FOURIER INTEGRAL OPERATORS. I

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Preface

Pseudo-differential operators have been developed as a tool for the study of elliptic differential equations. Suitably extended versions are also applicable to hypoelliptic equations, but their value is rather limited in genuinely non-elliptic problems. In this paper we shall therefore discuss some more general classes of operators which are adapted to such applications. For these operators we shall develop a calculus which is almost as smooth as that of pseudo-differential operators. It also seems that one gains some more insight into the theory of pseudo-differential operators by considering them from the point of view of the wider classes of operators to be discussed here so we shall take the opportunity to include a short exposition.

Pseudo-differential operators as well as our Fourier integral operators are intended to make it possible to handle differential operators with variable coefficients roughly as one would handle differential operators with constant coefficients using the Fourier transformation. For example, the inhomogeneous Laplace equation

$$\Delta u = f \in C_0^\infty(\mathbf{R}^n)$$

is for $n > 2$ solved by

$$u(x) = -(2\pi)^{-n} \int e^{i\langle x, \xi\rangle} |\xi|^{-2} \hat{f}(\xi)\, d\xi,$$

where

$$\hat{f}(\xi) = \int e^{-i\langle x, \xi\rangle} f(x)\, dx$$

is the Fourier transform of f. To be able to solve arbitrary elliptic equations with variable coefficients one is led to consider more general operators of the form

$$Af(x) = (2\pi)^{-n} \int e^{i\langle x, \xi\rangle} a(x, \xi) \hat{f}(\xi)\, d\xi, \tag{0.1}$$

where a behaves as a sum of homogeneous functions when $\xi \to \infty$. These are the (classical) pseudo-differential operators. On the other hand, suppose that we want to solve the Cauchy problem

$$\Delta u - \partial^2 u/\partial t^2 = 0;\ u = 0,\quad \partial u/\partial t = f \in C_0^\infty(\mathbf{R}^n) \text{ when } t = 0.$$

Then the solution is given by

$$u(x,t) = (2\pi)^{-n} \int e^{i(\langle x,\xi\rangle + t|\xi|)} (2i|\xi|)^{-1} \hat{f}(\xi)\, d\xi - (2\pi)^{-n} \int e^{i(\langle x,\xi\rangle - t|\xi|)} (2i|\xi|)^{-1} \hat{f}(\xi)\, d\xi. \quad (0.2)$$

Each of the terms on the right-hand side is similar to (0.1) except for the fact that the function $\langle x, \xi\rangle$ in the exponent has been replaced by $\langle x, \xi\rangle \pm t|\xi|$. This is a homogeneous function of ξ with critical points as a function of ξ where $x = \pm t\xi/|\xi|$, thus $|x|^2 = t^2$ which is the light cone. The function $\langle x, \xi\rangle$, on the other hand, has no critical point except when $x = 0$. These observations reflect the fact that the fundamental solution of the wave equation is singular on the light cone whereas the fundamental solution of the Laplacean is singular only at the origin.

As a generalization of (0.1), (0.2) it is natural to consider operators of the form

$$Af(x) = \int e^{iS(x,\xi)} a(x,\xi)\, \hat{f}(\xi)\, d\xi. \quad (0.3)$$

Lax [21] showed that for any strictly hyperbolic equation the solution of the Cauchy problem is for small values of the time variable a sum of operators of this form where S is obtained by solving the characteristic equation with initial data $x \to \langle x, \xi\rangle$. Related global results were proved by Ludwig [22]. A more systematic study of operators of the form (0.3) was made by Maslov [23] under the hypothesis that det $S''_{x\xi} \neq 0$, and his results have subsequently been extended and applied by Eškin [9], Egorov [7, 8] in connection with studies of non-elliptic pseudo-dfferential operators.

Introduction of the definition of the Fourier transform in (0.3) gives formally

$$Af(x) = \iint e^{i\phi(x,y,\xi)} a(x,y,\xi)\, f(y)\, dy\, d\xi, \quad (0.4)$$

where $\phi(x,y,\xi) = S(x,\xi) - \langle y,\xi\rangle$, and a is independent of y of course. Quite general operators of the form (0.4) were discussed by the author [14] and the term Fourier integral operator was introduced for them. The purpose was a study of the asymptotic properties of the eigenfunctions of elliptic operators, which is actually a problem involving a related hyperbolic operator. A more systematic development with applications to differential operators of principal type with real principal part was given in mimeographed lecture notes from the

Nordic Summer School of Mathematics 1969 (see also [15, section 5]). Originally this paper was intended as a finished version of those notes but in fact it has been completely revised and very much extended in order to take into account the very important observation of Egorov [7] that if A is an operator of the form (0.3) and P, Q are pseudo-differential operators with $PA = AQ$, then the principal symbols of P and Q are related by the canonical transformation corresponding to the generating function S. Now it turns out that with any operator of the form (0.4) where ϕ satisfies a certain regularity condition one can also associate a canonical transformation and prove that the class of operators of the form (0.4) is determined by the canonical transformation alone. It is then possible to develop a fairly complete calculus of such operators where the result of Egorov is imbedded in a natural way. As a result one can for example give a reinterpretation of the result of Lax [21] mentioned above which is valid globally in the time variable. The results indicated in [15] concerning operators of principal type with real principal part can also be made global under suitable convexity assumptions weaker than those discussed in [17, Chapter VIII]. These applications are left for the second part of the paper which is being written in collaboration with J. J. Duistermaat. However, we wish to call attention to the papers of Egorov [8] and Nirenberg–Trèves [25] which use operators of the form (0.3) in a very essential way in studies concerning existence and regularity theorems for general operators of principal type.

The work of Egorov is actually an application of ideas from Maslov [23] who stated at the International Congress in Nice that his book actually contains the ideas attributed here to Egorov [7] and Arnold [1] as well as a more general and precise operator calculus than ours. Since the book is highly inaccessible and does not appear to be quite rigorous we can only pass this information on to the reader, adding a reference to the explanations of Maslov's work given by Buslaev [5]. In this context we should also mention that the "Maslov index" which plays an essential role in Chapters III and IV was already considered quite explicitly by J. Keller [18]. It expresses the classical observation in geometrical optics that a phase shift of $\pi/2$ takes place at a caustic. The purpose of the present paper is not to extend the more or less formal methods used in geometrical optics but to extract from them a precise operator theory which can be applied to the theory of partial differential operators. In fact, we only use the simplest expansions which occur in geometrical optics, and a wealth of other ideas remain to be investigated.

The plan of the paper is as follows. Chapter I presents generalities concerning Fourier integral operators. Actually this is mainly a more systematic version of the introductory chapter of [14]. In Chapter II we review the calculus of pseudo-differential operators from this more general point of view and give some applications. The kernels of pseudo-differential operators are certain distributions in a product $X \times X$ with singularities only on the diagonal. With any manifold X and submanifold Y there is similarly associated in a natural way a class of distributions with singularities only on Y which is discussed at the end of Chapter II. Actually, these distributions are connected with the normal bundle of Y in the sense that they have symbols living on that bundle. In Chapter III we study more general classes of distributions which are associated with any conic Lagrangean submanifold of $T^*(X)$. If X is replaced by a product $X \times Y$ one can interpret these as classes of operators from functions on Y to functions on X. In particular there is such a class of operators associated with any canonical diffeomorphism of $T^*(Y)\setminus 0$ on $T^*(X)\setminus 0$. Composition of such operators corresponds to composition of the canonical transformations. Pseudo-differential operators are obtained when $X = Y$ and the canonical transformation is the identity. This general operator calculus contains the result of Egorov [7] referred to above and also leads immediately to estimates for the norm of the operators. It is developed in Chapter IV.

A summary of the results of this paper has been given in [16] which can also be read as an introduction giving additional background material.

Finally I would like to thank J. J. Duistermaat for many discussions concerning symplectic geometry which have improved the exposition.

I. Oscillatory integrals

1.0. Introduction

In this chapter we shall give precise definitions of integrals of the form (0.4) and discuss some of their most elementary properties. Concerning the amplitude a in (0.4) we shall usually make essentially the same hypotheses as in earlier studies of pseudo-differential operators (see [13]). The basic facts are collected in section 1.1. In section 1.2 we can then give a precise definition of the corresponding integrals of the form (0.4) by means of essentially the same methods as in [14]. However, the hypotheses of sections 1.1 and 1.2 are somewhat too special for some purposes. In section 1.3 we shall therefore relax the conditions on the amplitude a in (0.4). Basic facts concerning operators of the form (0.4) are then given in section 1.4. Under suitable additional assumptions concerning the phase function ϕ we shall give much more precise results in Chapters II and IV.

For standard notation not explained in the text we refer to Hörmander [17].

1.1. Symbols

The theory of distributions gives a meaning to the Fourier transform

$$\hat{f}(\xi) = \int e^{-i\langle x, \xi\rangle} f(x)\, dx$$

when $f(x) = O(|x|^m)$ for some m as $x \to \infty$. The definition of (0.4) is somewhat more delicate so we have to impose suitable growth conditions of this type on all derivatives of the function $a(x, y, \theta)$. There is no reason for us at this time to consider the variables x and y separately, so in the following definition we consider complex valued functions a defined in $X \times \mathbf{R}^N$ where X is an open subset of $\mathbf{R}^n$ (we allow n to be 0).

Definition 1.1.1. Let m, ϱ, δ be real numbers with $0 \leqslant \varrho \leqslant 1$, $0 \leqslant \delta \leqslant 1$. Then we denote by $S^m_{\varrho,\delta}(X \times \mathbf{R}^N)$ the set of all $a \in C^\infty(X \times \mathbf{R}^N)$ such that for every compact set $K \subset X$ and all multiorders α, β the estimate

$$|D_x{}^\beta D_\theta{}^\alpha a(x, \theta)| \leqslant C_{\alpha,\beta,K}(1 + |\theta|)^{m - \varrho|\alpha| + \delta|\beta|}, \quad x \in K, \theta \in \mathbf{R}^N, \tag{1.1.1}$$

is valid for some constant $C_{\alpha,\beta,K}$. The elements of $S^m_{\varrho,\delta}$ are called symbols of order m and type ϱ, δ. If $\varrho + \delta = 1$ we also use the notation S^m_ϱ instead of $S^m_{\varrho,\delta}$, and when $\varrho = 1$, $\delta = 0$ we sometimes write only S^m and talk about symbols of order m. If (1.1.1) is only valid for large $|\theta|$, we say that $a \in S^m_{\varrho,\delta}$ for large $|\theta|$. Finally we set

$$S^\infty_{\varrho,\delta} = \bigcup_m S^m_{\varrho,\delta}, \quad S^{-\infty}_{\varrho,\delta} = \bigcap_m S^m_{\varrho,\delta}.$$

By the conic support of a, denoted cone supp a we denote the closure in $X \times \mathbf{R}^N$ of $\{(x, t\theta);\ (x, \theta) \in \operatorname{supp} a,\ t \geqslant 0\}$.

