ANALYTIC CONTINUATION ACROSS A LINEAR BOUNDARY

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Introduction

To begin with we recall the following classical theorem concerning analytic continuation across a linear segment:

Let $Q = Q_{a,b}$ denote the rectangle $\{x + iy;\ |x| < a,\ |y| < b\}$ and let $Q^{\pm}$ be its intersection with the open upper and lower halfplane respectively. Two functions $f^{\pm}$ holomorphic in $Q^{\pm}$ are analytic continuations of each other across $(-a, a)$ if they have continuous and identical boundary values on $(-a, a)$.

Although the stated conditions are both necessary and sufficient the theorem is nevertheless inadequate in most nontrivial situations, the reason being that the two functions involved usually appear in a form which does not a priori imply either continuity or boundedness at any point on the common boundary. In most cases the a priori knowledge of $f^{\pm}$ consists of a growth limitation at $(-a, a)$ of the form

$$|f^{\pm}(x + iy)| \leqslant e^{h(|y|)}, \tag{1}$$

where $h(t)$ is a given function increasing steadily to ∞ as t tends to 0. The analytic continuation problem for functions satisfying (1) will be divided into two parts, referred to as the convergence problem which is closely related to a theorem by Runge, and the problem of mollification, to be treated in Chapter I and II respectively. The solutions of both are imperative for the formation of a general theory and both have solutions if and only if

$$\int_0^{\delta} \log h(y)\, dy < \infty. \tag{2}$$

If $h(t)$ increases sufficiently slowly to ∞, or more explicitly, if (1) is replaced by

$$|f^{\pm}(x + iy)| = O(|y|^{-k}), \tag{3}$$

valid for some $k > 0$, then the problem falls within the scope of Schwartz's distribution theory with this solution: $f^{\pm}$ have boundary values in the distribution sense if and only if (3) holds, and the functions are analytic continuations of each other if and only if the two distributions agree on intervals $[-c, c]$, $c < a$. The particular order of magnitude expressed in (3) is however not inherent in the problem but due to the basic use of derivatives and primitives in the distribution theory.

In the complex space C^m, $m > 1$, the domains of holomorphy add a new element to the continuation problem. In a third chapter we consider first a convexity notion of point sets in R^m, later to be applied when the two regions in C^m have a common flat boundary of dimension m. Theorem III, the main result of the chapter, is of a rather old date but seems to have survived the years in complete anonymity. It was proved by the author in 1958 after Gårding had made me aware of a related problem actual at the time in quantum field theory and called the edge of the wedge problem. The result was never published but it was presented at a seminar at The Institute for Advanced Study during the fall term 1958 and later at the Colloquium on Function Theory in Bombay 1960. A last chapter contains a brief review of an extended distribution theory [3] as compared with the methods used in this paper.

The problems considered in this paper have all been treated earlier by the author usually in a less complete form in articles on specific problems or in lectures and seminars. This more comprehensive exposition grew out of a series of lectures on the subject given the fall term 1970. Other references will be given in each chapter. Concerning the edge of the wedge problem the reader is referred to a recently published expository article by W. Rudin [8] and to the bibliography contained there.

We conclude this introduction by recalling some results on analytic functions which satisfy an inequality of the form

$$|f(x+iy)| \leqslant \text{const } e^{a|y|} \tag{4}$$

either in the upper or the lower halfplane or, if f is entire, in the whole complex plane. If f is regular for $y > 0$, then $\log|f(z)| - ay$ is subharmonic and bounded from above and consequently majorized by its Poisson integral. The familiar conclusion is that unless $f \equiv 0$, $\log|f(x)|$ has to be Poisson summable, by which we mean summable on $(-\infty, \infty)$ with respect to the measure $dx/(1+x^2)$. We note at this instance that (4) is satisfied by the Fourier transform of any measure with support in the interval $[-a, a]$.

We shall also resort to the following result. If $k(x)$ is an even, positive and Poisson summable function which increases steadily for $x > 0$, then for each given $a > 0$, there

exists a continuous function $\varphi(t)$ with support in $[-a, a]$, such that its Fourier transform has the properties: $\hat{\varphi}(0) = 1$ and

$$|\hat{\varphi}(x)| \leqslant \text{const } e^{-k(x)}, \quad -\infty < x < \infty, \tag{5}$$

where the constant only depends on k and a. When applying this result we may assume that $\varphi(t) \geqslant 0$, since the function $\lambda |\varphi(t)|^2$, due to the relation

$$\int e^{-k(x-\xi)-k(\xi)} d\xi < 2 e^{-k(x/2)} \int e^{-k(\xi)} d\xi,$$

will possess all the requested properties if λ and k are appropriately chosen.

I. On a theorem by Runge

Let Q, Q^+ and Q^- denote the rectangular regions considered in the introduction. If f^+ and f^- are analytic functions regular in Q^+ and Q^- respectively, then a classical theorem by Runge asserts the existence of a sequence f_n of functions holomorphic in Q and converging to f^+ in Q^+ and to f^- in Q^-. The f_n can obviously not remain bounded in a neighborhood of any point of the segment $(-a, a)$ unless f^+ and f^- are analytic continuations of each other across that point. The main problem of this chapter is to characterize the growth limitations at $(-a, a)$ which can be tolerated by the functions without ruining the approximating property in Runge's theorem. This question is most conveniently studied in the topology of a weighted supremum norm.

To this purpose let $w(y)$ be continuous in $[-b, b]$; $w(y) > 0$ for $y \neq 0$ and $w(0) = 0$. Assume moreover that $w(y)$ decreases steadily as y approaches 0 through positive and negative values, Let $C_w(Q)$ be the Banach space of complex valued functions f such that the product $w(y) f(z)$ is continuous in the closure of Q and vanishes on $[-a, a]$. The norm will be

$$\|f\|_w = \sup_{x+iy \in Q} w(y) |f(x+iy)|. \tag{6}$$

Define
$$A_w(Q) = \{f; f \in C_w(Q), f \text{ holomorphic in } Q\},$$

$$A_w(Q^\pm) = \{f; f \in C_w(Q), f \text{ holomorphic in } Q^+ \cup Q^-\}.$$

The conditions already imposed on w imply that $A_w(Q^\pm)$ is closed and thus a subspace of $C_w(Q)$. The same is not unconditionally true of $A_w(Q)$ and our primary objective is to determine the closure of that set.

THEOREM I. *Under the condition*

$$\int_0^\delta \log \log \frac{1}{w(y)} \, dy + \int_{-\delta}^0 = \infty \tag{7}$$

the closure of $A_w(Q)$ equals $A_w(Q^\pm)$ and so does the closure of polynomials in the metric (6). If (7) is finite, then $A_w(Q)$ is closed and consequently a proper subspace of $A_w(Q^\pm)$.

In the proof of this and some subsequent theorems we shall avail ourselves of some elementary but important properties and interrelations of positive monotonic functions on the real axis. Let $h(y)$ be a positive and monotonic decreasing function of $y > 0$ tending to ∞ at the origin. The lower Legendre envelope of h will be denoted $Lh = k(x)$ and defined by the relation

$$k(x) = \inf_{y > 0} (h(y) + xy), \quad x > 0. \tag{8}$$

As a lower envelope of linear functions, $k(x)$ is concave and it increases to ∞ with x. If $h(y)$ is only defined on a finite interval $(0, b]$ we extend its domain by setting $h(y) = h(b)$ for $y \geqslant b$. It should be noted that this modification does not influence the value of $k(x)$ for large x.

If $k(x)$ is a positive function for $x > 0$, tending to ∞ with x, but not necessarily monotonic increasing, we shall consider its upper Legendre envelope Uk defined as

$$Uk = \sup_{x > 0} (k(x) - xy), \quad y > 0. \tag{9}$$

This function is obviously convex and tends to ∞ at the origin. It should be noted that the upper envelope of $Lh = k(x)$ equals the largest convex minorant of the original function $h(y)$.

LEMMA I. *Let $h(y)$ be a decreasing positive function of $y > 0$, and let $h^*(y)$ be its largest convex minorant on $(0, \infty)$ and $k(x)$ its lower Legendre envelope. Then the integrals*

$$\int_0^\delta \log h(y)\, dy, \quad \int_0^\delta \log h^*(y)\, dy, \quad \int_1^\infty \frac{k(x)}{x^2}\, dx, \tag{10}$$

are simultaneously convergent or divergent.

The statement concerning the first two integrals is trivial. If h is continuous, which we may assume, then the set where $h(y) > h^*(y)$ is open and thus formed by disjoint open intervals. If $(a, a+t)$ is one of these intervals we shall have

$$\int_a^{a+t} \log h(y)\, dy \leqslant t \log h(a), \qquad \int_a^{a+t/2} \log h^*(y)\, dy > \frac{t}{2} \log \frac{h(a)}{2},$$

which proves our assertion.

Assume next that the first integral in (10) converges, and define $h_1(y) = h(y)/y$. Then

$$k(x) = \inf_{y > 0} y(h_1(y) + x) \leqslant 2\, \eta(x)\, x, \tag{11}$$

where $\eta(x)$ is the solution of the equation $h_1(\eta) = x$, if $h_1(\eta_1) = 1$, then

$$\int_1^\infty \frac{k(x)}{x^2}\,dx \leqslant 2\int_1^\infty \frac{\eta(x)}{x}\,dx = -2\int_0^{\eta_1} \eta\,d\log h_1(\eta) = 2\int_0^{\eta_1} \log h_1(\eta)\,d\eta < \infty.$$

Let now the third integral converge, and assume that the derivative $k'(x)$ is continuous and strictly decreasing. Define $\xi(y)$ by the relation $k'(\xi) = y$. Then

$$h^*(y) = \sup_{x>0}\,(k(x) - xy) < k(\xi(y)),$$

and

$$\int_0^\delta \log h^*(y)\,dy < \int_0^\delta \log k(\xi(y))\,dy = \int_{\xi_1}^\infty \log k(\xi)\,dk'(\xi) = -\delta\log k(\xi_1) + \int_1^\infty \frac{k'^2(\xi)}{k(\xi)}\,d\xi,$$

where $k'(\xi_1) = \delta$. Since k is concave the last integrand above is majorized by $k(\xi)/\xi^2$ and this completes the proof since the validity of the inequalities is not affected by the qualitative assumptions made on k'.

