ASYMPTOTIC BEHAVIOUR OF EIGEN FUNCTIONS ON A SEMISIMPLE LIE GROUP: THE DISCRETE SPECTRUM

BY

P. C. TROMBI and V. S. VARADARAJAN(1)

University of California, Los Angeles, Calif., USA

1. Introduction

Let G be a connected noncompact real form of a simply connected complex semisimple Lie group. For many questions of Fourier Analysis on G it is useful to have a good knowledge of the behaviour, at infinity on G, of the matrix coefficients of the irreducible unitary representations of G. In this paper we restrict ourselves to the discrete series of representations of G, and study the rapidity with which the corresponding matrix coefficients decay at infinity on the group.

Let K be a maximal compact subgroup of G. Given any p, with $1 \leqslant p \leqslant 2$, we denote by $\mathcal{E}_p(G)$ the set of all equivalence classes of irreducible unitary representations of G whose K-finite matrix coefficients are in $L^p(G)$; $\mathcal{E}_2(G)$ is then the discrete series of G, while $\mathcal{E}_{p'}(G) \subseteq \mathcal{E}_p(G)$ for $1 \leqslant p' \leqslant p \leqslant 2$. We assume that rk $(G) =$ rk (K) so that $\mathcal{E}_2(G)$ is nonempty. Let Ξ and σ be the spherical functions on G defined in [15]. Then it follows from the work in [14] that, if $\omega \in \mathcal{E}_2(G)$ and if f is a K-finite matrix coefficient of (a representation belonging to) ω, one can find constants $c > 0$, $\gamma > 0$, $q \geqslant 0$ (depending on f) such that

$$|f(x)| \leqslant c\Xi(x)^{1+\gamma}(1+\sigma(x))^q \quad (x \in G). \tag{1.1}$$

Given $\omega \in \mathcal{E}_2(G)$ and a number $\gamma > 0$, we shall say that ω is *of type* γ if the K-finite matrix coefficients of ω satisfy (1.1) for suitable $c > 0$, $q \geqslant 0$. For a fixed $\omega \in \mathcal{E}_2(G)$ it is then natural to ask what is the largest $\gamma > 0$ for which ω is of type γ. In particular, it is natural to ask for necessary and sufficient conditions in order that $\omega \in \mathcal{E}_p(G)$ $(1 \leqslant p < 2)$.

Let $\mathfrak{g}$ be the Lie algebra of G, and $\mathfrak{g}_c \supseteq \mathfrak{g}$ the complexification of $\mathfrak{g}$. Let $B \subseteq K$ be a Cartan subgroup of G; $\mathfrak{b}$, the Lie algebra of B; and $\mathfrak{b}_c = \mathbf{C}\cdot\mathfrak{b}$. Let $\mathcal{L}_{\mathfrak{b}}$ be the additive group of all

(1) The work of this author was supported at various stages by NSF Grant No. GE-18127, and by a fellowship of the Alfred P. Sloan Foundation. This support is gratefully acknowledged.

integral elements in the dual $\mathfrak{b}_c^*$ of $\mathfrak{b}_c$, and $\mathcal{L}_{\mathfrak{b}}'$, the subset of all regular elements of $\mathcal{L}_{\mathfrak{b}}$. Let $W(\mathfrak{b}_c)$ be the Weyl group of $(\mathfrak{g}_c, \mathfrak{b}_c)$, and $W(G/B)$ the subgroup of $W(\mathfrak{b}_c)$ that comes from G. For $\lambda \in \mathcal{L}_{\mathfrak{b}}'$, let $\omega(\lambda)$ be the equivalence class in $\mathcal{E}_2(G)$ constructed by Harish-Chandra ([14], Theorem 16). Let P be a positive system of roots of $(\mathfrak{g}_c, \mathfrak{b}_c)$, and let P_n (resp. P_k) be the set of all noncompact (resp. compact) roots in P. For any $\alpha \in P$, let H_α be the image of α in $\mathfrak{b}_c$ under the canonical isomorphism of $\mathfrak{b}_c^*$ with $\mathfrak{b}_c$; let $\bar{H}_\alpha$ be the unique element of $\mathbf{R}\cdot H_\alpha$ such that $\alpha(\bar{H}_\alpha)=2$; and let

$$k(\beta) = \tfrac{1}{2} \sum_{\alpha \in P} |\alpha(\bar{H}_\beta)| \quad (\beta \in P \cup (-P)). \tag{1.2}$$

One of our main results (Theorem 8.1) asserts that if $\gamma > 0$ and $\lambda \in \mathcal{L}_{\mathfrak{b}}'$ are given, then, for $\omega(\lambda)$ to be of type γ it is necessary that

$$|\lambda(\bar{H}_\beta)| \geqslant \gamma k(\beta) \quad (\forall \beta \in P_n) \tag{1.3}$$

and sufficient that

$$|(s\lambda)(\bar{H}_\beta)| \geqslant \gamma k(\beta) \quad (\forall \beta \in P_n,\ \forall s \in W(\mathfrak{b}_c)); \tag{1.4}$$

in particular, (1.4) is the necessary and sufficient condition that $\omega(s\lambda)$ be of type γ for all $s \in W(\mathfrak{b}_c)$.

Fix p, $1 \leqslant p < 2$. Let $\omega \in \mathcal{E}_2(G)$. We then prove that $\omega \in \mathcal{E}_p(G)$ if and only if it is of type γ for some $\gamma > (2/p) - 1$ (Theorem 7.5). It follows from this and Theorem 8.1 that for $\omega(\lambda)$ to be in $\mathcal{E}_p(G)$ it is necessary that

$$|\lambda(\bar{H}_\beta)| > \left(\frac{2}{p} - 1\right) k(\beta) \quad (\forall \beta \in P_n) \tag{1.5}$$

and sufficient that

$$|(s\lambda)(\bar{H}_\beta)| > \left(\frac{2}{p} - 1\right) k(\beta) \quad (\forall \beta \in P_n,\ \forall s \in W(\mathfrak{b}_c)); \tag{1.6}$$

as before, (1.6) is necessary and sufficient that $\omega(s\lambda) \in \mathcal{E}_p(G)$ for all $s \in W(\mathfrak{b}_c)$ (Theorem 8.2).

For any $x \in G$, let $D(x)$ be defined in the usual manner as the coefficient of t^l in $\det(\mathrm{Ad}(x) - 1 + t)$, where $l = \mathrm{rk}(G)$ and t is an indeterminate. For any Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ let $D_{\mathfrak{h}}$ and $G_{\mathfrak{h}}$ be as in [13], p. 110. Fix $\omega \in \mathcal{E}_2(G)$, and let Θ_ω be the character of ω. Then, for ω to be of type γ it is actually necessary (Theorem 8.1) that, for each Cartan subalgebra $\mathfrak{h}$, there should exist a constant $c(\mathfrak{h}) > 0$, such that,

$$|D(x)|^{\frac{1}{2}} |\Theta_\omega(x)| \leqslant c(\mathfrak{h}) |D_{\mathfrak{h}}(x)|^{-\gamma/2} \quad (x \in G_{\mathfrak{h}}). \tag{1.7}$$

The condition (1.7) is stricter than (1.3); to deduce (1.3) from this it is enough to specialize $\mathfrak{h}$ suitably. It appears likely that the validity of (1.7) for all Cartan subalgebras $\mathfrak{h}$ would also be sufficient to ensure that ω is of type γ. We have not been able to prove this.

The space $\mathcal{E}_1(G)$ was first introduced by Harish-Chandra [5] (cf. also [2], [16], [17]) in which, among other things, he obtained sufficient conditions for $\omega(\lambda)$ to be in $\mathcal{E}_1(G)$, when G/K is Hermitian symmetric and $\omega(\lambda)$ belongs to the so-called holomorphic discrete series; we verify in § 9 that these conditions are the same as (1.5) (with $p=1$). It follows from this that if G/K is Hermitian symmetric and $\omega(\lambda)$ belongs to the holomorphic discrete series, the conditions (1.5) (with $p=1$) are necessary and sufficient for $\omega(\lambda)$ to be in $\mathcal{E}_1(G)$. At the same time, this leads to examples of $\lambda \in \mathfrak{L}'_{\mathfrak{b}}$ for which $\omega(\lambda) \in \mathcal{E}_1(G)$ but $\omega(s\lambda) \notin \mathcal{E}_1(G)$ for some $s \in W(\mathfrak{b}_c)$; in other words, the equivalence classes in $\mathcal{E}_2(G)$ that correspond to the same infinitesimal character may be of different types. In the general case when G/K is not assumed to be Hermitian symmetric, Harish-Chandra had obtained certain sufficient conditions in order that $\omega(s\lambda) \in \mathcal{E}_1(G)$ for all $s \in W(\mathfrak{b}_c)$ ([9], [10], [11]); these are also discussed in § 9.

We are greatly indebted to Professors Harish-Chandra and Ranga Rao for many stimulating conversations dealing with harmonic analysis on semi simple Lie groups. We are also grateful to Professor Harish-Chandra for giving us permission to discuss his unpublished work [10], [11], and, more specifically, for suggesting the possibility of obtaining estimates such as Lemma 8.4.

2. Notation and preliminaries

G, K will be as in § 1 with rk $(G)=$ rk (K). We will assume that $G \subseteq G_c$, where G_c is a simply connected complex analytic group with Lie algebra $\mathfrak{g}_c$. $\mathfrak{k}$ is the Lie algebra of K and B, $\mathfrak{b}$, $\mathfrak{b}_c$ will be as in § 1. θ will denote the Cartan involution induced on G, as well as $\mathfrak{g}$, by K; and $\mathfrak{g}=\mathfrak{k}+\mathfrak{s}$, the Cartan decomposition. For $X \in \mathfrak{g}$, we put $\|X\|^2 = -\langle X, \theta X\rangle$, $\langle \cdot, \cdot \rangle$ being the Killing form. $\mathfrak{g}$ becomes a real Hilbert space under $\|\cdot\|$. $\mathfrak{g}=\mathfrak{k}+\mathfrak{a}+\mathfrak{n}$ $(\mathfrak{a} \subseteq \mathfrak{s})$, and $G=KAN$, are Iwasawa decompositions, with $A=\exp \mathfrak{a}$, $N=\exp \mathfrak{n}$; if $X \in \mathfrak{s}$ and $x=\exp X$, we write $X=\log x$. $\Delta=\Delta(\mathfrak{g}, \mathfrak{a})$ is the set of roots of $(\mathfrak{g}, \mathfrak{a})$; Δ^+, the set of positive roots; $\Sigma=\{\alpha_1, ..., \alpha_d\}$, the simple roots; and $\mathfrak{g}_\lambda$ $(\lambda \in \Delta)$ the root subspaces. $\mathfrak{a}^+$ is the positive chamber in $\mathfrak{a}$, and $A^+=\exp \mathfrak{a}^+$. $\varrho(H)=\text{tr}\,(\text{ad}\, H)_{\mathfrak{n}}(H \in \mathfrak{a})$, the suffix denoting restriction to $\mathfrak{n}$. $\mathfrak{l}$ denotes a θ-stable Cartan subalgebra with $\mathfrak{l} \cap \mathfrak{s}=\mathfrak{a}$. For any Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, we write $\mathfrak{h}_c$ for $\mathbf{C}\cdot\mathfrak{h}$, $W(\mathfrak{h}_c)$ for the Weyl group of $(\mathfrak{g}_c, \mathfrak{h}_c)$, and $\mathfrak{L}_{\mathfrak{h}}$ for the additive group of all integral elements of $\mathfrak{h}_c^*$. The spherical functions σ and Ξ on G are defined as in [15]. It is known that for suitable constants $c_0>0$, $r_0 \geqslant 0$,

$$e^{-\varrho(\log h)} \leqslant \Xi(h) \leqslant c_0 e^{-\varrho(\log h)}(1+\sigma(h))^{r_0} \quad (h \in A^+) \tag{2.1}$$

In particular, $\Xi^2(1+\sigma)^{-r} \in L^1(G)$ if $r>2r_0+d$. $\mathfrak{G}$ denotes the universal enveloping algebra of $\mathfrak{g}_c$; $\mathfrak{K}$, $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{L}$ etc. are the subalgebras of $\mathfrak{G}$ generated by $(1, \mathfrak{k})$, $(1, \mathfrak{a})$, $(1, \mathfrak{b})$, $(1, \mathfrak{l})$ etc.

The elements of $\mathfrak{G}$ act in the usual manner as differential operators from both left and right. We shall use Harish-Chandra's notation to denote differential operators; thus, if f is a C^∞ function on a C^∞ manifold M, and E is a differential operator acting from the left (resp. right), we write $f(x; E)$ (resp. $f(E; x)$) to denote $(Ef)(x)$ (resp. $(fE)(x)$) $(x \in M)$. $\circ$ denotes composition of differential operators. $\mathfrak{Z}$ is the center of $\mathfrak{G}$.

A subalgebra $\bar{\mathfrak{p}}$ of $\mathfrak{g}$ is called *parabolic* if $\mathbf{C}\cdot\bar{\mathfrak{p}}$ contains a Borel subalgebra of $\mathfrak{g}_c$. Let $\bar{\mathfrak{p}}$ be parabolic, $\bar{\mathfrak{n}}$, its nilradical. Write $\bar{\mathfrak{m}}_1 = \bar{\mathfrak{p}} \cap \theta(\bar{\mathfrak{p}})$. Then $\bar{\mathfrak{m}}_1$ is reductive in $\mathfrak{g}$, rk $(\bar{\mathfrak{m}}_1) =$ rk $(\mathfrak{g})$, and $\bar{\mathfrak{p}} = \bar{\mathfrak{m}}_1 + \bar{\mathfrak{n}}$ is a direct sum. Put $\bar{\mathfrak{a}} =$ center $(\bar{\mathfrak{m}}_1) \cap \mathfrak{s}$. Then $\bar{\mathfrak{m}}_1$ is the centralizer of $\bar{\mathfrak{a}}$ in $\mathfrak{g}$, and $\bar{\mathfrak{a}}$ is called the *split component* of $\bar{\mathfrak{p}}$. Let $F \subsetneqq \Sigma$ and let $\mathfrak{a}_F$ be the set of common zeros of members of F. Write $\mathfrak{m}_{1F}$ for the centralizer of $\mathfrak{a}_F$ in $\mathfrak{g}$, $\mathfrak{m}_F$ for the orthogonal complement of $\mathfrak{a}_F$ in $\mathfrak{m}_{1F}$, and Δ_F for the roots of $(\mathfrak{m}_{1F}, \mathfrak{a})$; we put $\Delta_F^+ = \Delta^+ \cap \Delta_F$. If $\mathfrak{n}_F = \sum_{\lambda \in \Delta^+ \setminus \Delta_F^+} \mathfrak{g}_\lambda$, then $\mathfrak{p}_F = \mathfrak{m}_F + \mathfrak{a}_F + \mathfrak{n}_F$ is parabolic, $\neq \mathfrak{g}$, and $\mathfrak{a}_F$ is its split component; and, given a parabolic subalgebra $\mathfrak{p} \neq \mathfrak{g}$ of $\mathfrak{g}$, there exists a unique $F \subsetneqq \Sigma$ such that for some $k \in K$, $\mathfrak{p}^k = \mathfrak{p}_F$. We write $\mathfrak{M}_{1F}$, $\mathfrak{M}_F$ and $\mathfrak{A}_F$ for the subalgebras of $\mathfrak{G}$ generated by $(1, \mathfrak{m}_{1F})$, $(1, \mathfrak{m}_F)$ and $(1, \mathfrak{a}_F)$ respectively. $\mathfrak{Z}_F$ is the center of $\mathfrak{M}_{1F}$. We put, for $H \in \mathfrak{a}$,

$$\varrho^F(H) = \tfrac{1}{2}\,\mathrm{tr}\,(\mathrm{ad}\,H)_{\mathfrak{n}_F}, \quad \varrho_F(H) = \tfrac{1}{2}\,\mathrm{tr}\,(\mathrm{ad}\,H)_{\mathfrak{m}_F \cap \mathfrak{n}}, \quad \beta_F(H) = \min_{\lambda \in \Sigma \setminus F} \lambda(H) \tag{2.2}$$

Then $\varrho = \varrho_F + \varrho^F$, $\varrho_F \mid \mathfrak{a}_F = 0$, $\varrho^F \mid \mathfrak{a} \cap \mathfrak{m}_F = 0$. Also let

$$\mathfrak{a}_F^+ = \{H : H \in \mathfrak{a}_F, \beta_F(H) > 0\}, \quad A_F^+ = \exp \mathfrak{a}_F^+. \tag{2.3}$$

Let M_{1F} denote the centralizer of $\mathfrak{a}_F$ in G; $A_F = \exp \mathfrak{a}_F$ and $N_F = \exp \mathfrak{n}_F$. Then $P_F = M_{1F}N_F$ is the normalizer of $\mathfrak{p}_F$ in G, and is called the *parabolic subgroup* corresponding to $\mathfrak{p}_F$. Let M_F denote the intersection of the kernels of all continuous homomorphisms of M_{1F} into the positive reals. Then $M_{1F} = M_F A_F$ and the map $m, a, n \mapsto man$ of $M_F \times A_F \times N_F$ into P_F is an analytic diffeomorphism; moreover, $G = KM_{1F}K$. In general, the group M_F is neither semisimple nor connected. Under our assumption that G is a matrix group, it is however not difficult to show that (i) M_F/M_F^0 is finite, M_F^0 being the connected component of M_F containing the identity (ii) if $\bar{M}_F$ and C_F are the analytic subgroups of M_F^0, defined respectively by the derived algebra and center of $\mathfrak{m}_F$, then they are both closed, $\bar{M}_F$ is a semisimple matrix group while C_F is compact, and $M_F^0 = \bar{M}_F\, C_F$. This circumstance makes it possible to extend to M_F most of the results valid for semisimple matrix groups. We shall make use of such extensions without explicit comment. $K_F = K \cap M_F = K \cap M_{1F}$ is a maximal compact subgroup of M_F. We denote by Ξ_F the fundamental spherical function on M_F, and extend it to M_{1F} by setting $\Xi_F(ma) = \Xi_F(m)$ $(m \in M_F, a \in A_F)$. Finally, we write d_F for the homomorphism of M_{1F} into the positive reals given by

$$d_F(ma) = e^{\varrho^F(\log a)} \quad (m \in M_F, a \in A_F). \tag{2.4}$$

The parabolic subgroup P_F is called *cuspidal* if $\mathrm{rk}\,(M_F)=\mathrm{rk}\,(K_F)$. P_F is cuspidal if and only if there is a θ-stable Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{h}\cap\mathfrak{s}=\mathfrak{a}_F$ ([15], § 5; cf. also [1]).

Let $W(\mathfrak{l}_c)_F$ denote the subgroup of $W(\mathfrak{l}_c)$ generated by the reflexions corresponding to the roots of $(\mathbf{C}\cdot\mathfrak{m}_{1F}, \mathfrak{l}_c)$. Let $I(W(\mathfrak{l}_c))$ (resp. $I(W(\mathfrak{l}_c)_F)$) be the subalgebra of all elements of $\mathfrak{L}$ invariant under $W(\mathfrak{l}_c)$ (resp. $W(\mathfrak{l}_c)_F$). We then have a canonical isomorphism $\mu_{\mathfrak{g}/\mathfrak{l}}$ (resp. $\mu_{\mathfrak{m}_{1F}/\mathfrak{l}}$) of $\mathfrak{Z}$ onto $I(W(\mathfrak{l}_c))$ (resp. $\mathfrak{Z}_F$ onto $I(W(\mathfrak{l}_c)_F)$ ([12], § 12). Suppose $z\in\mathfrak{Z}$. Then there is a unique element $z_1\in\mathfrak{Z}_F$ such that $z\equiv z_1 \pmod{\mathfrak{G}\mathfrak{n}_F}$. It is known that $z-z_1\in\theta(\mathfrak{n}_F)\mathfrak{G}\mathfrak{n}_F$; and that, if we write $\mu_F(z)=d_F\circ z_1\circ d_F^{-1}$, then μ_F is an algebra injection of $\mathfrak{Z}$ into $\mathfrak{Z}_F$, and $\mu_{\mathfrak{g}/\mathfrak{l}}(z)=\mu_{\mathfrak{m}_{1F}/\mathfrak{l}}(\mu_F(z))$ for all $z\in\mathfrak{Z}$)[13], § 10). It follows from this that $\mathfrak{Z}_F$ is a free finite module over $\mu_F(\mathfrak{Z})$ of rank equal to the index of $W(\mathfrak{l}_c)_F$ in $W(\mathfrak{l}_c)$. We shall denote by r_F this index ([12], § 12).

Let $\{H_1, ..., H_d\}$ be the basis of $\mathfrak{a}$ dual to $\{\alpha_1, ..., \alpha_d\}$. For $1\leqslant j\leqslant d$, let $F_j=\Sigma\diagdown\{\alpha_j\}$. We shall write P_j for the parabolic subgroup P_{F_j}, and in general (when this is not likely to cause confusion), we shall replace the suffix F_j by j in denoting the objects associated with F_j; thus $M_j=M_{F_j}$, $d_j=d_{F_j}$ etc.

We shall now give a brief outline of the proofs of our main results. Let $\lambda\in\mathfrak{L}_{\mathfrak{b}}'$ and let $O_{\mathfrak{l}}=W(\mathfrak{l}_c)(\lambda\circ y)$ where $y\in G_c$ is such that $y\cdot\mathfrak{l}_c=\mathfrak{b}_c$. Let $\bar{\gamma}>0$, let $\omega\in\mathcal{E}_2(G)$ be of type $\bar{\gamma}-\varepsilon$ for every $\varepsilon>0$, and let φ be a K-finite matrix coefficient of ω. For any $j=1, ..., d$ we consider the parabolic subgroup $P_j=M_{1j}N_j$, and transcribe the differential equations $z\varphi=\mu_{\mathfrak{g}/\mathfrak{l}}(z)(\Lambda)\varphi$ ($z\in\mathfrak{Z}$, $\Lambda\in O_{\mathfrak{l}}$) to M_{1j} (§ 4). It turns out that these differential equations are perturbations of the equations satisfied by suitable $\mathfrak{Z}_j$-eigenfunctions on M_{1j} (§ 5). This fact enables us to prove that for any $m\in M_{1j}$, the limit

$$\lim_{t\to+\infty} d_j(m\exp tH_j)^{1+\bar{\gamma}}\varphi(m\exp tH_j)=\varphi_{j,\bar{\gamma}}(m) \tag{2.5}$$

exists, and depends only on the component of m in M_j; and that the restriction of $\varphi_{j,\bar{\gamma}}$ to M_j belongs to the linear span of the K_j-finite matrix coefficients of certain classes $\omega_1, ..., \omega_r$ from $\mathcal{E}_2(M_j)$, whose infinitesimal characters can be computed from a knowledge of $O_{\mathfrak{l}}$ (§ 7). In particular, $\varphi_{j,\bar{\gamma}}=0$ if P_j is not cuspidal. Moreover, by carefully following up the various estimates, we obtain the following estimate

$$|\varphi(h)-d_j(h)^{-(1+\bar{\gamma})}\varphi_{j,\bar{\gamma}}(h)|\leqslant \text{const.}\,\Xi(h)^{1+\bar{\gamma}+\beta_0\mu} \tag{2.6}$$

for all $h\in A_j^+(\mu)$; here $0<\mu<1$, $A_j^+(\mu)$ is the sectorial region defined by (7.2), and $\beta_0>0$ is a constant independent of λ, μ, φ (Theorem 7.3).

Suppose now that λ satisfies (1.4). Then $|\Lambda(H_j)|\geqslant\gamma\varrho(H_j)$ for all $\Lambda\in O_{\mathfrak{l}}$ and j for which P_j is cuspidal (Lemma 8.3). Let $\bar{\gamma}$ be the supremum of all $\gamma'>0$ for which ω is of type γ'.

If $\bar{\gamma}<\gamma$, an examination of the differential equations satisfied by the $\varphi_{j,\bar{\gamma}}$ shows that $\varphi_{j,\bar{\gamma}}=0$ for cuspidal P_j, hence for all $j=1, ..., d$. (2.6) then implies that ω is of type γ' for some $\gamma'>\bar{\gamma}$, a contradiction. So $\bar{\gamma}\geqslant\gamma$, and a simple argument based on an induction on dim (G) completes the proof that ω is of type γ.

Suppose that $\omega\in\mathcal{E}_p(G)$ for some $p(1\leqslant p<2)$. Then ω is of type $\bar{\gamma}=(2/p)-1$ (Corollary 3.4) and (2.6) is valid for any K-finite matrix coefficient φ of ω. It follows from this that $\varphi_{j,\bar{\gamma}}=0$, $1\leqslant j\leqslant d$, and hence that ω is of type $\gamma'>\bar{\gamma}$ (Theorem 7.5).

We then consider the converse problem. Let $\omega\in\mathcal{E}_2(G)$ be of type $\gamma>0$, let Θ be the character of ω, and let π be a unitary representation belonging to ω. Denoting by $\mathcal{E}(K)$ the set of all equivalence classes of irreducible unitary representations of K, we obtain the following estimate from the work in § 3 and elementary properties of the discrete series (Lemma 5.6): there exist constants $C>0$, $r\geqslant 0$ such that for all $x\in G$, $\mathfrak{d}\in\mathcal{E}(K)$, and unit vectors e, e' in the space of π that transform under $\pi(K)$ according to $\mathfrak{d}$,

$$|(\pi(x)e, e')|\leqslant Cc(\mathfrak{d})^r\,\Xi(x) \tag{2.7}$$

(here $c(\mathfrak{d})$ is defined as in [14], § 3). Using (2.7) as uniform initial estimates in the differential equations for the functions $x\mapsto(\pi(x)e, e')$, and employing a method that is essentially one of successive approximation, we improve (2.7) and obtain the following: given any $\varepsilon>0$, we can find constants $C_\varepsilon>0$, $r_\varepsilon\geqslant 0$ such that

$$|(\pi(x)\,e, e')|\leqslant C_\varepsilon\, c(\mathfrak{d})^{r_\varepsilon}\,\Xi(x)^{1+\gamma-\varepsilon} \tag{2.8}$$

for all $x\in G$, $\mathfrak{d}\in\mathcal{E}(K)$, e, e' as before (Theorem 7.3). From (2.8) we obtain the following continuity property of Θ (Lemma 8.4): for each $\varepsilon>0$ we can find $\xi_\varepsilon\in\mathfrak{K}$ such that for all $f\in C_c^\infty(G)$

$$|\Theta(f)|\leqslant\sup_G\,\Xi^{-1+\gamma-\varepsilon}\,|\xi_\varepsilon f|. \tag{2.9}$$

We now imitate the arguments of § 19 of [14] to pass from (2.9) to estimates for the values of Θ on the various Cartan subgroups of G (Lemma 8.7); these lead to (1.7) in a direct manner.