Somewhat incorrectly we shall also say that a set $M \subset X \times \mathbf{R}^N$ is conic if $(x, \theta) \in M$ implies $(x, t\theta) \in M$ when $t > 0$. The conic support of a is thus the smallest closed conic subset of $X \times \mathbf{R}^N$ such that a vanishes in the complement.

Example 1.1.2. If $a \in C^\infty$ and a is a homogeneous function of degree m with respect to θ for large $|\theta|$, then a is a symbol of order m (and type 1, 0).

Example 1.1.3. If a is semi-homogeneous in the sense that

$$a(x, \theta_1 t^{m_1}, \ldots, \theta_N t^{m_N}) = t^m a(x, \theta_1, \ldots, \theta_N)$$

for some $m_j > 0$ and $m \in \mathbf{R}$, and if $a \in C^\infty$ for $\theta \neq 0$, then a is for large θ a symbol of degree $\max_j m/m_j$ and type $\min_{j,k} m_j/m_k$, 0.

Example 1.1.4. If $\chi \in \mathcal{S}(\mathbf{R}^n)$, the Schwartz space, then $a(x, \theta) = \chi(x|\theta|^\varepsilon)$ is in $S^0_{1,\varepsilon}(\mathbf{R}^n \times \mathbf{R}^N)$ for large $|\theta|$.

Example 1.1.5. If $0<t<1$, the function $(x, \theta)\to \exp ic(x)|\theta|^{1-t}$, where c is a real valued function, is in $S^0_{\varrho,\delta}$ for large $|\theta|$ if and only if $\varrho\leq t$, $\delta\geq 1-t$, or c is constant and $\varrho\leq t$, δ arbitrary, or $c=0$ and both ϱ and δ arbitrary.

PROPOSITION 1.1.6. *$S^m_{\varrho,\delta}(X\times \mathbf{R}^N)$ is a Fréchet space with the topology defined by taking as seminorms the best constants $C_{\alpha,\beta,K}$ which can be used in* (1.1.1). *This space increases when δ and m increase and ϱ decreases. If $a\in S^m_{\varrho,\delta}$ it follows that*

$$a^{(\alpha)}_{(\beta)}=(iD_\theta)^\alpha(iD_x)^\beta a\in S^{m-\varrho|\alpha|+\delta|\beta|}_{\varrho,\delta},$$

and if $b\in S^{m'}_{\varrho,\delta}$ it follows that $ab\in S^{m+m'}_{\varrho,\delta}$.

The proof is obvious. Note that to prove (1.1.1) for ab one needs to know only that (1.1.1) holds for a and for b when the differentiations involved are of order $\leq|\alpha+\beta|$. This is important in some proofs by induction.

It follows immediately from Definition 1.1.1 that $S^m_{\varrho,\delta}$ is invariant for diffeomorphisms in the x variable, so the definition makes sense also if X is a manifold. In order to be able to consider more general fiber spaces than $X\times\mathbf{R}^N$ over X we shall need some fact concerning the action on symbols of more general maps. In doing so it is useful to work locally so we first extend Definition 1.1.1 somewhat. Thus let Γ be an open conic set $\subset X\times\mathbf{R}^N$. If K is a compact subset of Γ we set $K^c=\{(x, t\theta);\ (x,\theta)\in K,\ t\geq 1\}$. A function in $C^\infty(\Gamma)$ is now said to be in $S^m_{\varrho,\delta}(\Gamma)$ if (1.1.1) is valid when $(x,\theta)\in K^c$ for any choice of the compact set K. This agrees with Definition 1.1.1 when $\Gamma=X\times\mathbf{R}^N$. Note that if Γ does not meet $X\times 0$ then there are no restrictions on the growth of the derivatives of $a(x,\theta)$ when $\theta\to 0$.

Let $\Gamma_1\subset X\times\mathbf{R}^M$, $\Gamma_2\subset Y\times\mathbf{R}^N$ be open conic sets disjoint from $X\times 0$ and $Y\times 0$, and let ψ: $\Gamma_1\to\Gamma_2$ be a C^∞ map which is positively homogeneous of degree one, that is, commutes with multiplication by positive scalars in the fibers of $X\times\mathbf{R}^M$ and $Y\times\mathbf{R}^N$.

PROPOSITION 1.1.7. *Under the preceding hypotheses we have $a\circ\psi\in S^m_{\varrho,\delta}(\Gamma_1)$ for all $a\in S^m_{\varrho,\delta}(\Gamma_2)$ provided that either*

(i) *$\varrho+\delta=1$; or*

(ii) *$\varrho+\delta\geq 1$ and ψ is fiber preserving, that is the projection of $\psi(x,\theta)$ on Y depends only on x; or*

(iii) *ϱ and δ are unrestricted but ψ is the direct product of a map $Y\to X$ and a homogeneous map from a cone in $\mathbf{R}^M$ to a cone in $\mathbf{R}^N$.*

Proof. A part of the last statement was already pointed out above. In the general proof we shall use the notation x, ξ for the variables in $X\times\mathbf{R}^M$ and y, η for the variables in

$Y\times\mathbf{R}^N$. Then $\psi(x,\xi)=(y(x,\xi),\ \eta(x,\xi))$ where y and η are homogeneous of degree 0 and 1 with respect to ξ. If K is a compact subset of Γ_1 we have for some positive constants C_1, C_2

$$C_1|\xi|\leqslant|\eta(x,\xi)|\leqslant C_2|\xi| \tag{1.1.2}$$

when $(x,\xi)\in K$ for by hypothesis $\eta(x,\xi)\neq 0$ then. Since (1.1.2) is homogeneous it is also valid when $(x,\xi)\in K^c$. Writing $b=a\circ\psi$, $b^k=(\partial a/\partial\eta_k)\circ\psi$, $b_k=(\partial a/\partial y_k)\circ\psi$, we have

$$\left.\begin{aligned}\partial b/\partial x_j&=\sum b_k\partial y_k/\partial x_j+\sum b^k\partial\eta_k/\partial x_j\\ \partial b/\partial\xi_j&=\sum b_k\partial y_k/\partial\xi_j+\sum b^k\partial\eta_k/\partial\xi_j.\end{aligned}\right\} \tag{1.1.3}$$

Here $\partial y_k/\partial x_j$, $\partial\eta_k/\partial x_j$, $\partial y_k/\partial\xi_j$, $\partial\eta_k/\partial\xi_j$ are homogeneous of degree 0, 1, -1, 0 with respect to ξ. The estimates (1.1.1) for $(x,\xi)\in K^c$ now follow immediately when $|\alpha+\beta|\leqslant 1$ if we use (1.1.2). (Note that in case (ii) we have $\partial y_k/\partial\xi_j=0$ and in case (iii) also $\partial\eta_k/\partial x_j=0$.) Assuming that (1.1.1) is proved when $|\alpha+\beta|\leqslant k$ for any symbol a, we conclude that (1.1.1) is also valid when $|\alpha+\beta|=k+1$ if we use the remark concerning the multiplicative properties of symbols made immediately after Proposition 1.1.6.

Remark. If one takes for a one of the examples 1.1.4 and 1.1.5 it is easy to see that the hypotheses on ϱ and δ in the proposition cannot be improved (apart from a case where $\partial\eta_k/\partial x_j=0$ and $\varrho+\delta\leqslant 1$ which we have omitted as of no interest). The hypothesis on homogeneity could be somewhat relaxed to $\psi\in S^1_{1,0}$ and an appropriate substitute for (1.1.2).

We can also compose symbols to the left with suitable functions:

PROPOSITION 1.1.8. *Let $a_1,\ldots,a_k$ be real valued functions in $S^0_{\varrho,\delta}(X\times\mathbf{R}^N)$ and let f be a C^∞ function in a neighborhood in $\mathbf{R}^k$ of all limit points of $(a_1(x,\theta),\ldots,a_k(x,\theta))$ when $\theta\to\infty$ while x may vary in X. Then it follows that $(x,\theta)\to f(a_1(x,\theta),\ldots,a_k(x,\theta))$ is in $S^0_{\varrho,\delta}(X\times\mathbf{R}^N)$ for large θ.*

Proof. Choose C so large that f is C^∞ in a neighborhood of the closure of $\{(a_1(x,\theta),\ldots,a_k(x,\theta)),\ x\in X,\ |\theta|>C\}$. For $|\theta|>C$ it is then clear that $f(a_1,\ldots,a_k)$ is bounded. Since

$$\partial f(a_1,\ldots,a_k)/\partial(x,\xi)=\sum_1^k(\partial f/\partial a_j)\,(\partial a_j/\partial(x,\xi))$$

and $\partial f/\partial a_j$ are bounded functions, it is clear that $f(a_1,\ldots,a_k)$ satisfies the estimates (1.1.1) when $|\alpha+\beta|\leqslant 1$. As in the proof of Proposition 1.1.7 it follows by induction that they are valid when $|\alpha+\beta|\leqslant j$ for $j=1,2,\ldots$, for we can use the multiplicative properties of symbols in Proposition 1.1.6 and the remark following that statement.

Proposition 1.1.7 makes it possible to define $S^m_{\varrho,\delta}(V)$ when $\varrho+\delta\geqslant 1$ if V is a *cone bundle* over a manifold X in the sense that the following three conditions are fulfilled:

(i) V, X are manifolds(¹) and we are given a C^∞ projection p: $V\to X$ with surjective differential.

(ii) There is given a C^∞ action of $\mathbf{R}_+$ on V which preserves the fibers.

(iii) Every point in V has a neighborhood N_0 invariant under the action of $\mathbf{R}_+$ such that there is a fiber preserving diffeomorphism $\varkappa$: $N_0\to\Gamma$ commuting with the group actions, where Γ is an open set in $\mathbf{R}^n\times(\mathbf{R}^N\setminus 0)$ invariant under the group action $(t, (x,\theta))\to(x, t\theta)$; $t\in\mathbf{R}_+$, $x\in\mathbf{R}^n$, $\theta\in\mathbf{R}^N$, and with the projection $(x,\theta)\to x$. Here $n=\dim X$.

If N_1 and N_2 are two such neighborhoods with diffeomorphisms $\varkappa_1$ and $\varkappa_2$, then $\varkappa=\varkappa_1\circ\varkappa_2^{-1}$ is a diffeomorphism $\Gamma_{21}\to\Gamma_{12}$ where Γ_{21} and Γ_{12} are open conic subset of Γ_2 and Γ_1 respectively, and $\varkappa$ satisfies the hypotheses in part (ii) of Proposition 1.1.7. Composition with $\varkappa$ therefore maps $S^m_{\varrho,\delta}(\Gamma_{12})$ to $S^m_{\varrho,\delta}(\Gamma_{21})$ continuously if $\varrho+\delta\geqslant 1$. For such ϱ and δ we define $S^m_{\varrho,\delta}(V)$ as the set of functions a on V for which $a\circ\varkappa^{-1}$ is in $S^m_{\varrho,\delta}(\Gamma)$ (and vanishes near $\mathbf{R}^n\times 0$) if $\varkappa$ is a local trivialization with the properties listed in (iii). By the preceding remarks it suffices to make this hypothesis for a set of such neighborhoods N_j which covers V.

