LINEAR EXTREMAL PROBLEMS FOR ANALYTIC FUNCTIONS

BY

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I. Introduction

In recent years the classical Schwarz lemma has been generalized in at least two important ways. The first way is the following. Let D be any plane domain. Let ξ be any point in D. Denote by $\mathcal{C}$ the family of all single-valued analytic functions $f(z)$ on D such that $|f(z)| \leqslant 1$ and $f(\xi)=0$. Define $M=\sup\{|f'(\xi)|; f \in \mathcal{C}\}$. Since $\mathcal{C}$ is a nonvoid normal family, there will be at least one F in $\mathcal{C}$ such that $F'(\xi)=M$. The problem is to study such extremal functions F. Generally speaking, the basic work on this problem when D is of finite connectivity was done by L. Ahlfors [1, 2], P. Garabedian [7], and H. Grunsky [9, 10, 11, 12]. Ahlfors and Grunsky proved results about the uniqueness and the boundary behavior of the extremal function. Garabedian treated the problem as a so-called dual extremal problem and expressed the extremal function in terms of the Szegö kernel function. The case in which D is an arbitrary plane domain (possibly of infinite connectivity) was first studied in detail around 1960 by L. Carleson [5] pp. 73–82 and S. Ja. Havinson [14]. Both of these authors established the uniqueness of the extremal function and Havinson went on to discuss the behavior of the extremal function in great depth.

The second type of generalization consists essentially of replacing the expression $f'(\xi)$ above by an arbitrary linear functional $\mathcal{L}[f]$ and by replacing the auxiliary condition $f(\xi)=0$ by a more general restriction on the zeros of $f(z)$. For an obvious reason, the problem is now called a linear extremal problem. In the case of finite connectivity, such linear extremal problems have been studied, for example, by S. Ja. Havinson [13] and P. Lax [20].

Thus far, a detailed study of the corresponding general linear extremal problems on arbitrary plane domains—not to mention open Riemann surfaces—seems to be lacking.

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It is basically with such problems that we concern ourselves in this paper. Our principal aim is to show that such general extremal problems, at least for plane domains, can be profitably investigated in a unified manner from the standpoint of the theory of open Riemann surfaces and modern potential theory. Another aim is to get some indication of the situation on open Riemann surfaces.

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II. Some basic notation

In the interest of clarity, we will now set down some basic notation. Let S denote the Riemann sphere or extended plane. Write $X = S - \{\infty\}$ = finite plane. Suppose that W = open Riemann surface. Let $H(W)$ = the family of real-valued harmonic functions on W. $HB(W) = \{u \in H(W) \mid u \text{ is bounded}\}$. $HP(W) = \{u \in H(W) \mid u \geqslant 0\}$. $\mathcal{S}(W)$ = the family of extended real-valued superharmonic functions on W. It is assumed that superharmonic functions can take the value $+\infty$ but not the value $-\infty$. $\mathcal{S}P(W) = \{v \in \mathcal{S}(W) \mid v \geqslant 0\}$. $A(W)$ = family of single-valued analytic functions on W. $AB(W) = \{f \in A(W) \mid f \text{ is bounded}\}$. $M(W)$ = family of single-valued meromorphic functions on W. A plane domain D is said to be in class $\mathcal{A}_p$ iff D is bounded and ∂D consists of p mutually disjoint analytic Jordan curves. Here $1 \leqslant p < \infty$. Finally, let U denote the open unit disk.

III. The fundamental extremal problem

In keeping with our expressed intention to study general linear extremal problems on arbitrary plane domains, it is convenient to formulate our basic extremal problem in a setting which is rather general. The formulation that we shall use here has its origins in the work of H. Grunsky [9, 10, 11, 12], S. Ja. Havinson [13], and P. Lax [20].

First of all, suppose that D is an arbitrary plane domain. In our investigation, the case $D = S$ is a trivial special case. We may therefore exclude this case from consideration. Furthermore, by means of an auxiliary linear fractional mapping, we might as well suppose too that $\infty \notin D$.

We now make the following list of assumptions:

(i) D is any plane domain, $\infty \notin D$;

(ii) $\{\xi_k\}_{k=1}^{\infty}$ is a sequence of distinct points in D;

(iii) $\{n_k\}_{k=1}^{\infty}$ is a sequence of non-negative integers;

(iv) $\chi \in H(D)$;

(v) K is a bounded closed set, $K \subseteq D$;

(vi) λ is a totally finite complex Borel measure on K;

(vii) for $h \in C(K)$,

$$\mathfrak{L}[h] = \int_K h\,d\lambda;$$

(viii) for $w \in X - K$,

$$k(w) = \frac{1}{2\pi i}\int_K \frac{1}{w-z}\,d\lambda(z).$$

Here $C(K)$ denotes the complex linear space of functions continuous on K.

Fundamental extremal problem. Let (i)–(viii) apply. Define

$$\mathfrak{C} = \{f \in A(D) \mid f(\xi_k) = 0, \text{ multiplicity} \geqslant n_k, \ln |f(z)| \leqslant \chi(z)\}. \tag{1}$$

Suppose that $\mathfrak{C}$ is nontrivial and that $M = \sup \{|\mathfrak{L}[f]|; f \in \mathfrak{C}\} \neq 0$. Surely the fact that $\mathfrak{C}$ is a normal family guarantees the existence of $F \in \mathfrak{C}$ such that $\mathfrak{L}[F] = M$. The problem is to discuss the extremal functions F.

Before turning to the statement of our fundamental theorem, a few remarks about the extremal problem are in order. First, condition $\ln |f| \leqslant \chi$ as above is immediately checked to be equivalent to a restriction on the boundary behavior of f. The use of such a defining condition for class $\mathfrak{C}$ is admittedly a bit arbitrary, but is very much in line with the works to which we have referred in section I. More will be said about this point later.

Secondly, probably the most basic nontrivial question about the extremal functions F is that of uniqueness. In this paper we shall concern ourselves for the most part with the uniqueness question. It is to this problem that we have succeeded in discovering a reasonably complete answer.

Thirdly, another very basic question is that of the boundary behavior of the extremal functions F (in some sense). We shall not pursue this question to any real depth in this paper.

Finally, we wish to point out that we shall discuss various generalizations of this fundamental extremal problem in later sections of this paper. One of these generalizations will concern the replacement of the plane domain D by an open Riemann surface W.

IV. Statement of the fundamental theorem

Several items of additional notation are needed in order to state our theorem in a concise manner. First, define $B = S - K$ and denote by B_j the components of open set B. Next, define $h_1(D) = \{u \in H(D) \mid u = u_1 - u_2$ for some u_1, u_2 in $HP(D)\}$. Let E be a closed totally disconnected subset of S. We will write $E \in N_G$ iff E has logarithmic capacity zero (that is, Cap $(E) = 0$). And we write $E \in N_B$ iff $AB(S-E)$ consists only of constants. Equivalently, $E \in N_B$ iff E is an AB null-set (see, for instance, Sario and Oikawa [29] pp. 255, 261).

THEOREM 1. *Let assumptions* (*i*)–(*viii*) *apply. Suppose that the fundamental extremal problem is nontrivial. Then, uniqueness holds provided at least one of the following conditions holds:*

(a) *D is of finite connectivity;*

(b) *$B_j \cap \partial D \notin N_B$ whenever $B_j \cap \partial D$ is nonempty;*

(c) *$\chi \in h_1(D)$ and no nonvoid $B_j \cap \partial D$ lies in $N_B - N_G$;*

(d) *$P(z)k(z) = Q(z)$ near ∂D for some P and Q, $P \not\equiv 0$, $P \in AB(D)$, and $Q \in A(D)$.*

Furthermore, explicit counterexamples show conditions (*b*) *and* (*c*) *are best possible.*

The proof of this theorem is rather long. For this reason we shall proceed in several stages. Along the way a number of side-results interesting in themselves will emerge.

As an indication of the strength of Theorem 1, consider the first generalization of the classical Schwarz lemma in section I. It is easy to see that WLOG $K = \{|z-\xi| \leqslant \varepsilon\}$, small $\varepsilon > 0$, $\partial D \notin N_B$, and $k(w) = (1/2\pi i)\,(w-\xi)^{-2}$. Uniqueness of the extremal function is now an easy consequence of either (b) or (d) of Theorem 1.

V. Outline of basic finite connectivity techniques

In order to study the situation on arbitrary plane domains, it is necessary to make important use of the techniques used for the case $D \in \mathcal{A}_p$. For this reason, and in the interest of clarity, we shall include here in outline form a development of the techniques used by P. Lax [20] to treat the case $D \in \mathcal{A}_p$. In so doing, we will correct an erroneous theorem in P. Lax [20] p. 447.

We shall use the list of assumptions in section III. In addition, suppose that: $D \in \mathcal{A}_p$; χ is in $HB(D)$ and has continuous boundary values; there is an integer N such that $n_k = 0$ for $k > N$.

As is customary, let $f(\xi)$ denote the Fatou nontangential boundary values for $f(z) \in AB\,(D)$. Then, it follows at once that

$$\mathbb{C} = \{f \in AB(D) \mid f(\xi_k) = 0, \text{ mult } \geqslant n_k, |f(\xi)| \leqslant e^{\chi(\xi)} \text{ on } \partial D\}. \tag{2}$$

Two good references for such techniques are G. M. Golusin [8] and I. I. Priwalow [23].

Now suppose that $f \in \mathbb{C}$ and that Γ is a path in D approximating ∂D. Then, as an easy calculation shows,

$$\mathcal{L}[f] = \int_{\Gamma} f(z)\, k(z)\, dz. \tag{3}$$

But suppose that $s(z)$ is analytic on $\bar{D}$ except for poles at the points ξ_k with order $\leqslant n_k$. Of course, $n_k = 0$ eventually. Then,

$$\mathcal{L}[f] = \int_{\Gamma} f(z)\, [k(z) - s(z)]\, dz. \tag{4}$$

Formula (4) points the way to the appropriate dual extremal problem.

We recall the following basic abstract lemma.

LEMMA. *Let E be a complex normed linear space with complex subspace T. Let $k \in E$. Set $m_1 = \inf\{\|k - s\|: s \in T\}$. Let $m_2 = \sup\{|l(k)|\}$, where l ranges over all complex linear functionals on E which vanish on T and have norm $\leqslant 1$. Then $m_1 = m_2$.*

Proof. Let l be a complex linear functional on E, $l[T] = 0$, $\|l\| \leqslant 1$. Choose $s \in T$, $\|k - s\| < m_1 + \varepsilon$. Then, $|l(k)| = |l(k - s)| \leqslant \|l\| \cdot \|k - s\| < m_1 + \varepsilon$. At once, $m_2 \leqslant m_1$. Clearly WLOG $m_1 > 0$. Let V be the subspace of E generated by k and T. Define linear functional l_0 on V by $l_0[T] = 0$, $l_0[k] = m_1$. Easily $\|l_0\|_V = 1$. By the Hahn-Banach theorem, l_0 extends to a linear functional l_0 on E with $\|l_0\|_E = 1$ (see, for example, Hewitt and Ross [18] p. 455). So, $m_1 \leqslant m_2$. ∎

In order to apply this lemma, we first assume WLOG that the extremal problem for $\mathcal{L}$ over $\mathfrak{C}$ is nontrivial. Next, set $\mathcal{E} = C(\partial D)$ and $\mathfrak{J}$ = complex linear space of boundary values of analytic functions $s(z)$ on $\bar{D}$ except for poles at the points ξ_k with order $\leqslant n_k$. The norm on $\mathcal{E}$ will be taken to be

$$\|u\| = \int_{\partial D} e^{\chi(z)} |u(z)|\, |dz|. \tag{5}$$

For the "k" of the lemma we use $k(w)$. Let m_1 and m_2 be the inf and sup for the associated dual (or reciprocal) extremal problem; $m_1 = m_2$. Recall too that $M = \sup\{|\mathcal{L}[f]|; f \in \mathfrak{C}\}$. It is easy to see that (4) yields $M \leqslant m_1 = m_2$ as $\Gamma \to \partial D$. Of course, $M > 0$. Next, let l_0 be one of the linear functionals on $\mathcal{E}$ solving the abstract maximum problem of the lemma for k, $\mathcal{E}$, and $\mathfrak{J}$. See the proof of the lemma. We may assume $l_0[\mathfrak{J}] = 0$, $\|l_0\| = 1$, and $l_0[k] = m_1 = m_2$. Define:

$$u(\beta; z) = \frac{1}{2\pi i}\, \frac{1}{\beta - z}, \quad \beta \in \partial D,\ z \notin \partial D. \tag{6}$$

For $z \notin \partial D$, $u(\beta; z) \in \mathcal{E}$. If $z \in \operatorname{ext} D$, $u(\beta; z) \in \mathfrak{J}$. Define:

$$F(z) = l_0[u(\beta; z)], \quad z \notin \partial D. \tag{7}$$

By use of the Riesz representation theorem (see, for example, Hewitt and Ross [18] pp. 129, 168), it is straightforward to check that F is analytic in each component of $S-\partial D$, $F[\text{ext}\, D]=0$, $\mathcal{L}[F]=m_1=m_2$, and F has a zero of multiplicity $\geqslant n_k$ at ξ_k. Next, let $z\in D$ be near ∂D. Let z^* be the reflection of z in ∂D. It follows now that

$$|F(z)|=|F(z)-F(z^*)|=|l_0[u(\xi;z)]-l_0[u(\xi;z^*)]|\leqslant \|l_0\|\cdot\|u(\xi;z)-u(\xi;z^*)\|$$
$$=\frac{1}{2\pi}\int_{\partial D} e^{\chi(\xi)}\left|\frac{1}{\xi-z}-\frac{1}{\xi-z^*}\right||d\xi|. \tag{8}$$

This last integral clearly behaves essentially like a Poisson integral. It follows that $\limsup_{z\to\xi}|F(z)|\leqslant e^{\chi(\xi)}$ for all $\xi\in\partial D$. By the maximum principle and (2) we find at once that $F\in\mathcal{C}$. By the above, it follows finally that $M=m_1=m_2$. We have thus exhibited the dual extremal problem corresponding to $\mathcal{L}$ over $\mathcal{C}$.

