# APPROXIMATION OF THE DIRICHLET PROBLEM ON A HALF SPACE

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There have been two general methods used most frequently to study the convergence of finite difference approximations of elliptic boundary value problems—most results in this area are based on an application of either the maximum principle or a variational principle. In this paper we attempt to develop a third approach to this problem. Our philosophy is to imitate as closely as possible the methods that have been developed to handle the differential equation itself. Of course the first step in this program is to study boundary value problems on a half space. Here we consider approximations of the Dirichlet problem on a half space $H$ for an elliptic differential operator of arbitrary (even) order; we do *not* assume that $\partial H$ is aligned with respect to the grid of the difference equation. For a certain class of difference schemes we give a necessary and sufficient condition for the convergence of the approximation. This condition, which involves only the symbols of the operators in the equation and not the operators themselves, is completely analogous to the so-called "covering condition" imposed on the boundary conditions of elliptic differential equations. (See for example [4], p. 125]. The accuracy of the difference schemes considered here is too limited for them to be important computationally, but we hope that our methods may serve as a first step towards a general theory for difference equations, not requiring an intermediate variational formulation and without the limitations associated with the maximum principle.

## § 1. Formulation of the results

Suppose $P(D)$ is an elliptic differential operator on $\mathbf{R}^n$, homogeneous of order $2m$, with constant real coefficients. Consider an approximation to $P(D)$ by a difference operator

$$Q_h(D) = h^{-2m} \sum_{j \in \mathbf{Z}^n} c_j \exp\,(ih\langle j, D\rangle) = h^{-2m} \sum_{j \in \mathbf{Z}^n} c_j T_{hj} \quad \text{(a finite sum)}$$

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where $D=-i(\partial/\partial x)$ and $T_y$ is the translation $T_y\phi(x)=\phi(x+y)$. We shall assume that $Q_h(\xi)$, the symbol of the difference operator, is real; thus $c_{-j}=\bar{c}_j$. We shall say that $Q_h(D)$ is *elliptic* if $Q_h(\xi)>0$ for $\xi\in\mathbf{R}^n\sim 2\pi h^{-1}\mathbf{Z}^n$ and that $Q_h(D)$ is *consistent* with $P(D)$ if

$$\lim_{h\to 0} Q_h(\xi)=P(\xi)$$

for every $\xi\in\mathbf{R}^n$. The symbol $Q_h(\xi)$ is a multiply periodic function on $\mathbf{R}^n$ with period $2\pi/h$; when convenient we shall regard $Q_h(\xi)$ as a function on the torus $\mathbf{T}^n$.

We study the Dirichlet problem

$$(P(D)+\lambda)u=F \text{ in } H,\ \left(\frac{\partial}{\partial N}\right)^k u=f^{(k)} \text{ on } \partial H \text{ for } k=0,1\ldots,m-1, \tag{1.1}$$

where $H=\{x\in\mathbf{R}^n\colon \langle x,N\rangle\geqslant 0\}$, $N$ being a unit vector in $\mathbf{R}^n$. If $\lambda>0$, then for any $F\in S(H)$, the Schwartz space, and for any $f^{(k)}\in S(\partial H)$, (1.1) has a unique solution $u\in S(H)$ – see for example [4]. Now $Q_h(D)$ is a non-local operator, and for $x$ near $\partial H$ the domain of dependence of $Q_h(D)v(x)$ will include points of $\mathbf{R}^n\sim H$. Therefore in approximating (1.1) by a difference equation, we modify the main equation

$$(Q_h(D)+\lambda)v=F \tag{1.2}$$

near the boundary. Choose $j_0\in\mathbf{Z}^n$ such that

$$c_{j_0}\neq 0 \text{ and } \langle j_0,N\rangle=\max\{\langle j,N\rangle : c_j\neq 0\}; \tag{1.3}$$

let $a=\langle j_0,N\rangle$ and let $S=\{x\in\mathbf{R}^n\colon 0\leqslant\langle x,N\rangle<a\}$. We suppose that in the boundary layer $hS$ we are given a difference operator with constant coefficients

$$q_h(D)=\sum_{j\in\mathbf{Z}^n} b_j T_{hj} \quad \text{(a finite sum)} \tag{1.4}$$

and a family of linear maps $\mu_h\colon \oplus_0^{m-1} S(\partial H)\to L^2(hS)$. As our approximation of (1.1), we impose (1.2) for $\langle x,N\rangle\geqslant ah$; and we supplement this equation by the boundary condition

$$q_h(D)v=\mu_h[f] \quad \text{in } hS,$$

where $f=(f^{(0)},\ldots,f^{(m-1)})$ is a $m$-tuple formed from the boundary data of (1.1). Perhaps the simplest example of such a scheme is based on using the first $m$ terms of a Taylor series to approximate $u$ near $\partial H$: let $q_h(D)=I$ and let

$$\mu_h[f](x'+tN)=\sum_{k=0}^{m-1}\frac{t^k}{k!}f^{(k)}(x') \tag{1.5}$$

for $x' \in \partial H$ and $0 \leqslant t < ah$. We shall assume in (1.4) that $b_j = 0$ for $\langle j, N\rangle < 0$, so that for $x \in hS$ the domain of dependence of $q_h(D)v(x)$ is entirely contained in $H$.

If $v \in L^2(\mathbf{R}^n)$, we define a discrete analogue of a Sobolev norm,

$$\|v : L^2(\mathbf{R}^n)\|_h^2 = \sum_{|\alpha| \leqslant m} h^{-2|\alpha|} \int_{\mathbf{R}^n} dx\, |\Delta_h^\alpha v|^2.$$

Here we use a multi-index notation for the finite differences $\Delta_{hk} v(x) = f(x + he_k) - f(x)$. For any measurable subset $\Omega \subset \mathbf{R}^n$ we define

$$\|v : L^2(\Omega)\|_h = \inf \{\|\tilde{v} : L^2(\mathbf{R}^n)\|_h : \tilde{v} = v \text{ in } \Omega\}.$$

This norm, which we often abbreviate to $\|v\|_h$, provides an estimate for the smoothness of $v$ that is not dependent on the non-local differences $\Delta_h^\alpha$ fitting nicely at $\partial\Omega$. We shall call a boundary scheme $(q_h(D), \mu_h)$ *consistent* (with Dirichlet boundary conditions) if for all $\phi \in \mathcal{S}(H)$

$$\lim_{h \to 0} \|q_h(D)\,\phi - \mu_h[p(D)\,\phi] : L^2(hS)\|_h = 0, \tag{1.6}$$

where $p(D)\phi = (\phi, \ldots, (\partial/\partial N)^{m-1}\phi)$ is the Dirichlet data of $\phi$ on $\partial H$, and we shall call a difference scheme *convergent* if the following two conditions are satisfied.