Proof of Theorem I. In order to prove the first part of the theorem it is sufficient to show that polynomials are dense in $A_w(Q^\pm)$. Each linear functional on $C_w(Q)$ has the form

$$\int f(z)\,w(y)\,d\mu(z),$$

where $d\mu$ is a Radon measure with support in the closure of Q. As a first step in the proof we shall show that if polynomials are orthogonal to the measure $w d\mu$, then the same is true for the restriction of that measure to the upper and to the lower halfplane. For arbitrary complex ζ we have

$$0 = \int e^{i\zeta z} w(y)\,d\mu(z) = \int_{y>0} + \int_{y<0} \equiv F^+(\zeta) + F^-(\zeta). \tag{12}$$

Both functions on the right are entire and of exponential type. $F^+(\xi)$ is bounded for real $\xi > 0$, and $F^-(\xi)$ for $\xi < 0$. Both are thus bounded on the whole real axis and must therefore satisfy the inequality (4). Without loss of generality we assume that the first integral in (7) diverges. Writing $h(y) \equiv -\log w(y)$, $y > 0$, we obtain for $\xi > 0$

$$|F^+(\xi)| \leqslant \int_{y>0} e^{-\xi y - h(y)}|d\mu(z)| \leqslant \|\mu\|e^{-k(\xi)}, \tag{13}$$

where $k(x)$ stands for the lower Legendre envelope of h. By Lemma I applied to $h(y)$ it follows that

$$\int_1^\infty \frac{k(\xi)}{\xi^2}\, d\xi = \infty.$$

Hence, $\log |F^+(\xi)|$ is not Poisson summable and $F^+ = F^- = 0$ follows, proving our statement.

In order to finish the proof we have to show that the measure $w\, d\mu$ annihilates all $f \in A_w(Q^\pm)$. To this purpose define $T_\varepsilon z = z_1 + (1-\varepsilon)(z-z_1)$, $0 < \varepsilon < 1$, where z_1 stands for the center of Q^+. By a simple estimate we find that

$$w(y)|f(T_\varepsilon z)| \leqslant \text{const}\, \|f\|_w, \quad z \in Q^+.$$

Thus, by dominated convergence,

$$\lim_{\varepsilon = +0} \int_{y>0} f(T_\varepsilon z)\, w(y)\, d\mu(z) = \int_{y>0} f(z)\, w(y)\, d\mu(z). \tag{14}$$

Since the function $f(T_\varepsilon z)$ is analytic in the closure of Q^+ it can be approached uniformly there by polynomials, proving that (14) vanishes. The analogous result holds for Q^- and polynomials are therefore dense in $A_w(Q^\pm)$, and the Runge approximation property holds under condition (7).

The second part of Theorem I together with the main result in the next chapter depends essentially on the solvability of a certain generalized Dirichlet problem relating to the function $m(y)$ defined as the logarithm of max $h(\pm y)$.

LEMMA II. *Let $m(y)$ be an even function, bounded from below, decreasing and summable in $(0, b]$. Then $m(y)$ has a majorant $m_1(y)$ with the same properties and such that the following holds: There exists a simply connected region D contained in Q, and containing each rectangle $P_\zeta = \{x + iy;\ |x| < |\xi|,\ |y| < |\eta|\}$ for $\zeta = \xi + i\eta \in \partial D$. Furthermore, the generalized Dirichlet problem for D with boundary values $= e^{m_1(\eta)}$ at $\zeta = \xi + i\eta \in \partial D$ has a solution u satisfying*

$$0 < u(x + iy) < 2\, e^{m_1(\eta)} + c, \tag{15}$$

in P_ζ for $\zeta \in \partial D$, c being a constant depending on Q and m.

We shall show first that the majorant can be taken equal to $m(y)$ itself if these conditions are satisfied: (i) $m(y) \in C^2(0, b]$; (ii) $-m'(y)y^3$ is decreasing in $(0, b]$; (iii) $-m'(y)y \geqslant \max(1/3, a/2b)$. We recognize that only the second condition is significant since the first always can be fulfilled by a smoothening, and the last by adding to m a term $c \log (b/y)$.

Instead of trying to estimate u for different regions D we shall construct a continuous

superharmonic function U such that the set $\{x+iy;\ U(x+iy)=e^{m(y)}\}$ forms the boundary of a region D with the prescribed properties. Then the Dirichlet problem will have a solution u which automatically will satisfy the same inequalities.

By (iii) there exists a positive number $\lambda \leqslant b$ such that

$$\int_0^\lambda -m'(y)\,y\,dy = \frac{a}{2}. \tag{16}$$

Let now $g(x)$ be the solution in $[0, a)$ of the differential equation

$$\frac{g(x)}{2} = \frac{d}{dx}\,m\left(\frac{1}{g(x)}\right), \quad g(0) = \frac{1}{\lambda}. \tag{17}$$

and consequently

$$\frac{x}{2} = \int_0^x m'\left(\frac{1}{g}\right)\frac{1}{g}\,d\left(\frac{1}{g}\right) = \int_{1/g(x)}^\lambda -m'(y)\,y\,dy.$$

The relation

$$\tfrac{1}{2} = -m'\left(\frac{1}{g}\right)\frac{g'}{g^3} \tag{18}$$

together with (ii) show that g' is positive and increasing with x. We also note that

$$\tfrac{1}{2}\int_0^x g(x)\,dx = m\left(\frac{1}{g(x)}\right) - m(\lambda), \quad x\in[0, a). \tag{19}$$

The definition of g is now extended to $(-a, a)$ by setting $g(-x)=g(x)$. Therefore $g(x)$ and the function

$$f(x) = \exp\left(\tfrac{1}{2}\int_0^x g(x)\,dx\right)$$

are even, convex and positive in $(-a, a)$ and tend to ∞ at $\pm a$. These properties are of course also shared by fg^2. As a step in the construction of U we set

$$U_0(x+iy) = f(x)\,(2 - y^2 g^2(x))$$

and define

$$D = \{x+iy;\ |x| < a,\ |y| < 1/g(x)\} \tag{20}$$

This region is contained in the rectangle $Q_{a,\lambda}$, and has the geometric properties prescribed in the lemma. At points $\zeta = \xi + i\eta \in \partial D$, we have $U_0(\zeta) = \exp(m(\eta) - m(\lambda))$, and in the inscribed rectangle P_ζ, $U_0(z) \leqslant 2U_0(\zeta)$. For $x > 0$,

$$\Delta U_0 = 2f'' - 2fg^2 - y^2(fg^2)'' \leqslant 2f'' - 2fg^2 = fg^2\left(\frac{g'}{g^2} - \frac{3}{2}\right), \tag{21}$$

where the parenthesis is $\leqslant 0$ due to (iii) and (18). Hence, U_0 is superharmonic in the

strips $0 < x < a$ and $-a < x < 0$ but not in D because $\partial U_0/\partial x$ has a jump on the imaginary axis equal to

$$2\,\frac{\partial U_0}{\partial x}\,(+0, y) = \frac{1}{\lambda}\left(2 - \frac{y^2}{\lambda^2} - 4y^2 g'(0)\right) < \frac{2}{\lambda},$$

and thus positive for small y. A superharmonic U satisfying all the conditions is however easily obtained by choosing

$$U(z) = e^{m(\lambda)}\left(U_0(z) + \frac{1}{\pi\lambda}\int_{-\lambda}^{\lambda} G(z, i\eta)\, d\eta\right), \tag{22}$$

where G is the Green function for D. The potential integral vanishes on ∂D and is bounded by some constant c_1 in D. The inequalities (15) are therefore verified by U with $c = c_1 \exp m(\lambda)$, and the same holds true for the harmonic function u.

It still remains to be proved that $m(y)$ has a summable majorant satisfying (ii), or equivalently, that $m(\xi^{-1/2}) = \psi(\xi)$ has a concave majorant on (c, ∞), $c = b^{-2}$, summable with respect to the measure $\xi^{-3/2}d\xi$. Due to the relation $\psi(\xi) = O(\xi^{-1/2})$ it follows that $\psi(\xi)$ has a least concave majorant $\psi^*(\xi)$ on (c, ∞). Let $\omega_n = (\xi_n, \xi_n')$ be the open disjoint intervals forming the set $\{\xi;\ \xi > c,\ \psi^*(\xi) > \psi(\xi)\}$ and set $\eta_n = \psi(\xi_n), \eta_n' = \psi(\xi_n')$. In the Cartesian plane (ξ, η) let $d\sigma$ denote the measure $\xi^{-3/2}dy$. To each interval ω_n we assign the strip $S_n = \{(\xi, \eta);\ \xi_n' < \xi < \infty,\ \eta_n < \eta < \eta_n'\}$, and the triangle Δ_n with vertices at (ξ_n, η_n), (ξ_n', η_n) and (ξ_n', η_n'). Since the S_n are disjoint and located between the graph of ψ and the ξ-axis, we have

$$\sum \sigma(S_n) \leqslant \int_c^\infty \psi(\xi)\, \xi^{-\frac{3}{2}}\, d\xi.$$

Because ψ is increasing and ψ^* linear in ω_n it follows that

$$\int_{\omega_n} (\psi^* - \psi)\, \xi^{-\frac{3}{2}}\, d\xi \leqslant \sigma(\Delta_n).$$

By computation,
$$\frac{\sigma(\Delta_n)}{\sigma(S_n)} = \frac{\sqrt{\xi_n'} - \sqrt{\xi_n}}{\sqrt{\xi_n'} + \sqrt{\xi_n}} < 1,$$

which proves the stated summability.

Let us now return to the proof of Theorem I and show that the set $A_w(Q)$ is closed if (7) is finite. Let $\{f_n\}_1^\infty \subset A_w(Q)$ be a Cauchy sequence and assume for simplicity that $\|f_n\| \leqslant 1$. Hence

$$\log |f_n(x + iy)| \leqslant e^{m_1(y)}, \quad x + iy \in Q, \tag{23}$$

and harmonic majoration yields

$$\log |f_n(x+iy)| \leqslant u(x+iy), \quad x+iy \in D, \tag{24}$$

proving that the f_n are uniformly bounded in Q outside neighborhoods of the points $\pm a$. The sequence therefore converges pointwise in D to a function which necessarily belongs to $A_w(Q)$.

II. Mollification of analytic functions

A scalar function f defined on an Abelian group G is most conveniently regularized by a convolution operator

$$\varphi * f(x) = \int_G \varphi(x-\xi) f(\xi) \, d\xi = \int_G \varphi(\xi) f(x-\xi) \, d\xi.$$

The function φ will be called a mollifier if it is continuous and $\geqslant 0$, has compact support and

$$\int_G \varphi(\xi) \, d\xi = 1.$$

The usefulness of a convolution derives mainly from the fact that it inherits those properties of its components which are invariant under the group operation. Another useful property is that the operator norm is 1 in all translation invariant metrics.

