VECTOR FIELDS WITH FINITE SINGULARITIES

BY

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I. Introduction

In this paper we give some generalizations of the famous theorem of H. Hopf which states that the number of singularities of a tangent vector field on a compact smooth manifold is equal to the Euler characteristic. Instead of a single vector field we consider r vector fields $u_1, \dots, u_r$ and we are interested in their "singularities", that is, the set Σ of points on the manifold at which they become linearly dependent. In general Σ will have dimension $r-1$, it is a cycle(1) and its homology class is the $(n-r+1)$th Stiefel-Whitney class of the manifold. This is the standard primary obstruction theory and it provides one way of generalizing the classical Hopf Theorem. However, this theory says nothing about Σ if $\dim \Sigma < r-1$. In this paper following E. Thomas [20] we shall generalize the Hopf theorem by considering the other extreme case in which Σ is finite, so that $\dim \Sigma = 0$. General homotopy theory tells us that we are now involved in higher order obstruction theory and that the situation is much more complicated, as we shall now explain.

For each point $A \in \Sigma$ we have a local obstruction(2)

$$O_A(u_1, \dots, u_r) \in \pi_{n-1}(V_{n,r})$$

where $V_{n,r} = SO(n)/SO(n-r)$ is the Stiefel manifold of orthogonal r-frames in $\mathbf{R}^n$. In local coordinates $(x_1, \dots, x_n)$ with origin A, O_A is just the homotopy class of the map of a small sphere $\sum x_i^2 = \varepsilon$ into(3) $W_{n,r} = GL(n, \mathbf{R})/GL(n-r, \mathbf{R})$ given by $x \mapsto u_1(x), \dots, u_r(x)$. The vanishing of O_A is the necessary and sufficient condition that we can deform $u_1, \dots, u_r$

(1) With integer or mod 2 coefficients depending on the parity of r.

(2) Thomas calls O_A the index at A. Since our methods involve using the index theory of elliptic operators we prefer a different terminology.

(3) As is well-known $V_{n,r} \to W_{n,r}$ is a homotopy equivalence (equivalently every r-frame can be naturally orthogonalized).

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near A (remaining fixed outside a ball of radius ε) so that this singularity disappears. O_A is independent of the choice of coordinates but does depend on the orientation.(1) Hence for an oriented manifold we can form the global obstruction $\Sigma_i O_{A_i}(u_1, \dots, u_r) \in \pi_{n-1}(V_{n,r})$ where we sum over all the singular points A_i. For $r=1, \pi_{n-1}(V_{n,r}) = \pi_{n-1}(S^{n-1}) \cong Z$, the local obstruction is an integer—the multiplicity of the singularity. In this case, because the obstruction is primary we know that

(a) the global obstruction is independent of the vector field u,

(b) the vanishing of the global obstruction is the necessary and sufficient condition for the existence of a vector field without singularities.

Moreover, the Hopf theorem identifies this global obstruction with the Euler characteristic. In the general case (for $r > 1$) the obstruction is not primary, so that (a) and (b) need not hold. A general theorem of Hopf type, identifying the global obstruction, is therefore not to be expected. Instead, what we shall do is to define certain homomorphic images $\theta\pi_{n-1}(V_{n,r})$ of the homotopy groups $\pi_{n-1}(V_{n,r})$ and then identify the sum $\Sigma_i \theta O_{A_i}(u_1, \dots, u_r)$ with a global invariant of the manifold. In particular this will show that the image under θ of the global obstruction is independent of the vector fields $u_1, \dots, u_r$. Of course, such a result is only of genuine interest if θ is sufficiently non-trivial. In fact, it will turn out that θ is an isomorphism for $r \leqslant 3 \leqslant n-r$ and is a projection onto a large direct factor for n divisible by a suitable power of 2 (depending on r). Moreover, there is some indication that our results are best possible in the sense that $\theta\pi_{n-1}(V_{n,r})$ may be the largest homomorphic image in which the global obstruction becomes independent of the vector fields.

The global invariants of manifolds which occur in our generalized Hopf theorems are all simple combinations of the Euler characteristic E, the Hirzebruch signature S and the (real) Kervaire semi-characteristic(2) R. Now E and S are indices of certain elliptic differential operators on the manifold and R is a "mod 2 index" of a certain skew-adjoint elliptic operator. This analytical interpretation of E, S, R was used in [5] to prove weak theorems of Hopf type, namely that the existence of r vector fields *without singularities* implies the vanishing of certain global invariants of the manifold. In the present paper we essentially refine the methods of [5] to derive the corresponding strong theorems (allowing finite singularities).

The basic idea, explained already in [5; § 5], is to pass from elliptic operators to their symbols which are elements of certain K-groups. The index theorem, in its various forms,

(1) Reversing the orientation replaces O_A by $-\tau O_A$ where τ is the involution induced on the homotopy groups of $V_{n,r}$ by the outer automorphism of $SO(n)$.

(2) Defined as $\sum_p \dim_{\mathbf{R}} H^{2p}(X; \mathbf{R}) \bmod 2$, for $\dim X \equiv 1 \bmod 4$.

asserts that the analytical index (E, S or R) can be computed purely in terms of K-theory from these symbols. Roughly speaking, we can say that these symbols are certain K-theory characteristic classes and that their indices E, S, R are the corresponding K-theory characteristic numbers. The existence of r vector fields without singularities implies divisibility by some 2^ν for the symbols of our operators and hence for their indices—this gives the weak theorems of [5]. If the r vector fields have finite singularities $\{A_i\}$ then our symbols are only divisible by 2^ν outside the A_i and so we may expect to get relative K-theory characteristic classes modulo 2^ν for the pair $(X, X - \bigcup_i A_i) \sim \Sigma_i (X, X - A_i)$. Then for each i we will get a local characteristic number modulo 2^ν. On the one hand this is some function of the local obstruction. On the other hand it is clear that the sum of these local numbers modulo 2^ν will equal the global index modulo 2^ν of our original operator. This will then give the Hopf type theorem we want.

The preceding heuristic discussion would suggest that we introduce K-theory with coefficients in the integers mod 2^ν by using a Moore space. In fact, there is a more natural choice of coefficient theory which gives better results and that is to consider the functors $KR^*(X \times P_k, X \times P_l)$ where P_k is real projective k-space, and k, l are appropriate integers. More precisely, writing n in the form $4k - s$ we shall define homomorphisms

$$\theta^s : \pi_{n-1}(V_{n,r}) \to KR^s(P_{r+s-1}, P_{s-1}).$$

The groups $A_r^s = KR^s(P_{r+s-1}, P_{s-1})$ are all tabulated in § 3 and, as we have already indicated, θ^s will be an isomorphism for $r \leqslant 3 \leqslant n - r$. Moreover, we shall define homomorphisms $\gamma_r^s : B^s \to A_r^s$ where $B^0 = Z \oplus Z$, $B^1 = 0$, $B^2 = Z$, $B^3 = Z_2$. With this notation our main theorem reads as follows:

THEOREM 1.1. *Let X be a compact oriented smooth manifold of dimension $n = 4k - s$, and let $u_1, \dots, u_r$ be r tangent vector fields with finite singularities $\{A_i\}$. Then*

$$\sum_i \theta^s O_{A_i}(u_1, \dots, u_r) = \gamma_r^s(b^s(X))$$

where $b^s(X) \in B^s$ is defined as follows:

$$b^0 = E \oplus \tfrac{1}{2}(E - (-1)^k S)$$
$$b^1 = 0$$
$$b^2 = \tfrac{1}{2} E$$
$$b^3 = R$$

where E, S, R are respectively the Euler characteristic, the Hirzebruch signature and the (real) Kervaire semi-characteristic of X.

This theorem includes, as special cases, many results obtained in this direction by E. Thomas and D. Frank. For a full survey of the subject we refer to the expository article [20] by Thomas. The paper [5] can also be read as an introduction to our present more detailed work.

An interesting fact about the homomorphism θ^s is that it is essentially the same as the Hurewicz homomorphism $\pi_* \to KR_*$ for the homology functor KR_*. This will be established in the final section of the paper. Although we shall not use this fact anywhere, it does provide a further justification for our choice of coefficient theory.

The proof of Theorem (1.1) occupies the next three sections. In § 2 we construct our basic relative characteristic classes and end up with the abstract K-theory version of Theorem (1.1). That is to say, the equation holds in the (uncomputed) group $A_r^s = KR^s(P_{r+s-1}, P_{s-1})$ and the global invariant $b^s(X)$ is also not explicitly computed. These computations are carried out in sections 3 and 4. In § 3 we compute all the groups A_r^s using an elegant result of G. Segal. In § 4 we relate our characteristic classes with symbols of explicit operators and then apply the index theorem to derive the identification of $b^s(X)$ given in Theorem (1.1).

In § 5 we study the local homomorphism θ^s for particular values of n, r and prove the non-triviality statements referred to earlier. In particular, for low values of r, this makes Theorem (1.1) quite explicit and these cases are then tabulated.

Sections 6 and 7 are devoted to various extensions and refinements of Theorem (1.1) in the two specially interesting dimensions $4k$, $4k+1$. In each case we derive a formula for a field of oriented 2-planes with finite singularities, the method of proof being essentially as in (1.1). In dimension $4k$, for r vector fields with r divisible by 4 there is an extra power of 2 which provides a strengthened variant of (1.1). In dimension $4k+1$ we also consider, for the first time, non-orientable manifolds. We show that, if $w_1^2 = 0$, there is an analogue of Theorem (1.1) involving a semi-characteristic based on cohomology with coefficients in a local coefficient system. Finally, we show in § 6, how Theorem (1.1) can be used to define an interesting invariant for vector fields without singularities on $(4k-1)$-manifolds.

2. The basic construction

All K-theory of locally compact spaces is K-theory with compact support in the sense of [9]. We recall that an element in $K(X)$ for X locally compact is given by a complex of vector bundles

$$0 \longrightarrow E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \dots \longrightarrow E^n \longrightarrow 0 \qquad (2.1)$$

which is exact outside a set $U \subset X$ such that the closure $\bar{U}$ is compact. If $A \subset X$ such that $A \subset X - U$ then (2.1) gives an element in $K(X-A) = K(X, A)$.

Now consider a double complex $\{D^{i,j}\}_{i \leqslant n_1, j \leqslant n_2}$, of vectorbundles with differentials $d_0^{ij}: D^{i,j} \to D^{i+1,j}$ and $d_1^{ij}: D^{i,j} \to D^{i,j+1}$. Suppose that (D^{*j}, d_0^{*j}) is exact outside $U_0 \subset X$ and that (D^{i*}, d_1^{i*}) is exact outside U_1 for every i and j. Let $A \subset X - U_0$ and $B \subset X - U_1$, and consider the total complex (E^s, d^s) where $E^s = \sum_{i+j=s} D^{ij}$ and d^s is given by the formula

$$d^s x = d_0^{ij} x + (-1)^i d_1^{i,j} x \text{ for } x \in D^{ij}.$$

This complex is exact outside $U = U_0 \cap U_1$, so if $\bar{U}_0 \cap \bar{U}_1$ is compact, we get an element in $K(X - (A \cup B)) = K(X, A \cup B)$.

In particular a commutative square

$$\begin{array}{ccc} D^{0,0} & \xrightarrow{d_0^{0,0}} & D^{1,0} \\ {\scriptstyle d_1^{0,0}}\downarrow & & \downarrow{\scriptstyle d_1^{1,0}} \\ D^{0,1} & \xrightarrow{d_0^{0,1}} & D^{1,1} \end{array} \tag{2.2}$$

where d_0^* are isomorphisms outside U_0 and d_1^* are isomorphisms outside U_1, gives an element in $K(X - (A \cup B))$ provided $\bar{U}_0 \cap \bar{U}_1$ is compact. In fact, the square (2.2) is equivalent to the complex

$$0 \longrightarrow D^{0,0} \xrightarrow{\alpha} D^{0,1} \oplus D^{1,0} \xrightarrow{\beta} D^{1,1} \longrightarrow 0 \tag{2.3}$$

where $\alpha = d_1^{0,0} \oplus d_0^{0,0}$ and $\beta = d_0^{0,1} - d_1^{1,0}$. This in turn is equivalent to the triple

$$(D^{0,0} \oplus D^{1,1}, D^{0,1} \oplus D^{1,0}, \gamma),$$

where γ is given by the matrix

$$\begin{pmatrix} d_1^{0,0} & (d_0^{0,1})^* \\ d_0^{0,0} & -(d_1^{1,0})^* \end{pmatrix}. \tag{2.4}$$

Our basic construction depends on a certain square like (2.2) which arises in the following way. Consider the real vectorspace $V = \mathbf{R}^{2n}$, and let

$$\Lambda^*(V) = \sum_{p=0}^{2n} \Lambda^p(V) \otimes \mathbf{C}$$

be the total complex exterior vectorspace, but with the *Clifford* multiplication (see [5, §2]). If $v \in \Lambda^*(V)$, L_v and R_v denote left- and right-multiplication by v respectively. The endomorphism $\tau = (i)^{n^2} L_\omega$, where $\omega = e_1 \dots e_{2n}$ is the volume element, satisfies $\tau^2 = 1$. Let

Λ_+^* and Λ_-^* denote the eigenspaces of τ. Over the tangent space $T\mathbf{R}^{2n} = \mathbf{R}^{2n} \oplus \mathbf{R}^{2n}$ consider the square with trivial vectorbundles:

$$\begin{tikzcd} \Lambda_+^{\mathrm{ev}} \arrow[r, "iL_v"] \arrow[d, "R_u"'] & \Lambda_-^{\mathrm{odd}} \arrow[d, "R_u"] \\ \Lambda_+^{\mathrm{odd}} \arrow[r, "iL_v"] & \Lambda_-^{\mathrm{ev}} \end{tikzcd} \tag{2.5}$$

Over a point (u, v), with u in the base and v in the fibre of $T\mathbf{R}^{2n}$, the vertical maps in (2.5) are R_u and the horizontal maps are iL_v. In this way we obtain an element in $K(T\mathbf{R}^{2n}) = K^{4n}(\text{point})$, and we shall now show that it is the canonical generator. From (2.3) and (2.4) it follows that the element is given by the triple

$$(\Lambda_+^{\mathrm{ev}} \oplus \Lambda_-^{\mathrm{ev}}, \Lambda_+^{\mathrm{odd}} \oplus \Lambda_-^{\mathrm{odd}}, \gamma)$$

where γ is given over a point (u, v) by the matrix

$$\begin{pmatrix} R_u & iL_v \\ iL_v & R_u \end{pmatrix}. \tag{2.6}$$

Thus the element is defined by the Clifford module Λ^* for $\mathrm{Cliff}(\mathbf{R}^{2n} \oplus \mathbf{R}^{2n})$, where the action $\Lambda^{\mathrm{ev}} \to \Lambda^{\mathrm{odd}}$ is given by (2.6) and the action $\Lambda^{\mathrm{odd}} \to \Lambda^{\mathrm{ev}}$ is given by the conjugate matrix. It is easily verified that this is $a+1$-module in the sense of [6], and hence our element is the canonical generator of $K(\mathbf{C}^{2n})$.