(i) For any $\lambda > 0$, if $h > 0$ is sufficiently small, then

$$\begin{aligned} (Q_h(D) + \lambda)v &= G \quad \text{in } H \sim hS \\ q_h(D)v &= g \quad \text{in } hS \end{aligned} \tag{1.7}$$

is uniquely soluble in $L^2(H)$ for any $G \in L^2(H \sim hS)$, $g \in L^2(hS)$.

(ii) For any $F \in \mathcal{S}(H)$, $f^{(k)} \in \mathcal{S}(\partial H)$, the solution $v_h$ of

$$\begin{aligned} (Q_h(D) + \lambda)v_h &= F \quad \text{in } H \sim hS \\ q_h(D)v_h &= \mu_h[f] \quad \text{in } hS \end{aligned} \tag{1.8}$$

converges to the solution $u$ of (1.1) in the norm $\|\cdot : L^2(H)\|_h$.

In the following theorem, the main result of this paper, we use an extension of the symbol $Q_h(\xi)$ to complex values of the argument. Since $Q_h(\xi) = h^{-2m}Q_1(h\xi)$, it is sufficient to extend only in the case $h = 1$; also, it is more convenient to work with the normalized operator $\hat{Q}(D) = T_{j_0}Q_1(D)$, where $j_0 \in \mathbf{Z}^n$ satisfies (1.3). For $\xi \in T^m$ and $0 \leqslant s < \infty$ let

$$\hat{Q}(\xi, s) = \sum_j c_j \exp\{i\langle j + j_0, \xi\rangle - \langle j + j_0, N\rangle s\}. \tag{1.9}$$

Then $\hat{Q}(\xi, 0)$ is the symbol of $\hat{Q}(D)$ and $\hat{Q}(\xi + tN, s)$ is an analytic function of $t + is$. Moreover by the choice of $j_0$, every term in (1.9) possesses a limit as $s\to\infty$, uniformly in $\xi$, which is constant along the line $\{tN\}$, so we may regard $\hat{Q}$ as a function on

$$\mathcal{M} = T^n \times [0,\infty) \cup \mathbf{T}^n/G \times \{\infty\},$$

where $G$ is the closure of the line $\{tN\}$ in the torus $\mathbf{T}^n$. Similarly we may extend the symbol of $q_h(D)$ to a function $\hat{q}$ on $\mathcal{M}$; here no normalization is required, as $b_j=0$ for $\langle j, N\rangle<0$.

THEOREM 1: *Suppose $Q_h(D)$ is an elliptic difference operator consistent with $P(D)$. The difference scheme $(Q_h(D), q_h(D), \mu_h)$ is convergent for all $\mu_h$ which give consistent boundary conditions if and only if $\hat{Q}$ and $\hat{q}$ do not vanish simultaneously in $\mathcal{M}$.*

We shall say that $q_h(D)$ is *elliptic* with respect to $Q_h(D)$ if $\hat{q}$ and $\hat{Q}$ do not vanish simultaneously in $\mathcal{M}$. The following theorem, which asserts that the difference equation may be solved *stably*, is the basis of the proof that ellipticity is a sufficient condition for convergence.

THEOREM 2: *Suppose $Q_h(D)$ is an elliptic difference operator consistent with $P(D)$. If $q_h(D)$ is elliptic with respect to $Q_h(D)$, then for all small $h$, (1.7) is uniquely soluble in $L^2(H)$ and moreover*

$$\|v\colon L^2(H)\|_h \leqslant C\{\|G\colon L^2(H\sim hS)\| + \|g\colon L^2(hS)\|_h\}$$

*for some constant $C$ independent of $h$.*

Theorems 1 and 2 are proved in § 4 and § 3 respectively. These proofs are based in part on the properties of certain difference operators acting on solutions of the homogeneous equation $(Q_h(D)+\lambda)v=0$ which are studied in § 2.

The principal restriction on the class of difference equations we consider is the assumption that $q_h(D)$ does not depend on the location of $x$ in $hS$. It is natural to assume that $q_h(D)$ is translationally invariant along directions parallel to $\partial H$, but one would like to allow a fairly general dependence on $\langle x, N\rangle$. Our assumption limits the accuracy of the difference equation to lowest order. It is also a restriction that $q_h(D)$ depends on $h$ only through the spacing of its translations. However this is less significant, for it is reasonable to suppose $q_h(D)$ is homogeneous in $h$, and in an approximation of Dirichlet boundary conditions it would be unnatural to have a positive degree of homogeneity.

We note the relation

$$Q_{\alpha h}(D) + \alpha_\alpha^{-2m}\lambda = \alpha^{-2m} J_\alpha^{-1}[Q_h(D)+\lambda]J_\alpha \tag{1.10}$$

for $\alpha>0$, where $J_\alpha$ is the dilation

$$J_\alpha v(x) = \alpha^{-n/2} v(x/\alpha) \tag{1.11}$$

needed in (1.10) to compensate for the fact that the spacing of the translations in $Q_h(D)$ and in $Q_{\alpha h}(D)$ are different. Of course $J_\alpha$ is an isometry of $L^2(H)$ onto itself, and both $J_\alpha$ and its inverse are uniformly bounded with respect to the discrete Sobolev norms. Throughout this paper we consider only the equation $(Q_h(D)+1)v = G$ where $\lambda = 1$, since the solution of the equation for other positive values of $\lambda$ may be reduced to this case by homogeneity.