3. Some estimates of the Sobolev type

In this section we obtain estimates for certain supremum norms of a function $f\in C^\infty(G)$ in terms of the L^p-norms of f and its derivatives (Theorem 3.3). These are analogous to the classical Sobolev estimates. Our proofs make no use of the assumption that rk $(G)=$rk (K). We put

$$J(h)=\prod_{\lambda\in\Delta^+}(e^{\lambda(\log h)}-e^{-\lambda(\log h)})^{\dim(\mathfrak{g}_\lambda)}\quad(h\in A^+). \tag{3.1}$$

Then we can normalize the Haar measures on G and A so that $dx = J(h)\,dk_1\,dh\,dk_2$, i.e., for all $f \in L^1(G)$,

$$\int_G f\,dx = \int_{K\times A^+\times K} f(k_1hk_2)\,J(h)\,dk_1\,dh\,dk_2. \tag{3.2}$$

In Lemmas 3.1 and 3.2 V will denote a real Hilbert space of finite dimension d, with norm denoted by $\|\cdot\|$. dx is a Lebesgue measure on V. For $x \in V$ and $r>0$, $B(x, r)$ denotes the closed ball with center x and radius r. We fix p with $1 \leqslant p < \infty$, a nonempty open set $U \subseteq V$ and a $w \in C^\infty(U)$ such that $w(x)>0$ for all $x \in U$. $\|\cdot\|_p$ denotes the usual norm on $L^p(V, dx)$. $\mathcal{S}$ is the symmetric algebra over the complexification of V; elements of $\mathcal{S}$ act in the usual manner as differential operators on $C^\infty(U)$, and for $\xi \in \mathcal{S}$, $f \mapsto \xi f$ denotes the corresponding differential operator. For $\xi \in \mathcal{S}$ and $f \in C^\infty(U)$, let

$$\mu_\xi(f) = \left(\int_U |\xi f|^p\,w\,dx\right)^{1/p}. \tag{3.3}$$

H_w is the space of all $f \in C^\infty(U)$ with $\mu_\xi(f) < \infty$ for all $\xi \in \mathcal{S}$. Each μ_ξ is a seminorm on H_ξ. We write $\mathcal{N}$ for the collection of all finite sums of the μ_ξ. Since w is bounded away from 0 on compact subsets of U, the usual form of Sobolev's lemma implies that for any compact set $W \subseteq U$ and any $\xi \in \mathcal{S}$, $f \mapsto \sup_{x\in W}|f(x;\xi)|$ is a seminorm on H_w that is continuous in the topology induced by $\mathcal{N}$. It follows easily from this that H_w, equipped with the topology induced by $\mathcal{N}$, is a Frechet space. Let H_0 be the space of all $f \in C^\infty(U)$ with $\sup_{x\in U}|f(x;\xi)| < \infty$ for each $\xi \in \mathcal{S}$. H_0 is also a Frechet space under the collection of seminorms $f \mapsto \sup_{x\in U}|f(x;\xi)|$ $(\xi \in \mathcal{S})$.

LEMMA 3.1. *Let notation be as above. Fix a real function ε on U such that $0<\varepsilon(x)\leqslant 1$, and $B(x, \varepsilon(x)) \subseteq U$, for all $x \in U$. Let*

$$\omega(x) = \inf\{w(y)\colon y \in B(x, \varepsilon(x))\}. \tag{3.4}$$

Then, there exists an integer $k \geqslant 0$, and seminorm $\nu \in \mathcal{N}$, such that for all $f \in H_w$, and all $x \in U$,

$$|f(x)| \leqslant \varepsilon(x)^{-k}\omega(x)^{-1/p}\nu(f). \tag{3.5}$$

Proof. For any $a>0$ let $u_a \in C_c^\infty(V)$ be the function

$$u_a(x) = \begin{cases} ca^{-d}\exp\left(-a^2/(a^2-\|x\|^2)\right) & \text{if } \|x\|<a \\ 0 & \text{if } \|x\| \geqslant a \end{cases}$$

where c is such that $\int_V u_a\,dx = 1$ for all $a>0$. For $x \in V$ and $r>0$ let $\varphi_{x,r} = 1_{B(x,\frac{1}{2}r)} * u_{r/4}$ (here 1_E is the characteristic function of E, and $*$ denotes convolution). Then $\varphi_{x,r} \in C_c^\infty(V)$,

$0 \leqslant \varphi_{x,r} \leqslant 1$, $\varphi_{x,r}=1$ on $B(x, r/4)$ and $\operatorname{supp} \varphi_{x,r} \subseteq B(x, 3r/4)$; moreover, it is easy to see that, for any homogeneous element $\zeta \in S$ of degree m, there is a constant $c(\zeta)>0$, such that, for all $x, y \in V$ and all $r>0$,

$$|\varphi_{x,r}(y;\zeta)| \leqslant c(\zeta) r^{-m}. \tag{3.6}$$

By the classical Sobolev's lemma, we can find $\zeta_1, \ldots, \zeta_q \in S$ such that, for all $\psi \in C_c^\infty(V)$ and all $y \in V$,

$$|\psi(y)| \leqslant \sum_{1\leqslant i\leqslant q} \|\zeta_i \psi\|_p.$$

Replacing ψ by $f\varphi_{x,\varepsilon(x)}$ we find, for $f \in H_w$ and $x \in U$,

$$|f(x)| \leqslant \sum_{1\leqslant i\leqslant q} \|\zeta_i(f\varphi_{x,\varepsilon(x)})\|_p. \tag{3.7}$$

By Liebniz's formula, we can find homogeneous elements $\xi_{ij}, \eta_{ij} \in S$ $(1 \leqslant i \leqslant q,\ 1 \leqslant j \leqslant r)$ such that, for all $u, v \in C^\infty(U)$, $\zeta_i(uv) = \Sigma_{1\leqslant j\leqslant r}(\xi_{ij}u)(\eta_{ij}v)$ for $1 \leqslant i \leqslant q$. We use this in (3.7) with $f=u$, $\varphi_{x,\varepsilon(x)}=v$. Setting

$$c = \max_{i,j} c(\eta_{ij}), \quad k = \max_{i,j} \deg(\eta_{ij})$$

and observing that $w(y) \geqslant \omega(x)$ for all $y \in \operatorname{supp}(\eta_{ij}\varphi_{x,\varepsilon(x)})$, we get, from (3.6) and (3.7),

$$|f(x)| \leqslant c\varepsilon(x)^{-k}\omega(x)^{-1/p} \sum_{1\leqslant i\leqslant q}\ \sum_{1\leqslant j\leqslant r} \mu_{\xi_{ij}}(f).$$

Lemma 3.1 follows at once from this.

LEMMA 3.2. *Let notation be as above. Suppose there are nonzero real linear functions* $\lambda, \ldots, \lambda_N$ *on* V, *and constants* $c>0$, $r\geqslant 0$, *such that,* $U=\{x: x\in V,\ \lambda_j(x)>0 \text{ for } 1\leqslant i\leqslant N\}$, *and*

$$w(x) \geqslant c(1 + (\min_{1\leqslant i\leqslant N} \lambda_i(x))^{-1})^{-r} \quad (x \in U). \tag{3.8}$$

Then $H_w \subseteq H_0$, *and the natural inclusion is continuous. This is in particular the case, if,* $w(x) = \Pi_{1\leqslant j\leqslant N}(1-e^{-\lambda_i(x)})$ $(x \in U)$.

Proof. We begin the proof with the following remark. Suppose φ is a C^∞ function on $(0, \alpha)$, $\alpha > 1$, and that, for suitable constants $L_m > 0$ $(m=0, 1, \ldots)$ and an integer $q \geqslant 0$, φ satisfies the inequalities

$$|\varphi^{(m)}(t)| \leqslant L_m t^{-q} \quad (0<t\leqslant 1,\ m=0, 1, \ldots);$$

we may then conclude that

$$|\varphi^{(m)}(t)| \leqslant 2^q \sum_{0\leqslant i\leqslant q+1} L_{m+i} \quad (0<t\leqslant 1,\ m=0, 1, \ldots). \tag{3.9}$$

This is trivial if $q=0$. Now, for $0<t\leqslant 1$,

$$|\varphi^{(m)}(t)| \leqslant \int_t^1 |\varphi^{(m+1)}(s)|\, ds + |\varphi^{(m)}(1)|. \tag{3.10}$$

If $q=1$, (3.10) gives $|\varphi^{(m)}(t)| \leqslant L_m + L_{m+1}|\log t|$, $0<t\leqslant 1$, $m=0, 1, \ldots$; applying (3.10) again with these estimates, we get (3.9). If $q>1$, (3.10) gives $|\varphi^{(m)}(t)| \leqslant (L_m+L_{m+1})t^{-(q-1)}$, $0<t\leqslant 1$, $m=0, 1, \ldots$; induction on q now proves (3.9).

This said, we come to the proof of the lemma. Write $c_1 = 2 \max_{1\leqslant i\leqslant N}(1+\|\lambda_i\|)$ and define

$$\varepsilon(x) = \frac{1}{c_1} \min(1, \lambda_1(x), \ldots, \lambda_N(x)) \quad (x\in U). \tag{3.11}$$

Then, for $x\in U$ and $y\in B(x, \varepsilon(x))$, $|\lambda_i(y-x)| \leqslant \frac{1}{2}\lambda_i(x)$ for $1\leqslant i\leqslant N$, so that $\lambda_i(y)\geqslant \frac{1}{2}\lambda_i(x)$ for $1\leqslant i\leqslant N$. It follows from this that $B(x, \varepsilon(x))\subseteq U$ for $x\in U$ and that, with $c_2 = c\cdot 2^{-r}$,

$$\omega(x) \geqslant c_2 \varepsilon(x)^r \quad (x\in U). \tag{3.12}$$

We now apply Lemma 3.1. Let k and ν be as in that lemma. Put $\nu_1 = c_2^{-1/p}\nu$ and let b any integer $\geqslant k+r/p$. Then (3.12) and (3.5) imply that $|f(x)| \leqslant \varepsilon(x)^{-b}\nu_1(f)$ for all $f\in H_w$, $x\in U$. For $\xi\in \mathfrak{S}$, let $\nu_\xi(f) = \nu_1(\xi f)$ $(f\in H_w)$. Then $\nu_\xi \in \mathfrak{N}$, and we have, for all $f\in H_w$, $x\in U$,

$$|f(x;\xi)| \leqslant \varepsilon(x)^{-p}\nu_\xi(f). \tag{3.13}$$

Choose and fix $u_0\in U$. Let $f\in H_w$, $\xi\in\mathfrak{S}$, $x\in U$, and let φ be the function defined by $\varphi(t) = f(x+tu_0;\xi)$ for $t\geqslant 0$ (note that $x+tu_0\in U$ for all $t\geqslant 0$). Clearly $\varphi\in C^\infty(0,\infty)$ and $\varphi^{(m)}(t) = f(x+tu_0; u_0^m\xi)$ $(t>0, m=0, 1, \ldots)$. On the other hand it is easy to see from (3.11) that $\varepsilon(x+tu_0)\geqslant t\varepsilon(u_0)$ for all t with $0<t\leqslant 1$. Hence, by (3.13),

$$|\varphi^{(m)}(t)| \leqslant \varepsilon(u_0)^{-b}\nu_{u_0^m\xi}(f)\, t^{-b} \quad (0<t\leqslant 1, m=0, 1, \ldots).$$

Let

$$\bar\nu_\xi = \varepsilon(u_0)^{-b}\, 2^b \sum_{0\leqslant m\leqslant b+1} \nu_{u_0^m\xi}.$$

Then the remark made at the beginning of the proof implies

$$|f(x;\xi)| \leqslant \bar\nu_\xi(f) \quad (f\in H_w, x\in U). \tag{3.14}$$

(3.14) gives the first assertion of the lemma. If $w = \Pi_{1\leqslant i\leqslant N}(1-e^{-\lambda_i})$, w satisfies (3.8) with $c=1$, $r=N$. This proves the lemma.

Fix p, $1\leqslant p<\infty$. Let $\mathcal{H}^p = \mathcal{H}^p(G)$ be the space of all $f\in C^\infty(G)$ such that $bfa\in L^p(G)$ for all $a, b\in\mathfrak{G}$. Exactly as in the case of the space H_w considered above, we use the classical Sobolev lemma to conclude that $\mathcal{H}^p$ is a Frechet space under the seminorms $f\mapsto \|bfa\|_p$ $(a, b\in\mathfrak{G})$. $\mathcal{H}_{0,p} = \mathcal{H}_{0,p}(G)$ is the space of all $f\in C^\infty(G)$ with $\sup_G \Xi^{-2/p}|bfa| < \infty$

for all $a, b \in \mathfrak{G}$; it is a Frechet space with respect to the seminorms $f \mapsto \sup_G \Xi^{-2/p}|bfa|$ $(a, b \in \mathfrak{G})$.

THEOREM 3.3. *Let $\mathcal{H}^p$ and $\mathcal{H}_{0,p}$ be as above. Then $\mathcal{H}^p \subseteq \mathcal{H}_{0,p}$, and the natural inclusion is continuous.*

Proof. Let J be as in (3.1). For any continuous function g on A^+, let $\|g\|_{J,p}$ denote the L^p-norm of g with respect to the measure Jdh. Let H_J denote the space of all $g \in C^\infty(A^+)$ for which $\|ag\|_{J,p} < \infty$ for all $a \in \mathfrak{A}$. Let w be the function $\Pi_{\lambda \in \Delta^+}(1-e^{-2\lambda})^{\dim(\mathfrak{g}_\lambda)}$ on $\mathfrak{a}^+$. Then, for any $\varphi \in C^\infty(\mathfrak{a}^+)$ and $a \in \mathfrak{A}$, with $a' = e^{(2/p)\varrho} \circ a \circ e^{-(2/p)\varrho}$,

$$\int_{\mathfrak{a}^+} |\varphi(H; a)|^p J(\exp H)\, dH = \int_{\mathfrak{a}^+} |(e^{(2/p)\varrho}\varphi)(H; a')|^p w(H)\, dH.$$

Lemma 3.2 (with $V = \mathfrak{a}$, $U = \mathfrak{a}^+$, w as above) and the above formula then give us the following: there exist $a_1, \dots, a_r \in \mathfrak{A}$ such that

$$|g(h)| \leqslant e^{-(2/p)\varrho(\log h)} \sum_{1 \leqslant i \leqslant r} \|a_i g\|_{J,p} \quad (g \in H_J, h \in A^+).$$

From (2.1) we then obtain

$$|g(h)| \leqslant \Xi(h)^{(2/p)} \sum_{1 \leqslant i \leqslant r} \|a_i g\|_{J,p} \quad (g \in H_J, h \in A^+). \tag{3.15}$$

For any $g \in C^\infty(G)$, $k_1, k_2 \in K$, let $g_{k_1,k_2}(h) = g(k_1 h k_2)$ $(h \in A^+)$. Given $a \in \mathfrak{A}$, we can find $c_1, \dots, c_m \in \mathfrak{G}$ and analytic functions $\beta_1, \dots, \beta_m$ on K such that

$$a\, g_{k_1,k_2} = \sum_{1 \leqslant i \leqslant m} \beta_i(k_2)(c_i g)_{k_1,k_2} \tag{3.16}$$

for all $g \in C^\infty(G)$, $k_1, k_2 \in K$. (3.16) and (3.2) show that if $f \in \mathcal{H}^p$, $f_{k_1,k_2} \in H_g$ for almost all $(k_1, k_2) \in K \times K$. Applying (3.15) to the f_{k_1,k_2} and using (3.16) with $a = a_i$ we get the following result: we can find a constant $c > 0$, and $b_1, \dots, b_q \in \mathfrak{G}$, such that for any $f \in \mathcal{H}^p$, the inequality

$$\sup_{h \in A^+} \Xi(h)^{-2/p} |f(k_1 h k_2)| \leqslant c \sum_{1 \leqslant j \leqslant q} \|(b_j f)_{k_1,k_2}\|_{J,p} \tag{3.17}$$

is satisfied for almost all $(k_1, k_2) \in K \times K$. Replacing f by $\xi f \eta$ $(\xi, \eta \in \mathfrak{K})$ in (3.17), we get, after an integration over $K \times K$, the following result: for any $\xi, \eta \in \mathfrak{K}$, $f \in \mathcal{H}^p$ and $h \in A^+$,

$$\left(\iint_{K \times K} |f(\eta; k_1 h k_2; \xi)|^p\, dk_1\, dk_2\right)^{1/p} \leqslant c\, \Xi(h)^{2/p} \sum_{1 \leqslant j \leqslant q} \|b_j \xi f \eta\|_p. \tag{3.18}$$

On the other hand, from the harmonic analysis on $K \times K$ we have the following familiar result: there are $\xi_i, \eta_i \in \mathfrak{K}$ $(1 \leqslant i \leqslant r)$ such that for all $\varphi \in C^\infty(K \times K)$, $(u_1, u_2) \in K \times K$,

$$|\varphi(u_1 : u_2)| \leqslant \sum_{1 \leqslant i \leqslant r} \left(\iint_{K \times K} |\varphi(\eta_i; k_1 : k_2; \xi_i)|^p \, dk_1 \, dk_2 \right)^{1/p}.$$

Combining this and (3.18) we then have

$$|f(k_1 h k_2)| \leqslant c\, \Xi(h)^{2/p} \sum_{1 \leqslant i \leqslant r} \sum_{1 \leqslant j \leqslant q} \|b_j \xi_i f \eta_i\|_p$$

for all $f \in \mathcal{H}^p$, $k_1, k_2 \in K$, $h \in A^+$. So, for $f \in \mathcal{H}^p$ and $u, v \in \mathfrak{G}$,

$$\sup_G \Xi^{-2/p} |ufv| \leqslant c \sum_{1 \leqslant i \leqslant r} \sum_{1 \leqslant j \leqslant q} \|b_j \xi_i u f v \eta_i\|_p. \tag{3.19}$$

Theorem 3.3 follows at once from (3.19).

COROLLARY 3.4. *If* $1 \leqslant p < 2$, *then any* $\omega \in \mathcal{E}_p(G)$ *is of type* $(2/p) - 1$. *If* $1 \leqslant p' \leqslant p$, *then* $\mathcal{E}_{p'}(G) \subseteq \mathcal{E}_p(G) \subseteq \mathcal{E}_2(G)$.

Proof. Let $1 \leqslant p < 2$, $\omega \in \mathcal{E}_p(G)$, and f, a K-finite matrix coefficient of ω. By Theorem 1 of [14] we can find $\alpha, \beta \in C_c^\infty(G)$ such that $f = \alpha * f * \beta$. Consequently, given $a, b \in \mathfrak{G}$, there exist $\alpha', \beta' \in C_c^\infty(G)$ such that $bfa = \alpha' * f * \beta'$. So $f \in \mathcal{H}^p$ and hence $\sup_G \Xi^{-2/p} |f| < \infty$. This proves that ω is of type $(2/p) - 1$. The second statement follows now on noting that for $1 \leqslant q' < q \leqslant 2$, $\Xi^{2/q'} \in L^q(G)$.

Remark. Let $\mathcal{C}^p = \mathcal{C}^p(G)$ be the space of all $f \in C^\infty(G)$ for which $\sup_G \Xi^{-2/p}(1+\sigma)^r |bfa| < \infty$ for all $a, b \in \mathfrak{G}$ and $r \geqslant 0$, topologized in the obvious way. It is then not difficult to deduce from Theorem 3.3 the following result: $\mathcal{C}^p$ is precisely the space of all $f \in C^\infty(G)$ for which $(1+\sigma)^r (bfa) \in L^p(G)$ for all $a, b \in \mathfrak{G}$, $r \geqslant 0$, and its topology is exactly the one induced by the seminorms $f \mapsto \|(1+\sigma)^r (bfa)\|_p$ $(a, b \in \mathfrak{G}, r \geqslant 0)$. We do not prove this here since we make no use of it in what follows.

4. Differential operators on $C^\infty(G: V: \tau)$

Let φ be a K-finite eigenfunction (for $\mathfrak{Z}$), and $P_F = M_{1F} N_F$ $(F \subsetneqq \Sigma)$, a parabolic subgroup. For studying the behavior of $\varphi(ma)$, when $a \in A_F^+$ and tends to infinity, while m varies in M_{1F}, we use Harish-Chandra's idea, of replacing the differential equations on G, by differential equations on M_{1F}. We shall find it convenient to work with vector valued functions.

Let V be a complex finite dimensional Hilbert space, the scalar product and norm of which are denoted by $(\cdot, \cdot)$ and $\|\cdot\|$. By a *unitary double representation* of K in V we mean a pair $\tau=(\tau_1, \tau_2)$ such that (i) τ_1 (resp. τ_2) is a representation (resp. antirepresentation) of K in V, and $\tau_j(k)$ is unitary for all $k\in K$, $j=1, 2$ (ii) $\tau_1(k_1)$ and $\tau_2(k_2)$ commute for all $k_1, k_2\in K$. We allow the $\tau_1(k)$ to act on vectors of V from the left, and the $\tau_2(k)$ to act from the right. We write τ_1 (resp. τ_2) for the corresponding representation (resp. antirepresentation) of $\mathfrak{K}$. A map $f: G\mapsto V$ is called *τ-spherical* if $f(k_1xk_2)=\tau_1(k_1)f(x)\tau_2(k_2)$ for all $x\in G$, $k_1, k_2\in K$; $C^\infty(G: V:\tau)$ denotes the space of all τ-spherical f of class C^∞. Note that $C^\infty(G: V:\tau)$ is invariant under $\mathfrak{Z}$.

Recall that $\mathfrak{g}$ is a Hilbert space. If we write $x^\dagger$ for $\theta(x^{-1})$ $(x\in G)$, then Ad (x) and Ad $(x^\dagger)$ are adjoints of each other.

Fix $F\subsetneqq\Sigma$. For $m\in M_{1F}$, let $\gamma_F(m)=\|\mathrm{Ad}\,(m^{-1})_{\mathfrak{n}_F}\|$. Then $\gamma_F(m)=\|\mathrm{Ad}\,(\theta(m))_{\mathfrak{n}_F}\|$ also. Put

$$\begin{cases} M'_{1F}=\{m: m\in M_{1F}, (Ad(m^{-1})-\mathrm{Ad}(m^\dagger))_{\mathfrak{n}_F} \text{ is invertible}\} \\ M^+_{1F}=\{m: m\in M_{1F}, \gamma_F(m)<1\}. \end{cases} \tag{4.1}$$

Define $b_F(m)$ and $c_F(m)$ for $m\in M'_{1F}$ by

$$b_F(m)=(\mathrm{Ad}(m^{-1})-\mathrm{Ad}(m^\dagger))^{-1}_{\mathfrak{n}_F},\ c_F(m)=\mathrm{Ad}(m^{-1})_{\mathfrak{n}_F}b_F(m). \tag{4.2}$$

It is easily verified that $M^+_{1F}\subseteq M'_{1F}$, and that for $m\in M^+_{1F}$,

$$c_F(m)=-\sum_{r\geqslant 1}(\mathrm{Ad}(m^\dagger m)_{\mathfrak{n}_F})^{-r},\ b_F(m)=-\mathrm{Ad}\,\theta(m)_{\mathfrak{n}_F}\sum_{r\geqslant 0}(\mathrm{Ad}(m^\dagger m)_{\mathfrak{n}_F})^{-r}, \tag{4.3}$$

the series converging since $\|\mathrm{Ad}\,(m^\dagger m)^{-1}_{\mathfrak{n}_F}\|\leqslant\gamma_F(m)^2<1$ (cf. [8] § 2). Note that $\gamma_F(\exp H)=e^{-\beta_F(H)}$ $(H\in Cl(\mathfrak{a}^+))$.

LEMMA 4.1. *Let E be the projection of $\mathfrak{g}$ on $\mathfrak{k}$ modulo $\mathfrak{s}$. Then for all $X\in\mathfrak{n}_F$, $m\in M'_{1F}$, we have*

$$\theta X=-2\mathrm{Ad}\,(m^{-1})\,Eb_F(m)X+2Ec_F(m)X.$$

Proof. Let $h\in M'_{1F}\cap A$, $\lambda\in\Delta^+\setminus\Delta^+_F$, $X\in\mathfrak{g}_\lambda$. Write $X=Y+Z$, $Y\in\mathfrak{k}$, $Z\in\mathfrak{s}$. A simple calculation shows that

$$(e^{\lambda(\log h)}-e^{-\lambda(\log h)})\,\theta X=2Y^{h^{-1}}-2e^{-\lambda(\log h)}Y.$$

This gives the result we want when $m=h$. The general case follows from the above special case, since $M'_{1F}=K_F(A\cap M'_{1F})K_F$, while $c_F(u_1mu_2)=\mathrm{Ad}\,(u_2^{-1})_{\mathfrak{n}_F}c_F(m)\,\mathrm{Ad}\,(u_2)_{\mathfrak{n}_F}$ and $b_F(u_1mu_2)=\mathrm{Ad}\,(u_1)_{\mathfrak{n}_F}b_F(m)\,\mathrm{Ad}\,(u_2)_{\mathfrak{n}_F}$, for $u_1, u_2\in K_F$, $m\in M_{1F}$.

LEMMA 4.2. *Let $\{Y_1, ..., Y_p\}$ be a basis for $(\mathfrak{n}_F+\theta(\mathfrak{n}_F))\cap\mathfrak{k}$. Let $S_{0,F}$ be the algebra generated (without 1) by the matrix coefficients of c_F and b_F. Then, given $X\in\mathfrak{n}_F$, we can find $f_i, h_i\in S_{0,F}$ $(1\leqslant i\leqslant p)$ such that $\theta X=\Sigma_{1\leqslant i\leqslant p}(f_i(m)\,Y_i^{m^{-1}}+h_i(m)\,Y_i)$ $(m\in M'_{1F})$.*

Proof. Let $\{X_1, ..., X_q\}$ be a basis for $\mathfrak{n}_F$, and $(c_{\alpha\beta}(m))$, $(b_{\alpha\beta}(m))$ the matrices of $c_F(m)$ and $b_F(m)$ respectively, with respect to it. Let $EX_\alpha = \Sigma_{1\leqslant i\leqslant p} a_{\alpha i} Y_i$, $X = \Sigma_{1\leqslant\alpha\leqslant q} x_\alpha X_\alpha$. We obtain Lemma 4.2 from Lemma 4.1 by routine calculation with $f_i = -2\Sigma_{1\leqslant\alpha,\beta\leqslant q} x_\beta a_{\alpha i} b_{\alpha\beta}$ and $h_i = 2\Sigma_{1\leqslant\alpha,\beta\leqslant q} x_\beta a_{\alpha i} c_{\alpha\beta}$ $(1\leqslant i\leqslant p)$.