We shall be concerned only with the case that G equals a Euclidean space R^m of dimension $m \geqslant 1$. By $\varrho(\varphi)$ we shall denote the radius of the smallest ball centered at the origin and containing the support of φ. A sequence $\{\varphi_n\}_1^\infty$ will be referred to as a mollifier sequence if $\varrho(\varphi_n) \to 0$. The following definition will be used concerning families $F(\Omega)$ of locally summable functions on an open subset Ω of R^m.

Definition. $F(\Omega)$ is said to be *mollifiable* if there exists a mollifier sequence $\{\varphi_n\}_1^\infty$ with these properties: To each compact subset K of Ω can be assigned an integer $N(K)$ so that for each fixed $n \geqslant N(K)$ the set $\varphi_n * F(\Omega)$ consists of functions equicontinuous on K and bounded there by some constant c_n.

It should be noted that the requirement of equicontinuity is redundant in the sense that if $\{\varphi_n\}$ mollifies $F(\Omega)$ to boundedness on compacts for fixed n, then $\{\varphi_n * \varphi_n\}$ yields both boundedness and equicontinuity. Similarly, if the sets $F_\nu(\Omega_\nu)$, $\nu = 1, 2, ..., q$, are mollified by the sequences $\{\varphi_{n,\nu}\}$, $\nu = 1, 2, ..., q$, then they are simultaneously mollified by $\{\psi_n\}$ with $\psi_n = \varphi_{n,1} * \varphi_{n,2} * ... * \varphi_{n,q}$.

In the sequel the space C^m of m complex variables will be considered as the Cartesian product $R^m \times R^m$ where the two copies of R^m carry the real and the imaginary part of the vector $z = x + iy$. We shall be concerned with regions of the particular form

$$\Omega + i\Sigma = \{z = x + iy; \ x \in \Omega, \ y \in \Sigma\} \tag{25}$$

where Ω and Σ are connected open subsets of R^m, and Σ has $y = 0$ as boundary point. The set $\{z = x + iy; \ x \in \Omega, \ y = 0\}$ will represent the linear boundary where analytic continuation will take place. As earlier in the paper, $h(t)$ will stand for a decreasing function of $t > 0$ tending to infinity at $t = 0$. We shall write $A_h(\Omega + i\Sigma)$ for the set of functions analytic and single valued in $\Omega + i\Sigma$ and subjected there to the majoration

$$|f(x + iy)| \leqslant e^{h(t(y))}, \quad t(y) = \text{dist}\ (y, \partial\Sigma). \tag{26}$$

This set will be called mollifiable in x if the family

$$F(\Omega) = \{g(x); \ g(x) = f(x + iy), \ y \in \Sigma, \ f \in A_h(\Omega + i\Sigma)\}$$

can be mollified in accordance with the given definition. The main problem of this chapter is to decide in terms of h and Σ whether a set $A_h(\Omega + i\Sigma)$ is mollifiable or not.

The one-dimensional case

For functions of one complex variable it is sufficient to consider the set $A_h(Q^+)$ of functions analytic in Q^+ and satisfying (26) with $t(y) = y$. The following result is of basic importance for the general problem.

THEOREM II. *$A_h(Q^+)$ is mollifiable if and only if $h(t)$ satisfies condition (2).*

The proof is based on Lemma II together with a certain representation of f described in the following lemma.

LEMMA III. *If h satisfies the previous condition, then each $f \in A_h(Q^+)$ can be written*

$$f(z) = g_1(z)f_1(z) + g_2(z), \tag{27}$$

where f_1 is analytic off the segment $[-a, a]$, vanishes of second order at ∞, and satisfies

$$\int_{-\infty}^{\infty} |f_1(x + iy)|\, dx \leqslant e^{h_2(y)}, \quad y > 0, \tag{28}$$

with $$h_2(y) = h_1\left(\frac{y}{2}\right) + \log^+\frac{1}{y} + \text{const}, \tag{29}$$

h_1 being the majorant of h figuring in Lemma II. The functions g_1, g_2 are analytic in a region containing $(-a, a)$ and have fixed bounds there.

Let u be the function harmonic in the region D of Lemma II, $m = \log h$, and let v be its

conjugate normalized by the condition $v(0) = 0$. D is contained in a rectangle $\{x + iy;$ $|x| < a, |y| < \lambda\}$ for some $\lambda \leqslant b$. Denote by γ the part of ∂D located in the upper half-plane, and let γ_ε, $0 < \varepsilon < \lambda$, be the modification of γ obtained by replacing the portion above $y = \varepsilon$ by the linear segment joining the points $\pm \xi + i\varepsilon$. Define $F = u + iv$, and

$$c_1 = \frac{1}{2\pi i} \int_\gamma e^{-F(\zeta)} f(\zeta)\, d\zeta. \tag{30}$$

Since the integrand is bounded by 1 and the length of γ is $< 2a + 2\lambda$ we have $|c_1| < (a + \lambda)/\pi$. In (30) as in the following formulas the integrations are made in the direction of increasing ξ. Define

$$f_1(z) = \frac{1}{2\pi i} \int_{\gamma_\varepsilon} (e^{-F(\zeta)} f(\zeta) - c_1)\, \frac{d\zeta}{\zeta - z}, \tag{31}$$

where ε is chosen so small that z lies outside the region limited by γ_ε and $(-a, a)$. For $x + iy \in D$, $y > 0$, we obtain by the Cauchy representation

$$e^{-F(z)} f(z) - c_1 = \frac{1}{2\pi i} \int_{\gamma_\varepsilon} - \frac{1}{2\pi i} \int_\gamma \equiv f_1(z) + f_2(z), \tag{32}$$

where f_2 is analytic off γ. A decomposition of the form (27) is thus obtained with $g_1 = \exp(F)$, $g_2 = (f_2 + c_1) \exp(F)$, and both functions are analytic in D. Let $\delta(z)$ denote the distance from z to $[-a, a]$. If $\delta(z) \geqslant 2b$ or $y < 0$ we choose γ as integration path in (31) and obtain

$$|f_1(z)| = \frac{\text{const}}{(1 + |z|)\, \delta(z)}.$$

If $\delta(z) < 2b$ and $y > 0$ the choice $\varepsilon = y/2$ yields

$$|f_1(z)| \leqslant \frac{\text{const}}{y}\, e^{h_1(y/2)},$$

and (28) follows with a constant in (29) uniformly bounded if a is bounded.

The integral

$$\Phi_1(\xi) = \int f_1(z)\, e^{-i\xi z}\, dz$$

extended over any line $\mathrm{Im}(z) = y > 0$ represents a function vanishing for $\xi < 0$ and independent of y. Consequently,

$$|\Phi_1(\xi)| \leqslant \inf_{y > 0} e^{h_2(y) + y\xi} = e^{k_2(\xi)}, \quad \xi > 0, \tag{33}$$

where k_2 stands for the lower Legendre envelope of h_2.

In order to prove that $A_h(Q^+)$ is mollifiable it is sufficient to show that for any $a' < a$ there exists a mollifier φ with $\varrho(\varphi) < a - a'$ such that $\varphi * f(x+iy)$ remains uniformly bounded throughout A_h and for $|x| \leqslant a'$, $0 < y \leqslant b$. To this purpose let $\alpha > 0$ be so small that the closed rectangle $P = \{x+iy; \ |x| \leqslant a' + \alpha, \ |y| \leqslant 2\alpha\}$ is contained in D. Since g_2 is bounded in P, and f is bounded for $|x| \leqslant a'$, $\alpha < y \leqslant b$, it suffices to prove that

$$\int \varphi(x-t) g_1(t+iy) f_1(t+iy) \, dt \tag{34}$$

is bounded for $|x| \leqslant a'$, $0 < y < \alpha$. We are going to apply the Parseval relation to (34) and denote by $\psi(\xi, x, y)$ the Fourier transform in the variable t of $\varphi(x-t) g_1(t+iy)$. Since $\Phi_1(\xi) \exp(-y\xi)$ is the transform of $f_1(t+iy)$, we find that (34) equals

$$\frac{1}{2\pi} \int_0^\infty \psi(-\xi, x, y) \, \Phi_1(\xi) \, e^{-\xi y} \, d\xi. \tag{35}$$

To conclude the proof we need this lemma.

LEMMA IV. *Let γ be a closed interval, α a positive number and $g(t)$ a function analytic and bounded by a constant M in the region $\{t; \ \mathrm{dist}(t, \gamma) < \alpha\}$. Let φ be a continuous function with support in an interval $\{t; \ |t - t_0| \leqslant \theta\alpha\}$ with $t_0 \in \gamma$, $\theta < 1$, and let the Fourier transform of φ satisfy*

$$|\hat{\varphi}(\xi)| \leqslant e^{-2K(\xi)}, \tag{36}$$

where $K(\xi)$ is concave for $\xi > 0$, even and Poisson summable. Then there exists a constant $C(K, \theta, \alpha)$ such that the Fourier transform of $\varphi(t) g(t)$ is majorized by

$$C(K, \theta, \alpha) \, M \, e^{-K(\xi)}. \tag{37}$$

Consider first the case $t_0 = 0$. Then,

$$\psi(t) \equiv \varphi(t) \, g(t) = \sum_0^\infty c_n t^n \varphi(t), \quad |c_n| \leqslant M \alpha^{-n}, \tag{38}$$

and consequently
$$\hat{\psi}(\xi) = \sum_0^\infty c_n (i)^n D^n \hat{\varphi}(\xi). \tag{39}$$

If $K(\xi, \eta)$ is the Poisson integral of K, then harmonic majoration applied to both half-planes yields

$$\log |\hat{\varphi}(\xi + i\eta)| \leqslant \beta |\eta| - 2K(\xi, |\eta|) \tag{40}$$

with $\beta = \theta\alpha$. By obvious reasons, $2K(\xi, \eta) \geqslant K(\xi, 0) = K(\xi)$. The minimum of $2K(\xi, \eta)$ on

the disk $\{\zeta;\ |\zeta-\xi|\leqslant r\}$ is $\geqslant K(0)$ if $r\geqslant\xi>0$, and $\geqslant K(\xi-r)$ if $r\leqslant\xi$. Assume $\xi>0$ and de-note by $m(r,\xi)$ the maximum of $|\hat{\varphi}(\zeta)|$ on the same disk. Hence, for $r\leqslant\xi$,

$$|D^n\hat{\varphi}(\xi)|\leqslant n!\, m(r,\xi)\, r^{-n}\leqslant n!\, r^{-n}\exp\,[\beta r-K(\xi-r)].\tag{41}$$