If $n = 2k$ is even then $\Lambda_\pm^*$ are Real vectorspaces, i.e. invariant under conjugation, so if we give $T\mathbf{R}^{4k}$ the antipodal involution along the fibres, the square (2.5) gives rise naturally to an element in $KR(T\mathbf{R}^{4k})$.

The square (2.5) will be the basis of our general construction of characteristic classes or symbols. Its particular form is motivated by consideration of the elliptic operators occurring on $\mathbf{R}^{2n}$. In fact, the horizontal rows are the symbols of two basic elliptic operators on $\mathbf{R}^{2n}$ (see § 4) and the vertical arrows define an isomorphism between these two operators outside the origin. This background may help to explain the lack of symmetry between the role of u and v components in (2.5).

We now proceed to consider the global analogue of the local square (2.5). Let E denote an arbitrary real oriented vectorbundle of dimension $4k$ over a compact space X, and suppose we have r sections $\{u_1, \dots, u_r\}$ of E which are linearly independent over a closed set $Y \subset X$. Giving E a Riemannian metric, we can assume that the sections are actually orthonormal over Y, and this is sometimes convenient, but for the moment we need not make this assumption.

Let iE denote the Real space E (in the sense of [3]) with antipodal involution on the fibres. Using the metric we can form the Real bundles $\Lambda^*_\pm(E)$, which we pull back over the Real space $iE \times P_{r-1}$, where P_{r-1} is $(r-1)$-dimensional projective space with trivial involution.

Now consider a point $(v, x) \in E \times S^{r-1}$, where v is in the fibre over $y \in X$, and put $x(y) = \sum_{i=1}^{r} x_i u_i(y)$. Again using left and right Clifford-multiplication, we can consider the square

$$\begin{array}{ccc} \Lambda^{\mathrm{ev}}_+(E) & \xrightarrow{iL_v} & \Lambda^{\mathrm{odd}}_-(E) \\ {\scriptstyle R_{x(y)}}\downarrow & & \downarrow{\scriptstyle R_{x(y)}} \\ \Lambda^{\mathrm{odd}}_+(E) & \xrightarrow{iL_v} & \Lambda^{\mathrm{ev}}_-(E) \end{array} \tag{2.7}$$

This defines a square of Real vectorbundles and homomorphisms over $iE \times S^{r-1}$. Clearly the maps are Z_2-equivariant with respect to the antipodal involution on S^{r-1} and the action on the bundles, defined by the trivial action on the upper row and multiplication by (-1) on the lower. Hence over $iE \times P_{r-1}$ we have the square

$$\begin{array}{ccc} \Lambda^{\mathrm{ev}}_+(E) & \xrightarrow{iL_v} & \Lambda^{\mathrm{odd}}_-(E) \\ {\scriptstyle R_{x(y)}}\downarrow & & \downarrow{\scriptstyle R_{x(y)}} \\ \Lambda^{\mathrm{odd}}_+(E)\otimes H & \xrightarrow{iL_v} & \Lambda^{\mathrm{ev}}_-(E)\otimes H \end{array} \tag{2.8}$$

where(1) H denotes the Hopf bundle over P_{r-1}. The horizontal maps are isomorphisms over $(E-X) \times P_{r-1}$ and the vertical maps are isomorphisms over $(E|Y) \times P_{r-1}$. Hence we obtain an element

$$\alpha_E(u_1, \dots, u_r) \in KR((iE|X-Y) \times P_{r-1}). \tag{2.9}$$

If furthermore $\{u_1, \dots, u_s\}$ are linearly independent over the whole of X, then the vertical maps of (2.8) are isomorphisms over $E \times P_{s-1} \cup (E|Y) \times P_{r-1}$, and thus the square defines an element in $KR((iE|X-Y) \times (P_{r-1}-P_{s-1}))$.

When the dimension n of E is not divisible by 4 we form the bundle $F = E \oplus \mathbf{R}^s$ where $n+s=4k$. Then F has $r+s$ sections $\{u_{-s+1}, \dots, u_0, u_1, \dots, u_r\}$, where the first s are linearly independent over X. Hence if $\{u_1, \dots, u_r\}$ are linearly independent over Y, then the above construction yields an element

(1) When necessary we write H_r to distinguish between the Hopf bundles over different projective spaces.

$$\begin{array}{c} \alpha_E^s(u_1, \ldots, u_r) \in KR((iF|X-Y) \times (P_{r+s-1} - P_{s-1})) \\ \| \\ KR^s((iE|X-Y) \times (P_{r+s-1} - P_{s-1})). \end{array} \tag{2.10}$$

This element is our basic characteristic class in its most general form. We shall now specialize it to define global and local invariants as required for the general Hopf theorem.

If X is a $4k-s$ dimensional oriented compact manifold with boundary Y and $E=TX$, the tangent bundle of X, then we shall write

$$\alpha_X^s(u_1, \ldots, u_r) = \alpha_{TX}^s(u_1, \ldots, u_r). \tag{2.11}$$

This element lies in $KR^s((iTX|X-Y) \times (P_{r+s-1} - P_{s-1}))$. Taking the index in the sense of [9] we get an element

$$\operatorname{ind} \alpha_X^s(u_1, \ldots, u_r) \in KR^s(P_{r+s-1}, P_{s-1}). \tag{2.12}$$

In particular, if $Y=\varnothing$ so that X is closed, (2.12) defines a *global invariant* for the manifold:

$$\operatorname{ind} \alpha_{X,r}^s \in KR^s(P_{r+s-1}, P_{s-1}). \tag{2.13}$$

Notice that for $Y=\varnothing$ the elements in (2.11) and (2.13) are restrictions of elements in(1) $KR^s(iTX \times (P_\infty - P_{s-1}))$ and $KR^s(P_\infty, P_{s-1})$ respectively. In fact, we can take r arbitrarily large when $Y=\varnothing$.

Passing now to the local situation let $X=B^q$, the unit ball in $\mathbf{R}^q$, and $Y=S^{q-1}$ the unit sphere. Put $n=4k-s$ and let $V_{n,r}$ denote the Stiefel manifold of r-frames in $\mathbf{R}^n$. Take E to be the trivial n-dimensional bundle over X, and let $u \in \pi_{q-1}(V_{n,r})$ be given by the frame $u=\{u_1, \ldots, u_r\}$. Then the construction (2.10) gives an element

$$\theta^s(u) = \alpha_{\mathbf{R}^n}^s(u_1, \ldots, u_r)$$

which lies in

$$KR^s(i\mathbf{R}^n \times (B^q - S^{q-1}) \times (P_{r+s-1} - P_{s-1})) = KR^{4k,q}(P_{r+s-1}, P_{s-1}).$$

Using the periodicity map, this last group is $KR^{4k-q}(P_{r+s-1}, P_{s-1})$.

Thus we obtain a map

$$\theta^s : \pi_{q-1}(V_{n,r}) \rightarrow KR^{4k-q}(P_{r+s-1}, P_{s-1}). \tag{2.14}$$

We shall show later that θ^s is indeed a homomorphism. Notice that for $q=n$, $\theta^s(u)$ can also be defined by(2) (2.12), thus giving a map

(1) For our purposes it is not necessary to worry about the different definitions of K for infinite-dimensional spaces. We simply use the notation $K(P_\infty)$ as shorthand for $\varprojlim K(P_n)$.

(2) We recall that the index for $\mathbf{R}^n$ coincides with the periodicity map.

$$\theta^s : \pi_{n-1}(V_{n,r}) \to KR^s(P_{r+s-1}, P_{s-1}). \tag{2.15}$$

This is the local invariant which enters in Theorem (1.1).

Remark. It is well-known (see [4]) that a stunted projective space $P_{r+s,r} = P_{r+s-1}/P_{s-1}$ is naturally homeomorphic to the Thom complex P_{r-1}^{sH} of the bundle sH_r over P_{r-1}. In fact, there is the natural identification

$$P_{s-1}^{sH} \to P_{r+s-1}/P_{s-1}$$

defined as follows. A point in sH over $x = \{x_1, \dots, x_r\} \in P_{r-1}$ is given by s functionals $\varphi_1, \dots, \varphi_s$ on the line spanned by x. Hence the coordinates $\{x_1, \dots, x_r, \varphi_1(x), \dots, \varphi_s(x)\}$ define a point in P_{r+s-1}. The map

$$sH_r \to P_{r+s-1} - P_{s-1}$$

thus defined clearly extends to the required homeomorphism.

The element (2.10) thus lies in the group $KR^s((iE|X-Y) \times sH_r)$.

Note the following important properties of the characteristic class (2.10).

Naturality. If $f: (X', Y') \to (X, Y)$ is a continuous function, then

$$\bar{f}^* \alpha_E^s (u_1, \dots, u_r) = \alpha_{f^*E}^s (f^* u_1, \dots, f^* u_r) \tag{2.16}$$

where $\bar{f}: f^*E|X' - Y' \to E|X - Y$ is the obvious map. In particular, taking $X = BSO(n)$, $Y = BSO(n-r)$ and $E = E(n)$, the universal bundle over X we get a universal element [1]

$$\alpha_{n,r}^s \in KR^s(MSO(n)/MSO(n-r) \wedge P_{r+s-1}/P_{s-1})$$

where $MSO(n)$ denotes the Thom complex of $E(n)$.

Multiplicativity. Let E and E' be vector bundles over X and X' respectively, and let $\{u_1, \dots, u_r\}$ and $\{u_1', \dots, u_r'\}$ be sections of E, E', linearly independent over Y and Y' respectively. Then $\{u_1 + u_1', \dots, u_r + u_r'\}$ are sections of $E \times E'$ over $X \times X'$, linearly independent over $X \times Y' \cup Y \times X'$. Furthermore,

$$\alpha_{E \times E'}^{s+s'} (u_1 + u_1', \dots, u_r + u_r') = (H_r)^{ss'} \alpha_E^s (u_1, \dots, u_r)\, \alpha_{E'}^{s'} (u_1', \dots, u_r') \tag{2.17}$$

where the multiplication is carried out by the composite map

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[1] We use $\wedge$ for reduced product.

$$\begin{gathered} KR^s((iE|X-Y)\times sH_r)\otimes KR^{s'}((iE'|X'-Y')\times s'H_r) \\ \downarrow \\ KR^{s+s'}((iE\times iE'|(X-Y)\times(X'-Y'))\times sH_r\times s'H_r) \\ \downarrow \Delta^* \\ KR^{s+s'}((iE\times iE'|(X-Y)\times(X'-Y'))\times(s+s')H_r). \end{gathered}$$

Here Δ^* is induced by the diagonal

$$P_{r-1}^{(s+s')H} \to P_{r-1}^{sH}\wedge P_{r-1}^{s'H}$$

or equivalently by the map

$$P_{r+s+s'-1}/P_{s+s'-1} \to P_{r+s-1}/P_{s-1}\wedge P_{r+s'-1}/P_{s'-1}$$

induced by projecting $\mathbf{R}^{r+s+s'}=\mathbf{R}^r\oplus\mathbf{R}^s\oplus\mathbf{R}^{s'}$ onto the two subspaces $\mathbf{R}^r\oplus\mathbf{R}^s$ and $\mathbf{R}^r\oplus\mathbf{R}^{s'}$. The multiplication by $(H_r)^{ss'}$ is understood analogously.

The proof of (2.17) follows from the multiplicativity of the exterior algebra. We omit the details, but remark that the factor $(H_r)^{ss'}$ occurs as a consequence of the fact that the bundles

$$(E\oplus\mathbf{R}^s)\times(E'\oplus\mathbf{R}^{s'}) \text{ and } (E\times E')\oplus\mathbf{R}^{s+s'}$$

differ in orientation by the sign $(-1)^{ss'}$.

Consider the special case $X=\text{point}$, $Y=\varnothing$, $E=0$ and $s\equiv 0 \bmod 4$. Then

$$\lambda_{sH}=\alpha_0^s(0,\dots,0)\in KR^s(P_{r-1}^{sH}) \tag{2.18}$$

is the Thom class of the Spin-bundle $sH_r=\mathbf{R}^s\otimes H_r$.

To see this first observe that the tensor product of a Spin $(4l)$-bundle and any line bundle again has a natural Spin-structure. In fact, the isomorphism

$$SO(4l)\times_{O(1)}O(1)\simeq SO(4l)$$

is covered by the isomorphism

$$\operatorname{Spin}(4l)\times_{O(1)}O(1)\simeq\operatorname{Spin}(4l)$$

where $O(1)=\{\pm 1\}$ acts on $\operatorname{Spin}(4l)$ by left multiplication by the volume element ω. In particular, for $s=4l$ we have an explicit $\operatorname{Spin}^c(s,s)$ structure for the bundle $i\mathbf{R}^s\oplus\mathbf{R}^s\otimes H_r$, and analogous to the discussion following (2.5) it is easily verified that our square defining $\alpha_0^s(0,\dots,0)$ indeed gives the Thom class as defined in [3]. Notice that the natural identification

$$P_{r-1}^{sH}\simeq P_{r+s-1}/P_{s-1}$$

induces a natural order of the s copies of H_r, and so in turn a natural Spin-structure on sH_r.

Periodicity. By the Thom isomorphism theorem it follows that for $s' \equiv 0 \bmod 4$

$$KR^s((iE|X-Y)\times sH_r) \simeq KR^{s+s'}((iE|X-Y\times(s+s')H_r)$$

where the isomorphism is induced by multiplication by $\lambda_{s'H}$. Furthermore, according to (2.17)

$$\alpha_E^{s+s'}(u_1,\dots,u_r) = \alpha_E^s(u_1,\dots,u_r)\,\lambda_{s'H}. \tag{2.19}$$

It follows that the element defined in (2.10) is essentially independent of the choice of s modulo 4. The same assertion is therefore also true for the invariants (2.11) – (2.15).

Remark. One can get rid of the involution on E in (2.10) by the following trivial observation. The diagonal homomorphism

$$\Delta: O(n) \to O(n)\times O(n) \to O(2n)$$

actually maps into $SO(2n)$, and the restriction of Δ to $SO(n)$ has a unique lifting

$$SO(n) \to \mathrm{Spin}\,(2n)$$

which takes the identity to the identity. Hence, the double of any vectorbundle has a natural orientation, and the double of any oriented vectorbundle has a natural Spin-structure. So let V be any oriented vectorbundle of dimension n over a space X. V is given the trivial involution and iV the antipodal. Then by the Thom isomorphism

$$KR^s(iV) \simeq KR^{s+2n}(V\oplus V\oplus iV) \simeq KO^{s+2n}(V).$$

Now that we have defined our general characteristic class (2.10) it is quite trivial to establish the following version of (I.1):

THEOREM 2.20. *Let X be a closed oriented manifold of dimension $n = 4k - s$, and let $\{u_1,\dots,u_r\}$ be a set of vector fields, linearly independent except at the finite set of points $\{A_1,\dots,A_l\}$. Then in the group $KR^s(P_{r+s-1})$ we have the formula for the global invariant*

$$\mathrm{ind}\,\alpha_{X,r}^s = \sum_{i=1}^{l} \theta^s O_{A_i}(u_1,\dots,u_r)$$

where $O_A(u_1,\dots,u_r) \in \pi_{n-1}(V_{n,r})$ is the local obstruction to extending the vector fields.