Write $\mathfrak{k}_F = \mathfrak{m}_{1F}\cap\mathfrak{k}$, $\mathfrak{s}_F = \mathfrak{m}_{1F}\cap\mathfrak{s}$. Then $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}_F + \theta(\mathfrak{n}_F)$ is a direct sum. Let $\boldsymbol{\lambda}$ be the symmetrizer map of $S(\mathfrak{g}_c)$ onto $\mathfrak{G}$ and let $\mathfrak{S}_F = \boldsymbol{\lambda}(S(\mathfrak{s}_F))$. Then $\mathfrak{G} = \theta(\mathfrak{n}_F)\mathfrak{G} + \mathfrak{S}_F\mathfrak{K}\mathfrak{k} + \mathfrak{S}_F$ is also a direct sum. For $b\in\mathfrak{G}$, let $\nu_i(b)$ $(i=0, 1, 2)$ be the respective components of b in $\theta(\mathfrak{n}_F)\mathfrak{G}$, $\mathfrak{S}_F\mathfrak{K}\mathfrak{k}$ and $\mathfrak{S}_F$. Define $\nu_F(b) = \nu_1(b) + \nu_2(b)$. It follows easily from the Poincaré-Birkhoff-Witt theorem that $\deg \nu_i(b) \leqslant \deg(b)$ $(i=0, 1, 2)$, and that we can write $\nu_F(b) = \Sigma_{1\leqslant j\leqslant r}\eta_j\zeta_j$, where $\eta_j\in\mathfrak{S}_F$, $\zeta_j\in\mathfrak{K}$, $\deg(\eta_j) + \deg(\zeta_j) \leqslant \deg(b)$ $(1\leqslant j\leqslant r)$.

LEMMA 4.3. *Let $b\in\mathfrak{G}$ and $\deg(b) = r$. Define $S_{0,F}$ as in Lemma 4.2. Then we can select $\xi_i, \zeta_i\in\mathfrak{K}$, $\eta_i\in\mathfrak{M}_{1F}$, $g_i\in S_{0,F}$ $(1\leqslant i\leqslant s)$ such that* (i) $\deg(\eta_i)\leqslant r-1$, $\deg(\xi_i) + \deg(\eta_i) + \deg(\zeta_i) \leqslant r$ $(1\leqslant i\leqslant s)$ (ii) *for all $m\in M'_{1F}$,*

$$b = \nu_F(b) + \sum_{1\leqslant i\leqslant s} g_i(m)\,\xi_i^{m^{-1}}\eta_i\zeta_i. \tag{4.4}$$

Proof. We use induction on r. The case $r=0$ is trivial. Let $r=1$, $b = Y\in\mathfrak{g}$. If $Y\in\mathfrak{k}+\mathfrak{s}_F$, then $\nu_F(Y) = Y$ and we have (4.4) with $g_i\equiv 0$; if $Y = \theta X$ for some $X\in\mathfrak{n}_F$, then $\nu_F(Y) = 0$, and Lemma 4.2 implies what we want. Let $r\geqslant 2$ and assume that the lemma has been proved for elements of degree $\leqslant r-1$. If $b\in\mathfrak{S}_F\mathfrak{K}$, then $\nu_F(b) = b$ and we have (4.4) with $g_i\equiv 0$. So it is enough to consider the case $b\in\theta(\mathfrak{n}_F)\mathfrak{G}$. We may obviously assume that $b = \theta X\cdot\bar{b}$ where $X\in\mathfrak{n}_F$ and $\deg(\bar{b})\leqslant r-1$. Note that $\nu_F(b) = 0$. By the induction hypothesis, we can find $\bar{\xi}_j$, $\bar{\zeta}_j\in\mathfrak{K}$, $\bar{\eta}_j\in\mathfrak{M}_{1F}$, $\bar{g}_j\in S_{0,F}$ such that the appropriate conditions on degrees are satisfied, and for all $m\in M'_{1F}$,

$$\bar{b} = \nu_F(\bar{b}) + \sum_{1\leqslant j\leqslant \bar{s}} \bar{g}_j(m)\,\bar{\xi}_j^{m^{-1}}\bar{\eta}_j\bar{\zeta}_j.$$

Write $\nu_F(\bar{b}) = \Sigma_{1\leqslant k\leqslant q} u_k v_k$ where $u_k\in\mathfrak{S}_F$, $v_k\in\mathfrak{K}$, $\deg(u_k) + \deg(v_k)\leqslant r-1$ for $1\leqslant k\leqslant q$. Substituting for θX from Lemma 4.2 we find, after a simple calculation, the following result, valid for $m\in M'_{1F}$:

$$b = \sum_{1\leqslant i\leqslant p} h_i(m)[Y_i, \bar{b}] + \sum_{1\leqslant i\leqslant p}\sum_{1\leqslant k\leqslant q}(f_i(m)Y_i^{m^{-1}}u_k v_k + h_i(m)u_k v_k Y_i)$$
$$+ \sum_{1\leqslant i\leqslant p}\sum_{1\leqslant j\leqslant \bar{s}} \bar{g}_j(m)\{f_i(m)(Y_i\bar{\xi}_j)^{m^{-1}}\bar{\eta}_j\bar{\zeta}_j + h_i(m)\,\bar{\xi}_j^{m^{-1}}\bar{\eta}_j\bar{\zeta}_j Y_i\}.$$

Applying the induction hypothesis to $[Y_i, \bar{b}]$ (which is permissible as $\deg([Y_i, \bar{b}])\leqslant r-1$), and substituting in the above expression for b, we obtain (4.4) without much difficulty.

LEMMA 4.4. *For* $z \in \mathfrak{Z}$, $\nu_F(z) = d_F^{-1} \circ \mu_F(z) \circ d_F$.

Proof. $\nu_F(z)$ is the unique element of $\mathfrak{S}_F \mathfrak{K}$ such that $z - \nu_F(z) \in \theta(\mathfrak{n}_F)\mathfrak{G}$. On the other hand, $d_F^{-1} \circ \mu_F(z) \circ d_F \in \mathfrak{M}_{1F} \subseteq \mathfrak{S}_F \mathfrak{K}$, while $z - d_F^{-1} \circ \mu_F(z) \circ d_F \in \theta(\mathfrak{n}_F)\mathfrak{G}\mathfrak{n}_F$, for $z \in \mathfrak{Z}$. This proves the lemma.

We choose and fix elements $v_1 = 1, v_2, ..., v_{r_F} \in \mathfrak{Z}_F$ such that

$$\mathfrak{Z}_F = \sum_{1 \leq i \leq r_F} \mu_F(\mathfrak{Z}) v_i \quad \text{(direct sum)}. \tag{4.5}$$

Let $S_{0,F}$ be as in Lemma 4.2. We denote by S_F the algebra generated (without 1) by functions of the form ηg ($\eta \in \mathfrak{M}_{1F}$, $g \in S_{0,F}$). The following is then the main result of this section.

THEOREM 4.5. (i) *Let* $b \in \mathfrak{G}$ *and let* $g_i, \xi_i, \eta_i, \zeta_i$ *be as in Lemma* 4.3. *Write* $\nu_F(b) = \sum_{1 \leq j \leq r} \eta_j \bar{\zeta}_j$ ($\eta_j \in \mathfrak{M}_{1F}$, $\bar{\zeta}_j \in \mathfrak{K}$). *Then for arbitrary* V, τ *and* $\varphi \in C^\infty(G: V: \tau)$ *we have, for* $m \in M'_{1F}$,

$$\varphi(m; b) = \sum_{1 \leq j \leq r} \varphi(m; \eta_j) \tau_2(\bar{\zeta}_j) + \sum_{1 \leq i \leq s} g_i(m) \tau_1(\xi_i) \varphi(m; \eta_i) \tau_2(\zeta_i).$$

(ii) *Fix* $v \in \mathfrak{Z}_F$ *and let* z_i $(1 \leq i \leq r_F)$ *be the unique elements of* $\mathfrak{Z}$ *such that* $v = \sum_{1 \leq i \leq r_F} v_i \mu_F(z_i)$ *Then, there exist* $\xi_j, \zeta_j \in \mathfrak{K}$, $\eta_j \in \mathfrak{M}_{1F}$, $g_j \in S_F$ $(1 \leq j \leq q)$ *with the following property: for arbitrary* V, τ, $\varphi \in C^\infty(G: V: \tau)$, *and* $m \in M'_{1F}$,

$$\varphi(m; v \circ d_F) = \sum_{1 \leq i \leq r_F} \varphi(m; v_i \circ d_F \circ z_i) + \sum_{1 \leq j \leq q} g_j(m) \tau_1(\xi_j) \varphi(m; \eta_j \circ d_F) \tau_2(\zeta_j).$$

Proof. If $\varphi \in C^\infty(G: V: \tau)$, $\xi, \zeta \in \zeta \in \mathfrak{K}$, $\eta \in \mathfrak{G}$, $x \in G$, then $\varphi(x; \xi^{x^{-1}} \eta \zeta) = \tau_1(\xi) \varphi(x; \eta) \tau_2(\zeta)$. (4.4) then leads at once to (i). We shall now prove (ii). By Lemmas 4.3 and 4.4 we can select $\xi_{ij}, \zeta_{ij} \in \mathfrak{K}$, $\eta_{ij} \in \mathfrak{M}_{1F}$, $g_{ij} \in S_{0,F}$ such that for all $m \in M'_{1F}$, $1 \leq i \leq r_F$,

$$z_i = d_F^{-1} \circ \mu_F(z_i) \circ d_F - \sum_{1 \leq j \leq s} g_{ij}(m) \xi_{ij}^{m^{-1}} \eta_{ij} \zeta_{ij} \tag{4.6}$$

so that, for arbitrary $V, \tau, \varphi \in C^\infty(G: V: \tau)$, and m, i as above,

$$\varphi(m; d_F \circ z_i) = \varphi(m; \mu_F(z_i) \circ d_F) - d_F(m) \sum_{1 \leq j \leq s} g_{ij}(m) \tau_1(\xi_{ij}) \varphi(m; \eta_{ij}) \tau_2(\zeta_{ij}).$$

From this we calculate $\varphi(m; v \circ d_F)$ to be

$$\sum_{1 \leq i \leq r_F} \varphi(m; v_i \circ d_F \circ z_i) + \sum_{1 \leq i \leq r_F} \sum_{1 \leq j \leq s} \tau_1(\xi_{ij}) \varphi(m; v_i \circ g_{ij} \circ \tilde{\eta}_{ij} \circ d_F) \tau_2(\zeta_{ij}) \tag{4.7}$$

where $\tilde{\eta}_{ij} = d_F \circ \eta_{ij} \circ d_F^{-1}$. By the definition of S_F, we can find $w_k \in \mathfrak{M}_{1F}$, $h_{ijk} \in S_F$ $(1 \leq k \leq t)$ such that $v_i \circ g_{ij} = \sum_{1 \leq k \leq t} h_{ijk} \circ w_k$ for all i, j. Substituting in (4.7) we get the required result.

Remarks 1. We note that, in (ii), g_j, ξ_j, η_j, ζ_j *do not depend on* V *and* τ. This enables us to keep track of the way in which our subsequent estimates for φ vary with V and τ.

2. The results of this section do not need the assumption rk (G) = rk (K) for their validity.

5. The differential equations for Ψ and certain initial estimates

We fix $F \subsetneqq \Sigma$. We select a complex Hilbert space T of dimension r_F, an orthonormal basis $\{e_1, \ldots, e_{r_F}\}$ of it, and identify endomorphisms of T with their matrices in this basis. Given V and $\tau = (\tau_1, \tau_2)$ as in § 4, we define $\underline{V} = V \otimes T$, $\underline{\tau}_1(k) = \tau_1(k) \otimes 1$, $\underline{\tau}_2(k) = \tau_2(k) \otimes 1$ $(k \in K)$. $\underline{V}$ is a Hilbert space in the usual way, and $\underline{\tau} = (\underline{\tau}_1, \underline{\tau}_2)$ is a unitary double representation of K in $\underline{V}$. τ_F and $\underline{\tau}_F$ are the double representations of K_F obtained by restricting τ and $\underline{\tau}$ respectively to K_F.

Given $v \in \mathfrak{Z}_F$, there are unique $z_{v:ij} \in \mathfrak{Z}$ such that

$$vv_j = \sum_{1 \leqslant i \leqslant r_F} \mu_F(z_{v:ij})\, v_i \quad (1 \leqslant j \leqslant r_F). \tag{5.1}$$

For $\Lambda \in \mathfrak{l}_c^*$ let $\Gamma(\Lambda : v)$ be the endomorphism of T with matrix $(\mu_{\mathfrak{g}/\mathfrak{l}}(z_{v:ji})(\Lambda))_{1 \leqslant i, j \leqslant r_F}$; then $\Gamma(s\Lambda : v) = \Gamma(\Lambda : v)$ $(s \in W(\mathfrak{l}_c))$ and $v \mapsto \Gamma(\Lambda : v)$ is a representation of $\mathfrak{Z}_F$ in T. It is known that $\Gamma(\Lambda : v)$ has the numbers $\mu_{\mathfrak{m}_{1F}/\mathfrak{l}}(v)(s\Lambda)$ $(s \in W(\mathfrak{l}_c))$ as its eigenvalues, and that it is semisimple if Λ is regular. Let $\mathfrak{l}_c^{*\prime}$ be the set of all regular $\Lambda \in \mathfrak{l}_c^*$. Since $\mathfrak{a}_F \subseteq \mathfrak{Z}_F$, it is then clear that for $\Lambda \in \mathfrak{l}_c^{*\prime}$ and $H \in \mathfrak{a}_F$, $\Gamma(\Lambda : H)$ is semisimple with eigenvalues $(s\Lambda)(H)$ $(s \in W(\mathfrak{l}_c))$. In fact, the following lemma is valid (cf. [7] § 3, [8] Lemma 19).

LEMMA 5.1. *Let* $\overline{P}$ *be a positive system of roots of* $(\mathfrak{g}_c, \mathfrak{l}_c)$ *and* $\overline{P}_F$ *the subset of* $\overline{P}$ *vanishing on* $\mathfrak{a}_F$. *Write* $\varpi = \Pi_{\alpha \in \overline{P}} H_\alpha$, $\varpi_F = \Pi_{\alpha \in \overline{P}_F} H_\alpha$. *Let* $s_1 = 1, s_2, \ldots, s_{r_F}$ *be a complete system of representatives of* $W(\mathfrak{l}_c)/W(\mathfrak{l}_c)_F$. *Let* $u_j = \mu_{\mathfrak{m}_{1F}/\mathfrak{l}}(v_j)$, $1 \leqslant j \leqslant r_F$ *and let* $e_k(\Lambda)$ *be the element* $\Sigma_{1 \leqslant j \leqslant r_F} u_j(s_k^{-1}\Lambda) e_j$ *of* T. *Then, if* $\Lambda \in \mathfrak{l}_c^{*\prime}$, *the* $e_j(\Lambda)$ *form a basis of* T, *and* $\Gamma(\Lambda : v) e_j(\Lambda) = \mu_{\mathfrak{m}_{1F}/\mathfrak{l}}(s_j^{-1}\Lambda) e_j(\Lambda)$ $(v \in \mathfrak{Z}_F, 1 \leqslant j \leqslant r_F)$. *Moreover, there is an* $r_F \times r_F$ *matrix* E *with entries in the quotient field of* $I(W(\mathfrak{l}_c)_F)$ *having the following properties:* (i) $(\varpi/\varpi_F) E$ *has entries in* $I(W(\mathfrak{l}_c)_F)$ (ii) *for* $\Lambda \in \mathfrak{l}_c^{*\prime}$, $E(s_k^{-1}\Lambda)$ *are the projections* $T \to \mathbf{C} \cdot e_k(\Lambda)$ *corresponding to the direct sum* $T = \Sigma_{1 \leqslant k \leqslant r_F} \mathbf{C} \cdot e_k(\Lambda)$.

Fix $v \in \mathfrak{Z}_F$. By Theorem 4.5 we can choose $\xi_{jk}^v, \zeta_{jk}^v \in \mathfrak{K}$, $\eta_{jk}^v \in \mathfrak{M}_{1F}$, $g_{jk}^v \in \mathfrak{s}_F$ $(1 \leqslant j \leqslant r_F$, $1 \leqslant k \leqslant q)$ such that for arbitrary $V, \tau, \varphi \in C^\infty(G : V : \tau)$, and $m \in M'_{1F}$,

$$\varphi(m; vv_j \circ d_F) = \sum_{1 \leqslant i \leqslant r_F} \varphi(m; v_i \circ d_F \circ z_{v:ij}) + \sum_{1 \leqslant k \leqslant q} g_{jk}^v(m)\, \tau_1(\xi_{jk}^v)\, \varphi(m; \eta_{jk}^v \circ d_F)\, \tau_2(\zeta_{jk}^v). \tag{5.2}$$

We now define the differential operator $D_v^{\underline{\tau}}$ on $C^\infty(M'_{1F} : \underline{V})$ by setting, for all $\underline{f} = \sum_{1 \leqslant j \leqslant r_F} f_j \otimes e_j$ $(f \in C^\infty(M'_{1F} : V))$,

$$D_v^\tau f = \sum_{1\leqslant i\leqslant r_F} D_{v:i}^\tau f_1 \otimes e_i \tag{5.3}$$

where, for $f\in C^\infty(M'_{1F}: V)$ and $m\in M'_{1F}$,

$$(D_{v:i}^\tau f)(m) = \sum_{1\leqslant k\leqslant q} g_{ik}^v(m)\,\tau_1(\xi_{ik}^v)\,f(m;\eta_{ik}^v)\,\tau_2(\zeta_{ik}^v).$$

The following lemma is then immediate.

LEMMA 5.2. *Let notation be as above. For $\varphi\in C^\infty(G: V:\tau)$ let*

$$\Phi(m) = \Sigma_{1\leqslant j\leqslant r_F}\,\varphi(m; v_j\circ d_F)\otimes e_j.$$

Assume that for some $\Lambda\in\mathfrak{l}_c^*$, $z\varphi = \mu_{\mathfrak{g}/\mathfrak{l}}(z)(\Lambda)\varphi$ for all $z\in\mathfrak{Z}$. Then, for $v\in\mathfrak{Z}_F$ and $m\in M'_{1F}$,

$$\Phi(m; v) = (1\otimes\Gamma(\Lambda: v))\,\Phi(m) + \Phi(m; D_v^\tau). \tag{5.4}$$

Moreover, let $\gamma\geqslant 0$ and let $\Psi = d_F^\gamma\Phi$. For $\eta\in\mathfrak{M}_{1F}$ and $v\in\mathfrak{Z}_F$, let $'\eta = d_F^{-\gamma}\circ\eta\circ d_F^\gamma$, $'D_{v,\eta}^\tau = d_F^\gamma\circ({}'\eta D_v^\tau)\circ d_F^{-\gamma}$. Then, for $m\in M'_{1F}$,

$$\Psi(m; v\eta) = (1\otimes\Gamma(\Lambda: {}'v))\Psi(m;\eta) + \Psi(m;\ {}'D_{v,\eta}^\tau). \tag{5.5}$$

If $m\in M_{1F}^+$, $H\in\mathfrak{a}_F^+$, then $m\exp tH\in M_{1F}^+$ for $t\geqslant 0$; also $'H = H+\gamma\varrho(H)1$. So Lemma 5.2 gives

LEMMA 5.3. *Let notation be as above. Fix $H\in\mathfrak{a}_F^+$, $\eta\in\mathfrak{M}_{1F}$. For $m\in M_{1F}^+$ let $F_m = F_{m,H,\eta}$ and $G_m = G_{m,H,\eta}$ be the functions on $[0,\infty)$ defined by*

$$F_m(t) = \Psi(m\exp tH;\eta),\quad (G_m(t) = \Psi(m\exp tH;\ {}'D_{H,\eta}^\tau). \tag{5.6}$$

Then, on $(0,\infty)$

$$\frac{dF_m}{dt} = \{1\otimes(\Gamma(\Lambda: H) + \gamma\varrho(H)\,1)\}\,F_m + G_m. \tag{5.7}$$

Choose an orthonormal basis $\{X_1,\dots,X_a\}$ of $\mathfrak{k}$, Put

$$\Omega = 1-(X_1^2+\dots+X_a^2),\quad |\tau| = (1+\|\tau_1(\Omega)\|)\,(1+\|\tau_2(\Omega)\|). \tag{5.8}$$

LEMMA 5.4. *Fix $v\in\mathfrak{Z}_F$, $\eta\in\mathfrak{M}_{1F}$. Then there exist $r = r_{v,\eta}\geqslant 0$, $\omega_k = \omega_{k,v,\eta}\in\mathfrak{M}_{1F}$ $(1\leqslant k\leqslant q = q_{v,\eta})$ such that for arbitrary V, τ, and $f\in C^\infty(M_{1F}^+; V)$, and all $m\in M_{1F}^+$,*

$$\|f(m;\eta\circ D_v^\tau)\| \leqslant \gamma_F(m)\,(1-\gamma_F(m))^{-r}\,|\tau|^r \sum_{1\leqslant k\leqslant q}\|f(m;(m;\omega_k)\|.$$

Proof. It is clear from the definition of $D_{v:i}^\tau$ and D_v^τ that for f, m as above,

$$\|f(m;\eta\circ D_v^\tau)\| \leqslant \sum_{1\leqslant i\leqslant r_F}\sum_{1\leqslant k\leqslant q}\|\tau_1(\xi_{ik}^v)\|\,\|\tau_2(\zeta_{ik}^v)\|\,\|f_1(m;\eta\circ g_{ik}^v\circ\eta_{ik}^v)\|.$$

Now we can select $\eta_{ikj}^v\in\mathfrak{M}_{1F}$, $g_{ikj}^v\in\mathfrak{s}_F$ such that $\eta\circ g_{ik}^v\circ\eta_{ik}^v = \sum_{1\leqslant j\leqslant r} g_{ikj}^v\circ\eta_{ikj}^v$ for all i, k. So we get

$$\|f(m;\eta\circ D_v^\tau)\| \leqslant \sum_{i,j,k} |g_{ikj}^v(m)|\, \|\tau_1(\xi_{ik}^v)\|\, \|\tau_2(\zeta_{ik}^v)\|\, \|f(m;\eta_{ikj}^v)\|. \qquad (*)$$

Observe now that given any $g \in S_F$, there are constants $c(g)>0$, $q(g)\geqslant 0$ such that for all $m \in M_{1F}^+$

$$|g(m)| \leqslant c(g)\gamma_F(m)(1-\gamma_F(m))^{-q(g)}. \tag{5.9}$$

Indeed, this is immediate from Lemma 7 of [8] if $g=vh$ for some $v\in\mathfrak{M}_{1F}$ and some matrix coefficient h of c_F. On the other hand, we see from (4.3) that $b_F(m) = -\mathrm{Ad}\,(\theta(m))_{\mathfrak{n}_F}(1-c_F(m))$, so that our claim is true for derivatives of matrix coefficients of b_F also. The estimate (5.9) now follows from the definition of S_F. Furthermore, we have the following elementary result from the representation theory of K: given $\xi\in\mathfrak{K}$ of degree s, there is a constant $a(\xi)>0$ such that, for any finite dimensional unitary representation β of K, $\|\beta(\xi)\|\leqslant a(\xi)\|\beta(\Omega)\|^{s/2}$. Using this and (5.9) in (*) we get the lemma.

Let $\|\cdot\|$ be a norm on $\mathfrak{l}_c^*$. Given $\Lambda\in\mathfrak{l}_c^*$ and τ, put

$$\begin{aligned} &|\tau,\Lambda| = (1+\|\tau_1(\Omega)\|)(1+\|\tau_2(\Omega)\|)(1+\|\Lambda\|) \\ &\mathcal{E}(\Lambda\colon G\colon\tau) = \{\varphi\colon \varphi\in C^\infty(G\colon V\colon\tau),\ z\varphi=\mu_{\mathfrak{g}/\mathfrak{l}}(z)(\Lambda)\varphi \quad \text{for all } z\in\mathfrak{Z}\}. \end{aligned} \tag{5.10}$$

As usual, $L^2(G\colon V)$ is the Hilbert space of functions $f\colon G\to V$ with $\|f\|_2^2=\int_G\|f(x)\|^2dx<\infty$. Note that $\mathcal{E}(\Lambda\colon G\colon\tau)\cap L^2(G\colon V)\neq\{0\}$ if and only if $\Lambda\in\mathfrak{L}_{\mathrm{I}}'$ [14]. Also it follows from Theorem 1 of [14] that if $f\in\mathcal{E}(\Lambda\colon G\colon\tau)\cap L^2(G\colon V)$, then $bfa\in L^2(G\colon V)$ for all $a,b\in\mathfrak{G}$.

LEMMA 5.5. *Let* $r\geqslant 0$; $a,b\in\mathfrak{G}$ *such that* $\deg(a)+\deg(b)\leqslant r$. *Then* $\exists$ *a constant* $C=C_{a,b}>0$ *such that for arbitrary* τ, $\Lambda\in\mathfrak{L}_{\mathrm{I}}'$, *and* $f\in\mathcal{E}(\Lambda\colon G\colon\tau)\cap L^2(G\colon V)$,

$$\|bfa\|_2\leqslant C|\tau,\Lambda|^r\|f\|_2. \tag{5.11}$$

Proof. Extend $\{X_1,\dots,X_a\}$ to an orthonormal basis $\{X_1,\dots,X_n\}$ for $\mathfrak{g}$, and let $q=-(X_1^2+\dots+X_n^2)$, $\omega=-(X_1^2+\dots+X_a^2)+(X_{a+1}^2+\dots+X_n^2)$. Then ω is the Casimir of G, $q=-\omega+2\Omega-2$, and $\mu_{\mathfrak{g}/\mathfrak{l}}(\omega)(\Lambda)=\langle H_\Lambda,H_\Lambda\rangle-c$ for all $\Lambda\in\mathfrak{L}_{\mathrm{I}}'$, c being a constant. So we can select a $c_0\geqslant 1$ such that $2+|\mu_{\mathfrak{g}/\mathfrak{l}}(\omega)(\Lambda)|\leqslant c_0^2(1+\|\Lambda\|)^2$ for all $\Lambda\in\mathfrak{L}_{\mathrm{I}}'$. Now, if π is any unitary representation of G in a Hilbert space $\mathfrak{H}$ and ψ is a differentiable vector for π, $-(\pi(X_i)^2\psi,\psi)=\|\pi(X_i)\psi\|^2\geqslant 0$ $(1\leqslant i\leqslant n)$, so that $\|\pi(X_i)\psi\|^2\leqslant(\pi(q)\psi,\psi)$. We apply this to the case when $\mathfrak{H}=L^2(G\colon V)$, π is the right regular representation of G in $\mathfrak{H}$, and $\psi=f\in\mathcal{E}(\Lambda\colon G\colon\tau)\cap\mathfrak{H}$; as $f=\alpha*f*\beta$ for suitable $\alpha,\beta\in C_c^\infty(G)$ by Theorem 1 of [14], f is surely differentiable for π. Thus, for $1\leqslant i\leqslant n$, $\|X_if\|_2^2\leqslant-(\omega f,f)-2(f,f)+2(\Omega f,f)\leqslant(2+|\mu_{\mathfrak{g}/\mathfrak{l}}(\omega)(\Lambda)|)\|f\|_2^2+2|(\Omega f,f)|$. But $|(\Omega f,f)|=|\int_G(f(x)\tau_2(\Omega),f(x))dx|\leqslant\|\tau_2(\Omega)\|\,\|f\|_2^2$. So we get the estimate $\|X_if\|_2\leqslant c_0|\tau,\Lambda|\,\|f\|_2$ from which we get $\|Xf\|_2\leqslant n\|X\|c_0|\tau,\Lambda|\,\|f\|_2$

for all $X \in \mathfrak{g}$. A similar estimate holds for $\|fX\|_2$. We have thus proved the lemma when $\deg(a)+\deg(b) \leqslant 1$.