Let V_j be a cone bundle over X_j, $j=1,2$, and let ψ be a fiber preserving C^∞ map $V_1\to V_2$ commuting with the action of $\mathbf{R}_+$. If $a\in S^m_{\varrho,\delta}(V_2)$, $\varrho+\delta\geqslant 1$, it follows from part (ii) of Proposition 1.1.7 that $a\circ\psi\in S^m_{\varrho,\delta}(V_1)$. If $\varrho+\delta=1$, part (i) of the same result shows that the same conclusion is valid even if ψ is not fiber preserving.

Let V be a cone bundle over X and let X be a fiber space over another manifold Y so that we have a C^∞ map p_Y: $X\to Y$ with surjective differential. Then V is also a cone bundle over Y if we replace the projection p: $V\to X$ by $p_Y p$: $V\to Y$. To prove this it suffices to note that if U is an open set in $\mathbf{R}^n$, Γ an open cone in $\mathbf{R}^N\setminus 0$, and p the projection.

$$U\times\Gamma\ni(x,\theta)\to x'=(x_1,\dots,x_\nu)\in\mathbf{R}^\nu,$$

where $\nu\leqslant n$, then $U\times\Gamma$ is a cone bundle over $\mathbf{R}^\nu$. This follows from the fact that we have the local homogeneous diffeomorphism

$$(x,\theta)\to(x', x''|\theta|, \theta)\in\mathbf{R}^\nu\times\mathbf{R}^{N+n-\nu}$$

where $x''=(x_{\nu+1},\dots,x_n)$. We denote this cone bundle by V_Y. The map $V\to V_Y$ (but not the map $V_Y\to V$) is then fiber preserving, so if $\varrho+\delta\geqslant 1$ we have the inclusions

$$S^m_{\varrho,1-\varrho}(V)\subset S^m_{\varrho,\delta}(V_Y)\subset S^m_{\varrho,\delta}(V).$$

(¹) By a manifold we shall always mean a paracompact C^∞ manifold.

When Y is a point the left inclusion becomes an equality. Note that in that case V_Y is V with the projection p forgotten entirely. A cone bundle over a point will also be called a *conic manifold* in this paper.

With V still denoting a cone bundle over a manifold X we assume that we have a complex vector bundle W over V in which there is given an equivariant C^∞ action of $\mathbf{R}_+$. (cf. Atiyah [2].) Thus the projection on V of the vector bundle map t: $W \to W$, $t \in \mathbf{R}_+$, is assumed to be the already given map t: $V \to V$. We can now introduce in a natural way a class $S^m_{\varrho,\delta}(V, W)$ of sections of W over V. For in a conic neighborhood of any point $v_0 \in V$ we can find a basis $s_1, \ldots, s_N$ of C^∞ sections of W which are invariant under $\mathbf{R}_+$. In fact, it suffices to choose them on a manifold transversal to the orbit of $\mathbf{R}_+$ at v_0 and extend them by homogeneity. Two such bases differ by multiplication with a $N \times N$ matrix of functions homogeneous of degree 0 in a conic neighborhood of v_0. A section s of W over V is now said to be in $S^m_{\varrho,\delta}(V, W)$ if for the local representation $s = \sum a_j s_j$ with such bases we have $a_j \in S^m_{\varrho,\delta}$ in a conic neighborhood of v_0. It is clear that the definition does not depend on the choice of bases.

All remarks made previously concerning the behavior of $S^m_{\varrho,\delta}$ under mappings carry over with obvious modifications to $S^m_{\varrho,\delta}(V, W)$. We leave the statements for the reader in order not to burden the exposition further. In the following results we only consider symbols in $X \times \mathbf{R}^N$ for simplicity in the statements but it should be clear that they carry over easily to symbols in the spaces $S^m_{\varrho,\delta}(V, W)$.

Now we recall an elementary but important completeness property of the space of symbols, proved for example in [13, Theorem 2.7].

PROPOSITION 1.1.9. *Let $a_j \in S^{m_j}_{\varrho,\delta}(X \times \mathbf{R}^N)$, $j = 0, 1, 2, \ldots$ and assume that $m_j \to -\infty$ as $j \to \infty$. Set $m'_k = \max_{j \geq k} m_j$. Then one can find $a \in S^{m'_0}_{\varrho,\delta}(X \times \mathbf{R}^N)$ such that for every k*

$$a - \sum_{j<k} a_j \in S^{m'_k}_{\varrho,\delta}(X \times \mathbf{R}^n). \tag{1.1.4}$$

The function a is uniquely determined modulo $S^{-\infty}(X \times \mathbf{R}^N)$ and has the same property relative to any rearrangement of the series $\sum a_j$. We write $a \sim \sum a_j$.

The condition (1.1.4) involves bounds on all derivatives of the function on the left. In order to simplify a verification of (1.1.4) it is therefore useful to have the following result, which is Theorem 2.9 in [13].

PROPOSITION 1.1.10. *Let $a_j \in S^{m_j}_{\varrho,\delta}(X \times \mathbf{R}^n)$, $j = 0, 1, 2, \ldots$ and assume that $m_j \to -\infty$*

when $j\to\infty$. Let $a\in C^\infty(X\times\mathbf{R}^N)$ and assume that for all multiorders α,β and compact sets $K\subset X$ we have for some C and μ depending on α,β and K

$$|a_{(\beta)}^{(\alpha)}(x,\theta)|\leqslant C(1+|\theta|)^\mu,\quad x\in K.$$

If there exist numbers $\mu_k\to-\infty$ such that for arbitrary K and k

$$|a(x,\theta)-\sum_{j<k}a_j(x,\theta)|\leqslant C_{K,k}(1+|\theta|)^{\mu_k},\quad x\in K,$$

it follows that $a\in S^m_{\varrho,\delta}(X\times\mathbf{R}^N)$ where $m=\sup_j m_j$, and that $a\sim\sum a_j$.

Finally we shall make some remarks on the topology of the Fréchet space $S^m_{\varrho,\delta}(X\times\mathbf{R}^N)$. Recall that a set $M\subset S^m_{\varrho,\delta}$ is bounded if (1.1.1) is valid with $C_{\alpha,\beta,K}$ independent of a when $a\in M$. On a bounded set in $S^m_{\varrho,\delta}$ the topology of pointwise convergence, the topology of $C^\infty(X\times\mathbf{R}^N)$ and the toplogy of $S^{m'}_{\varrho,\delta}(X\times\mathbf{R}^N)$, $m'>m$, all coincide. This is an immediate consequence of Ascoli's theorem.

PROPOSITION 1.1.11. *Let $a\in S^m_{\varrho,\delta}(X\times\mathbf{R}^N)$ and let $\chi\in\mathcal{S}(\mathbf{R}^N)$ be equal to 1 at 0. If $a_\varepsilon(x,\theta)=\chi(\varepsilon\theta)a(x,\theta)$, it follows that $a_\varepsilon\in S^{-\infty}_{\varrho,\delta}(X\times\mathbf{R}^N)$ and that $a_\varepsilon\to a$ in $S^{m'}_{\varrho,\delta}(X\times\mathbf{R}^N)$ when $\varepsilon\to 0$ if $m'>m$.*

Proof. It suffices to note that the functions $(x,\theta)\to\chi(\varepsilon\theta)$ form a bounded set in $S^0_{1,0}$ when $0\leqslant\varepsilon\leqslant 1$ (see Example 1.1.4), for the continuity of multiplication of symbols then shows that the functions a_ε form a bounded set too.

In particular, we can take χ with compact support. Then we obtain

COROLLARY 1.1.12. *Let L be a linear map from functions in $C^\infty(X\times\mathbf{R}^N)$ vanishing for large $|\theta|$ to a Fréchet space F such that, for every $m\in\mathbf{R}$, the map L is continuous for the topology induced by $S^m_{\varrho,\delta}(X\times\mathbf{R}^N)$. Then there is a unique extension of L to $S^\infty_{\varrho,\delta}(X\times\mathbf{R}^N)$ which is continuous on $S^m_{\varrho,\delta}(X\times\mathbf{R}^N)$ for every m.*

1.2. Oscillatory integrals

We shall now discuss the definition of integrals of the form

$$I_\phi(au)=\iint e^{i\phi(x,\theta)}a(x,\theta)\,u(x)\,dx\,d\theta,\quad u\in C_0^\infty(X),\tag{1.2.1}$$

where $a\in S^m_{\varrho,\delta}(X\times\mathbf{R}^N)$. From now on we assume that $\varrho>0$ and that $\delta<1$. For the sake of simplicity it will be assumed that ϕ is real valued and positively homogeneous of degree 1 with respect to θ, and that $\phi \in C^\infty$ for $\theta \neq 0$. However, this hypothesis could easily be relaxed (see also [14] for a somewhat weaker hypothesis).

The integral (1.2.1) is absolutely convergent for every $a \in S^m_{\varrho,\delta}(X \times \mathbf{R}^N)$ provided that $m+N<0$. In particular, it is well defined if $a(x,\theta)=0$ for large $|\theta|$. We wish to extend the definition of (1.2.1) to arbitrary $a \in S^\infty_{\varrho,\delta}$ using Corollary 1.1.12. This is not always possible —for example it cannot be done if ϕ vanishes in an open set—but we shall prove that the definition of (1.2.1) is always possible if ϕ has no critical point with $\theta \neq 0$. The proof depends on partial integrations in (1.2.1). In order to avoid having to split (1.2.1) into a sum of terms where integration by parts with respect to a fixed variable will do, it is convenient to use the following

LEMMA 1.2.1. *If ϕ has no critical point (x,θ) with $\theta \neq 0$, then one can find a first order differential operator*

$$L = \Sigma\, a_j \partial/\partial\theta_j + \Sigma\, b_j \partial/\partial x_j + c$$

with $a_j \in S^0(X \times \mathbf{R}^N)$ and $b_j, c \in S^{-1}(X \times \mathbf{R}^N)$ such that ${}^tLe^{i\phi}=e^{i\phi}$ if ${}^tL$ is the adjoint of L.

Proof. By hypothesis the sum

$$|\theta|^2\, \Sigma\, (\partial\phi/\partial\theta_j)^2 + \Sigma\, (\partial\phi/\partial x_j)^2$$

is homogeneous of degree 2 with respect to θ and $\neq 0$ for $\theta \neq 0$. Let ψ be the reciprocal of this sum which is then homogeneous of degree -2 and C^∞ for $\theta \neq 0$. With $\chi \in C_0^\infty(\mathbf{R}^N)$ chosen so that $\chi = 1$ near 0, we set

$$M = \Sigma\, a'_j \partial/\partial\theta_j + \Sigma\, b'_j \partial/\partial x_j + \chi,$$

where $\quad a'_j = -i(1-\chi)\psi|\theta|^2 \partial\phi/\partial\theta_j \in S^0, \quad b'_j = -i(1-\chi)\psi\partial\phi/\partial x_j \in S^{-1}.$

The coefficients are chosen so that $Me^{i\phi}=e^{i\phi}$, so $L = {}^tM$ has the required properties since

$$a_j = -a'_j, \quad b_j = -b'_j, \quad c = \chi - \textstyle\sum \partial a'_j/\partial\theta_j - \Sigma\, \partial b'_j/\partial x_j \in S^{-1}.$$

The lemma is proved.