Proof. Around every point A_i consider a disk $B_i \subset X$ and let S_i be the bounding sphere. By assumption the vector fields are linearly independent over $Y = X - (\bigcup_{i=1}^{l} B_i)$, and we consider

$$\alpha_X^s(u_1, \dots, u_r) \in KR^s((iTX, iTX\,|\,Y) \times (P_{r+s-1}, P_{s-1})).$$

Now we have the commutative diagram

$$\begin{array}{ccc}
KR^s((iTX, iTX\,|\,Y) \times (P_{r+s-1}, P_{s-1})) & \xrightarrow{j^*} & KR^s(iTX \times (P_{r+s-1}, P_{s-1})) \\
\simeq \big\downarrow & & \big\downarrow \text{ind} \\
\bigoplus_{i=1}^{l} KR^s((iTB_i, (iTB_i\,|\,S_i)) \times (P_{r+s-1}, P_{s-1})) & \xrightarrow[\text{ind}]{} & KR^s(P_{r+s-1}, P_{s-1})
\end{array}$$

and clearly $j^* \alpha_X^s(u_1, \dots, u_r) = \alpha_{X,r}^s$. Also, as remarked earlier, the index of the restriction of $\alpha_X^s(u_1, \dots, u_r)$ to B_i is exactly $\theta^s(O_{A_i}(u_1, \dots u_r))$. The formula follows.

3. Calculation of the groups A_s^r

In this section we shall tabulate the groups $A_r^s = KR^s(P_{r+s-1}, P_{s-1})$. They can be calculated by various methods but the most convenient way is probably to use a result of G. Segal and some associated exact sequences. We shall explain these sequences but the actual details of the computations will be omitted.

Let C_r denote the Clifford algebra of $\mathbf{R}^r$ with the quadratic form $-(x_1^2 + \dots + x_r^2)$. Clifford algebras are discussed in detail in [6], where they are listed in Table 2. It is seen that they are matrix algebras, or the sum of two copies of a matrix algebra, over $\mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$. For any compact space X let $M_r(X)$ denote the Grothendieck group of bundles of Z_2-graded C_r-modules over X, and $M_r^*(-)$ is the corresponding cohomology theory. In fact M_r^* is either real, complex or quaternionic K-theory (or a direct sum of 2 copies of those). $M_r(X)$ can be defined for Real spaces if we require the C_r-module structure to commute with the anti-linear involution on the Real bundles.

Now let G be the multiplicative group consisting of ± 1. There is a natural map

$$M_r(X) \rightarrow KR_G(X \times \mathbf{R}^r) \tag{3.2}$$

where $KR_G(X \times \mathbf{R}^r)$ is equivariant Real K-theory of $X \times \mathbf{R}^r$ with G acting on $\mathbf{R}^r$ by multiplication. In fact if $F = (F^0, F^1)$ is a Z_2-graded C_r-module, then Clifford multiplication by $v \in \mathbf{R}^r$ defines a homomorphism $\varphi: p^*F^0 \rightarrow p^*F^1$ where $p: X \times \mathbf{R}^r \rightarrow X$ is the projection. The triple $(p^*F^0, p^*F^1, \varphi)$ defines an element in $KR_G(X \times \mathbf{R}^r)$.

THEOREM 3.3. (G. Segal). *The map*

$$M_r^*(X) \to KR_G^*(X \times \mathbf{R}^r),$$

defined as above, is an isomorphism.

This theorem may be proved by induction on r using the five lemma for two exact sequences. The first is the exact sequence in KR_G-theory for the pair $(X \times \mathbf{R}^r \times I, X \times \mathbf{R}^r \times \dot{I})$ where again G acts by multiplication on $\mathbf{R}^r \times I$:

$$\ldots \to KR_G^i(X \times \mathbf{R}^{r+1}) \to KR_G^i(X \times \mathbf{R}^r) \to KR^{i-r}(X) \to \ldots. \tag{3.4}$$

The second sequence is established using the classifying spaces of Fredholm operators in Hilbert space as discussed in [8]:

$$\ldots \to M_{r+1}^i(X) \to M_r^i(X) \to KR^{i-r}(X) \to \ldots. \tag{3.5}$$

As an application, the exact sequence in KR_G-theory for the triple $(X \times B^r \times B^s, X \times (B^r \times B^s)^{\cdot}, X \times B^r \times S^{s-1})$ gives the following exact sequence:

$$\ldots \to M_{r+s}^i(X) \to M_s^i(X) \to KR^i(X \times (P_{r+s-1} - P_{s-1})) \to \ldots. \tag{3.6}$$

In particular for $X = \mathrm{pt}$ we get the exact sequence

$$\ldots \to M_{r+s}^i(\mathrm{pt}) \to M_s^i(\mathrm{pt}) \to KR^i(P_{r+s-1}, P_{s-1}) \to \ldots \tag{3.7}$$

from which we can calculate $A_r^s = KR^s(P_{r+s-1}, P_{s-1})$. We just have to do this for $s = 0$, 1, 2 and 3. The result is the following.

$s = 0$: $A_r^0 = KR(P_{r-1}) = Z \oplus Z_{a_r}$, where a_r is a power of 2, namely the least integer n such that $\mathbf{R}^n$ is a module for C_{r-1}. According to [6] a_r is given by the following table

r	1	2	3	4	5	6	7	8
a_r	1	2	4	4	8	8	8	8

and $a_{r+8} = 16\,a_r$.

$s = 1$: $A_r^1 = KR^1(P_r, P_0)$ is given by the table

r mod 8	1	2	3	4	5	6	7	8
A_r^1	Z	Z_2	$Z_2 \oplus Z_2$	Z_2	Z	0	0	0

$s = 2$: $A_r^2 = KR^2(P_{r+1}, P_1)$ is given by the tables

r	1	2	3
A_r^2	Z	$Z \oplus Z_2$	$Z \oplus Z_2$

and for $r > 3$

$r \bmod 8$	1	2	3	4	5	6	7	8
A_r^2	$Z\oplus Z_2$	$Z\oplus Z_2\oplus Z_2$	$Z\oplus Z_2$	Z	Z	Z	Z	Z

$s=3$:

$$A_r^3 = KR^3(P_{r+2}, P_2) = \begin{cases} Z_2 & r \not\equiv 1 \bmod 4 \\ Z\oplus Z_2 & r\equiv 1 \bmod 4 \ r>1 \\ Z & r=1. \end{cases}$$

Remark 3.8. $KR(P_{r-1})$ has generators 1 and $(H-1)$ for Z and Z_{a_r} respectively. Hence the map $KR(P_r) \to KR(P_{r-1})$ takes Z to Z and maps $Z_{a_{r+1}}$ onto Z_{a_r}.

For $s=2$ the map $A_{r+1}^2/\mathrm{Tor} \to A_r^2/\mathrm{Tor}$ is an isomorphism except for $r=2$ where it is multiplication by 2.

Finally for $s=3$ the map $A_r^3 \to A_2^3$ is the projection onto Z_2.

Hence we get the following table for $A_\infty^s = KR^s(P_\infty, P_{s-1})$:

s	0	1	2	3
A_∞^s	$Z\oplus \hat{Z}_2$	0	Z	Z_2

Here $\hat{Z}_2 = \varprojlim_n Z_{2^n}$ is the group of 2-adic integers. Note that $A_\infty^2 \to A_1^2$ is multiplication by 2.

Note that the groups $M_s^s(\mathrm{pt})$ are given by the table

s	0	1	2	3
$M_s^s(\mathrm{pt})$	$Z\oplus Z$	0	Z	Z_2

Moreover, the map $M_s^s(\mathrm{pt}) \to A_\infty^s$ arising in the sequence (3.7) is the identity for $s\neq 0$ and for $s=0$ it is given by

$$(a,b) \mapsto a-bH = (a-b)-b(H-1) \tag{3.9}$$

(note that $(H-1)$ generates the 2-adic factor in $A_\infty^0 = KR(P_\infty)$). The groups $M_s^s(\mathrm{pt})$ will be essentially the B^s groups of Theorem (1.1) and the natural composite homomorphism $M_s^s(\mathrm{pt}) \to A_\infty^s \to A_r^s$ will be the γ_r^s of (1.1). However, for $s=0$ we will change the basis of $Z\oplus Z$ by the map $(a,b)\mapsto ((a-b), -b)$ so that $\gamma_r^0: Z\oplus Z \to Z\oplus Z_{a_r} = A_r^0$ will be the identity on the first factor and the natural projection on the second factor. In other words B^0 is essentially the subgroup of $A_\infty^0 = KR(P_\infty)$ with given generators 1 and $(H-1)$.

Returning now to the $(4k-s)$-manifold X of Theorem (2.20) we recall that the global invariant ind $\alpha_{X,r}^s \in A_r^s$ is actually the image of an element ind $\alpha_{X,\infty}^s \in A_\infty^s$. To complete the proof of Theorem (1.1) we will show:

ind $\alpha_{X,\infty}^s$ is the image of an element $b^s(X) = \operatorname{ind} \beta_X^s \in M_s^s(\mathrm{pt})$, where $\beta_X^s \in M_s^s(iTX)$; (3.10)

$$\text{ind}\ \beta_X^s \text{ is given by the required combinations of } E, S, R \text{ as in (1.1).} \tag{3.11}$$

(3.10) will be established now and (3.11) will then be dealt with in the next section by identifying β_X^s with the symbol of an appropriate differential operator.

Now (3.10) will follow from the commutative diagram

$$\begin{tikzcd} M_s^s(iTX) \arrow[r] \arrow[d, "\text{ind}"] & KR^s(iTX \times (P_\infty - P_{s-1})) \arrow[d, "\text{ind}"] \\ M_s^s(pt) \arrow[r] & KR^s(P_\infty, P_{s-1}) \end{tikzcd} \tag{3.12}$$

provided we can construct an element $\beta_X^s \in M_s^s(iTX)$ which maps to $\alpha_{X,\infty}^s$ in the top row (this homomorphism comes from (3.6) with X replaced by iTX and $r = \infty$). But the basic square (2.7) which was used to construct $\alpha_{X,\infty}^s$ can also be used to construct β_X^s. In fact, if $E = TX \oplus \mathbf{R}^s$ then $\Lambda_+^*(E)$ and $\Lambda_-^*(E)$ are both Z_2-graded C_s-modules. The grading is given by even and odd forms and the action of C_s is generated by the right Clifford multiplications (the vertical arrows in (2.7)). The horizontal arrows of (2.7) give an isomorphism of $\Lambda_+^*(E)$ with $\Lambda_-^*(E)$ when lifted up to the non-zero vectors of E. This isomorphism is an isomorphism of Z_2-graded Real [1] C_s-modules because of the commutativity of (2.7). Hence we get an element $\beta_X^s \in M_s(iE) \cong M_s^s(iTX)$ as required.

The map $M_s^s(iTX) \to KR^s(iTX \times (P_\infty - P_{s-1}))$ of (3.12) may be viewed as the composition

$$M_s(iE) \to KR_G(iE \times \mathbf{R}^s) \to KR(iE \times sH)$$

where H is the Hopf bundle on P_∞. Comparing the definitions of β_X^s and $\alpha_{X,\infty}^s$ it is then clear that they correspond under this homomorphism.

4. Identification of the global invariants

In this section we shall relate the elements $\beta_X^s \in M_s^s(iTX)$ defined in § 3 with the symbols of explicit elliptic differential operators. We consider separately the four values of s. The case $s = 1$ is trivial because $A_\infty^1 = 0$. The case $s = 2$ is also rather easy because $A_\infty^2 \to A_1^2$ is $Z \xrightarrow{2} Z$. This means that we only have to check Theorem (1.1) in this case for $r = 1$ and this is the classical Hopf Theorem. It is of course possible to prove this analytically by relating the element

[1] E is given the antipodal involution as usual and the factor i in (2.7) ensures that we then have Real isomorphism.

$$\beta_X^2 \in M_2^2(iTX) = K^2(iTX) \cong K(TX)$$

with the symbol of the operator $d+\delta:\Omega^{\mathrm{ev}} \to \Omega^{\mathrm{odd}}$ on real differential forms ($\delta = d^*$ the Riemannian adjoint of d). Note that $R_\omega^2 = -1$ in this case so that Ω has a complex structure, and $\frac{1}{2}E$ is the index of $d+\delta$ viewed as complex linear operator.

We consider next the case $s=0$. Then

$$\beta_X^0 \in M_0^0(iTX) = KR(iTX) \oplus KR(iTX)$$

is just the direct sum of the two elements given by the top and bottom rows of our basic square. Thus $\beta_X^0 = \beta^+ \oplus \beta^-$ where

$$\begin{aligned} \beta^+ &= (\Lambda_+^{\mathrm{ev}}\,(TX), \Lambda_-^{\mathrm{odd}}(TX), \varphi^+) \\ \beta^- &= (\Lambda_+^{\mathrm{odd}}(TX), \Lambda_-^{\mathrm{ev}}\,(TX), \varphi^-) \end{aligned} \tag{4.1}$$

and $\varphi^\pm$ over a point $v \in TX$ is the map iL_v. Now iL_v is just the symbol of the operator $d+\delta$ on differential forms. Hence β^+, β^- are just the symbols of the operators

$$\begin{aligned} B^+ &: \Omega_+^{\mathrm{ev}} \to \Omega_-^{\mathrm{odd}}, \\ B^- &: \Omega_+^{\mathrm{odd}} \to \Omega_-^{\mathrm{ev}} \end{aligned}$$

obtained by restricting $d+\delta$ to these four subspaces of the differential forms ($\Omega_\pm$ denote of course the $\pm$ 1-eigenspaces of L_ω acting on Ω). To compute the indices of B^+ and B^- it is easiest to compute $\operatorname{ind}(B^+ \oplus B^-)$ and $\operatorname{ind}(B^+ \oplus (B^-)^*)$, where

$$(B^-)^* : \Omega_-^{\mathrm{ev}} \to \Omega_+^{\mathrm{odd}}$$

is the adjoint of B^-. Clearly we have $\operatorname{ind}(B^+ \oplus (B^-)^*) = \dim_{\mathbf{C}} \sum H^{2p} - \dim_{\mathbf{C}} \sum H^{2p+1} = E$ where H^p is the space of harmonic p-forms (and so by the Hodge theory is isomorphic to $H^p(X, \mathbf{C})$). On the other hand $B^+ \oplus B^- : \Omega_+ \to \Omega_-$ is shown in [10] to have index [1] $(-1)^k S$. Adding and subtracting (and using $\operatorname{ind}(B^-)^* = -\operatorname{ind} B^-$) we get

$$\begin{aligned} \operatorname{ind} B^+ &= \tfrac{1}{2}(E + (-1)^k S) \\ \operatorname{ind} B^- &= -\tfrac{1}{2}(E - (-1)^k S). \end{aligned}$$

Since $\operatorname{ind} \alpha_{X,\infty}^0$ is the image of $\operatorname{ind} \beta_X^0$ under the map $M_0^0(\mathrm{pt}) \to KR(P_\infty)$ given by (3.9) it follows that

$$\operatorname{ind} \alpha_{X,\infty}^0 = E \oplus \tfrac{1}{2}(E - (-1)^k S)(H-1). \tag{4.2}$$

This completes the proof of Theorem (1.1) for the case $s=0$.