Assume the lemma for $r=m$. Let $a', b' \in \mathfrak{G}$ with $\deg(a')+\deg(b') \leqslant m$. Let $\mathfrak{G}_1$ (resp. $\mathfrak{G}_2$) be the subspace of $\mathfrak{G}$ of all elements of degree $\leqslant \deg(a')$ (resp. $\deg(b')$), and let $(a_i)_{1\leqslant i \leqslant R}$ (resp. $(b_j)_{1 \leqslant j \leqslant S}$) be a basis of $\mathfrak{G}_1$ (resp. $\mathfrak{G}_2$) such that the matrices $(\alpha_{ij}(k))$ (resp. $\beta_{ij}(k)$) $(k \in K)$ of the adjoint representation of K in $\mathfrak{G}_1$ (resp. $\mathfrak{G}_2$) are unitary. Let U be a Hilbert space with an orthonormal basis $(u_{ij})_{1\leqslant i \leqslant R, 1 \leqslant j \leqslant S}$, and define the unitary double representation $\nu=(\nu_1, \nu_2)$ of K in U by setting $\nu_1(k)u_{pq} = \sum_{1\leqslant i\leqslant R}\alpha_{pi}(k^{-1})u_{iq}$, $u_{pq}\nu_2(k) = \sum_{1\leqslant j\leqslant S}\beta_{qj}(k)u_{pj}$ $(k \in K,\ 1 \leqslant p \leqslant R,\ 1 \leqslant q \leqslant S)$. Given V, τ, f as above, let $\tilde{V} = V \otimes U$, $\tilde{\tau} = \tau \otimes \nu$, and $F(x) = \sum_{1\leqslant i \leqslant R, 1\leqslant j \leqslant S} f(a_i; x; b_j) \otimes u_{ij}$ $(x \in G)$. It is easily seen that $F \in \mathcal{E}(\Lambda: G: \tilde{\tau}) \cap L^2(G: \tilde{V})$. So by the earlier result, $\|XF\|_2 + \|FX\|_2 \leqslant c_X|\tilde{\tau}, \Lambda| \, \|F\|_2$ for $X \in \mathfrak{g}$, $c_X > 0$ depending only on X. Thus, for $1 \leqslant i \leqslant R$, $1 \leqslant j \leqslant S$, $X \in \mathfrak{g}$,

$$\|Xb_j f a_i\|_2 + \|b_j f a_i X\|_2 \leqslant c_X |\tilde{\tau}, \Lambda| \sum_{1\leqslant p \leqslant R,\, 1 \leqslant q \leqslant S} \|b_q f a_p\|_2.$$

We estimate the right side of this inequality by the induction hypothesis applied to $\|b_q f a_p\|_2$, and by the (easily proved) fact that for a suitable constant $c' > 0$, $|\tilde{\tau}, \Lambda| \leqslant c' |\tau, \Lambda|$ for all Λ, τ. This gives the lemma for $r = m+1$.

From Lemma 5.5 and Theorem 3.3 we get

LEMMA 5.6. *Given $a, b \in \mathfrak{G}$, there are constants $C = C_{a,b} > 0$ and $r = r_{a,b} \geqslant 0$ such that for arbitrary V, τ, Λ, $f \in \mathcal{E}(\Lambda: G: \tau) \cap L^2(G: V)$,*

$$\|f(a; x; b)\| \leqslant C|\tau, \Lambda|^r \, \Xi(x) \|f\|_2 \quad (x \in G). \tag{5.12}$$

LEMMA 5.7. *Given $\eta \in \mathfrak{M}_{1F}$, there are constants $C = C_\eta > 0$, $r = r_\eta \geqslant 0$ such that for arbitrary V, τ, Λ, $m \in M^+_{1F}$, $\varphi \in \mathcal{E}(\Lambda: G: \tau) \cap L^2(G: V)$ and Φ as in Lemma 5.2,*

$$\|\Phi(m; \eta)\| \leqslant C|\tau, \Lambda|^r d_F(m) \Xi(m) \|\varphi\|_2. \tag{5.13}$$

Proof. Let $(\eta v_j)' = d_F^{-1} \circ (\eta v_j) \circ d_F$. The lemma follows from Lemma 5.6 and the inequality

$$\|\Phi(m; \eta)\| \leqslant d_F(m) \sum_{1 \leqslant j \leqslant r_F} \|\varphi(m; (\eta v_j)')\|. \tag{5.14}$$

LEMMA 5.8. (i) *There are constants $c_1 > 0$, $r_1 \geqslant 0$ such that for all $m \in M^+_{1F}$, $d_F(m)\Xi(m) \leqslant c_1 \Xi_F(m)(1+\sigma(m)^{r_1})$;* (ii) *given $H \in \mathfrak{a}^+_F$, there is a constant $c_2(H) > 0$ such that $m \exp tH \in M^+_{1F}$ for any $m \in M_{1F}$ and $t \geqslant c_2(H)\sigma(m)$;* (iii) *given $H \in \mathfrak{a}^+_F$, $\gamma \geqslant 0$, $0 < \varepsilon < 1$, there are constants $a = a_{H,\gamma}$, $0 < a < 1$, and $c(\varepsilon) = c_{H,\gamma}(\varepsilon) > 0$, such that, for $m \in M^+_{1F}$ and $t \geqslant 0$,*

$$d_F(m \exp tH)^{1+\gamma}\,\Xi(m \exp tH)^{1+\gamma-\varepsilon a} \leqslant c(\varepsilon)\,d_F(m)^{1+\gamma}\,\Xi(m)^{1+\gamma-\varepsilon}\,e^{\varepsilon t}. \tag{5.15}$$

Proof. (i) and (iii) follow quickly from (2.1) and the relation $M_{1F}^+ \subseteq K_F Cl(A^+) K_F$. For (ii) see [14], p. 69.

LEMMA 5.9. *Let $H \in \mathfrak{a}_F^+$, $\eta \in \mathfrak{M}_{1F}$. Then we can select $r = r_{H,\eta} \geqslant 0$, $q = q_{H,\eta} \geqslant 1$ and $\omega_s \in \mathfrak{M}_{1F}$ $(1 \leqslant s \leqslant q)$ such that for arbitrary V, τ, Λ, $\varphi \in \mathcal{E}(\Lambda\colon G\colon \tau)$, the functions F_m and G_m defined by* (5.6) *satisfy the following inequalities, for all $m \in M_{1F}^+$ and $t \geqslant 0$:*

$$\|F_m(t)\| \leqslant d_F(m \exp tH)^{1+\gamma} \sum_{1 \leqslant s \leqslant q} \|\varphi(m \exp tH;\omega_s)\|$$

$$\|G_m(t)\| \leqslant \gamma_F(m)\,(1-\gamma_F(m \exp tH))^{-r}\,|\tau|^r\,e^{-t\beta_F(H)}\,d_F(m \exp tH)^{1+\gamma} \sum_{1 \leqslant s \leqslant q} \|\varphi(m \exp tH;\omega_s)\|. \tag{516}$$

Proof. Write $e_t = \exp tH$. Then (5.14) gives, for m, t as above,

$$\|F_m(t)\| \leqslant d_F(me_t)^{1+\gamma} \sum_{1 \leqslant j \leqslant r_F} \|\varphi(me_t;('\eta v_j)')\|.$$

Further, $G_m(t) = d_F(me_t)^\gamma\,\Phi(me_t;'\eta D_H^r)$ can be estimated by Lemma 5.4. Write, in the notation of that lemma, $\bar q = q_{H,'\eta}$, $\bar r = r_{H,'\eta}$, $\zeta_k = \omega_{k,H,'\eta}$; then $\|G_m(t)\|$ is majorized by

$$\gamma_F(me_t)\,(1-\gamma_F(me_t))^{-\bar r}\,|\tau|^{\bar r}\,d_F(me_t)^\gamma \sum_{1 \leqslant k \leqslant \bar q} \|\Phi(me_t;\zeta_k)\|;$$

as $\gamma_F(me_t) \leqslant e^{-t\beta_F(H)}\gamma_F(m)$, we find from (5.14) that $\|G_m(t)\|$ is majorized by

$$\gamma_F(m)\,(1-\gamma_F(me_t))^{-r}\,|\tau|^r\,e^{-t\beta_F(H)}\,d_F(me_t)^{1+\gamma} \sum_{j,k} \|\varphi(me_t;(\zeta_k v_j)')\|.$$

Our lemma follows at once from these estimates.

Remark. Except Lemmas 5.5 and 5.6, the results of this section do not need the assumption rk (G) = rk (K) for their validity.

6. A lemma on ordinary differential equations

In this §, X is a finite dimensional Banach space with norm $\|\cdot\|$; Γ is a semisimple endomorphism of X with only real eigenvalues; $S = S(\Gamma)$ is the set of eigenvalues of Γ, and $[S]$ is the number of elements of S; for $c \in S$, X_c is the eigensubspace and E_c is the spectral projection, corresponding to c. We define

$$C = \max_{c \in S} \|E_c\| \quad \alpha = \min\left(\tfrac{1}{2}, \min_{c \in S, c \neq 0} |c|\right). \tag{6.1}$$

LEMMA 6.1. *Let f and g be functions of class C^1 defined on an interval of the form $(-h, \infty)$ $(h>0)$, with values in X. Suppose that $df/dt = \Gamma f + g$ on $(0, \infty)$, and that, for each ε with $0<\varepsilon<1$, there is a constant $C_\varepsilon>0$ for which*

$$\|f(t)\| \leqslant C_\varepsilon e^{\varepsilon t}, \|g(t)\| \leqslant C_\varepsilon e^{\varepsilon t - t} \quad (t \geqslant 0). \tag{6.2}$$

Then $f_\infty = \lim_{t\to+\infty} f(t)$ exists, lies in X_0, and for all $t \geqslant 0$, $0<\varepsilon\leqslant \frac{1}{2}$

$$\|f_\infty\| \leqslant 3CC_\varepsilon, \|f(t) - f_\infty\| \leqslant 3[S]CC_\varepsilon e^{\varepsilon t - \alpha t}. \tag{6.3}$$

Proof. For $c \in S$ put $f_c(t) = E_c f(t)$, $g_c(t) = E_c g(t)$. Then $df_c/dt = cf_c + g_c$ on $(0, \infty)$, and we have, for $t \geqslant 0$ and $0<\varepsilon<1$,

$$\|f_c(t)\| \leqslant CC_\varepsilon e^{\varepsilon t}, \|g_c(t)\| \leqslant CC_\varepsilon e^{\varepsilon t - t}. \tag{6.4}$$

We consider three cases.

Case 1: $c>0$. Then, for $0 \leqslant t < t'$, we have

$$e^{-ct'} f_c(t') - e^{-ct} f_c(t) = e^{-ct} \int_0^{t'-t} e^{-cu} g_c(t+u)\, du.$$

Taking $\varepsilon < \min(c, 1)$ in (6.4) we find that $e^{-ct'} f_c(t') \to 0$ as $t' \to +\infty$ while

$$\int_0^\infty e^{-cu} \|g_c(t+u)\|\, du < \infty.$$

So $f_c(t) = -\int_0^\infty e^{-cu} g_c(t+u)\, du$, from which we get, on using (6.4),

$$\|f_c(t)\| \leqslant (1+\alpha-\varepsilon)^{-1} CC_\varepsilon e^{\varepsilon t - t} \quad (c>0, t \geqslant 0). \tag{6.5}$$

Case 2: $c<0$. We have, for $t \geqslant 0$,

$$f_c(t) = e^{ct} f_c(0) + \int_0^t e^{cu} g_c(t-u)\, du.$$

From (6.4) we find that the integrand is majorized by $CC_\varepsilon e^{\varepsilon t - t} e^{cu+u-\varepsilon u}$ which is $\leqslant CC_\varepsilon e^{\varepsilon t - t} e^{(1-\alpha)u}$, as $c \leqslant -\alpha$. We then find

$$\|f_c(t)\| \leqslant (1 + 1/(1-\alpha))\, CC_\varepsilon e^{\varepsilon t - \alpha t} \quad (c<0, t \geqslant 0). \tag{6.6}$$

Case 3: $c=0$. Since $df_0/dt = g_0$ and $\int_0^\infty \|g_0(u)\|\, du < \infty$, we see that $f_\infty = \lim_{t\to+\infty} f_0(t)$ exists, lies in X_0, and, for $t \geqslant 0$,

$$f_\infty = f_0(t) + \int_0^\infty g_0(t+u)\, du. \tag{6.7}$$

Taking $t=0$ in (6.7) and using (6.4) we find easily that

$$\|f_\infty\| \leqslant (1+1/(1-\varepsilon))\, CC_\varepsilon; \tag{6.8}$$

moreover, for $t \geqslant 0$, (6.7) and (6.4) give

$$\|f_0(t)-f_\infty\| \leqslant (1-\varepsilon)^{-1} CC_\varepsilon e^{\varepsilon t - t} \quad (0<\varepsilon<1). \tag{6.9}$$

On the other hand, we have

$$\|f(t)-f_\infty\| \leqslant \|f_0(t)-f_\infty\| + \sum_{c\in S,\, c\neq 0} \|f_c(t)\|. \tag{6.10}$$

From (6.5), (6.6), (6.8)–(6.10), we see that $f(t) \to f_\infty$ as $t \to +\infty$, and that (6.3) is true for $t \geqslant 0$, $0<\varepsilon\leqslant \frac{1}{2}$.

7. The functions $\varphi_{j,\gamma}$ associated with a φ of type (Λ, τ, γ)

Let $\gamma>0$ and V, τ as in §§ 4, 5. A function φ: $G\to V$ is said to be of *type* (Λ, τ, γ) if $\varphi \in \mathcal{E}(\Lambda\colon G\colon \tau)$ and if, given $b\in \mathfrak{G}$, $\varepsilon>0$, we can choose a constant $B_\varepsilon = B_\varepsilon(b\colon \varphi)>0$ such that

$$\|\varphi(x; b)\| \leqslant B_\varepsilon \Xi(x)^{1+\gamma-\varepsilon} \quad (x \in G). \tag{7.1}$$

Such φ lie in $L^2(G\colon V)$; conversely, it follows from the work of [14] that any $\varphi \in \mathcal{E}(\Lambda\colon G\colon \tau) \cap L^2(G\colon V)$ is of type (Λ, τ, β) for some $\beta>0$. In this § we shall make a close study of functions of type (Λ, τ, γ).

We recall the sets F_j and the parabolic subgroups $P_j = M_j A_j N_j$ defined in § 2 $(1 \leqslant j \leqslant d)$. For any $\mu>0$ we put

$$A_j^+(\mu) = \{h\colon h\in A^+,\ \alpha_j(\log h) > \mu\varrho(\log h)\} \tag{7.2}$$

for $1\leqslant j\leqslant d$. Then $A_j^+(\mu) \subseteq A_j^+(\mu')$ if $0<\mu'\leqslant \mu$, and $A+ \subseteq \bigcup_{1\leqslant j\leqslant d} A_j^+(\mu)$ for sufficiently small μ. To see the latter, let Q be the compact set $\{h\colon h\in Cl A^+,\ \|\log h\|=1\}$, and let $c_1 = \inf_{h\in Q} \varrho(\log h)$, $c_2 = \sup_{h\in Q}\varrho(\log h)$, and $c_3 = \Sigma_{1\leqslant i\leqslant d}\varrho(H_i)$; if $h \in A^+$, then $\log h = \Sigma_{1\leqslant j\leqslant d}\alpha_j(\log h) H_j$, so that for $h\in Q\cap A^+$ one has $c_1 \leqslant c_3 \max_{1\leqslant j\leqslant d}\alpha_j(\log h)$, proving that $\alpha_j(\log h) > (c_1/2c_2c_3)\varrho(\log h)$ for some j. In other words,

$$A^+ \subseteq \bigcup_{1\leqslant j\leqslant d} A_j^+(\mu) \quad (0<\mu<c_1/2\,c_2 c_3). \tag{7.3}$$

As mentioned in § 2, we write $d_j = d_{F_j}$, $r_j = r_{F_j}$ etc.

THEOREM 7.1. *Let $\Lambda \in \mathfrak{L}_I'$, $\gamma>0$, V, τ as usual, and let φ be of type (Λ, τ, γ). Let $1\leqslant j\leqslant d$. Then, for any $m\in M_{1j}$*

$$\varphi_{j,\gamma}(m) = \lim_{t\to+\infty} d_j(m \exp tH_j)^{1+\gamma} \varphi(m \exp tH_j)$$

exists. Moreover, we can write $\varphi_{j,\gamma}=\sum_{1\leqslant i\leqslant r_j}\varphi_{j,\gamma,i}$ *where* $\varphi_{j,\gamma,i}(ma)=\varphi_{j,\gamma,i}(m)$ *for* $m\in M_j$, $a\in A_j$, *and* $\varphi_{j,\gamma,i}|M_j$ *is of type* $(s_i\Lambda|\mathfrak{l}\cap\mathfrak{m}_j, \tau_{F_j},\gamma)$ [1] $(1\leqslant i\leqslant r_j)$; *in particular,*

$$\mu_{F_j}(z)(d_j^{-\gamma}\varphi_{j,\gamma})=\mu_{\mathfrak{g}/\mathfrak{l}}(z)(\Lambda)(d_j^{-\gamma}\varphi_{j,\gamma})\quad (z\in\mathfrak{Z})$$

$\varphi_{j,\gamma}=0$ *if* P_j *is not cuspidal. If* $\varphi_{j,\gamma}\neq 0$, *we can find* $s\in V(\mathfrak{l}_c)$ *such that* $(s\Lambda)(H_j)=-\gamma\varrho(H_j)$.

Proof. Define Ψ as in Lemma 5.2. For any $\eta\in\mathfrak{M}_{1j}$ and $m\in M_{1j}^+$, let F_m and G_m be as in Lemma 5.3, with $F=F_j$ and $H=H_j$. Then $dF_m/dt=A_jF_m+G_m$ on $(0,\infty)$ where $A_j=1\otimes(\Gamma(\Lambda: H_j)+\gamma\varrho(H_j)1)$. We obtain easily from (5.15), (5.16) and (7.1) the following result (note that $\beta_{F_j}(H_j)=1$): if $Q\subseteq M_{1j}^+$ is a compact set and $0<\varepsilon<1$, there is a constant $C_{Q,\varepsilon}>0$ such that

$$\|F_m(t)\|\leqslant C_{Q,\varepsilon}e^{\varepsilon t},\quad \|G_m(t)\|\leqslant C_{Q,\varepsilon}e^{\varepsilon t-t} \tag{7.4}$$

for $m\in Q$, $t\geqslant 0$. Further, as $\Lambda\in\mathfrak{L}_{\mathfrak{l}}'$, A_j is a semisimple endomorphism of V whose eigenvalues are the real numbers $(s\Lambda)(H_j)+\gamma\varrho(H_j)$ $(s\in W(\mathfrak{l}_c))$. Let $T_0=\{u: u\in T, \Gamma(\Lambda: H_j)u+\gamma\varrho(H_j)u=0\}$. Then, by Lemma 6.1, we can find $\Theta_\eta(m)\in V\otimes T_0$ such that $F_m(t)=\Psi(m\exp tH_j;\eta)\to\Theta_\eta(m)$ as $t\to+\infty$, for each $m\in M_{1j}^+$, $\eta\in\mathfrak{M}_{1j}$. Moreover, using (7.4), we infer from that lemma the existence of a constant $\alpha>0$ such that, for any compact set $Q\subseteq M_{1j}^+$ and any ε $(0<\varepsilon\leqslant\frac{1}{2})$, we have

$$\|\Psi(m\exp tH_j;\eta)-\Theta_\eta(m)\|\leqslant D_{Q,\varepsilon}\,e^{\varepsilon t-\alpha t}\quad (t\geqslant 0, m\in Q) \tag{7.5}$$

for suitable constants $D_{Q,\varepsilon}$. Let $\Psi_t(m)=\Psi(m\exp tH_j)$. Then the estimates (7.5) show that for any $\eta\in\mathfrak{M}_{1j}$, $\eta\Psi_t\to\Theta_\eta$ uniformly on compact subsets of M_{1j}^+. Thus Θ_1 is of class C^∞ and $\Theta_\eta=\eta\Theta_1$ for $\eta\in\mathfrak{M}_{1j}$.

Now $\Theta_1(m\exp tH_j)=\Theta_1(m)$ for $m\in M_{1j}^+$, $t\geqslant 0$. On the other hand, given any compact set $Q\subseteq M_{1j}$, there is $t_0>0$ such that $m\exp tH_j\in M_{1j}^+$ for $m\in Q$, $t\geqslant t_0$ (Lemma 5.8). It follows easily from this that we can extend Θ_1 uniquely to a function $\Theta\in C^\infty(M_{1j}: V\otimes T_0)$ such that $\Theta(ma)=\Theta(m)$ for all $m\in M_{1j}$, $a\in A_j$. Obviously

$$\Theta(m;\eta)=\lim_{t\to+\infty}\Psi(m\exp tH_j;\eta)\quad (m\in M_{1j}, \eta\in\mathfrak{M}_{1j}). \tag{7.6}$$

From (7.6) we see that Θ is $\underline{\tau}_{F_j}$-spherical. Suppose $\Theta\neq 0$. Since the values of Θ are in $V\otimes T_0$, we have $T_0\neq\{0\}$. So, for some $s\in W(\mathfrak{l}_c)$, $(s\Lambda)(H_j)+\gamma\varrho(H_j)=0$. Let $v\in\mathfrak{Z}_j$, $m\in M_{1j}$. Then we get from (5.5) (with $\eta=1$), for all sufficiently large t,

$$\Psi(m\exp tH_j; v)=(1\otimes\Gamma(\Lambda: 'v))\Psi(m\exp tH_j)+\Psi(m\exp tHj;\, d_j^{\gamma}\circ D_{\cdot v}^{\tau}\circ d_j^{-\gamma}). \tag{7.7}$$

---

[1] The s_i are as in Lemma 5.1 with $F=F_j$. Also M_j, is in general neither connected nor semisimple, and we should remember the remarks made in § 2.