If a vanishes for large $|\theta|$, we can integrate by parts in (1.2.1) after replacing $e^{i\phi}$ by ${}^tLe^{i\phi}$. This gives

$$I_\phi(au) = \iint e^{i\phi(x,\theta)} L(a(x,\theta)\, u(x))\, dx\, d\theta$$

or after iteration

$$I_\phi(au) = \iint e^{i\phi(x,\theta)} L^k(a(x,\theta)\, u(x))\, dx\, d\theta, \quad k=0,1,2,\ldots. \tag{1.2.2}$$

Now L is a continuous map of $S^m_{\varrho,\delta}$ into $S^{m-t}_{\varrho,\delta}$ if $t=\min(\varrho, 1-\delta)$. Hence L^k maps $S^m_{\varrho,\delta}$

continuously into $S^{m-kt}_{\varrho,\delta}$. If $m-kt< -N$, the integral (1.2.2) is thus defined and continuous on all of $S^m_{\varrho,\delta}(X\times\mathbf{R}^N)$. In view of Corollary 1.1.12 we have therefore proved

PROPOSITION 1.2.2. *If ϕ has no critical points and $\varrho>0$, $\delta<1$, then the definition of the integral* (1.2.1) *can be extended in one and only one way to all $a\in S^\infty_{\varrho,\delta}(X\times\mathbf{R}^N)$ and $u\in C_0^\infty(X)$ so that $I_\phi(au)$ is a continuous function of $a\in S^m_{\varrho,\delta}$ for every fixed m. The linear form A: $u\to I_\phi(au)$ is a distribution of order $\leqslant k$ if $a\in S^m_{\varrho,\delta}$ and $m-k\varrho<-N$, $m-k(1-\delta)<-N$.*

For the extended form I_ϕ we have the representation (1.2.2) if k is sufficiently large. According to Proposition 1.1.11 we also have

$$I_\phi(au)=\lim_{\varepsilon\to 0}\iint e^{i\phi(x,\theta)}a(x,\theta)\,\chi(\varepsilon\theta)\,u(x)\,dx\,d\theta \tag{1.2.3}$$

if $\chi\in\mathcal{S}$ and $\chi(0)=1$. We shall keep the notation (1.2.1) for the continuous extension of the form I_ϕ which we have just defined and refer to the generalized integral as an *oscillatory integral.*

If ϕ and a are continuous functions of a parameter t with values in $C^\infty(X\times(\mathbf{R}^N\setminus\{0\}))$ and $S^m_{\varrho,\delta}(X\times\mathbf{R}^N)$ respectively, then an inspection of the proof of Lemma 1.2.1 and Proposition 1.2.2 shows that $I_\phi(au)$ is a continuous function of t. Note that if a is a continuous function of t with values in $C^\infty(X\times\mathbf{R}^N)$ whose range is a bounded subset of $S^m_{\varrho,\delta}(X\times\mathbf{R}^N)$, then a is a continuous function of t with values in $S^{m'}_{\varrho,\delta}(X\times\mathbf{R}^N)$ when $m'>m$. These remarks allow us to pass to the limit in the oscillatory integral (1.2.2) if there is continuous dependence on a parameter. In particular we can differentiate with respect to parameters under the integral sign.

Now let ϕ be a C^∞ function in $X\times Y\times(\mathbf{R}^N\setminus\{0\})$ where X and Y are open subsets of some Euclidean spaces, and assume that ϕ has no critical point even when considered as a function in $X\times(\mathbf{R}^N\setminus\{0\})$ depending on the parameter $y\in Y$. If $a\in S^m_{\varrho,\delta}(X\times Y\times\mathbf{R}^N)$, $\varrho>0$, $\delta<1$, and $u\in C^\infty(X\times Y)$, we can then prove a Fubini theorem

$$\iiint e^{i\phi(x,y,\theta)}a(x,y,\theta)\,u(x,y)\,dx\,dy\,d\theta=\int dy\left(\iint e^{i\phi(x,y,\theta)}a(x,y,\theta)\,u(x,y)\,dx\,d\theta\right). \tag{1.2.4}$$

Indeed, this follows if we introduce a factor $\chi(\varepsilon\theta)$ as in (1.2.3) in both sides and then let $\varepsilon\to 0$.

After these remarks we return to the oscillatory integral (1.2.1). Let X_ϕ be the open set of all $x\in X$ such that the function $\theta\to\phi(x,\theta)$ has no critical point $\theta\neq 0$. If $u\in C^\infty(X_\phi)$, we can regard x as a parameter and rewrite (1.2.1) in the form

$$I_\phi(au) = \int A(x)\,u(x)\,dx, \quad u \in C_0^\infty(X_\phi),$$

where

$$A(x) = \int e^{i\phi(x,\theta)} a(x,\theta)\,d\theta, \quad x \in X_\phi. \tag{1.2.5}$$

By our preceding remarks on oscillatory integrals, A is a continuous function of $x \in X_\phi$, and since we can differentiate under the sign of (oscillatory) integration as often as we like, we conclude that $A \in C^\infty(X_\phi)$. If we recall that the singular support (written sing supp) of a distribution is the completment of the largest open set where it is a C^∞ function, we have proved

PROPOSITION 1.2.3. *For the distribution A: $u \to I_\phi(au)$ defined by* (1.2.1) *we have*

$$\text{sing supp } A \subset \{x \in X;\ \phi'_\theta(x,\theta) = 0 \text{ for some } \theta \neq 0\}. \tag{1.2.6}$$

The formula (1.2.5) also makes sense for all $x \in X$ provided that ϕ has no critical point as a function of θ in cone supp a, for this is clearly all that is required in the proof of Proposition 1.2.3. Thus we have the following simple result which shows that the singularities of the distribution A are uniquely determined by the behavior of the symbol a in a conical neighborhood of the set of points where ϕ is critical with respect to the θ variables.

PROPOSITION 1.2.4. *If $a \in S_{\varrho,\delta}(X \times \mathbf{R}^N)$ and a vanishes in some conic neighborhood of the set*

$$C = \{(x,\theta);\, x \in X,\, \theta \in \mathbf{R}^N \setminus \{0\},\, \phi'_\theta(x,\theta) = 0\}, \tag{1.2.7}$$

then the distribution $u \to I_\phi(au)$ defined by (1.2.1) *is a C^∞ function.*

If one looks more carefully into the proof of Proposition 1.2.2 one finds easily that the conclusion of Proposition 1.2.4 remains valid if we assume only that for some C and ε with $\varepsilon < \min(\varrho, 1/2)$ we have $a(x,\theta) = 0$ when $|\phi'_\theta(x,\theta)|\,|\theta|^\varepsilon < C$. The proof is left to the reader, but we shall prove a stronger result under some hypotheses on ϕ which guarantee that C is a smooth manifold.

Let Γ be an open conic set in $X \times \mathbf{R}^N$ and let ϕ be a positively homogeneous function of degree 1 with respect to θ which is in C^∞ and has no critical point in $\Gamma \setminus (X \times \{0\})$. Such a function will be called a *phase function* from now on. It is clear that the definition of (1.2.1) given above is still valid for such a function ϕ provided that we require that cone supp $a \subset \Gamma \cup (X \times \{0\})$.

We shall say that ϕ is *non-degenerate* if at any point in the set C defined by (1.2.7) the differentials $d(\partial\phi/\partial\theta_j)$, $j = 1, \ldots, N$, are linearly independent. This implies of course that

C is a manifold of dimension dim X. For such phase functions we can improve Proposition 1.2.4.

PROPOSITION 1.2.5. *Let ϕ be a non-degenerate phase function in $\Gamma \subset X \times \mathbf{R}^N$ and let* $a \in S^m_{\varrho,\delta}(X \times \mathbf{R}^N)$, cone supp $a \subset \Gamma \cup (X \times 0)$. *We assume that $\varrho > \delta$ and that either ϕ is a linear function of θ or that $\varrho + \delta = 1$. Then the distribution $u \to I_\phi(au)$ defined by* (1.2.1) *is a C^∞ function if a vanishes of infinite order on*

$$C = \{(x, \theta) \in \Gamma;\ \phi'_\theta(x, \theta) = 0\}.$$

If a just vanishes on C we can find $b \in S^{m+\delta-\varrho}_{\varrho,\delta}(X \times \mathbf{R}^N)$ with cone supp $b \subset \Gamma \cup (X \times 0)$ such that $I_\phi(au) = I_\phi(bu)$, $u \in C_0^\infty(X)$.

For the proof we need a lemma.

LEMMA 1.2.6. *Let $\phi_1, \ldots, \phi_k$ be real valued C^∞ functions in $\Gamma \setminus (X \times \{0\})$ which are homogeneous of degree 0, and assume that the differentials $d\phi_j$, $j = 1, \ldots, k$ are linearly independent in*

$$C = \{(x, \theta) \in \Gamma,\ \theta \neq 0,\ \phi_j(x, \theta) = 0,\ j = 1, \ldots, k\}.$$

Let $a \in S^m_{\varrho,\delta}(X \times \mathbf{R}^N)$ where we assume that $\varrho + \delta = 1$ unless $\phi_1, \ldots, \phi_k$ are functions of x only. If $a = 0$ in a neighborhood of $X \times \{0\}$ and a vanishes (*of infinite order*) *on C*, cone supp $a \subset$ $\Gamma \cup (X \times \{0\})$, *one can find $a_j \in S^{m+\delta}_{\varrho,\delta}(X \times \mathbf{R}^N)$, $j = 1, \ldots, k$, with* cone supp $a_j \subset \Gamma \cup (X \times \{0\})$ *such that* (a_j *vanishes of infinite order on C and*)

$$a = \sum_1^k a_j \phi_j. \tag{1.2.8}$$

Proof that the lemma implies the proposition. If we apply the lemma with $\phi_j = \partial\phi/\partial\theta_j$, which are functions independent of θ precisely when ϕ is a linear function of θ, an integration by parts gives

$$I_\phi(au) = \iint e^{i\phi(x,\theta)} \sum_1^N i\partial a_j(x, \theta)/\partial\theta_j u(x)\, dx\, d\theta.$$

Here the new amplitude function is of order $m + \delta - \varrho$ and in case a vanishes of infinite order on C it will also vanish of infinite order there. If the argument is repeated k times we find that a can be replaced by a symbol of order $m + k(\delta - \varrho) \to -\infty$, $k \to \infty$, so the distribution $u \to I_\phi(au)$ is a C^∞ function.