We next study $M=\inf\{\|k-s\|;\, s\in\mathfrak{J}\}$ under (5). By use of the fact that $D\in\mathcal{A}_p$, some reasonably elementary estimates involving the Cauchy integral formula, and a normal family argument, it is found that there exists some $s_0(z)$ which is analytic on D except for poles of order $\leqslant n_k$ at the ξ_k such that

$$\lim_{\Gamma\to\partial D}\int_\Gamma e^{\chi(z)}|k(z)-s_0(z)||dz|=M, \tag{9}$$

where we assume that Γ always denotes an analytic path approximating ∂D sufficiently well; of course, $\Gamma\subseteq D$. Intuitively, (9) is just as would be expected. For the details, see Lax [20] pp. 444–446.

The following remark about boundary behavior is probably not out of place at this point. Equation (9) shows that the integrals $\int_\Gamma |s_0(z)||dz|$ are uniformly bounded as $\Gamma\to\partial D$. It is now possible to generalize some classical results on simply-connected domains concerning either the Hardy class H_1 or the Smirnov class E_1, as, for example, in Priwalow [23], to deduce that s_0 has nontangential boundary values a.e., say, $s_0(\xi)$, such that

$$M=\int_{\partial D} e^{\chi(\xi)}|k(\xi)-s_0(\xi)||d\xi|. \tag{10}$$

See also the paper of S. Ja. Havinson [13] in this regard. We will not, however, use this remark in our development.

The crucial point is now derived as follows. We know that for any extremal function $F\in\mathcal{C}$

$$M=\mathcal{L}[F]=\int_\Gamma F(z)\,[k(z)-s_0(z)]\,dz$$

so that

$$M = \int_\Gamma F(z)\,[k(z)-s_0(z)]\,dz \leqslant \int_\Gamma |F(z)\,[k(z)-s_0(z)]\,dz| \leqslant \int_\Gamma e^{\chi(z)}\,|k(z)-s_0(z)|\,|dz| \to M$$

as $\Gamma \to \partial D$. It is now to be expected that

$$0 = \lim_{\Gamma\to\partial D} \int_\Gamma |\operatorname{Im}\{F(z)\,[k(z)-s_0(z)]\,dz\}|.$$

This indicates that in some sense $F(z)\,[k(z)-s_0(z)]\,dz$ is real on ∂D. Presumably, then, some sort of generalized Schwarz reflection principle would apply after a local conformal mapping of ∂D onto a line segment on the real axis. Hence, it is expected that $F(z)[k(z)-s_0(z)]$ continues across ∂D.

To make this rigorous, Lax uses the following type of generalized Schwarz reflection principle, whose proof he omits.

LEMMA. *Let* $E=\{z\,|\,-2<\operatorname{Re}(z)<2,\ 0<\operatorname{Im}(z)<2\}$. *Let* $u\in H(E)$ *have:*
(i) $T(y)=\int_{-2}^{2}|u(x+iy)|\,dx$ *uniformly bounded for* $0<y<2$;
(ii) $T(y)\to 0$ *as* $y\to 0$.
Then u *can be continued harmonically across the real axis by Schwarz reflection.*

Proof. It may be appropriate to sketch an elementary proof of the lemma at this point. It clearly suffices to study the situation near the origin. Write $Q_n=\{z\,|\,z\in U,\ \operatorname{Im}(z)>1/n\}$ for $1\leqslant n\leqslant\infty$. By (i), $\int_0^2 T(y)\,dy \neq \infty$ so that WLOG $\infty \neq \int_0^\pi |u(e^{i\theta})|\,d\theta$. For $z_0\in Q_\infty$ and $n\to\infty$,

$$u(z_0)=\frac{1}{2\pi}\int_{\partial Q_n} u(\xi)\,\frac{\partial g}{\partial n_\xi}\,(\xi; z_0; Q_n)\,|d\xi|.$$

A study of the limiting behavior of $g(z; z_0; Q_n)$ as $n\to\infty$ shows that

$$u(z_0)=\frac{1}{2\pi}\int_0^\pi u(\xi)\,\frac{\partial g}{\partial n_\xi}\,(\xi; z_0; Q_\infty)\,d\theta,$$

where $\xi=e^{i\theta}$. But, $g(z; z_0; Q_\infty)=g(z; z_0; U)-g(z; \bar z_0; U)$, $z\in Q_\infty$. The Poisson formula for $u(z_0)$ results. ∎

We thus conclude, as in Lax [20], pp. 447–449, that $F(k-s_0)$ continues analytically across ∂D. Further, $F(k-s_0)dz\geqslant 0$ along ∂D. Recall now that $\ln x\leqslant x-1$ for $0\leqslant x<\infty$. By means of

$$\lim_{\Gamma\to\partial D}\int_\Gamma \left\{\frac{e^\chi}{|F|}-1\right\}|F(k-s_0)|\,|dz|=0$$

we thus conclude that

$$\lim_{\Gamma\to\partial D}\int_\Gamma [\chi-\ln|F|]\,|F(k-s_0)|\,|dz|=0.$$

Denote the components of ∂D by Γ_j for $0\leqslant j\leqslant p-1$. By means of the lemma, it now follows that near each Γ_j either $F(k-s_0)\equiv 0$ or else $F(k-s_0)\not\equiv 0$ and $\chi-\ln|F|$ continues harmonically across Γ_j except for a finite number of exceptional points under Schwarz reflection.

THEOREM 2. *Let the assumptions (i)–(viii) apply and suppose additionally that $D\in\mathcal{A}_p$, $\chi\in HB(D)$ with Hölder continuous boundary values*(1) *β, and $n_k=0$ eventually. Suppose too that the fundamental extremal problem for $\mathfrak{L}$ over $\mathfrak{C}$ is nontrivial. The extremal problem then has a unique solution $F\in\mathfrak{C}$ with $\mathfrak{L}[F]=M$. In addition, there exists $s_0(z)$ which is analytic on D and continuous on $\bar D$ except for poles of order $\leqslant n_k$ at the points ξ_k such that* (10) *holds. Further, $F(k-s_0)$ continues across ∂D and $F(k-s_0)\,dz\geqslant 0$ along ∂D. For each component Γ_j of ∂D, there are exactly two possibilities:*

(i) *$k\not\equiv s_0$ near Γ_j, F continuous near Γ_j, $|F|=e^\chi$ on Γ_j; or,*

(ii) *$k\equiv s_0$ near Γ_j, with nothing asserted about F near Γ_j.*

Proof. Suppose first of all that $k\not\equiv s_0$ near Γ_j. Then, as above, $F(k-s_0)\not\equiv 0$ continues across Γ_j and, up to a finite number of exceptional points, $\chi-\ln|F|$ remains harmonic across Γ_j under Schwarz reflection. Suppose now for simplicity that Γ_j is the outer component of ∂D and that $\Gamma_j=\partial U$ (or else use an auxiliary conformal mapping). Determine $g\in A(U)$ such that $Re(g)=\chi$ on ∂U. By a well-known theorem of Priwalow, g is Hölder continuous on $\bar U$ (see, for example, Golusin [8] p. 364).

Next, $F(k-s_0)$ is analytic on ∂U. WLOG $F\neq 0$ in $\{1-\delta<|z|<1\}$ for small δ. In this annulus, write $F(k-s_0)=[Fe^{-g}]\,[e^g(k-s_0)]$. Trivially, Fe^{-g} is bounded near ∂U and has Fatou boundary values of unit modulus a.e. For perspective, it is useful to recall the behavior of $\exp\{(z+1)/(z-1)\}$. Away from the zeros of $F(k-s_0)$ on ∂U, one directly verifies that Fe^{-g} remains analytic across ∂U (use Schwarz reflection and the behavior of $\chi-\ln|F|$). By means of (9) it is apparent that $e^g(k-s_0)$ behaves essentially like an H_1 function near ∂U. We can now apply a theorem of Szegö and Smirnov on the factorization of H_1 functions to the functions Fe^{-g}, $e^g(k-s_0)$, and $F(k-s_0)$ locally. Standard techniques allow us to conclude at once that Fe^{-g} remains analytic on ∂U. For the theorem of Szegö and Smirnov, see, for example, Priwalow [23] pp. 75, 78 and for the method see, for instance, Havinson [13] pp. 25–27.

(1) Note: if E is a compact subset of X, we say $f\in C(E)$ is Hölder continuous iff f satisfies a Lipschitz condition of order α on E for some $\alpha>0$.

It follows at once that case (i) holds for Γ_j. Case (ii) is trivial.

To prove uniqueness, notice that $k \not\equiv s_0$ near some Γ_j because $M \neq 0$. By virtue of case (i) and $F(k-s_0)\,dz \geq 0$ on Γ_j, we immediately see that the values of F along Γ_j are predetermined. Uniqueness then follows by a well-known theorem of Lusin-Riesz-Priwalow. ∎

Remark 1. Theorem 2 is the correct form of Theorem III of Lax [20] p. 447.

Remark 2. Suppose that $\mathcal{C} = \{f \in A(U) \mid \ln |f| \leq \chi\}$ where

$$\chi(z) = \sum_{n=2}^{\infty} \frac{r^n \sin n\theta}{n \ln n} \quad (z = re^{i\theta}).$$

Let $\mathcal{L}[f] = f(0)$ and consider extremal problem $\mathcal{L}$ over $\mathcal{C}$. The unique extremal function is

$$F(z) = \exp\left\{-i \sum_{n=2}^{\infty} \frac{z^n}{n \ln n}\right\}$$

since $\chi = \ln |F|$. But F is not continuous on $\bar{U}$. It thus follows that Theorem 2 is not strictly true in general if χ is just continuous on ∂D. (Compare Lax [20] p. 439.)

VI. Two examples in finite connectivity

In this section we wish to present two examples which illustrate Theorem 2, with both cases (i) and (ii), quite well.

Example 1. Let $D = \{1 < |z| < R\}$, $1 < R < \infty$, and let $\chi = 0$. Select $1 < c < R$ and take $K = \{|z| = c\}$. Define $\mathcal{L}[h] = \int_K h\,dz$ for $h \in C(K)$. Let $M = \sup\{|\mathcal{L}[f]|; f \in \mathcal{C}\}$ where $\mathcal{C} = \{f \in A(D) \mid \ln |f| \leq \chi\}$. By the Cauchy integral theorem, for $f \in \mathcal{C}$,

$$\mathcal{L}[f] = \int_{|z|=r} f(z)\,dz, \text{ any } 1 < r < R.$$

Trivially, $|\mathcal{L}[f]| \leq 2\pi r$, $f \in \mathcal{C}$, so that $|\mathcal{L}[f]| \leq 2\pi$, $f \in \mathcal{C}$. But equality holds for $F(z) = 1/z \in \mathcal{C}$. Here $|F| = 1$ on $|z| = 1$ yet $|F| = 1/R < 1$ on $|z| = R$. Also, an elementary calculation shows that $k \equiv -1$ near $|z| = 1$ while $k \equiv 0$ near $|z| = R$. By Theorem 2, it follows that $s_0(z) \equiv 0$ near $|z| = R$ and hence for all $z \in D$. This agrees with requirement (10).