A simple argument based on Lemma 5.4 shows that the second term on the right of (7.7) tends to 0 as $t \to +\infty$. Changing v to $d_j^{\gamma} \circ v \circ d_j^{-\gamma}$, we get from (7.6) and (7.7),

$$v(d_j^{-\gamma}\Theta) = (1 \otimes \Gamma(\Lambda : v))\,(d_j^{-\gamma}\Theta) \quad (v \in \mathfrak{Z}_j) \tag{7.8}$$

Observe that, if $v = \mu_{F_j}(z)$ $(z \in \mathfrak{Z})$, then $z_{v:rs} = \delta_{rs} z$ in (5.1), so that $\Gamma(\Lambda: \mu_{F_j}(z)) = \mu_{\mathfrak{g}/\mathfrak{l}}(z)(\Lambda)\cdot 1$. (7.8) then gives

$$\mu_{F_j}(z)\,(d_j^{-\gamma}\Theta) = \mu_{\mathfrak{g}/\mathfrak{l}}(z)\,(d_j^{-\gamma}\Theta) \quad (z \in \mathfrak{Z}). \tag{7.9}$$

Let $E(s_k^{-1}\Lambda)$ be as in Lemma 5.1, and let $\Theta_k = (1 \otimes E(s_k^{-1}\Lambda))\,\Theta$. Then $\Theta = \sum_{1 \leqslant k \leqslant r_j} \Theta_k$; moreover, from (7.8) we have

$$v(d_j^{-\gamma}\Theta_k) = \mu_{\mathfrak{m}_{1j}/\mathfrak{l}}(v)\,(s_k^{-1}\Lambda)\,(d_j^{-\gamma}\Theta_k) \quad (v \in \mathfrak{Z}_j,\ 1 \leqslant k \leqslant r_j). \tag{7.10}$$

We shall now estimate Θ. Fix $\eta \in \mathfrak{M}_{1j}$. Let E_0 be the spectral projection $V \to V \otimes T_0$: Then from (5.6), (5.7), and (6.7) (with $t=1$) we have, for all $m \in M_{1j}^{+}$,

$$\Theta(m;\eta) = E_0 F_m(1) + \int_1^{\infty} E_0 G_m(u)\,du. \tag{7.11}$$

Estimating the right side of (7.11) using (5.16), we easily obtain the following result: let ω_k $(1 \leqslant k \leqslant q)$ be as in Lemma 5.9; then there is a constant $C > 0$ such that for all $m \in M_{1j}^{+}$,

$$\|\Theta(m;\eta)\| \leqslant C d_j(m \exp H_j)^{1+\gamma} \sum_{1 \leqslant k \leqslant q} \|\varphi(m \exp H_j;\, \omega_k)\|$$
$$+\, C \sum_{1 \leqslant k \leqslant q} \int_1^{\infty} e^{-u} d_j(m \exp uH_j)^{1+\gamma}\, \|\varphi(m \exp uH_j;\, \omega_k)\|\,du.$$

If we now use (5.15) and (7.1) to estimate the right side of this inequality, we get the following result: given δ with $0 < \delta < 1$, there is a constant $A_{\eta,\delta} > 0$ such that

$$\|\Theta(m^+;\eta)\| \leqslant A_{\eta,\delta}\, d_j(m^+)^{1+\gamma}\, \Xi(m^+)^{1+\gamma-\delta} \quad (m^+ \in M_{1j}^{+}). \tag{7.12}$$

On the other hand, if c_1 and $c_2 = c_2(H_j)$ are as in (i) and (ii) of Lemma 5.8, then, for any $m \in M_j$, $m^+ = m \exp c_2\sigma(m) H_j \in M_{1\,j}^{+}$ and $\Theta(m;\eta) = \Theta(m^+;\eta)$; so, from (7.12) we get, for all $m \in M_j$, writing $A'_{\eta,\delta} = A_{\eta,\delta}\, c_1^{1+\gamma}$ and $r_2 = r_1(1+\gamma)$,

$$\|\Theta m;\eta)\| \leqslant A'_{\eta,\delta}\, \Xi_j(m^+)^{1+\gamma-\delta}(1+\sigma(m^+))^{r_2} d_j(m^+)^{\delta}. \tag{*}$$

But $\Xi_j(m^+) = \Xi_j(m)$, $d_j(m^+) = e^{c_2'\sigma(m)}$ $(c_2' = c_2\varrho(H_j))$, and there are constants $c_3 > 0$, $c_4 > 0$, such that $\Xi_j(m) \leqslant c_3 e^{-c_4\sigma(m)}$, $(1+\sigma(m^+)) \leqslant c_3(1+\sigma(m))^2$ $(m \in M_j)$. Let $0 < \varepsilon < 1$. Then, writing $A_{\eta,\varepsilon,\delta} = c_3^{\varepsilon/2 + r_2} A'_{\eta,\delta}$, we get from (*), for all $m \in M_j$ and $0 < \delta < \varepsilon/2$,

$$\|\Theta(m;\eta)\| \leqslant A_{\eta,\varepsilon,\delta}\Xi_j(m)^{1+\gamma-\varepsilon}\{e^{-(\varepsilon/2)c_4\sigma(m)}(1+\sigma(m))^{2r_3}e^{\delta c_3\sigma(m)}\}.$$

It is clear that there is a $\delta=\delta(\varepsilon)$ with $0<\delta<\varepsilon/2$, such that the supremum of the expression within $\{...\}$, as m varies in M_j, is finite. Choosing $\delta=\delta(\varepsilon)$, we find the following: given ε, $0<\varepsilon<1$, there is $B_{\eta,\varepsilon}>0$ such that

$$\|\Theta(m;\eta)\| \leqslant B_{\eta,\varepsilon}\Xi_j(m)^{1+\gamma-\varepsilon} \quad (m\in M_j). \tag{7.13}$$

Let $\Theta(m)=\sum_{1\leqslant s\leqslant r_j}\theta_s(m)\otimes e_s$, $\Theta_i(m)=\sum_{1\leqslant s\leqslant r_j}\theta_{i,s}(m)\otimes e_s$ $(m\in M_{1j})$, and put $\varphi_{j,\gamma}=\theta_1$, $\varphi_{j,\gamma,i}=\theta_{i,1}$ $(1\leqslant i\leqslant r_j)$. Then it is obvious that $d_j(m\exp tH_j)^{1+\gamma}\varphi(m\exp tHj)\to\varphi_{j,\gamma}(m)$ as $t\to+\infty$, for each $m\in M_{1j}$. From the properties of Θ and Θ_i it is moreover immediate that $\varphi_{j,\gamma,i}(ma)=\varphi_{j,\gamma,i}(m)$ for $m\in M_j$, $a\in A_j$, that $\varphi_{j,\gamma}=\sum_{1\leqslant i\leqslant r_j}\varphi_{j,\gamma,i}$, and that the $\varphi_{j,\gamma,i}$ are τ_{F_j}-spherical. If we remember that $d_j=1$ on M_j, we may conclude from (7.10) and (7.13) that $\varphi_{j,\gamma,i}|M_j$ is of type $(s_i\Lambda|\mathfrak{m}_j\cap\mathfrak{l},\tau_{F_j},\gamma)$ $(1\leqslant i\leqslant r_j)$. Finally (7.9) leads to the required differential equations for $d_j^{-\gamma}\varphi_{j,\gamma}$.

Now, if P_j is not cuspidal, M_j cannot admit any nonzero eigenfunction (for the center of $\mathfrak{M}_j$) in $L^2(M_j)$. So, in this case, we must have $\varphi_{j,\gamma,i}=0$ for $1\leqslant i\leqslant r_j$, proving that $\varphi_{j,\gamma}=0$. If $\varphi_{j,\gamma}\not\equiv 0$, then $\Theta\not\equiv 0$ and so, as we saw earlier, $(s(\Lambda)(H_j)+\gamma\varrho(H_j)=0$ for some $s\in W(\mathfrak{l}_c)$. This completes the proof of the theorem.

We now turn to the problem of obtaining estimates for $\varphi-\varphi_{j,\gamma}$. With later applications in mind we shall formulate the estimates so as to take into account the variation of τ and Λ.

LEMMA 7.2. *Fix j $(1\leqslant j\leqslant d)$. Then* (i) $\{\Lambda(H_j):\Lambda\in\mathfrak{L}_{\mathrm{I}}\}=\mathcal{D}_j$ *is a discrete additive subgroup of* $\mathbf{R}$ (ii) *there are constants $C_0>0$, $q_0\geqslant 0$ with the following property: if $E(s_k^{-1}\Lambda)$ are as in Lemma* 5.1,

$$\sum_{1\leqslant k\leqslant r_j}\|E(s_k^{-1}\Lambda)\|\leqslant C_0(1+\|\Lambda\|)^{q_0} \quad (\forall\Lambda\in\mathfrak{L}_{\mathrm{I}}'). \tag{7.14}$$

Proof. If $\Lambda\in\mathfrak{L}_{\mathrm{I}}$, Λ is a linear combination with rational coefficients of the roots of $(\mathfrak{g}_c,\mathfrak{l}_c)$. Hence $\Lambda|\mathfrak{a}$ is a linear combination with rational coefficients of $\alpha_1,...,\alpha_d$, proving that $\Lambda(H_j)$ is rational. As $\mathfrak{L}_{\mathrm{I}}$ is finitely generated, we may conclude that $\mathcal{D}_j$ is a finitely generated subgroup of the rationals. Hence $\mathcal{D}_j$ is discrete. To prove (ii) observe that $(\varpi/\varpi_{F_j})E$ has polynomial entries (Lemma 5.1), and so there are constants $C_1>0$, $q_0\geqslant 0$ such that

$$|\varpi(\Lambda)/\varpi_{F_j}(\Lambda)|\,\|E(\Lambda)\|\leqslant C_1(1+\|\Lambda\|)^{q_0} \quad (\Lambda\in\mathfrak{l}_c^*).$$

On the other hand, there is a constant $c_1>0$ such that $|\langle\Lambda,\beta\rangle|\geqslant c_1>0$ for all roots β of $(\mathfrak{g}_c,\mathfrak{l}_c)$ and all regular $\Lambda\in\mathfrak{L}_{\mathrm{I}}$, and so there is a constant $c_2>0$ such that $|\varpi(\Lambda)/\varpi_{F_j}(\Lambda)|\geqslant c_2>0$ for all $\Lambda\in\mathfrak{L}_{\mathrm{I}}'$. This leads to (ii).

THEOREM 7.3. (i) *Let $\gamma>0$. Given any $\varepsilon>0$, and $a, b\in\mathfrak{G}$, there are constants $D_\varepsilon=D_{\varepsilon,a,b,\gamma}>0$, and $q_\varepsilon=q_{\varepsilon,a,b,\gamma}\geqslant 0$, such that, for arbitrary V, τ, and φ of type (Λ,τ,γ), we have*

$$\|\varphi(a; x; b)\|\leqslant D_\varepsilon|\tau,\Lambda|^{q_\varepsilon}\|\varphi\|_2\Xi(x)^{1+\gamma-\varepsilon}\quad (x\in G). \tag{7.15}$$

(ii) *Let $\gamma>0$. Then there exists $\beta_0=\beta_0(\gamma)>0$ with the following property: given any μ with $0<\mu<1$, we can select constants $L_{\mu,\gamma}>0$ and $p_{\mu,\gamma}\geqslant 0$ such that for $1\leqslant j\leqslant d$, $h\in A_j^+(\mu)$, and for arbitrary V, τ, and φ of type (Λ,τ,γ), one has the following estimate*

$$\|\varphi(h)-d_j(h)^{-(1+\gamma)}\varphi_{j,\gamma}(h)\|\leqslant L_{\mu,\gamma}|\tau,\Lambda|^{p_{\mu,\gamma}}\|\varphi\|_2\Xi(h)^{1+\gamma+\beta_0\mu}. \tag{7.16}$$

Proof. We note first that it is enough to prove (i) with $a=b=1$. Suppose in fact that this has been done. Let $q'_\varepsilon\geqslant 0$ and $D'_\varepsilon>0$ be such that for arbitrary V, τ, Λ, and f of type (Λ,τ,γ),

$$\|f(x)\|\leqslant D'_\varepsilon|\tau,\Lambda|^{q'_\varepsilon}\|f\|_2\Xi(x)^{1+\gamma-\varepsilon}\quad (x\in G).$$

Let $a, b\in\mathfrak{G}$, and $\deg(a)+\deg(b)\leqslant p$. Given f of type (Λ,τ,γ), we define F as in Lemma 5.5 and use the notation therein (with $a=a'$, $b=b'$, $p=m$). Since F is of type $(\Lambda,\tilde\tau,\gamma)$, we have, for each $\varepsilon>0$,

$$\|F(x)\|\leqslant D'_\varepsilon|\tilde\tau,\Lambda|^{q'_\varepsilon}\|F\|_2\Xi(x)^{1+\gamma-\varepsilon}\quad (x\in G).$$

Let $a=\sum_{1\leqslant i\leqslant R}c_ia_i$, $b=\sum_{1\leqslant j\leqslant S}d_jb_j$ $(c_i, d_j\in\mathbf{C})$ and let $Q=(\sum|c_id_j|^2)^{\frac12}$. Then $\|f(a; x; b)\|\leqslant Q\|F(x)\|$, and so, for $x\in G$ and $\varepsilon>0$,

$$\|f(a; x; b)\|\leqslant QD'_\varepsilon|\tilde\tau,\Lambda|^{q'_\varepsilon}\Xi(x)^{1+\gamma-\varepsilon}\Big(\sum_{i,j}\|b_jfa_i\|_2^2\Big)^{\frac12}.$$

This gives (7.15) in view of (5.11) and the fact that $|\tilde\tau,\Lambda|\leqslant c|\tau,\Lambda|$ for some constant $c>0$ independent of τ and Λ.

It is convenient to prove (i) and (ii) together. We begin by choosing a number γ_0, $0\leqslant\gamma_0\leqslant\gamma$, with the following property: given $b\in\mathfrak{G}$ and $\varepsilon>0$, there are constants $L(b:\varepsilon)>0$ and $p(b:\varepsilon)\geqslant 0$ such that for arbitrary Λ, τ, and φ of type (Λ,τ,γ), and each $\varepsilon>0$,

$$\|\varphi(x; b)\|\leqslant L(b:\varepsilon)|\tau,\Lambda|^{p(b:\varepsilon)}\|\varphi\|_2\Xi(x)^{1+\gamma_0-\varepsilon}\quad (x\in G). \tag{7.17}$$

It is clear from Lemma 5.6 that such numbers γ_0 exist; for example, 0. We now proceed as in the proof of Theorem 7.1. Let $1\leqslant j\leqslant d$, Φ, as in Lemma 5.2, and $\Psi^0=d_j^{\gamma_0}\Phi$. For $v\in\mathfrak{Z}_j$, put $\hat v=d_j^{-\gamma_0}\circ v\circ d_j^{\gamma_0}$. Define, for $m\in M_{1j}^+$, the functions F_m^0 and G_m^0 on $(0,\infty)$ by

$$F_m^0(t)=\Psi^0(m\exp tH_j),\quad G_m^0(t)=\Psi^0(m\exp tH_j; d_j^{\gamma_0}\circ D^{\tau}_{\hat H_j,1}\circ d_j^{-\gamma_0}).$$

Let $A_{j,\Lambda}=1\otimes(\Gamma(\Lambda:H_j)+\gamma_0\varrho(H_j)1)$. Then, we have, on $(0,\infty)$

$$\frac{dF_m^0}{dt}=A_{j,\Lambda}F_m^0+G_m^0.$$

Arguing as in Theorem 7.1 we conclude that $\Theta^0(m)=\lim_{t\to+\infty}\Psi^0(m\exp tH_j)$ exists for each $m\in M_{1j}$. Write $\Theta^0(m)=\Sigma_{1\leqslant k\leqslant r_j}\theta_k^0(m)\otimes e_k$, and put $\varphi_{j,\gamma_0}=\theta_1^0$.

We shall now estimate $\Psi^0-\Theta^0$ using (6.3) (with $A_{j,\Lambda}$ instead of Γ). To this end we shall find bounds for the constants C, C_ε, α defined in (6.1) and (6.2).

Let $S_{j,\Lambda}$ be the set of eigenvalues of $A_{j,\Lambda}$, and, for $c\in S_{j,\Lambda}$, let $E_{c,j,\Lambda}$ be the corresponding spectral projection. Then it follows from Lemmas 5.1 and 7.2 that $S_{j,\Lambda}\subseteq\mathcal{D}_j+\gamma_0\varrho(H_j)$ and that for any $c\in S_{j,\Lambda}$

$$E_{c,j,\Lambda}=1\otimes\sum_{k:(s_k^{-1}\Lambda+\gamma_0\varrho)(H_j)=c}E(s_k^{-1}\Lambda)\quad(\Lambda\in\mathfrak{L}_I'). \tag{7.18}$$

Since $\bigcup_{1\leqslant j\leqslant d}(\mathcal{D}_j+\gamma_0\varrho(H_j))$ is a discrete subset of $\mathbf{R}$, we can select $\alpha_0=\alpha_0(\gamma_0)$ such that (i) $0<\alpha_0\leqslant\frac{1}{2}$ (ii) if $c\neq 0$ and $c\in\bigcup_{1\leqslant j\leqslant d}(\mathcal{D}_j+\gamma_0\varrho(H_j))$, then $|c|>\alpha_0$. With this choice of α_0, we have

$$c\in S_{j,\Lambda},\,c\neq 0\Rightarrow|c|>\alpha_0\quad(\Lambda\in\mathfrak{L}_I',\,1\leqslant j\leqslant d). \tag{7.19}$$

Moreover, from (7.14) and (7.18), there are constants $C_1>0$, $q_1\geqslant 0$, such that

$$\|E_{c,j,\Lambda}\|\leqslant C_1(1+\|\Lambda\|)^{q_1}\quad(\Lambda\in\mathfrak{L}_I',\,1\leqslant j\leqslant d,\,c\in S_{j,\Lambda}). \tag{7.20}$$

Also $[S_{j,\Lambda}]\leqslant r_j$.

It remains to determine bounds for the C_ε. We use Lemma 5.9 with $H=H_j$, with F_j instead of F, and F_m^0, G_m^0 and γ_0 instead of F_m, G_m and γ. Let q, r, ω_s $(1\leqslant s\leqslant q)$ be as in that lemma; moreover, let $a_0=a_{H_j,\gamma_0}$ and $c_0(\varepsilon)=c_{H_j,\gamma_0}(\varepsilon)$ $(0<\varepsilon<1)$ be the constants satisfying (5.15). Then (5.15), (5.16), and (7.17) give us the estimates

$$\|F_m^0(t)\|\leqslant C_\varepsilon e^{\varepsilon t},\quad\|G_m^0(t)\|\leqslant C_\varepsilon e^{\varepsilon t-t} \tag{7.21}$$

for all $m\in M_{1j}^+$, $t\geqslant 0$, $0<\varepsilon<1$, where $C_\varepsilon=C_{\varepsilon,m,j,\Lambda,\tau}$ is defined as follows, with $p_\varepsilon'=r+\max_{1\leqslant s\leqslant q}p(\omega_s:\varepsilon a_0)$:

$$C_\varepsilon=c_0(\varepsilon)\,|\tau,\Lambda|^{p_\varepsilon'}\Big(\sum_{1\leqslant s\leqslant q}L(\omega_s:\varepsilon a_0)\Big)(1-\gamma_j(m))^{-p_\varepsilon'}d_j(m)^{1+\gamma_0}\Xi(m)^{1+\gamma_0-\varepsilon}\|\varphi\|_2. \tag{7.22}$$

We now observe that for any $m'\in M_{1j}$, $\|\varphi_{j,\gamma_0}(m')\|\leqslant\|\Theta^0(m')\|$ and

$$\|\varphi(m')-d_j(m')^{-(1+\gamma_0)}\varphi_{j,\gamma_0}(m')\|\leqslant d_j(m')^{-(1+\gamma_0)}\|\Psi^0(m')-\Theta^0(m')\|.$$

Define $p''(\varepsilon)=p_\varepsilon'+q_1$ where p_ε' is as above and q_1 is as in (7.20). Put

$$K(\varepsilon)=3C_1c_0(\varepsilon)r_j\Big(\sum_{1\leqslant s\leqslant q}L(\omega_s:\varepsilon\alpha_0)\Big) \tag{7.23}$$

where C_1 is as in (7.20). From Lemma 6.1 we then get the following estimate (α_0 is as in (7.19)): for arbitrary Λ, τ, φ of type (Λ,τ,γ), $m\in M_{1j}^+$, $t\geqslant 0$, and $0<\varepsilon<\frac{1}{2}\alpha_0$,

$$\begin{aligned}\|\varphi(m\exp tH_j)-d_j(m\exp tH_j)^{-(1+\gamma_0)}\varphi_{j,\gamma_0}(m\exp tH_j)\|\\ \leqslant K(\varepsilon)\,|\tau,\Lambda|^{p''(\varepsilon)}(1-\gamma_j(m))^{-p''(\varepsilon)}\,\Xi(m)^{1+\gamma_0-\varepsilon}\|\varphi\|_2\,e^{-\frac{1}{2}\alpha_0 t-(1+\gamma_0)\varrho(H_j)t}.\end{aligned} \tag{7.24}$$

Moreover, as $\varphi_{j,\gamma_0}(m)=\varphi_{j,\gamma_0}(m\exp H_j)$, we obtain from (6.3) the following estimate for $\varphi_{j,\gamma_0}(m)$: let

$$K'(\varepsilon)=K(\varepsilon)\left(1-\frac{1}{e}\right)^{-p''(\varepsilon)}d_j(\exp H_j)^{1+\gamma_0}; \tag{7.25}$$

then, for $m\in M_{1j}^+$, $0<\varepsilon<\frac{1}{2}\alpha_0$,

$$\|\varphi_{j,\gamma_0}(m)\|\leqslant K'(\varepsilon)\,|\tau,\Lambda|^{p''(\varepsilon)}\,\Xi(m\exp H_j)^{1+\gamma_0-\varepsilon}d_j(m)^{1+\gamma_0}\|\varphi\|_2. \tag{7.26}$$

We now convert (7.24) and (7.26) into uniform estimates for $\|\varphi(h)-d_j(h)^{-(1+\gamma_0)}\varphi_{i,\gamma_0}(h)\|$ as h varies over $A_j^+(\mu)$. Let ${}_j\mathfrak{a}$ be the null space of α_j, so that $\mathfrak{a}={}_j\mathfrak{a}+\mathfrak{a}_j$ is a direct sum. If $H\in\mathfrak{a}$, $H={}_jH+\alpha_j(H)H_j$ where ${}_jH\in{}_j\mathfrak{a}$; if $H\in\mathfrak{a}^+$, then ${}_jH\in Cl(\mathfrak{a}^+)$. Suppose now $h=\exp H\in A_j^+(\mu)$ (cf. (7.2)), where $0<\mu<1$ and $\alpha_j(\log h)>2$. Then $h=m\exp tH$, where $t=\frac{1}{2}\alpha_j(H)>1$ and $m=\exp({}_jH+\frac{1}{2}\alpha_j(H)H_j)$. Clearly $m\in M_{1j}^+$ and $\gamma_j(m)\leqslant 1/e$. We now substitute these choices for m and t in (7.24). We also select, for any ε with $0<\varepsilon<\frac{1}{2}$, a constant $d(\varepsilon)>0$ such that $\Xi(h')^{1+\gamma_0-\varepsilon}\leqslant d(\varepsilon)e^{-(1+\gamma_0-2\varepsilon)\varrho(\log h')}$ for all $h'\in Cl(A^+)$. Defining

$$K_1(\varepsilon)=K(\varepsilon)\left(1-\frac{1}{e}\right)^{-p''(\varepsilon)}d(\varepsilon), \tag{7.27}$$

we obtain from (7.24) the following estimate: for arbitrary Λ,τ,φ of type (Λ,τ,γ), $h\in A_j^+(\mu)$ with $\alpha_j(\log h)>2$, and $0<\varepsilon<\frac{1}{2}\alpha_0$,

$$\|\varphi(h)-d_j(h)^{-(1+\gamma_0)}\varphi_{j,\gamma_0}(h)\|\leqslant K_1(\varepsilon)\,|\tau,\Lambda|^{p''(\varepsilon)}\,\|\varphi\|_2\,e^{-(1+\gamma_0-2\varepsilon+(\alpha_0\mu/4))\varrho(\log h)};$$

in deriving this we must remember that $t=\frac{1}{2}\alpha_j(\log h)>(\mu/2)\,\varrho(\log h)$. So, remembering (2.1) we find, for arbitrary Λ,τ,φ of type (Λ,τ,γ), and ε with $0<\varepsilon\leqslant(\alpha_0\mu/16)$,

$$\|\varphi(h)-d_j(h)^{-(1+\gamma_0)}\varphi_{j,\gamma_0}(h)\|\leqslant K_1(\varepsilon)\,|\tau,\Lambda|^{p''(\varepsilon)}\,\|\varphi\|_2\,\Xi(h)^{1+\gamma_0+(\alpha_0\mu/8)}, \tag{7.28}$$

for all $h\in A_j^+(\mu)$ with $\alpha_j(\log h)>2$. On the other hand, let $Q_\mu=\{h: h\in A_j^+(\mu),\ \alpha_j(\log h)\leqslant 2\}$. Then $Cl(Q_\mu)$ is compact, and so we can find, for each ε with $0<\varepsilon\leqslant(\alpha_0\mu/16)$, a constant $K(\varepsilon:\mu)>0$ such that for all $h\in Q_\mu$,

$$L(1:\varepsilon)\,\Xi(h)^{1+\gamma_0-\varepsilon}+K'(\varepsilon)\,\Xi(h\exp H_j)^{1+\gamma_0-\varepsilon}\leqslant K(\varepsilon:\mu)\,\Xi(h)^{1+\gamma_0+(\alpha_0\mu/8)}.$$

Taking into account (7.17) $a=b=1$ we have, from (7.26) and the above inequality, for all $h\in Q_\mu$ and $0<\varepsilon\leqslant(\alpha_0\mu/16)$,

$$\|\varphi(h)-d_j(h)^{-(1+\gamma_0)}\varphi_{j,\gamma_0}(h)\|\leqslant K(\varepsilon:\mu)\,|\tau,\Lambda|^{p_\varepsilon}\,\|\varphi\|_2\,\Xi(h)^{1+\gamma_0+(\alpha_0\mu/8)} \tag{7.29}$$

where $p_\varepsilon=p(1:\varepsilon)+p''_\varepsilon$. Let $\varepsilon_\mu=(\alpha_0\mu/16)$ and write

$$\beta_0=\tfrac{1}{8}\alpha_0,\ p_\mu=p_{\varepsilon_\mu},\ L_\mu=K(\varepsilon_\mu:\mu)+K_1(\varepsilon_\mu). \tag{7.30}$$

Then, on combining (7.28) and (7.29), we obtain the following result. Given μ, with $0<\mu<1$, we have, for arbitrary Λ, τ, φ of type (Λ,τ,γ), and $h\in A_j^+(\mu)$ $(1\leqslant j\leqslant d)$,

$$\|\varphi(h)-d_j(h)^{-(1+\gamma_0)}\varphi_{j,\gamma_0}(h)\|\leqslant L_\mu\,|\tau,\Lambda|^{p_\mu}\,\|\varphi\|_2\,\Xi(h)^{1+\gamma_0+\beta_0\mu}. \tag{7.31}$$

We must remember that (7.31) has been proved under the sole assumption that, for each $b\in\mathfrak{G}$ and $\varepsilon>0$, (7.17) is satisfied by all φ of type (Λ,τ,γ). Note also that L_μ and p_μ depend on γ_0 and γ.

We are now in a position to prove (i) with $a=b=1$. Let Z be the set of all numbers γ' with $0\leqslant\gamma'\leqslant\gamma$ such that (i) is true for all φ of type (Λ,τ,γ) with γ' replacing γ in the estimate (7.15). From Lemma 5.6 it follows that $0\in Z$, so that Z is nonempty. Let $\gamma_0=\sup_{\gamma'\in Z}\gamma'$. Then, for any $\varepsilon>0$, there is a $\gamma_\varepsilon\in Z$ such that $\gamma_0-\varepsilon/2<\gamma_\varepsilon\leqslant\gamma_0$. A simple argument then proves that given $b\in\mathfrak{G}$ and $\varepsilon>0$, we can select constants $L(b:\varepsilon)>0$, $p(b:\varepsilon)\geqslant 0$ such that (7.17), and hence (7.31), is true for all φ of type (Λ,τ,γ), Λ, τ being arbitrary. If $\gamma_0\geqslant\gamma$, we already obtain (i) (with $a=1$ to be sure, but this is enough, in view of our earlier remarks). We shall now prove that $\gamma_0<\gamma$ leads to a contradiction. Suppose $0\leqslant\gamma_0<\gamma$. If φ is of type (Λ,τ,γ), then we know from Theorem 7.1 that for any $m\in M_{1j}$, $\varphi_{j,\gamma}(m)=\lim_{t\to+\infty}d_j(m\exp tH_j)^{1+\gamma}\varphi(m\exp tH_j)$ exists. On the other hand, as $\gamma-\gamma_0>0$, $d_j(m\exp tH_j)^{-(\gamma-\gamma_0)}\to 0$ as $t\to+\infty$, for each $m\in M_{1j}$. Therefore we have $\varphi_{j,\gamma_0}=0$, $1\leqslant j\leqslant d$. So, from (7.31) we have, for arbitrary Λ, τ, φ of type (Λ,τ,γ), $h\in A_j^+(\mu)$ $(0<\mu<1,\ 1\leqslant j\leqslant d)$

$$\|\varphi(h)\|\leqslant L_\mu\,|\tau,\Lambda|^{p_\mu}\,\|\varphi\|_2\,\Xi(h)^{1+\gamma_0+\beta_0\mu}. \tag{7.32}$$

Choose μ_0 with $0<\mu_0<1$ such that $A^+\subseteq\bigcup_{1\leqslant j\leqslant d}A_j^+(\mu_0)$ (cf. (7.3)) and write $L_0=L_{\mu_0}$, $p_0=p_{\mu_0}$, $\delta_0=\beta_0\mu_0$. Then (7.32) gives us the following result: for arbitrary Λ, τ, and φ of type (Λ,τ,γ).

$$\|\varphi(x)\|\leqslant L_0\,|\tau,\Lambda|^{p_0}\,\|\varphi\|_2\,\Xi(x)^{1+\gamma_0+\delta_0}\quad(x\in G). \tag{7.33}$$

It is clear from (7.33) that $\gamma_0+\delta_0\in Z$, contradicting the definition of γ_0. The proof of (i) is thus complete.