Proof of the lemma. It is sufficient to find a local solution of (1.2.8) and then apply a partition of unity on the sphere bundle in $\Gamma \subset X \times \mathbf{R}^N$, extended to a system of homogeneous functions of degree 0, to obtain a global solution. If $(x_0, \theta_0) \in \Gamma$, $\theta_0 \neq 0$, the existence of a solution in a conic neighborhood is obvious unless $(x_0, \theta_0) \in C$ as we shall now assume. The functions $\phi_1, \ldots, \phi_k$ are then independent functions on the unit sphere bundle at $(x_0, \theta_0/|\theta_0|)$ so we can find additional homogeneous functions $\phi_{k+1}, \ldots, \phi_l$ of degree 0, $l = \dim X + N - 1$, all vanishing at (x_0, θ_0) so that $\phi_1, \ldots, \phi_l$ is a local coordinate system at $(x_0, \theta_0/|\theta_0|)$ on the sphere bundle. But then the map

$$(x, \theta) \to (\phi_1, \ldots, \phi_l, |\theta|) \in \mathbf{R}^l \times \mathbf{R}_+$$

is a homogeneous diffeomorphism of a conic neighborhood of (x_0, θ_0) on $U \times \mathbf{R}_+$ where U is an open ball in $\mathbf{R}^l$ with center at 0. By part (i) of Proposition 1.1.7 symbols of type ϱ, δ are preserved by such maps if $\varrho + \delta = 1$ so when this is assumed we have reduced the proof to the case when $\phi_1, \ldots, \phi_k$ are equal to the first coordinates $x_1, \ldots, x_k$ in X and X is a ball with center at 0. If $\phi_1, \ldots, \phi_k$ are independent of θ the same result is achieved by a substitution which only affects the x variables and thus preserves symbols of type ϱ, δ for arbitrary ϱ, δ. The lemma now follows from Taylor's formula which gives, since a vanishes when $x_1 = \ldots = x_k = 0$

$$a(x, \theta) = \sum_1^k x_j \int_0^1 a_{(j)}(tx_1, \ldots, tx_k, x_{k+1}, \ldots, \theta)\, dt,$$

where $a_{(j)} = \partial a / \partial x_j \in S^{m+\delta}_{\varrho, \delta}$. The proof is complete.

Summing up, when ϕ is non-degenerate the singularities of the distribution $u \to I_\phi(au)$ only depend on the Taylor expansion of a on the set C defined by (1.2.7), provided that suitable assumptions are made concerning ϱ and δ. In Chapters II and III we shall study the consequences of this more closely and also discuss how essential the choice of the phase function ϕ really is.

1.3. Singular symbols and oscillatory integrals

The definition of the oscillatory integral (1.2.1) given in the preceding section did not fully use the hypothesis that a is a symbol in the sense of section 1.1. Indeed, we only used the fact that for some first order differential operator L with the properties stated in Lemma 1.2.1 we can conclude that $L^k(au)$ is an integrable function for sufficiently large values of k. This we shall exploit in what follows.

In some constructions of fundamental solutions one needs to be able to define integrals of the form

$$\iint e^{i\phi(x,\theta)} a(x, \theta)/q(x, \theta)\, dx\, d\theta, \tag{1.3.1}$$

where q is homogeneous with respect to θ of degree m, say, and may have simple real zeros.

These form an obvious difficulty in defining the integral. To bypass this singularity we would like to integrate over a suitable cycle in the complex domain instead of over $\mathbf{R}^n$. Assume for simplicity that ϕ, q and a are analytic with respect to θ in a neighborhood of the real domain and that there is a vector η such that $\langle \operatorname{grad}_\theta q(x, \theta), \eta\rangle \neq 0$ when $q(x, \theta) = 0$ and $\theta \in \mathbf{R}^N \diagdown 0$, $x \in X$. (It will in fact be necessary to let the direction of η vary and in case the data a, q, ϕ are not analytic make suitable "almost analytic" continuations of them. These questions will be discussed elsewhere, but here we only wish to motivate what follows.) Our hypotheses imply that

$$|q(x, \theta + i\eta)| = |\theta|^m |q(x, \theta/|\theta| + i\eta/|\theta|)| \geqslant c|\theta|^{m-1}$$

for large $|\theta|$, so if we replace θ by $\theta + i\eta$ in (1.3.1) we shall for large θ no longer have any infinities in the integrand. We shall now examine to what extent the function $(x, \theta) \to a(x, \theta + i\eta)/q(x, \theta + i\eta)$ has the properties of a symbol. We have for example

$$\partial q^{-1}(x, \theta + i\eta)/\partial x_j = -q^{-2}\partial q/\partial x_j$$

and we can only be certain that this can be bounded by $|\theta|^{m-2(m-1)} = |\theta|^{2-m}$. Pursuing this argument one will find that $1/q(x, \theta + i\eta) \in S^{1-m}_{0,1}$ which does not suffice for application of Proposition 1.2.2. However, we can say more about the action on q^{-1} of some operators

$$L = \Sigma\, a_j(x, \theta)\partial/\partial\theta_j + \Sigma\, b_j(x, \theta)\partial/\partial x_j + c \tag{1.3.2}$$

with $a_j \in S^0$ and $b_j, c \in S^{-1}$. Indeed, if $\Sigma\, a_j \partial q/\partial\theta_j + \Sigma\, b_j \partial q/\partial x_j = 0$ when $q = 0$ we obtain $|L^0 q| \leqslant C|q|\,|\theta|^{-1}$ if L^0 is the principal part of L, and this leads easily to a proof that $L(1/q(x, \theta + i\eta)) \in S^{-m}_{0,1}$. More generally, application of k operators of this type will always give an element in $S^{1-m-k}_{0,1}$.

We have made the preceding discussion rather brief for it has to be reexamined after a precise definition of "almost analytic" continuation has been introduced. However, the preceding arguments should suffice to motivate the interest of the following developments.

Let $\mathcal{L}$ be the set of all first order differential operators of the form (1.3.2) with $a_j \in S^0(X \times \mathbf{R}^N)$ and $b_j, c \in S^{-1}(X \times \mathbf{R}^N)$. This is a module over the ring $S^0(X \times \mathbf{R}^N)$. If $a \in S^m_\varrho(X \times \mathbf{R}^N)$ and $L \in \mathcal{L}$ we have $La \in S^{m-\varrho}_\varrho(X \times \mathbf{R}^N)$ in view of Proposition 1.1.6. By iteration it follows that $L_1 \ldots L_k a \in S^{m-k\varrho}_\varrho(X \times \mathbf{R}^N)$ if $L_1, \ldots, L_k \in \mathcal{L}$. Conversely, if we just know that $L_1 \ldots L_k a \in S^{m-k\varrho}_0(X \times \mathbf{R}^N)$ for all $L_1, \ldots, L_k \in \mathcal{L}$, taking these operators to be differentiations with respect to x_j and θ_i variables we conclude that $a \in S^m_\varrho(X \times \mathbf{R}^N)$. This connects our earlier definitions with the following one.

Definition 1.3.1. If F is a subset of $\mathcal{L}$ we denote by ${}^F S^m_\varrho(X \times \mathbf{R}^N)$ the set of all $a \in S^m_0(X \times \mathbf{R}^N)$ such that for arbitrary $L_1, \ldots, L_k \in F$, $k = 1, 2, \ldots$ we have

$$L_1 \dots L_k a \in S_0^{m-k\varrho}(X \times \mathbf{R}^N). \tag{1.3.3}$$

It is clear by the definition that $La \in {}^F S_\varrho^{m-\varrho}$ if $L \in F$ and $a \in {}^F S_\varrho^m$. If we set $\mathcal{L}^m = S^m \mathcal{L}$ we also have $\mathcal{L}^m \, {}^F S_\varrho^{m'} \subset {}^F S_\varrho^{m+m'}$. For let $L \in \mathcal{L}^m$, $a \in {}^F S_\varrho^{m'}$. Then we have $a \in S_0^{m'}$ so $La \in S_0^{m+m'}$. Furthermore, if $L_j \in F$,

$$L_1 \dots L_k La = LL_1 \dots L_k a + \sum_{j=1}^{k} L_1 \dots L_{j-1} [L_j, L] L_{j+1} \dots L_k a.$$

We shall prove by induction over k that this is in $S_0^{m+m'-k\varrho}$. This is clear for the first term since $L_1 \dots L_k a \in S_0^{m'-k\varrho}$. Since $L_{j+1} \dots L_k a \in {}^F S_\varrho^{m'-(k-j)\varrho}$ and the commutator $[L_j, L] \in \mathcal{L}^{m-1}$, the inductive hypothesis shows that the terms in the sum are in $S^{m-1+m'-(k-j)\varrho-(j-1)\varrho} \subset S_0^{m+m'-k\varrho}$, which proves the assertion. If $b \in S^m$ it follows similarly that $ba \in {}^F S_\varrho^{m'+m}$, for with the same notation we have

$$L_1 \dots L_k ba = bL_1 \dots L_k a + \sum_1^k L_1 \dots L_{j-1} [L_j, b] L_{j+1} \dots L_k a, \tag{1.3.4}$$

and $[L_j, b] \in S^{m-1}$, so the same proof by induction can be applied. It follows in particular that if F' is the S^0 module generated by F and S^{-1}, then ${}^{F'} S_\varrho^m = {}^F S_\varrho^m$. We can therefore always assume without restriction that F is a module containing S^{-1}. If $L_j \in F$ it follows then that the principal part $L_j^0 \in F$, hence $[L_j, b] = L_j^0 b \in {}^F S_\varrho^{m-\varrho}$ if $b \in {}^F S_\varrho^m$. An induction proof based again on (1.3.4) therefore gives that $ba \in {}^F S_\varrho^{m+m'}$ if $b \in {}^F S_\varrho^m$ and $a \in {}^F S_\varrho^{m'}$. Summing up, we have proved

PROPOSITION 1.3.2. *If F' is the S^0 submodule of $\mathfrak{L}$ generated by F and S^{-1} (considered as a set of differential operators of order 0), then ${}^F S_\varrho^m = {}^{F'} S_\varrho^m$. If $a \in {}^F S_\varrho^m$, $b \in {}^F S_\varrho^{m'}$, we have $ab \in {}^F S_\varrho^{m+m'}$, and if $L \in \mathcal{L}$* we have $La \in {}^F S_\varrho^m$.

Note that the proposition shows that $L_1 \dots L_k a \in S^{m-j\varrho}$ if $L_1, \dots, L_k \in \mathfrak{L}$ and j of these operators belong to F. This could also have been taken as a definition and saved much of the proof of Proposition 1.3.2.

If ϕ is a real valued function $\in S^1(X \times \mathbf{R}^N)$ (possibly only for large $|\theta|$) and if for some $L \in F$ we have ${}^t L e^{i\phi} = e^{i\phi}$, we can define the oscillatory integral (1.2.1) for all $a \in {}^F S_\varrho^m(X \times \mathbf{R}^N)$. It is a distribution of order $\leqslant k$ if $m - k\varrho < -N$. Indeed the proof of Proposition 1.2.2 does not require any change.

