Then we write $d = \deg(\Gamma\alpha\Gamma)$. Under the assumption (3.6.1), d is the degree of the covering

$$\mathfrak{H}^*/(\alpha^{-1}\Gamma\alpha \cap \Gamma) \to \mathfrak{H}^*/\Gamma,$$

and the algebraic correspondence T maps a point $\varphi(z)$ onto the points $\varphi(\alpha_\nu(z))$. Let us further assume that $\alpha \notin \{\pm 1\}\Gamma$. Then $\pm 1 \notin \Gamma\alpha\Gamma$, and T is different from the diagonal.

A point $\varphi(z)$ on V with $z \in \mathfrak{H}^*$ may be called a "fixed point" of T if (and only if) $z \in \Gamma\alpha\Gamma z$. However, we have to take account of "the branches of the correspondence" T passing through $\varphi(z)$. Therefore we consider all $z_0 \in \mathfrak{H}^*$ such that $z_0 \in \Gamma\alpha\Gamma z_0$, and fix a complete set of representatives Ξ_0 for such z_0 under Γ-equivalence. Then let $\Xi = \Xi(\Gamma\alpha\Gamma)$ denote the set of all couples $(z_0, \Gamma\beta)$ with $z_0 \in \Xi_0$ and $\Gamma\beta \subset \Gamma\alpha\Gamma$ such that $\Gamma\beta z_0 = \Gamma z_0$. We call Ξ a *representative set of fixed points of* $\Gamma\alpha\Gamma$. (The number of elements of Ξ is not necessarily equal to the intersection number of T with Δ.)

We are going to define a complex number $I(\xi)$ for each $\xi = (z_0, \Gamma\beta) \in \Xi$. Choose β so that $\beta(z_0) = z_0$, and call ξ *elliptic, hyperbolic*, or *parabolic*, according to the type of β (which depends only on ξ).

(I) *Elliptic case.* Put $\sigma = \begin{bmatrix} \bar{z}_0 & z_0 \\ 1 & 1 \end{bmatrix}$. By Lemma 3.4, we have $\sigma^{-1}\beta\sigma = \begin{bmatrix} \bar{\lambda} & 0 \\ 0 & \lambda \end{bmatrix}$, $h(\beta, z_0) = \eta$ with $|\lambda| = |\eta| = 1$. Then we put

$$I(\xi) = \eta^{-1}\lambda^{-2e\mu}/(1 - \lambda^{-2e}),$$

where $e = e_p$, $\mu = \mu_p$ with $p = \varphi(z_0)$. Note that $\lambda^{2e} \neq 1$ since $\beta \notin \{\pm 1\} \cdot \Gamma$. By virtue of Lemma 3.4, $I(\xi)$ depends only on ξ, and not on the choice of β.

(II) *Hyperbolic case.* Let β be an arbitrary hyperbolic element of $SL_2(\mathbf{R})$, and z_0 a fixed point of β on $\mathbf{R} \cup \{\infty\}$. Take $\varrho \in SL_2(\mathbf{R})$ so that $\varrho(\infty)=z_0$. Then $\varrho^{-1}\beta\varrho = \begin{bmatrix} \lambda^{-1} & x \\ 0 & \lambda \end{bmatrix}$ with real numbers λ and x. We call z_0 *the upper fixed point* or *the lower fixed point of* β, according as $|\lambda|>1$ or $|\lambda|<1$. This does not depend on the choice of ϱ. If z_0 is the upper fixed point of β, then the other fixed point is the lower fixed point.

Now suppose that $\beta \in \Gamma\alpha\Gamma$, and z_0 is a cusp of Γ. Take an element ϱ^* of $\mathfrak{G}_m$ with ϱ as its projection to $SL_2(\mathbf{R})$. Then we have

$$\varrho^{*-1}\beta^*\varrho^* = \left(\begin{bmatrix} \lambda^{-1} & x \\ 0 & \lambda \end{bmatrix}, \eta\right)$$

with a complex number η such that $|\eta| = |\lambda|^m$. Now, for $\xi = (z_0, \Gamma\beta)$, we put

$$I(\xi) = \begin{cases} -\eta^{-1} & \text{if } \mu_p = 1 \text{ and } |\lambda|>1, \\ 0 & \text{otherwise,} \end{cases}$$

where $p=\varphi(z_0)$. Since Γ and $\beta\Gamma\beta^{-1}$ are commensurable, λ^2 must be a rational number. (See also Lemma 4.2 below.)

(III) *Parabolic case.* Let z_0 be a cusp, and let δ be an element of Γ that generates $\{\gamma \in \Gamma \mid \gamma(z_0)=z_0\}/(\Gamma \cap \{\pm 1\})$. Take the above ϱ so that $\varrho^{-1}\delta\varrho = \varepsilon \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ with $\varepsilon = \pm 1$, and take an element ϱ^* of $\mathfrak{G}_m$ whose projection to $SL_2(\mathbf{R})$ is ϱ. Then

$$\varrho^{*-1}L(\delta)\,\varrho^* = \left(\varepsilon\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, e^{2\pi i\mu}\right),$$

where $\mu = \mu_p$ with $p=\varphi(z_0)$. Now let $\xi = (z_0, \Gamma\beta)$ with a parabolic β such that $\beta(z_0)=z_0$. Then

$$\varrho^{*-1}\beta^*\varrho^* = \left(c\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}, \eta\right)$$

with $c=\pm 1$, $x \in \mathbf{R}$, $|\eta|=1$. We put then

$$I(\xi) = \eta^{-1}e^{2\pi i\mu x}/(1-e^{2\pi i x}).$$

In each of the three cases, the number $I(\xi)$ is independent of the choice of β.

We are now ready to state our main result:

3.7. THEOREM. *Let* $[\Gamma\alpha\Gamma]^*$ *and* $[\Gamma\alpha^{-1}\Gamma]_*$ *be as in* 3.5, *under the assumption* (3.6.1). *Suppose that* $\alpha \notin \{\pm 1\}\Gamma$, *and the divisor* $\mathfrak{a}$ *defined in* 3.3 *satisfies* (2.4.10). *Then*

$$\operatorname{tr}([\Gamma\alpha\Gamma]^* \mid F(\mathfrak{a})) - \operatorname{tr}([\Gamma\alpha^{-1}\Gamma]_* \mid F'(\mathfrak{b})) = \Sigma_{\xi\in\Xi} I(\xi),$$

where Ξ *and* $I(\xi)$ *are defined as in* 3.6.

The proof will be completed in 3.11.

3.8. As a preliminary step, let us make a few observations about the field of rationality for automorphic forms, although these are actually dispensable. Let $\Gamma_1 = \Gamma \cap \alpha^{-1}\Gamma\alpha$, and let (V_1, φ_1) be a model of $\mathfrak{H}^*/\Gamma_1$. Then V_1 is birationally equivalent with the curve T. Obviously the restriction of L or L' to Γ_1 are also a proper lifting of Γ_1, hence we can define $\mathfrak{F}_1 = \{F_1, F_1', Z_1, \mathfrak{z}_1\}$ for V_1 and Γ_1. Now there are two projection maps π and π' of V_1 onto V defined by $\pi\circ\varphi_1 = \varphi$ and $\pi'\circ\varphi_1 = \varphi\circ\alpha$. Fix any non-zero $f_0 \in F$ and $f_1 \in F_1$. Then we see that both f_1/f_0 and $(f_0 \mid \alpha^*)/f_0$ belong to $\mathbf{C}(V_1)\circ\varphi_1$. Fix any field of rationality k_0 for $\mathfrak{F}$, $\mathfrak{F}_1$, π, π', f_0, f_1/f_0, and $(f_0 \mid \alpha^*)/f_0$. Then if $k_0 \subset k$, we see that

$$F(k) = F \cap F_1(k),\ F'(k) = F' \cap F_1'(k); \tag{3.8.1}$$

$$f \in F(k) \Rightarrow f \mid \alpha^* \in F_1(k). \tag{3.8.2}$$

In fact, if $f \in F(k)$, then $f = (r\circ\varphi)f_0$ with $r \in k(V)$, so that $f = (r\circ\pi\circ\varphi_1)f_1\cdot(f_1/f_0)^{-1} \in F_1(k)$ and $f \mid \alpha^* = (r\circ\varphi\circ\alpha)(f_0 \mid \alpha^*)f_0^{-1}f_0 = (r\circ\pi'\circ\varphi_1)(f_0 \mid \alpha^*)f_0^{-1}f_0 \in F_1(k)$, q.e.d.