By virtue of (i), estimates of the form (7.17) are now true with γ replacing γ_0. But then the estimates (7.31) are also true, with γ replacing γ_0. This gives (ii).

Theorem 7.3 is completely proved.

COROLLARY 7.4. *Fix $\gamma>0$ and a φ of type (Λ, τ, γ). Then, given $a, b \in \mathfrak{G}$, there are constants $C>0$, $q\geqslant 0$ such that*

$$\|\varphi(a; x; b)\| \leqslant C\Xi(x)^{1+\gamma}(1+\sigma(x))^q \quad (x\in G). \tag{7.34}$$

Proof. As usual we come down to the case $a=b=1$. We use induction on dim (G). Choose μ_0, $0<\mu_0<1$, such that $A^+\subseteq \bigcup_{1\leqslant j\leqslant d} A_j^+(\mu_0)$, and let $K_0=L_{\mu_0}|\tau, \Lambda|^{p_{\mu_0}}\|\varphi\|_2$, $\delta_0=\beta_0\mu_0$ where L_μ and p_μ are as in (7.31). Then (7.31) implies that for all $h\in A^+$

$$\|\varphi(h)\| \leqslant K_0\Xi(h)^{1+\gamma+\delta_0} + \sum_{1\leqslant j\leqslant d} d_j(h)^{-(1+\gamma)}\|\varphi_{j,\gamma}(h)\|. \tag{7.35}$$

Now $\varphi_{j,\gamma}=0$ if P_j is not cuspidal. Consider j such that P_j is cuspidal, and write $\varphi_{j,\gamma}=\sum_{1\leqslant i\leqslant r_j}\varphi_{j,\gamma,i}$ as in Theorem 7.1. Since $\varphi_{j,\gamma,i}|M_j$ is of type $(s_i\Lambda|\mathfrak{m}_j\cap\mathfrak{l}, \tau_{F_j}, \gamma)$ and dim $(M_j)<$ dim (G), the induction hypothesis is applicable [1] and so we can find constants $C>0$, $q\geqslant 0$ such that

$$\|\varphi_{j,\gamma}(m)\| \leqslant C\Xi_j(m)^{1+\gamma}(1+\sigma(m))^q \quad (m\in M_j,\ 1\leqslant j\leqslant d). \tag{7.36}$$

If $h\in A^+$ and we write $h=h_1h_2$ where $h_1\in M_j\cap A$, $h_2\in A_j$, then $\lambda(\log h_1)\geqslant 0$ for all $\lambda\in\Delta_{F_j}^+$, while there is a constant $c_j>0$ independent of h such that $1+\sigma(h_1)\leqslant c_j(1+\sigma(h))$. Therefore, as $\varphi_{j,\gamma}(h)=\varphi_{j,\gamma}(h_1)$, we find from (7.36) and (2.1) the following result: there are constants $C_1>0$, $q_1\geqslant 0$ such that for all $h\in A^+$, $1\leqslant j\leqslant d$,

$$\|\varphi_{j,\gamma}(h)\| \leqslant C_1 e^{-\varrho_{F_j}(\log h)(1+\gamma)}(1+\sigma(h))^{q_1}. \tag{7.37}$$

From (7.37), (7.35) and (2.1) we obtain, for all $h\in A^+$

$$\|\varphi(h)\| \leqslant K_0\Xi(h)^{1+\gamma+\delta_0} + dC_1\Xi(h)^{1+\gamma}(1+\sigma(h))^{q_1}. \tag{7.38}$$

This leads to the corollary easily.

THEOREM 7.5. (i) *Let $1\leqslant p<2$ and $\bar\gamma=(2/p)-1$. If $\varphi\in\mathcal{E}(\Lambda: G: \tau)\cap L^2(G: V)$, then $\varphi\in L^p(G: V)$ if and only if φ is of type (Λ, τ, γ) for some $\gamma>\bar\gamma$.*

(ii) *Let $1\leqslant p<2$. Then there is $\varepsilon_0=\varepsilon_0(p)>0$, and, for each a, $b\in\mathfrak{G}$, constants $C_{a,b}>0$, $q_{a,b}\geqslant 0$, such that for arbitrary V, τ, Λ, and $\varphi\in\mathcal{E}(\Lambda: G: \tau)\cap L^p(G: V)$,*

[1] Cf. the remarks made made in § 2 concerning M_F. If $d=1$, M_j is compact, $\Xi_j(m)\equiv 1$, and (7.36) is trivial. So the case $d=1$, which starts the induction, is simple to handle, and in fact, its proof is contained in the given proof.

$$\|\varphi(a; x; b)\| \leqslant C_{a,b} |\tau, \Lambda|^{q_{a,b}} \|\varphi\|_2 \Xi(x)^{(2/p)+\varepsilon_0} \quad (x \in G). \tag{7.39}$$

Proof. (i) If φ is of type (Λ, τ, γ) with $\gamma > (2/p) - 1$, then $\|\varphi(x)\|^p \leqslant$ const. $\Xi(x)^\beta$ for all $x \in G$, β being a constant > 2. So $\varphi \in L^p(G: V)$.

Conversely, let $\varphi \in \mathcal{E}(\Lambda: G: \tau) \cap L^p(G: V)$. Arguing as in Corollary 3.4 we see that $a\varphi b \in L^p(G: V)$ for all $a, b \in \mathfrak{G}$. Hence by Theorem 3.3, $\sup_{x \in G} \Xi(x)^{-2/p} \|\varphi(a: x; b)\| < \infty$ for all $a, b \in \mathfrak{G}$. So φ is of type $(\Lambda, \tau, \bar{\gamma})$.

We shall now prove that $\varphi_{j,\bar{\gamma}} = 0$, $1 \leqslant j \leqslant d$. Fix j and write $\psi = \varphi_{j,\bar{\gamma}}$. Choose μ such that $0 < \mu < 1$ and $A_j^+(\mu)$ is nonempty. We then obtain from (7.16) (with $\bar{\gamma}$ replacing γ) the following result: there are constants $C > 0$, $\delta > 0$ such that, for all $h \in A_j^+(\mu)$,

$$d_j(h)^{-(2/p)} \|\psi(h)\| \leqslant \|\varphi(h)\| + Ce^{-((2/p)+\delta)\varrho(\log h)} \tag{7.40}$$

Let J be as in (3.1). Then $J(h) \leqslant e^{2\varrho(\log h)}$ for all $h \in A^+$, and so, each of the functions appearing in the right of (7.40) belongs to $L^p(A^+, Jdh)$. So, if we write $\mathfrak{a}_\mu = \{H: H \in \mathfrak{a}^+, \alpha_j(H) > \max(1, \mu\varrho(H))\}$, then $\mathfrak{a}_\mu$ is nonempty, and

$$\int_{\mathfrak{a}_\mu} \|\psi(\exp H)\|^p d_j(\exp H)^{-2} J(\exp H)\, dH < \infty, \tag{7.41}$$

dH being a Lebesgue measure on $\mathfrak{a}$. On the other hand, if we put

$$^*J(h) = \prod_{\lambda \in \Delta_{F_j}^+} (e^{\lambda(\log h)} - e^{-\lambda(\log h)})^{\dim(\mathfrak{g}_\lambda)} \quad (h \in A^+), \tag{7.42}$$

it is easily seen that there is a constant $c_0 > 0$ for which $J(\exp H) \geqslant c_0 d_j(\exp H)^2 \, {}^*J(\exp H)$ for all $H \in \mathfrak{a}_\mu$. (7.41) then gives us

$$\int_{\mathfrak{a}_\mu} \|\psi(\exp H)\|^p \, {}^*J(\exp H)\, dH < \infty. \tag{7.43}$$

Let ${}_j\mathfrak{a}$ be the null space of α_j. Select $H_0 \in \mathfrak{a}_\mu$, and write $H_0 = H_0' + s_0 H_j$, where $H_0' \in {}_j\mathfrak{a}$. If we put

$$U = \{H': H' \in {}_j\mathfrak{a},\ \alpha_i(H') > 0 \text{ for } i \neq j,\ \tfrac{1}{2}\varrho(H_0') \leqslant \varrho(H') \leqslant 2\varrho(H_0')\}$$

then an easy verification shows that U is a neighborhood of H_0' in ${}_j\mathfrak{a}$ and that $H' + sH_j \in \mathfrak{a}_\mu$ whenever $H' \in U$ and $s \geqslant 2s_0$. Writing dH' for the Lebesgue measure on ${}_j\mathfrak{a}$, we then get from (7.43)

$$\int_U \int_{2s_0}^\infty \|\psi(\exp H' \exp tH_j)\|^p \, {}^*J(\exp H' \exp tH_j)\, dH'\, dt < \infty. \tag{7.44}$$

But the integrand in (7.44) is *independent of* t. So $\psi(\exp H' \exp tH_j) = 0$, for $H' \in U$, $t \geqslant 2s_0$. As ψ is analytic, $\psi | A = 0$; and the fact that ψ is τ_{F_j}-spherical then implies that $\psi = 0$.

It now follows from (7.16) (with $\gamma=\bar{\gamma}$ and $\varphi_{j,\bar{\gamma}}=0$) that for suitable constants $C>0$, $\delta>0$, $\|\varphi(x)\|\leqslant C\Xi(x)^{2/p+\delta}$ for all $x\in G$. (i) follows from this.

To prove (ii), select μ_0 such that $0<\mu_0<1$ and $A^+\subseteq\bigcup_{1\leqslant j\leqslant d}A_j^+(\mu_0)$, and take $\gamma=\bar{\gamma}$, $\mu=\mu_0$ and $\varphi\in L^p(G\colon V)\cap\mathcal{E}(\Lambda\colon G\colon\tau)$ in (7.16). If $K_0=L_{\mu_0,\bar{\gamma}}$, $p_0=p_{\mu_0,\bar{\gamma}}$, $\varepsilon_0=\beta_0\mu_0$, we obtain the following result: for arbitrary $\Lambda,\tau,\varphi\in\mathcal{E}(\Lambda\colon G\colon\tau)\cap L^p(G\colon V)$

$$\|\varphi(x)\|\leqslant K_0|\tau,\Lambda|^{p_0}\|\varphi\|_2\,\Xi(x)^{(2/p)+\varepsilon_0}\quad(x\in G). \tag{7.45}$$

This proves (ii) with $a=b=1$. The case of arbitrary $a,b\in\mathfrak{G}$ is then deduced from this in the usual manner. This proves the theorem.

8. Estimates for the matrix coefficients of the discrete series

Let P,P_n and $k(\beta)$ $(\beta\in P\cup -P)$ be as in § 1. It is obvious that $k(\beta)=k(-\beta)=k(s\beta)>0$ $(s\in W(\mathfrak{b}_c))$, and that $k(\beta)$ does not depend on P. Moreover, for fixed β, if $P^{\beta,+}$ (resp. $P^{\beta,-}$) is the set of all $\alpha\in P$ with $\langle\alpha,\beta\rangle\geqslant 0$ (resp. $\langle\alpha,\beta\rangle<0$), $P_\beta=P^{\beta,+}\cup(-P^{\beta,-})$, and $\delta_\beta=\frac{1}{2}\Sigma_{\alpha\in P_\beta}\alpha$, then it is easily seen that P_β is a positive system and $k(\beta)=\delta_\beta(\bar{H}_\beta)$. This shows that $k(\beta)$ is an integer for all β. For any Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, we define the function $D_{\mathfrak{h}}$ and the set $G_{\mathfrak{h}}$ as in [13] (p. 110). The function D is a in § 1. If $\mathfrak{h}_j$ $(j=1,2)$ are Cartan subalgebras of $\mathfrak{g}_c$, and O_j is a $W(\mathfrak{h}_j)$-orbit in $\mathfrak{h}_j^*$, we say that O_1 and O_2 *correspond* if there is a $y\in G_c$ such that $y\cdot\mathfrak{h}_1=\mathfrak{h}_2$ and $O_2\circ y=O_1$.

Let $\lambda\in\mathcal{L}_{\mathfrak{b}}'$ and $\gamma>0$. Suppose π is a representation in $\omega(\lambda)$, and that, for some $q\geqslant 0$ and a pair ψ_0, ψ_0' of nonzero K-finite vectors in the space of π,

$$\sup_{x\in G}\Xi(x)^{-(1+\gamma)}(1+\sigma(x))^{-q}|(\pi(x)\psi_0,\psi_0')|<\infty; \tag{8.1}$$

then a simple argument, based on Theorem 1 of [14] and the irreducibility of π, shows that (8.1) is true when ψ_0 and ψ_0' are replaced by any other pair ψ, ψ' of K-finite vectors, with the same choice of γ and q. Thus, in this case, $\omega(\lambda)$ is of type γ in the sense of the definition in § 1. The purpose of this section is to obtain proofs of the following theorems.

THEOREM 8.1. *Let $\lambda\in\mathcal{L}_{\mathfrak{b}}'$, $\omega=\omega(\lambda)$, and let Θ_ω be the character of $\omega(\lambda)$. Fix $\gamma>0$. Then, in order that ω be of type γ, it is necessary that for each Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$,*

$$\sup_{x\in G_{\mathfrak{h}}}|D_{\mathfrak{h}}(x)|^{\gamma/2}|D(x)|^{\frac{1}{2}}|\Theta_\omega(x)|<\infty; \tag{8.2}$$

in particular, it is necessary that

$$|(\lambda)(\bar{H}_\beta)|\geqslant\gamma k(\beta)\quad(\forall\beta\in P_n). \tag{8.3}$$

Moreover, in order that $\omega(s\lambda)$ be of type γ for all $s \in W(\mathfrak{b}_c)$, it is necessary and sufficient that

$$|(s\lambda)(\bar{H}_\beta)| \geqslant \gamma k(\beta) \quad (\forall \beta \in P_n, \forall s \in W(\mathfrak{b}_c)). \tag{8.4}$$

THEOREM 8.2. *Fix p, $1 \leqslant p < 2$. If $\omega \in \mathcal{E}_2(G)$, then $\omega \in \mathcal{E}_p(G)$ if and only if it is of type γ for some $\gamma > (2/p) - 1$. Let $\lambda \in \mathcal{L}'_{\mathfrak{b}}$, $\omega = \omega(\lambda)$. Then, in order that $\omega \in \mathcal{E}_p(G)$ it is necessary that for some $\gamma > (2/p) - 1$, (8.2) should be satisfied for all Cartan subalgebras $\mathfrak{h}$ of $\mathfrak{g}$; in particular, it is necessary that*

$$|\lambda(\bar{H}_\beta)| > \left(\frac{2}{p} - 1\right) k(\beta) \quad (\forall \beta \in P_n). \tag{8.5}$$

In order that $\omega(s\lambda) \in \mathcal{E}_p(G)$ for all $s \in W(\mathfrak{b}_c)$, it is necessary and sufficient that

$$|(s\lambda)(\bar{H}_\beta)| > \left(\frac{2}{p} - 1\right) k(\beta) \quad (\forall \beta \in P_n, \forall s \in W(\mathfrak{b}_c)). \tag{8.6}$$

We begin with the proof that (8.4) is sufficient for $\omega(s\lambda)$ to be of type γ for all $s \in W(\mathfrak{b}_c)$. We need a lemma.

LEMMA 8.3. *Let Q be the set of all j with $1 \leqslant j \leqslant d$ such that the parabolic subgroup P_j is cuspidal. Given $\beta \in P_n$ and $j \in Q$, let us write $\beta \sim j$, if there is some $y \in G_c$ and some $t \neq 0$ in $\mathbf{R}$, such that, $\mathfrak{b}_c^y = \mathfrak{l}_c$, $\bar{H}_\beta^y = tH_j$, and $k(\beta) = |t| \varrho(H_j)$. Then, for any $\beta \in P_n$, there is $j \in Q$ such that $\beta \sim j$; and, for any $j \in Q$, there is $\beta \in P_n$ such that $\beta \sim j$. In particular, if $\lambda \in \mathcal{L}'_{\mathfrak{b}}$, $O_{\mathfrak{b}} = W(\mathfrak{b}_c) \cdot \lambda$, and $O_{\mathfrak{l}}$ is the $W(\mathfrak{l}_c)$-orbit in $\mathfrak{l}_c^*$ that corresponds to $O_{\mathfrak{b}}$, then*

$$\{|\mu(\bar{H}_\beta)|/k(\beta) : \mu \in O_{\mathfrak{b}}, \beta \in P_n\} = \{|\Lambda(H_j)|/\varrho(H_j) : \Lambda \in O_{\mathfrak{l}}, j \in Q\}.$$

Proof. Let $\beta \in P_n$. Let $\mathfrak{b}(\beta)$ be the null space of β. Select $H_0 \in \mathfrak{b}(\beta)$ such that β is the only root in P that vanishes at H_0. Let $\mathfrak{z}$ be the centralizer of H_0 in $\mathfrak{g}$, and $\mathfrak{z}_1$, the derived algebra of $\mathfrak{z}$. Then $\dim(\mathfrak{z}_1) = 3$, $\theta(\mathfrak{z}_1) = \mathfrak{z}_1$, and the noncompactness of β implies that $\mathfrak{z}_1$ is isomorphic to $\mathfrak{sl}(2, \mathbf{R})$. It follows (cf. also [13], § 24) from this that we can find H$'$, X', $Y' \in \mathfrak{z}_1$ such that (i) $[H', X'] = 2X'$, $[H', Y'] = -2Y'$, $[X', Y'] = H'$ (ii) $H' \in \mathfrak{s}$, $Y' = -\theta X'$, $X' - Y' = i\bar{H}_\beta$. Since $\mathfrak{b}(\beta)$ is the center of $\mathfrak{z}$, $\mathfrak{b}$ and $\mathfrak{b}(\beta) + \mathbf{R} \cdot H' = \mathfrak{h}$ are two θ-stable Cartan subalgebras of $\mathfrak{z}$ (and $\mathfrak{g}$), and so, we can find $y_0 \in G_c$ such that, y_0 centralizes $\mathfrak{b}(\beta)$, $y_0 \cdot \mathfrak{b}_c = \mathfrak{h}_c$, and $\bar{H}_\beta^{y_0} = H'$. Let Δ' be the set of roots of $(\mathfrak{g}_c, \mathfrak{h}_c)$, and $P' = P \circ y_0^{-1}$. Then P' is a positive system for Δ' and $k(\beta) = \frac{1}{2}\Sigma_{\alpha' \in P'} |\alpha'(H')|$, so that, we must have $k(\beta) = \frac{1}{2}\Sigma_{\alpha' \in \Delta', \alpha'(H') > 0} \alpha'(H')$. On the other hand, let $\mathfrak{m}'$ be the centralizer of H' in $\mathfrak{g}$, and let $\mathfrak{n}'$ be the space spanned by the eigensubspaces of ad H' that correspond to its positive eigenvalues. It is easy to see that $\mathfrak{p}' = \mathfrak{m}' + \mathfrak{n}'$ is a parabolic subalgebra of $\mathfrak{g}$; our previous expression for $k(\beta)$ now gives $k(\beta) = \frac{1}{2} \operatorname{tr} (\operatorname{ad} H')_{\mathfrak{n}'}$. Also $\mathbf{R} \cdot H' = \mathfrak{h} \cap \mathfrak{s}$ is the split component of $\mathfrak{p}'$. Choose $F \subsetneqq \Sigma$ and $k \in K$

such that $(\mathfrak{p}')^k = \mathfrak{p}_F$. Clearly $\mathfrak{a}_F = (\mathfrak{h} \cap \mathfrak{s})^k = \mathfrak{h}^k \cap \mathfrak{s}$. $\mathfrak{p}_F$ is thus cuspidal and $\dim(\mathfrak{a}_F) = 1$, so that $F = F_j$ for some $j \in Q$. It follows from the construction of $\mathfrak{p}'$ that $H'^k = tH_j$ for some $t > 0$, and so $k(\beta) = t\varrho(H_j)$. Write $\mathfrak{h}_j = \mathfrak{h}^k$. Let M_{jc} be the complex analytic subgroup of G_c defined by $\mathbf{C}\cdot\mathfrak{m}_j$. Then there is $z \in M_{jc}$ such that $\mathfrak{h}_{jc}^z = \mathfrak{l}_c$. Define $y = zky_0$. Then $\mathfrak{b}_c^y = \mathfrak{l}_c$, $\bar{H}_\beta^y = tH_j^z = tH_j$, and $k(\beta) = t\varrho(H_j)$. This proves that $\beta \sim j$.

Conversely, let $j \in Q$. Let $\mathfrak{h}_j$ be a θ-stable Cartan subalgebra of $\mathfrak{m}_j$ such that $\mathfrak{h}_j \cap \mathfrak{s} = \mathbf{R}\cdot H_j$. If M_{jc} is as in the previous paragraph, we can find $z \in M_{jc}$ such that $\mathfrak{h}_{jc}^z = \mathfrak{l}_c$. As $\mathfrak{h}_j$ is not conjugate to $\mathfrak{b}$ in G, we can find a root α' of $(\mathfrak{g}_c, \mathfrak{h}_c)$ that is real valued on $\mathfrak{h}_j$ ([6], Lemma 33). It is obvious that $H_{\alpha'} \in \mathbf{R}\cdot H_j$, and so, replacing α' by $-\alpha'$ if necessary, we may assume that $\bar{H}_{\alpha'} = tH_j$ for some $t > 0$. It follows from the definition of $\mathfrak{n}_j$ that $t\varrho(H_j) = \frac{1}{2}\sum_{\gamma' \in \Delta', \langle\gamma', \alpha'\rangle > 0} \gamma'(\bar{H}_{\alpha'})$, where Δ' is the set of roots of $(\mathfrak{g}_c, \mathfrak{h}_{jc})$. If P' is a positive system in Δ', we have then $t\varrho(H_j) = \frac{1}{2}\sum_{\gamma' \in P'} |\gamma'(\bar{H}_{\alpha'})|$. On the other hand, a simple argument, based on the facts that $\mathfrak{h}_j$ is θ-stable and α' is real valued on $\mathfrak{h}_j$, enables us to select nonzero $X_{\pm\alpha'} \in \mathfrak{g}$, such that, $X_{\pm\alpha'}$ are root vectors corresponding to $\pm\alpha'$, $X_{-\alpha'} = -\theta X_{\alpha'}$, and $[X_{\alpha'}, X_{-\alpha'}] = \bar{H}_{\alpha'}$. Write $\mathfrak{b}_1 = (\mathfrak{h}_j \cap \mathfrak{k}) + \mathbf{R}\cdot(X_{\alpha'} - X_{-\alpha'})$. Then $\mathfrak{b}_1 \subseteq \mathfrak{k}$, and $\mathfrak{b}_1$ and $\mathfrak{h}_j$ are Cartan subalgebras of the centralizer of $\mathfrak{h}_j \cap \mathfrak{k}$ in $\mathfrak{g}$. Select $y_1 \in G_c$ centralizing $\mathfrak{h}_j \cap \mathfrak{k}$ such that $\mathfrak{b}_{1c}^{y_1} = \mathfrak{h}_{jc}$. Then $\alpha_1 = \alpha' \circ y_1$ is a non compact root of $(\mathfrak{g}_c, \mathfrak{b}_{1c})$, $P'' = P' \circ y_1$ is a positive system of roots of $(\mathfrak{g}_c, \mathfrak{b}_{1c})$, and, $t\varrho(H_j) = \frac{1}{2}\sum_{\gamma \in P''} |\gamma(\bar{H}_{\alpha_1})|$. Select $k \in K$ such that $\mathfrak{b}^k = \mathfrak{b}_1$ and write $\beta_1 = \alpha_1 \circ k$. Then β_1 is noncompact and so $\beta = \varepsilon\beta_1 \in P_n$ where $\varepsilon = \pm 1$. If $y = zy_1k$, then $\mathfrak{b}_c^y = \mathfrak{l}_c$, $\bar{H}_\beta^y = \varepsilon H_{\alpha'}^z = \varepsilon tH_j$, $k(\beta) = t\varrho(H_j)$. So $\beta \sim j$. The second statement of the lemma is an immediate consequence of the first.

At this stage we can complete the proof that (8.4) is sufficient for $\omega(s\lambda)$ to be of type γ for all $s \in W(\mathfrak{b}_c)$. Fix s, λ; let $O_\mathfrak{b} = W(\mathfrak{b}_c)\cdot\lambda$, and $O_\mathfrak{l}$, the corresponding $W(\mathfrak{l}_c)$-orbit in $\mathfrak{l}_c^*$; and let $\Lambda \in O_\mathfrak{l}$. Let π be a representation in $\omega(s\lambda)$ acting in a Hilbert space $\mathcal{H}$. Let $\mathfrak{d}$ be an equivalence class of irreducible representations of K that occurs in $\pi|K$. We write $\mathcal{H}_\mathfrak{d}$ for the corresponding subspace of $\mathcal{H}$ and denote by $P_\mathfrak{d}$ the orthogonal projection $\mathcal{H} \to \mathcal{H}_\mathfrak{d}$. Denote by $V_\mathfrak{d}$ the algebra of endomorphisms of $\mathcal{H}_\mathfrak{d}$, and, for $k \in K$, $v \in V_\mathfrak{d}$, let $\tau_{\mathfrak{d},1}(k)v = \pi_\mathfrak{d}(k)v$, $v\tau_{\mathfrak{d},2}(k) = v\pi_\mathfrak{d}(k)$, where $\pi_\mathfrak{d}(k) = \pi(k)|\mathcal{H}_\mathfrak{d}$. Then $v \to |||v|||^2 = \operatorname{tr}(vv^\dagger)$ ($\dagger$ denotes adjoints) converts $V_\mathfrak{d}$ into a Hilbert space, and $\tau_\mathfrak{d} = (\tau_{\mathfrak{d},1}, \tau_{\mathfrak{d},2})$ is a unitary double representation of K in $V_\mathfrak{d}$. If we define $\varphi_\mathfrak{d}(x) = \varphi(x) = P_\mathfrak{d}\pi(x)P_\mathfrak{d}$ (considered as an element of $V_\mathfrak{d}$) for $x \in G$, it is clear that $\varphi \in \mathcal{E}(\Lambda: G: \tau)$ in the notation of § 7. In view of Corollary 7.4, it is sufficient to prove that φ is of type (Λ, τ, γ). Let γ_0 be the supremum of all numbers $\gamma' \geq 0$ such that φ is of type (Λ, τ, γ'). It is obvious from the definition in § 7 (cf. (7.1)) that φ is of type $(\Lambda, \tau, \gamma_0)$ also. We assert that for some j_0 with $1 \leq j_0 \leq d$, $\varphi_{j_0,\gamma_0} \neq 0$. Otherwise, if $\varphi_{j,\gamma_0} = 0$ for $1 \leq j \leq d$, the estimates (7.16) (with $\gamma = \gamma_0$) would imply the existence of constants $C > 0$, $\delta > 0$ such that $|||\varphi(x)||| \leq C\Xi(x)^{1+\gamma_0+\delta}$ for all $x \in G$; this would show that φ is of type $(\Lambda, \tau, \gamma_0+\delta)$, contradicting the definition of γ_0. From Theorem 7.1 we now conclude that P_{j_0} is cuspidal, i.e., $j_0 \in Q$, and that there exists $\Lambda' \in O_{\mathrm{I}}$ such that $\Lambda'(H_{j_0}) = -\gamma_0 \varrho(H_{j_0})$. But then the last statement of Lemma 8.3 implies at once the existence of $\beta \in P_n$ and $\mu \in O_{\mathfrak{b}}$ such that $|\mu(\bar{H}_\beta)| = \gamma_0 k(\beta)$. So, by (8.4), $\gamma_0 \geqslant \gamma$. Since φ is of type $(\Lambda, \tau, \gamma_0)$, it must be of type (Λ, τ, γ) also. This proves what we wanted.