Example 2. For this example we make use of some simple results in the Caratheodory theory of prime ends (see, for instance, Collingwood and Lohwater [6] pp. 167–182 or Golusin [8] pp. 29–39).

Choose a doubly-connected domain G in the w-plane whose inner boundary component is $|w| = 1$ and whose outer component is a very large square with a prime end of the third kind adjoined to one of the sides. The impression of the prime end can be taken to be a line segment I along one of the sides of the square. G will be conformally equivalent to a domain $D=\{1<|z|<R\}$ with $\{|z|=1\}\leftrightarrow\{|w|=1\}$. Let the canonical conformal mapping be $z=h(w)$, $w=g(z)$. Let L be a simple closed very smooth curve in G separating the components of ∂G. Take $K=h[L]$.

We now define $\mathcal{C}=\{f\in A(D)\,|\,\ln|f|\leqslant\chi\}$ with $\chi=0$. Set $\mathfrak{L}[p]=\int_K p(z)\,d\lambda(z)=\int_L p[h(w)]dw$ for $p\in C(K)$. We wish to study extremal problem $\mathfrak{L}$ over $\mathcal{C}$. A calculation as in example 1 readily shows that the unique extremal function is essentially $F(z)=1/g(z)$. The impression I of the prime end used in G corresponds to a single point of $|z|=R$ under $w=g(z)$, $z=h(w)$. It follows that $F(z)$ is badly discontinuous along $|z|=R$ and, in addition, $|F|$ will be bounded away from 1 here.

It is of interest to calculate k and s_0 for D. An easy calculation shows

$$k(z_0)=-\frac{1}{2\pi i}\int_K\frac{g'(z)}{z-z_0}dz.$$

By Theorem 2, $k\equiv s_0$ near $|z|=R$. Simple use of the classical theory of Cauchy-type integrals (as, for example, in Priwalow [23] pp. 130–144) and the uniqueness theorem of Lusin-Riesz–Priwalow will show that $k(z)-s_0(z)\equiv -g'(z)$ for all $z\in D\cap\{\text{int }K\}$. Requirement (10) is then clearly satisfied.

VII. Development of the fundamental theorem, part one

We shall adhere in general to the notation of Sario and Oikawa [29] and shall moreover make reasonably free use of the results found in both Ahlfors and Sario [3] and Sario and Oikawa [29].

Recall that open set $B=S-K$ decomposes into components B_j. For convenience, let B_∞ denote that component of B which includes ∞. See section IV.

THEOREM 1(b). *Let the assumptions (i)–(viii) of section III apply. Suppose that whenever $B_j\cap\partial D$ is nonvoid, then $B_j\cap\partial D\notin N_B$. Then uniqueness holds for the fundamental extremal problem.*

Remark. Our proof will involve ideas from topology and functional analysis. For such matters, we refer to Kelley [19], Hewitt and Ross [18], and Yosida [30].

Proof. We take a proper analytic exhaustion $\{D_n\}_{n=1}^\infty$ of D such that: (i) each D_n, $n\geqslant 1$, is a domain in $\bigcup_{p=1}^\infty A_p$; (ii) $K\subseteq D_n\subseteq \bar{D}_n\subseteq D_{n+1}$, $n\geqslant 1$; (iii) $\bar{D}_n\subseteq D$, $n\geqslant 1$; (iv) $D=\bigcup_{n=1}^\infty D_n$. Since $\mathcal{C}$ is nontrivial, WLOG $\xi_k\to\partial D$ as $k\to\infty$. Hence, at most finitely many ξ_k lie in any given D_n. Also, WLOG $\xi_k \notin$ all ∂D_n. Define subfamily $\mathcal{C}_n$ of $A(D_n)$ in complete analogy to $\mathcal{C}$. Each $\mathcal{C}_n$ has

$$M_n = \sup \{ |\mathfrak{L}[f]|; f \in \mathcal{C}_n \} \geq M > 0. \tag{11}$$

A simple normal family argument implies that $M_n \to M$. Apply now Theorem 2 to extremal problem $\mathfrak{L}$ over $\mathcal{C}_n$. Let F_n denote the unique function in $\mathcal{C}_n$ with $\mathfrak{L}[F_n] = M_n$. Let $S_n(z)$ be analytic on D_n and continuous on $\bar{D}_n$ except for poles of order $\leq n_k$ at those ξ_k in D_n with

$$M_n = \int_{\partial D_n} e^{\chi(z)} |k(z) - S_n(z)| |dz|. \tag{12}$$

By use of Theorem 2 and the Schwarz reflection principle, it follows that for each nonvoid $B_j \cap \partial D_n$ there are two possibilities: (a) $k \not\equiv S_n$ near $B_j \cap \partial D_n$, $F_n(k - S_n)\, dz \geq 0$ along $B_j \cap \partial D_n$ with $S_n(z)$ analytic across $B_j \cap \partial D_n$, $F_n(z)$ analytic across $B_j \cap \partial D_n$, and $\ln |F_n(z)| = \chi(z)$ along $B_j \cap \partial D_n$; or (b) $k \equiv S_n$ near $B_j \cap \partial D_n$, so that $S_n(z)$ is analytic across $B_j \cap \partial D_n$.

We now study extremal problem $\mathfrak{L}$ over $\mathcal{C}$. $\mathcal{C}$ is a normal family. The usual techniques allow us to assume WLOG that $F_n(z) \rightrightarrows F(z)$ on D compacta as $n \to \infty$ and that F is an extremal function for $\mathfrak{L}$ over $\mathcal{C}$. Surely, $\mathfrak{L}[F] = M$. Let $F_\infty \in \mathcal{C}$ have $\mathfrak{L}[F_\infty] = M$. We must prove that $F(z) \equiv F_\infty(z)$ on D.

LEMMA 1. *$F(z)\, S_n(z)$ is eventually uniformly bounded on D compacta.*

Proof. Select any large integer N. For $\zeta \in \partial D_N$ and $n > N$,

$$\frac{1}{2\pi i} \int_{\partial D_n} \frac{F(z)\,[k(z) - S_n(z)]}{z - \zeta}\, dz - \frac{1}{2\pi i} \int_{\partial D_1} \frac{F(z)\,[k(z) - S_n(z)]}{z - \zeta}\, dz = F(\zeta)\,[k(\zeta) - S_n(\zeta)]. \tag{13}$$

By use of (12) the first integral is uniformly bounded. The second integral is just $(1/2\pi i) \int_{\partial D_1} [F(z)\, k(z)/(z - \zeta)]\, dz$. At once, $F(k - S_n)$ is uniformly bounded along ∂D_N as $n \to \infty$. Hence, FS_n must be uniformly bounded along ∂D_N as $n \to \infty$, which in turn yields the lemma by the maximum modulus principle. ∎

From Lemma 1, it follows that $\{S_n(z)\}_{n=1}^{\infty}$ is a normal family of meromorphic functions. We can assume therefore WLOG that

$$S_n(z) \rightrightarrows S(z) \text{ as } n \to \infty \tag{14}$$

on D compacta in the spherical metric. Clearly, $S(z) \in M(D)$ with its only poles at the ξ_k with order $\leq n_k$.

LEMMA 2. *For each $n \geq 1$, $k(z) \equiv S(z)$ along ∂D_n is impossible.*

Proof. By (3) and (4) we find that

$$M = \mathfrak{L}[F] = \int_{\partial D_n} F(z)\,[k(z) - S_m(z)]\,dz,\ \ m \to \infty,$$

for each $n \geqslant 1$. But $M \neq 0$. The lemma is now clear. ∎

We now begin the nontrivial portion of the proof. Let $B(D)$ denote the complex Banach space consisting of all bounded continuous complex-valued functions on D with sup norm. Define the following linear functionals on $B(D)$:

$$U_n[f] = \int_{\partial D_n} f e^{\chi} |k - S_n| |dz|; \tag{15}$$

$$V_n[f] = \int_{\partial D_n} f F(k - S_n)\,dz; \tag{16}$$

$$W_n[f] = \int_{\partial D_n} f F_\infty (k - S_n)\,dz, \tag{17}$$

where $f \in B(D)$ and directed boundaries (as usual) are used. U_n is a nonnegative bounded linear functional on $B(D)$. V_n and W_n are complex bounded linear functionals on $B(D)$.

$$U_n[1] = M_n \to M. \tag{18}$$

We may now apply Alaoglu's theorem (see Hewitt and Ross [18] p. 458 or Yosida [30] p. 137). It is thus possible to find a subnet $\{n\alpha\}$ of $\{n\}_{n=1}^{\infty}$ (see Kelley [19]) such that

$$U_{n\alpha} \xrightarrow{w^*} U. \tag{19}$$

The weak-star convergence refers of course to Banach space $B(D)$. We fix this subnet $\{n\alpha\}$. Recall that $n\alpha$ becomes arbitrarily large eventually with respect to $\alpha : \alpha \geqslant \alpha_N$ implies $n\alpha \geqslant N$. (This is the definition of a subnet.) Note: since $B(D)$ is separable, it is possible to use a subsequence $\{n\alpha\}$; we shall, however, not make use of this fact.

The linear functionals on $B(D)$ possess lattice properties as, for example, in Hewitt and Ross [18] pp. 166–184, 461–463.

LEMMA 3. *We have:*

$$V_{n\alpha} \xrightarrow{w^*} U, \quad W_{n\alpha} \xrightarrow{w^*} U. \tag{20}$$

Proof. It is sufficient to show that $W_{n\alpha} \xrightarrow{w^*} U$. Note here that $n\alpha$ depends only on U_n, not on F or F_∞. A moment's thought shows that it will further suffice to show that every subnet of $\{W_{n\alpha}\}$ has a subnet w^* convergent to U. So, choose any subnet $\{W_{n\alpha\beta}\}$ of $\{W_{n\alpha}\}$. For $f \in B^+(D)$,

$$|W_{n\alpha}|(f) = \int_{\partial D_{n\alpha}} f|F_\infty||k-S_{n\alpha}||dz| \leqslant \int_{\partial D_{n\alpha}} fe^{\chi}|k-S_{n\alpha}||dz| = U_{n\alpha}[f]. \tag{21}$$

By Alaoglu's theorem, WLOG $W_{n\alpha\beta}$ and $|W_{n\alpha\beta}|$ are w^* convergent for $B(D)$.

Write
$$W_{n\alpha\beta} \xrightarrow{w^*} W, \quad |W_{n\alpha\beta}| \xrightarrow{w^*} Q. \tag{22}$$

We claim that
$$|W| \leqslant Q. \tag{23}$$

To see this, let $f \in B^+(D)$. We must show that $|W|(f) \leqslant Q(f)$. By definition, $|W|(f) = \sup\{|W(g)|; g \in B(D), |g| \leqslant f\}$. It therefore suffices to show that $g \in B(D)$, $|g| \leqslant f$ implies $|W(g)| \leqslant Q(|g|) \leqslant Q(f)$. But, this is an easy consequence of (22). Thus, (23) holds.