Let $\Gamma\alpha\Gamma = \bigcup_\nu \Gamma\alpha_\nu$ be a disjoint union. Then we see easily that

(3.8.3) *For* $f \in F(k)$, *let* r *be an element of* $k(V_1)$ *such that* $r\circ\varphi_1 = (f \mid \alpha^*)/f$. *Then* $(\mathrm{Tr}_{k(V_1)/k(V)}(r)\circ\varphi)f = f \mid [\Gamma\alpha\Gamma]^*$, *where* Tr *is defined with respect to the injection* $k(V) \to k(V)\circ\pi \subset k(V_1)$.

This shows especially that $[\Gamma\alpha\Gamma]^*$ maps $F(k)$ into itself.

3.9. For each field k of rationality for $\mathfrak{F}$, define $E(k)$ as in 2.1, and let E denote the union of $E(k)$ for all fields k of rationality for $\mathfrak{F}$. Then E is a one-dimensional vector space over $\mathbf{C}(V \times V)$. With each $X = A \otimes f \otimes g \in E$ with $A \in \mathbf{C}(V \times V)$, $f \in F$, and $g \in F'$, we associate a meromorphic function $X(z, w)$ on $\mathfrak{H} \times \mathfrak{H}$ by

$$X(z, w) = A(\varphi(z), \varphi(w))f(z)g(w) \quad ((z, w) \in \mathfrak{H} \times \mathfrak{H}).$$

This does not depend on the choice of A, f, g for a given X, and

$$X(\gamma(z), \delta(w)) = X(z, w)j(\gamma, z)j'(\delta, w) \quad \text{for } (\gamma, \delta) \in \Gamma \times \Gamma.$$

In this way E can be identified with the set of all meromorphic functions $X(z, w)$ on $\mathfrak{H} \times \mathfrak{H}$ such that $X(z, w)f(z)^{-1}g(w)^{-1}$ is an element of $\mathbf{C}(V \times V)$ for $0 \neq f \in F$, $0 \neq g \in F'$.

Let $\Gamma\alpha\Gamma = \bigcup_\nu \Gamma\alpha_\nu$ be as before. We now define $X \mid T$ to be an element of E such that

$$(X \mid T)(z, w) = \Sigma_\nu X(\alpha_\nu(z), w)h(\alpha_\nu, z)^{-1}.$$

More algebraically, we have

$$X\,|\,T=\mathrm{Tr}_{\mathbf{C}(V_1\times V)/\mathbf{C}(V\times V)}(A')\otimes f\otimes g,$$

where A' is an element of $\mathbf{C}(V_1\times V)$ such that

$$A'(\varphi_1(z),\,\varphi(w))=A(\varphi(\alpha(z)),\,\varphi(w))(f\,|\,\alpha^*)(z)/f(z).$$

In view of (3.8.3), this shows that

(3.9.1) $\quad X\,|\,T\in E(k)$ *if* $X\in E(k)$ *and* k *contains the field* k_0 *of* 3.8.

Suppose that the diagonal of $V\times V$ is not contained in the pole of X. Then we see that $X(z,z)$ is a Γ-automorphic form of weight 2 in the ordinary sense. Therefore $X(z,z)dz$ can be viewed as a differential form on V, hence the residue $\mathrm{Res}_p(X(z,z)dz)$ at each $p\in V$ is meaningful. We write

$$X(z,z)dz=X_{z=w}dz.$$

It can easily be seen that

(3.9.2) $\quad X_{z=w}dz$ *is* k*-rational if* $X\in E(k)$.

3.10. We take the field k_0 of 3.8 so that the points of R, $\mathfrak{a}$, $\mathfrak{b}$, and T are all rational over k_0, and take an extension k of k_0 which is algebraically closed and has an infinite transcendence degree over k_0. With this k as the basic field, we define objects f_i, g_j, u, v, G_p, S_p, A_p, B, and q_i as in § 2. Put

$$n=\dim F(\mathfrak{a}),\quad r=\dim F'(\mathfrak{b}).$$

In § 2, we chose an arbitrary $\{g_i\}$ dual to $\{f_i\}$. Here we fix a k_0-rational point q of $V-R$, which is neither a fixed point of T, nor contained in the image or the inverse image of R by T. Then we choose $\{g_i\}$ so that

(3.10.1) $$g_{jp}=0\quad\text{for } q\neq p\in P(k).$$

This is possible by virtue of Proposition 1.9. Note also that the set of points $\{q_i\}$ is disjoint with the image and the inverse image of $\{q\}\cup R$ by T and also with the fixed points of T, since the q_i are generic points of V over k_0. We can also choose u and v so that

(3.10.2) $$\nu_p(u)=\mu_p,\ \nu_p(v)=\mu'_p\quad\text{for all } p\in R\cup\varphi(\Xi_0).$$

For brevity, let us write $T^*=[\Gamma\alpha\Gamma]^*$. To compute $\mathrm{tr}\,(T^*\,|\,F(\mathfrak{a}))$, put $f_j\,|\,T^*=\Sigma_{i=1}^n a_{ij}f_i$ with $a_{ij}\in k$. Since $\{g_i\}$ is dual to $\{f_i\}$, we have

$$a_{ij}=\Sigma_{p\in P(k)}\mathrm{Res}_p\,((f_j\,|\,T^*)g_{ip}dz)=\mathrm{Res}_q\,((f_j\,|\,T^*)g_{iq}dz)$$

by (3.10.1). By (2.4.4), we have

(3.10.3) $$\Sigma_{i=1}^n f_i\otimes g_{iq}=A_q+B.$$

By (2.4.9), a non-constant divisor $\mathfrak{P}$ of $k(V\times V)$ is contained in the pole of B if and only if $\mathfrak{P}$ is the diagonal Δ. By (3.10.3), A_q has the same property. Since T is different from the diagonal, both $(B|T)_{z=w}dz$ and $(A_q|T)_{z=w}dz$ are meaningful, hence

$$\text{(3.10.4)} \qquad \mathrm{tr}\,(T^*|F(\mathfrak{a}))=\Sigma_{i=1}^{n}a_{ii}=\mathrm{Res}_q\,((A_q|T)_{z=w}dz)+\mathrm{Res}_q\,((B|T)_{z=w}dz).$$

Now by (2.4.7), (2.4.9), (2.8.1), (2.8.2), we have

$$\text{(3.10.5}_{\mathrm{a}}) \qquad \nu_{V\times p}(B)\geqslant 0 \qquad (p\in V-R\cup\{q\}),$$

$$\text{(3.10.5}_{\mathrm{b}}) \qquad \nu_{V\times p}(B)\geqslant \mu_p' \qquad (p\in R),$$

$$\text{(3.10.5}_{\mathrm{c}}) \qquad \nu_{p\times V}(B)\geqslant 0 \qquad (p\in V-R\cup\{q_1,\ldots,q_r\}),$$

$$\text{(3.10.5}_{\mathrm{d}}) \qquad \nu_{p\times V}(B)\geqslant \mu_p \qquad (p\in R),$$

$$\text{(3.10.5}_{\mathrm{e}}) \qquad \nu_{q_i\times V}(B)\geqslant -1 \qquad (i=1,\ldots,r),$$

$$\text{(3.10.5}_{\mathrm{f}}) \qquad \nu_{\Delta}(B)=-1 \qquad (\Delta\text{: diagonal}),$$

$$\text{(3.10.5}_{\mathrm{g}}) \qquad \nu_{\mathfrak{P}}(B)\geqslant 0 \qquad (\mathfrak{P}\text{ non-contant}, \neq\Delta).$$

(In § 2, we considered only k-rational prime divisors. However, since B is k-rational, we see easily that the above inequalities hold for any points or divisors which are not necessarily k-rational.)