We shall now fix $\lambda \in \mathcal{L}'_{\mathfrak{b}}$, assume that $\omega = \omega(\lambda)$ is of type $\gamma > 0$, and prove that (8.2) and (8.3) are satisfied. Put $\Theta = \Theta_\omega$. Ω is as in (5.8).

LEMMA 8.4. *Assume, as above, that ω is of type γ. Then, given any ε with $0<\varepsilon<\gamma$, we can find a constant $C = C_\varepsilon > 0$ and an integer $p = p_\varepsilon \geqslant 0$ such that, for all $f \in C_c^\infty(G)$,*

$$|\Theta(f)| \leqslant C \sup_G \Xi^{-1+\gamma-\varepsilon} |\Omega^p f|. \tag{8.7}$$

Proof. Let π be a representation in ω acting in the Hilbert space $\mathcal{H}$, and let $\mathcal{E}(K)$ (resp. $\mathcal{E}_\pi$) denote the set of all equivalence classes of irreducible unitary representations of K (resp. occurring in the reduction of $\pi|K$). Given $\mathfrak{d} \in \mathcal{E}_\pi$, let $\mathcal{H}_{\mathfrak{d}}$, $V_{\mathfrak{d}}$, $P_{\mathfrak{d}}$, $\tau_{\mathfrak{d}}$ and $\varphi_{\mathfrak{d}}$ have the same meaning as in the preceding discussion, so that $\varphi_{\mathfrak{d}}$ is of type $(\Lambda, \tau_{\mathfrak{d}}, \gamma)$. Write $n(\mathfrak{d}) = \dim(\mathcal{H}_{\mathfrak{d}})$ $(\mathfrak{d} \in \mathcal{E}_\pi)$; then, there is a constant $c_0 > 0$ such that $n(\mathfrak{d}) \leqslant c_0 \dim(\mathfrak{d})^2$ for all $\mathfrak{d} \in \mathcal{E}_\pi$. For $\mathfrak{d} \in \mathcal{E}(K)$, let $c(\mathfrak{d})$ denote the scalar into which the element Ω is mapped by representations from $\mathfrak{d}$. Then $c(\mathfrak{d})$ is real, $\geqslant 1$, and it is not difficult to show that there are constants $c_1 > 0$, $r_1 > 0$ for which

$$\sum_{\mathfrak{d} \in \mathcal{E}(K)} c(\mathfrak{d})^{-r_1} < \infty, \quad \dim(\mathfrak{d}) \leqslant c_1 c(\mathfrak{d})^{r_1} \quad (\forall \mathfrak{d} \in \mathcal{E}(K)) \tag{8.8}$$

(cf. [14], §4). Since $\tau_{\mathfrak{d},1}(\Omega) = \tau_{\mathfrak{d},2}(\Omega) = c(\mathfrak{d}) \cdot \text{identity}$, $\|\tau_{\mathfrak{d},1}(\Omega)\| = \|\tau_{\mathfrak{d},2}(\Omega)\| = c(\mathfrak{d})$ $(\mathfrak{d} \in \mathcal{E}_\pi)$. So, in view of (5.10) we can choose a constant $c = c_\Lambda > 0$ such that $|\tau_{\mathfrak{d}}, \Lambda| \leqslant c c(\mathfrak{d})^2$ for all $\mathfrak{d} \in \mathcal{E}_\pi$.

Given any ε with $0 < \varepsilon < \gamma$, we can select by virtue of (i) of Theorem 7.3, constants $D'_\varepsilon > 0$, $q'_\varepsilon \geqslant 0$ such that for all $\mathfrak{d} \in \mathcal{E}_\pi$ and all $x \in G$,

$$|||\varphi_{\mathfrak{d}}(x)||| \leqslant D'_\varepsilon |\tau_{\mathfrak{d}}, \Lambda|^{q'_\varepsilon} \|\varphi_{\mathfrak{d}}\|_2 \Xi(x)^{1+\gamma-(\varepsilon/2)}. \tag{8.9}$$

On the other hand, if $e_1, \dots, e_{n(\mathfrak{d})}$ is an orthonormal basis for $\mathcal{H}_{\mathfrak{d}}$, we have

$$|||\varphi_{\mathfrak{d}}(x)|||^2 = \sum_{1 \leqslant i,j \leqslant n(\mathfrak{d})} |(\pi(x) e_j, e_i)|^2 \quad (x \in G),$$

from which it follows that $\|\varphi_{\mathfrak{d}}\|_2 = d_\omega^{-\frac{1}{2}} n(\mathfrak{d})$, d_ω being the formal degree of ω. From (8.8), (8.9), and the earlier estimates for $|\tau_{\mathfrak{d}}, \Lambda|$ and $n(\mathfrak{d})$ we then obtain the following result: given any ε with $0<\varepsilon<\gamma$, we can find a constant $D_\varepsilon > 0$ and an integer $q_\varepsilon \geqslant 0$ such that, for all $\mathfrak{d} \in \mathcal{E}_\pi$ and all $x \mathcal{E} G$,

$$n(\mathfrak{d})\,|||\varphi_{\mathfrak{d}}(x)|||\leqslant D_\varepsilon c(\mathfrak{d})^{q_\varepsilon}\Xi(x)^{1+\gamma-(\varepsilon/2)}. \tag{8.10}$$

Let $f\in C_c^\infty(G)$. Then

$$\Theta(f)=\sum_{\mathfrak{d}\in\mathcal{E}_\pi}\int_G f(x)\,\mathrm{tr}\,(\varphi_{\mathfrak{d}}(x))\,dx,$$

the series converging absolutely. Now, for any integer $p\geqslant 0$ and $x\in G, \varphi_{\mathfrak{d}}(x;\Omega^p)=c(\mathfrak{d})^p\varphi_{\mathfrak{d}}(x)$; so, for such p,

$$\Theta(f)=\sum_{\mathfrak{d}\in\mathcal{E}_\pi}c(\mathfrak{d})^{-p}\int_G f(x;\Omega^p)\,\mathrm{tr}\,(\varphi_{\mathfrak{d}}(x))\,dx.$$

On the other hand, if $\mathfrak{d}\in\mathcal{E}_\pi$ and $x\in G$, $|\mathrm{tr}\,(\varphi_{\mathfrak{d}}(x))|\leqslant n(\mathfrak{d})\,|||\varphi_{\mathfrak{d}}(x)|||$, so that $|\mathrm{tr}\,(\varphi_{\mathfrak{d}}(x))|\leqslant D_\varepsilon c(\mathfrak{d})^{q_\varepsilon}\Xi(x)^{1+\gamma-(\varepsilon/2)}$, by (8.10). Choosing $p=p_\varepsilon=q_\varepsilon+r_1$ in the last formula for $\Theta(f)$, and writing $C'_\varepsilon=D_\varepsilon=\sum_{\mathfrak{d}\in\mathcal{E}_\pi}c(\mathfrak{d})^{-r_1}$, we have,

$$|\Theta(f)|\leqslant C'_\varepsilon\int_G\Xi(x)^{1+\gamma-(\varepsilon/2)}\,|f(x;\Omega^p)|\,dx. \tag{8.11}$$

Put $C_\varepsilon=C'_\varepsilon\int_G\Xi(x)^{2+(\varepsilon/2)}dx$. Then (8.11) leads to (8.7). This proves the lemma.

By a simple modification of the argument above that led to (8.10) we obtain the following result from (7.39): let $1\leqslant p<2$, and π, an irreducible unitary representation of G in a Hilbert space $\mathcal{H}$ such that the equivalence class of π belongs to $\mathcal{E}_p(G)$. Then, there are constants $C>0$, $r\geqslant 0$ such that, with $\varepsilon_0>0$ as in (ii) Theorem 7.5,

$$|(\pi(x)\psi,\psi')|\leqslant Cc(\mathfrak{d})^r c(\mathfrak{d}')^r\,\Xi(x)^{(2/p)+\varepsilon_0} \tag{8.12}$$

for all $x\in G$, all $\mathfrak{d},\mathfrak{d}'\in\mathcal{E}_\pi$, and arbitrary unit vectors $\psi\in\mathcal{H}_{\mathfrak{d}}$, $\psi'\in\mathcal{H}_{\mathfrak{d}'}$. The estimate (8.12) leads at once to the following two corollaries. For deducing the first of these we must recall that if $\psi\in\mathcal{H}$ is a differentiable vector for π, then $\sum_{\mathfrak{d}\in\mathcal{E}_\pi}\|P_{\mathfrak{d}}\psi\|\,c(\mathfrak{d})^m<\infty$ for every $m>0$ ([14], § 3).

COROLLARY 8.5. *Let $1\leqslant p<2$. Let π be an irreducible unitary representation of G in a Hilbert space $\mathcal{H}$ such that the equivalence class of π is in $\mathcal{E}_p(G)$. Then, if ψ, ψ' are two differentiable vectors for π, and $\varepsilon_0>0$ is as in Theorem* 7.5, (ii), *we can find a constant $C=C_{\psi,\psi'}>0$ such that*

$$|(\pi(x)\psi,\psi')|\leqslant C\,\Xi(x)^{(2/p)+\varepsilon_0}\quad(x\in G).$$

In particular, the function $x\mapsto|(\pi(x)\psi,\psi')|$ lies in $L^p(G)$.

COROLLARY 8.6. *Let $1\leqslant p<2$. Let π be an irreducible unitary representation in a Hilbert space $\mathcal{H}$ such that the equivalence class of π belongs to $\mathcal{E}_p(G)$. Then, there are constants $c>0$, $r\geqslant 0$, such that, for arbitrary $\mathfrak{d},\mathfrak{d}'\in\mathcal{E}_\pi$, and $\psi\in\mathcal{H}_{\mathfrak{d}}$, $\psi'\in\mathcal{H}_{\mathfrak{d}'}$, with $\|\psi\|=\|\psi'\|=1$,*

$$\int_G |(\pi(x)\psi, \psi')|^p dx \leqslant cc(\mathfrak{d})^r c(\mathfrak{d}')^r. \tag{8.13}$$

Consider a θ-stable Cartan subalgebra $\mathfrak{h}$ that is not conjugate to $\mathfrak{b}$ under G. Let $A_\mathfrak{h}$ be the corresponding Cartan subgroup; $A'_\mathfrak{h}$, the set of regular points of $A_\mathfrak{h}$; $G_\mathfrak{h} = (A'_\mathfrak{h})^G$. Write $\mathfrak{h}_2 = \mathfrak{h} \cap \mathfrak{s}$, $A_1 = A_\mathfrak{h} \cap K$, $A_2 = \exp \mathfrak{h}_2$. Then $A_\mathfrak{h} = A_1 A_2$ is a direct product, and we write a_i, for the component in A_i, of $a \in A_\mathfrak{h}$. Given $\mu \in \mathfrak{L}_\mathfrak{h}$, ξ_μ denotes the corresponding character of $A_\mathfrak{h}$. Let A_1^+ be a connected component of A_1, $\mathfrak{h}'_2$ be the set of all $H \in \mathfrak{h}_2$ such that $\alpha(H) \neq 0$ for any root α of $(\mathfrak{g}_c, \mathfrak{h}_c)$ that is not identically zero on $\mathfrak{h}_2$, and let $\mathfrak{h}_2^+$ be a connected component of $\mathfrak{h}'_2$; write $A_2^+ = \exp \mathfrak{h}_2^+$. Fix a positive system Q^+ of roots of $(\mathfrak{g}_c, \mathfrak{h}_c)$, such that, if α is a root and $\alpha | \mathfrak{h}_2 \not\equiv 0$, then $\alpha \in Q^+$ if and only if $\alpha(H) > 0$ for all $H \in \mathfrak{h}_2^+$. Let

$$\delta^+ = \tfrac{1}{2} \sum_{\alpha \in Q^+} \alpha, \quad \Delta_\mathfrak{h}^+ = \xi_{-\delta^+} \prod_{\alpha \in Q^+} (\xi_\alpha - 1). \tag{8.14}$$

$\delta^+ | \mathfrak{h}_2$ actually depends only on $\mathfrak{h}_2^+$. In fact, let $\mathfrak{z}$ be the centralizer of $\mathfrak{h}_2$ in $\mathfrak{g}$, and, more generally, for any $\nu \in \mathfrak{h}_2^*$, let $\mathfrak{g}_\nu$ be the space of all $X \in \mathfrak{g}$ with $[H, X] = \nu(H) X$ for all $H \in \mathfrak{h}_2$; if $\mathfrak{n}^+ = \sum_{\nu : \nu(H) > 0 \forall H \in \mathfrak{h}_2^+} \mathfrak{g}_\nu$, then $\mathfrak{p}^+ = \mathfrak{z} + \mathfrak{n}^+$ is a parabolic subalgebra, and

$$\delta^+(H) = \tfrac{1}{2} \operatorname{tr} (\operatorname{ad} H)_{\mathfrak{n}^+} \quad (H \in \mathfrak{h}_2). \tag{8.15}$$

Define the function $\Phi_\mathfrak{h}$ on $A'_\mathfrak{h}$ by $\Phi_\mathfrak{h}(a) = \Delta_\mathfrak{h}^+(a)\Theta(a)$ $(a \in A'_\mathfrak{h})$, Θ (and ω) being as in Lemma 8.4. If $\alpha \in Q^+$ is real on $\mathfrak{h}$, it is not difficult to verify that $\xi_\alpha - 1$ has no zero in $A_1^+ A_2^+$. Writing $A_\mathfrak{h}^+ = A_1^+ A_2^+ \cap A'_\mathfrak{h}$ we may therefore conclude that $\Phi_\mathfrak{h} | A_\mathfrak{h}^+$ extends to an analytic function on $A_1^+ A_2^+$ ([12], Lemma 31). Let $O_\mathfrak{h}$ be the $W(\mathfrak{h}_c)$-orbit in $\mathfrak{h}_c^*$ that corresponds to $W(\mathfrak{b}_c) \cdot \lambda$. It is then clear that for suitable constants c_μ^+ $(\mu \in O_\mathfrak{h})$ we have the following formula:

$$\Phi_\mathfrak{h}(a) = \sum_{\mu \in O_\mathfrak{h}} c_\mu^+ \xi_\mu(a_1) e^{\mu(\log a_2)} \quad (a \in A_\mathfrak{h}^+). \tag{8.16}$$

LEMMA 8.7. *Let $\omega = \omega(\lambda)$ be of type γ, $\Theta = \Theta_\omega$, and let notation be as above. Then*

$$\mu \in O_\mathfrak{h}, c_\mu^+ \neq 0 \Rightarrow (\mu + \gamma\delta^+)(H) \leqslant 0 \text{ for all } H \in \mathfrak{h}_2^+. \tag{8.17}$$

Proof. It is clearly enough to prove the following implication:

$$\mu \in O_\mathfrak{h}, c_\mu^+ \neq 0 \Rightarrow (\mu + (\gamma - \varepsilon)\delta^+)(H) \leqslant 0 \text{ for all } H \in \mathfrak{h}_2^+, \tag{8.18}$$

for every ε with $0 < \varepsilon < \gamma$. In what follows we fix ε $(0 < \varepsilon < \gamma)$, write $\varkappa = \gamma - \varepsilon$, and select $C > 0$, $p \geqslant 0$ such that $|\Theta(f)| \leqslant C \sup_G \Xi^{-1+\varkappa} |\Omega^p f|$ for all $f \in C_c^\infty(G)$. Let $A_\mathfrak{h}^\sim$ be the normalizer of $A_\mathfrak{h}$ in G, and let W_A be the image of $A_\mathfrak{h}^\sim / A_\mathfrak{h}$ in $W(\mathfrak{h}_c)$.

Proceeding as in § 19 of [14] we construct a map $\beta \mapsto f_\beta$ of $C_c^\infty(A'_\mathfrak{h})$ into $C_c^\infty(G_\mathfrak{h})$ with the following properties:

(i) for $\beta \in C_c^\infty(A'_\mathfrak{h})$ and $a \in A'_\mathfrak{h}$, writing $\bar{G} = G/A_\mathfrak{h}$,

$$\Delta_\mathfrak{h}^+(a)^{\text{conj}} \int_{\bar{G}} f_\beta(a^{\bar{x}})\, d\bar{x} = \sum_{s \in W_A} \varepsilon(s)\, \beta(a^s); \tag{8.19}$$

here, $x \mapsto \bar{x}$ is the natural map of G onto $\bar{G}$, $d\bar{x}$ is an invariant measure on $\bar{G}$.

(ii) there is a compact set $X = X^{-1} \subseteq G$ such that $\operatorname{supp}(f_\beta) \subseteq (\operatorname{supp} \beta)^X$ for all $\beta \in C_c^\infty(A'_\mathfrak{h})$.

(iii) Let $\mathfrak{R}$ be the algebra of functions on $A'_\mathfrak{h}$ generated by 1 and all the $\eta_\alpha = (1-\xi_\alpha)^{-1}$ (α any root of $(\mathfrak{g}_c, \mathfrak{h}_c)$), and let $\mathfrak{H}$ be the subalgebra of $\mathfrak{G}$ generated by $(1, \mathfrak{h})$; then, given any $u \in \mathfrak{G}$, there exist $u_{is} \in \mathfrak{H}$, $g_{is} \in \mathfrak{R}$ ($s \in W_A$, $1 \leqslant i \leqslant q$) such that, for all $\beta \in C_c^\infty(A'_\mathfrak{h})$, $a \in A'_\mathfrak{h}$, $x \in X$,

$$|f_\beta(a^x; u)| \leqslant |\xi_{\delta^+}(a)|^{-1} \sum_{1 \leqslant i \leqslant q} \sum_{s \in W_A} |g_{is}(a)|\, |\beta(a^s; u_{is})|. \tag{8.20}$$

It follows from (8.19) that $\Theta(f_\beta) = \int_{A'_\mathfrak{h}} \Phi_\mathfrak{h}(a)\, \beta(a)\, da$ for all $\beta \in C_c^\infty(A'_\mathfrak{h})$, provided da is suitably normalized. On the other hand, by (ii) above, we have, for all $\beta \in C_c^\infty(A_\mathfrak{h}^+)$,

$$\sup_G \Xi^{-1+\varkappa} |\Omega^p f_\beta| = \sup_{a \in A_\mathfrak{h}^+,\, x \in X} \Xi(a^x)^{-1+\varkappa} |f_\beta(a^x; \Omega^p)|,$$

and we can estimate the right side of this relation by (8.20). Observing that there is a constant $c > 0$ with $c^{-1}\Xi(y) \leqslant \Xi(x_1 y x_2) \leqslant c\Xi(y)$ for all $y \in G$, $x_1, x_2 \in X$, we then get the following result: there are $v_{is} \in \mathfrak{H}$, $h_{is} \in \mathfrak{R}$ ($1 \leqslant i \leqslant r$, $s \in W_A$) such that for all $\beta \in C_c^\infty(A_\mathfrak{h}^+)$,

$$\left| \int_{A_\mathfrak{h}^+} \Phi_\mathfrak{h}(a)\, \beta(a)\, da \right| \leqslant \sum_{i,s} \sup_{a \in A_\mathfrak{h}^+} \left(\Xi(a)^{-1+\varkappa} |\xi_{\delta^+}(a)|^{-1} |h_{is}(a)|\, |\beta(a^s; v_{is})| \right). \tag{8.21}$$

Now, each element of W_A is induced by some element of K, and hence $\Xi(a^s) = \Xi(a)$ ($a \in A_\mathfrak{h}$, $s \in W_A$)(1). On the other hand, from (8.15), and the fact that the parabolic subalgebra $\mathfrak{p}^+$ defined there is conjugate to some $\mathfrak{p}_F$ through an element of K, we conclude that $1 \leqslant \Xi(\exp H) e^{\delta^+(H)} \leqslant c_0(1 + \|H\|)^{r_0}$ for all $H \in \mathfrak{h}_2^+$, c_0 and r_0 being as in (2.1). So

$$1 \leqslant |\xi_{\delta^+}(a)|\, \Xi(a) \leqslant c_0(1 + \sigma(a))^{r_0} \quad (a \in A_\mathfrak{h}^+). \tag{8.22}$$

(1) Suppose $x \in A_\mathfrak{h}^\sim$ induces $s \in W_A$. Writing $x = k \exp Z$ ($k \in K$, $Z \in \hat{\mathfrak{s}}$) one finds that $\exp 2Z = \theta(x^{-1})\, x \in A_\mathfrak{h}^\sim$, so that $Z \in \mathfrak{h}_2$. This shows that $k \in A_\mathfrak{h}^\sim$ and induces s.

Finally, since $\mathfrak{R}$ is stable under the action of W_A, the functions f_{is}: $a \mapsto h_{is}(a^{s^{-1}})$ $(a \in A'_{\mathfrak{h}}$, $s \in W_A)$ belong to $\mathfrak{R}$. Using these observations in (8.21) we find after a simple calculation, the following estimate, valid for $\beta \in C_c^\infty(A_{\mathfrak{h}}^+)$:

$$\left|\int_{A_{\mathfrak{h}}^+} \Phi_{\mathfrak{h}}(a)\beta(a)\,da\right| \leqslant c_0^\varkappa \sum_{i,s} \sup_{a \in A_{\mathfrak{h}}^+} \left((1+\sigma(a))^{r_0\varkappa} |\xi_{\delta^+}(a)|^{-\varkappa} |f_{is}(a)| \, |\beta(a; v_{is})|\right).$$

Since ξ^+: $a \to |\xi_{\delta^+}(a)|^\varkappa$ is a character of $A_{\mathfrak{h}}$, it follows that ${\xi^+}^{-1} \circ v_{is} \circ \xi^+$ are well defined elements of $\mathfrak{H}$. Replacing β by $\beta\xi^+$ in the above estimate, we finally obtain the following result: there exist $m \geqslant 1$, $v_j \in \mathfrak{H}$, $h_j \in \mathfrak{R}$ $(1 \leqslant j \leqslant r)$ such that, for all $\beta \in C_c^\infty(A_{\mathfrak{h}}^+)$,

$$\left|\int_{A_{\mathfrak{h}}^+} \Phi_{\mathfrak{h}}(a)\,\xi^+(a)\,\beta(a)\,da\right| \leqslant \sum_{1 \leqslant j \leqslant r} \sup_{a \in A_{\mathfrak{h}}^+} \left((1+\sigma(a))^m |h_j(a)| \, |\beta(a; v_j)|\right). \tag{8.23}$$

The estimate (8.23) is the analogue of Lemma 32 of [14] with the function

$$\Phi_{\mathfrak{h}}\xi^+ : a \mapsto \sum_{\mu \in O_{\mathfrak{h}}} c_\mu^+ \xi_\mu(a_1)\, e^{(\mu + \varkappa\delta^+ \ (\log a_2)}$$

in the place of Φ. If we now argue as in [14]. we obtain (8.18) in exactly the same way as Lemma 34 is deduced from Lemma 32 in [14]. This proves the lemma.

It follows from (8.16) and (8.17) that, if $\omega = \omega(\lambda)$ is of type γ, and $\mathfrak{h} = \theta(\mathfrak{h})$ is as above, then there is a constant $c_{\mathfrak{h}}^+ > 0$ such that

$$|D(a)|^{\frac{1}{2}} |\Theta(a)| \leqslant c_{\mathfrak{h}}^+ |\xi_{\delta^+}(a)|^{-\gamma} \quad (a \in A_{\mathfrak{h}}^+). \tag{8.24}$$

Let Q_I^+ be the set of all roots $\alpha \in Q^+$ with $\alpha|\mathfrak{h}_2 = 0$, and let ν be the number of elements in $Q^+ \setminus Q_I^+$. If $a \in A_{\mathfrak{h}}^+$ and $\alpha \in Q^+ \setminus Q_I^+$, we have $|1 - \xi_{-\alpha}(a)| \leqslant 1 + e^{-\alpha(\log a_2)} < 2$, while, for $a \in A_{\mathfrak{h}}$ and $\alpha \in Q_I^+$, $|\xi_\alpha(a)| = 1$. Hence, for $a \in A_{\mathfrak{h}}^+$,

$$\begin{aligned} |D_{\mathfrak{h}}(a)| &= \prod_{\alpha \in Q^+ \setminus Q_I^+} |1 - \xi_\alpha(a)| \, |1 - \xi_{-\alpha}(a)| \\ &= \prod_{\alpha \in Q^+ \setminus Q_I^+} |1 - \xi_{-\alpha}(a)|^2 |\xi_\alpha(a)| \leqslant 2^{2\nu} \prod_{\alpha \in Q^+} |\xi_\alpha(a)| = 2^{2\nu} |\xi_{\delta^+}(a)|^2. \end{aligned}$$

Writing $c(A_{\mathfrak{h}}^+) = 2^{\nu\gamma} c_{\mathfrak{h}}^+$, we then obtain from (8.24)

$$|D(a)|^{\frac{1}{2}} |\Theta(a)| \leqslant c(A_{\mathfrak{h}}^+) |D_{\mathfrak{h}}(a)|^{-\gamma/2} \quad (a \in A_{\mathfrak{h}}^+). \tag{8.25}$$

Since there are only finitely many sets of the form $A_{\mathfrak{h}}^+$ (for a given $\mathfrak{h}$), and since their union is dense in $A'_{\mathfrak{h}}$, we conclude from (8.25) that for $\omega = \omega(\lambda)$ to be of type γ, (8.2) must be true for all $\mathfrak{h}$.