Next,

$$M = \int_{\partial D_{n\alpha\beta}} F_\infty(z)\,k(z)\,dz = \int_{\partial D_{n\alpha\beta}} F_\infty(k-S_{n\alpha\beta})\,dz = W_{n\alpha\beta}(1) = |W_{n\alpha\beta}(1)|$$
$$\leqslant |W_{n\alpha\beta}|(1) \leqslant U_{n\alpha\beta}(1) \to U(1) = M. \tag{24}$$

Hence, by (23),
$$M = W(1) \leqslant |W|(1) \leqslant Q(1) \leqslant U(1) = M. \tag{25}$$

It follows that $W = |W| = Q = U$. Indeed, if $f \in B^+(D)$, $0 \leqslant f \leqslant 1$, then $|W|\ (f) \leqslant Q(f)$, $|W|(1-f) \leqslant Q(1-f)$, so $|W|(1) \leqslant Q(1)$ with equality iff $|W|(f) = Q(f)$ and $|W|(1-f) = Q(1-f)$. Similarly for the others. ∎

We now keep $\alpha \geqslant \alpha_N$ where N is large so that $n\alpha \geqslant N$. Then:

$$\frac{1}{2\pi i}\int_{\partial D_{n\alpha}} \frac{P(z)[F-F_\infty](k-S_{n\alpha})}{z-\zeta}\,dz - \frac{1}{2\pi i}\int_{\partial D_1} \frac{P(z)[F-F_\infty](k-S_{n\alpha})}{z-\zeta}\,dz$$
$$= P(\zeta)[F(\zeta)-F_\infty(\zeta)][k(\zeta)-S_{n\alpha}(\zeta)], \tag{26}$$

for any $P \in A(D)$ and ζ between ∂D_1 and $\partial D_{n\alpha}$. It is now apparent that whenever $P \in A(D)$, $\alpha \geqslant \alpha_N$, and ζ lies between ∂D_1 and $\partial D_{n\alpha}$, then:

$$V_{n\alpha}\left[\frac{P(z)}{z-\zeta}\right] - W_{n\alpha}\left[\frac{P(z)}{z-\zeta}\right] - \frac{1}{2\pi i}\int_{\partial D_1} \frac{P(z)(F-F_\infty)(k-S_{n\alpha})}{z-\zeta}\,dz$$
$$= P(\zeta)[F(\zeta)-F_\infty(\zeta)][k(\zeta)-S_{n\alpha}(\zeta)]. \tag{27}$$

Let us now assume that $P \in AB(D)$. Lemma 3 now applies if ζ is held fixed. Thus,

$$-\frac{1}{2\pi i}\int_{\partial D_1} \frac{P(z)(F-F_\infty)(k-S)}{z-\zeta}\,dz = P(\zeta)[F(\zeta)-F_\infty(\zeta)][k(\zeta)-S(\zeta)], \tag{28}$$

for $P \in AB(D)$ and $\zeta \in D - \bar{D}_1$. It is to be observed that $P(z)/(z-\zeta)$ is bounded and continuous near ∂D and, for application of Lemma 3, this is clearly all that matters.

Since $\infty \notin D$ we see that $B_\infty \cap \partial D$ is nonvoid. Either $B_\infty \cap \partial D$ is totally disconnected or it contains a continuum. If the latter is the case, we can WLOG assume from the start that D is bounded with outer boundary component, for example, the unit circle. The crucial point here is of course that N_B is preserved under auxiliary conformal mapping (see, for instance, Sario and Oikawa [29] p. 261).

Next, let Γ be a component of ∂D_1 along which $k \not\equiv S$. Suppose $\Gamma \subseteq B_j$. Since $D_n \to D$ was assumed to be a proper analytic exhaustion, every component of $S - D_n$ has nonempty intersection with ∂D, as in the Kerekjarto–Stoilow compactification. Let $F = \text{int}\ \Gamma$ when Γ is an inner component of ∂D_1; otherwise, let $F = \text{ext}\ \Gamma$. Since $F \cap \partial D$ is nonvoid, $B_j \cap \partial D$ will be nonvoid and hence $\notin N_B$. We may assume WLOG that $F \cap \partial D \notin N_B$ (see, for example, Sario and Oikawa [29] pp. 261, 289). We note that if $\infty \in F \cap \partial D$, then by assumption $B_\infty \cap \partial D$ is totally disconnected. Since $F \cap \partial D \notin N_B$, it follows in this case that for some large integer N we have $\{F \cap \partial D\} \cap \{|z| \leqslant N\} \notin N_B$ (see, for instance, Sario and Oikawa [29] p. 289).

If $F = \text{int}\,\Gamma$, write $H = F \cap \partial D$. If $\infty \in F \cap \partial D$, write $H = \{|z| \leqslant N\} \cap \{F \cap \partial D\} \notin N_B$ (in this case H is totally disconnected). And if $F = \text{ext}\ \Gamma$ and D has outer boundary ∂U, let H be a small arc along ∂U. Let W denote the unbounded component of open set $S - H$. W is a well-defined domain such that $\infty \in W$, $W \notin O_{AB}$. Define

$$\mathcal{E} = \{f \in AB(W) \mid f(\infty) = 0,\ |f| \leqslant 1 \text{ on } W\}. \tag{29}$$

Consider the classical extremal problem $f \in \mathcal{E}$, $|f'(\infty)| = \max$. A normal family argument shows that there is at least one solution $P \in \mathcal{E}$. Clearly we must have $P'(\infty) \neq 0$, $\limsup_{z \to H} |P(z)| = 1$, and $\sup_{z \in \partial D_1} |P(z)| < 1$. One might also refer to Sario and Oikawa [29] p. 176. Since $D \subseteq W$, $P \in AB(D)$. By (28),

$$-\frac{1}{2\pi i} \int_{\partial D_1} \frac{P(z)^n}{z - \zeta} (F - F_\infty)(k - S)\, dz = P(\zeta)^n [F(\zeta) - F_\infty(\zeta)][k(\zeta) - S(\zeta)], \tag{30}$$

for $\zeta \in D - \bar{D}_1$ and $n \geqslant 1$. Choose any ζ close to H so that $k(\zeta) - S(\zeta) \neq 0$, ∞ and $|P(z)/P(\zeta)| < 1 - \varepsilon$ for all $z \in \partial D_1$. Here $\varepsilon > 0$ is small. Hence,

$$\begin{aligned} |F(\zeta) - F_\infty(\zeta)|\,|k(\zeta) - S(\zeta)| &\leqslant \frac{1}{2\pi} \int_{\partial D_1} \left|\frac{P(z)}{P(\zeta)}\right|^n \frac{1}{|z - \zeta|} |F - F_\infty|\,|k - S|\,|dz| \\ &\leqslant \frac{1}{2\pi} \int_{\partial D_1} (1 - \varepsilon)^n \frac{2e^{\chi(z)}|k - S|}{|z - \zeta|}\,|dz|, \end{aligned} \tag{31}$$

all $n \geqslant 1$. Letting $n \to \infty$ yields $F(\zeta) = F_\infty(\zeta)$, which, because of our relative freedom in the choice of ζ, implies $F \equiv F_\infty$ on D. This concludes the proof of Theorem 1(b). ∎

It is now relatively easy to prove part (d) of Theorem 1.

THEOREM 1(d). *Let the assumptions (i)–(viii) of section III hold. Suppose that the fundamental extremal problem is nontrivial. Assume that* $P(z)k(z)=Q(z)$ *near* ∂D *for some* P *and* Q *such that* $P\not\equiv 0$, $P\in AB(D)$, *and* $Q\in A(D)$. *Then uniqueness holds.*

Proof. We refer to the proof of Theorem 1(b). Clearly, $B_j\cap\partial D$ nonvoid implies $P(z)k(z)=Q(z)$ for $z\in B_j\cap D$. Clearly, $Q\not\equiv 0$. Select a component Γ of ∂D_1 along which $k\not\equiv S$; let $\Gamma\subseteq B_j$. Suppose that ζ lies between ∂D_1 and $\partial D_{n\alpha}$. WLOG $\zeta\neq$ all ξ_k.

$$\frac{1}{2\pi i}\int_{\partial D_{n\alpha}}\frac{P(z)}{z-\zeta}F(z)\,[k(z)-S_{n\alpha}(z)]\,dz=Q(\zeta)\,F(\zeta)-P(\zeta)\,F(\zeta)\,S_{n\alpha}(\zeta), \tag{32}$$

via the Cauchy integral formula. Similarly for any other extremal function F_∞. Hence

$$V_{n\alpha}\left[\frac{P(z)}{z-\zeta}\right]=2\pi i\,F(\zeta)\,[Q(\zeta)-P(\zeta)\,S_{n\alpha}(\zeta)]; \tag{33}$$

$$W_{n\alpha}\left[\frac{P(z)}{z-\zeta}\right]=2\pi i\,F_\infty(\zeta)\,[Q(\zeta)-P(\zeta)\,S_{n\alpha}(\zeta)]. \tag{34}$$

If we now keep ζ near ∂D_1 and in B_j, at once

$$U\left[\frac{P(z)}{z-\zeta}\right]=2\pi i\,F(\zeta)\,[Q(\zeta)-P(\zeta)\,S(\zeta)]; \tag{35}$$

$$U\left[\frac{P(z)}{z-\zeta}\right]=2\pi i\,F_\infty(\zeta)\,[Q(\zeta)-P(\zeta)\,S(\zeta)]. \tag{36}$$

But, surely, $Q(\zeta)-P(\zeta)\,S(\zeta)\not\equiv 0$ near $B_j\cap\partial D_1$. Hence, $F\equiv F_\infty$ near $B_j\cap\partial D_1$ whence $F\equiv F_\infty$.

VIII. Some removable singularity theorems via potential theory

Our basic reference for potential theory in Euclidean space will be L. Helms [17].

The first result of this section concerns meromorphic functions of bounded characteristic (Nevanlinna class) on arbitrary plane domains. We adhere to the notation and concepts contained in Sario and Nakai [27] pp. 269–280. In addition, we will often make use of results found in Sario and Noshiro [28] chapter 3. It is to be recalled that if W is an open Riemann surface, $MB^*(W)$ denotes the family of all $f\in M(W)$ for which $\ln^+|f|$ admits a superharmonic majorant. We define $AB^*(W)=A(W)\cap MB^*(W)$.

THEOREM 3. *Let* V *be any subdomain of* S. *Let* E *be a bounded closed totally disconnected set,* $E\subseteq V$. *Suppose that* Cap $(E)=0$. *Then* $MB^*(V-E)=MB^*(V)$.

Proof. If $V \in O_G$, trivially $V - E \in O_G$. This case is now trivial since $O_G = O_{HP} = O_{MB^*} = O_{AB^*}$ for plane domains. So, WLOG, $V \notin O_G$. Let $R = V - E$. Suppose that $f \in MB^*(R)$, f nonconstant. Let $R \cap f^{-1}(\infty) = \{a_1, a_2, \ldots\}$. Let $R \cap f^{-1}(0) = \{b_1, b_2, \ldots\}$. Let v_n be the order of a_n and μ_n the order of b_n. It follows that

$$\ln |f(z)| = \sum_n v_n g(z; a_n; R) - \sum_n \mu_n g(z; b_n; R) + h_1(z) - h_2(z), \tag{37}$$

$z \in R$, where
$$\sum_n v_n g(z; a_n; R) \not\equiv \infty; \quad \sum_n \mu_n g(z; b_n; R) \not\equiv \infty, \tag{38}$$

and h_1 and h_2 are in $HP(R)$. (See Sario and Nakai [27] p. 271). But, $g(z; \xi; R) = g(z; \xi; V)$ for $(z, \xi) \in R \times R$. Thus, $a_n \to \partial V$ and $b_n \to \partial V$ whenever $n \to \infty$. Since Cap $(E) = 0$, h_1 and h_2 admit unique superharmonic extensions to V (see Helms [17] pp. 130, 150). Call these extensions h_1 and h_2; these are nonnegative. An easy application of the Riesz decomposition theorem for superharmonic functions ((Helms [17] pp. 116, 105) shows that there are totally finite Borel measures μ_1 and μ_2 concentrated on E such that:

$$h_1(z) = \int_E g(z; \xi; V)\, d\mu_1(\xi) + H_1(z); \tag{39}$$

$$h_2(z) = \int_E g(z; \xi; V)\, d\mu_2(\xi) + H_2(z), \tag{40}$$

where H_1 and H_2 are now in $HP(V)$, $z \in V$. Of course, H_j is just the greatest harmonic minorant of $h_j \in SP(V)$. Let $\mu = \mu_1 - \mu_2$. μ is a totally finite signed Borel measure concentrated on E. Recall the Hahn decomposition for μ and $|\mu|$. Define $A = \{x \in E \mid |\mu|(x) \geq 1\}$. Notice that $|\mu|(x) = |\mu(x)|$. Set A is finite; let $A = \{x_1, \ldots, x_p\}$. Consider, say, $x_1 \in A$. By the regularity of μ and $|\mu|$ we easily determine a nested sequence of analytic Jordan curves $\{\Gamma_n\}_{n=1}^{\infty}$ such that $\Gamma_n \subseteq R$, $\Gamma_n \to \{x_1\}$ in the Frechét metric, with

$$\mu(G_n \cap E) \to \mu(x_1); \quad |\mu|(G_n \cap E) \to |\mu|(x_1); \tag{41}$$

$$\mu(G_n \cap E - \{x_1\}) \to 0; \quad |\mu|(G_n \cap E - \{x_1\}) \to 0, \tag{42}$$

with $G_n = \operatorname{int} \Gamma_n$. WLOG $\bar{G}_1$ is free of a_n's and b_n's. Now, for $z \in R$,

$$\ln |f(z)| = H_1(z) - H_2(z) + \sum v_n g(z; a_n; V) - \sum \mu_n g(z; b_n; V) + \int_E g(z; \xi; V)\, d\mu(\xi). \tag{43}$$

In n denotes the inner normal, then as $k \to \infty$,

$$\frac{1}{2\pi} \int_{\Gamma_k} \frac{\partial}{\partial n} \ln |f(z)|\, |dz| = \mu[G_k \cap E] = \text{integer}, \tag{44}$$

where $\Gamma_k = \partial G_k$. At once, $\mu(x_1) =$ integer and $\mu[G_n \cap E - \{x_1\}] \equiv 0$ for n large. Similarly for the other $x_j \in A$.