The choice $r=n/\beta$ together with Stirling's formula gives the result,

$$|D^n\hat{\varphi}(\xi)|<3\sqrt{n+1}\,\beta^n\exp\,[-K(\xi-n/\beta)],\tag{42}$$

where $K(\xi-n/\beta)$ has to be replaced by $K(0)$ if $n\geqslant\beta\xi$. Writing in the first series below, $\theta^n=\theta^{n/2}\exp\,(-\delta n)$, $\delta=(\log\theta)/2$, we get

$$|\hat{\varphi}(\xi)|\,e^{K(\xi)}<3M\big\{\sum_{n\leqslant\beta\xi}\sqrt{n+1}\,\theta^{n/2}\exp\,[K(\xi)-K(\xi-n/\beta)-\delta n]$$
$$+\exp\,[K(\xi)-K(0)]\sum_{n>\beta\xi}\sqrt{n+1}\,\theta^n\big\}.\tag{43}$$

The exponent $K(\xi)-K(\xi-n/\beta)-\delta n$, considered as a function of the continuous variable n, is convex in the interval $[0,\beta\xi]$ and its maximum $A(\xi)$ there is consequently assumed at one of the endpoints. Hence, $A(\xi)$ is the largest of the numbers 0 and $K(\xi)-K(0)-\delta\beta\xi$. Since $K(\xi)=o(\xi)$ it follows that the second quantity above is $\leqslant 0$ if $\xi\geqslant\xi_0$, and $A(\xi)$ is thus bounded for $\xi>0$ by a constant depending only on K, θ and α. By symmetry the same bound holds for $\xi<0$. The first series in (43) is therefore majorized by a constant with the prescribed properties, and the same is true of the second series by a similar argument. The previous proof is valid for $0\neq t_0\in\gamma$, and the lemma is thus established.

The function $g(t)=g_1(t+iy)$ satisfies the conditions of Lemma IV for $|y|\leqslant\alpha$, with $\gamma=[-a',a']$, M being the maximum of $|g_1|$ in the rectangle P, and α the number figuring in the definition of P. Let the mollifier φ satisfy the conditions (36) and $\varrho(\varphi)\leqslant\theta\alpha<\alpha$. By Lemma IV and (33) it now follows that (35) is bounded under the given restriction on $x,\,y$, if

$$\int e^{-K(\xi)+k_2(\xi)}d\xi$$

is finite, which can be realized by an appropriate choice of K. This finishes the proof of the sufficiency of the condition in Theorem II.

The necessity of the condition will be proved by construction of an example showing that $A_h(Q^+)$ contains an f which cannot be mollified if h violates the summability condition. Let again $k(t)$ be the lower Legendre envelope of $h(y)$, and define

$$f(z)=\int_1^\infty e^{itz+k(t)}\,\frac{dt}{t^2}.\tag{44}$$

Whether or not k is Poisson summable, we always have $k(t) = o(t)$, and f is thus analytic in the upper halfplane and satisfies

$$|f(x+iy)| \leqslant \sup_{t \geqslant 1} e^{-ty+k(t)} \leqslant e^{h(y)}, \quad y > 0.$$

If f were mollifiable in x on any interval $(-a, a)$ there would exist a mollifier φ such that $\varphi * f(x+iy)$ remained bounded for $y > 0$ and for $|x|$ sufficiently small, and the same would hold if φ is replaced by

$$\psi(x) = \int \varphi(x + \xi) \, \varphi(\xi) \, d\xi.$$

Since $\hat{\psi}(t) = |\hat{\varphi}(t)|^2$ we would have

$$\psi * f(0+iy) = \int_1^\infty e^{-ty+k(t)} |\hat{\varphi}(t)|^2 \frac{dt}{t^2}. \tag{45}$$

If h violates (2), Lemma I asserts that k is not Poisson summable. Together with (45) and the inequality between the arithmetic and geometric means this leads to the contradiction

$$\int_{-\infty}^\infty \log |\hat{\varphi}(t)| \frac{dt}{1+t^2} = -\infty,$$

finishing the proof of Theorem II.

The previous results make it clear how, and how far, the classic continuation theorem can be extended in the case of one complex variable:

COROLLARY OF THEOREMS I AND II. *Two functions $f^\pm$, analytic in $Q^\pm$ respectively, are analytic continuations of each other across $(-a, a)$ if and only if there exists a sequence $\{\varphi_n\}$ mollifying $f^\pm$ on $(-a, a)$ and such that the mollified functions $f^\pm$ agree on intervals $[-a', a']$ for $a' < a$ and satisfy an inequality*

$$|f_n^\pm(x+iy)| \leqslant e^{h(|y|, a')}$$

where $h(y, a')$ is decreasing in y and (2) is verified.

The multidimensional case

We shall begin this section by considering some geometric notions and later show how Theorem II by means of finite compositions of one-dimensional convolutions can be applied to the multidimensional mollification problem.

If y is a point $\in R^m$ and r a positive number, $B(y, r)$ will denote the open ball of radius r centered at y. We shall be concerned with a certain kind of truncated circular cones $C(y_1, y_2, r)$, $r < |y_1 - y_2|$, defined as the union of all linear open segments (y_1, y) for $y \in B(y_2, r)$. The linear segment joining y_1, y_2 will be called the axis of the cone, and the angle $\theta = 2 \arcsin r/|y_1 - y_2|$ its opening.

Definition. A connected open subset Γ of R^m is said to have the *interior cone property* if there exist positive constants τ, θ, and a finite set of unit vectors $\{u_\nu\}_1^q$ such that the following holds: For each $y \in \Gamma$ there exists at least one truncated circular cone contained in Γ and with opening $\geqslant \theta$ and otherwise such that its axis contains y, has length $|y_1 - y_2| \geqslant \tau$ and is parallel with one of the vectors $\{u_\nu\}_1^q$.

As an example let Γ be an open convex cone. The interior cone property holds with $q = 1$ if the sole vector u_1 is chosen in Γ. The conditions are met with $\theta/2 = $ minimum angle between u_1 and the generators of Γ, and with an arbitrary $\tau > 0$. Each open convex region Γ has the interior cone property and explicit bounds for q, τ and θ can be obtained in terms of the dimension m and ϱ_1, ϱ_2, if Γ is bounded and $\varrho_1 < \varrho_2$ are numbers such that for some y_0, $B(y_0, \varrho_1) \subset \Gamma \subset B(y_0, \varrho_2)$. The previous definition is thus satisfied by finite unions of convex regions, but in general not by infinite unions, nor by any region with a boundary containing a cusp, and hence not unconditionally by starshaped regions.

COROLLARY OF THEOREM II. *The set $A_h(\Omega + i\Gamma)$ is mollifiable if h satisfies* (2) *and Γ has the interior cone property.*

It is sufficient to show the existence of a probability measure $d\mu$ with support in an arbitrarily small ball centered at the origin such that

$$\int f(x + iy - \xi)\, d\mu(\xi) \tag{46}$$

is uniformly bounded in $K + i\Gamma$, K being a given compact $\subset \Omega$. Assume dist $(K, \partial\Omega) > \delta$, and let $\{u_i\}_1^q$ be the unit vectors in the previous definition related to Γ. The integral

$$\int f\left(x + iy - \sum_1^q \lambda_i u_i\right) \prod_1^q \varphi(\lambda_i)\, d\lambda_i \tag{47}$$

is obviously of the form (46), and if the support of φ is contained in $[-\varepsilon, \varepsilon]$ then the support of $d\mu$ lies in the closure of $B(0, q\varepsilon)$. Let ν be the index of the particular vector u_i corresponding to the truncated cone with axis passing the point $y \in \Gamma$. We shall pay special attention to the result of the integration with respect to $d\lambda_\nu$ in (47), assumed to be made first. Write $\lambda_\nu = \xi$ and $x + \Sigma_{i \neq \nu}\, \lambda_i u_i = x_0$. Then,

$$f(x+iy-\sum_1^q \lambda_i u_i) = f(x_0+iy-\xi u_\nu) = F(\xi). \tag{48}$$

Assume $\varepsilon < \delta/q$, where q, as later τ and θ, refer to the parameters of the interior cone property of Γ. $F(\xi+i\eta)$ is analytic in a rectangle $P = \{\xi+i\eta; \ -\varepsilon < \xi < \varepsilon, \ \eta_1 < \eta < \eta_2\}$ with $\eta_2-\eta_1 \geqslant \tau$ and satisfies there the inequality

$$\log |F(\xi+i\eta)| \leqslant h(c(\eta-\eta_1)), \ \ c = \sin \theta/2,$$

Since ε, τ, and c are fixed, there exists by Theorem II a mollifier φ such that

$$\left| \int \varphi(\xi) F(\xi) \, d\xi \right| \leqslant c(\varepsilon), \tag{49}$$

where the bound $c(\varepsilon)$ is finite for $\varepsilon > 0$, in general tending to ∞ as $\varepsilon \to 0$. Since the integrations with respect to the other variables λ_i only can decrease the value of (49), the corollary is established.

The following remark concerning the existence of unique boundary values will be useful. Assume $A_h(\Omega+i\Gamma)$ mollifiable and let K be a compact $\subset \Omega$. Then there exists a φ such that for $g \in \varphi * A_h$,

$$|g(x+iy)| + |\mathrm{grad}_x \, g(x+iy)| \leqslant c, \ \ x+iy \in K+i\Gamma.$$

By the Cauchy-Riemann equations, $|\mathrm{grad}_x \, g| = |\mathrm{grad}_y \, g|$. Consequently, $|g(x+iy) - g(x+iy')| \leqslant c[y, y', \Gamma]$, where the bracket stands for the inner distance in Γ defined as the lower bound of the length of an arc joining y, y' within Γ. Call a boundary point y_0 simple if for each sequence $\{y_i\}_1^\infty \subset \Gamma$ the convergence $y_i \to y_0$ implies

$$\lim_{i,j=\infty} [y_i, y_j, \Gamma] = 0.$$

Then $g(x+iy)$ has a unique and continuous limit on K as y tends to a simple boundary point.