## § 2. A study of certain operators on the space of solutions of the homogeneous equation

If $y \in \mathbf{R}^n$ satisfies $\langle y, N\rangle \geqslant 0$, let $R_y$ be the restriction of the translation $T_y$ to $L^2(H)$. The translations $\{R_y: \langle y, N\rangle \geqslant 0\}$ form a commutative semi-group, where of course $R_y R_{y'} = R_{y+y'}$. Moreover

$$\mathfrak{X}_h = \{v \in L^2(H): (Q_h(D)+1)v = 0 \quad \text{in } H \sim hs\}$$

is an invariant subspace of $R_y$ for $\langle y, N\rangle \geqslant 0$, being the kernal of

$$R_{hj_0}(Q_h(D)+1) = h^{-2m} \sum_j c_{j-j_0} R_{hj} + R_{hj_0}$$

which commutes with $R_y$.

Let $\mathcal{A}$ be the Banach algebra of functions on $\mathbf{T}^n$ with absolutely convergent Fourier series whose Fourier coefficients $\{a_j: j \in \mathbf{Z}^n\}$ vanish for $\langle j, N\rangle < 0$. By the above remarks we may define a representation $\varrho_h$ of $\mathcal{A}$ on $\mathfrak{X}_h$: let

$$\varrho_h(\Sigma a_j e^{i\langle j,\cdot\rangle}) = \Sigma a_j R_{hj} \,|\, \mathfrak{X}_h.$$

It is clear that $\varrho_h$ is norm-decreasing, where for $\psi \in \mathcal{A}$ we take $\|\psi\|$ to be the absolute sum of the Fourier coefficients of $\psi$ and

$$\|\varrho_h(\psi)\| = \sup_v \{\|\varrho_h(\psi)v\|_h: v \in \mathfrak{X}_h \text{ and } \|v\|_h \leqslant 1\}.$$

The proofs of §§ 3 and 4 will use the properties of the operators $\varrho_h(\psi)$ which are stated be ow in Lemmas 2.1 and 2.2. In these lemmas, if $\psi = \Sigma a_j e^{i\langle j,\cdot\rangle} \in \mathcal{A}$ we define $\hat{\psi}$ as the extension of $\psi$ to a function on $\mathfrak{M}$,

$$\hat{\psi}(\xi, s) = \Sigma a_j \exp\{i\langle j, \xi\rangle - \langle j, N\rangle s\}$$

for $0 \leqslant s < \infty$ and $\hat{\psi}(\xi + G, \infty) = \lim_{s\to\infty} \hat{\psi}(\xi, s)$, and we define $\mathcal{N} = \hat{Q}^{-1}(0) \subset \mathcal{M}$. Below we identify $\mathcal{M}$ with the maximal ideal space of $\mathcal{A}$; that is, we show that every non-zero homomorphism of $\mathcal{A}$ into $\mathbf{C}$ is of the form $\psi \to \hat{\psi}(\eta)$ for some $\eta \in \mathcal{M}$. On an intuitive level, Lemmas 2.1 and 2.2 state that for small $h$, $\mathcal{N}$ is a good approximation of the maximal ideal space of the Banach algebra of operators on $\mathfrak{X}_h$ generated by $\varrho_h(\mathcal{A})$.

LEMMA 2.1: *Suppose $\psi \in \mathcal{A}$; if $\hat{\psi}$ is non-zero on $\mathcal{N}$, then $\varrho_h(\psi)$ is invertible for all small $h$ and*

$$\limsup_{h\to 0} \|\varrho_h(\psi)^{-1}\| < \infty.$$

LEMMA 2.2: *Suppose $\psi \in \mathcal{A}$ and $\eta_0 \in \mathcal{N}$; for any $\varepsilon > 0$, if $h$ is sufficiently small, $\varrho_h(\psi)$ has an approximate eigenvalue in the disk of radius $\varepsilon$ centered at $\hat{\psi}(\eta_0)$.*

*Proof of Lemma 2.1*: We begin the proof by determining the maximal ideal space of $\mathcal{A}$. Of course $\mathcal{A}$ is isomorphic to $l^1(H \cap \mathbf{Z}^n)$ by the Fourier transform, and we quote the results of Arens and Singer [1] concerning the latter algebra. These authors define a character of the semi-group $H \cap \mathbf{Z}^n$ as a continuous, non-zero homomorphism of $H \cap \mathbf{Z}^n$ into the unit disk, and they show that the maximal ideal space of $l^1(H \cap \mathbf{Z}^n)$ is homeomorphic with the space of characters of $H \cap \mathbf{Z}^n$, given the topology of uniform convergence on compact sets. They also show that any character $\zeta$ of $H \cap \mathbf{Z}^n$ admits a polar decomposition

$$\zeta(j) = |\zeta(j)|\, e^{i\langle \xi, j\rangle}$$

where $\xi \in \mathbf{T}^n$.

For each $\eta \in \mathcal{M}$ we define a character of $H \cap \mathbf{Z}^n$ as follows. If $\eta = (\xi, s)$ where $\xi \in \mathbf{T}^n$ and $0 \leqslant s < \infty$, let

$$e^{i\langle \eta, j\rangle} = e^{i\langle \xi, j\rangle - \langle j, N\rangle s}. \tag{2.1}$$

With the convention that $e^{-\infty} = 0$ we may also use (2.1) to define $e^{i\langle \eta, j\rangle}$ when $\eta = (\xi + G, \infty)$ —although $e^{i\langle \xi + G, j\rangle}$ is not defined for all $j$, we need only define this expression when $\langle j, N\rangle = 0$, and this is possible since the dual group of $\partial H \cap \mathbf{Z}^n$ is $\mathbf{T}^n/G$. We claim that any character of $H \cap \mathbf{Z}^n$ is of the form (2.1). Suppose $\zeta$ is a character of $H \cap \mathbf{Z}^n$. If $\langle j, N\rangle \leqslant \langle j', N\rangle$, then

$$|\zeta(j')| = |\zeta(j' - j)|\, |\zeta(j)| \leqslant |\zeta(j)|.$$

It follows that there is an order reversing homomorphism $\pi\colon \Sigma \to [0, 1]$, where

$$\Sigma = \{\langle j, N\rangle : j \in H \cap \mathbf{Z}^n\} \subset [0, \infty),$$

such that $|\zeta(j)| = \pi(\langle j, N\rangle)$. Since any such homomorphism is given by $\pi(\sigma) = e^{-s\sigma}$ for some

$s \in [0, \infty]$, we see from the polar decomposition of $\zeta$ that it is included in (2.1). Thus the characters of $H \cap \mathbf{Z}^n$ are in one-to-one correspondance with $\mathcal{M}$, so we may speak of $\mathcal{M}$ as the maximal ideal space of $\mathcal{A}$.