Similar methods show that $|\mu|(x_0) = |\mu(x_0)| = 0$ for all $x_0 \in E - A$. Define:

$$\left.\begin{aligned} \delta(M) &= \mu[M \cap A] \\ \eta(M) &= \mu[M \cap (E-A)] \\ \mu &= \eta + \delta \\ |\mu| &= |\eta| + |\delta| \end{aligned}\right\} \tag{45}$$

Now, if $\delta(x_j) = c_j$, $1 \leqslant j \leqslant p$, then for $z \in R$,

$$\ln |f(z)| = H_1 - H_2 + \sum v_n g(z; a_n; V) - \sum \mu_n g(z; b_n; V) + \sum_{j=1}^{p} c_j g(z; x_j; V) + \int_E g(z; \xi; V)\, d\eta(\xi). \tag{46}$$

Choose any $h \in C(E)$. Take an exhaustion $R_n = V - E_n$ of R towards E. Each E_n is therefore the disjoint union of a finite number of analytic Jordan regions as in the Kerekjarto-Stoilow compactification. (See, for example, Ahlfors and Sario [3] pp. 81–90 or Sario and Nakai [27] pp. 250–252.) For large n, by uniform continuity, the variation of h on each component of E_n will be $< \varepsilon$. Of course $|\eta|(E) \neq \infty$. Let us write E_n in components as $E_n = \bigcup \{E_n^\alpha \mid 1 \leqslant \alpha \leqslant \alpha_n\}$. Choose $z_n \in E \cap E_n^\alpha$. Then:

$$\left| \int_E h\,d\eta - \sum_{\alpha=1}^{\alpha_n} h(z_n^\alpha)\, \eta(E_n^\alpha) \right| \leqslant \varepsilon |\eta|(E). \tag{47}$$

Suppose that $\eta(E_n^\alpha) \neq 0$ for some large n and $1 \leqslant \alpha \leqslant \alpha_n$. Construct a sequence $E_n^\alpha \geqslant E_{n+1}^{t_{n+1}} \geqslant E_{n+2}^{t_{n+2}} \geqslant \dots$ such that $\eta(E_{n+j}^{t_{n+j}}) \neq 0$, each $j \geqslant 1$. Of course, $1 \leqslant t_{n+j} \leqslant \alpha_{n+j}$. But, just as in the Kerekjarto-Stoilow compactification,

$$\bigcap_{j=1}^{\infty} E_{n+j}^{t_{n+j}} = \{x_0\} \subseteq E. \tag{48}$$

By use of a formula similar to (44) except over $\partial E_{n+j}^{t_{n+j}}$ we see that each $\eta(E_{n+j}^{t_{n+j}}) =$ integer. However, from $\eta(E_{n+j}^{t_{n+j}}) \to \eta(x_0) = 0$, we conclude that $\eta(E_{n+j}^{t_{n+j}}) = 0$ for all large j. Contradiction. It follows at once that

$$\left| \int_E h(z)\, d\eta(z) \right| \leqslant \varepsilon |\eta|(E), \tag{49}$$

all $\varepsilon > 0$. It follows at once that $\eta = 0$. So, for $z \in R$, with $H_j \in HP(V)$,

$$\ln |f(z)| = H_1 - H_2 + \sum v_n g(z; a_n; V) - \sum \mu_n g(z; b_n; V) + \sum_{j=1}^{p} c_j g(z; x_j; V). \tag{50}$$

But now, since Cap $(E)=0 \Rightarrow E\in N_B$, we see at once that $f\in M(V)$ (see, for example, Sario and Oikawa [29] pp. 255, 260). At once, $\ln^+|f|$ admits a superharmonic majorant on V, whence $f\in MB^*(V)$. Thus, $MB^*(R)\subseteq MB^*(V)$. The opposite inclusion is trivial. ∎

Suppose now that V is any subdomain of S. Suppose that $E\subseteq V$, with E bounded, closed and totally disconnected. Let Cap $(E)=0$. Suppose that $\chi\in HP(R)$, where $R=V-E$. We wish to now investigate the properties of $f\in A(R)$ provided $\ln|f|\leqslant\chi$. First of all, suppose that $V\in O_G$. Then, $R\in O_G$ and by virtue of $O_G=O_{HP}=O_{HB}$ for plane regions, we find that both χ and f are constants. Therefore, WLOG, $V\notin O_G$. Furthermore, we might as well asume $\infty\notin V$. It is not out of place to point out that χ could conceivably behave like an Evans–Selberg potential near E (see Sario and Noshiro [28] pp. 113–114 or Helms [17] p. 152).

In any case, $\chi\in HP(R)$ extends uniquely to a superharmonic function on V. And χ will be in $SP(V)$. (Helms [17] 130, 150, 152.) We now apply the Riesz decomposition to χ on V (as in Helms [17] pp. 116, 105). There thus exists a totally finite Borel measure μ concentrated on E such that

$$\chi(z)=G_V\mu+h=\int_E g(z;\xi;V)\,d\mu(\xi)+h(z), \tag{51}$$

with $h\in H(V)$. At this point, it is important to notice that μ is unique and that h is merely the greatest harmonic minorant of χ over V. (See Helms [17] pp. 92, 101, 105, 112, 115–117.)

For purposes of the next theorem, we point out that similar remarks hold when $\chi\in h_1(R)$.

THEOREM 4. *Let V be a subdomain of S, $\infty\notin V$, $V\notin O_G$. Let E be a bounded closed totally disconnected set, $E\subseteq V$, with* Cap $(E)=0$. *Take $R=V-E$ and $\chi\in h_1(R)$. Then, one can write $\chi=G_V\mu+\mathcal{Y}$ where μ is a totally finite Borel signed measure concentrated on E and $\mathcal{Y}\in h_1(V)$. Both μ and $\mathcal{Y}$ are unique. Suppose that $f\in A(R)$ with $\ln|f|\leqslant\chi$. Then, $f\in MB^*(V)$ and*

$$\ln|f(z)|\leqslant\sum_{x\in A}[\mu(x)]\,g(z;x;V)+\mathcal{Y}(z),\quad z\in V,\,(^1) \tag{52}$$

where A denotes the set of atoms of μ. If $f\not\equiv 0$, then in the Hahn decomposition $\mu=\mu^+-\mu^-$, μ^- will be concentrated on a finite subset of A.

Proof. We shall first assume that $\chi\in HP(R)$. In this case, the first part of the theorem has already been discussed. Therefore, suppose $f\in A(R)$ with $\ln|f|\leqslant\chi$. First of all, clearly, $\ln|f|\leqslant\ln^+|f|\leqslant\chi$ so that $f\in AB^*(V-E)$. By Theorem 3, however, we immediately conclude that $f\in MB^*(V)$.

(1) $[x]$ denotes the greatest integer function.

Equation (52) is trivial when $f \equiv 0$. Therefore assume $f \not\equiv 0$. We know that $f \in MB^*(V)$ and that $f^{-1}(\infty) \cap V \subseteq E$. Let $f^{-1}(0) \cap E = \{b_1, \ldots, b_r\}$ and $f^{-1}(0) \cap R = \{b_{r+1}, \ldots\}$. Let μ_n denote the multiplicity of b_n. As usual, for $z \in R$,

$$\ln |f(z)| + \sum_{n \geq r+1} \mu_n g(z; b_n; R) \leq \chi(z), \tag{53}$$

where we recall that $g(z; \xi; V) = g(z; \xi; R)$ for $(z, \xi) \in R \times R$. As in (51), write

$$\chi(z) - \ln |f(z)| - \sum_{n \geq r+1} \mu_n g(z; b_n; V) = \int_E g(z; \xi; V)\, d\eta(\xi) + q(z), \tag{54}$$

$z \in V$, $\eta =$ a totally finite Borel measure concentrated on E, $q \in HP(V)$. But,

$$\ln |f(z)| = \sum_{k=1}^{m} \delta_k g(z; y_k; V) - \sum_{n \geq 1} \mu_n g(z; b_n; V) + h_1(z), \tag{55}$$

$h_1 \in H(V)$, $\{y_1, \ldots, y_m\} =$ poles of $f(z)$, $\delta_k =$ order of pole y_k, because $f \in MB^*(V)$ (Sario and Nakai [27] pp. 270–271). At once, $z \in V$,

$$\chi = \sum_{k=1}^{m} \delta_k g(z; y_k; V) - \sum_{n=1}^{r} \mu_n g(z; b_n; V) + h_1(z) + G_V \eta + q(z). \tag{56}$$

By virtue of (51) and its uniqueness, at once:

$$\left.\begin{aligned} h_1(z) + q(z) &= h(z) \\ \sum_{k=1}^{m} \delta_k g(z; y_k) - \sum_{n=1}^{r} \mu_n g(z; b_n) + G_V \eta &= G_V \mu \end{aligned}\right\} \tag{57}$$

Let $y_k \in E$ be a proper pole of $f(z)$. By (57), $1 \leq \delta_k \leq \delta_k + \eta(y_k) = \mu(y_k)$ so that $\delta_k \leq [\mu(y_k)]$. Next, let $b_n \in E$. Here $-\mu_n + \eta(b_n) = \mu(b_n)$. $\eta(b_n) \geq 0$ implies $-\mu_n \leq [\mu(b_n)]$. Of course, μ_n is an integer. At all other points x_0 of E, $\eta(x_0) = \mu(x_0)$. By (55),

$$\begin{aligned} \ln |f(z)| = \sum_{k=1}^{m} \delta_k g(z; y_k) - \sum_{n \geq 1} \mu_n g(z; b_n) + h_1(z) &\leq \sum_{k=1}^{m} \delta_k g(z; y_k) - \sum_{n=1}^{r} \mu_n g(z; b_n) + h_1(z) \\ &\leq \sum_{k=1}^{m} [\mu(y_k)] g(z; y_k) + \sum_{n=1}^{r} [\mu(b_n)] g(z; b_n) + h(z) \end{aligned} \tag{58}$$

since $q \in HP(V)$ and $h_1 + q = h$. It is now easy to check that

$$\ln |f(z)| \leq h(z) + \sum_{x \in E} [\mu(x)] g(z; x; V), \quad z \in V. \tag{59}$$

Equation (52) follows at once. The theorem is thus proved for the case $\chi \in HP(R)$.

Now consider the case $\chi \in h_1(R)$. Use of (51) and the Hahn decomposition easily yields the first part of the theorem. Now, let $f \in A(R)$, $\ln |f| \leqslant \chi$. Of course, WLOG $f \not\equiv 0$. Write $\chi = \chi_1 - \chi_2$ with $\chi_j \in HP(R)$ so that $\ln^+ |f| \leqslant \chi_1$. Theorem 3 then implies $f \in MB^*(V)$. To prove the remaining portion of the theorem, proceed roughly verbatim relative to the preceding case. Note, in particular, the counterpart to (57). The required results follow at once. ∎

We will make important use of Theorem 4 in the proof of Theorem 1(c) to follow. We have introduced these removability theorems in advance of our proof for Theorem 1(c) in order not to break the continuity of thought. Removability theorems as above will be seen to be precisely what is called for in the proof of Theorem 1(c).