---

[1] In [10] the forms are split by the involution $(-1)^k L_\omega$: the point is that on $2k$-forms we have $L_\omega = (-1)^k *$, where $*$ is the duality operator of the metric.

Finally we come to the case $s=3$. Then

$$\beta_X^3 \in M_3^3(iTX) \cong KH^3(iTX) \cong KR^{-1}(iTX)$$

(where KH stands for symplectic K-theory for Real spaces, as in [7]). On the other hand, as described in [5], the semi-characteristic $R(X)$ is the mod 2 index of the skew-adjoint operator $T = L_\omega \circ (d+\delta)$ acting on the space Ω^{ev}. Furthermore, it is shown in [5] that T can be replaced by a 0-order skew-adjoint elliptic operator. As shown in [5] and [8] such operators have a symbol in $KR^{-1}(iTX)$. In general, a skew-adjoint Fredholm operator A of a real Hilbert space V gives rise to the elliptic family over P_1, given over a point $\{x,y\} \in P_1$ by the operator $xI + yA: V \to V \otimes H$, where H is the Hopf-bundle over P_1. In our case the symbol $\sigma(T) \in KR^{-1}(iTX) = KR(iTX \oplus \mathbf{R})$ is defined by the triple $(\Lambda^{\mathrm{ev}}(TX), \Lambda^{\mathrm{ev}}(TX), \psi)$ where ψ is given over $v \oplus t \in TX \oplus \mathbf{R}$ by $iL_{\omega v} + tI$.

The proof of Theorem (1.1) for the case $s=3$ will now follow from the following lemma:

LEMMA 4.3. $\beta_X^3 = -\sigma(T) \in KR^{-1}(iTX)$.

Proof. Since dim $X = 4k-3$ is odd we can find a nowhere zero vector field and hence write $TX = E \oplus \mathbf{R}$ where E is an oriented $(4k-4)$-dimensional bundle. Then $KR^{-1}(iTX) \cong KR(iE)$ by periodicity. We will now show that β_X^3 and $-\sigma(T)$ both correspond to the element $\beta_E^+ + \beta_E^- \in KR(iE)$, where $\beta^\pm$ are the elements given in (4.1) (with TX now replaced by E). Consider first the element β_X^3. From the naturality properties of our basic square it is clear that $\beta_X^3 \in M_3^3(iTX) \cong M_3^4(iE)$ is the restriction of $\beta_E^4 \in M_4^4(iE)$. On the other hand the restriction $M_4^4(iE) \to M_3^4(iE)$ can be identified with the homomorphism

$$KR(iE) \oplus KR(iE) \to KR(iE)$$

given by $(a, b) \mapsto a+b$, and the periodicity (mod 4) of our basic square shows that β_E^4 corresponds to

$$\beta_E^0 \in M_0^0(iE) \cong KR(iE) \oplus KR(iE).$$

Since $\beta_E^0 = \beta_E^+ \oplus \beta_E^-$ the identification of β_X^3 with $\beta_E^+ + \beta_E^-$ in $KR(iE)$ follows.

We turn now to the element $-\sigma(T)$. The image of $-(\beta_E^+ + \beta_E^-)$ under the periodicity isomorphism $KR(iE) \cong KR(iE \oplus \mathbf{C}) = KR^{-1}(iTX)$ is given by the square over $iE \oplus \mathbf{C}$

$$\begin{tikzcd}
\Lambda^+(E) \arrow[r, "iL_e"] \arrow[d, "\lambda"'] & \Lambda^-(E) \arrow[d, "\lambda"] \\
\Lambda^+(E) \arrow[r, "iL_e"] & \Lambda^-(E)
\end{tikzcd}$$

where $e \in E_x$, $x \in X$ and $\lambda \in \mathbf{C}$. According to § 2 this is equivalent to the triple $(\Lambda^+(E) \oplus \Lambda^-(E), \Lambda^+(E) \oplus \Lambda^-(E), \phi)$, where ϕ over the point $(e, \lambda) \in E \oplus \mathbf{C}$ is given by the matrix

$$\begin{pmatrix} \lambda & iL_e \\ iL_e & -\bar{\lambda} \end{pmatrix}.$$

Letting ω' denote the volume element of E, we have

$$\phi_{(e,\lambda)} = \tfrac{1}{2}(iL_e + \lambda I)(I + L_{\omega'}) + \tfrac{1}{2}(iL_e - \bar{\lambda} I)(I - L_{\omega'}) = iL_e + (\operatorname{Re}\lambda) L_{\omega'} + i(\operatorname{Im}\lambda) I.$$

Now fix $x \in X$ and let $e_0 \in (TX)_x$ be the section at x spanning the line complementary to E_x. Identify $i\mathbf{R}e_0$ with the *imaginary* axis of $\mathbf{C}$. Also $\Lambda^* \simeq \Lambda^{\mathrm{ev}}(TX)$ by the inclusion of $\Lambda^{\mathrm{ev}}(E)$ and by $L_{e_0}: \Lambda^{\mathrm{odd}}(E) \to \Lambda^{\mathrm{ev}}(TX)$. With these identifications our element in $KR(iTX \oplus \mathbf{R}) = KR(iE \oplus \mathbf{C})$ is given by the triple $(\Lambda^{\mathrm{ev}}(TX), \Lambda^{\mathrm{ev}}(TX), \psi')$ where ψ' is defined over $(v, \operatorname{Re}\lambda) \in TX \oplus \mathbf{R}$, $v = e + (\operatorname{Im}\lambda) e_0 \in (TX)_x$, by

$$\psi'_{(v, \operatorname{Re}\lambda)} = (iL_{\omega v} + (\operatorname{Re}\lambda) I) \circ L_{(-\omega e_0)}.$$

But this is clearly isomorphic (under $L_{-\omega e_0}$) to the symbol for the operator $T = L_\omega \circ (d + \delta)$. This ends the proof of the lemma.

Lemma (4.3) involves an identity between certain universal symbols. It is quite instructive to state it in terms of the universal groups. $\beta_E^\pm$ come from universal elements $\beta^\pm$ in the equivariant group $KR_{SO(4l)}(i\mathbf{R}^{4l})$, where $l = k - 1$. Then one can verify the following statements:

(a) $KR_{SO(4l)}(i\mathbf{R}^{4l}) \to KR_{SO(4l)}(\mathrm{pt}) = RSO(4l)$ is injective and its image is the ideal generated by the representations

$$\sigma^+ = \Lambda_+^{\mathrm{ev}} - \Lambda_-^{\mathrm{odd}}, \quad \sigma^- = \Lambda_+^{\mathrm{odd}} - \Lambda_-^{\mathrm{ev}}$$

(b) $\sigma^\pm$ are the images of $\beta^\pm$.

(c) The restriction

$$KR_{SO(4l+1)}^{-1}(i\mathbf{R}^{4l+1}) \to KR_{SO(4l)}^{-1}(i\mathbf{R}^{4l+1}) \cong KR_{SO(4l)}(i\mathbf{R}^{4l})$$

is injective and its image is the $RSO(4l+1)$-submodule generated by $\beta^+ + \beta^-$.

From (c) and Lemma (4.3) it follows that $\sigma(T)$ comes from a universal symbol which generates the universal group $KR_{SO(4l+1)}^{-1}(i\mathbf{R}^{4l+1})$ as $RSO(4l+1)$-module.

5. The local invariant

The map
$$\theta^s : \pi_{n-1}(V_{n,r}) \rightarrow KR^s(P_{r+s-1}, P_{s-1})$$
as defined in § 2 is our invariant for the local obstruction for extending a set of r vector fields with finite singularities. It is therefore important to know how nontrivial θ^s is. We shall show that θ^s is an isomorphism for $r \leqslant 3 \leqslant n-r$. Furthermore, if n is divisible by the 2-power a_r and r is *not* divisible by 4, then θ^0 is the projection onto a direct summand of the form $Z \oplus Z_{a_r}$, and a_r is the highest possible 2-primary order of an element in $\pi_{n-1}(V_{n,r})$. For instance, it is known (see [18]) that for $k>1, \pi_{8k-1}(V_{8k-1,5}) = Z \oplus Z_8$, hence θ^0 is an isomorphism in this case. If r is divisible by 4 there is a modification of the above statement.

More generally consider the map defined by (2.14):
$$\theta^s : \pi_{q-1}(V_{n,r}) \rightarrow KR^{n+s-q}, (P_{r+s-1}, P_{s-1}) \tag{5.1}$$
where $n+s$ is divisible by 4. We start with some elementary properties of (5.1). For two homotopy elements $u \in \pi_{q-1}(V_{n,r})$ and $u' \in \pi_{q'-1}(V_{n',r})$ one can form the join $u * u' \in \pi_{q+q'-1}(V_{n+n',r})$ in the sense of James [15].

LEMMA 5.2. *The map θ^s has the following properties.*

1) *θ^s is a homomorphism with respect to addition.*

2) *Let $u \in \pi_{q-1}(V_{n,r})$ and $u' \in \pi_{q'-1}(V_{n',r})$. Then*
$$\theta^s(u)\, \theta^{s'}(u') = (H_r)^{ss'} \theta^{s+s'}(u * u').$$

3) *Let $\iota_n \in \pi_n(S^n)$ be the generator, then $\theta^s(\iota_n)$ is the canonical generator of $KR^s(P_s, P_{s-1})$.*

4) *Let $u \in \pi_q(V_{n,r})$ and $g \in \pi_{q'}(S^q)$, then*
$$\theta^s(u \circ g) = g^*(\sigma_q)\, \theta^s(u)$$
where $\sigma_q \in \widetilde{KR}^q(S^q)$ is the canonical generator.

Proof. 1) Simply follows using (2.16) for the pinching map
$$(B^q, S^{q-1}) \rightarrow (B^q \vee B^q, S^{q-1} \vee S^{q-1}).$$

2) Is immediate from (2.17).

Now 3) was verified in § 2 for n divisible by 4. For general n we can apply 2) and deduce that
$$(\theta^s(\iota_n))^4 = \theta^{4s}(\iota_{4n})$$

showing that $\theta^s(\iota_n)$ is always a generator of $KR^s(P_s, P_{s-1}) \cong Z$. The identification of sign can be checked explicitly from the definitions but it also follows from the interpretation of θ^s as a Hurewicz homomorphism which will be given in § 8.

Finally 4) is obvious from (2.16).

Now consider the homotopy exact sequence for the fibration $V_{n-1,r-1} \to V_{n,r} \to S^{n-1}$:

$$\dots \to \pi_{q-1}(V_{n-1,r-1}) \to \pi_{q-1}(V_{n,r}) \to \pi_{q-1}(S^{n-1}) \to \dots . \tag{5.3}$$

If $n+s=n-1+s+1$ is divisible by 4, θ^{s+1} and θ^s maps (5.3) into the exact sequence for the triple $(P_{r+s-1}, P_s, P_{s-1})$, and we clearly have a commutative diagram

$$\begin{tikzcd}
\pi_{q-1}(V_{n-1,r-1}) \arrow[r] \arrow[d, "\theta^{s+1}"] & \pi_{q-1}(V_{n,r}) \arrow[r] \arrow[d, "\theta^s"] & \pi_{q-1}(S^{n-1}) \arrow[d, "\theta^s"] \\
KR^{n+s-q}(P_{r+s-1}, P_s) \arrow[r] & KR^{n+s-q}(P_{r+s-1}, P_{s-1}) \arrow[r] & KR^{n+s-q}(P_s, P_{s-1})
\end{tikzcd} \tag{5.4}$$

Furthermore, for the boundary operators

$$\partial : \pi_{q-1}(S^{n-1}) \to \pi_{q-2}(V_{n-1,r-1}) \text{ and}$$
$$\delta : KR^{n+s-q}(P_s, P_{s-1}) \to KR^{n+s-q+1}(P_{r+s-1}, P_s)$$

we have

LEMMA 5.5. *The following diagram commutes*

$$\begin{tikzcd}
\pi_{q-1}(S^{n-1}) \arrow[r, "\partial"] \arrow[d, "\theta^s"] & \pi_{q-2}(V_{n-1,r-1}) \arrow[d, "\theta^{s+1}"] \\
KR^{n+s-q}(P_s, P_{s-1}) \arrow[r, "-\delta"] & KR^{n+s-q+1}(P_{r+s-1}, P_s)
\end{tikzcd}$$

Proof. Let $u \in \pi_{q-1}(S^{n-1}) = \pi_{q-1}(V_{n,r}, V_{n-1,r-1})$ be given by r vector fields $\{u_1, u_2, \dots, u_r\}$ of the vector bundle $B^{q-1} \times \mathbf{R}^n$, such that over $S^{q-2} \subset B^{q-1}$, u_r is constantly the vector $e_n = (0, \dots, 0, 1)$ in $\mathbf{R}^n$. Let $\overline{W}$ denote the mapping cone of the inclusion $P_s/P_{s-1} \subseteq P_{r+s-1}/P_{s-1}$. Write $W = P_{r+s-1} \cup P_s \times I$ so $\overline{W} = W/P_s' \cup P_{s-1} \times I$, where $P_s' = P_s \times \{1\}$ and $P_s \times \{0\}$ is identified with $P_s \subset P_{r+s-1}$. Consider the trivial $n+s$-dimensional bundle over $B^{q-1} \times W$ and make a construction as in (2.10) using the constant last $s+1$ vector fields of $\mathbf{R}^{n+s}$ over P_s', and the constant last s vector fields over $P_{s-1} \times I$. In this way we get an element

$$\alpha \in KR^{n+s}((B^{q-1}, S^{q-2}) \times (W, P_{r+s-1} \cup P_s' \cup P_{s-1} \times I)).$$

The restriction of α to

$$KR^{n+s}((B^{q-1}, S^{q-2}) \times (P_s \times I, P_s \cup P'_s \cup P_{s-1} \times I)) = KR^{n+s,q}(P_s, P_{s-1})$$

is easily seen to be $-\theta^s(u)$. Hence $-\delta\theta^s(u)$ is simply the restriction α' of α to

$$KR^{n+s}((B^{q-1}, S^{q-2}) \times (W, P'_s \cup P_{s-1} \times I)).$$

Now because u_r over S^{q-2} is constantly e_n, we see that α' is naturally defined in

$$KR^{n+s}((B^{q-1}, S^{q-2}) \times (W, P_s \times I)) = KR^{n+s,q-1}(P_{r+s-1}, P_s)$$

where it exactly represents the element $\theta^{s+1}\partial u$.