Note that $\hat{Q}$ is the Gelfand representation of $\hat{Q}(\cdot, 0) \in \mathcal{A}$. Let $\mathcal{J}$ be the closure of the principal ideal in $\mathcal{A}$ generated by $\hat{Q}(\cdot, 0)$. It follows from an elementary computation that $\mathcal{N} = \hat{Q}^{-1}(0)$ is the maximal ideal space of $\mathcal{A}/\mathcal{J}$. (See for example Theorem 3.1.17 on p. 116 of Rickart [5].)

Suppose $\psi$ is an element of $\mathcal{A}$ such that $\hat{\psi}$ is non-zero on $\mathcal{N}$. Then $\psi$ is invertible mod $\mathcal{J}$, so there exists $\psi' \in \mathcal{A}$ such that $\psi\psi' - I \in \mathcal{J}$; since $\mathcal{J}$ is the closure of $\mathcal{A}\,\hat{Q}(\cdot, 0)$,

$$\|\psi\psi' - I - \chi\hat{Q}(\cdot, 0)\| \leq \tfrac{1}{3}$$

for some $\chi \in \mathcal{A}$. Now

$$\|\varrho_h(\psi\psi' - I)\| \leq \|\varrho_h(\psi\psi' - I - \chi\hat{Q}(\cdot, 0))\| + \|\varrho_h(\chi\hat{Q}(\cdot, 0))\|. \tag{2.2}$$

But

$$\varrho_h(\widehat{Q}(\cdot, 0)) = -h^{2m} R_{hj_0} | \mathcal{X}_h.$$

Therefore if $h^{2m} \leq \frac{1}{3}\|\chi\|^{-1}$, it follows from (2.2) that $\varrho_h(\psi)\varrho_h(\psi') = I + A$ for some operator $A$ with $\|A\| \leq \frac{2}{3}$. Thus $\varrho_h(\psi)$ is invertible and

$$\|\varrho_h(\psi)^{-1}\| \leq 3\|\varrho_h(\psi')\| \leq 3\|\psi'\|.$$

The proof is now complete.

It is easily verified that the topology on $\mathcal{M}$ induced by the Gelfand representation of $\mathcal{A}$ is the quotient topology on $\mathbf{T}^n \times [0, \infty)$ obtained by identifying points of the form $(\xi, \infty)$ and $(\xi + \gamma, \infty)$ for $\gamma \in G$.

*Proof of Lemma 2.2*: Let $\mathcal{M}_0 = \mathbf{T}^n \times (0, \infty)$ be the manifold densely contained in $\mathcal{M}$. If $U$ is the (open) upper half plane, we define a smooth immersion $\phi: \partial H \times U \to \mathcal{M}_0$ by

$$\phi(\xi', t + is) = (\xi' + tN + 2\pi\mathbf{Z}^n, s)$$

Using this map we may pull back the function

$$(Q + h^{2m})^\wedge(\eta) = \hat{Q}(\eta) + h^{2m} e^{i\langle j_0, \eta\rangle}$$

defined on $\mathcal{M}$ to a function $\phi^*(Q + h^{2m})^\wedge(\xi', z)$ on $\partial H \times U$. Of course this function is analytic and almost periodic in $z$.

We show below that if for some $\eta \in \mathcal{M}_0$

$$(Q + h^{2m})^\wedge(\eta) = 0, \tag{2.3}$$

then $\hat{\psi}(\eta)$ is an approximate eigenvalue of $\varrho_h(\psi)$. But we claim that any neighborhood of the given point $\eta_0 \in \mathcal{N}$ contains solutions of (2.3) which belong to $\mathcal{M}_0$, providing $h$ is small.

Note that (2.3) is a small perturbation of the equation $\hat{Q}(\eta)=0$ which defines $\mathcal{N}$, and by ellipticity $\phi^*\hat{Q}(\xi', z)$ cannot vanish identically for any $\xi'$. Thus the claim follows from the analyticity of $\phi^*\hat{Q}$, if $\eta_0 \in \mathcal{N} \cap \mathcal{M}_0$. However, by Lemma 2.3 below, $\mathcal{N} \cap \mathcal{M}_0$ is dense in $\mathcal{N}$, so the claim follows for general $\eta_0 \in \mathcal{N}$ by a two-epsilon argument. In this way we obtain approximate eigenvalues of $\varrho_h(\psi)$ close to $\hat{\psi}(\eta_0)$.

Suppose $\eta \in \mathcal{M}_0$ satisifies (2.3). We shall exhibit approximate eigenfunctions of $\varrho_h(\psi)$ in $\mathfrak{X}_h$ as linear superpositions of exponentials,

$$v(x) = \int_{\partial H} d\xi' \, w(\xi') \, e^{i\langle \xi', x\rangle} \, e^{iz(\xi')\langle x, N\rangle} \tag{2.4}$$

where $w \in C_c^\infty(\partial H)$ and $\operatorname{Im} z(\xi') \geqslant \delta > 0$ so that $v \in L^2(H)$. Equation (2.4) defines an element of $\mathfrak{X}_h$ if for each $\xi' \in \operatorname{supp} w$, $z(\xi')$ is a root of $\phi^*(Q+h^{2m})^\wedge(\xi', z)=0$. Now if $\eta = \phi(\xi_0', z_0)$, then $z_0$ is a zero of $\phi^*(Q+h^{2m})^\wedge(\xi_0', z)$ with $\operatorname{Im} z_0 > 0$. By analyticity there is a neighborhood $\mathcal{O}$ of $\xi_0'$ such that for $\xi' \in \mathcal{O}$, $\phi^*(Q+h^{2m})^\wedge(\xi', z)$ has a root $z(\xi')$ near $z_0$. Thus if we take $\operatorname{supp} w \subset \mathcal{O}$ we may obtain an element of $\mathfrak{X}_h$ from (2.4). But

$$\|[\varrho_k(\psi) - \hat{\psi}(\eta)]v\| \leqslant \sup_{\xi \in \operatorname{supp} w} |\phi^*\hat{\psi}(\xi', z(\xi')) - \phi^*\hat{\psi}(\xi_0', z_0)| \, \|v\| \tag{2.5}$$