By (2.4.6), we have $\nu_{V\times q}(A_q)\geqslant 0$. Now let $q=\varphi(z_0)$ with a point z_0 of $\mathfrak{H}$. For any $\beta\in\Gamma\alpha\Gamma$, put $p=\varphi(\beta(z_0))$. By (3.10.3) and (3.10.5$_{\mathrm{c}}$), we have $\nu_{p\times V}(A_q)\geqslant 0$. It follows that $A_q(\beta(z), z)$ is finite at $z=z_0$ for every $\beta\in\Gamma\alpha\Gamma$. (One cannot have $p=q$ because of our choice of q.) Therefore $\mathrm{Res}_q\,((A_q|T)_{z=w}dz)=0$. On the other hand, $(B|T)_{z=w}dz$ is a differential form on V, hence

$$\Sigma_{p\in V}\,\mathrm{Res}_p\,((B|T)_{z=w}dz)=0.$$

Therefore (3.10.4) becomes

$$\text{(3.10.6)} \qquad \mathrm{tr}\,(T^*|F(\mathfrak{a}))=\mathrm{Res}_q\,((B|T)_{z=w}dz)=-\Sigma_{p\neq q}\,\mathrm{Res}_p\,((B|T)_{z=w}dz).$$

(By (3.8.1, 2), $(B|T)_{z=w}dz$ is k-rational, but we do not need this fact.)

3.11. Our task is thus to compute $\mathrm{Res}_p\,((B|T)_{z=w}dz)$ for each $p\neq q$. Let us first show that the residue can be non-trivial only when either p is a fixed point of T, or p belongs to the inverse image of $\{q_1,\ldots,q_r\}$ under T. Let $p=\varphi(z_0)$, $p'=\varphi(\beta(z_0))$ with any $z_0\in\mathfrak{H}^*$ and any $\beta\in\Gamma\alpha\Gamma$. Suppose $p\neq q$ and $p'\notin\{p, q_1,\ldots,q_r\}$. Take $u_1\in F$ and $v_1\in F'$ so that $\nu_{p'}(u_1)=\mu_{p'}$ and $\nu_p(v_1)=\mu_p'$. Put $B(z,w)=B_1(z,w)u_1(z)v_1(w)$. Then

$$\text{(3.11.0)} \qquad B(\beta(z), z)h(\beta, z)^{-1}dz=B_1(\beta(z), z)(u_1|\beta^*)(z)v_1(z)dz.$$

By (3.10.5), B_1 is finite at $(\beta(z_0), z_0)$. If z_0 is not a cusp, we have $\nu_p(v_1dz)=-\mu_p>-1$, and

$u_1|\beta^*$ is finite at z_0. If z_0 is a cusp, then $\nu_p(v_1 dz) = -\mu_p \geqslant -1$, and $u_1|\beta^*$ vanishes at z_0, since $\beta(z_0)$ is also a cusp, and $\nu_{p'}(u_1) = \mu_{p'} > 0$. Therefore, in either case, the form (3.11.0) measured by a local parameter on V at p has order > -1, hence the desired conclusion.

To compute the residue at a fixed point p of T, take $z_0 \in \Xi_0$ so that $p = \varphi(z_0)$, and consider $\xi = (z_0, \Gamma\beta) \in \Xi$ such that $\beta(z_0) = z_0$.

(I) First suppose that β is elliptic, hence $z_0 \in \mathfrak{H}$. Let $\mathfrak{D}$ denote the unit disc, and put

$$\sigma = \begin{bmatrix} \bar{z}_0 & z_0 \\ 1 & 1 \end{bmatrix}, \ \sigma(s) = (\bar{z}_0 s + z_0)/(s+1) \quad \text{for } s \in \mathfrak{D},$$

and define a holomorphic function $\varkappa$ on $\mathfrak{D}$ by

$$\varkappa(0) = 1, \ \varkappa(s) = (s+1)^m \quad (s \in \mathfrak{D}).$$

Then σ maps $\mathfrak{D}$ onto $\mathfrak{H}$, and $\sigma(0) = z_0$. By Lemma 3.4, if $\beta \in \Gamma\alpha\Gamma$ and $\beta(z_0) = z_0$, we have

$$\sigma^{-1}\beta\sigma = \begin{bmatrix} \bar{\lambda} & 0 \\ 0 & \lambda \end{bmatrix}, \ h(\beta, z_0) = \eta$$

with $|\lambda| = |\eta| = 1$. Moreover we can easily verify that

$$h(\beta, \sigma(s)) = \eta \cdot \varkappa(\lambda^{-2}s)/\varkappa(s). \tag{3.11.1}$$

Let us write e, μ, μ' for e_p, μ_p, μ'_p with $p = \varphi(z_0)$. By (3.10.2), we can put $u(\sigma(s)) = s^{e\mu}u_0(s)$, $v(\sigma(s)) = s^{e\mu'}v_0(s)$ with functions u_0 and v_0 which are holomorphic and $\neq 0$ at the origin. Put $B = B_0 \otimes u \otimes v$ with $B_0 \in \mathrm{C}(V \times V)$. Further put $\psi = \varphi \circ \sigma$ and

$$\begin{aligned} D(s, t) &= (s^e - t^e)s^{-e\mu}t^{-e\mu'}B(\sigma(s), \sigma(t)) \\ &= (s^e - t^e)B_0(\psi(s), \psi(t))u_0(s)v_0(t) \quad ((s, t) \in \mathfrak{D} \times \mathfrak{D}). \end{aligned}$$

Now s^e is a local parameter at z_0. Therefore, by (3.10.5), we see that $D(s, t)$ is holomorphic at $(0, 0)$. Consider the differential form

$$B_0(\psi(s), \psi(t))u(\sigma(s))v(\sigma(s))d\sigma(s) = (s^e - t^e)^{-1}D(s, t)s^{e\mu+e\mu'}v_0(s)v_0(t)^{-1}d\sigma(s).$$

By Proposition 2.9, viewing t as a constant, the residue of this form at $s^e = t^e$ is 1. Since $e\mu + e\mu' = e - 1$ we have

$$s^{e\mu+e\mu'}d\sigma(s) = e^{-1}\sigma'(s)d(s^e), \tag{3.11.2}$$

hence $e^{-1}\sigma'(t)D(t, t) = 1$, especially

$$e^{-1}\sigma'(0)D(0, 0) = 1. \tag{3.11.3}$$

Putting $z = \sigma(s)$, we have, by (3.11.1, 2),

$$
\begin{aligned}
B(\beta(z), z)h(\beta, z)^{-1}dz &= B(\sigma(\lambda^{-2}s), \sigma(s))h(\beta, \sigma(s))^{-1}d\sigma(s)\\
&= (\lambda^{-2e}s^e - s^e)^{-1}(\lambda^{-2}s)^{e\mu}s^{e\mu'}D(\lambda^{-2}s, s)\eta^{-1}\varkappa(\lambda^{-2}s)^{-1}\varkappa(s)d\sigma(s)\\
&= \eta^{-1}\lambda^{-2e\mu}(\lambda^{-2e}-1)^{-1}s^{-e}e^{-1}\sigma'(s)D(\lambda^{-2}s, s)\varkappa(\lambda^{-2}s)^{-1}\varkappa(s)d(s^e).
\end{aligned}
$$

By (3.11.3), the residue of the last form at $s^e=0$ is $-I(\xi)$ with $\xi=(z_0, \Gamma\beta)$, hence

$$
\operatorname{Res}_p[(B|T)_{z=w}dz] = -\Sigma_\xi I(\xi), \tag{3.11.4}
$$

the sum being taken over all $\xi=(z_0, \Gamma\beta)$ with the fixed point z_0 in question.