Example 1.3.3. Let Σ be a closed conic C^∞ submanifold of $X \times (\mathbf{R}^N \setminus \{0\})$ and let F be the set of all $L \in \mathcal{L}$ whose principal part defines a vector field tangential to Σ at every point in Σ. (Equivalently, if L^0 is the principal part of an operator in F, then $L^0 a$ vanishes on Σ if a does. Thus we are considering a generalization of the situation discussed at the beginning of the section.) Then there exists an operator $L \in F$ with the required properties if and only if *neither* ϕ *nor the restriction of* ϕ *to* Σ *has a critical point.* Indeed, as in the proof of Lemma 1.2.1 we have $L \in F$ if and only if $M = {}^tL \in F$, and if we write $M = \Sigma\, a_j \partial/\partial\theta_j + \Sigma\, b_j \partial/\partial x_j + c$ the problem is to choose the vector field (a, b) tangential to Σ on Σ and such that

$$i(\Sigma\, a_j \partial\phi/\partial\theta_j + \Sigma\, b_j \partial\phi/\partial x_j) + c = 1. \tag{1.3.5}$$

Since $a_j \in S^0$ and $b_j,\, c \in S^{-1}$ this requires that ϕ has no critical point and that the restriction to Σ has no critical point. Conversely, when this is true we can find a solution of (1.3.5) in a conic neighborhood of any point, such that $c = 0$ and a_j, b_j are homogeneous of degree 0 and -1 respectively, by just choosing a vector field tangential to Σ which does not annihilate ϕ at the point in question. Application of a partition of unity, introducing the solution $c = 1$, $a_j = b_j = 0$ near 0, then gives the assertion.

A special case is obtained when $\phi(x, \theta) = \langle x, \theta\rangle$ and for a splitting of the θ variables in $\theta' = (\theta_1, ..., \theta_{N-k})$, $\theta'' = (\theta_{N-k+1}, ..., \theta_N)$ we have $\Sigma = \{(x, \theta', \theta''); (\theta', \theta'') \neq 0 \text{ and } \theta' = 0 \text{ or } \theta'' = 0\}$. This occurs in the study of the multiplicative properties of the index of elliptic pseudo-differential operators (cf. Palais [26], pp. 206–209).

We shall use the notation ${}^{\Sigma}S^m_\varrho$ instead of ${}^{F}S^m_\varrho$ when F is defined by Σ as in Example 1.3.3. This should cause no ambiguity.

When $a \in {}^{\Sigma}S^m_\varrho$, the singularities of the distribution defined by (1.2.1) may be caused either by points with $\phi'_\theta = 0$ or by points in Σ with $\phi'_\theta \neq 0$. We shall investigate the latter contributions. In doing so we assume that the manifold Σ is transversal to the fibers $x = \text{constant}$ so that the sets $\Sigma_x = \{\theta; (x, \theta) \in \Sigma\}$ are manifolds of the same codimension as Σ at every point. Let $(x_0, \theta_0) \in \Sigma$, $\phi'_\theta(x_0, \theta_0) \neq 0$, and let k be the codimension of Σ there. Choose a labelling of the θ coordinates so that with $\theta' = (\theta_1, ..., \theta_{N-k})$ and $\theta'' = (\theta_{N-k+1}, ..., \theta_N)$ the plane $dx = d\theta' = 0$ is transversal to Σ. Then we have $\theta'_0 \neq 0$, for Σ_{x_0} being a cone, the vector $(0, \theta''_0)$ would otherwise lie in the tangent plane of Σ_{x_0} at θ_0. In a neighborhood of (x_0, θ_0) the manifold Σ is therefore of the form $\theta'' = \psi(x, \theta')$ where ψ is homogeneous of degree 1 with respect to θ' and defined in a conical neighborhood of θ'_0. In the integral

$$\langle A, u\rangle = \iint e^{i\phi(x,\theta)} a(x, \theta)\, u(x)\, dx\, d\theta$$

we assume that the support of a belongs to such a small conic neighborhood of (x_0, θ_0) that we can introduce $\theta'' - \psi(x, \theta')$ as a new variable instead of θ'' there. This transforms Σ to the manifold Σ^0: $\theta'' = 0$. The new amplitude $a_1(x, \theta) = a(x, \theta', \theta'' + \psi(x, \theta'))$ will belong to

${}^{\Sigma^0}S^m_\varrho$ and vanish when $|\theta''|>|\theta'|$, say. Let F^0 be the corresponding subspace of $\mathcal{L}$ defined in Example 1.3.3. The operators in F^0 are then those which when $\theta''=0$ do not involve differentiation with respect to the θ'' variables.—The new phase function $\phi_1(x,\theta)=\phi(x,\theta',\theta''+\psi(x,\theta'))$ will be defined in a conic neighborhood of cone supp a_1, we have $\phi'_{1\theta}(x_0,\theta'_0,0)\neq 0$ and

$$\langle A,u\rangle=\iint e^{i\phi_1(x,\theta)}a_1(x,\theta)\,u(x)\,dx\,d\theta,$$

If $\phi'_{1\theta'}(x_0,\theta'_0,0)\neq 0$ and the support of a is so small that this inequality remains valid in cone supp a_1, we can prove that $A\in C^\infty$ by using the proof of Prop. 1.2.3. In fact, the derivatives of a_1 with respect to the θ' variables are as well behaved as if a_1 were in S^m_ϱ. If, on the other hand, $\phi'_{1\theta'}(x_0,\theta'_0,0)=0$ we have instead $\phi'_{1\theta''}(x_0,\theta'_0,0)\neq 0$ and we may assume that this is true in a neighborhood of cone supp a_1. Now we can write

$$\langle A,u\rangle=\iint e^{i\phi_1(x,\theta',0)}b(x,\theta')\,u(x)\,dx\,d\theta',$$

where
$$b(x,\theta')=\int e^{i(\phi_1(x,\theta',\theta'')-\phi_1(x,\theta',0))}a_1(x,\theta',\theta'')\,d\theta''.$$

Here we may assume that $a_1=0$ when $|\theta'|<1$, for modification of a_1 on a compact set only changes A by a C^∞ term. We wish to prove that $b\in S^{m+k}_\varrho$. Since $|\theta'|\leqslant|\theta|<2|\theta'|$ in cone supp a_1 and $|a_1(x,\theta)|\leqslant C(1+|\theta|)^m$, it is clear that $|b(x,\theta')|\leqslant C(1+|\theta'|)^{m+k}$. To estimate the derivatives we note that

$$\partial(\phi_1(x,\theta',\theta'')-\phi_1(x,\theta',0))/\partial x_j=\sum_{N-k+1}^{N}a_{jil}(x,\theta)\,\theta_i\partial\phi_1(x,\theta)/\partial\theta_l=|\theta'|\,L_j\phi_1,$$

where $L_j\in F^0$. A similar result without the factor $|\theta'|$ is valid for the derivative with respect to θ_j when $j\leqslant N-k$. Now we obtain after an integration by parts

$$\partial b(x,\theta')/\partial x_j=\int e^{i(\phi_1(x,\theta',\theta'')-\phi_1(x,\theta',0))}|\theta'|\,({}^tL_j+|\theta'|^{-1}\partial/\partial x_j)\,a_1(x,\theta)\,d\theta''$$

and a similar formula for $\partial b(x,\theta')\partial\theta_j$ when $j\leqslant N-k$. Here ${}^tL_j+|\theta'|^{-1}\partial/\partial x_j$ belongs to F^0 (after suitable modification when $|\theta'|<1$). Application of $\partial/\partial\theta_j$, $j=1,\ldots,N-k$, or $\partial/\partial x_j$ to b is thus equivalent to operation on a_1 by an operator in F^0 followed by multiplication by $|\theta'|$ in the case of $\partial/\partial x_j$. This gives immediately that $b\in S^{m+k}_\varrho$. Summing up, we have proved

THEOREM 1.3.4. *Let $a\in{}^{\Sigma}S^m_\varrho$ vanish in a conic neighborhood of the set where $\phi'_\theta=0$; assume that in a neighborhood of* cone supp a *the restriction of ϕ to Σ has no critical point and that the manifold Σ can be expressed in the form $\theta''=\psi(x,\theta')$ where $\theta'=(\theta_1,\ldots,\theta_{N-k})$ and $\theta''=(\theta_{N-k+1},\ldots,\theta_N)$. Then the distribution A defined by*

$$\langle A, u\rangle = \iint e^{i\phi(x,\theta)} a(x,\theta)\, u(x)\, dx\, d\theta, \quad u \in C_0^\infty(X)$$

can also be defined by means of the phase function $\phi(x, \theta', \psi(x, \theta'))$ *and an amplitude* $b \in S_\varrho^{m+k}$.

From the preceding result one should not conclude that there is no need to consider singular symbols. The new features are caused by points in the set Σ where $\phi'_\theta = 0$. We leave for the reader to construct an example of this for example by means of the special case of Example 1.3.3 mentioned at the end of it. We shall also encounter natural examples in part II.

As in section 1.2 it is easy to extend the preceding discussion to operators depending on parameters. Indeed, let ϕ_t be a continuous function of a parameter $t \in T$ with values in $C^\infty(X \times (\mathbf{R}^N \setminus \{0\}))$ and let Σ_t be a conic submanifold of $X \times (\mathbf{R}^N \setminus \{0\})$ which also depends continuously on t. This means that locally in $T \times X \times (\mathbf{R}^N \setminus \{0\})$ we can define Σ_t by equations $q_t^1(x, \theta) = \ldots = q_t^k(x, \theta) = 0$ where q^j is a continuous function of t whose values are C^∞ functions of (x, θ) with the differentials of $q^1, \ldots, q^k$ linearly independent. We assume that neither Σ_t nor the restriction of ϕ_t to Σ_t has any critical point. Let F be the set of all continuous maps L: $T \to \mathcal{L}$ such that the corresponding vector field is tangential to Σ_t for every t. It is easy to see that one can choose $L \in F$ so that $L_t \exp i\phi_t = \exp i\phi_t$ for every t. Defining ${}^\Sigma S_\varrho^m$ in the obvious way using the operators in F, we conclude that

$$\iint e^{i\phi_t(x,\theta)} a_t(x,\theta)\, u(x)\, dx\, d\theta, \quad u \in C_0^\infty(X),$$

is a continuous function of t. Again this allows passage to the limit under the integral sign, differentiation with respect to parameters and so on.