By further restricting the support of $w$ we may make the right hand side of (2.5) small, so we see that $\hat{\psi}(\eta)$ is an approximate eigenvalue of $\varrho_h(\psi)$.

LEMMA 2.3: *$\mathcal{N} \cap \mathcal{M}_0$ is dense in $\mathcal{N}$.*

*Proof*: We observe that

$$\mathcal{N} \sim (\mathcal{N} \cap \mathcal{M}_0) = (\mathcal{N} \cap \mathbf{T}^n \times \{0\}) \cup (\mathcal{N} \cap \mathbf{T}^n/G \times \{\infty\})$$

By ellipticity $\mathcal{N} \cap \mathbf{T}^n \times \{0\}$ contains only the origin. Now by consistency $\phi^*\hat{Q}(0, z)$ has a zero of order $2m$ at the origin, so for small $\xi' \neq 0$, $\phi^*\hat{Q}(\xi', z)$ has $2m$ zeros near the origin. Half of these zeros must belong to the upper half plane. These latter zeros correspond to points of $\mathcal{N} \cap \mathcal{M}_0$ close to the origin.

Suppose that $\eta_0 = (\xi_0 + G, \infty) \in \mathcal{N}$, or that

$$\lim_{\operatorname{Im} z \to \infty} \phi^*\hat{Q}(\xi_0', z) = 0.$$

If the almost periodic function $f(z) = \phi^*\hat{Q}(\xi_0', z)$ has zeros arbitrarily high in the upper half plane, these yield immediately points in $\mathcal{N} \cap \mathcal{M}_0$ close to $\eta_0$. We may therefore assume that $f(z)$ is non-zero for $\operatorname{Im} z > s_0$. We shall use Lemma A in the appendix to show that in any neighborhood of $\xi_0'$ there exist points $\xi'$ such that $\phi^*\hat{Q}(\xi', z)$ has zeros high in the upper half plane. Of course these zeros correspond to points of $\mathcal{N} \cap \mathcal{M}_0$ close to $\eta_0$.

In our choice of $j_0$ we arranged that at least one term in the Fourier series of $\hat{Q}(\cdot, \infty)$ on $\mathbf{T}^n/G$ was non-zero. Thus $\hat{Q}(\cdot, \infty)$ cannot vanish identically on $\mathbf{T}^n/G$. But $\hat{Q}(\cdot, \infty)$ is a polynomial, so it cannot vanish on any open subset of $\mathbf{T}^n/G$.

It follows from (A.3) that for any sufficiently large $s$ there is an $\varepsilon > 0$ such that $|f(z)| \geqslant \varepsilon$ on the line $\operatorname{Im} z = s$. By continuity for $\xi'$ close to $\xi_0'$

$$|\phi^* \hat{Q}(\xi', z) - \phi^* \hat{Q}(\xi_0', z)| \leqslant \varepsilon/2 \tag{2.6}$$

if $\operatorname{Im} z = s$; moreover, since $\hat{Q}(\cdot, \infty)$ does not vanish on any open subset of $\mathbf{T}^n/G$, we may choose $\xi'$ close to $\xi_0'$ such that

$$\lim_{\operatorname{Im} z \to \infty} \phi^* \hat{Q}(\xi', z) \neq 0. \tag{2.7}$$

Given any neighborhood $O$ of $\xi_0'$, choose $\xi' \in O$ to satisfy (2.6) and (2.7) and let $g(z) = \phi^*\hat{Q}(\xi', z)$. Then $f$ and $g$ satisfy the hypotheses of Lemma A, so by (2.7), $g$ has infinitely many zeros above the line $\operatorname{Im} z = s$. This completes the proof.

## § 3. Existence, uniqueness, and stability

We prove Theorem 2 in this section. First we solve the difference equation in the special case

$$(Q_h(D) + 1) w = G \quad \text{in } H \sim hS \tag{3.1}$$
$$w = 0 \quad \text{in } hS$$

where $q_h(D) = I$ and the boundary condition is homogeneous. An explicit solution of this equation may be obtained with the Wiener-Hopf technique. We then transform the general problem (1.7) to a homogeneous equation by the change of variable $v' = v - w$, where $w$ is the solution of (3.1). Finally we complete the proof of Theorem 2 by solving (1.7) in the homogeneous case.

The Wiener-Hopf factorization of $Q_h(\xi) + 1$ does not present any problem. Since $Q_h(\xi)$ is non-negative, $\log [Q_h(\xi) + 1]$ is a smooth function on $\mathbf{T}^n$ whose Fourier coefficients $\{a_j: j \in \mathbf{Z}^n)$ are rapidly decreasing. Therefore we may write

$$\log [Q_h(\xi) + 1] = \Psi_+(h; \xi) + \Psi_-(h; \xi),$$

where the Fourier coefficients of $\Psi_\pm$ vanish for $\pm\langle j, N\rangle > 0$. We define $Q_\pm(h; \xi) = \exp \Psi_\pm(h; \xi)$. Note that $Q_+(h; \xi)$ and $Q_-(h; \xi)$ are complex conjugates and their product is $Q_h(\xi) + 1$; thus