PROPOSITION 5.6. *Let $r \leqslant 3$. The map*

$$\theta^s : \pi_{q-1}(V_{n,r}) \rightarrow KS^{n+s-q}(P_{r+s-1}, P_{s-1})$$

is an isomorphism for $n-3 \leqslant q \leqslant n+3-r$ and $n \geqslant r+3$. In particular

$$\theta^s : \pi_{n-1}(V_{n,r}) \rightarrow KR^s(P_{r+s-1}, P_{s-1})$$

is an isomorphism for $r \leqslant 3$ and $n \geqslant r+3$.

Proof. For $r=1$ this follows from 3)–4) of Lemma (5.2), using the well-known fact that KR-theory detects the Hopf-map $\eta : S^3 \rightarrow S^2$ and its square $\eta^2 : S^4 \rightarrow S^2$. The remaining cases now follow by induction on r using Lemma (5.5) and the five lemma.

We shall now compute θ^s in some other special cases. Again let $\iota_n \in \pi_n(S^n)$ be the generator, and consider the homotopy exact sequence for the fibration $V_{n,r} \rightarrow V_{n+1,r+1} \rightarrow S^n$. Denote $\partial\iota_n \in \pi_{n-1}(V_{n,r})$ by $d_{n,r}$.

PROPOSITION 5.7. *$\theta^s(d_{n,r}) \in KR^s(P_{r+s-1}, P_{s-1})$ comes from $KR^s(P_\infty, P_{s-1})$ and the value is*

$n \equiv 0 \bmod 4$: $\theta^0(d_{n,r}) = 1 + H \in KR(P_{r-1})$.

$n \equiv 1 \bmod 4$: *$\theta^3(d_{n,r})$ is the image in A_r^3 of the generator of $A_\infty^3 = Z_2$.*

$n \equiv 2 \bmod 4$: *$\theta^2(d_{n,r})$ is the image in A_r^2 of the generator of $A_\infty^2 = Z$.*

$n \equiv 3 \bmod 4$: $\theta^1(d_{n,r}) = 0$.

Proof. $d_{n,r}$ is the local obstruction at ∞ for the r vector fields on $S^n = \mathbf{R}^n \cup \infty$ given by the standard r-frame of $\mathbf{R}^n$. The proposition now follows by applying our main Theorem (1.1) and using the values of E, S, R for S^n.

For the remainder of this section we assume $n = 4k$ unless otherwise specified. Again, let C_{r-1} denote the Clifford algebra generated by $\{e_1, \dots, e_{r-1}\}$ subject to the relations

$e_i^2 = -1$ and $e_i e_j = -e_j e_i$ for $i \neq j$, We shall sometimes think of C_{r-1} as the even part C_r^0 of C_r, i.e., generated by $\{e_0 e_1, \ldots, e_0 e_{r-1}\}$, where $\{e_0, \ldots, e_{r-1}\}$ generate C_r. Also the number a_r is the dimension of an irreducible C_{r-1}-module, or equivalently, the dimension of the even part of a graded C_r-module.

Suppose n is divisible by a_r, i.e. $\mathbf{R}^n \simeq M$, a module for C_{r-1}. M defines a *Clifford element* $c_{n,r} \in \pi_{n-1}(V_{n,r})$ by the frame

$$c_{n,r}(x) = \{x, e_1 x, e_2 x, \ldots, e_{r-1} x\}.$$

This, of course, depends on the choice of coordinates for the module M, and, if r is divisible by 4, there are different possibilities for the choice of module. For instance, if $n = t a_r$ where t is odd and 4 divides r there are exactly two modules Δ^+ and Δ^-. We shall always choose the coordinates such that the orientation is induced from the complex structure defined by restricting to $C_1 \subset C_{r-1}$. This determines $c_{n,r}$ uniquely for $4 \nmid r$. But if $n = t a_r$ as above and $4 \mid r$, then we get two different elements $c_{n,r}^+$ and $c_{n,r}^-$.

LEMMA 5.8. *Let $c_{n,r} \in \pi_{n-1}(V_{n,r})$ be a Clifford element as defined above. Then*

$$\theta^0(c_{n,r}) = 1 \in KR(P_{r-1}).$$

Proof. Strictly speaking $\theta^0(c_{n,r})$ lies in $KR^{n,n}(P_{r-1})$ or $KR(iT(B^n - S^{n-1}) \times P_{r-1})$ and is given by the square

$$\begin{tikzcd} \Lambda_+^{\mathrm{ev}} \arrow[r, "iL_v"] \arrow[d, "R_{x(y)}"'] & \Lambda_-^{\mathrm{odd}} \arrow[d, "R_{x(y)}"] \\ \Lambda_+^{\mathrm{odd}} \otimes H \arrow[r, "iL_v"] & \Lambda_-^{\mathrm{ev}} \otimes H \end{tikzcd} \tag{5.9}$$

where $y \in B^n, v \in \mathbf{R}^n$ and $x \in S^{r-1}$. Here $x(y)$ is given by the Clifford module structure of C_{r-1} on $\mathbf{R}^n$. Identifying $C_{r-1} = C_r^0$, S^{r-1} is the unit sphere in $\mathbf{R}^r$ with the basis $\{1, e_0 e_1, \ldots, e_0 e_{r-1}\}$. In this way $x(y) = \varrho(x)(y)$ where $\varrho: \mathrm{Spin}(r) \to SO(n)$ is the Spin-representation defined by M, and $x \in S^{r-1} \subset \mathrm{Spin}(r)$.

Now lift ϱ to $\tilde{\varrho}: \mathrm{Spin}(r) \to \mathrm{Spin}(n)$, i.e., $\varrho = \pi \circ \tilde{\varrho}$, where $\pi: \mathrm{Spin}(n) \to SO(n)$ is the covering map. This is clearly possible for $r > 2$, in which case Spin (r) is simply connected. For $r = 2$ the irreducible C_{r-1} module occurs with even multiplicity because $n = 4k$, so we can lift according to Remark following (2.19). For $r = 1$ the lemma follows from Lemma (5.2), so we assume $r > 1$. The lifting is uniquely determined if we require $\tilde{\varrho}(1) = 1$. Let $\varepsilon \in \mathrm{Ker}\,(\mathrm{Spin}(r) \to SO(r))$ denote the element $-1 \in C_r^0$. Clearly $\varrho(\varepsilon) = -1 \in SO(n)$ and therefore $\tilde{\varrho}(\varepsilon) = \pm\omega$, where $\omega \in \mathrm{Spin}(n)$ is the volume element. From our orientation conventions it follows easily that $\tilde{\varrho}(\varepsilon) = \omega$ (it is enough to consider $r=2$). Hence $\tilde{\varrho}(-x) = \tilde{\varrho}(\varepsilon x) = \omega\,\tilde{\varrho}(x)$.

We thus have a map $\tilde{\varrho}: S^{r-1} \to \mathrm{Spin}(n)$ such that $x(y) = \tilde{\varrho}(x)^{-1}\, y\, \tilde{\varrho}(x)$ for $x \in S^{r-1}$ and $y \in \mathbf{R}^n$. Furthermore, $\tilde{\varrho}$ is equivariant with respect to the involutions: multiplication by -1 on S^{r-1} and multiplication by ω on $\mathrm{Spin}(n)$.

It is now easy to check that right-multiplication by $\tilde{\varrho}(x)^{-1}$ transforms the square (5.9) into the square

$$\begin{tikzcd} \Lambda_+^{\mathrm{ev}} \arrow[r, "iL_v"] \arrow[d, "R_y"'] & \Lambda_-^{\mathrm{odd}} \arrow[d, "R_y"] \\ \Lambda_+^{\mathrm{odd}} \arrow[r, "iL_v"] & \Lambda_-^{\mathrm{ev}} \end{tikzcd}$$

which represent the periodicity class in $KR^{n,n}(P_{r-1})$.

It follows from Proposition (5.7) and Lemma (5.8) that $\theta^0: \pi_{n-1}(V_{n,r}) \to KR(P_{r-1})$ is onto for n divisible by a_r. In fact it is (with certain modifications) the projection onto a direct summand. Before showing this, let us define a modification of θ for r divisible by 4. Put

$$\pi'_{n-1}(V_{n,r}) = \mathrm{Im}\,[\pi_{n-1}(V_{n,r+1}) \to \pi_{n-1}(V_{n,r})]. \tag{5.10}$$

We shall define a map

$$\theta': \pi'_{n-1}(V_{n,r}) \to KR(P_r). \tag{5.11}$$

Consider the commutative diagram with exact columns

$$\begin{tikzcd} \pi_{n-1}(S^{n-r-1}) \arrow[r, "\theta^0"] \arrow[d] & KR(P_r, P_{r-1}) \arrow[d] \\ \pi_{n-1}(V_{n,r+1}) \arrow[r, "\theta^0"] \arrow[d] & KR(P_r) \arrow[d] \\ \pi'_{n-1}(V_{n,r}) \arrow[r, "\theta^0"] \arrow[d] & KR(P_{r-1}) \\ 0 & \end{tikzcd} \tag{5.12}$$

According to 4) in Lemma (5.2) the map in the top row of (5.12) is just the Hurewicz map $\pi_{n-1}(S^{n-r-1}) \to \widetilde{KR}^{n-r-1}(S^{n-1})$. For $r>0$ the image of this map consists of torsion (as one sees by taking Chern characters). However, $KR(P_r, P_{r-1}) \simeq Z$ for r divisible by 4, so the image is actually zero. It follows that the map

$$\pi_{n-1}(V_{n,r+1}) \rightarrow KR(P_r)$$

factors through $\pi'_{n-1}(V_{n,r})$, thus defining the desired θ'.

PROPOSITION 5.13. *Let $n = 4k$ be divisible a_r, and assume $r < n/2$.*

1) *Suppose $4 \nmid r$. Then*

$$\theta : \pi_{n-1}(V_{n,r}) \rightarrow KR(P_{r-1}) \simeq Z \oplus Z_{a_r}$$

is a projection onto a direct summand, and Z_{a_r} is the highest 2-primary cyclic summand in $\pi_{n-1}(V_{n,r})$.

2) *Suppose $4 \mid r$ and let n be divisible by $a_{r+1} = 2a_r$. Then*

$$\theta' : \pi'_{n-1}(V_{n,r}) \rightarrow KR(P_r) \simeq Z \oplus Z_{a_{r+1}}$$

is the projection onto a direct summand, and $Z_{a_{r+1}}$ is the highest 2-primary cyclic summand in $\pi_{n-1}(V_{n,r})$.

Proof. Consider the Adams spectral sequence for $P_{n+1,r+1} = P_n/P_{n-r-1}$. It follows from [1], Theorem 2, that the highest 2-primary order of $\pi_{n-1}(P_{n+1,r+1})$, for $r < n/2$, is at most $2a_r$ in case $4 \nmid r$ and at most $2a_{r+1}$ for $4 \mid r$.(1) It is well known (see [16]) that $\pi_{n-1}(V_{n+1,r+1}) \simeq \pi_{n-1}(P_{n+1,r+1})$ for $r < n/2$.

The fibration $p : V_{n,r} \rightarrow S^{n-1}$ is split by the Clifford element $c_{n,r}$. Hence we have the following diagram with exact columns and rows

$$\begin{tikzcd}
0 \arrow[dr] & & \pi_n(S^n) \arrow[d] \arrow[dr] & \\
0 \arrow[r] & \pi_{n-1}(V_{n-1,r-1}) \arrow[r] \arrow[dr] & \pi_{n-1}(V_{n,r}) \arrow[r, shift left] \arrow[d, "j_*"] & \pi_{n-1}(S^{n-1}) \arrow[l, shift left, "p_*"] \\
& & \pi_{n-1}(V_{n+1,r+1}). &
\end{tikzcd} \tag{5.14}$$

Now consider $x = d_{n,r} - 2c_{n,r}$ in $\pi_{n-1}(V_{n,r})$. By (5.7) and (5.8) $\theta(x) = H - 1$ so x has order at least a_r. On the other hand, if $4 \nmid r$, $j_* x = -2 j_* c_{n,r}$ has 2-primary order at most a_r. Hence $j_* a_r x = 0$ modulo odd torsion; i.e., $a_r x = \partial(m\iota_n)$ modulo odd torsion for $m \in Z$. But

(1) We are indebted to Elmer Rees for showing this fact to us.

in that case $0 = p_* a_r x = 2m\iota_{n-1}$, so $m = 0$. We conclude that the 2-primary order of x is a_r, from which 1) follows easily. The proof of 2) is analogous.

Since θ^s is an isomorphism for $r \leqslant 3$ and $n \geqslant r+3$ (Proposition (5.6)), Theorem (1.1) gives a complete answer to the problem of expressing the sum of the local obstructions in terms of global invariants, in case $r \leqslant 3$ and X is oriented. Emery Thomas has already listed the result for $r = 2$ in [20], Table 1. Let us make the corresponding table for $r = 3$:

$\dim X \geqslant 6$	$\pi_{n-1}(V_{n,3})$	$\operatorname{ind} \alpha_X$
$4k$	$Z \oplus Z_4$	$E(X) \oplus \frac{1}{2}(E(X) - (-1)^k S(X))$
$4k+1$	Z_2	$R(X)$
$4k+2$	$Z \oplus Z_2$	$\frac{1}{2} E(X) \oplus 0$
$4k+3$	$Z_2 \oplus Z_2$	0

In particular we have the following necessary and sufficient conditions for having 3 linearly independent vector fields on a simply connected manifold(¹)

$\dim X \geqslant 6$			
$4k$	$w_{4k-2}(X) = 0$	$E(X) = 0$	$S(X) \equiv 0 \bmod 8$
$4k+1$	$\delta^* w_{4k-2}(X) = 0$	$R(X) = 0$	
$4k+2$	$w_{4k}(X) = 0$	$E(X) = 0$	
$4k+3$	$\delta^* w_{4k} = 0$(²)		

Here, as usual, w_i is the i-th Stiefel-Whitney class and $\delta^*: H^i(X, Z_2) \to H^{i+1}(X, Z)$ is the Bockstein homomorphism.

Let us mention as a curiosity, that we also can give necessary and sufficient conditions for the existence of 5 vector fields on 3-connected manifolds of dimension $8k$, $k > 1$. We have already mentioned that $\theta^0: \pi_{8k-1}(V_{8k,5}) \to Z \oplus Z_8$ is an isomorphism. Hence the conditions are

$$w_{8k-4}(X) = 0, \quad E(X) = 0 \quad \text{and} \quad S(X) \equiv 0 \mod 16.$$

6. More on dimension $4k$

In this section we shall concentrate on dimension $4k$ and obtain a few refinements and extensions of the basic theorem.

First we consider the problem of fields of oriented 2-planes on X with finite singularities. Thus we suppose given an oriented 2-dimensional sub-bundle of TX outside a

(¹) This assumption can be removed for $\dim X \neq 4k+1$.

(²) As shown by Massey this condition is always fulfilled.

finite set of points $\{A_i\}$. If we fix a Riemannian metric on X such a field of 2-planes is described by a section u of $\Lambda^2(TX)$ outside the $\{A_i\}$, normalized so that the Clifford product $uu = -1$. If u_1, u_2 are a local oriented orthogonal basis for the 2-plane, then $u = u_1 u_2$ in the Clifford algebra.