IX. Development of the fundamental theorem, part two

Before turning to the proof of Theorem 1(c), we wish to recall a very useful result which is by now reasonably well-known. Becuase of its importance in our investigation, we state this result as a theorem.

THEOREM 5. (*Generalized Nevanlinna-Frostman theorem*). *Let D be a subdomain of S, $D \notin O_G$. Let K be a bounded closed subset of S with* Cap $(K) \neq 0$. *Suppose that $f \in M(D)$ and that $f(D) \cap K$ is void. Then, $f \in MB^*(D)$.*

Proof. We merely refer, for example, to Heins [15] pp. 426–428, Rudin [26] pp. 48–49, or Sario and Noshiro [28] p. 92. For the original version, see Golusin [8] pp. 282–284 or Nevanlinna [21] pp. 272–276. ∎

THEOREM 1(c). *Suppose that assumptions (i)–(viii) of section III hold, all nonvoid $B_j \cap \partial D \notin N_B - N_G$, extremal problem $\mathfrak{C}$ over C is nontrivial, and that $\chi \in h_1(D)$. Then uniqueness holds.*

Proof. We first prove the result under the condition that χ is bounded above. Clearly $D \notin O_{AB}$ WLOG. It follows that some $B_j \cap \partial D \notin N_B$ (see Sario and Oikawa [29] p. 289). We may assume WLOG that $B_\infty \cap \partial D \notin N_B$. Suppose now that the result is false. Let N be the number of j for which $B_j \cap \partial D$ is nonvoid. Of course, $N \geqslant 1$. WLOG N is minimal. Now, $N \neq 1$ or else the proof of Theorem 1(b) applies. Select any component Γ of ∂D_1 along which $k \not\equiv S$. We shall refer to the proof of Theorem 1(b). Suppose $\Gamma \subseteq B_j$. Therefore, $B_j \cap \partial D$ is nonvoid. In the usual way, now, $k \not\equiv S$ for all components of $B_j \cap \partial D_1$. If $B_j \cap \partial D \notin N_B$ the proof of Theorem 1(b) again applies. It follows therefore that $B_j \cap \partial D \in N_B$. Note that $j \neq \infty$. By hypothesis, then, $B_j \cap \partial D \in N_G$. Consider $\{\text{int } \Gamma\} \cap \partial D$ and $E = D \cup \{\text{int } \Gamma\}$. Because $\{\text{int } \Gamma\} \cap \partial D \in N_B$, it follows that $F(z) \in AB(E)$. Let the distinct zeros of $F(z)$

on $\{\text{int } \Gamma\} \cap \partial D$ be $A_1, \ldots, A_s$. Each A_h is a boundary component of D. At this point we recall that for plane regions $O_G = O_{HP} = O_{HB}$ and $N_G = N_{HB}$ (see, for example, Sario and Oikawa [29] pp. 259–261). It is now easy to check that χ must remain harmonic on $\{\text{int } \Gamma\} - \{A_1, \ldots, A_s\}$. Then,

$$\ln |F(z)| \leqslant \chi(z), \quad z \in E - \{A_1, \ldots, A_s\} \tag{60}$$

We can now find $e_j \geqslant 0$ such that:

$$\chi(z) = -\sum_{h=1}^{s} e_h g(z; A_h; E) + \mathcal{Y}(z), \tag{61}$$

with $\mathcal{Y}$ bounded above and harmonic on E. Define:

$$N_h = \begin{cases} e_h, & e_h \text{ integral} \\ 1 + [e_h], & e_h \text{ nonintegral} \end{cases}.$$

Also, let

$$\mathcal{C}_E = \{f \in AB(E) \mid \ln |f| \leqslant \mathcal{Y},\ f(\xi_k) = 0,\ \text{mult} \geqslant n_k,\ f(A_h) = 0,\ \text{mult} \geqslant N_h\}.$$

We wish to show that $\mathcal{C}$ is precisely the restriction of $\mathcal{C}_E$ to D. To see this, let $\varphi(z) \in \mathcal{C}$. As seen above, $\varphi \in AB(E)$. By continuity, $\ln |\varphi(z)| \leqslant \chi(z)$ for $z \in E$. By (61), clearly $\varphi(A_h) = 0$, mult $\geqslant N_h$. Since $\chi \leqslant \mathcal{Y}$, $\varphi \in \mathcal{C}_E$. Now, conversely, let $\psi \in \mathcal{C}_E$. We must show that $\ln |\psi(z)| \leqslant \chi(z)$. Or,

$$\ln |\psi(z)| + \sum_{h=1}^{s} e_h g(z; A_h; E) \leqslant \mathcal{Y}(z). \tag{62}$$

But, (62) is clear since $\ln |\psi(z)| + \sum_{h=1}^{s} N_h g(z; A_h; E) + \sum_{k=1}^{\infty} n_k g(z; \xi_k; E) \leqslant \mathcal{Y}(z)$ and $e_h \leqslant N_h$.

We now solve the fundamental extremal problem for $\mathcal{L}$ over $\mathcal{C}$ and $\mathcal{C}_E$. These are of course equivalent. Repeating the above process on each component of $B_j \cap \partial D_1$ clearly leads us to a contradiction of the fact that N is minimal. This proves Theorem 1(c) for the case in which χ is bounded above.

Remark. It seems appropriate to note that equation (61) was a very critical part of the above proof. If we knew only that $B_j \cap \partial D \in N_B$ we could not necessarily derive such a decomposition for χ. For instance, χ might be an equilibrium potential when $B_j \cap \partial D \in N_B - N_G$.

We now continue with the proof of Theorem 1(c) in the case $\chi \in h_1(D)$. If χ is bounded above, the previous part of the proof applies. Hence, WLOG, $D \notin O_G = O_{HB} = O_{HP}$. There is thus at least one nonvoid $B_j \cap \partial D$ with positive capacity (see Sario and Oikawa [29] p. 259). By hypothesis, then, $B_j \cap \partial D \notin N_B$. It follows that $D \notin O_{AB}$. We may assume WLOG that

$B_\infty \cap \partial D \notin N_B$. We now proceed much as in the preceding case. Suppose the result is false. If N denotes the number of j for which $B_j \cap \partial D$ is nonvoid, then $N \geqslant 1$ and WLOG N is minimal. Surely $N \neq 1$ or else the proof of Theorem 1(b) applies. Select any component Γ of ∂D_1 along which $k \not\equiv S$. Let $\Gamma \subseteq B_j$. $k \not\equiv S$ for all components of $B_j \cap \partial D_1$. Clearly, $B_j \cap \partial D$ is nonvoid. If $B_j \cap \partial D \notin N_B$, the proof of Theorem 1(b) again applies. Hence, $B_j \cap \partial D \in N_B$ and $j \neq \infty$. By our hypothesis, $B_j \cap \partial D \in N_G$. Let $\Delta = B_j \cap \partial D$ and $V = D \cup \Delta$. Surely, $V \notin O_{AB}$ because $B_\infty \cap \partial D \notin N_B$. Then, of course, $V \notin O_G$. Write $\chi = \chi_1 - \chi_2$ with $\chi_j \in HP(D)$. By (51),

$$\chi_j(z) = G_V \eta_j + \mathcal{Y}_j(z), \quad j = 1, 2, \tag{63}$$

where η_j is a totally finite Borel measure concentrated on Δ, and where $\mathcal{Y}_j \in HP(V)$ represents the greatest harmonic minorant of $\chi_j \in SP(V)$. Recall too the uniqueness of $\mathcal{Y}_j$ and η_j. By Theorem 4, for any $f \in \mathcal{C}$,

$$\ln|f(z)| \leqslant \sum_{x \in \Delta} [\eta_1(x) - \eta_2(x)]\, g(z; x; V) + \mathcal{Y}_1(z) - \mathcal{Y}_2(z) \tag{64}$$

for $z \in V$ and, to be sure, f will be in $MB^*(V)$. Select $\{\zeta_1, \ldots, \zeta_s\} \subseteq \Delta$ so that (64) becomes

$$\ln|f(z)| \leqslant \sum_{k=1}^{s} \delta_k\, g(z; \zeta_k; V) + \mathcal{Y}_1(z) - \mathcal{Y}_2(z), \quad z \in V. \tag{65}$$

The δ_k are nonzero integers dependent solely upon χ_1 and χ_2.

Consider now the proof of the preceding case when $\chi \leqslant 0$. For (65) we have $\mathcal{Y}_1 = 0$ and all $\delta_k \leqslant -1$. Because of this, we were able to transform extremal problem $\mathfrak{L}$ over $\mathcal{C}$ on D to an equivalent extremal problem $\mathfrak{L}$ over $\mathcal{C}_V$ on V, generally speaking, where we knew that uniqueness held. In the present case, however, the δ_k can be positive integers so that the previous argument fails. An obvious modification of this method is to study $f(z)M(z)$ where $M(z) = \prod_{k=1}^{s} (z - \zeta_k)^{\delta_k}$:

$$\ln|f(z)\, M(z)| \leqslant \ln|M| + \sum_{k=1}^{s} \delta_k\, g(z; \zeta_k; V) + \mathcal{Y}_1(z) - \mathcal{Y}_2(z), \tag{66}$$

$z \in V$. We recall that $MB^*(V)$ is a field (see, for example, Sario and Nakai [27] p. 270). Consider $z - \zeta_k$ for $z \in V$. Clearly this omits a set of positive capacity. Hence, by the generalized Nevanlinna-Frostman theorem, $z - \zeta_k \in MB^*(V)$. Thus, $M(z) \in MB^*(V)$. So,

$$\ln|M(z)| = -\sum_{k=1}^{s} \delta_k\, g(z; \zeta_k; V) + h(z),$$

for $z \in V$, $h(z) \in h_1(V)$ (see, for instance, Sario and Nakai [27] p. 271). Thus,

$$\ln|f(z)\, M(z)| \leqslant h(z) + \mathcal{Y}_1(z) - \mathcal{Y}_2(z) \in h_1(V) \tag{67}$$

Thus, $f \in \mathcal{C}$ implies $fM \in A(V)$ with $\ln |fM| \leqslant h + \mathcal{Y}_1 - \mathcal{Y}_2$ on V. Conversely, define

$$\mathcal{C}_1 = \{q \in A(V) \mid q(\xi_k) = 0, \text{ mult} \geqslant n_k, \ln |q| \leqslant h + \mathcal{Y}_1 - \mathcal{Y}_2\}. \tag{68}$$

Suppose now that $q \in \mathcal{C}_1$. Then, $q/M \in A(D)$ and

$$\ln |q/M| \leqslant \sum_{k=1}^{s} \delta_k\, g(z; \zeta_k; V) + \mathcal{Y}_1 - \mathcal{Y}_2 = \sum_{x \in \Delta} [\eta_1(x) - \eta_2(x)]\, g(z; x; V) + \mathcal{Y}_1 - \mathcal{Y}_2$$

$$\leqslant \int_\Delta g(z; x; V)\, d(\eta_1 - \eta_2)(x) + \mathcal{Y}_1 - \mathcal{Y}_2$$

by means of a careful application of Theorem 4. (Notice above all that if $\eta = \eta_1 - \eta_2$, then η^- is concentrated at finitely many points.) It follows that $q/M \in \mathcal{C}$ whenever $q \in \mathcal{C}_1$, while $fM \in \mathcal{C}_1$ whenever $f \in \mathcal{C}$. With

$$\mathcal{L}_1[v] = \int_K (v/M)\, d\lambda, \quad v \in C(K), \tag{69}$$

we thus have $\mathcal{L}_1[v] = \mathcal{L}[v/M]$, all $v \in C(K)$, and so extremal problems $\mathcal{L}$ over $\mathcal{C}$ and $\mathcal{L}_1$ over $\mathcal{C}_1$ are equivalent. However, by minimality of N, extremal problem $\mathcal{L}_1$ over $\mathcal{C}_1$ has uniqueness. Contradiction. ■

X. Development of the fundamental theorem, part three

After providing proofs for Theorems 1(b), (c), and (d) it is now possible to handle the case of finite connectivity rather easily. It will be seen that although the techniques needed are reasonably elementary, they are not exactly trivial.