Remark. It seems natural to ask whether or not a more general result could emerge by weakening the conditions on h. If h is lower semicontinuous and $\log h \in L^1(0, \delta)$, then there still exists a region D such that the Dirichlet problem in Lemma II has a solution. The boundary ∂D does not however need to be rectifiable any longer. As a consequence the second part of Theorem I remains valid under the new conditions, whereas the truth of Theorem II remains in doubt due to the use of the Cauchy integral.

It should be noted that the majoration connected with Lemma II, and the problems of mollification and analytic continuation in one complex variable was treated in [2] and [3] by means of Fourier analysis.

III. Analytic continuation in C^m, $m \geqslant 2$

The continuation problem in C^m contains a variety of situations among which we shall only consider one of the main cases, the symmetric case, characterized by the symmetry $D^\pm = \Omega \pm i\Sigma$ of the two regions involved. Here and forthwith Ω and Σ are always open connected sets, $\Sigma \cap (-\Sigma) = \varnothing$ and Σ has $y = 0$ as boundary point. If $f^\pm$ are holomorphic in $D^\pm$ and ω is an open subset of Ω, analytic continuation of $f^\pm$ into each other "across" ω means something different than in the one dimensional case, the set $D^+ \cup D^- \cup \{x + iy;\ x \in \omega,\ y = 0\}$ being no longer an open connected subset of C^m. In the present situation $f^\pm$ are by definition analytic continuations of each other across $\omega \subset \Omega$ if there exists an open set Δ in C^m and a function f holomorphic in Δ and such that $f^\pm$ agree with f in the sets $D^\pm \cap \Delta$ respectively. This problem has an interesting property lacking in the one dimensional case and due to the new character of holomorphic convexity. If namely $f^\pm$, or their mollified functions, agree on a set ω, then this set has in general an extension $\tilde\omega$ depending on ω, Ω and Σ such that $f^\pm$ are analytic extensions across $\tilde\omega$. The study of $\tilde\omega$ is the main object of this chapter.

We now introduce some convexity notions associated to an open convex cone Γ. If a, b is an ordered pair of points and Γ an open convex cone in R^m, we define

$$\Gamma(a, b) = (a + \Gamma) \cap (b - \Gamma). \tag{50}$$

If $b - a \in \Gamma$, then $\Gamma(a, b)$ is an open convex set symmetric with respect to the point $\frac{1}{2}(a + b)$ and containing the open segment (a, b). If $b - a \notin \Gamma$, then $\Gamma(a, b) = \varnothing$.

Definition. (i) An open set ω is Γ-*convex* if $\Gamma(a, b) \subset \omega$ whenever $[a, b] \subset \omega$; (ii) the Γ-*convex hull* $\tilde\omega$ *of an open set* ω is the least open Γ-convex set containing it.

The convexity notion defined above will only be applied to open sets. We note the following consequences of the definition. Γ-convexity is closed under finite intersections of open sets, and the same is true of finite unions provided the sets are mutually disjoint. The components of a disconnected Γ-convex set are Γ-convex.

The Γ-convex hull $\tilde\omega$ of an arbitrary open set ω equals the interior of the intersection of all open Γ-convex sets containing ω. In view of a later application we shall also need this constructive definition of $\tilde\omega$. Let $H\omega$ denote the union of all sets $\Gamma(a, b)$ for $[a, b] \subset \omega$. Set $H^{n+1}\omega = H(H^n\omega)$, $n \geqslant 1$, and define

$$\omega^* = \bigcup_1^\infty H^n \omega.$$

This set is by construction open, Γ-convex and it contains ω. Thus $\tilde\omega \subset \omega^*$. Since, however, each Γ-convex set containing ω also contains $H^n \omega$, $n \geqslant 1$, it follows that $\tilde\omega \supset \omega^*$ and the two sets are identical.

We recognize that in definition (i) the implication $\Gamma(a, b) \subset \omega$ remains true under the following weaker assumptions on a, b. Assume $a, b \subset \omega$, $b - a \in \Gamma$, and let there exist sequences $\{a_n\}_1^\infty$, $\{b_n\}_1^\infty$, converging to a and b respectively and such that $[a_n, b_n] \subset \omega$, $b_n - a_n \in \Gamma$. Then $\bigcup_1^\infty \Gamma(a_n, b_n)$ will contain the open segment (a, b) implying $[a, b] \subset \omega$.

As a consequence we find that $\Gamma(a, b) \subset \omega$ if a, b are endpoints of a Jordan arc γ contained in ω and such that $x - a \in \Gamma$ for $x \in \gamma$, $x \neq a$. We have $[a, x] \subset \omega$ for $x \in \gamma$ sufficiently close to a. If therefore $[a, b]$ were not contained in ω there would exist a point $b' \in \gamma$, $b' \neq a, b$, such that $[a, b'] \not\subset \omega$ but $[a, x] \subset \omega$ for x belonging to the open arc limited by a and b'. The previous remark leads to the contradiction $[a, b'] \subset \omega$ proving our statement.

Fig. 1 shows the typical shape a connected Γ-convex set in the plane when Γ equals the first quadrant.
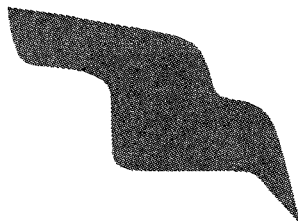

Fig. 1.

By a mollification $\varphi \times f$ the domain of f shrinks. It should therefore be pointed out that if $\omega_\varepsilon = \{x; x \in \omega, \text{dist}(x, \partial\omega) > \varepsilon\}$ we shall always have Γ-hull $\omega = \bigcup_{\varepsilon>0} \Gamma$-hull ω_ε. Because if x belongs to the hull, then there exists an index n such that $x \in H^n \omega$, which together with the relation $H^n \omega = \bigcup_{\varepsilon>0} H^n \omega_\varepsilon$ confirms our statement.

Continuation in the special symmetric case

The heading of this section refers to the case that $D^\pm = \Omega \pm i\Gamma_r$, where Γ is an open convex cone and Γ_r, $0 < r \leqslant \infty$, is the intersection of Γ and the open ball $B(0, r)$. In the following theorem we assume $f^\pm \in A_h$ $(\Omega \pm i\Gamma_r)$ where h satisfies (2), and therefore $f^\pm$ are mollifiable.

THEOREM III. *Let there exist a mollifier sequence $\{\varphi_n\}$ such that $f_n^\pm = \varphi_n \times f^\pm$ agree on compact subsets of an arbitrarily thin open set ω containing a segment (a', b') where $b' - a' \in \Gamma$. If*

$$\omega \subset \Gamma(a', b') \subset \Omega,$$

then $f^{\pm}$ are analytic continuations across $\tilde{\omega} = \Gamma(a', b')$ and this property is not shared by any other open set containing $\tilde{\omega}$.

If $f_n^{\pm}$ agree on compact sets contained in an arbitrary open set $\omega \subset \Omega$, then $f^{\pm}$ are analytic continuations across $\tilde{\omega} = \Gamma$-hull ω, provided $\tilde{\omega} \subset \Omega$.

We begin by proving the optimal property of $\tilde{\omega} = \Gamma(a', b')$ stated in the first part of the theorem, its proof being free of technical elements and conducive to conveying insight in the problem. Let $g(t)$ be a continuous and summable function on R, vanishing for $t \geqslant 0$ and >0 for $t < 0$ and such that

$$g(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{g(t)}{t-z} \, dt,$$

is bounded with continuous boundary values on the real axis for $y \to \pm 0$. Let $\{\xi_n\}$ be a sequence of unit vectors $\in R^m$ such that the halfspaces $H_n = \{x; \langle x, \xi_n \rangle > 0\}$ contain Γ and meet $\partial\Gamma$ along generators. Assume that $\{\xi_n\}$ is so dense that no point $x \notin \overline{\Gamma}$ is contained in the intersection of the H_n. The series

$$F(z) = \sum_n 2^{-n} \{g(\langle z - a', \xi_n \rangle) + g(\langle b' - z, \xi_n \rangle)\}$$

represents a function holomorphic in an open connected set containing

$$(R^m + i\Gamma) \cup (R^m - i\Gamma) \cup \{x + iy; \, x \in \Gamma(a', b'), \, y = 0\}.$$

We also have

$$\lim_{\Gamma \ni y \to 0} (F(x + iy) - F(x - iy)) = \sum_n 2^{-n} \{g(\langle x - a', \xi_n \rangle) + g(\langle b' - x, \xi_n \rangle)\}$$

where the series is $= 0$ for $x \in \Gamma(a', b')$ and >0 for $x \notin \overline{\Gamma}(a', b')$. This proves that under the stipulated conditions analytic continuation in the real space R^m cannot be extended beyond $\tilde{\omega} = \Gamma(a', b')$.

The main statement in the theorem will first be proved under the condition that any two generators of Γ form an angle $\leqslant \theta < \pi$. If $[a, b]$ is a given segment carried by (a', b') we choose δ so small that the set $V = \{x; \, \text{dist}\, (x, [a, b]) < \delta\}$ is contained in ω. If n is sufficiently large $f_n^{\pm}$ will agree on V. We shall prove that this implies the $f_n^{\pm}$ agree on $\Gamma(a, b)$ and are analytic continuations across that set. In other words, we shall show the existence of a function f_n holomorphic in an open set $\Delta \subset C^m$ containing $\Gamma(a, b)$ and such that $f_n^{\pm} = f_n$ on the set $\Delta \cap (\Gamma(a, b) \pm i\Gamma)$ respectively. The proof is based on a certain analytic function ψ of one variable now to be defined.