$$|Q_+(h; \xi)| = |Q_-(h; \xi)| = [Q_h(\xi) + 1]^{\frac{1}{2}}$$

Since $Q_h(D)$ is elliptic, it follows that there exists a constant $C$, independent of $h$, such that

$$C^{-1}(1+s_h(\xi)^m) \leqslant |Q_\pm(h;\xi)| \leqslant C(1+s_h(\xi)^m), \tag{3.2}$$

where $s_h(\xi)^2$ is the symbol of the difference analogue of the Laplacian,

$$s_k(\xi)^2 = h^{-2}\sum_{k=1}^{n} \sin^2(\tfrac{1}{2}h\xi_k).$$

Let $Q_\pm(h; D)$ be the difference operator on $\mathbf{R}^n$ whose symbol is $Q_\pm(h;\xi)$—a multiplication operator in Fourier transform space. Because of (3.2), $Q_\pm(h, D)$ is invertible on $L^2(\mathbf{R}^n)$ for any $h>0$. Indeed, since

$$\left\{\int_{\mathbf{R}^n} d\xi\,(1+s_h(\xi)^m)^2\,|\tilde{v}(\xi)|^2\right\}^{\frac{1}{2}}$$

is a norm on $L^2(\mathbf{R}^n)$ equivalent to the discrete Sobolev norm $\|\cdot\|_h$, one sees that

$$C^{-1}\|v: L^2(\mathbf{R}^n)\|_h \leqslant \|Q_\pm(h;D)v\| \leqslant C\|v: L^2(\mathbf{R}^n)\|_h \tag{3.3}$$

for some constant $C$ independent of $h$.

In attempting to use the Fourier transform to solve (3.1) on a half space, one encounters the usual difficulties of a Wiener-Hopf equation. We remark that with our conventions $L^2(H\sim hS)$ is an invariant subspace of $Q_+(h; D)$. Thus the standard Wiener-Hopf solution of (3.1) is

$$w = \frac{1}{Q_+(h;D)}\,E_h\,\frac{1}{Q_-(h;D)}\,G, \tag{3.4}$$

where $E_h$ is multiplication by the characteristic function of $H\sim hS$. Although we are interested in $w$ only on $H$, in fact (3.4) defines $w$ as a function on $\mathbf{R}^n$ which vanishes on $\mathbf{R}^n\sim H$. It follows immediately from (3.3) that $\|w: L^2(H)\|_h \leqslant C\|G\|$. (Note that only $Q_+(h; D)^{-1}$ contributes to the smoothness of $w$—the smoothness of $Q_-(h;D)^{-1}G$ is destroyed by the projection $E_h$.) Therefore (3.1) may be solved stably for any $G\in L^2(H\sim hS)$, and this solution is unique.

We remark that the equation

$$\begin{aligned}(Q_h(D)+1)w &= 0 \quad \text{in } H\sim hS\\ w &= g \quad \text{in } hS,\end{aligned} \tag{3.5}$$

where the inhomogeneity appears in the boundary condition, may be reduced to (3.1) by the standard trick: extend $g$ to a function $\phi\in L^2(\mathbf{R}^n)$ and let $w' = w-\phi$. Since $\|g: L^2(hS)\|_h$ is defined as a quotient norm, we may choose $\phi$ so that $\|\phi: L^2(\mathbf{R}^n)\|_h = \|g: L^2(hS)\|_h$. One easily computes that

$$w' = -\frac{1}{Q_+(h;D)}\,E_h\,\frac{1}{Q_-(h;D)}\,(Q_h(D)+1)\,\phi = -\frac{1}{Q_+(h;D)}\,E_h Q_+(h;D)\,\phi.$$

It follows from (3.3) that the unique solution of (3.5) satisfies $\|w\|_h \leq C\|g\|_h$.

Suppose $q_h(D) = \Sigma b_j T_{hj}$ is a boundary difference operator. We have assumed that $b_j = 0$ for $\langle j, N\rangle < 0$. Hence the symbol of $q_h(D)$ determines a function $q(\xi) = \Sigma b_j e^{i\langle j,\xi\rangle}$ in the algebra $\mathcal{A}$ of § 2 whose Gelfand representation is $q$. Also note that $\varrho_h(q) = q_h(D)\,|\,\mathfrak{X}_h$. The following lemma, which shows that solutions to the homogeneous problem

$$\begin{aligned} (Q_h(D)+1)v &= 0 \quad \text{in } H \sim hS \\ q_h(D)v &= g \quad \text{in } hS \end{aligned} \tag{3.6}$$

may be obtained from an inverse of $\varrho_h(q)$, allows us to apply the results of § 2 in solving (3.6).

Lemma 3.1: *Equation* (3.6) *is uniquely soluble in* $L^2(H)$ *for any* $g \in L^2(hS)$ *if and only if* $\varrho_h(q)$ *is invertible on* $\mathfrak{X}_h$. *Moreover* (3.6) *may be solved stably if and only if*

$$\limsup_{h\to 0} \|\varrho_h(q)^{-1}\| < \infty.$$

*Proof*: First suppose that $\varrho_h(q)$ is invertible. If $g \in L^2(hS)$, let $w$ be the solution of (3.5). Then $v = \varrho_h(q)^{-1}w$ belongs to $\mathfrak{X}_h$ and satisfies the boundary condition $q_h(D)v\,|\,hS = w\,|\,hS = g$. Thus (3.6) has at least one solution. If $v$ and $v'$ are both solutions of (3.6), then $\varrho_h(q)v$ and $\varrho_h(q)v'$ are both solutions of (3.5). Since the solution of (3.5) is unique, $\varrho_h(q)v = \varrho_h(q)v'$, and by invertibility $v = v'$.

On the other hand, suppose (3.6) is uniquely soluble for all data $g$. If $w \in \mathfrak{X}_h$, let $v$ be the solution of (3.6) with boundary data $g = w\,|\,hS$. Then $\varrho_h(q)\,v$ and $w$ are both solutions of (3.5) with the same boundary data. Hence $\varrho_h(q)v = w$, so $\varrho_h(q)$ is surjective. If $\varrho_h(q)v = 0$, then $v$ is a solution of (3.6) with homogeneous boundary data, so $v = 0$. Thus $\varrho_h(q)$ is also injective, and therefore invertible.