1.4. Definition of Fourier integral operators

Let X, Y be open sets in $\mathbf{R}^{n_X}$ and $\mathbf{R}^{n_Y}$ and let ϕ be a real valued function of $(x, y, \theta) \in X \times Y \times \mathbf{R}^N$ which is positively homogeneous of degree 1 with respect to θ and infinitely differentiable for $\theta \neq 0$. With a symbol $a \in S_{\varrho,\delta}^m(X \times Y \times \mathbf{R}^N)$, $\varrho > 0$, $\delta < 1$, we wish to consider the operator defined by the integral

$$Au(x) = \iint e^{i\phi(x,y,\theta)} a(x,y,\theta)\, u(y)\, dy\, d\theta, \quad u \in C_0^\infty(Y), x \in X, \tag{1.4.1}$$

or a weak form of (1.3.1)

$$\langle Au, v\rangle = \iiint e^{i\phi(x,y,\theta)} a(x,y,\theta)\, v(x)\, u(y)\, dx\, dy\, d\theta, \quad u \in C_0^\infty(Y), v \in C_0^\infty(X). \tag{1.4.2}$$

To give a meaning to these integrals we apply the results on oscillatory integrals (depending on parameters in the case (1.4.1)) proved in section 1.2. The conclusions are as follows:

THEOREM 1.4.1. (i) *If ϕ has no critical point as a function of (x, y, θ) with $\theta \neq 0$ then the oscillatory integral* (1.4.2) *exists and is a continuous bilinear form for the C_0^k topologies on u, v if*

$$m - k\varrho < -N, \quad m - k(1-\delta) < -N. \tag{1.4.3}$$

When (1.4.3) *is valid we thus obtain a continuous linear map A from $C_0^k(Y)$ to $\mathcal{D}'^k(X)$ which has a distribution kernel $K_A \in \mathcal{D}'^k(X \times Y)$ given by the oscillatory integral*

$$K_A(u) = \iiint e^{i\phi(x,y,\theta)} a(x,y,\theta)\, u(x,y)\, dx\, dy\, d\theta, \quad u \in C_0^\infty(X \times Y). \tag{1.4.4}$$

(ii) *If for each fixed x the function ϕ has no critical point (y, θ) with $\theta \neq 0$, then* (1.4.1) *is defined as an oscillatory integral. When* (1.4.3) *is valid we obtain a continuous map $A: C_0^k(Y) \to C(X)$. By differentiation under the integral sign it follows that A is also a continuous map from $C_0^k(Y)$ to $C^j(X)$ if*

$$m + N + j < k\varrho, \quad m + N + j < k(1-\delta). \tag{1.4.5}$$

(iii) *If for each fixed y the function ϕ has no critical point (x, θ) with $\theta \neq 0$, then the adjoint of A has the properties listed in* (ii) *so A is a continuous map of $\mathcal{E}'^j(Y)$ into $\mathcal{D}'^k(X)$ when* (1.4.5) *is fulfilled. In particular, A defines a continuous map from $\mathcal{E}'(Y)$ to $\mathcal{D}'(X)$.*

(iv) *Let R_ϕ be the open set of all $(x, y) \in X \times Y$ such that $\phi(x, y, \theta)$ has no critical point $\theta \neq 0$ as a function of θ. Then the oscillatory integral*

$$K_A(x,y) = \int e^{i\phi(x,y,\theta)} a(x,y,\theta)\, d\theta, \quad (x,y) \in R_\phi, \tag{1.4.6}$$

defines a function in $C^\infty(R_\phi)$ which is equal to the distribution (1.4.4) *in R_ϕ. If $R_\phi = X \times Y$, it follows that A is an integral operator with a C^∞ kernel, so A is a continuous map of $\mathcal{E}'(Y)$ into $C^\infty(X)$.*

The proof is an immediate consequence of Proposition 1.2.2 and the remarks following it.

Example 1.4.2. Pseudo-differential operators correspond to the function $\phi(x, y, \theta) = \langle x - y, \theta \rangle$ $(n_X = n_Y = N)$. Then (i), (ii), (iii) are fulfilled and R_ϕ is the complement of the diagonal if we take $X = Y$. We shall study this case extensively in Chapter II.

Example 1.4.3. In the introduction we saw that the study of the Cauchy problem for the wave equation leads to the function

$$\phi(x, t; y, \theta) = \langle x-y, \theta\rangle + t|\theta|.$$

Here $n_X - 1 = n_Y = N$ and the variable in X is denoted by (x, t). Then (i), (ii), (iii) are fulfilled and $\complement R_\phi$ consists of all $(x, t; y)$ with $|x-y|^2 = t^2$. This means that (x, t) lies on the light cone with vertex at $(y, 0)$.

Definition 1.4.4. A real valued function ϕ of $(x, y, \theta) \in X \times Y \times \mathbf{R}^N$ which is a C^∞ function for $\theta \neq 0$ and positively homogeneous of degree 1 with respect to θ will be called an operator phase function if for each fixed x (or y) it has no critical point (y, θ) (or (x, θ)) with $\theta \neq 0$.

When ϕ is an operator phase function the hypotheses of parts (i), (ii), (iii) of Theorem 1.4.1 are thus fulfilled. Let C_ϕ denote the complement of R_ϕ in $X \times Y$, that is, the projection on $X \times Y$ of the conic set

$$C = \{(x, y, \theta) \in X \times Y \times (\mathbf{R}^N \smallsetminus \{0\}),\ \phi'_\theta(x, y, \theta) = 0\}. \tag{1.4.7}$$

From (iv) in Theorem 1.4.1 it follows then that

$$\operatorname{sing\,supp} Au \subset C_\phi \operatorname{supp} u,\quad u \in \mathscr{E}'(Y) \tag{1.4.8}$$

where the right-hand side is defined by considering C_ϕ as a relation between points in Y and in X, thus

$$C_\phi K = \{x;\ (x, y) \in C_\phi \text{ for some } y \in K\}.$$

In fact, if $K = \operatorname{supp} u$ and K' is a compact subset of X which does not intersect $C_\phi K$, we have $K' \times K \subset R_\phi$ so we can find neighborhoods $\Omega' \supset K'$, $\Omega \supset K$ such that $\Omega' \times \Omega \subset R_\phi$. Hence $Au \in C^\infty(\Omega')$ which proves (1.4.8). Using (ii) in Theorem 1.4.1 we can improve (1.4.8) further. For if Ω is any neighborhood of sing supp u, we can make a decomposition $u = v + w$ where supp $v \subset \Omega$ and $w \in C^\infty$. Since $Aw \in C^\infty$ we obtain

$$\operatorname{sing\,supp} Au = \operatorname{sing\,supp} Av \subset C_\phi \operatorname{supp} v,$$

so we have proved

THEOREM 1.4.5. *If* $u \in \mathscr{E}'(Y)$, *then*

$$\operatorname{sing\,supp} Au \subset C_\phi \operatorname{sing\,supp} u. \tag{1.4.9}$$

Example 1.4.6. For pseudo-differential operators (see Example 1.4.2) this means that

$$\operatorname{sing\,supp} Au \subset \operatorname{sing\,supp} u,$$

which is usually called the pseudo-local property.

Note that Propositions 1.2.4 and 1.2.5 can be applied to show that modulo operators with C^∞ kernel the operator A is determined by the behavior of the symbol a at the set C defined by (1.4.7).

If we replace the results of section 1.2 by those of section 1.3 we obtain an analogue of Theorem 1.4.1 for operators with singular symbols. Let Σ be a C^∞ conic submanifold of $X \times Y \times (\mathbf{R}^N \setminus \{0\})$, and let $a \in {}^\Sigma S^m_\varrho(X \times Y \times \mathbf{R}^N)$.

THEOREM 1.4.7. (i) *If neither ϕ nor its restriction to Σ has a critical point (x, y, θ) with $\theta \neq 0$ then the oscillatory integral* (1.4.2) *exists and is a continuous bilinear form for the C^k_0 topologies on u, v if*

$$m - k\varrho < -N. \tag{1.4.10}$$

When (1.4.10) *is valid we thus obtain a continuous linear map A from $C^k_0(Y)$ to $\mathcal{D}'^k(X)$ which has a distribution kernel $K_A \in \mathcal{D}'(X \times Y)$ given by the oscillatory integral* (1.4.4).

(ii) *Assume that the fibers $x =$ constant intersect Σ transversally so that the corresponding intersection Σ_x is a manifold of the same codimension as Σ at every point. If for each fixed x the function ϕ as well as its restriction to Σ_x has no critical point (y, θ) with $\theta \neq 0$, then* (1.4.1) *is defined as an oscillatory integral. When* (1.4.10) *is valid we obtain a continuous map A: $C^k_0(Y) \to C(X)$. By differentiation under the integral sign it follows that A is also a continuous map from $C^k_0(Y)$ to $C^j(X)$ if*

$$m + N + j < k\varrho. \tag{1.4.11}$$

(iii) *Assume that the fibers $y =$ constant intersect Σ transversally so that the corresponding intersection Σ'_y is a manifold of the same codimension as Σ at every point. If for each fixed y the function ϕ as well as its restriction to Σ'_y has no critical point (x, θ) with $\theta \neq 0$, then the adjoint of A has the properties listed in* (ii) *so A is a continuous map from $\mathcal{E}'^j(Y)$ to $\mathcal{D}'^k(X)$ when* (1.4.11) *is fulfilled. In particular, A defines a continuous map from $\mathcal{E}'(Y)$ to $\mathcal{D}'(X)$.*

(iv) *Let R_ϕ be the open set of all $(x, y) \in X \times Y$ such that Σ intersects the fiber over (x, y) transversally in $\Sigma_{x,y}$ and neither ϕ nor its restriction to $\Sigma_{x,y}$ has a critical point $\theta \neq 0$. Then the oscillatory integral* (1.4.6) *defines a function in $C^\infty(R_\phi)$ which is equal to the distribution* (1.4.4) *in R_ϕ.*

If we let C_ϕ be the complement of R_ϕ it follows again, when ϕ satisfies the hypotheses in (ii) and (iii), that (1.4.9) is valid.

Example 1.4.8. Let $q \in C^\infty(X \times Y \times (\mathbf{R}^N \setminus \{0\})$ be homogeneous with respect to the last variable, grad $q \neq 0$ when $q = 0$, and let Σ be the zeros of q. Then the conditions in Theorem 1.4.7 reduce to

grad $\phi \neq 0$ and the covectors grad ϕ, grad q are linearly independent when $q = 0$.

Here gradients are to be taken with respect to (x, y, θ) in condition (i), (y, θ) in condition (ii), (x, θ) in condition (iii) and θ in the definition of R_ϕ in (iv). In particular, let $\phi(x, y, \theta) = \langle x-y, \theta\rangle$ and assume that $\operatorname{grad}_\theta q \neq 0$ when $q=0$. The conditions (i), (ii), (iii) are then fulfilled over a neighborhood of the diagonal.

II. Pseudo-differential operators and related classes of distributions

2.0. Introduction

In this chapter we shall review the calculus of pseudo-differential operators using the techniques developed in Chapter I. This leads to several simplifications. In particular, in this framework it is easy to prove the invariance of pseudo-differential operators using an idea of Kuranishi (see also Friedrichs [10] and Nirenberg [24]). The simplicity of this proof also allows one to give a direct global definition of pseudo-differential operators on a manifold which is particularly useful when one has to discuss operators depending on parameters. A related definition has been discussed by Bokobza [4].

The kernel of a pseudo-differential operator in X has its singular support in the diagonal of $X \times X$. In section 2.4 we shall see that oscillatory integrals with a linear phase function lead one to attach to every submanifold Y of a manifold X a class of distributions with singular support in Y. With these we can associate a "principal symbol". The discussion of these distributions and indeed the whole chapter is to a large extent a preparation for Chapters III and IV where the corresponding questions for general non-degenerate phase functions will be studied.