Let $G_{n,2} = SO(n)/SO(n-2) \times SO(2)$ denote the Grassmannian of oriented 2-planes in $\mathbf{R}^n$. Then the projection $p: V_{n,2} \to G_{n,2}$ has fibre $S^1 = SO(2)$ and so $p_*: \pi_{n-1}(V_{n,2}) \to \pi_{n-1}(G_{n,2})$ is an isomorphism for $n \geqslant 3$. Now for $n = 4k$ $(k > 1)$ we have $\pi_{n-1}(V_{n,2}) \cong Z \oplus Z_2$, the isomorphism being induced by

$$\theta: \pi_{n-1}(V_{n,2}) \to KR(P_1) = Z \oplus \widetilde{KR}(P_1)$$

(see Proposition (5.6)). Hence $\pi_{n-1}(G_{n,2}) \cong Z \oplus Z_2$ and we may ask for a formula giving the sum of the local obstructions for the field u. Thomas [19] has determined which integers can occur as the sum of the local Z-obstructions. We shall now derive a formula for the sum of the local Z_2-obstructions, showing in particular that this part is independent of the choice of u.

Going back to our basic square (2.8) for $r = 2$ we write

$$x_1 u_1(y) + x_2 u_2(y) = u_1(y)\,(x_1 - x_2 u(y))$$

where $u = u_1 u_2$. This exhibits (2.8) as a composition of two squares [1]

$$\begin{tikzcd}
\Lambda_+^{\mathrm{ev}}(E) \arrow[r, "iL_v"] \arrow[d, "R_{u_1}"'] & \Lambda_-^{\mathrm{odd}}(E) \arrow[d, "R_{u_1}"] \\
\Lambda_+^{\mathrm{odd}}(E) \arrow[r, "iL_v"] \arrow[d] & \Lambda_-^{\mathrm{ev}}(E) \arrow[d] \\
\Lambda_+^{\mathrm{odd}}(E) \otimes H \arrow[r, "iL_v"] & \Lambda_-^{\mathrm{ev}}(E) \otimes H
\end{tikzcd}$$

Applied locally this expresses our homomorphism

$$\theta: \pi_{n-1}(V_{n,2}) \to KR(P_1)$$

as a sum of two homomorphisms θ_1 and θ_2. Now the top square is independent of x and, by (5.8), θ_1 projects onto the Z-summand of $KR(P_1)$. Hence θ_2 is the projection onto $Z_2 = \widetilde{KR}(P_1)$. But since θ_2 only involves u (and not u_1, u_2) it can be used to define a re-

(1) It is well known that composition is stably homotopic to direct sum.

lative characteristic class for a global 2-field u. The corresponding global index is, in the notation of §4, ind $B^-(1-H)$. Using the formula for ind B^- given in §4 we obtain finally the following result:

THEOREM 6.1. *Let X be a closed oriented manifold of dimension $4k$, $k>1$. Let u be a 2-plane field with a finite number of singularities. Then the sum of the local Z_2-obstructions is* $\frac{1}{2}(E(X)-(-1)^k S(X)) \bmod 2$.

When the number of vector fields is divisible by 4, we get a theorem analogous to Theorem (2.20) using the local index map (5.11) defined on the subgroup $\pi'_{n-1}(V_{n,r}) \subset \pi_{n-1}(V_{n,r})$.

THEOREM 6.2. *Let X be a closed oriented manifold of dimension $n=4k$, and let $\{u_1, \dots, u_r\}$ be a set of vector fields as in Theorem* (2.20). *Suppose r is divisible by* 4 *and suppose that at every singular point $A_i \in X$ the local obstruction $O_{A_i}(u_1, \dots, u_r)$ lies in $\pi'_{n-1}(V_{n,r})$. Then in the group $KR(P_r)$ we have the formula*

$$\operatorname{ind} \alpha^0_{X,r+1} = \sum_{i=1}^{l} \theta'(O_{A_i}(u_1, \dots, u_r)).$$

For the proof of this theorem first observe the following. Suppose X is the union of two manifolds W and W' with common boundary $Y=W \cap W'$ and let $\{u_0, u_1, \dots, u_r\}$ be any set of vector fields (r not necessarily divisible by 4) independent over Y. Then clearly by excision

$$\operatorname{ind} \alpha^0_{X,r+1} = \operatorname{ind} \alpha^0_W(u_0, \dots, u_r) + \operatorname{ind} \alpha^0_{W'}(u_0, \dots, u_r) \tag{6.3}$$

in $KR(P_r)$.

In particular for the proof of (6.2) put $W=\bigcup_{i=1}^{l} B_i$ and $W'=X-(\bigcup_{i=1}^{l} \mathring{B}_i)$ and $Y=\bigcup_{i=1}^{l} S_i$. Choose a vector field u_0 on X, never linearly dependent on $\{u_1, \dots, u_r\}$ over Y (this is possible by assumption). Formula (6.2) then follows from (6.3) and the following proposition.

PROPOSITION 6.4. *Let W be a compact oriented manifold with boundary ∂W, and let $\{u_0, u_1, \dots, u_r\}$ be a set of vector fields, linearly independent along ∂W. Suppose r is divisible by* 4. *Then*

1) ind $\alpha^0_W(u_0, u_1, \dots, u_r) \in KR(P_r)$ *depends only on $\{u_1, \dots, u_r\}$ and not on u_0.*
2) *If furthermore $\{u_1, \dots, u_r\}$ are linearly independent over all of W, then*

$$\operatorname{ind} \alpha^0_W(u_0, \dots, u_r) = 0 \text{ in } KR(P_r).$$

Proof. The first statement follows formally from the second. Thus define a manifold

$\tilde{W}$ by gluing a collar $(\partial W)\times I$ on W such that ∂W is identified with $\partial W\times 0$. Now suppose u_0 and u_0' are two vector fields on ∂W so that both sets $\{u_0, u_1, \dots, u_r\}$ and $\{u_0', u_1, \dots, u_r\}$ are linearly independent on ∂W. We then define a set $\{v_0, v_1, \dots, v_r\}$ of vector fields on $\tilde{W}$ such that v_0 is u_0' on $\partial W\times 1$ but u_0 on $W\subset \tilde{W}$, and such that v_i for $i>0$ is the natural extension of u_i to $\tilde{W}$, constant along the interval I. By excision

$$\operatorname{ind}\alpha_{\tilde{W}}(v_0, \dots, v_r) = \operatorname{ind}\alpha_{\partial W\times I}(v_0, \dots, v_r) + \operatorname{ind}\alpha_W(u_0, u_1, \dots, u_r).$$

Also $\tilde{W}$ is diffeomorphic to W in such a way that $\operatorname{ind}\alpha_{\tilde{W}}(v_0, \dots, v_r) = \operatorname{ind}\alpha_W(u_0', u_1, \dots, u_r)$. Hence statement 1) is a consequence of the fact that $\operatorname{ind}\alpha_{\partial W\times I}(v_0, v_1, \dots, v_r)=0$ according to 2).

In the proof of the second statment assume for convenience that $\{u_1, \dots, u_r\}$ are orthonormal over W and that $\{u_0, u_1, \dots, u_r\}$ are orthonormal over ∂W. Setting $T=TW$ for short, we find that $\alpha_W(u_0, \dots, u_r)\in KR((iT, iT\,|\,\partial W)\times P_r)$ is the image of an element $b\in KR((iT, iT\,|\,\partial W)\times (P_r, P_{r-1}))$ under the map

$$j^*: KR((iT, iT\,|\,\partial W)\times (P_r, P_{r-1})) \;\to\; KR((iT, iT\,|\,\partial W)\times P_r)$$

(see §2).

Now the pair (P_r, P_{r-1}) is relatively homeomorphic to the pair (B^r, S^{r-1}) of the ball and the sphere in $\mathbf{R}^r$. Let BW and SW denote the ball- and sphere-bundle of TW, and let $\pi: BW\to W$ be the projection. The vector fields $\{u_0, u_1, \dots, u_r\}$ then define a map

$$f: (W, \partial W)\times (B^r, S^{r-1}) \;\to\; (BW, SW)$$

such that the induced map

$$\bar{f}: (iT, iT\,|\,\partial W)\times (B^r, S^{r-1}) \;\to\; (\pi^*(iT), \pi^*(iT)\,|\,SW)$$

has the following property. The bundle $\pi^*(iT) = T\oplus iT$ over W has a Thom class

$$\lambda_T\in KR(\pi^*(iT), \pi^*(iT)\,|\,SW) \text{ and } \bar{f}^*\lambda_T = b.$$

The map f is defined by the formula

$$f(w, y) = (\sqrt{1-|y|^2})\, u_0(w) + y_1\, u_1(w) + \dots + y_r\, u_r(w)$$

for $w\in W$ and $y=(y_1, \dots y_r)\in B^r$.

At this point let us recall the definition of topological index [9]. Embed W in $\mathbf{R}^{n+q}$ with normal bundle N of dimension q. The index map

$$KR((iT, iT\,|\,\partial W)\times P_r) \;\to\; KR^{n+q,n+q}(P_r)$$

is defined as the composite map of the Thom isomorphism

$$\Phi: KR((iT, iT\,|\,\partial W)\times P_r) \;\to\; KR((N\oplus iN, (N\oplus iN)\,|\,\pi^{-1}\partial W)\times P_r)$$

and the map induced by the Thom map

$$g:(\mathbf{R}^{n+q,\,n+q})^+ \;\to\; (N\oplus iN)^+/(N\oplus iN\,|\,\pi^{-1}\partial W).$$

We have analogous maps for P_r replaced by (P_r, P_{r-1}).

By naturality
$$\operatorname{ind}\alpha_W(u_0,\dots,u_r) = j^*\circ g^*\circ\Phi(b) \tag{6.5}$$
where
$$j^*: KR(\mathbf{R}^{n+q,\,n+q}\times(P_r,P_{r-1})) \;\to\; KR^{n+q,\,n+q}(P_r).$$

We also have the commutative diagram

$$\begin{tikzcd}
KR((iT, iT\,|\,\partial W)\times(P_r,P_{r-1})) \arrow[d, "\Phi"] & KR(T\oplus iT) \arrow[l, "\tilde f^*"'] \arrow[d, "\Phi"] \\
KR((N\oplus iN, N\oplus iN\,|\,\pi^{-1}\partial W)\times(P_r,P_{r-1})) \arrow[d, "g^*"] & KR(T\oplus iT\oplus N\oplus iN) \arrow[l, "\tilde f^*"'] \arrow[d, equal] \\
KR^{n+q,\,n+q}(P_r,P_{r-1}) & KR(\mathbf{R}^{n+q,\,n+q})\times W
\end{tikzcd}$$

where $\tilde f$ is induced from f in the obvious way.

Clearly $\Phi(\lambda_T)$ is the Thom class of the trivial bundle $\mathbf{R}^{n+q,\,n+q}\times W$, and therefore induced from the sphere $(\mathbf{R}^{n+q,\,n+q})^+$. It follows that $g^*\circ\Phi(b)$ is induced by the equivariant map

$$p\circ\tilde f\circ g:(\mathbf{R}^{n+q,\,n+q})^+\wedge S^r \;\to\; (\mathbf{R}^{n+q,\,n+q})^+$$

where $p:(\mathbf{R}^{n+q,\,n+q}\times W)^+\to(\mathbf{R}^{n+q,\,n+q})^+$ is the projection. But, as we observed for the definition of (5.11), the image of $(p\circ\tilde f\circ g)^*$ in KR-theory consists of torsion. On the other hand, $KR^{n+q,\,n+q}(P_r,P_{r-1})\cong Z$ for r divisible by 4. Hence $g^*\circ\Phi(b)=0$, and consequently $\operatorname{ind}\alpha_W(u_0,u_1,\dots,u_r)=0$ by (6.5).

The Theorems (1.1) and (6.2) have the following corollary due to Mayer [17] and, in a weaker form, to Frank [14].

COROLLARY 6.6. *Let X be a closed oriented manifold of dimension $4k$ and suppose X admits r linearly independent vector fields. Then the signature $S(X)$ is divisible by $2a_r$ for $4\nmid r$ and it is divisible by $2a_{r+1}=4a_r$ for $4\mid r$.*

Explicitly the signature is divisible by b_r where b_r is given by the table

r	1	2	3	4	5	6	7	8
b_r	2	4	8	16	16	16	16	32

and $b_{r+8} = 16 \cdot b_r$

We conclude this section with a different application of our characteristic class (2.9). Suppose X is the boundary of an oriented manifold W of dimension $n = 4k$ and let $\{u_1, \dots, u_r\}$ be linearly independent vector fields on X. We shall define an invariant $\theta_X(u_1, \dots, u_r)$ in $KR(P_r)/(1+H) = Z_{2a_{r+1}}$, which does not depend on the choice of the bounding manifold W.

First let v denote the outward pointing normal vector field on X, and consider

$$\operatorname{ind} \alpha_W(v, u_1, u_2, \dots, u_r) \in KR(P_r). \tag{6.7}$$

Now let W' be another manifold with $\partial W' = X$ and consider the closed manifold $\tilde{W} = W \cup (-W')$ such that $X = W \cap (-W')$. Here $-W'$ denotes as usual W' with the opposite orientation. By excision we have

$$\operatorname{ind} \alpha_W(v, u_1 \dots, u_r) + \operatorname{ind} \alpha_{-W'}(v, u_1, \dots, u_r) = \operatorname{ind} \alpha_{\tilde{W}, r+1}. \tag{6.8}$$

The outward pointing normal on X with respect to W' is $-v$, so we actually want to compare (6.7) with the element $\operatorname{ind} \alpha_{W'}(-v, u_1, \dots, u_r)$. The change of orientation amounts in $KR(P_r)$ to multiplication by H. Furthermore, the shift from v to $-v$ corresponds to the map induced in $KR(P_r)$ by the map of P_r which changes the sign of the first coordinate. This, however, induces the identity, and so

$$\operatorname{ind} \alpha_{-W'}(v, u_1, \dots, u_r) = H \operatorname{ind} \alpha_{W'}(-v, u_1, \dots, u_r).$$

Hence using (4.2), we deduce from (6.8):

$$\operatorname{ind} \alpha_W(v, u_1, \dots, u_r) - \operatorname{ind} \alpha_{W'}(-v, u_1, \dots, u_r) \equiv (-1)^{k+1} S(\tilde{W}) \cdot 1 \mod (1+H) \tag{6.9}$$

where $S(\tilde{W})$ is the signature of the closed $4k$-dimensional manifold $\tilde{W}$. As noted by Novikov the signature can be defined for manifolds with boundary, so that $S(\tilde{W}) = S(W) - S(W')$ (see [10]). It follows from (6.9), that the invariant

$$\theta_X(u_1, \dots, u_r) = \operatorname{ind} \alpha_W(v, u_1, \dots, u_r) + (-1)^k S(W) \in KR(P_r)/(1+H) \tag{6.10}$$

is independent of the choice of W.