These considerations may easily be extended to cover the question of stability in solving (3.6), so the proof is complete.

It is now trivial to prove Theorem 2. By the reduction presented at the beginning of this section, it suffices to solve (1.7) in the homogeneous case (3.6). Suppose that $q_h(D)$ is elliptic with respect to $Q_h(D)$. Then $\hat{q}$ is non-vanishing on $\mathcal{N} = Q^{-1}(0)$, so by Lemma 2.1, $\varrho_h(q)$ is invertible for small $h$. Of course, by Lemma 3.1, this implies that (3.6) is uniquely soluble for small $h$. It also follows from these lemmas that (3.6) may be solved stably.

This completes the proof of Theorem 2.

## § 4. Proof of the main theorem

In this section we prove Theorem 1. The proof that ellipticity is a sufficient condition for convergence involves only a simple application of Theorem 2. Indeed, suppose $q_h(D)$ is elliptic with respect to $Q_h(D)$. It follows from Theorem 2 that for any $v \in L^2(H)$

$$\|v: L^2(H)\|_h \leq C\{\|(Q_h(D)+1)v: L^2(H \sim hS)\| + \|q_h(D)v: L^2(hS)\|_h\}. \tag{4.1}$$

Let $\mu_h$ be a consistent data map; given $F \in \mathcal{S}(H)$ and $f^{(k)} \in \mathcal{S}(\partial H)$, let $u$ be the solution of (1.1) and let $v_h$ be the solution of (1.8). Then by (4.1)

$$\|u - v_h\|_h \leq C\{\|(Q_h(D)+1)u - (P(D)+1)u\| + \|q_h(D)u - \mu_h[p(D)u]\|_h\}; \tag{4.2}$$

here we have used $$(Q_h(D)+1)v_h = F = (P(D)+1)u$$

and an analogous equation for $q_h(D)v_h$ to simplify (4.2). Now $u$ belongs to $\mathcal{S}(H)$. Since $Q_h(D)$ is consistent with $P(D)$, the first term on the right of (4.2) tends to zero with $h$; the second term of (4.2) also tends to zero by the consistency of the boundary conditions. Thus $v_h$ converges to $u$ in the discrete Sobolev norm. This shows that ellipticity is a sufficient condition for convergence.

Before continuing the proof of Theorem 1 we show that for any boundary difference operator $q_h(D)$ there is a data map $\mu_h$ which gives consistent boundary conditions. Indeed if $q_h(D) = \Sigma b_j T_{hj}$, then for $f \in \oplus_0^{m-1} \mathcal{S}(\partial H)$ let

$$\mu_h[f](x' + tN) = \sum_{j \in \mathbf{Z}^n} b_j \sum_{k=0}^{m-1} \frac{(t + h\langle j, N\rangle)^k}{k!} f^{(k)}(x' + hj'), \tag{4.3}$$

where we write $j' = j - \langle j, N\rangle N$. With this definition, if $\phi \in \mathcal{S}(H)$, then $\mu_h[p(D)\phi]$ approximates $q_h(D)\phi$ to $O(h^m)$ in the boundary layer $hS$. However, it follows immediately from the definition of $\|\cdot\|_h$ that for any $g \in L^2(hS)$

$$\|g: L^2(hS)\|_h \leq C h^{-(m-\frac{1}{2})} \left\{ \int_{\partial H} dx' \int_0^a dt\, |g(x' + htN)|^2 \right\}^{\frac{1}{2}}.$$

Thus if $\phi \in \mathcal{S}(H)$

$$\|q_h(D)\phi - \mu_h[p(D)]\phi : L^2(hS)\|_h = O(h^{\frac{1}{2}}).$$

Therefore (4.3) gives rise to a consistent boundary approximation.

If the difference equation (1.8) is convergent for some $\mu_h$, then in particular the homogeneous equation (3.6) may be solved for any $g \in L^2(hS)$, if $h$ is small enough. In the following lemma we show that if the difference equation is convergent for *all* consistent $\mu_h$, then (3.6) may be solved stably.

LEMMA 4.1: *If the difference equation is convergent for all consistent* $\mu_h$, *then there exists a constant* $C$ *such that for small* $h$ *the solution of* (3.6) *satisfies*

$$\|v\|_h \leqslant C\|g\|_h. \tag{4.4}$$

*Proof*: Suppose that (4.4) does not hold; that is, suppose there exists a sequence $\{h_k\}$ decreasing to zero and a sequence $\{g_k\}$, where $g_k \in L^2(h_k S)$, such that

$$\|v_k\|_{h_k} = 1 \text{ but } \|g_k\|_{h_k} \to 0.$$

Here of course $v_k$ is the solution of (3.6) with data $g_k$. For the remainder of the proof we shall omit the subscript "$h_k$" from these norms. We define a data map $\nu_h$: $\oplus_0^{m-1} S(\partial H) \to L^2(hS)$ as follows: choose a non-zero linear functional $l$ on $\oplus_0^{m-1} S(\partial H)$; if $h_k \leqslant h < h_{k-1}$, let

$$\nu_h[f] = l(f)\, J_{h/h_k} g_k,$$

where $J_\alpha$ is the dilation (1.11). Then if $\mu_h$ is a consistent data map, so is $\mu_h + \nu_h$, since $\|g_k\| \to 0$. Choose $f \in \oplus_0^{m-1} S(\partial H)$ such that $l(f) = 1$, and let $w_h$, $w_h'$ be the solution of (3.6) with boundary data $\mu_h[f]$, $(\mu_h + \nu_h)[f]$ respectively. Then

$$\|w_{h_k} - w'_{h_k}\| = \|v_k\| = 1,$$

so not both $w_h$ and $w_h'$ can converge to the solution of the continuous problem. Therefore if (4.4) does not hold, the difference scheme cannot converge for all consistent $\mu_h$. This completes the proof of the lemma.