(II) Suppose $p=\varphi(z_0)$ with a cusp z_0 of Γ, $\beta(z_0)=z_0$ with a hyperbolic element $\beta\in\Gamma\alpha\Gamma$. We may assume $z_0=\infty$, and take $t(z)=e^{2\pi iz}$ as a local parameter. Again we write μ and μ' for μ_p and μ'_p. By virtue of (3.10.5), if we put

$$
H(t(z), t(w)) = t(\mu z)^{-1}t(\mu' w)^{-1}(t(z)-t(w))B(z, w),
$$

then H is holomorphic at $(0, 0)$. Define B_0 as in (I). Then

$$
\begin{aligned}
&B_0(\varphi(z), \sigma(w))u(z)v(z)dz\\
&\qquad= (t(z)-t(w))^{-1}H(t(z), t(w))v(z)t(\mu' z)^{-1}v(w)^{-1}t(\mu' w)(2\pi i)^{-1}dt(z).
\end{aligned}
$$

Viewing $t(w)$ as a constant, this has residue 1 at $t(z)=t(w)$, by virtue of Proposition 2.9, hence $H(t(w), t(w))=2\pi i$, especially

$$
H(0, 0) = 2\pi i. \tag{3.11.5}
$$

Now we can put $\beta=\begin{bmatrix}\lambda^{-1} & 0\\ 0 & \lambda\end{bmatrix}$ and $h(\beta, z)=\eta$ with $\lambda\in\mathbf{R}$ and $\eta\in\mathbf{C}$. We have seen that λ^2 is a rational number. Put $\varkappa=\lambda^{-2}$. Then $\beta(z)=\varkappa z$, and

$$
\begin{aligned}
&B(\beta(z), z)h(\beta, z)^{-1}dz \\
&\qquad= \eta^{-1}t(z)^{-1}t((\varkappa\mu+\mu')z)(t(\varkappa z)-t(z))^{-1}H(t(\varkappa z), t(z))(2\pi i)^{-1}dt(z).
\end{aligned} \tag{3.11.6}
$$

If $\varkappa<1$, we have

$$
t((\varkappa\mu+\mu')z)/(t(\varkappa z)-t(z)) = t(\mu'(1-\varkappa)z)/(1-t((1-\varkappa)z)),
$$

hence the residue, or more precisely the coefficient of $t(z)^{-1}dt(z)$ of (3.11.6), is either 0 or η^{-1} according as $\mu'>0$ or $\mu'=0$, by virtue of (3.11.5). If $\varkappa>1$,

$$
t((\varkappa\mu+\mu')z)/(t(\varkappa z)-t(z)) = -t((\varkappa-1)\mu z)/(1-t((\varkappa-1)z)),
$$

hence the "residue" of (3.11.6) is 0. Thus (3.11.4) holds also for hyperbolic ξ.

(III) Still with $z_0=\infty$, suppose β parabolic. We can put

$$
\beta^* = \left(\varepsilon\begin{bmatrix}1 & x\\ 0 & 1\end{bmatrix}, \eta\right)
$$

with $\varepsilon=\pm 1$, $x\in\mathbf{R}$, $|\eta|=1$. Put $\zeta_1=e^{2\pi ix}$, $\zeta_2=e^{2\pi i\mu x}$. With the same $t(z)$ and H as in (II), we have

$$B(\beta(z), z)h(\beta, z)^{-1}dz=\eta^{-1}(\zeta_1 t-t)^{-1}\zeta_2 H(\zeta_1 t, t)(2\pi i)^{-1}dt,$$

hence the residue at $t=0$ is $\eta^{-1}\zeta_2/(\zeta_1-1)=-I(\xi)$.

(IV) Suppose $r(=\dim F'(\mathfrak{b}))>0$, and p belongs to the inverse image of $\{q_1, \ldots, q_r\}$ under T. Let $q_j=\varphi(z_j)$ for $j=1, \ldots, r$. Then the sum of $\mathrm{Res}_p[(B|T)_{z=w}dz]$ at all such p is equal to

$$\textstyle\sum_{j=1}^r \sum_{\Gamma\delta w=\Gamma z_j}\mathrm{Res}_w[B(\delta(z), z)h(\delta, z)^{-1}dz],$$

where the second Σ is extended over all $\Gamma\delta\subset\Gamma\alpha\Gamma$ such that $\Gamma\delta w=\Gamma z_j$; w is any point satisfying $\Gamma\delta w=\Gamma z_j$. Take a set of representatives $\{\beta\}$ so that $\Gamma\alpha^{-1}\Gamma=\cup\,\Gamma\beta=\cup\,\beta\Gamma$. Then $\Gamma\alpha\Gamma=\bigcup\Gamma\beta^{-1}$, and the above sum becomes

$$\begin{aligned}(3.11.7)\qquad &\textstyle\sum_{j=1}^r\sum_\beta \mathrm{Res}_{\beta(z_j)}[B(\beta^{-1}(z), z)h(\beta^{-1}, z)^{-1}dz]\\ &=\textstyle\sum_{j=1}^r\sum_\beta \mathrm{Res}_{z_j}[B(z, \beta(z))h'(\beta, z)^{-1}dz],\end{aligned}$$

where h' is defined by (3.5.4). Fix an element g of $F'(\mathfrak{b})$ such that $\nu_{q_i}(g)=0$ for $i=1, \ldots, r$, and define an element B_g of $\mathbf{C}(V\times V)$ by $B=B_g\otimes u\otimes g$, and further define $c_j\in\mathbf{C}(V)$ as in Proposition 2.10 with g in place of w. Put

$$H_j(z, w)=(z-z_j)B_g(\varphi(z), \varphi(w))u(z)g(w).$$

Since $\varphi(z_j)\neq\varphi(\beta(z_j))$, we see, by (3.10.5), that $H_j(z, w)$ is holomorphic at $(z_j, \beta(z_j))$. Therefore, by Proposition 2.10, viewing w as a constant, we obtain

$$c_j(\varphi(w))=\mathrm{Res}_{z_j}[(z-z_j)^{-1}H_j(z, w)g(z)g(w)^{-1}dz],$$

hence $H_j(z_j, w)g(z_j)/g(w)=c_j(\varphi(w))$, especially, putting $a_j(z)=c_j(\varphi(z))g(z)$, we have

$$(3.11.8)\qquad H_j(z_j, \beta(z_j))=a_j(\beta(z_j))/g(z_j).$$

Therefore (3.11.7) equals

$$\begin{aligned}(3.11.9)\qquad &\textstyle\sum_{j=1}^r\sum_\beta \mathrm{Res}_{z_j}[(z-z_j)^{-1}H_j(z, \beta(z))h'(\beta, z)^{-1}dz]\\ &=\textstyle\sum_{j=1}^r\sum_\beta H_j(z_j, \beta(z_j))h'(\beta, z_j)^{-1}\\ &=\textstyle\sum_{j=1}^r g(z_j)^{-1}\sum_\beta a_j(\beta(z_j))h'(\beta, z_j)^{-1}\\ &=\textstyle\sum_{j=1}^r g(z_j)^{-1}b_j(z_j),\end{aligned}$$

where we put $b_j=a_j|[\Gamma\alpha^{-1}\Gamma]_*$. By Proposition 2.10, $\{a_j\}$ is a basis of $F'(\mathfrak{b})$, and $b_i=-\sum_{j=1}^r(b_i/g)(z_j)\cdot a_j$, hence

$$\mathrm{tr}([\Gamma\alpha^{-1}\Gamma]_*|F'(\mathfrak{b}))=-\textstyle\sum_{j=1}^r(b_j/g)(z_j),$$

which is exactly (-1) times (3.11.9).

Combining the results of (I, II, III, IV) together, we obtain Theorem 3.7.

3.12. *Remark.* In this section we have considered only a special type of divisors $\mathfrak{a}$ and $\mathfrak{b}$, while a more general case was discussed in § 2. Actually we could state our theorem in such a general case, provided that $[\Gamma\alpha\Gamma]^*$ (resp. $[\Gamma\alpha^{-1}\Gamma]_*$) maps $F(\mathfrak{a})$ (resp. $F'(\mathfrak{b})$) into itself. In general, however, it is not easy to obtain a criterion for this requirement. A discussion is given in Eichler [3] for a question of the same type in a somewhat different formulation.

4. The trace formula: second formulation

4.1. We shall now express the sum $\Sigma_{\xi\in\Xi} I(\xi)$ of Theorem 3.7 in a more group-theoretical fashion. We do this not only for its own sake, but also to weaken the condition (2.4.10) under which the formula was proved. We shall introduce certain equivalence classes C in $\Gamma\alpha\Gamma$, and define a complex number $J(C)$ for each $C\subset\Gamma\alpha\Gamma$. Then the sum $\Sigma_{\xi\in\Xi} I(\xi)$ will be expressed as $\Sigma_{C\subset\Gamma\alpha\Gamma} J(C)$. To define $J(C)$, first put, for each $\beta\in\Gamma\alpha\Gamma$,

$$Z_\Gamma(\beta) = \{\gamma\in\Gamma \mid \gamma\beta = \beta\gamma\}.$$

Let $\Phi(\Gamma\alpha\Gamma)$ denote the subset of $\Gamma\alpha\Gamma$ consisting of:

all scalar elements of $\Gamma\alpha\Gamma$,

all elliptic elements of $\Gamma\alpha\Gamma$,

all hyperbolic elements of $\Gamma\alpha\Gamma$ whose upper fixed points are cusps of Γ (see 3.6, (II)),

all parabolic elements of $\Gamma\alpha\Gamma$ whose fixed points are cusps of Γ.