If $X = S^{n-1}$, $n = 4k$, then θ_X is induced by the restriction of

$$\theta^0 : \pi_{n-1}(V_{n, r+1}) \to KR(P_r)$$

to the set $p_*^{-1}(\iota_{n-1})$ where $p_*:\pi_{n-1}(V_{n,r+1}) \to \pi_{n-1}(S^{n-1})$. In particular, if a_{r+1} divides n, then the Clifford element $c_{n,r+1}$ defines r vector fields on S^{n-1}. According to Lemma (5.8)

$$\theta_{S^{n-1}}(c_{n,r+1}) = 1. \tag{6.11}$$

Consider the map $X \to X \vee S^{n-1}$ defined by pinching the boundary of a disk embedded in X. This map defines an action of $\pi_{n-1}(V_{n-1,r})$ on the set of r-frames on X. Thus if $\{u_1, \dots, u_r\}$ is a set of vector fields on X and $y \in \pi_{n-1}(V_{n-1,r})$, we get a new set of vector fields $\{u_1, \dots, u_r\}^y$. Mapping $\pi_{n-1}(V_{n-1,r})$ into $\pi_{n-1}(V_{n,r+1})$ we have defined θ^0 on this group, and it is immediate that

$$\theta_X(\{u_1, \dots, u_r\}^y) = \theta_X(u_1, \dots, u_r) + \theta^0(y). \tag{6.12}$$

We now have

PROPOSITION 6.13. *Let X be a closed n-1-dimensional oriented manifold, and suppose $n=4k$ is divisible by a_{r+1}. If X has one r-frame field then it has at least a_{r+1} non-homotopic r-frames. If furthermore X is an oriented boundary, then these r-frames are distinguished by the invariant θ_X.*

Proof. The manifold $Y = X \cup (-X)$ is clearly a boundary (of $X \times I$), and we let $\pi_{n-1}(V_{n-1,r})$ act on the end $X \times 1$. Using (6.12), and the fact (Proposition (5.13)) that there is an element $x \in \pi_{n-1}(V_{n-1,r})$ with $\theta^0(x) \equiv 2 \mod H+1$, we easily verify the first statement. The second statement is equally obvious.

7. More on dimension $4k+1$

We now proceed to further refinements in dimension $4k+1$. First let us remark that the skew-adjoint operator T, with index the semi-characteristic, can be used to obtain information about 2-plane fields.

Again let X be a closed oriented manifold of dimension $n=4k+1$ and let $T = L_\omega \circ (d+\delta)$ be the skew-adjoint operator on the space $\Omega^{ev}(X)$ of real exterior forms. Suppose that TX admits a 2-plane field u over $Y \subset X$. Clearly Clifford multiplication by u defines an automorphism of the symbol for T, so a construction analogous to (2.10) yields a characteristic class

$$\bar{\alpha}_X(u) \in KR^{-1}((iTX \mid X - Y) \times (P_1 - P_0)). \tag{7.1}$$

Obviously $\bar{\alpha}_X \in KR^{-1}((iTX) \times (P_1 - P_0))$ is simply $\sigma(T)(H-1)$, and the index in $\widetilde{KR}^{-1}(P_1) = KR^{-2}(\text{pt})$ is $R(X)\eta^2$, where $\eta \in \widetilde{KR}^{-1}(\text{pt})$ is the generator. Also $\bar{\alpha}$ defines a local invariant $\bar{\theta}: \pi_{n-1}(G_{n,2}) \to Z_2$, which is in fact an isomorphism. We therefore obtain

THEOREM 7.2. *Let X be a closed oriented manifold of dimension $n=4k+1$ and let u be an oriented 2-plane field with singularities at a finite number of points. Then the sum of the local obstructions is the semi-characteristic $R(X)$.*

Until now we have supposed that X at least was orientable. We shall now extend the definition of the semi-characteristic to a wider class of manifolds and thereby solve the problem raised by Thomas [20] concerning the index of a 2-frame field on a $4k+1$-dimensional manifold satisfying $w_1^2=0$. Thomas has shown that if $w_1^2\neq 0$ then both 0 and 1 in Z_2 occur as the index for a 2-frame field. On the other hand he has also shown, that the sum of the local obstructions is independent of the choice of 2-frame, provided $w_1^2=0$. Manifolds satisfying this condition are studied in [13], and we recall some of the basic facts.

Let $\varrho:\tilde{X}\to X$ denote the orientation covering and let Θ denote the associate real bundle. Consider the Bockstein exact sequence for the coefficient sequence

$$Z_2\to Z_4\to Z_2.$$

The condition $w_1^2=0$ implies that there is a Z_4-covering $\Gamma: Y\to X$, such that if g is the corresponding automorphism of Y of order 4, then Y/g^2 is the orientation covering $\tilde{X}$ of X.

Now Z_4 acts on $\mathbf{C}$ by multiplication with i. Hence Γ gives rise to a coefficient system L of complex numbers, and $L\otimes_{\mathbf{C}} L=\Theta\otimes_{\mathbf{R}}\mathbf{C}$. The cohomology $H^*(X,L)$ with coefficients in L has a cup-product

$$H^*(X,L)\otimes H^*(X,L)\ \to\ H^*(X,\Theta\otimes_{\mathbf{R}}\mathbf{C}).$$

This defines a Poincaré pairing

$$H^{n-p}(X,L)\otimes H^p(X,L)\ \to\ H^n(X,\Theta\otimes_{\mathbf{R}}\mathbf{C})=\mathbf{C}.$$

It therefore makes sense to define the semi-characteristic

$$R_L(X)=\tfrac{1}{2}\left(\sum_{i=0}^{n}\dim_{\mathbf{C}} H^i(X,L)\right)\quad \mathrm{mod}\ 2$$

for n odd. From now on, we shall assume $n=4k+1$. $R_L(X)$ is again the index for an elliptic operator. L is a flat complex line bundle so $H^*(X,L)$ is the homology of the de Rham complex $\Omega^*(X,L)$ of sections of $\Lambda^*(TX)\otimes L$. $\Omega^*(X,L)$ is just the forms α on the 4-fold covering Y satisfying $g^*\alpha=i\alpha$.

Again $d+\delta$ defines an operator $D_L:\Omega^{\mathrm{ev}}(X,L)\to\Omega^{\mathrm{odd}}(X,L)$ and clearly $R_L(X)=\dim_{\mathbf{C}}\operatorname{Ker} D_L$. Let L^{-1} denote the conjugate line bundle of L. Clifford multiplication then induces

$$\Omega^*(X,\Theta)\otimes\Omega^*(X,L^{-1}) \rightarrow \Omega^*(X,L)$$

or equivalently an *anti-linear* map

$$\Omega^*(X,\Theta)\otimes\Omega^*(X,L) \rightarrow \Omega^*(X,L).$$

In particular left multiplication by the volume form $\omega\in\Omega^n(X,\Theta)$ induces an anti-linear map $L_\omega:\Omega^{\text{odd}}(X,L)\rightarrow\Omega^{\text{ev}}(X,L)$ and the composite operator $T_L=L_\omega\circ D_L$ is a *skew-adjoint anti-linear* operator of $\Omega^{\text{ev}}(X,L)$.

As explained in [8] and [12] such operators have an index in $KR^{-2}(\text{pt})=Z_2$ equal to the complex dimension of the kernel. In fact, if $A:V\rightarrow V$ is an elliptic skew-adjoint anti-linear operator of a complex Hilbert space V; then we get a family of elliptic operators over CP_1. Over a point $\{x,y\}\in CP_1$ this family is defined by $xI+yA:V\rightarrow V\otimes H$.

Hence T_L has a symbol $\sigma(T_L)$ in $KR^{-2}(iTX)$ and ind $\sigma(T_L)=R_L(X)\eta^2$ in $KR^{-2}(\text{pt})=Z_2$.

Notice that if X is oriented and L is the trivial line bundle then T_L is just the complexification of the operator T and $\sigma(T_L)=\sigma(T)\eta$.

At this point we can clearly construct a relative characteristic symbol for an oriented 2-plane in analogy with (7.1). But unfortunately the index map of this element will end up in $KR^{-2}(P_1)=0$. Suppose even we had a 2-plane field u without singularities. Right multiplication with u would then just give a linear endomorphism R_u of $\Omega^{\text{ev}}(X,L)$ with square -1, and this would imply nothing about the dimension of the kernel of T_L.

On the other hand, if we have a *twisted* 2-plane field, i.e., a plane field with *oriented complement*, then this gives an element $u\in\Omega^2(X,\Theta)$ and R_u is an *anti-linear* endomorphism of square -1. Hence Ker T_L is quaternionic and therefore of even complex dimension.

Notice that in the oriented case, there is no difference between twisted and untwisted fields; but in the non-orientable case the analysis forces us to restrict to twisted fields.

In order to construct the corresponding characteristic class we need an involution also on the projective space. Thus conjugation in $\mathbf{R}^n=i\mathbf{R}^p\oplus\mathbf{R}^q$ induces an involution on P_{n-1}, and we shall denote this Real space by $P^{p,q}$. The Hopf bundle H is clearly equivariant, so $H\otimes\mathbf{C}$ has a natural Real structure over $P^{p,q}$. Stunted projective spaces with involution occur naturally as Thom complexes with involution. For example, $P^{1,n}/P^{1,0}=P^{iH}_{n-1}$.

Now return to a manifold X of dimension $4k+1$ satisfying $w_1^2=0$ and choose a coefficient system L as above. Suppose that u is a twisted 2-plane field over $Y\subset X$, i.e. u is a section over Y of $\Lambda^2(TX)\otimes\Theta$, satisfying $u^2=-1$. For $x_0\in\mathbf{R}$ and $x_1\in\mathbf{C}$ such that $x_0^2+|x_1|^2=1$, we form as usual $x(y)=x_0+x_1u(y)$. Also let $x'(y)=x_0-x_1u(y)$. Because $R_{u(y)}$ is anti-linear and $R^2_{u(y)}=-1$, we have $R_{x'(y)}\circ R_{x(y)}=1$. Hence $R_{x(y)}$ is an automorphism of the symbol for T_L, and we therefore get a characteristic class

$$\alpha_X^L(u) \in KR^{-2}((iTX|X-Y)\times(P(\mathbf{R}\oplus\mathbf{C})-P(\mathbf{R}))) = KR^{-2}((iTX|X-Y)\times(P^{1,2}-P^{0,1})). \quad (7.3)$$

The class $\alpha_X^L \in KR^{-2}((iTX)\times(P^{1,2}-P^{0,1}))$ is just $\sigma(T_L)(H-1)$, hence the index of α_X^2 in $KR^{-2}(P^{1,2})$ is $R_L(X)(H-1)\eta^2$. The element $(H-1)\eta^2 \in KR^{-2}(P^{1,2}) \simeq \widetilde{KR}^0(P_2) \simeq Z_4$ generates the subgroup of order 2. Also we have a local invariant $\tilde{\theta}: \pi_{n-1}(G_{n,2}) \to Z_2$ which is an isomorphism. We thus get the following extension of Theorem (7.2):

THEOREM 7.4. *Let X be a closed manifold of dimension $n = 4k+1$ satisfying $w_1^2 = 0$, and let L be as above. Suppose u is a twisted 2-plane field with finite singularities. Then the sum of the local obstructions is $R_L(X)$.*

Remark. The above discussion for 2-plane fields can obviously be extended to p-plane fields with $p \equiv 2 \bmod 4$. We remark that *twisted* p-plane fields with $p \equiv 2 \bmod 4$ are in one-to-one correspondence with *oriented* $n-p$-plane fields, $n-p \equiv 3 \bmod 4$.

A particular case of a twisted 2-plane is given by a pair consisting of a vector field and an embedded line bundle isomorphic to Θ. The obstruction for having such a "twisted 2-frame field" with finite singularities is

$$w_{n-1}(TX-\Theta) = w_{n-1}(X) = w_{n-2}(X) \cup w_1(X).$$

But if follows from the Wu-formulas that $w_{n-2} \cup w_1 = 0$. Hence the obstruction for having a twisted 2-frame field with a finite number of singularities is the same as for an ordinary 2-frame field.

Nevertheless we can also calculate the sum of the local obstructions for an ordinary 2-frame field in the following way. In general let $\{u_1, \dots, u_r\}$ be a set of vector fields, linearly independent over $Y \subset X$. Right multiplication by $\omega \cdot u_i$ (which are sections of $\Lambda^{\mathrm{ev}}(X) \otimes \Theta$) give antilinear maps with square $+1$. In the usual way this gives rise to a characteristic class

$$\tilde{\alpha}_X^L(u_1, \dots, u_r) \in KR^{-2}((iTX|X-Y)\times P_{r-1}) \quad (7.5)$$

such that $\tilde{\alpha}_{X,r}^L \in KR^{-2}(iTX \times P_{r-1})$ is just $\sigma(T_L)(H-1)$. Hence the index of $\tilde{\alpha}_{X,r}^L$ in $KR^{-2}(P_{r-1})$ is $R_L(X)(H-1)\eta^2$. In particular for $r=3$ the index ends up in $KR^{-2}(P_2) \simeq Z_2$, and we also have a local index map $\tilde{\theta}: \pi_{n-1}(V_{n,3}) \to Z_2$. Now the natural map $\pi_{n-1}(V_{n,3}) \to \pi_{n-1}(V_{n,2})$ is an isomorphism and both groups are isomorphic to Z_2 (see, for example, Proposition (5.6)). Hence we have $\tilde{\theta}: \pi_{n-1}(V_{n,2}) \to Z_2$.

THEOREM 7.6. *Let X and L be as in Theorem (7.4) and let $\{u_1, u_2\}$ be 2 vector fields on X with singularities at a finite set of points $\{A_1, \dots, A_l\}$. Then*

$$\sum_{i=1}^{l} \bar{\theta}(O_{A_i}(u_1, u_2)) = R_L(X)$$

where $O_{A_i}(u_1, u_2) \in \pi_{n-1}(V_{n,2})$ *is the local obstruction.*

For the proof we shall use the same method as for the proof of Theorem (6.2). Thus it suffices to prove the following analogue of Proposition (6.4).

LEMMA 7.7. *Let* W *be a compact manifold with boundary* ∂W, *and suppose* $w_1^2(W) = 0$. *Let* L *be a complex coefficient system as before, and let* $\{u_0, u_1, u_2\}$ *be a set of vector fields, linearly independent over* ∂W. *If the set* $\{u_1, u_2\}$ *is linearly independent over all of* W, *then in* $KR^{-2}(P_2) = Z_2$

$$\operatorname{ind} \bar{\alpha}_W^L(u_0, u_1, u_2) = 0.$$

Proof: As in the proof of Proposition (6.4) $\bar{\alpha}_W^L(u_0, u_1, u_2) = j^* b_L$, where j is the map $P_2 \to P_2/P_1$ and

$$b_L \in KR^{-2}((iTW \mid W - \partial W) \times (P_2 - P_1))$$

is defined by a certain square over $iTW \times CP_1 \times P_2$. Conjugation defines a map $\mu: CP_1 \to CP_1$ which induces -1 in $KR^{-2}((iTW \mid W - \partial W) \times (P_2 - P_1))$. On the other hand $\mu^* b_L = b_{L^{-1}} = \Theta \otimes b_L$. Hence b_L is annihilated by the element $1 + \Theta \in KR(W)$. Since $1 + \Theta$ has augmentation 2, b_L has finite order. On the other hand, $\operatorname{ind} b_L \in KR^{-2}(P_2, P_1) \simeq Z$ so $\operatorname{ind} b_L = 0$. It follows that $\operatorname{ind} \bar{\alpha}_W^L(u_0, u_1, u_2) = 0$.