THEOREM 1 (a). *Let the assumptions (i)–(viii) of section III hold, let D be of finite connectivity, and assume extremal problem $\mathcal{L}$ over $\mathcal{C}$ is nontrivial. Then uniqueness holds.*

Proof. There are two cases: (a) ∂D is totally disconnected; and, (b) ∂D contains a continuum. We handle case (b) first. In this case, via an auxiliary conformal mapping, WLOG D is bounded. Let $\partial D = \Gamma_0 + \ldots + \Gamma_{p-1}$ and take Γ_0 to be the outer component. Define $\beta_k = (1/2\pi)\int_{\Gamma_k} d\chi^*$. Here χ^* is the harmonic conjugate of χ. Let $a_j \in \Gamma_j$, $1 \leqslant j \leqslant p-1$. Set

$$\mathcal{U} = \chi + \sum_{k=1}^{p-1} \beta_k \ln |z - a_k|. \tag{70}$$

At once, $\mathcal{U} = \operatorname{Re} h(z)$, $h(z) \in A(D)$. Write

$$\chi = \operatorname{Re} h(z) + \sum_{k=1}^{p-1} (N_k - \beta_k) \ln |z - a_k| - \ln |(z - a_1)^{N_1} \cdots (z - a_{p-1})^{N_{p-1}}|$$

for large positive integers $N_1, \ldots, N_{p-1}$. Thus, since D is bounded,

$$\chi(z) = \ln |P(z)| + \mathcal{Y}(z), \tag{71}$$

$P \in A(D)$, $P \neq 0, \infty$, and $\mathcal{Y} \in H(D)$ and is bounded above. Let

$$\mathcal{C}_1 = \{f \in A(D) \mid f(\xi_k) = 0, \text{ mult} \geq n_k, \ln |f| \leq \mathcal{Y}\};$$

$$\mathfrak{L}_1[v] = \int_K vP \, d\lambda, \quad v \in C(K).$$

It is easily checked that extremal problems $\mathfrak{L}_1$ over $\mathcal{C}_1$ and $\mathfrak{L}$ over $\mathcal{C}$ are equivalent. By Theorem 1(c), $\mathfrak{L}_1$ over $\mathcal{C}_1$ has uniqueness. Hence, so does $\mathfrak{L}$ over $\mathcal{C}$. This proves case (b).

For case (a) we use a function similar to (70) to prove that $D = \{f \in A(D) \mid \ln |f| \leq \chi\}$ is essentially a one-dimensional class. This reasonably straightforward proof is omitted. The uniqueness is now trivial. ∎

XI. Development of the fundamental theorem, part four

The following result will serve to show that in a very strong sense Theorems 1(b) and 1(c) are best possible. We shall use the notation of section III.

THEOREM 6. *Let E be any bounded closed totally disconnected subset of S such that $E \subseteq U$. Let $D = U - E$ and suppose that $E \in N_B - N_G$. It is then possible to choose $\chi \in HB(D)$ and a linear functional $\mathfrak{L}$ with K an analytic Jordan curve, $E \subseteq \operatorname{int} K$, such that the fundamental extremal problem for $\mathfrak{L}$ over $\mathcal{C} = \{f \in A(D) \mid \ln |f| \leq \chi\}$ is nontrivial yet fails to possess a unique extremal function.*

Proof. A well-known theorem of Kellogg asserts that the set of irregular boundary points of D is an F_σ set of inner logarithmic capacity zero (see, for example, Sario and Oikawa [29] p. 185). Since Cap $(E) \neq 0$, it follows that uncountably many points of E will be regular boundary points of D. By an auxiliary linear fractional mapping, WLOG the origin is a regular boundary point of D. We next note that the Perron-Wiener-Brelot process, as described, for example, in Ahlfors and Sario [3] pp. 138–142, Helms [17] chapter 8, or Sario and Nakai [27] pp. 249–250, is used to solve the Dirichlet problem on D.

If we now solve the Dirichlet problem on D for continuous boundary value $|\text{Re}(\xi)|$, $\xi \in \partial D$, then the solution $\chi(z) \in HB(D)$ will satisfy $\lim_{z \to 0} \chi(z) = 0$ since the origin is a regular point. When solving this Dirichlet problem by the Perron-Wiener-Brelot process, observe that $\text{Re}(z)$, $-\text{Re}(z)$, and 0 are three competing subharmonic functions. Hence, $\text{Re}(z) \leq \chi(z)$, $-\text{Re}(z) \leq \chi(z)$, and $0 \leq \chi(z)$.

Let $\mathcal{C} = \{f \in A(D) \mid \ln |f| \leq \chi\}$. Hence, e^z, e^{-z}, and 1 all lie in $\mathcal{C}$. It is easy to see that $E \in N_B$ implies $\mathcal{C} \subseteq AB(U)$. We may therefore let $\mathfrak{L}[f] = f(0)$ for $f \in \mathcal{C}$ using the Cauchy integral formula over $K = \{|z| = 1 - \delta\}$ in the obvious way. Of course, δ is small. At once, e^z, e^{-z}, and 1 are extremal functions for the extremal problem $\mathfrak{L}$ over $\mathcal{C}$. ∎

XII. Some generalizations

As has been mentioned previously in section III, the families of analytic functions over which we take extremums have been basically defined by a condition on the boundary behavior of the functions involved. For example, the condition $\ln|f(z)| \leqslant \chi(z)$, with χ harmonic, which we have used thus far, works quite well. Nonetheless, there are other natural conditions of a similar type which we might just as well use. One such condition arises from the study of Hardy H_p classes on arbitrary domains (see, for example, Heins [16] or Rudin [26]). This condition is $|f(z)|^p \leqslant \chi(z)$, with $0<p<\infty$ and χ harmonic. The crucial point here is that if $f(z)$ is analytic, then $|f(z)|^p$ is subharmonic.

With this motivation, let Φ be any function from $(0, \infty)$ into the reals such that Φ is continuous and strictly increasing with $\lim_{u\to\infty} \Phi(u) = \infty$. Let $\eta = \lim_{u\to 0}\Phi(u)$. Define $\psi = \Phi^{-1}$ so that ψ takes $[\eta, \infty]$ onto $[0, \infty]$. Suppose finally that whenever f is single-valued analytic on some plane domain G, then $\Phi[|f|]$ is subharmonic on G.

First of all, we have the following result.

THEOREM 7. *Let the assumptions (i)–(viii) of section III apply. Define*

$$\mathcal{C} = \{f \in A(D) \,|\, f(\xi_k) = 0, \text{ mult} \geqslant n_k, \; \Phi[\,|f(z)|\,] \leqslant \chi(z)\}. \tag{72}$$

Suppose that $\mathcal{C}$ is well-defined and that the fundamental extremal problem for $\mathfrak{L}$ over $\mathcal{C}$ is non-trivial. Uniqueness then holds if either of the following conditions holds:

(*a*) *no nonvoid $B_j \cap \partial D$ lies in N_B;*

(*b*) *$P(z)k(z) = Q(z)$ near ∂D for $P \in AB(D)$, $Q \in A(D)$, $P \not\equiv 0$.*

Proof. We begin by noting that the condition $\Phi[|f|] \leqslant \chi$ is a boundary condition by virtue of the maximum principle for subharmonic functions. If one now replaces $e^{\chi(z)}$ by $\psi[\chi(z)]$ at the appropriate places, the proofs of Theorems 1(b) and (d) go over almost verbatim. We note, however, one subtle point. In order to apply Theorem 2 strictly, we must have $\psi[\chi(z)]$ Hölder continuous, which, of course, it not always the case. To get around this difficulty, we merely use the Fatou nontangential boundary values where appropriate, since the functions concerned belong basically to Hardy class H_1 or Smirnov class E_1 (see, for example, Priwalow [23]). Also, refer to equation (10). We omit further details. ∎

We next seek to generalize Theorem 1 (c). This we now do, but we will by no means attempt to achieve maximum generality. As before, it will be convenient to prove an appropriate removable singularity theorem first. See also Parreau [21 a] pp. 182–183, 192–193.

THEOREM 8. *Let V be a subdomain of S, $\infty \notin V$, $V \notin O_G$. Let E be a bounded closed totally disconnected set, $E \subseteq V$, with* Cap $(E) = 0$. *Let $R = V - E$ and $\chi \in HP(R)$. Suppose next* *that* $\eta=0$ *and* $\Phi(u)/\ln u\to\infty$ *as* $u\to\infty$. *Suppose finally that* $f\in A(R)$ *with* $\Phi[|f(z)|]\leqslant\chi(z)$. *Then* $f\in A(V)$ *and* $\Phi[|f(z)|]\leqslant h(z)$ *in the notation of equation (51).*

Proof. Let Q be chosen so large that $\Phi(u)\geqslant\ln u-Q$, $0\leqslant u<\infty$. It follows that $\ln|f(z)|\leqslant\chi(z)+Q=G_V\mu+h(z)+Q$ for $z\in R$. By Theorem 4, $f\in MB^*(V)$ and

$$\ln|f(z)|\leqslant h(z)+Q+\sum_{x\in E}[\mu(x)]\,g(z;x;V),\ \ z\in V. \tag{73}$$

Let $\{x_1,\dots,x_s\}\subseteq E$ be the set of atoms of μ with mass $\geqslant 1$. By (73), $f^{-1}(\infty)\cap V\subseteq\{x_1,\dots,x_s\}$. Suppose, for example, that x_1 is actually a pole of $f(z)$. WLOG $x_1=0$. Let $N_\varepsilon=\{|z|<\varepsilon\}$ for small ε. For all small ε and some constant A,

$$\Phi[|f(z)|]\leqslant A+\int_E\ln\frac{1}{|z-\xi|}\,d\mu(\xi),\quad z\in N_\varepsilon. \tag{74}$$

A simple calculation shows that

$$\int_{N_\varepsilon}\ln\frac{1}{|z-\xi|}\,d\omega\leqslant\pi\varepsilon^2\ln(1/\varepsilon)+\tfrac{1}{2}\pi\varepsilon^2$$

where $d\omega=dx\,dy$ and $\xi\in X$. By (74),

$$\int_{N_\varepsilon}\Phi[|f(z)|]\,d\omega\leqslant A\pi\varepsilon^2+\int_E\int_{N_\varepsilon}\ln\frac{1}{|z-\xi|}\,d\omega\,d\mu(\xi). \tag{75}$$

Choose any large N. Then, for all small ε,

$$N\int_{N_\varepsilon}\ln\frac{1}{|z|}\,d\omega\leqslant\int_{N_\varepsilon}\Phi[|f(z)|]\,d\omega. \tag{76}$$

A simple calculation now shows that (75) and (76) imply $N\leqslant\mu(E)$. This contradicts $\mu(E)\neq\infty$. Hence, $f\in A(V)$. It is now easy to check that $\Phi[|f|]\leqslant\chi$ for all $z\in V$ by the usual properties of superharmonic functions. Since $h(z)$ is the greatest harmonic minorant of $\chi\in SP(V)$ and $\Phi[|f(z)|]$ is subharmonic on V, we immediately deduce that $\Phi[|f(z)|]\leqslant h(z)$ for $z\in V$. This completes the proof. ∎

We can now state our theorem.

THEOREM 9. *Let the hypotheses of Theorem 7 hold. Suppose additionally that* $\eta=0$, $\chi\in HP(D)$, *and* $\Phi(u)/\ln u\to\infty$ *as* $u\to\infty$. *Then uniqueness holds if either of the following holds:*

(a) *no nonvoid* $B_j\cap\partial D$ *lies in* N_B-N_G;

(b) *D has at most countably many boundary components.*

Proof. (b) is of course a consequence of (a). In order to prove (a) we proceed in a manner entirely similar to the second part of the proof of Theorem 1 (c). We use of course the proof of Theorem 7 (a) in place of the proof of Theorem 1 (b) now. Thus, WLOG, $N \geqslant 2$ is minimal, $B_\infty \cap \partial D \notin N_B$, $\Delta = B_j \cap \partial D \in N_G$, and $V = D \cup \Delta \notin O_G$. Of course, $\chi \in SP(V)$. In the notation of (51), define

$$\mathcal{C}_1 = \{f \in A(V) \mid f(\xi_k) = 0, \text{ mult} \geqslant n_k, \Phi[\,|f(z)|\,] \leqslant h(z)\}.$$

By use of Theorem 8, $\mathcal{C}$ is precisely the restriction of $\mathcal{C}_1$ to D. Hence, extremal problems $\mathcal{L}$ over $\mathcal{C}$ and $\mathcal{L}$ over $\mathcal{C}_1$ are equivalent. Since N is minimal, $\mathcal{L}$ over $\mathcal{C}_1$ has uniqueness. Contradiction. This proves the theorem. ∎

Remark. We wish to point out in particular that Theorems 7, 8, and 9 hold for the Hardy classes H_p, $0 < p < \infty$, with $\Phi(u) = u^p$.