We call two elements β and β' of $\Phi(\Gamma\alpha\Gamma)$ *equivalent* if:

$\beta = \beta'$ when β and β' are scalars,

$\gamma\beta\gamma^{-1} = \beta'$ for some $\gamma\in\Gamma$, when β and β' are elliptic or hyperbolic,

$\gamma\beta'\gamma^{-1}\in Z_\Gamma(\beta)\beta$ for some $\gamma\in\Gamma$, when β and β' are parabolic.

We denote by $\Phi(\Gamma\alpha\Gamma/\Gamma)$ the set of all equivalence classes in $\Phi(\Gamma\alpha\Gamma)$ in this sense. Let $\beta\in\Phi(\Gamma\alpha\Gamma)$. If β is elliptic or parabolic, then β has a unique fixed point z_0 in $\mathfrak{H}^*$. Then

$$Z_\Gamma(\beta) = \{\gamma\in\Gamma \mid \gamma(z_0) = z_0\}.$$

If β is hyperbolic, one has $Z_\Gamma(\beta) = \Gamma\cap\{\pm 1\}$.

Now we define, for each $C\in\Phi(\Gamma\alpha\Gamma/\Gamma)$, a complex number $J(C)$ as follows:

$$J(C) = \begin{cases} [\Gamma\cap\{\pm 1\}:1]^{-1}\eta^{-1}2^{-1}(m-1)\,v(\mathfrak{H}/\Gamma) & \text{if } \beta^* = (\pm 1, \eta), \\ [Z_\Gamma(\beta):1]^{-1}\eta^{-1}/(1-\lambda^{-2}) & \text{if } \beta \text{ is elliptic}, \\ -[\Gamma\cap\{\pm 1\}:1]^{-1}\eta^{-1}/(1-\lambda^{-2}) & \text{if } \beta \text{ is hyperbolic}, \\ \eta^{-1}e^{2\pi i\mu x}(2^{-1}-\mu) & \text{if } \beta \text{ is parabolic and } \beta\in\{\pm 1\}\cdot\Gamma, \\ \eta^{-1}e^{2\pi i\mu x}/(1-e^{2\pi i x}) & \text{if } \beta \text{ is parabolic and } \beta\notin\{\pm 1\}\cdot\Gamma. \end{cases}$$

In each case we pick any β from C, and define λ, η, μ, and x for β as in 3.6, (I, II, III). We have $|\lambda|>1$ for hyperbolic β, since we consider only the upper fixed point of β. Obviously $J(C)$ does not depend on the choice of β. Note also that $J(C)\neq 0$ even if β is hyperbolic and $\mu<1$.

4.2. Lemma. *Let β be a hyperbolic element of $\Gamma\alpha\Gamma$ with a cusp z_0 as its fixed point. Let δ be an element of Γ that generates*

$$\{\gamma\in\Gamma \mid \gamma(z_0)=z_0\}/(\Gamma\cap\{\pm 1\}).$$

Let ϱ^* be an element of $\mathfrak{G}_m$ whose projection ϱ to $SL_2(\mathbf{R})$ is such that $\varrho^{-1}\delta\varrho=\varepsilon\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$ with $\varepsilon=\pm 1$, and put

$$\varrho^{*-1}\delta^*\varrho^*=\left(\varepsilon\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}, e^{2\pi i\mu}\right),$$

$$\varrho^{*-1}\beta^*\varrho^*=\left(\begin{bmatrix}\lambda^{-1} & x\\ 0 & \lambda\end{bmatrix}, \eta\right).$$

Then λ^2 is a rational number. Moreover, put $\lambda^2=s/t$ with positive integers s and t such that $(s,t)=1$. Then

(i) $\beta\delta^s=\varepsilon^{t-s}\delta^t\beta$;

(ii) *$s-t$ is even, if $-1\notin\Gamma$ and $\varepsilon=-1$;*

(iii) $(s-t)\mu\in\mathbf{Z}$.

Proof. We have

$$\varrho^{-1}\beta^{-1}\varrho\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\varrho^{-1}\beta\varrho=\begin{bmatrix}1 & \lambda^2\\ 0 & 1\end{bmatrix},$$

hence the rationality of λ^2 follows from the commensurability of Γ with $\beta^{-1}\Gamma\beta$. Then (i) is immediate. If $-1\notin\Gamma$, $\varepsilon=-1$, and $s-t$ is odd, then $\delta^{-t}\beta\delta^s=-\beta$, which contradicts the assumption (3.6.1). To prove (iii), we may assume $\beta\delta^s=\delta^t\beta$. (If $\varepsilon=-1$ and $s-t$ is odd, then $-1\in\Gamma$. Take $-\delta$ in place of δ.) Then $\varrho^{*-1}(\delta^{-t}\beta\delta^s)^*\varrho^*=\varrho^{*-1}\beta^*\varrho^*$, hence $e^{2\pi i\mu(s-t)}\cdot\eta=\eta$, which proves (iii). (Note that (iii) is a consequence of (3.5.2).)

4.3. Lemma. *Let x be an indeterminate, and let ζ be a primitive k-th root of unity with a positive integer $k>1$. Then, for $b=0, 1, \ldots, k-1$, one has*

$$\textstyle\sum_{a=1}^{k}\zeta^{-ab}/(1-\zeta^a x)=kx^b/(1-x^k),$$

$$\textstyle\sum_{a=1}^{k-1}\zeta^{-ab}/(1-\zeta^a)=(k-1)/2-b.$$

The proof is easy, and therefore omitted.

4.4. For any subgroup Γ_1 of Γ of finite index, we can consider the restrictions of L and L' to Γ_1. Then we can define objects F_1, F_1', $\mathfrak{a}_1$, $\mathfrak{b}_1$ with respect to Γ_1 corresponding to F, F', $\mathfrak{a}$, $\mathfrak{b}$. If an element $\tau=(\alpha, h)$ of $\mathfrak{G}_m$ satisfies (3.5.1, 2), then it satisfies the same conditions with Γ_1 in place of Γ. Therefore $[\Gamma_1\alpha\Gamma_1]^*$ and $[\Gamma_1\alpha^{-1}\Gamma_1]_*$ are meaningful.

4.5. THEOREM. *Let $\tau=(\alpha, h)$ be an element of $\mathfrak{G}_m$ satisfying* (3.5.1, 2) *and* (3.6.1). *Suppose that Γ has a normal subgroup Γ_1 of finite index with the following properties:*

(i) $\deg(\Gamma\alpha\Gamma)=\deg(\Gamma_1\alpha\Gamma_1)$;

(ii) $\Gamma\alpha\Gamma_1=\Gamma_1\alpha\Gamma=\Gamma\alpha\Gamma$;

(iii) $F_1(\mathfrak{a}_1-p)\neq F_1(\mathfrak{a}_1)$ *for every* $p\in V_1=\mathfrak{H}^*/\Gamma_1$;

(iv) Γ_1 *and* α *satisfy* (3.6.1).

Then, without assuming (2.4.10) *for* $F(\mathfrak{a})$, *one has*

$$\operatorname{tr}([\Gamma\alpha\Gamma]^*\,|\,F(\mathfrak{a}))-\operatorname{tr}([\Gamma\alpha^{-1}\Gamma]_*\,|\,F'(\mathfrak{b}))=\Sigma_{C\in\Phi(\Gamma\alpha\Gamma/\Gamma)}J(C).$$

Proof. Let us first prove the case $\alpha=\pm 1$. Put $\alpha^*=(\alpha, t)$ with $|t|=1$. Then $(\alpha^{-1})_*=(\alpha, t)$, hence $f|[\Gamma\alpha\Gamma]^*=t^{-1}f$, $g|[\Gamma\alpha^{-1}\Gamma]_*=t^{-1}g$. Therefore our formula follows from (3.3.6) and Lemmas 3.4, 4.3.