We may now prove that ellipticity is also a necessary condition for convergence. Suppose the difference equation is convergent for all consistent $\mu_h$. It follows from Lemmas 3.1 and 4.1 that

$$\limsup_{h\to 0} \|\varrho_h(q)^{-1}\| < \infty.$$

Now if $\eta$ belongs to $\mathcal{N} = \hat{Q}^{-1}(0)$, then by Lemma 2.2, for any $\varepsilon > 0$, $\varrho_h(q)$ has an approximate eigenvalue $\lambda$ with $|\lambda| \leqslant |\hat{q}(\eta)| + \varepsilon$, providing $h$ is small. Therefore for small $h$, the spectral radius of $\varrho_h(q)^{-1}$ is at least $\{|\hat{q}(\eta)| + \varepsilon\}^{-1}$. Of course the spectral radius of $\varrho_h(q)^{-1}$ is dominated by the norm of this operator, so

$$\frac{1}{|\hat{q}(\eta)| + \varepsilon} \leqslant \limsup_{h\to 0} \|\varrho_h(q)^{-1}\| = C < \infty. \tag{4.5}$$

But since (4.5) holds for every $\varepsilon > 0$, we have $|\hat{q}(\eta)| \geqslant C^{-1} > 0$. Thus $\hat{q}$ is non-vanishing on $\hat{Q}^{-1}(0)$, so $\hat{q}$ and $\hat{Q}$ do not vanish simultaneously on $\mathcal{M}$.

The proof of our main theorem is now complete.

## APPENDIX

Our purpose here is to prove the lemma below. Informally, it asserts that the zeros at infinity of an analytic, almost periodic function are stable with respect to small perturbations. We begin by recalling certain results from the theory of almost periodic functions. (See Chapter VI, §§ 1–3 of Levin [3].)

Suppose $f$ is an analytic, almost periodic function in the upper half plane. If $f$ is non-vanishing in a strip $\{z: |\operatorname{Im} z - s| < \delta\}$, then the limit

$$\theta(s) = \lim_{T\to\infty} (2T)^{-1}\{\arg f(T+is) - \arg f(-T+is)\}$$

defining the mean winding number of $f$ along the line $\operatorname{Im} z = s$ exists. If the mean winding number of $f$ exists along two such lines, say $\operatorname{Im} z = s_1$, and $\operatorname{Im} z = s_2$ with $s_1 < s_2$, then

$$\theta(s_1) - \theta(s_2) = 2\pi D(s_1, s_2), \tag{A.1}$$

where $D(s_1, s_2)$ is the density of the zeros of $f$ in the strip $\{z: s_1 < \operatorname{Im} z < s_2\}$: that is, let $D(s_1, s_2; T)$ be the number of zeros of $f$ in the rectangle $\{z: s_1 < \operatorname{Im} z < s_2, |\operatorname{Re} z| < T\}$ and let $D(s_1, s_2) = \lim_{T\to\infty} (2T)^{-1} D(s_1, s_2; T)$. (Under the above hypothesis this limit exists.)

Suppose moreover that $f$ is bounded in the upper half plane. A leading term may be extracted from the Fourier series of $f$, say

$$f(z) = a_0 e^{i\lambda_0 z} + \sum_{k=1}^{\infty} a_k e^{i\lambda_k z} \tag{A.2}$$

where $a_0 \neq 0$ and $0 \leq \lambda_0 < \lambda_k$ for $k \geq 1$, if and only $f$ is non-vanishing for $\operatorname{Im} z$ sufficiently large, say $\operatorname{Im} z > s_0$. In this case

$$e^{-i\lambda_0 z}[f(z) - a_0 e^{i\lambda_0 z}] \to 0 \tag{A.3}$$

as $\operatorname{Im} z \to \infty$, uniformly in $\operatorname{Re} z$. Thus the mean winding number $\theta(s)$ exists for $s > s_0$ and) $\theta(s) = \lambda_0$. However, whether or not $f$ may be written in the form (A.2), as $\operatorname{Im} z \to \infty$, $f(z$ tends to a definite limit, uniformly in $\operatorname{Re} z$.

LEMMA A: *Let $f$ and $g$ be bounded, analytic, almost periodic functions in the upper half plane. Suppose that $f$ is non-vanishing for* $\operatorname{Im} z > s_0$ *but that*

$$\lim_{\operatorname{Im} z\to\infty} f(z) = 0. \tag{A.4}$$

*If $|g - f| < |f|$ along some line* $\operatorname{Im} z = s_1 > s_0$, *then either* $\lim g(z) = 0$ as $\operatorname{Im} z \to \infty$ *or $g$ has infinitely many zeros above this line.*

*Proof:* Since $f$ is non-vanishing for $\operatorname{Im} z > s_0$, we may extract a leading term $a_0 e^{i\lambda_0 z}$ from the Fourier series of $f$. By (A.4), $\lambda_0 > 0$. Thus $\theta_f(s_1) = \lambda_0 > 0$, where $\theta_f(s_1)$ is the mean winding number of $f$ along $\operatorname{Im} z = s_1$. But $|g - f| < |f|$ so $|\arg g - \arg f| < \pi$; therefore the mean winding number of $g$ along $\operatorname{Im} z = s_1$ also exists and $\theta_g(s_1) = \theta_f(s_1) > 0$.

Suppose $\lim g(z) \neq 0$ as $\operatorname{Im} z \to \infty$. Then $g(z)$ must be non-vanishing for large $\operatorname{Im} z$, say $\operatorname{Im} z > s_2 - \delta$. Thus a leading term $b_0 e^{i\mu_0 z}$ may also be extracted from the Fourier series of $g$, and moreover $\mu_0 = 0$. Hence the mean winding number $\theta_g(s_2)$ must vanish.

We have shown that $\theta_g(s_1) - \theta_g(s_2) > 0$. By (A.1) the zeros of $g$ in the strip $\{z: s_1 < \operatorname{Im} z < s_2\}$ have a positive density. This completes the proof.

## References

Reference [2] may be helpful to the reader unfamiliar with the Wiener-Hopf theory. In [6] Thomée proved the convergence of the difference scheme described in formula (1.5) of this paper; elliptic difference operators were first defined by him.


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