In order to complete the proof of Theorem 8.1 it remains to show how (8.3) may be obtained from (8.2) by choosing $\mathfrak{h}$ suitably. Let β be a noncompact root of $(\mathfrak{g}_c, \mathfrak{b}_c)$. We now specialize the Cartan subalgebra $\mathfrak{h}$ of the above discussion to be the one constructed at the beginning of the proof of Lemma 8.3. Let H' be as in that lemma, $y = \exp(-1)^{\frac{1}{2}}(\pi/4)(X'+Y')$. Then $H' \in \mathfrak{h}_2'$, and on defining $\mathfrak{h}_2^+ = \{tH' : t > 0\}$, we find at once that $\delta^+(H') = k(\beta)$. On the other hand, there are nonzero constants c_s $(s \in W(G/B))$ such that, for all $a \in A_{\mathfrak{h}}^+$,

$$\Delta_{\mathfrak{h}}^+(a)\,\Theta(a) = \sum_{s \in W(G/B)} c_s \xi_{(s\lambda)\circ y^{-1}}(a_1)\, e^{-|((s\lambda)\circ y^{-1})(\log a_2)|}. \tag{8.26}$$

This formula was established by Harish-Chandra in § 24 of [13] in the special case when $\mathrm{rk}\,(G/K) = 1$; in the more general case treated here, (8.26) can be established with only minor modifications in the arguments of [13]. In view of (8.26) and (8.24), we must have $|(\lambda \circ y^{-1})(H')| \geqslant \gamma \delta^+(H')$, i.e., $|\lambda(\bar{H}_\beta)| \geqslant \gamma k(\beta)$.

Theorem 8.1 is therefore completely proved. Theorem 8.2 follows at once from Theorem 8.1, since an ω in $\mathcal{E}_2(G)$ belongs to $\mathcal{E}_p(G)$ if and only if it is of type γ for some $\gamma > (2/p) - 1$ (cf. Theorem 7.5).

9. Examples and remarks

We shall now complement the results of the preceding sections with some examples and remarks.

We begin with a discussion of the condition (cf. [10], [11]) of Harish-Chandra which is sufficient for $\omega(s\lambda)$ to belong to $\mathcal{E}_1(G)$ for all $s \in W(\mathfrak{b}_c)$. Let $\lambda \in \mathcal{L}_{\mathfrak{b}}'$, $O_{\mathfrak{b}} = W(\mathfrak{b}_c) \cdot \lambda$, $O_{\mathfrak{l}}$ = the $W(\mathfrak{l}_c)$-orbit in $\mathfrak{l}_c^*$ that corresponds to $O_{\mathfrak{b}}$; and let $\mathfrak{o}$ be the subset of $\mathfrak{a}^*$ obtained by restricting the elements of $O_{\mathfrak{l}}$ to $\mathfrak{a}$. Given $\nu \in \mathfrak{a}^*$ we write $\nu \prec 0$ to mean $\nu(H_i) < 0$ for $1 \leqslant i \leqslant d$; here, the H_i are as in § 2. Let $\mathfrak{o}^-$ be the set of all $\nu \in \mathfrak{o}$ such that $\nu \prec 0$. Then Harish-Chandra's result is as follows: *In order that $\omega(s\lambda) \in \mathcal{E}_1(G)$ for all $s \in W(\mathfrak{b}_c)$ it is sufficient that $\nu + \varrho \prec 0$ for every $\nu \in \mathfrak{o}^-$.* To prove this it is enough to verify that this condition implies that $|(s\lambda)(\bar{H}_\beta)| > k(\beta)$ for all $s \in W(\mathfrak{b}_c)$, $\beta \in P_n$, or equivalently, that $|\Lambda(H_j)| > \varrho(H_j)$ for all $\Lambda \in O_{\mathfrak{l}}$ and $j \in Q$, by virtue of Lemma 8.3 (here Q is as in that lemma). This implication is an immediate consequence of the following lemma.

LEMMA 9.1. *Fix $\Lambda \in O_{\mathfrak{l}}$, $j \in Q$. Then there exists $\Lambda' \in O_{\mathfrak{l}}$ such that* (i) $|\Lambda'(H_j)| = |\Lambda(H_j)|$ (ii) $(\Lambda' | \mathfrak{a}) \in \mathfrak{o}^-$.

Proof. We use the notation of § 2. Let $\mathfrak{h}_j$ be a θ-stable Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{h}_j \cap \mathfrak{s} = \mathfrak{a}_j (= \mathbf{R} \cdot H_j)$. As in the proof of Lemma 8.3 we can select a root α' of $(\mathfrak{g}_c, \mathfrak{h}_{jc})$

and an element $z\in G_c$ centralizing H_j, such that $\mathfrak{h}_{jc}^z=\mathfrak{l}_c$ and $\bar{H}_{\alpha'}=cH_j$ for some $c\neq 0$. If $\alpha_1=\alpha'\circ z^{-1}$, then α_1 is a root of $(\mathfrak{g}_c, \mathfrak{l}_c)$ and $\bar{H}_{\alpha_1}=cH_j$. This shows that $\Lambda(H_j)\neq 0$ and $(s_{\alpha_1}\Lambda)(H_j)=-\Lambda(H_j)$. We may therefore assume without any loss of generality that $\Lambda(H_j)<0$.

Select a positive system Q^+ of roots of $(\mathfrak{g}_c, \mathfrak{l}_c)$ with the property that, if α is any root and $\alpha|\mathfrak{a}\neq 0$, then $\alpha\in Q^+$ if and only if $\alpha(H)>0$ for all $H\in\mathfrak{a}^+$. Let Q_j^+ be the set of all $\alpha\in Q^+$ with $\alpha(H_j)=0$, and let $\delta_j^+=\frac{1}{2}\sum_{\alpha\in Q_j^+}\alpha$. Q_j^+ is then a positive system of roots of $(\mathbf{C}\cdot\mathfrak{m}_{1j}, \mathfrak{l}_c)$, and $\delta_j^+|\mathfrak{a}=\varrho_{F_j}$. Let $\mathfrak{z}=[\mathfrak{m}_{1j}, \mathfrak{m}_{1j}]$, $\bar{\mathfrak{l}}=\mathfrak{z}\cap\mathfrak{l}$, and $\bar{\mathfrak{a}}=\mathfrak{z}\cap\mathfrak{a}$. As $\mathfrak{a}_j=\text{center }(\mathfrak{m}_{1j})\cap\mathfrak{s}$, it follows that $\bar{\mathfrak{a}}$ is precisely the orthogonal complement of $\mathfrak{a}_j$ in $\mathfrak{a}$, so that $\bar{\mathfrak{a}}=\mathfrak{m}_j\cap\mathfrak{a}$ also. Now Λ is regular and integral, and so, we can find an $s\in W(\mathfrak{l}_c)_{F_j}$ such that $(s\Lambda)(\bar{H}_\alpha)$ is an integer <0 for all $\alpha\in Q_j^+$. Then $s\cdot H_j=H_j$, and we can write $-s\Lambda=\Lambda_1+\delta_j^+$ where $\Lambda_1(\bar{H}_\alpha)\geqslant 0$ for every $\alpha\in Q_j^+$. On the other hand, if $\beta_1, ..., \beta_r$ are the simple roots in Q_j^+, it follows from a well known result that we can write $\Lambda_1|\bar{\mathfrak{l}}_c=\sum_{1\leqslant j\leqslant r}m_j(\beta_j|\bar{\mathfrak{l}}_c)$ where the m_j are all $\geqslant 0$. In particular $\Lambda_1|\bar{\mathfrak{a}}=\sum_{1\leqslant j\leqslant r}m_j(\beta_j|\bar{\mathfrak{a}})$. But the β_j vanish on $\mathfrak{a}_j$, and ϱ^{F_j} vanishes on $\bar{\mathfrak{a}}$; moreover, $(s\Lambda)(H_j)=\Lambda(H_j)$. So, on defining $t=-\Lambda(H_j)/\varrho^{F_j}(H_j)$, we find that $t>0$ and $s\Lambda|\mathfrak{a}=-\varrho_{F_j}-t\varrho^{F_j}-\sum_{1\leqslant j\leqslant r}m_j(\beta_j|\mathfrak{a})$. If $u=\min(1, t)$, $(s\Lambda)(H)\leqslant -u\varrho(H)$ for all $H\in Cl(\mathfrak{a}^+)$, so that $(s\Lambda)(H_i)<0$, $1\leqslant i\leqslant d$. We then have (i) and (ii) with $\Lambda'=s\Lambda$.

We assume next that G/K is Hermitian symmetric, and consider those members of $\mathcal{E}_2(G)$ which constitute the so-called holomorphic discrete series. For brevity, a positive system of roots of $(\mathfrak{g}_c, \mathfrak{b}_c)$ will be called *admissible* if every noncompact root in it is totally positive. We now assume that the positive system P is admissible. Let P_k be the set of compact roots in P. We write $\delta=\frac{1}{2}\sum_{\alpha\in P}\alpha$. Let $\lambda'\in\mathcal{L}_\mathfrak{b}$ be such that $\lambda'(\bar{H}_\alpha)\geqslant 0$ for all $\alpha\in P_k$ and $(\lambda'+\delta)(\bar{H}_\alpha)<0$ for all $\alpha\in P_n$. Then $\lambda=\lambda'+\delta\in\mathcal{L}'_\mathfrak{b}$; moreover, if $\pi_{\lambda'}$ is the representation associated with λ' constructed by Harish-Chandra in [3], [4], [5], then $\pi_{\lambda'}\in\omega(\lambda)$. Our aim now is to examine under what circumstances $\omega(\lambda)\in\mathcal{E}_1(G)$.

THEOREM 9.2. *Let G/K be Hermitian symmetric and let λ, P be as described above. The following statements are then equivalent:*

(i) $\omega(\lambda)\in\mathcal{E}_1(G)$

(ii) $|\lambda(\bar{H}_\beta)|>k(\beta)$ *for all* $\beta\in P_n$.

(iii) $\lambda(\bar{H}_\beta)<1-2\delta_n(\bar{H}_\beta)$ *for all* $\beta\in P_n$, *where* $2\delta_n=\sum_{\alpha\in P_n}\alpha$.

Proof. Theorem 8.2 gives the implication (i) $\Rightarrow$ (ii). In his paper [5] (Lemma 30) Harish-Chandra established the implication (iii) $\Rightarrow$ (i). It therefore remains to verify that (ii) $\Rightarrow$ (iii). Let $P'=-P_k\cup P_n$. If s_0 is the element of the Weyl group of $(\mathfrak{k}_c, \mathfrak{b}_c)$ such that $s_0\cdot P_k=-P_k$, it is clear that $s_0\cdot P=P'$. So P' is a positive system of roots of $(\mathfrak{g}_c, \mathfrak{b}_c)$. It is obvious that P' is also admissible and that $P'_n=P_n$. Let $(\beta_1, ..., \beta_l)$ be the simple system of roots of P', and let notation be such that $\beta_1, \dots, \beta_t$ are precisely the noncompact roots from among $\beta_1, \dots, \beta_l$. It is known that every $\alpha \in P'_k$ is a linear combination with non-negative integral coefficients of $\mathfrak{b}_{t+1}, \dots, \mathfrak{b}_l$ ([3], Lemma 13), so that, $\alpha(\bar{H}_{\beta_j}) \leqslant 0$ whenever $\alpha \in P'_k$ and $1 \leqslant j \leqslant t$. It is also known that, for any $\beta', \beta'' \in P_n$, $\beta''(\bar{H}_{\beta'}) \geqslant 0$ ([5], Lemma 10).

Assume that λ satisfies (ii). Since $k(\beta)$ is an integer and $\lambda(\bar{H}_\beta) < 0$ for $\beta \in P_n$, we have $\lambda(\bar{H}_\beta) \leqslant -k(\beta) - 1$ for all $\beta \in P_n$. We assert that $\lambda(\bar{H}_{\beta_j}) \leqslant -2\delta_n(\bar{H}_{\beta_j})$ for $1 \leqslant j \leqslant l$. Suppose $j > t$. Then $\beta_j \in -P_k$ so that $\lambda(\bar{H}_{\beta_j}) < 0$. But $s\delta_n = \delta_n$ for all s in the Weyl group of $(\mathfrak{k}_c, \mathfrak{b}_c)$, as P is admissible, so that $\delta_n(H_{\beta_j}) = 0$. Thus our assertion is true in this case. On the other hand, let $1 \leqslant j \leqslant t$. Then $\beta_j \in P_n$ and so $\lambda(\bar{H}_{\beta_j}) \leqslant -k(\beta_j) - 1$. Now

$$k(\beta_j) = \tfrac{1}{2} \sum_{\alpha \in P'} |\alpha(\bar{H}_{\beta_j})| = \tfrac{1}{2} \{ \sum_{\alpha \in P'_k} (-\alpha(\bar{H}_{\beta_j})) + \sum_{\alpha \in P'_n} \alpha(\bar{H}_{\beta_j}) \} = -\tfrac{1}{2} \sum_{\alpha \in P'} \alpha(\bar{H}_{\beta_j}) + 2\delta_n(\bar{H}_{\beta_j}).$$

But, as β_j is simple in P', $\frac{1}{2} \sum_{\alpha \in P'} \alpha(\bar{H}_{\beta_j}) = 1$. So

$$k(\beta_j) + 1 = 2\delta_n(\bar{H}_{\beta_j}) \quad (1 \leqslant j \leqslant t). \tag{9.1}$$

From (9.1) we obtain $\lambda(\bar{H}_{\beta_j}) \leqslant -2\delta_n(\bar{H}_{\beta_j})$ when $1 \leqslant j \leqslant t$. Our assertion is therefore proved.

We therefore have $\langle \lambda, \beta_j \rangle \leqslant -2\langle \delta_n, \beta_j \rangle$, $1 \leqslant j \leqslant l$. This implies that $\langle \lambda, \beta \rangle \leqslant -2\langle \delta_n, \beta \rangle$ for all $\beta \in P'$, in particular, for all $\beta \in P_n$. But then $\lambda(\bar{H}_\beta) \leqslant -2\delta_n(\bar{H}_\beta) < 1 - 2\delta_n(\bar{H}_\beta)$ for all $\beta \in P_n$, proving (iii).

We shall now use Theorem 9.2 to construct examples of $\lambda \in \mathcal{L}'_{\mathfrak{b}}$ such that $\omega(\lambda) \in \mathcal{E}_1(G)$, but $\omega(s\lambda) \notin \mathcal{E}_1(G)$ for some $s \in W(\mathfrak{b}_c)$. Let notation be as above. We shall assume that there are elements of $W(\mathfrak{b}_c)$ which transform a compact root into a noncompact root.(1) Let $c_1, \dots, c_l$ be integers > 0 such that $0 < -\delta(\bar{H}_{\beta_j}) \leqslant c_j \leqslant k(\beta_j)$ for $t < j \leqslant l$. Since $-\beta_j \in P$ $(t < j \leqslant l)$ and $k(\beta) \geqslant \delta(\bar{H}_\beta)$ $\forall \beta \in P$, it is possible to choose such c_i. Define $\lambda \in \mathfrak{b}_c^*$ by setting $\lambda(\bar{H}_{\beta_j}) = -c_j$, $1 \leqslant j \leqslant l$. It is obvious that $\lambda \in \mathcal{L}'_{\mathfrak{b}}$, and that $\lambda = \lambda' + \delta$, where $\lambda'(\bar{H}_\alpha) \geqslant 0$ for all $\alpha \in P_k$; and so, (iii) of Theorem 9.2 shows that $\omega(\lambda) \in \mathcal{E}_1(G)$ if $c_1, \dots, c_t$ are all sufficiently large. But, if j and $s \in W(\mathfrak{b}_c)$ are such that $t < j \leqslant l$, and $s\beta_j = \beta$ is a noncompact root $|(s\lambda)(\bar{H}_\beta)| = |\lambda(\bar{H}_{\beta_j})| \leqslant k(\beta_j) = k(\beta)$, so that $\omega(s\lambda) \notin \mathcal{E}_1(G)$.

Let us now return to the case of an arbitrary G. The estimates for the eigenfunctions for $\mathfrak{Z}$ which we have obtained have also taken into account the variation of the eigenvalues. We shall now indicate an application of these estimates.

Fix p with $1 \leqslant p < 2$. Let $\mathcal{C}(G)$ $(= \mathcal{C}^2(G)$ in the notation of the remark following Corollary 3.4) be the Schwartz space of G. Let ${}^0L^2(G))$ (resp. ${}^0L^2_p(G))$ be the smallest closed subspace of $L^2(G)$ containing all the K-finite matrix coefficients of the members

(1) It is not difficult to show that this is always the case unless $\mathfrak{g}$ is the direct sum of $[\mathfrak{k}, \mathfrak{k}]$ and a certain number of algebras isomorphic to $\mathfrak{sl}(2, \mathbf{R})$.

of $\mathcal{E}_2(G)$ (resp. $\mathcal{E}_p(G)$). Let ${}^0E$ (resp. ${}^0E_p$) be the orthogonal projection $L^2(G)\to{}^0L^2(G)$ (resp. $L^2(G)\to{}^0L^2_p(G)$). Harish-Chandra has proved ([15]) that if $f\in\mathcal{C}(G)$, ${}^0Ef\in\mathcal{C}(G)$ also, and that $f\mapsto{}^0Ef$ is continuous in the Schwartz topology. We shall now obtain an extension of this result.

THEOREM 9.3. *Let notation be as above. Then, for any $f\in\mathcal{C}(G)$, ${}^0E_pf\in\mathcal{C}^p(G)$, and the map $f\mapsto E_pf$ is continuous from $\mathcal{C}(G)$ into $\mathcal{C}^p(G)$.*

Proof. Let $\mathcal{L}(p)$ be the set of all $\lambda\in\mathcal{L}'_{\mathfrak{b}}$ such that $\lambda(\bar{H}_\alpha)>0$ for all $\alpha\in P_k$ and $\omega(\lambda)\in\mathcal{E}_p(G)$. Then $\lambda\mapsto\omega(\lambda)$ is a bijection of $\mathcal{L}(p)$ onto $\mathcal{E}_p(G)$. For each $\lambda\in\mathcal{L}(p)$ we select a Hilbert space $\mathcal{H}_\lambda$, a representation $\pi_\lambda\in\omega(\lambda)$ acting in $\mathcal{H}_\lambda$, and an orthonormal basis $\{e_{\lambda,i}: i\in N_\lambda\}$ of $\mathcal{H}_\lambda$, such that, each $e_{\lambda,i}$ lies in a subspace invariant and irreducible under $\pi_\lambda(K)$. Let Ω be as in (5.8). Then there are numbers $c_{\lambda,i}\geq 1$ such that $\pi_\lambda(\Omega)e_{\lambda,i}=c_{\lambda,i}e_{\lambda,i}$ $(i\in N_\lambda)$. Now, there is an integer $m\geq 1$ such that for any $\lambda\in\mathcal{L}'_{\mathfrak{b}}$ and any equivalence class $\mathfrak{d}$ of irreducible representations of K, the multiplicity of $\mathfrak{d}$ in $\pi_\lambda|K$ is $\leq m\cdot\dim(\mathfrak{d})$. It follows from this and (8.8), that there are constants $a>0$, $r\geq 0$ with the following property:

$$\sup_{\gamma\in\mathcal{L}(p)}\sum_{i\in N_\lambda} c_{\gamma,i}^{-r}=a<\infty.$$

Moreover, if ω is the Casimir of G, we have $\mu_{\mathfrak{g}/\mathfrak{b}}(\omega)(\lambda)=\|\lambda\|^2-\|\delta\|^2$ $(\delta=\frac{1}{2}\sum_{\alpha\in P}\alpha)$ for all $\lambda\in\mathcal{L}'_{\mathfrak{b}}$. So, if $z=\omega+(1+\|\delta\|^2)$, we have $z\in\mathfrak{Z}$, and $\mu_{\mathfrak{g}/\mathfrak{b}}(z)(\lambda)=1+\|\lambda\|^2$, $\lambda\in\mathcal{L}'_{\mathfrak{b}}$.

Let d_λ be the formal degree of $\omega(\lambda)$. We define

$$a_{\lambda,i,j}(x)=d_\lambda^{\frac{1}{2}}(\pi_\lambda(x)e_{\lambda,j},e_{\lambda,i})\quad(x\in G,\ i,j\in N_\lambda).\tag{9.3}$$

Then $\{a_{\lambda,i,j}:\lambda\in\mathcal{L}(p),\ i,j\in N_\lambda\}$ is an orthonormal basis for ${}^0L^2_p(G)$, and one has, for any $f\in L^2(G)$,

$${}^0E_pf=\sum_{\lambda\in\mathcal{L}(p)}\sum_{i,j\in N_\lambda}(f,a_{\lambda,i,j})\,a_{\lambda,i,j}.\tag{9.4}$$

Suppose now that $f\in\mathcal{C}(G)$. If $q>0$ is sufficiently large, $\int_G\Xi(1+\sigma)^{-q}|g|\,dy<\infty$ for each $g\in L^2(G)$. It follows easily from this that the function $x\mapsto\int_G f(xy)g(y)\,dy$ is of class C^∞ for each $g\in L^2(G)$. f is thus a weakly, and hence strongly, differentiable vector for the left regular representation. A similar result is true for the right regular representation also. Since ${}^0E_p$ commutes with both regular representations, ${}^0E_pf$ is also differentiable for both. In particular ${}^0E_pf$ is of class C^∞, and, for $u,v\in\mathfrak{G}$, $v({}^0E_pf)u={}^0E_p(vfu)$; so

$$v({}^0E_pf)\,u=\sum_{\lambda\in\mathcal{L}(p)}\sum_{i,j\in N_\lambda}(ufv,a_{\lambda,i,j})\,a_{\lambda,i,j}.\tag{9.5}$$

We shall now estimate the terms on the right of (9.5). Since $za_{\lambda,i,j}=(1+\|\lambda\|^2)\,a_{\lambda,i,j}$,

$\Omega^m a_{\lambda,i,j}\Omega^m = c^m_{\lambda,i}c^m_{\lambda,j}a_{\lambda,i,j}$, and since both f and $a_{\lambda,i,j}$ are in $\mathcal{C}(G)$, we have, for any integer $m \geqslant 0$,

$$(ufv, a_{\lambda,i,j}) = [c_{\lambda,i}c_{\lambda,j}(1+\|\lambda\|^2)]^{-m}(\Omega^m z^m ufv\Omega^m, a_{\lambda,i,j}). \tag{9.6}$$

On the other hand, we obtain without much difficulty, the following estimate, from (7.39): there are constants $C>0$, $q\geqslant 0$ such that

$$|a_{\lambda,i,j}(x)| \leqslant C[c_{\lambda,i}c_{\lambda,j}(1+\|\lambda\|^2)]^q\,\Xi(x)^{(2/p)+\varepsilon_0} \tag{9.7}$$

for all $\lambda\in\mathcal{L}(p)$, $i, j\in N_\lambda$, $x\in G$ ($\varepsilon_0>0$ as in (7.39)). So, combining (9.6) and (9.7) we have, for any integer $m\geqslant q$ and λ, i, j, x as above,

$$|(ufv, a_{\lambda,i,j})\,a_{\lambda,i,j}(x)| \leqslant C[c_i c_j(1+\|\lambda\|^2)]^{-(m-q)}\,\Xi(x)^{(2/p)+\varepsilon_0}\,\|\Omega^m z^m ufv\,\Omega^m\|_2. \tag{9.8}$$

Choose $m_0>q$ such that

$$C_0 = C\sum_{\lambda\in\mathcal{L}(p)}\sum_{i,j\in N_\lambda}[c_{\lambda,i}c_{\lambda,j}(1+\|\lambda\|^2)]^{-(m_0-q)} < \infty, \tag{9.9}$$

which is clearly possible in view of (9.2). We then have, from (9.5) and (9.8)

$$\sup_{x\in G}\,\Xi(x)^{-((2/p)+\varepsilon_0)}\,|({}^0E_p f)(u; x; v)| \leqslant C_0\,\|\Omega^m z^m ufv\,\Omega^m\|_2, \tag{9.10}$$

for all $f\in\mathcal{C}(G)$. Theorem 9.3 follows at once from (9.10).

References

[1]. BOREL, A., Linear algebraic groups. *Proceedings of Symposia in pure mathematics*, IX (1966), 3–19.

[2]. DIXMIER, J., Représentations intégrables due groupe de De Sitter. *Bull. Soc. Math. France*, 89 (1961), 9–41.

[3]. HARISH-CHANDRA, Representations of semisimple Lie groups, IV. *Amer. J. Math.*, 77 (1955), 743–777.

[4]. —— Representations of semisimple Lie groups, V. *Amer. J. Math.*, 78 (1956), 1–41.

[5]. —— Representations of semisimple Lie groups, VI. *Amer. J. Math.*, 78 (1956), 564–628.

[6]. —— Fourier Transforms on a semisimple Lie algebra, I. *Amer. J. Math.*, 79 (1957), 193–257.

[7]. —— Spherical functions on a semisimple Lie group, I. *Amer. J. Math.*, 80 (1958), 241–310.

[8]. —— Spherical functions on a semisimple Lie group, II. *Amer. J. Math.*, 80 (1958), 553–613.

[9]. —— Some results on differential equations and their applications. *Proc. Nat. Acad. Sci. USA*, 45 (1959), 1763–1764.

[10]. —— Some results on differential equations. (1960) (unpublished).

[11]. —— Differential equations and semisimple Lie groups. (1960) (unpublished).

[12]. —— Invariant eigendistributions on a semisimple Lie group. *Trans. Amer. Math. Soc.*, 119 (1965), 457–508.

[13]. —— Two theorems on semisimple Lie groups. *Ann. Math.*, 83 (1966), 74–128.

[14]. —— Discrete series for semisimple Lie groups, II. *Acta Mathematica*, 116 (1966), 1–111.
[15]. —— Harmonic Analysis on semisimple Lie groups. *Bull. Amer. Math. Soc.*, 76 (1970), 529–551.
[16]. LANGLANDS, R. P., Dimension of spaces of automorphic forms. *Proceedings of symposia in pure mathematics*, IX (1966), 253–257.
[17]. SCHMID, W., On a conjecture of Langlands. *Ann. Math.*, 93 (1971), 1–42.

Received August 9, 1971