Next let us prove the case $\Gamma=\Gamma_1$, assuming $\alpha\notin\{\pm 1\}\cdot\Gamma$. In this case, our task is to transform the sum $\Sigma_{\xi\in\Xi}I(\xi)$ into $\Sigma_C J(C)$. Let $\xi=(z_0, \Gamma\beta)$ be as in 3.6, and suppose that ξ is elliptic and $\beta(z_0)=z_0$ with $z_0\in\mathfrak{H}$. Let γ be a generator of $Z_\Gamma(\beta)$, and put $\sigma=\begin{bmatrix}\bar z_0 & z_0\\ 1 & 1\end{bmatrix}$. By Lemma 3.4, we can put

$$\sigma^{-1}\gamma\sigma=\begin{bmatrix}\bar\zeta & 0\\ 0 & \zeta\end{bmatrix},\ j(\gamma, z_0)=\zeta^{-2e\mu},$$

$$\sigma^{-1}\beta\sigma=\begin{bmatrix}\bar\lambda & 0\\ 0 & \lambda\end{bmatrix},\ j(\beta, z_0)=\eta,$$

where $e=e_p$, $\mu=\mu_p$ with $p=\varphi(z_0)$. Let C_a denote the class containing $\gamma^a\beta$ for $a=1, \ldots, k$, where $k=[Z_\Gamma(\beta):1]$. Thus ξ corresponds exactly to these k classes C_a. Now $k=2e$ or e according as k is even or odd, and in both cases one has

$$\begin{aligned}\textstyle\sum_{a=1}^k J(C_a)&=\textstyle\sum_{a=1}^k k^{-1}\eta^{-1}\zeta^{2ae\mu}/(1-\lambda^{-2}\zeta^{-2a})\\ &=\eta^{-1}\lambda^{-2e\mu}/(1-\lambda^{-2e})=I(\xi)\end{aligned}$$

by Lemma 4.3.

Next suppose that β is hyperbolic. Without losing generality, we may assume ∞ is the upper fixed point of β. Define δ, ε, λ, η, s, and t as in Lemma 4.2.

(We may assume $\varrho^*=1$, so that $\delta=\varepsilon\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$.) Put $\Gamma_\infty=\{\gamma\in\Gamma \mid \gamma(\infty)=\infty\}$. We have $s>t$, since $|\lambda|>1$. Let us first assume that $-1\notin\Gamma$ and $\varepsilon=-1$. Then

$$\{\sigma\in\Gamma\beta\delta^n \mid \sigma(\infty)=\infty\}=\{\delta^m\beta\delta^k \mid m\in\mathbf{Z}\}=\Gamma_\infty\beta\delta^k.$$

Consider all $\xi\in\Xi$ of the form $\xi=(\infty, \Gamma\beta\delta^k)$. Now $\delta^m\beta\delta^k=\delta^{-k}\delta^{m+k}\beta\delta^k$, hence we obtain from such a ξ a class C containing elements of the form $\delta^m\beta$. Suppose $\gamma\delta^m\beta\gamma^{-1}=\delta^n\beta$ with $\gamma\in\Gamma$. Then $\gamma\in\Gamma_\infty$, and such a γ exists if and only if $m\equiv n \pmod{s-t}$, by virtue of Lemma 4.2, (i), (ii). Thus there are exactly $s-t$ classes C_n represented by $\delta^n\beta$ with $n=1, \ldots, s-t$. On the other hand, if $\gamma\delta^m\beta\gamma^{-1}$ has ∞ as its upper fixed point, γ must be contained in Γ_∞, so that $\Gamma\gamma\delta^m\beta\gamma^{-1}=\Gamma\beta\delta^n$ with $n\in\mathbf{Z}$. Now $\Gamma\beta\delta^n=\Gamma\beta\delta^m$ if and only if $m\equiv n \pmod{s}$. Thus there are exactly s different $\xi_k=(\infty, \Gamma\beta\delta^k)$ for $k=1, \ldots, s$ corresponding to the C_n. Since $\lambda^{-2}-1=(t-s)/s$, we have

$$\sum_{n=1}^{s-t}J(C_n)=\sum_{n=1}^{s-t}\eta^{-1}e^{-2\pi i\mu n}/(\lambda^{-2}-1)=\begin{cases}-s\eta^{-1} & \text{if } \mu=1,\\ 0 & \text{if } \mu<1,\end{cases}$$

by virtue of Lemma 4.2, (iii). Thus $\Sigma_n J(C_n)=\Sigma_k I(\xi_k)$. The same conclusion holds also in the case $-1\in\Gamma$ or $\varepsilon=1$, by a similar and simpler argument.

Still with $\Gamma=\Gamma_1$, consider a parabolic $\xi=(z_0, \Gamma\beta)$. Then there is a unique C in $\Phi(\Gamma\alpha\Gamma/\Gamma)$ containing β, and conversely C determines ξ uniquely. According to our definition, we have $J(C)=I(\xi)$ trivially. This completes the proof in the case $\Gamma=\Gamma_1$.

Now let us consider the general case assuming $\alpha\notin\{\pm1\}\cdot\Gamma$. Fix a normal subgroup Γ_1 of Γ satisfying the conditions (i–iv). Let S be a set of representatives for Γ/Γ_1. Define P: $F_1\to F$ and P': $F_1'\to F'$ by

$$P=[\Gamma:\Gamma_1]^{-1}\Sigma_{\gamma\in S}L(\gamma),$$
$$P'=[\Gamma:\Gamma_1]^{-1}\Sigma_{\gamma\in S}L'(\gamma).$$

We see that, for any $\gamma\in\Gamma$, (3.5.1, 2) and (3.6.1) are satisfied by $\alpha^*\gamma^*$ and Γ_1, hence $[\Gamma_1\alpha\gamma\Gamma_1]^*$ and $[\Gamma_1\gamma^{-1}\alpha^{-1}\Gamma_1]_*$ are meaningful, and $[\Gamma_1\alpha\gamma\Gamma_1]^*=[\Gamma_1\alpha\Gamma_1]^*L(\gamma)$, $[\Gamma_1\gamma^{-1}\alpha^{-1}\Gamma_1]_*=L'(\gamma^{-1})[\Gamma_1\alpha^{-1}\Gamma_1]_*$. Since our formula is true for Γ_1, we have, for every $\gamma\in S$,

$$\operatorname{tr}([\Gamma_1\alpha\gamma\Gamma_1]^* \mid F_1(\mathfrak{a}_1))-\operatorname{tr}([\Gamma_1\gamma^{-1}\alpha^{-1}\Gamma_1]_* \mid F_1'(\mathfrak{b}_1))=\Sigma_{C_1}J(C_1),$$

where C_1 runs over all classes in $\Phi(\Gamma_1\alpha\gamma\Gamma_1/\Gamma_1)$. By our assumptions (i, ii), $[\Gamma\alpha\Gamma]^*$ (resp. $[\Gamma\alpha^{-1}\Gamma]_*$) is the restriction of $[\Gamma_1\alpha\Gamma_1]^*$ (resp. $[\Gamma_1\alpha^{-1}\Gamma_1]_*$) to F (resp. F'). Furthermore, P (resp. P') defines a projection map of $F_1(\mathfrak{a}_1)$ onto $F(\mathfrak{a})$ (resp. $F_1'(\mathfrak{b}_1)$ onto $F'(\mathfrak{b})$). Therefore

$$\begin{aligned}&\operatorname{tr}([\Gamma\alpha\Gamma]^* \mid F(\mathfrak{a}))-\operatorname{tr}([\Gamma\alpha^{-1}\Gamma]_* \mid F'(\mathfrak{b}))\\ &\quad=[\Gamma:\Gamma_1]^{-1}\Sigma_{\gamma\in S}\{\operatorname{tr}([\Gamma_1\alpha\gamma\Gamma_1]^* \mid F_1(\mathfrak{a}_1))-\operatorname{tr}([\Gamma_1\gamma^{-1}\alpha^{-1}\Gamma_1]_* \mid F_1'(\mathfrak{b}_1))\}\\ &\quad=[\Gamma:\Gamma_1]^{-1}\Sigma_D J(D),\end{aligned}$$

where D runs over all classes in $\bigcup_{\gamma\in S}\Phi(\Gamma_1\alpha\gamma\Gamma_1/\Gamma_1)$. Observe that $\Gamma\alpha\Gamma=\bigcup_{\gamma\in S}\Gamma_1\alpha\gamma\Gamma_1$, and this is a disjoint union by (i, ii). Let $C\in\Phi(\Gamma\alpha\Gamma/\Gamma)$. If C is elliptic or hyperbolic, it can easily be seen that C contains exactly $[\Gamma\colon\Gamma_1 Z_\Gamma(\beta)]$ classes D of $\bigcup_{\gamma\in S}\Phi(\Gamma_1\alpha\gamma\Gamma_1/\Gamma_1)$, where $\beta\in C$. Now we have

$$[\Gamma\colon\Gamma_1]=[\Gamma\colon\Gamma_1 Z_\Gamma(\beta)][Z_\Gamma(\beta)\colon 1][Z_{\Gamma_1}(\beta)\colon 1]^{-1},$$

hence $J(C)=[\Gamma\colon\Gamma_1]^{-1}\sum_{D\subset C}J(D)$.