Remark. It follows from Theorem 7.6 that if X has two vector fields with a finite number of singularities, then $R_L(X)$ does not depend on the choice of coefficient system L. This agrees with the following formula proved in [13]: If L and L' are two coefficient systems such that $L' = L \otimes \xi$ for a real line bundle ξ with characteristic class $y \in H^1(X, Z_2)$, then

$$R_{L'}(X) + R_L(X) = \langle y \cup w_{4k}(X), [X] \rangle.$$

8. The Hurewicz map in KR-theory for $P_{n,r}$

In view of Lemmas (5.2) and (5.5) it is not surprising that the map θ^s is closely related to the Hurewicz map in KR-theory for the stunted projective spaces $P_{n,r} = P_{n-1}/P_{n-r-1}$. In this section we shall establish this connection and comment on the relation with Adams' work on vector fields on spheres [2].

First recall that P_{n-1} is embedded naturally on $O(n)$ by identifying $x \in P_{n-1}$ with the reflection in the hyperplane orthogonal to $x \in \mathbf{R}^n$. In this way there is a map i: $P_{n,r} \to V_{n,r}$ defined by applying the reflection on the standard r-frame $\{e_0, \ldots, e_{r-1}\}$, where

$e_i = (0, \dots, 1, 0, \dots, 0)$ has 1 in the $i+1$'st coordinate. In particular the base point of $P_{n,r} = P_{n-1}/P_{n-r-1}$ maps to the standard r-frame in $V_{n,r}$. By convention $P_{n,n} = (P_{n-1})_+$ and the extra point corresponds in $O(n)$ to the identity.

Now again let s be chosen so that $n+s=4k$, and assume that a_r divides $(n+s)$. Hence $\mathbf{R}^{n+s}$ is a module for C_{r-1}. This means that $(n+s)H_r$ is trivial, where H_r is the Hopf bundle over P_{r-1}. It follows from [4] that the Thom complexes P_{r-1}^{sH} and $P_{r-1}^{(n-r)H} = P_{n,r}$ are S-dual, or precisely $s+(n-r)+(r-1) = n+s-1$-dual. Hence there is a duality isomorphism

$$D: KR_{q-1}(P_{n,r}) \xrightarrow[\simeq]{} KR^{n+s-q}(P_{r+s-1}, P_{s-1}).$$

THEOREM 8.1. *Let $\varkappa: \pi_{q-1}(P_{n,r}) \to KR_{q-1}(P_{n,r})$ be the Hurewicz map. Then the following diagram is commutative:*

$$\begin{tikzcd}
\pi_{q-1}(V_{n,r}) \arrow[r, "\theta^s"] & KR^{n+s-q}(P_{r+s-1}, P_{s-1}) \\
\pi_{q-1}(P_{n,r}) \arrow[u, "i_*"] \arrow[r, "\varkappa"] & KR_{q-1}(P_{n,r}) \arrow[u, "D"']
\end{tikzcd}$$

More precisely: we shall define an explicit S-duality (S-)map

$$\nabla: P_{n,r} \wedge P_{r+s,r} \to S^{n+s-1}. \tag{8.2}$$

Let Σ denote the suspension, and for a map $u: S^{q-1} \to P_{n,r}$ let $Du = \Sigma(\nabla \circ (u \wedge 1))$, i.e. $Du: \Sigma^q P_{r+s,r} \to S^{n+s}$ is the dual S-map of u. We shall prove

THEOREM 8.1′. *Let $u: S^{q-1} \to P_{n,r}$ and so $iu: S^{q-1} \to V_{n,r}$. Then*

$$\theta^s(iu) = (Du)^*(\lambda_{n+s,n+s})$$

in $KR^{n+s-q}(P_{r+s-1}, P_{s-1})$. Here $\lambda_{n+s,n+s}$ is the generator of $KR^{n+s,n+s}$ (pt).

Before defining ∇ let us consider what is the universal situation in this context. In the construction (2.10) use $X = CP_{n,r}$ the cone on $P_{n,r}$ and $Y = P_{n,r}$ the base of this cone. The trivial bundle of dimension n has r vector fields along $P_{n,r}$ determined by the map $i: P_{n,r} \to V_{n,r}$. Hence we get an element

$$\mu_{n,r}^s \in KR^{n+s}((CP_{n,r}/P_{n,r}) \wedge (P_{r+s-1}/P_{s-1})) = KR^{n+s}(\Sigma P_{n,r} \wedge P_{r+s,r}). \tag{8.3}$$

In order to prove Theorem (8.1′) it suffices to show

$$(\Sigma\nabla)^*(\lambda_{n+s,\,n+s}) = \mu^s_{n,r}. \tag{8.4}$$

In fact $\theta^s(iu) = (\Sigma u \wedge 1)^*(\mu^s_{n,r})$, so (8.4) gives

$$\theta^s(iu) = (\Sigma\nabla \circ (\Sigma u \wedge 1))^*(\lambda_{n+s,\,n+s}) = (Du)^*(\lambda_{n+s,\,n+s}).$$

Observe that if we increase n by a multiple of $\max(4, a_r)$, then $P_{n,r}$ is suspended the same number of times, and using (2.17) and Lemma (5.8), it follows that the elements $\mu^s_{n,r}$ correspond under the periodicity isomorphism. We can therefore assume n so large that $n+s$ is divisible by a_{r+s}, i.e. $\mathbf{R}^{n+s}$ is a module for C_{r+s-1}. (If $n+s = p\,a_r$ and $a_{r+s} = t\,a_r$ then increase n by $(mt-p)\,a_r$ for some m.) Under this assumption we shall construct an explicit map

$$\bar{\nabla}: V_{n,r} \times P_{r+s-1} \to S^{n+s-1} \tag{8.5}$$

which induces an S-duality map

$$\nabla^s_{n,r}: P_{n,r} \wedge P_{r+s,r} \to S^{n+s-1} \tag{8.6}$$

such that (8.4) is satisfied for $\nabla = \nabla^s_{n,r}$.

First assume $s=0$. So there is a C_{r-1}-module M of dimension n. $\mathbf{R}^r \subset C_{r-1}$ is spanned by $\{e_0 = 1, e_1, \dots, e_{r-1}\}$, and the Clifford-module structure induces an orthogonal pairing $(\cdot,\cdot): \mathbf{R}^r \times M \to M$. Choose $e_0 \in M$ of unit length and embed $\mathbf{R}^r$ in M by sending $a \in \mathbf{R}^r$ into (a, e_0). We can thus choose coordinates for $M \simeq \mathbf{R}^n$ so that $\{e_0, \dots, e_{r-1}\}$ is the standard r-frame. Hence $\mathbf{R}^n = \mathbf{R}^r \oplus \mathbf{R}^{n-r}$ and the pairing $(\cdot,\cdot): \mathbf{R}^r \times \mathbf{R}^n \to \mathbf{R}^n$ satisfies

1) $(e_i, e_j) = -(e_j, e_i)$ $\quad i \neq j$ and $0 < i, j \leqslant r-1$
2) $(e_i, e_i) = -e_0$ $\quad i = 1, \dots, r-1$
3) $(a, e_0) = a$ for $a \in \mathbf{R}^r$ and
$(e_0, b) = b$ for $b \in \mathbf{R}^n$.

Also define a conjugation in $\mathbf{R}^r$ by putting $\bar{e}_0 = e_0$ and $\bar{e}_i = -e_i$ for $i > 0$. It follows that $(\bar{a}, a) = |a|^2 e_0$ for $a \in \mathbf{R}^r$.

We are now in the position to define the map

$$\bar{\nabla}: V_{n,r} \times P(\mathbf{R}^r) \to S^{n-1}. \tag{8.7}$$

An element in $P(\mathbf{R}^r) = P_{r-1}$ is given by homogeneous coordinates $x = (x_0, \dots, x_{r-1}) \in \mathbf{R}^r$, and we assume $|x| = 1$. An element y in $V_{n,r}$ is given by an orthogonal r-frame $\{e_0(y), \dots, e_r(y)\}$ in $\mathbf{R}^n$. Put $x(y) = \sum_{i=1}^{r-1} x_i\, e_i(y)$. Now for $\{x\} \in P(\mathbf{R}^r)$ and $y \in V_{n,r}$ define

$$\bar{\nabla}(y, \{x\}) = (\bar{x}, x(y)). \tag{8.8}$$

Clearly this is an element in $S^{n-1} \subset \mathbf{R}^n$, and $\bar{\nabla}$ is obviously well defined. Notice that the base point $(*)$ of $V_{n,r}$ is the standard r-frame, so clearly $(*) \times P(\mathbf{R}^r)$ goes to the base point e_0 of S^{n-1}.

Now again let s be arbitrary, so $\mathbf{R}^{n+s}$ is a C_{r+s-1}-module. (8.7) gives the map

$$\bar{\nabla}: V_{n+s,\,r+s} \times P(\mathbf{R}^{r+s}) \to S^{n+s-1}.$$

Write $\mathbf{R}^{r+s} = \mathbf{R}^r \oplus \mathbf{R}^s$, so $\mathbf{R}^{n+s} = \mathbf{R}^r \oplus \mathbf{R}^s \oplus \mathbf{R}^{n-r}$ and $\mathbf{R}^n$ is the sum of the first and the last factor. Then $V_{n,r}$ is included in $V_{n+s,\,r+s}$ by adjoining the s vectors $\{e_r, \dots, e_{r+s-1}\}$ which span $\mathbf{R}^s \subset \mathbf{R}^{n+s}$. The map (8.5) is now simply defined as the composite

$$\bar{\nabla}: V_{n,r} \times P(\mathbf{R}^{r+s}) \to V_{n+s,\,r+s} \times P(\mathbf{R}^{r+s}) \to S^{n+s-1}. \tag{8.9}$$

Clearly $x = x(y)$ for $x \in \mathbf{R}^s = \operatorname{span}\{e_r, \dots, e_{r+s-1}\}$ so (8.9) induces a map

$$V_{n,r} \wedge (P_{r+s-1}/P_{s-1}) \to S^{n+s-1}.$$

Finally we compose with $i = P_{n,r} \to V_{n,r}$, thus completing the construction of

$$\nabla^s_{n,r}: P_{n,r} \wedge P_{r+s,\,r} \to S^{n+s-1}. \tag{8.10}$$

LEMMA 8.11. *The map $\nabla^s_{n,r}$ is an S-duality map.*

Proof. We shall use induction on r. Consider the diagram

$$\begin{tikzcd}[column sep=small]
 & P_{n,r} \wedge P_{r+s,\,r} \arrow[dr, "\nabla^s_{n,r}"] & \\
P_{n-1,\,r-1} \wedge P_{r+s,\,r} \arrow[ur, "j \wedge 1"] \arrow[dr, "1 \wedge j'"'] & & S^{n+s-1} \\
 & P_{n-1,\,r-1} \wedge P_{r+s,\,r-1} \arrow[ur, "\nabla^{s+1}_{n-1,\,r-1}"'] &
\end{tikzcd}$$

where $j: P_{n-1,\,r-1} \to P_{n,r}$ is the inclusion and $j': P_{r+s,\,r} \to P_{r+s,\,r-1}$ is the natural map squeezing the bottom cell to a point. This diagram is clearly commutative from the very definition of the ∇'s; hence by the induction hypothesis we just have to show that the product of the bottom cell of $P_{r+s,\,r}$ and the top cell of $P_{n,r}$ maps to S^{n+s-1} by a map of degree one. But this clearly reduces to the case $r = 1$. However, it is easy to see that the map

$$\nabla^s_{n,1}: S^{n-1} \times S^s = V_{n,1} \times P_s/P_{s-1} \to S^{n+s-1}$$

has degree one. In fact, the element $-e_0 \in S^{n+s-1}$ is a regular value with only one pre-image.

We can now complete the proof of Theorem (8.1) by showing formula (8.4). Notice that ∇ induces a map

$$\Sigma\nabla : (CP_{n,r}, P_{n,r}) \times (P_{r+s-1}, P_{s-1}) \rightarrow (B^{n+s}, S^{n+s-1})$$

and we shall compute the induced map in $KR^{n+s,0}$. Now assume $s=0$ for simplicity (the general case proceeds analogously), i.e.

$$\Sigma\nabla : (CP_{n,r}, P_{n,r}) \times P_{r-1} \rightarrow (B^n, S^{n-1}).$$

The generator of $KR^{n,0}(B^n, S^{n-1})$ is given by the square

$$\begin{array}{ccc} \Lambda_+^{\mathrm{ev}} & \xrightarrow{iL_v} & \Lambda_-^{\mathrm{odd}} \\ {\scriptstyle R_u}\downarrow & & \downarrow{\scriptstyle R_u} \\ \Lambda_+^{\mathrm{odd}} & \xrightarrow{iL_v} & \Lambda_-^{\mathrm{ev}} \end{array} \tag{8.12}$$

where $v \in \mathbf{R}^{n,0}$ and $u \in S^{n-1}$. Hence $(\Sigma\nabla)^*(\lambda_{n,n})$ is given over $CP_{n,r} \times P_{r-1} \times \mathbf{R}^{n,0}$ by the square (8.12) with $u = (\bar{x}, x(y))$ for $y \in P_{n,r}$ and $x \in P_{r-1}$. An argument similar to that of Lemma (5.8) transforms this square into the square

$$\begin{array}{ccc} \Lambda_+^{\mathrm{ev}} & \xrightarrow{iL_v} & \Lambda_-^{\mathrm{odd}} \\ {\scriptstyle R_{x(y)}}\downarrow & & \downarrow{\scriptstyle R_{x(y)}} \\ \Lambda_+^{\mathrm{odd}} \otimes H & \xrightarrow{iL_v} & \Lambda_-^{\mathrm{ev}} \otimes H \end{array}$$

which defines the element $\mu_{n,r}$. This proves formula (8.4) and ends the proof of Theorem (8.1).

Remark. Using Theorem (8.1′) we can reformulate Adams' proof [2] on the vector fields on spheres as follows:

Let $u : S^{n-1} \rightarrow V_{n,r} (n \equiv 0 \bmod 8)$ be a section of $V_{n,r} \rightarrow S^{n-1}$ and choose s such that a_r divides $n+s$. Consider the element $\theta^s(u) \in KR^s(P_{r+s-1}, P_{s-1})$. The restriction to $KR^s(P_s, P_{s-1})$ is the generator. On the other hand, Theorem (8.1′) shows that $\theta^s(u)$ is induced from a sphere, so the Adams operations ψ^k act on it as multiplication by k^s. A calculation shows that these two facts are incompatible if r is greater than or equal to the Radon-Hurwitz number.

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