Before closing this section we point out that the construction used in the proof of Theorem 6 carries over trivially to the present case if one merely replaces $|\mathrm{Re}\,(\xi)|$ by $\Phi[e^{|\mathrm{Re}(\xi)|}]$. Hence, Theorems 7 (a) and 9 (a) are (in a certain sense) best possible.

XIII. Several remarks on boundary behavior of extremal functions

Before proceeding to get an indication of the situation on open Riemann surfaces, it is perhaps useful to briefly mention a few items about the boundary behavior of extremal functions for the fundamental extremal problem formulated still on plane domains.

Let us first of all consider again the first generalization of the classical Schwarz lemma considered in section I. We will suppose from the start that $\infty \notin D$ and that $D \notin O_{AB}$ so that $M \neq 0$ and the extremal problem is nontrivial. Also, recall the remark in section IV on this matter. Let $F(z)$ denote the unique extremal function.

For the case $D \in \mathcal{A}_p$, $1 \leqslant p < \infty$, the boundary behavior of $F(z)$ is by now classical (see L. Ahlfors [1, 2], P. Garabedian [7], and H. Grunsky [10]). In this case one can write

$$\ln |F(z)| + \sum_{j=0}^{p-1} g(z; \xi_j; D) = 0, \tag{77}$$

for $z \in D$, where $\xi_0 = \xi$ and $\{\xi_1, \ldots, \xi_{p-1}\}$ is the set of remaining zeros of $F(z)$.

It is natural to inquire whether a formula like (77) holds in the general case. The answer is in general negative. To see this, suppose for example that $D = U - E$ with $E \in N_B - N_G$. If (77) were valid, the fact that $AB(U) = AB(D)$ and that E contains regular points for the Dirichlet problem by Kellogg's theorem (Sario and Oikawa [29] p. 185) easily lead to a contradiction. Thus, informally speaking, the question of a representation like (77) is properly posed only when D is AB maximal, that is, D has no AB removable boundary points. At the present time, the answer to this question seems unknown as one learns upon a comparison with L. Rubel [25] pp. 17–18. (Here the question is whether or not F is an inner function.) We might mention however that S. Ja. Havinson [14] has derived a number of results on the boundary behavior of F by means of techniques much like those we have used in section VII.

The question now arises as to what boundary behavior holds in the case of the general linear extremal problem. We shall consider only the case in which $\chi \in HB(D)$. A glance at Theorem 2 and the proof of Theorem 1 suggests that $k \not\equiv S$ near $B_j \cap \partial D$ indicates good boundary behavior along $B_j \cap \partial D$. This is, as might be expected, not strictly correct as can be seen via a construction as in section XI. It appears, therefore, that the question as to when $k \not\equiv S$ near $B_j \cap \partial D$ indicates good boundary behavior is reasonably posed only when $B_j \cap \partial D$ has no AB removable points.

A deeper question is the following. Suppose for simplicity that $j = \infty$ is the only value of j for which $B_j \cap \partial D$ is nonvoid. Suppose too that D is AB maximal, as explained above. Assume that the fundamental extremal problem for $\mathfrak{L}$ over C is nontrivial. By Theorem 1 (b), let $F(z)$ be the unique extremal function. The question is when can we write

$$\ln |F(z)| + \sum_{a \in A} g(z; a; D) = \chi(z), \tag{78}$$

where A is the set of zeros (by multiplicity) of $F(z)$. The case $D \in \mathcal{A}_p$ is of course basically a consequence of Theorem 2.

At this point, we wish to mention that while we can develop results for the general linear extremal problem analogous to the results of S. Ja. Havinson [14] mentioned above, we are not prepared at present to enter into a more detailed discussion of boundary behavior and representation formulas such as (78).

XIV. Extremal problems on open Riemann surfaces

In the case of arbitrary plane domains, we have seen that general linear extremal problems can be handled reasonably well by means of a study of some natural dual extremal problems and of several appropriate removable singularity theorems. It is our purpose in this section to show that similar general linear extremal problems on arbitrary open Riemann surfaces seemingly cannot be handled by the above techniques.

Let us begin by noting that the assumptions (i)–(vii) in section III make good sense if D is replaced by an open Riemann surface W. We notice that assumption (viii) is in general not applicable. In any case, the fundamental extremal problem of section III (with trivial modifications) is now well-formulated over arbitrary open Riemann surface W. We can ask the same questions as in section III. Of course, the existence of an extremal function is again trivial. Although the condition $\ln |f| \leqslant \chi$ can be replaced by the more general boundary condition $\Phi[|f|] \leqslant \chi$ as in section XII, we shall use only the former.

Suppose first of all that W is the interior of a compact bordered surface $\overline{W}$. L. Ahlfors [2] and H. Royden [24] have studied the present linear extremal problem on such W at least for the case $\chi \equiv$ constant and $\mathcal{L}[f] = f(b)$ with $b \in W$. Their results are entirely similar to those obtained for plane domains.

Suppose now that W is assumed to be of finite genus $g \geqslant 1$. A theorem of S. Bochner [4] pp. 419–421 allows us to assume WLOG that W is an open subregion of a compact Riemann surface W_0 also of genus g. The obvious approach is to use W_0 in the same role that was played by the Riemann sphere S in the planar case. In words, it should now be possible to apply the classical theory of compact Riemann surfaces and the well-known theorem of Behnke-Stein about analytic functions with prescribed zeros (Pfluger [22] p. 199) to determine suitable substitutes for the Cauchy integral formula, so strongly exploited above, and to then generalize the planar development given above in a reasonable fashion. Here it should perhaps be mentioned that the techniques of L. Ahlfors [2] and H. Royden [24] suggest in this regard that a duality between functions and first-order differentials is the way to proceed. Of course, the generalization of Theorem 2 will be straightforward. At any rate, these developments are perhaps best left for another paper.

For this reason, we concentrate on the case of infinite genus. Here it seems that the main obstacle in any attempt to generalize the planar development is the lack of a properly behaved substitute for the Cauchy integral formula.

Example 1. Let $\{a_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of positive reals such that $a_1 = 1/2$, $\lim_{n\to\infty} a_n = 1$, and $\sum(1-a_n) = \infty$. Let $\{\delta_n\}_{n=1}^{\infty}$ be chosen so that $a_1 < a_1 + \delta_1 < a_2 < a_2 + \delta_2 < a_3 < \ldots$ Let $W_1 = U - \bigcup_{n=1}^{\infty} [a_n, a_n + \delta_n]$ and $W_2 = W_1 - \{|z| \leqslant 1/4\}$. Connect W_1 and W_2 cross-wise along each $[a_n, a_n + \delta_n]$ to get an open Riemann surface W of infinite genus. W has exactly two Kerekjarto-Stoilow boundary components. W is of course a Myrberg-type surface (Sario and Nakai [27] pp. 53–54). As usual, then, $AB(W) = AB(U)$. Let ξ be the point of W over the origin. Consider

$$\mathcal{C} = \{f \in AB(W) \mid |f| \leqslant 1 \text{ on } W, f(\xi) = 0\}.$$

Choose any $b \in W$ over the positive real axis. Let π denote the projection map $W \to U$. Linear functional $\mathcal{L}[f] = f(b)$ is virtually as simple as possible. The extremal problem $\mathcal{L}$ over $\mathcal{C}$ clearly has unique solution $F(z) = \pi(z)$. But, $|F| = 1/4 \neq 1$ along the inner component of the ideal boundary of W. We therefore have an example of bad boundary behavior in virtually the simplest possible setting. For perspective, compare this extremal problem to the corresponding extremal problem on regular subregions of W.

Example 2. Let $V = U \cap \{\mathrm{Re}(z) > -\varepsilon\}$ for some small $\varepsilon > 0$. Let I be a short line segment along the imaginary axis through the origin. Let $\{a_n\}_{n=1}^{\infty}$ and $\{\delta_n\}_{n=1}^{\infty}$ be as in example 1. Let $W_1 = V - \bigcup_{n=1}^{\infty} [a_n, a_n + \delta_n]$ and $W_2 = W_1 - I$. Form Riemann surface W of infinite genus by connecting W_1 and W_2 crosswise along the $[a_n, a_n + \delta_n]$. As usual, $AB(W) = AB(V)$.

By use of the Wiener compactification, say, of W (Sario and Nakai [27] chapter 4) we solve the Dirichlet problem on W for boundary value $\ln(100/\varepsilon)$ on the outer boundary component of W and 0 on the inner component. Let the solution be χ. Of course, $\chi \in HB(W)$ is essentially a harmonic measure. Let π: $W \to V$ be the usual projection map. Let $\xi \in W$ have $\pi(\xi) = 2\varepsilon$. Let $\eta \in W$ have $\pi(\eta) = 0$. Write $\mathcal{L}[f] = f(\eta)$ for $f \in AB(W)$. Define

$$\mathcal{C} = \{f \in AB(W) \mid \ln|f| \leqslant \chi \text{ on } W, f(\xi) = 0\}.$$

We now recall that $AB(W) = AB(V)$. At once, $f_1(z) = (\pi(z) - 2\varepsilon)/(\pi(z) + 2\varepsilon)$ and $f_2(z) = e^{\pi(z)} f_1(z)$ belong to $\mathcal{C}$. Also, $|\mathcal{L}[f]| \leqslant 1$ for all $f \in \mathcal{C}$. However, equality holds for $f_1(z)$ and $f_2(z)$. Thus, extremal problem $\mathcal{L}$ over $\mathcal{C}$ does not possess a unique solution. Here again the setting is quite simple. One might again compare the situation on regular subregions which approximate W.

Example 3. (Carleman-Milloux type extremal problem). We first consider a planar extremal problem. Define

$$\mathcal{C} = \{f \in AB(U) \mid |f(z)| \leqslant 1 \text{ on } U, |f(x)| \leqslant \varepsilon \text{ along } -1 < x \leqslant -c\} \tag{79}$$

where $0 < c < 1$ and $0 < \varepsilon < 1$ are fixed. Let $\mathcal{L}[f] = f(0)$ for $f \in \mathcal{C}$. A trivial normal family argument shows that there exists at least one extremal function for extremal problem $\mathcal{L}$ over $\mathcal{C}$. The question is to discuss the properties of the extremal functions F. This problem is motivated by the classical Carleman-Milloux extremal problem, as presented, for example, in R. Nevanlinna [21] pp. 102–113.

We point out that this type of extremal problem is much different than those we have treated thus far on plane domains. This is apparent when it is noticed that (79) is defined not merely by a restriction on the boundary behavior of the functions, but also by a so-called interior condition. A general treatment of such problems seems to be an open problem.

We now show that such a Carleman-Milloux type extremal problem has a very simple equivalent fomulation on a suitable open Riemann surface of infinite genus. To see this, let $\{a_n\}_{n=1}^{\infty}$ and $\{\delta_n\}_{n=1}^{\infty}$ be as in example 1. Let $W_1 = U - \bigcup_{n=1}^{\infty} [a_n, a_n + \delta_n]$ and $W_2 = W_1 - [-1, -c]$. Form open surface W by joining W_1 and W_2 crosswise along the segments $[a_n, a_n + \delta_n]$.

W will be of infinite genus and will have exactly one ideal boundary component, Let $\pi: W \to U$ as usual. Of course, $AB(U) = AB(W)$. Let χ be the solution of the Dirichlet problem on W for boundary value 0 over ∂U and $\ln \varepsilon$ over $[-1, -c]$. (Use, for example, the Wiener compactification as in example 2.) χ is essentially a harmonic measure. Let $\xi \in W$, $\pi(\xi) = 0$ and

$$\mathcal{C}_1 = \{f \in AB(W) \mid \ln |f| \leqslant \chi\}.$$

Set $\mathcal{L}_1[f] = f(\xi)$ for $f \in \mathcal{C}_1$. At once, extremal problems $\mathcal{L}$ over $\mathcal{C}$ and $\mathcal{L}_1$ over $\mathcal{C}_1$ are equivalent.

Hence, as is now apparent, any general treatment of linear extremal problems (in our original sense) over open Riemann surfaces, must be powerful enough to handle the Carleman–Milloux type extremal problem above. That the method of dual extremal problems even applies is not at all clear.

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