It remains to prove the last equality for parabolic C. Let $\beta\in C\in\Phi(\Gamma\alpha\Gamma/\Gamma)$ with a parabolic β. We may assume $\beta(\infty)=\infty$. Put

$$\Gamma_\infty=\{\gamma\in\Gamma\,|\,\gamma(\infty)=\infty\},\quad \Gamma_{1\infty}=\Gamma_\infty\cap\Gamma_1.$$

Let us first consider the case $-1\notin\Gamma$. Then we may assume that Γ_∞ is generated by an element δ of the form $\delta=\varepsilon\begin{bmatrix}1&1\\0&1\end{bmatrix}$, with $\varepsilon=\pm1$. Put $k=[\Gamma_\infty\colon\Gamma_{1\infty}]$, and $\beta^*=\left(c\begin{bmatrix}1&x\\0&1\end{bmatrix},\,\eta\right)$. Let D be an element of $\bigcup_{\gamma\in S}\Phi(\Gamma_1\alpha\gamma\Gamma_1/\Gamma_1)$ contained in C. Then D contains an element of the form $\gamma\delta^m\beta\gamma^{-1}$ with $\gamma\in\Gamma$. It can easily be seen that $\gamma\delta^m\beta\gamma^{-1}$ and $\gamma'\delta^n\beta\gamma'^{-1}$ belong to the same D if and only if $\gamma^{-1}\gamma'\in\Gamma_1\Gamma_\infty$ and $m\equiv n \pmod{k}$. Let P be a set of representatives for $\Gamma/\Gamma_1\Gamma_\infty$. Then $[\Gamma\colon\Gamma_1]$ elements

$$\gamma\delta^n\beta\gamma^{-1}\quad(\gamma\in P;\ n=1,\dots,k)$$

form a set of representatives for all the classes D contained in C. If $\gamma\delta^n\beta\gamma^{-1}\in D$, then

$$J(D)=\eta^{-1}e^{-2\pi in\mu}e^{2\pi i\mu_1(x+n)/k}/(1-e^{2\pi i(x+n)/k}),$$

where μ_1 is defined with respect to Γ_1. We can put $k\mu-\mu_1=b$ with an integer b such that $0\leqslant b<k$. Then

$$\begin{aligned}\textstyle\sum_{D\subset C}J(D)&=[\Gamma\colon\Gamma_1\Gamma_\infty]\eta^{-1}e^{2\pi i(\mu-b/k)x}\textstyle\sum_{n=1}^{k}e^{-2\pi inb/k}/(1-e^{2\pi i(x+n)/k})\\&=k\cdot[\Gamma\colon\Gamma_1\Gamma_\infty]\eta^{-1}e^{2\pi i\mu x}/(1-e^{2\pi ix})\\&=[\Gamma\colon\Gamma_1]\,J(C)\end{aligned}$$

by Lemma 4.3. The case $-1\in\Gamma$ can be treated in a similar way, which concludes our proof.

5. Supplementary results and remarks

5.1. We observe that the couple $(F(\mathfrak{a}),\ F'(\mathfrak{b}))$ is almost symmetric. Therefore if $F'(\mathfrak{b})$ satisfies (2.4.10), i.e., if $F'(\mathfrak{b}-p)\neq F'(\mathfrak{b})$ for every $p\in V$, then we can repeat the whole discussion interchanging $F(\mathfrak{a})$ and $F'(\mathfrak{b})$, and obtain a formula of the type

$$\operatorname{tr}([\Gamma\alpha^{-1}\Gamma]_*\,|\,F'(\mathfrak{b}))-\operatorname{tr}([\Gamma\alpha\Gamma]^*\,|\,F(\mathfrak{a}))=\textstyle\sum_{C'}J'(C'),$$

where the sum is taken over all $C' \in \Phi(\Gamma\alpha^{-1}\Gamma/\Gamma)$. Let us now show that this becomes exactly -1 times the previous formula.

First we prove a formula corresponding to 3.7, with the sum $\Sigma_{\xi'} I'(\xi')$ extended over all $\xi' \in \Xi(\Gamma\alpha^{-1}\Gamma)$. In this case, since $0 \leqq \mu'_p < 1$, we have to define $I'(\xi')$ for $\xi' = (z_0, \Gamma\beta)$ with a parabolic β by

$$I'(\xi') = \begin{cases} \eta^{-1} & \text{if } \mu'_p = 0 \text{ and } |\lambda| < 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $\varrho_*^{-1}\beta_*\varrho_* = \left(\begin{bmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{bmatrix}, \eta\right)$ with a suitable $\varrho_* \in \mathfrak{G}_{2-m}$. Then we can repeat the discussion of § 4, and arrive at the desired conclusion. As a consequence, we obtain

5.2. THEOREM. *The formula of* 4.5 *holds also when the condition* (iii) *is replaced by the following*

(iii′) $\qquad F'_1(\mathfrak{b}_1 - p) \neq F'_1(\mathfrak{b}_1)$ *for all* $p \in V_1 = \mathfrak{H}^*/\Gamma_1$.

5.3. As a simple example, take the case where $m=2$, and L is defined by $L(\gamma) = (\gamma, (cz+d)^2)$ for $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$. Then we see that $L'(\gamma) = (\gamma, 1)$, $\mathfrak{b} = 0$, $F'(\mathfrak{b}) = \mathbf{C}$, and $F'(\mathfrak{b} - p) = \{0\} \neq F'(\mathfrak{b})$ for every $p \in V$. Therefore (iii′) is satisfied with $\Gamma_1 = \Gamma$, and the trace-formula is valid. In this case, $F(\mathfrak{a})$ is exactly the space of cusp forms of weight 2 in the ordinary sense. Therefore (2.4.10) is satisfied if and only if $F(\mathfrak{a}) \neq \{0\}$. Thus our discussion shows that the trace-formula holds even if $F(\mathfrak{a}) = \{0\}$.

5.4. There is still another symmetry between $F(\mathfrak{a})$ and $F'(\mathfrak{b})$. First, to indicate that $\mathfrak{a}$ and $\mathfrak{b}$ are defined with respect to L and L', put $\mathfrak{a} = \mathfrak{a}(L)$ and $\mathfrak{b} = \mathfrak{b}(L')$. Now, interchanging L and L', we can define $\mathfrak{a}(L')$ and $\mathfrak{b}(L)$. More explicitly,

$$\mathfrak{a}(L') = \mathfrak{b}(L') - \Sigma_{p \in S}\, p,$$

$$\mathfrak{b}(L) = \mathfrak{a}(L) + \Sigma_{p \in S}\, p,$$

where S is the set of all cusps $p \in R$ for which $\mu_p = 1$ (i.e., $\mu'_p = 0$). Then $F(\mathfrak{b}(L))$ is the space of all integral forms with respect to L, and $F'(\mathfrak{a}(L'))$ is the space of all cusp forms with respect to L'. Our formula applied to this case gives the difference

$$\operatorname{tr}([\Gamma\alpha\Gamma]^* \mid F(\mathfrak{b}(L))) - \operatorname{tr}([\Gamma\alpha^{-1}\Gamma]_* \mid F'(\mathfrak{a}(L'))). \tag{5.4.1}$$

We have of course $F(\mathfrak{a}(L)) \subset F(\mathfrak{b}(L))$ and $F'(\mathfrak{a}(L')) \subset F'(\mathfrak{b}(L'))$; the complementary parts are spanned by Eisenstein series. Therefore (5.4.1) gives the sum of the value given in

Theorem 4.5 and the traces of $[\Gamma\alpha\Gamma]^*$ and $[\Gamma\alpha^{-1}\Gamma]_*$ on Eisenstein series with respect to L and L'. As a special case of this fact, we obtain, from (3.3.6),