Let ε be a positive parameter and set

$$g(\xi) = \sqrt{\frac{2}{\pi\varepsilon}} \int_0^\xi \exp\left(-\frac{\xi^2}{2\varepsilon}\right) d\xi,$$

$$\varrho = \int_0^1 g(\xi)\, d\xi$$

$$\psi(\xi) = \varrho - \int_0^\xi g(\xi)\, d\xi.$$

When ε tends to 0, $g(\xi)$ converges uniformly to sign ξ for real ξ, $|\xi| \geqslant \delta > 0$, and $\varrho < 1$ tends to 1. We shall consider $\psi = u + iv$ in a rhombus P_α with vertices at ± 1, $\pm i\alpha$. On the real axis $u(\xi)$ is concave, and in $[-1, 1]$ we have $|\partial u/\partial\xi| \leqslant g(1) < 1$, $u(\pm 1) = 0$, $u(0) = \varrho$. Consequently,

$$\varrho \leqslant \frac{u(\xi)}{1 - |\xi|} \leqslant g(1) < 1, \quad \xi \in [-1, 1]. \tag{51}$$

By the Cauchy–Riemann equations we have for fixed ε,

$$v(\xi, \eta) = \eta\, \frac{\partial u(\xi)}{\partial \xi} + O(\eta^3), \quad u(\xi, \eta) = u(\xi) + O(\eta^2).$$

We can therefore choose α so small that the inequalities

$$\left|\frac{v(\xi, \eta)}{\eta}\right| \leqslant 1, \quad \left|\frac{u(\xi, \eta) - u(\xi)}{\eta}\right| \leqslant 1, \tag{52}$$

are satisfied in P_α. On combining (51) and (52) we obtain for $\zeta \in P_\alpha$,

$$u(\zeta) = (1 - |\xi|)(1 + \delta(\zeta)), \quad |\delta(\zeta)| \leqslant 1 - \varrho + \alpha, \tag{53}$$

$$v(\zeta) = \lambda(\zeta)\eta, \quad |\lambda(\zeta)| \leqslant 1. \tag{54}$$

We notice that on ∂P_α, $|u(\zeta)/\eta|$ is bounded by $1 + 1/\alpha$.

Assume for the sake of simplicity that $a = -b$ so that $\Gamma(a, b)$ is centered at the origin. Consider the function

$$F(z, \zeta) = f_n(\zeta b + \psi(\zeta)(x + iy)) \tag{55}$$

for $\operatorname{Re} z \in \Gamma(-b, b)$, $\zeta \in P_\alpha$. Here $f_n(z)$ is defined as $f_n^+(z)$ or $f_n^-(z)$ according to whether the imaginary part of the vector belongs to Γ or to $-\Gamma$. Write

$$X(\zeta) = \xi b + u(\zeta)x - v(\zeta)y,$$

$$Y(\zeta) = \eta b + v(\zeta)x + u(\zeta)y.$$

In the sequel $\partial^{\pm} P_{\alpha}$ will denote the intersection of ∂P_{α} with the upper and the lower half plane, and the notation $[x_1, x_2, ..., x_{n+1}]$ will stand for the polygonal path consisting of the segments $[x_{\nu}, x_{\nu+1}]$, $1 \leqslant \nu \leqslant n$. If $y = 0$ the mapping $\zeta \to X(\zeta)$ takes $[-1, 1]$ to a curve $\subset R^m$ confined to the 2-dimensional polygon $[-b, x, b, \varrho x, -b]$, and it is thus contained in V if $x \in V$. Whether $y = 0$ or $\neq 0$ it follows by (53), (54) that the same map takes P_{α} to a set with real part contained in $\Gamma(-b, b)$ provided

$$\begin{cases} x \in \Gamma(-b, b), \\ (1 - \varrho + \alpha)|x| + \alpha|y| < \text{dist } (x, \partial\Gamma(-b, b)) = r_1. \end{cases} \tag{56}$$

Assuming again $y = 0$ and writing $Y(\zeta) = \eta(b + v(\zeta)/\eta) = \eta Y'(\zeta)$, $s = r/\alpha$, we find that the condition $b^{\pm} x \in \Gamma_s$ implies $Y'(\zeta) \in \Gamma_s$ for $\zeta \in P_{\alpha}$ and consequently $Y(\zeta) \in \Gamma_r$ for $\zeta \in P_{\alpha}^+$ and $Y(\zeta) \in -\Gamma_r$ for $\zeta \in P_{\alpha}^-$. The set $(b - \Gamma_s) \cap (-b + \Gamma_s)$ is however equal to $\Gamma(-b, b)$ if α is so small that $2|b|\alpha < r \sin \theta$, which condition keeps the spherical boundary of Γ_r out of the picture. If, therefore, ε and α are chosen so small that all previous conditions are satisfied, and if $x \in V$, then the function $F(x, \zeta)$ is analytic in ζ for $\zeta \in P_{\alpha}^+ \cup P_{\alpha}^-$, continuous on the common boundary $[-1, 1]$ and hence holomorphic in P_{α} according to the classical theorem. By the Cauchy integral representation

$$F(x, 0) = f_n(\varrho x) = \frac{1}{2\pi i} \int_{\partial P} F(x, \zeta) \frac{d\zeta}{\zeta}. \tag{57}$$

Let us now return to the case $y \neq 0$. It has already been shown that $X(\zeta) \in \Gamma(-b, b)$ for $\zeta \in \partial P_{\alpha}$ if (56) is verified. Writing $Y(\zeta) = \eta(b + xv(\zeta)/\eta + yu(\zeta)/\eta)$ we find by (52) and (54) that $Y(\zeta) \in \Gamma$ for $\zeta \in \partial^+ P_{\alpha}$ and $Y(\zeta) \in -\Gamma$ for $\zeta \in \partial^- P_{\alpha}$ if

$$[b + x, b - x] \subset \Gamma, \quad (1 + 1/\alpha)|y| < \text{dist } ([b - x, b + x], \partial\Gamma). \tag{58}$$

It should be noted that the distance from a point on the segment $[b + x, b - x]$ to $\partial\Gamma$ is minimum at one of the endpoints of the segment. This implies that the right hand side of (58) equals the distance r_1 in (56).

Summing up, we have established that to each $\varepsilon > 0$, $\varepsilon < \varepsilon_1$ can be associated an $\alpha = \alpha(\varepsilon) > 0$ and a region

$$\Delta_{\varepsilon} = \{x + iy; \ x \in \Gamma(-b, b), \ (1 - \varrho + \alpha)|x| + (1 + 1/\alpha)|y| < \text{dist } (x, \partial\Gamma(-b, b))\} \tag{59}$$

such that $F(z, \zeta)$ is holomorphic in z for $z \in \Delta_{\varepsilon}$ provided $\zeta \in \partial P_{\alpha}$, $\zeta \neq \pm 1$.

Define now

$$J(z) = \frac{1}{2\pi i} \int_{\partial P_{\alpha}} F(z, \zeta) \frac{d\zeta}{\zeta}, \tag{60}$$

and let J_δ denote the integral when two arcs of ∂P_α of length δ centered at ± 1 are deleted from the path of integration. $J_\delta(z)$ is manifestly holomorphic in Δ_ε and the same must be true of $J(z)$ due to uniform convergence as $\delta \downarrow 0$. By virtue of (57), $J(x) = f_n(\varrho x)$ for x belonging to a certain open set $\subset R^m$ containing the segment (a, b). This implies that for each ε, $J(z)$ represents an analytic extension of $f_n^\pm(\varrho z)$ to Δ_ε. Consequently $f_n^\pm(z)$ possesses an analytic extension to the open set $\Delta = \bigcup_{0 < \varepsilon < \varepsilon_1} \varrho(\varepsilon)^{-1} \Delta_\varepsilon$, which contains $\Gamma(a, b)$ since $\varrho(\varepsilon) \uparrow 1$ as $\varepsilon \downarrow 0$. Because the summability condition (2) is satisfied by $h(t)$, Lemma I applies to functions of the form $F_n(\zeta) = f_n(x_0 + x\zeta)$, and we find as in the one-dimensional case, that $f_n(z)$ converges in Δ to a function holomorphic there and equal to $f^\pm$ in $\Delta \cap (\Gamma(a, b) \pm i\Gamma)$.

If $\partial\Gamma$ contains a whole straight line the sets $\Gamma(a, b)$ are no longer bounded. In this case we apply the previous proof to interior cones Γ' with maximal opening angle $< \pi$ and obtain the stated result by letting Γ' grow out to Γ. This finishes the proof of the first part of Theorem III since a and b can be taken arbitrarily close to a' and b' respectively.

The second part of the theorem is merely a corollary of the result already obtained combined with the definition and properties of the Γ-convex hull of an open set. The proof would consist of repetitions and is therefore deleted.

Theorem III does not take account of the case when the Γ-hull of ω is not contained in Ω. This case requires the definition of the Ω restricted Γ-hull of an open set $\omega \subset \Omega$. We replace the operator H defined previously, by H_Ω defined as the union of all $\Gamma(a, b)$ contained in Ω and such that $[a, b] \subset \omega$. Except for this modification the previous definition is unchanged. The result now is of course that $f^\pm$ are analytic continuations across the Ω restricted Γ-hull of ω.

The general symmetrical case

In its most advanced form the result usually referred to as "the edge of the wedge theorem" states that if $f^\pm$ are analytic in $\omega \pm i\Gamma_r$ and have boundary values in the sense of Schwartz's distributions which agree on compact subsets of ω, then $f^\pm$ are analytic continuations across ω.

Little or no attention used to be paid to the fact that the distributions involved only exist provided inequalities $|f^\pm(x + iy)| = O(|y|^{-k})$ are verified on compact subsets of ω. Another aspect of the problem which seems to have been overlooked concerns analytic extension in the real space R^m beyond the set where $f^\pm$ agree in some sense or another. There exists, however, a result due to H. Epstein [6], [8], about analytic extension in the imaginary space R^m, which combined with Theorem III will yield a more complete treatment of the symmetrical case. We formulate Epstein's theorem as follows:

Let Γ be an open cone consisting of finitely many connected components. Let Γ_r be the intersection of Γ with a ball $B(0, r)$ and denote by $\tilde{\Gamma}_r$ the ordinary convex hull of Γ_r. Assume that $f(z)$ is locally analytic in $\omega + i\Gamma_r$ and continuous on $\omega \cup (\omega + i\Gamma_r)$. Then there exists an open set $\Delta \subset C^m$ containing ω and such that $f(z)$ extends to a function locally analytic in $D = \Delta \cap (\omega + i\tilde{\Gamma}_r)$ and continuous on $D \cup \omega$.