(5.4.2) $$\dim F(\mathfrak{b}(L))-\dim F(\mathfrak{a}(L))+\dim F'(\mathfrak{b}(L'))-\dim F'(\mathfrak{a}(L'))$$
$$=\text{the number of cusps } p \text{ on } V \text{ for which } \mu'_p=0.$$

5.5. Let us now consider the case of modular forms of half integral weight. For a positive integer N, put

$$\Gamma_0(N)=\left\{\begin{bmatrix} a & b \\ c & d\end{bmatrix}\in SL_2(\mathbf{Z})\,|\,c\equiv 0 \pmod N\right\},$$

$$\Gamma_1(N)=\left\{\begin{bmatrix} a & b \\ c & d\end{bmatrix}\in \Gamma_0(N)\,|\,a\equiv d\equiv 1 \pmod N\right\},$$

and define functions $\theta(z)$ and $j(\gamma, z)$ for $\gamma\in\Gamma_0(4)$ by

$$\theta(z)=\textstyle\sum_{n=-\infty}^{\infty}\exp(2\pi i n^2 z),$$
$$j(\gamma, z)=\theta(\gamma(z))/\theta(z) \qquad (\gamma\in\Gamma_0(4)).$$

Then $j(\gamma, z)^4=(cz+d)^2$ for $\gamma=\begin{bmatrix} a & b \\ c & d\end{bmatrix}\in\Gamma_0(4)$, and hence the map $\gamma\mapsto(\gamma, j(\gamma, z))\in\mathfrak{G}_{\frac{1}{2}}$ defines a proper lifting of $\Gamma_0(4)$ of weight $\frac{1}{2}$. (For this and other facts on modular forms of half integral weight, the reader is referred to [7].) Now fix an odd positive integer k, a positive multiple N of 4, and a character χ modulo N such that $\chi(-1)=1$; put then

$$\begin{aligned} L(\gamma) &= (\gamma, \chi(d)j(\gamma, z)^k) \\ L'(\gamma) &= (\gamma, \chi(d)^{-1}j(\gamma, z)^{4-k}) \end{aligned} \qquad \text{for } \gamma=\begin{bmatrix} a & b \\ c & d\end{bmatrix}\in\Gamma_0(N).$$

These are obviously proper liftings of $\Gamma_0(N)$ of weight $k/2$ and $(4-k)/2$, respectivey. The elements of F and F' defined with these L and L' are exactly the modular forms considered in [7]. In this case as well as in the case of ordinary modular forms of integral weight, $[\Gamma\alpha\Gamma]^*$ has a certain commutative property with the map $f(z)\mapsto\overline{f(-\bar{z})}$, from which we can deduce a somewhat simpler form for the trace-formula; but we shall not go into details of this topic.

Let us now fix our attention to the case $k=3$, which is of special interest because both $F(\mathfrak{a})$ and $F'(\mathfrak{b})$ can be non-trivial. To simplify our discussion, we consider only the case $N=4M$ with an odd prime M.

5.6. PROPOSITION. *If $k=3$ and $N=4M$ with an odd prime M, then the condition (iii′) of 5.2 is satisfied by $\Gamma_1=\Gamma_1(N)$ and L' defined as above.*

Proof. Let (V_0, φ_0), (V, φ), and (V_1, φ_1) be models of $\mathfrak{H}^*/\Gamma_0(4)$, $\mathfrak{H}^*/\Gamma_0(N)$, and $\mathfrak{H}^*/\Gamma_1(N)$, respectively. Note that $F_1'(\mathfrak{b}_1) = F_1'(0)$ and $F_1'(\mathfrak{b}_1 - p) = F_1'(-p)$ for every $p \in V_1 = \mathfrak{H}^*/\Gamma_1(N)$. Therefore it is sufficient to show that for every $p \in V_1$, there is an element h of $F_1'(\mathfrak{b}_1)$ such that $\nu_p(h) < 1$. Let div_0, div, div_1 denote the divisors measured on V_0, V, V_1, respectively. Now $\Gamma_0(4)$ has three inequivalent cusps 0, ∞, $\frac{1}{2}$, but no elliptic elements. By (3.3.5), we have $\deg(\mathrm{div}_0(\theta)) = \frac{1}{4}$, from which we can easily conclude that $\mathrm{div}_0(\theta) = (\frac{1}{4}) \cdot \varphi_0(\frac{1}{2})$, which is actually a well known classical fact. There are exactly two points $\varphi(\frac{1}{2})$ and $\varphi((2M)^{-1})$ on V lying above $\varphi_0(\frac{1}{2})$ with ramification index 1 and M, respectively. Further, above each one of them, there are exactly $(M-1)/2$ points on V_1 with ramification index 2. Therefore

$$\mathrm{div}_1(\theta) = \textstyle\sum_{i=1}^{t} ((1/2)\, p_i + (M/2)\, q_i) \qquad (t = (M-1)/2)$$

with these points p_i and q_i. Put $g(z) = \theta(-1/Nz) z^{-\frac{1}{2}}$. By [7, Prop. 1.4], $g \in F_1'(\mathfrak{b}_1)$, and

$$\mathrm{div}_1(g) = \textstyle\sum_{i=1}^{t} ((M/2)\, p_i + (1/2)\, q_i).$$

Therefore, for every $p \in V_1$, we have either $\nu_p(\theta) < 1$ or $\nu_p(g) < 1$, q.e.d.

5.7. Let n be a positive integer, and let

$$\alpha = \begin{bmatrix} n^{-1} & 0 \\ 0 & n \end{bmatrix}, \quad \alpha^* = \tau = (\alpha, 1) \in \mathfrak{G}_{3/2}.$$

Then we see that the conditions (3.5.1, 2) and (3.6.1) are satisfied by α, τ, and $\Gamma = \Gamma_0(N)$ with the above L. Moreover, (i, ii, iv) of 4.5 are satisfied by $\Gamma_1 = \Gamma_1(N)$. Therefore, by 5.2, the trace-formula holds for $[\Gamma\alpha\Gamma]^*$ and $[\Gamma\alpha^{-1}\Gamma]_*$ in the present case with $k=3$. The operators $[\Gamma\alpha\Gamma]^*$ and $[\Gamma\alpha^{-1}\Gamma]_*$ differ from $T^N_{3,\chi}(n^2)$ and $T^N_{1,\bar{\chi}}(n^2)$ of [7] only by constant factors. In this case, it is plausible that $F'(\mathfrak{b})$ is one-dimensional and spanned by θ if χ is trivial and $N/4$ is a prime. In such a case, $\mathrm{tr}\,([\Gamma\alpha\Gamma]^* \mid F(\mathfrak{a}))$ is effectively computable.

5.8. We conclude our study by making some observations in the case $m=1$. Consider a lifting of the type

$$L(\gamma) = (\gamma, \chi(d)(cz+d)) \quad \left(\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)\right)$$

with a character χ modulo N such that $\chi(-1) = -1$. By an argument similar to the proof of 5.6, we can show that our trace formula holds for the ordinary Hecke operators on the space of modular forms of weight 1 with respect to L. Unfortunately, however, it can be verified that the difference

$$\mathrm{tr}\,([\Gamma\alpha\Gamma]^*\,|\,F(\mathfrak{a})) - \mathrm{tr}\,([\Gamma\alpha^{-1}\Gamma]_*\,|\,F'(\mathfrak{b}))$$

with a natural choice of α, say $\alpha = n^{-\frac{1}{2}}\begin{bmatrix}1 & 0\\ 0 & n\end{bmatrix}$, produces nothing particularly significant: it shows either that something which must be 0 is actually 0, or that the trace on the space of Eisenstein series is computable, which we could do anyway without the trace-formula. (Note that this is so even if $\chi^2 \neq 1$.) If we take an element of the form $\alpha\beta$ instead of α with a suitable element β of the normalizer of $\Gamma_0(N)$, say $\beta = N^{-\frac{1}{2}}\begin{bmatrix}0 & -1\\ N & 0\end{bmatrix}$, then the formula becomes somewhat more non-trivial. But still this gives only the trace on the space of cusp forms corresponding to L-functions of imaginary quadratic fields with abelian characters. In this way one can obtain at least, or at most, a certain characterization of such cusp forms.

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