We shall apply this result to the case $f^{\pm} \in A_h(\Omega \pm i\Sigma)$ where Σ is an open connected set with the origin as boundary point and such that for small $r > 0$ the set $\Sigma_r = \{x; |x| < r, (0, x) \subset \Sigma\}$ is not void and not containing a point x together with $-x$. Let Γ^r denote the cone $\{\lambda x; x \in \Sigma_r, \lambda > 0\}$. Γ^r is obviously increasing for decreasing r and possesses thus a limit Γ as $r \downarrow 0$. If Γ is disconnected, its different components define distinct boundary points which happen to have the same coordinates and should therefore be considered separately. By this reason we assmue that Γ^r is connected for r sufficiently small, say for $r < r_0$. We approximate Γ_r from within by finite unions γ of convex cones. According to the results in Chapter II the sets $A_h(\Omega \pm i\gamma_r)$ are mollifiable. Assume that for n sufficiently large the mollified functions f_n^+ agree on an open set $\omega \subset \Omega_0$ where Ω_0 has compact closure contained in Ω. By applying Epstein's theorem to f_n^+ and f_n^- separately, we conclude that these functions are analytic in $\Omega_0 \pm i\tilde{\gamma}_r$ respectively, where $\tilde{\gamma}_r$ is the convex hull of γ_r. By virtue of Theorem III $f_n^{\pm}$ agree on the $\tilde{\gamma}_r$-hull of ω and are analytic continuations across that set, provided it is contained in Ω_0. At this instance we should note that because the regions $\Delta \cap \Omega_0 \pm i\tilde{\gamma}_r$ are contained in the holomorphic hulls of $\Omega_0 \pm i\gamma_r$, the functions $f_n^{\pm}$ are bounded in the former regions by the same constants as in the latter. We finally obtain the result that $f^{\pm}$ are analytic continuations across the $\tilde{\Gamma}$-hull of ω by letting γ increase to Γ^r and then r decrease to 0.

Remarks. In case the reader might not have noticed it, we point out here some of the questions left unanswered in this chapter. Even though the optimal role of the sets $\Gamma(a, b)$ has been made clear, it does not follow automatically that the hull $\tilde{w}$ in the second part of Theorem II is the optimal set in the continuation problem. Nor is it obvious that the Ω restricted Γ-hull of ω is optimal if Ω is not convex in the ordinary sense.

To these remarks we add some specific applications of the results obtained. Let $w(\xi)$ be a positive measurable function in R^m, and let L_w^2 be the Hilbert space of functions square summable with respect to the measure $w(\xi)d\xi$, and with scalar product

$$(f, g) = \int f(\xi)\overline{g(\xi)}\, w(\xi)\, d\xi.$$

Let γ be an open symmetric cone $\subset R^m$, i.e. an open set such that $x \in \gamma$ implies $\lambda x \in \gamma$ for real $\lambda \neq 0$. Assume that L_w^2 contain the function

$$g(\xi) = \begin{cases} e^{r|\xi|}, & \xi \notin \gamma \\ 1, & \xi \in \gamma \end{cases} \tag{61}$$

for some $r > 0$. Let $\{h\}$ be the set of hyperplanes which meet $\bar{\gamma}$ only at the origin, and denote by h^+ and h^- the two open halfspaces separated by h. For the purpose of identification let $x_0 \subset \gamma$ be a fixed reference point and h^+ the halfspace containing x_0. Call two h_1, $h_2 \in \{h\}$ equivalent if $h_1^+ \cap \gamma = h_2^+ \cap \gamma$. Let finally $\{h_i\} \subset \{h\}$ be a set of mutually inequivalent hyperplanes and let $\Gamma(h_i)$ be the open convex cones

$$\Gamma(h_i) = \{x; x \in R^m, \langle \xi, x \rangle > 0, \forall \xi \subset \gamma \cap h_i^+\}.$$

The condition concerning the function (61) implies that all bounded continuous characters $e^{i\langle \xi, x \rangle}$ belong to L_w^2. We are interested in the closed subspace $S(\omega)$ of L_w^2 spanned by a collection of characters $E(\omega) = \{e^{i\langle \xi, x \rangle}; x \in \omega)\}$ where ω is a given open set. An application of Theorem III yields this result:

$$E(\tilde{\omega}) \subset S(\omega) \tag{62}$$

where $\tilde{\omega}$ is an extension of ω which can be defined as the closure of $\bigcup_1^\infty \omega_n$, where $\omega_1 = \bigcup_i \Gamma(h_i)$-hull of ω, $\omega_{n+1} = \bigcup_i \Gamma(h_i)$-hull of ω_n.

In order to establish (62), assume that $g \in L_w^2$ is orthogonal to $E(\omega)$. We shall therefore have,

$$0 = \int e^{i\langle \xi, x \rangle} \overline{g(\xi)} w(\xi) d\xi = \int_{h_i^+} + \int_{h_i^-} \equiv f_i^+ + f_i^-, \tag{63}$$

for $x \in \omega$ and for any h_i. The functions $f_i^\pm$ are obviously analytic and bounded in the regions $R^m \pm i\Gamma_r(h_i)$ and continuous on R^m. Theorem III ascertains that $f_i^+ + f_i^-$ vanishes on the $\Gamma(h_i)$-hull of ω, and iterated use of this result leads to the stated property.

If ω consists of a neighborhood of a convenient Jordan curve, for example a straight line with direction belonging to some $\Gamma(h_i)$, then $\tilde{\omega}$ would contain the whole space and the trigonometric polynomials

$$\sum c_\nu e^{i\langle \xi, x_\nu \rangle}, \quad x_\nu \in w,$$

would be dense in L_w^2.

We shall also show a case where the double cone $(x_0 + \Gamma) \cup (x_0 - \Gamma)$ plays a role similar to $\Gamma(a, b)$ in Theorem III. In [3] we considered properties of bounded continuous functions

on an open interval $\Omega \subset R$ in relation to functions analytic in a region D contained either in the upper or the lower halfplane and with boundary containing Ω. The approximation index $a(\lambda) = a(\lambda, f, D)$ was defined by the relation

$$e^{-a(\lambda)} = \inf_g \|f - g\| \tag{64}$$

when g is analytic in D, bounded there by e^λ and continuous on Ω. It was proved that if

$$\int_1^\infty \frac{a(\lambda)}{\lambda^2} \, d\lambda = \infty \tag{65}$$

then f vanishes identically on Ω if $f = 0$ on a set of positive measure. By the method used in [3] the result is easily extended as follows: To each $a(\lambda)$ satisfying (65) can be associated a function $h(t)$ increasing steadily to ∞ as $t \downarrow 0$ and such that if f has a sufficiently "strong" zero at a point x_0, in the specific sense that

$$|f(x_0 + x)| = O(e^{-ch(|x|)}), \quad c > 0, \tag{66}$$

then $f = 0$ on Ω.

The approximation index is a very useful and flexible tool. In the one-dimensional case it is largely independent of D, insofar as only the two possible orientations of D are relevant, namely if D is located in the upper or in the lower halfplane. In several dimensions the problems promise to be much more interesting since now D can be chosen as $\Omega + i\Gamma_r$ for any convex cone Γ, and with a more general D if that is desirable. We quote the following direct application of the one-dimensional result: If f is continuous on Ω and has an approximation index $a(\lambda)$ with respect to a region $\Omega + i\Gamma_r$, then f vanishes on the double cone $(x_0 + \Gamma) \cup (x_0 - \Gamma)$ if f has a "strong" zero at x_0 and (65) is satisfied. This in turn implies, of course, that $f = 0$ throughout Ω, if this set is connected.

IV. Distributions

It should be obvious at this stage that the problems considered in Chapters I and II must be related to an appropriate extension of Schwartz's distributions. Such an extension was actually presented at the American Mathematical Society Summer School in Stanford in 1961 [3]. It was based on certain convolution algebras dating back to 1938 [1]. The object of this chapter is to reproduce the pertinent definitions and properties of that theory and to relate it to the main problem of this paper.

Let $W(R^m)$ denote the collection of measurable subadditive functions $w(\xi)$ on R^m bounded in a neighborhood of the origin and satisfying

$$0 = w(0) \leqslant w(\xi + \eta) \leqslant w(\xi) + w(\eta), \quad \xi, \eta \in R^m.$$

To each w we assign a Banach space $A_w = A_w(R^m)$ consisting of functions $\varphi(x)$ admitting a representation

$$\varphi(x) = \int e^{i\langle x, \xi \rangle} \hat{\varphi}(\xi) \, d\xi,$$

where $\hat{\varphi}(\xi) \exp w(\xi)$ is summable. The norm in A_w is

$$\|\varphi\| = \|\varphi\|_w = \int |\hat{\varphi}(\xi)| \, e^{w(\xi)} \, d\xi.$$

A_w is an algebra under pointwise multiplication: $\|\varphi\psi\| \leqslant \|\varphi\| \, \|\psi\|$. By replacing w by λw, $\lambda > 0$, we obtain a family of algebras $A_{\lambda w}$ with norms $\|\varphi\|_{\lambda w} = \|\varphi\|_\lambda$. Define

$$\mathscr{A}_w = \mathscr{A}_w(R^m) = \{\varphi; \, \varphi \in \bigcap_{\lambda > 0} A_{\lambda w}, \, \text{supp } \varphi \text{ is compact}\}$$

If K is a compact subset of R^m, we define

$$\mathscr{A}_w(K) = \{\varphi; \, \varphi \in A_w, \, \text{supp } \varphi \subset K\}$$

Expressed in the usual terminology $\mathscr{A}_w(K)$ consists of testfunctions with support in K. The topology of the space $\mathscr{A}_w(K)$ is determined by the norms $\|\varphi\|_\lambda$ for $\lambda = 1, 2, \ldots$, considered as seminorms, and the topology of $\mathscr{A}_w(R^n)$ is defined as the inductive limit of the topologies of $\mathscr{A}_w(K_n)$, where K_n can be taken as a sequence of closed balls $B(0, r_n)$ with radii $r_n \uparrow \infty$.

Distributions of class w, denoted $\mathscr{A}_w'(R^m)$, is by definition the dual of $\mathscr{A}_w(R^m)$. In the particular case $w(\xi) = \log(1 + |\xi|)$ the spaces $\mathscr{D}$ and $\mathscr{D}'$ of Schwartz coincide with $\mathscr{A}_w$ and $\mathscr{A}_w'$ respectively. These notions would, of course, be void if $\mathscr{A}_w(K_n)$ only contained the identically vanishing function. This question is resolved by

THEOREM IV. *Let $w \in W(R^m)$, $m \geqslant 1$, and assume*

$$J_m(w) = \int_{|\xi| \geqslant 1} w(\xi) \, \frac{d\xi}{|\xi|^{m+1}} = \infty. \tag{67}$$

Then the integral $$p(\xi) = \int_1^\infty w(r\xi) \, \frac{dr}{r^2} \tag{68}$$

is $= \infty$ on an open halfspace of R^m and $\mathscr{A}_w(K)$ is empty for each compact K.

If $J_m(w) < \infty$, then there exists a concave function $k(r)$ on $r \geqslant 0$ such that

$$w(\xi) \leqslant k(|\xi|), \quad \int_1^\infty k(r) \, \frac{dr}{r^2} < \infty. \tag{69}$$

and the sets $\mathscr{A}_w(K)$ contain nontrivial functions whenever K has interior points.

In order to prove the first part of the theorem consider R^m as the Cartesian product $R^+ \times S^{m-1}$, and write $d\xi = r^{m-1} dr d\theta$, where $r = |\xi|$ and $d\theta$ is a measure on the unit sphere S^{m-1} in R^m. For the integral (67) we obtain

$$J_m(w) = \int_{S^{m-1}} p(\xi) d\theta(\xi).$$

If $m(\xi)$ is the least upper bound of w on the segment $[0, \xi]$, we shall have

$$p(\lambda \xi) \leqslant \lambda p(\xi), \quad \lambda \geqslant 1$$

$$p(\lambda \xi) \leqslant \lambda p(\xi) + (1 - \lambda) m(\xi), \quad 0 \leqslant \lambda \leqslant 1. \tag{70}$$

Since w is subadditive, the assumptions $p(\xi) < \infty$, $\xi \neq 0$, imply $p(\lambda \xi) < \infty$. Together with the inequality $p(\xi + \eta) \leqslant p(\xi) + p(\eta)$ this implies that the set $\Gamma = \{\xi; \xi \in R^m, p(\xi) < \infty\}$, is formed by a convex cone, and hence either contained in a closed halfspace or equal to the whole of R^m. Under the latter alternative $p(\xi)$ is finite on S^{m-1} which, due to the subadditivity, implies boundedness there, contradictory to the assumption (67). Therefore Γ is contained in a closed halfspace, on the complement of which $p(\xi) = \infty$.

Let now φ belong to an algebra $A_{\lambda w}$, say for $\lambda = 1$, and have compact support. If

$$q(\xi) = \int_0^\infty |\hat{\varphi}(r\xi)| e^{w(r\xi)} r^{m-1} dr, \quad \xi \in S^{m-1}$$

then

$$\int_{S^{m-1}} q(\xi) d\theta(\xi) = \|\varphi\|_w < \infty,$$

and we would have a.e. on S^{m-1},

$$p(\xi) + p(-\xi) = \infty, \quad q(\xi) + q(-\xi) < \infty. \tag{71}$$

Because of the inequality between the arithmetic and geometric means, the function $\log |\hat{\varphi}(r\xi)|$ cannot be Poisson summable in r for any ξ satisfying (71). As a function of r, $\hat{\varphi}(r\xi)$ is the restriction to the real axis of an entire function of exponential type bounded on R, and therefore vanishing identically in r for a.e. $\xi \in S^{m-1}$. This implies $\hat{\varphi} = 0$, which in turn proves that $\varphi = 0$.

The proof of the second part of Theorem IV rests on the following result which can be considered as a converse of Lemma I.

LEMMA V. *If $k(\xi) \in W(R)$ and is Poisson summable, then it possesses on R^+ a concave majorant which is Poisson summable.*

It should be noted that the corresponding statement is wrong for the set of monotonic increasing $k(\xi)$.

Since the maximum of $k(\pm\xi)$ still belongs to $W(R)$ we may assume that $k(\xi)$ is even, and consequently that $|k(\xi+\eta)-k(\xi)| \leqslant k(\eta)$, $\xi, \eta \in R$. Without loss of generality we shall also assume k continuous. It is readily seen that the function k_1 defined as the maximum of k on $[0, \xi]$, is still subadditive. We shall prove first that $k_1(\xi)\xi^{-2} \in L^1(1, \infty)$, and then that the least concave majorant $k_2(\xi)$ of $k_1(\xi)$ has the same summability.

Let (a, b) be one of the intervals forming the open set where $k_1(\xi) > k(\xi)$, and denote by γ_t the interval $(a, a+t)$, $0 \leqslant t \leqslant b-a$. Define

$$E_t = \{\xi; \; \xi \in \gamma_t, \; k(\xi) < k(a)/3\}$$

$$E_t' = \{\xi; \; \xi \in \gamma_t, \; k(\xi) \geqslant k(a)/3\}$$

$$E_t'' = \{\xi; \; \xi = a+\xi'-\xi'' \in \gamma_t, \; \xi', \xi'' \in E_t\}$$

We shall have on E_t'', $k(\xi) \geqslant k(a) - k(\xi') - k(\xi'') \geqslant k(a)/3$, implying $E_t'' \subset E_t'$. As for the measures of these sets we have: $|E_t''| \geqslant |E_t|$, $t = |E_t| + |E_t'| \leqslant |E_t''| + |E_t'| \leqslant 2|E_t'|$. It is readily seen that this implies

$$\int_a^b k(\xi) \frac{d\xi}{\xi^2} \geqslant \frac{k(a)}{6} \int_a^b \frac{d\xi}{\xi^2} = \frac{1}{6} \int_a^b k_1(\xi) \frac{d\xi}{\xi^2}.$$

Hence, k_1 is Poisson summable.

We have still to show that $k_2(\xi)$ has the proper summability. Assume that on (a, b), $k_2 - k_1$ is positive while vanishing at the endpoints, and define

$$\lambda = \frac{k_2(b) - k_2(a)}{b - a}.$$

Then,

$$k_2(\xi) \leqslant k_2(b) + \lambda(\xi - b), \; \xi \geqslant 0,$$

and by subadditivity of k_1,

$$k_2(b) = k_1(b) \leqslant k_1(b-\xi) + k_1(\xi). \cdot$$

Hence,

$$k_1(\xi) \geqslant k_2(b) - k_2(b-\xi) \geqslant \lambda\xi, \; 0 \leqslant \xi \leqslant b.$$

Since $k_1(\xi)$ is monotonic increasing we shall have

$$k_1(\xi) \geqslant \max(k_1(a), \lambda\xi), \; a \leqslant \xi \leqslant b.$$

We shall show that these inequalities imply that $k_2(\xi) \leqslant 2k_1(\xi)$ in (a, b). In order to

simplify we consider the normalized case: $a = 1$, $k_1(a) = 1$ and consequently $0 \leqslant \lambda \leqslant 1$. Hence for $\xi \in (a, b)$

$$\frac{k_2(\xi)}{k_1(\xi)} \leqslant \frac{1 + \lambda(\xi - 1)}{\max (1, \lambda \xi)} \leqslant 2 - \lambda \leqslant 2,$$

which finishes the proof of the Lemma V.

Returning to Theorem IV we first form the maximum $k_0(r)$ of w at the $2m$ points where the sphere $|\zeta| = r$ meets the coordinate axes. Since the restriction of w to each axis belongs to $W(R)$ it follows that the same is true of the even function $k_0(r)$, which therefore by Lemma V has a concave majorant $k(r)$ on R^+ with the proper summability. At a point ξ with the coordinates $\{\xi_\nu\}^m$ we shall have

$$w(\xi) \leqslant \sum_1^m k_0(|\xi_\nu|) \leqslant mk(|\xi|),$$

ending the proof.

Due to results previously obtained in this paper, the condition $J_m(w) < \infty$ thus ascertains the existence in $\mathscr{A}_w$ of mollifiers with support in balls $B(0, r)$ of arbitrarily small radius, which in turn leads to the existence of local units and a partition of the unity.

Let us now spell out explicitly what is meant by a statement like this: "The function $f(x + iy)$, analytic in $Q = \{x + iy; |x| < a, 0 < y < b\}$, has boundary values in the distribution sense on $(-a, a)$ as $y \downarrow 0$." The meaning is this: To each compact $K \subset (-a, a)$ can be associated a $w \in W(R)$ with $J_1(w) < \infty$ such that the distributions

$$T_y(\varphi) = \int f(x + iy)\, \varphi(x)\, dx \in \mathscr{A}_w' \tag{72}$$

converge pointwise for $\varphi \in \mathscr{A}_w(K)$ as $y \downarrow 0$, and uniformly in bounded set $\{\varphi; \varphi \in \mathscr{A}_w(K), \|\varphi\|_\lambda \leqslant a_\lambda, \lambda = 1, 2, ...\}$ where $\{a_\lambda\}_1^\infty$ are given positive numbers.

There is nothing wrong with the definition as such, but one can hardly omit to ask for a criterion by means of which it is possible to decide whether or not the distributions in question do converge in the prescribed sense.

In the theory of Schwartz the same question is hardly conspicuous since its answer is contained in the definition of the testfunctions. In the case at hand, T_y considered as elements of $\mathscr{D}'$, converges if and only if f has a primitive of finite order which is bounded on K for each $K \subset (a, b)$, and this result can be obtained by a finite number of partial integrations in (72). The testfunctions do therefore in this case effectively provide a test of convergence, whereas testfunctions $\varphi \in \mathscr{A}_w(K)$ do not give that service. This unsatisfactory situation is one of the reasons for this paper, and particularly for the study undertaken in Chapters I and II.

According to what is established about sets $A_h(\Omega + i\Sigma)$ of analytic functions, it follows that all $f \in A_h(\Omega + i\Sigma)$ have boundary values on Ω in the distribution sense if and only if A_h is a mollifiable set. The situation is not the same for individual functions. It is obvious that the existence of a boundary distribution implies that $f(x+iy)$ is mollifiable in x for $x \in \Omega$, but the converse is not true. Nor is it true that a function violating the stated growth condition cannot be mollifiable. Here is a simple counterexample. A Taylor series

$$f(z) = \sum_{\nu=1}^{\infty} c_{n_\nu} z^{n_\nu},$$

with $\sum_1^\infty 1/n_\nu < \infty$, can have a radius of convergence $= 1$ and a maximum modulus $M(r)$ larger than any given increasing function, for values of r arbitrarily close to 1, but still be mollifiable. The reason for this is the existence of mollifiers $\{\varphi_i\}$ with $\varrho(\varphi_i) \to 0$ and such that $\hat{\varphi}_i(n_\nu) = 0$ for all n_ν sufficiently large. Hence,

$$\varphi_i * f = \sum c_{n_\nu} \hat{\varphi}_i(n_\nu) z^{n_\nu}$$

is a polynomial and therefore f mollifiable. The proof is based on quite elementary properties of Weierstrass products.

The results obtained so far on the three main problems, convergence, mollification, and distributions dovetail perfectly in that they all reveal an inherent optimal order of magnitude, the same in all three cases, indicating the close relationship between analytic functions and harmonic analysis on Euclidean groups.

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