M. Sato [27] has introduced for his hyperfunctions on a manifold X a closed subset of the cosphere bundle which describes not only the location of the singularities but also some of their harmonic analysis. His definition involves a considerable amount of cohomological machinery but for a distribution one can use pseudo-differential operators to give a very simple definition of a set with similar properties. We call it the wave front set of the distribution. Section 2.5 is devoted to this concept which clarifies much of the contents of this paper and will be indispensable in part II.

2.1. The calculus of pseudo-differential operators

If X is an open set in $\mathbf{R}^n$ we shall write $L^m_{\varrho,\delta}(X)$ for the class of Fourier integral operators (1.4.1) with the phase function $\phi(x, y, \theta) = \langle x-y, \theta\rangle$ in $X \times X \times \mathbf{R}^n$ and a symbol $a \in S^m_{\varrho,\delta}(X \times X \times \mathbf{R}^n)$. As we shall see in a moment, this agrees with the definitions given in [13] at least when $\delta < \varrho$, so we shall call these operators pseudo-differential of type ϱ, δ.

A pseudo-differential operator

$$Au(x) = \iint e^{i\langle x-y,\theta\rangle} a(x,y,\theta)\, u(y) dy\, d\theta \tag{2.1.1}$$

is called *properly supported* if both projections supp $K_A \to X$ are proper, that is, if

$$\{(x, y) \in \operatorname{supp} K_A;\ x \in K \text{ or } y \in K\}$$

is compact for every compact set $K \subset X$. (Recall that K_A is the distribution kernel of A given by (1.4.4).) It is clear that Au can then be defined without restrictions on the support of u. Thus A maps $C^\infty(X)$ into $C^\infty(X)$ and $\mathcal{D}'(X)$ into $\mathcal{D}'(X)$. Furthermore, A maps $C_0^\infty(X)$ into $C_0^\infty(X)$ and $\mathcal{E}'(X)$ into $\mathcal{E}'(X)$. If χ is a function in $C^\infty(X \times X)$ which is equal to 1 in a neighborhood of supp K_A and the projections supp $\chi \to X$ are also proper, it is evident that the operator A is also defined by the symbol $a_1(x, y, \theta) = \chi(x, y) a(x, y, \theta)$. Note that

$$\{(x, y);\ x \text{ or } y \in K \text{ and } (x, y, \theta) \in \operatorname{supp} a_1 \text{ for some } \theta\}$$

is then relatively compact in $X \times X$ for every compact set $K \subset X$. We shall say that a symbol with this property is *proper*. Every pseudo-differential operator is the sum of one with a C^∞ kernel and one which is properly supported. This follows immediately if we choose $\chi \in C^\infty(X \times X)$ so that $\chi = 1$ in a neighborhood of the diagonal and χ is properly supported. In fact, the symbol $(1-\chi)a$ defines an operator with C^∞ kernel according to Proposition 1.2.4, and χa is a proper symbol.

When $\delta < \varrho$ we shall now derive an expression for a properly supported pseudo-differential operator A which will connect the definition used here with that given in [13]. Thus let A be defined by (2.1.1) where a is proper. We may then consider A as an operator from $C^\infty(\mathbf{R}^n)$ to $C^\infty(X)$—which of course strictly speaking is the composition of the restriction map to X and A. Applying A to the exponential function $e_\eta(y) = \exp i\langle y, \eta\rangle$, we obtain $A\, e_\eta(x) = \sigma_A(x, \eta)\, e_\eta(x)$ where

$$\sigma_A(x,\eta) = \iint a(x,y,\theta)\, e^{i\langle x-y,\theta\rangle + i\langle y-x,\eta\rangle} dy\, d\theta = \iint a(x, x+y, \eta+\theta)\, e^{-i\langle y,\theta\rangle} dy\, d\theta.$$

The oscillatory integral here may be interpreted as a repeated integral taken first with respect to y and then with respect to θ. We set

$$b(x, y, \eta) = a(x, x+y, \eta)$$

and introduce the Fourier transform

$$\hat{b}(x,\theta,\eta)=\int b(x,y,\eta)\,e^{-i\langle y,\theta\rangle}dy.$$

Since $a\in S^m_{\varrho,\delta}$ and a is proper, we obtain for every compact set $K\subset X$

$$|D_x^\alpha\theta^\beta D_\eta^\gamma\hat{b}(x,\theta,\eta)|\leqslant C(1+|\eta|)^{m+\delta(|\alpha|+|\beta|)-\varrho|\gamma|},\quad x\in K,\tag{2.1.2}$$

hence for any positive integer ν

$$|D_x{}^\alpha D_\eta^\gamma\hat{b}(x,\theta,\eta)|\leqslant C(1+|\eta|)^{m+\delta(|\alpha|+\nu)-\varrho|\gamma|}(1+|\theta|)^{-\nu}.\tag{2.1.3}$$

Now we have
$$\sigma_A(x,\eta)=\int\hat{b}(x,\theta,\eta+\theta)\,d\theta.$$

Since $\delta<1$, it follows from (2.1.3) that σ_A and any one of its derivatives can be bounded by some power of $(1+|\eta|)$. To obtain the asymptotic behavior of σ_A when $\eta\to\infty$ we form the Taylor expansion of $\hat{b}(x,\theta,\eta+\theta)$. In view of (2.1.3) we have

$$\Big|\hat{b}(x,\theta,\eta+\theta)-\sum_{|\alpha|<N}(iD_\eta)^\alpha\hat{b}(x,\theta,\eta)\,\theta^\alpha/\alpha!\Big|\leqslant C|\theta|^N\sup_{0<t<1}(1+|\eta+t\theta|)^{m+\delta\nu-\varrho N}(1+|\theta|)^{-\nu}.$$

Here ν may be chosen as any positive integer or 0. With $\nu=N$ we obtain the bound $C(1+|\eta|)^{m+(\delta-\varrho)N}$ if $|\theta|<|\eta|/2$, and if we choose ν large we get a bound by any power of $(1+|\theta|)^{-1}$ if $|\eta|<2|\theta|$. Hence Fourier's inversion formula gives

$$\Big|\sigma_A(x,\eta)-(2\pi)^n\sum_{|\alpha|\leqslant N}(iD_\eta)^\alpha D_y^\alpha b(x,y,\eta)/\alpha!\,|_{y=0}\Big|\leqslant C(1+|\eta|)^{m+n+(\delta-\varrho)N}.$$

In view of Proposition 1.1.10 it follows that $\sigma_A\in S^m_{\varrho,\delta}(X\times\mathbf{R}^n)$ and that

$$\sigma_A(x,\eta)\sim(2\pi)^n\sum(iD_\eta)^\alpha D_y^\alpha a(x,y,\eta)/\alpha!\,|_{y=x}.\tag{2.1.4}$$

If $u\in\mathcal{S}(\mathbf{R}^n)$ we have Fourier's inversion formula

$$u(y)=(2\pi)^{-n}\int e_\eta(y)\,\hat{u}(\eta)\,d\eta.$$

Since A is continuous from $C^\infty(\mathbf{R}^n)$ to $C^\infty(X)$, we can apply A under the sign of integration and obtain

$$Au(x)=(2\pi)^{-n}\int e^{i\langle x,\eta\rangle}\sigma_A(x,\eta)\,\hat{u}(\eta)\,d\eta,\quad u\in\mathcal{S}(\mathbf{R}^n),x\in X.\tag{2.1.5}$$

In the left-hand side we should interpret u as the restriction of u to X. Obviously (2.1.5) determines σ_A uniquely. Summing up, we have proved

THEOREM 2.1.1. *If A is a properly supported operator $\in L^m_{\varrho,\delta}(X)$, $\delta<\varrho$, then A can be written in one and only one way in the form* (2.1.5). *Here $\sigma_A \in S^m_{\varrho,\delta}(X\times\mathbf{R}^n)$ is asymptotically given by* (2.1.4) *and is called the symbol of A.*

Incidentally we note that for the phase function $\phi(x,y,\theta)=\langle x-y,\theta\rangle$ we have now given another proof of Proposition 1.2.5.

Conversely, every operator of the form (2.1.5) with $\sigma_A\in S^m_{\varrho,\delta}$ is in $L^m_{\varrho,\delta}(X)$ by our present definition, for (2.1.5) is equivalent to

$$Au(x)=(2\pi)^{-n}\iint e^{i\langle x-y,\eta\rangle}\sigma_A(x,\eta)\,u(y)\,dy\,d\eta,\quad u\in C_0^\infty\mathbf{R}^n). \tag{2.1.5$'$}$$

Hence the definitions used here are equivalent to those used in [13] when $\delta<\varrho$, which we assume from now on. If we note that $\sigma_A\in S^{-\infty}$ if and only if $K_A\in C^\infty$, the preceding theorem shows that the map $A\to\sigma_A$ defined there together with the map $\sigma_A\to A$ given by (2.1.5) leads to an isomorphism

$$L^m_{\varrho,\delta}/L^{-\infty}_{\varrho,\delta}\to S^m_{\varrho,\delta}/S^{-\infty}_{\varrho,\delta}.$$

We shall call $a(x,\theta)\in S^m_{\varrho,\delta}(X\times\mathbf{R}^n)$ a symbol of $A\in L^m_{\varrho,\delta}(X)$ when their equivalence classes correspond in this isomorphism.

The formula for the symbol of the transpose of A given for example in [13] is an immediate consequence of Theorem 2.1.1. Indeed, if A is properly supported and we define the transpose of A by $\langle Au,v\rangle=\langle u,{}^tAv\rangle$, then we obtain from (2.1.5)$'$

$${}^tAv(x)=(2\pi)^{-n}\iint e^{-i\langle x-y,\eta\rangle}\sigma_A(y,\eta)\,v(y)\,dy\,d\eta$$

or

$${}^tAv(x)=(2\pi)^{-n}\iint e^{i\langle x-y,\eta\rangle}\sigma_A(y,-\eta)\,v(y)\,dy\,d\eta.$$

Since ${}^tA$ is properly supported it follows from Theorem 2.1.1 that the symbol is given by

$$\sigma_{{}^tA}(x,\eta)\sim\sum(iD_\eta)^\alpha D_x^\alpha\sigma_A(x,-\eta)/\alpha!. \tag{2.1.6}$$

The formula extends immediately to the symbols for arbitrary $A\in L^m_{\varrho,\delta}$.

Using the adjoint operator we can also get another useful representation for a properly supported pseudo-differential operator, already used by Kohn and Nirenberg [19]. In fact, if

$${}^tAv(y)=(2\pi)^{-n}\int e^{i\langle y,\xi\rangle}\sigma'(y,\xi)\,\hat v(\xi)\,d\xi,$$

then

$$\langle Au,v\rangle=(2\pi)^{-n}\iint e^{i\langle y,\xi\rangle}\sigma'(y,\xi)\,\hat v(\xi)\,u(y)\,d\xi\,dy,$$

which means that Au is the Fourier transform of

$$\xi \to (2\pi)^{-n} \int e^{i\langle y,\xi\rangle} \sigma'(y,\xi)\, u(y)\, dy.$$

Writing $\tilde{\sigma}_A(y,\xi) = \sigma'(y,-\xi)$, we have then with the notation of oscillatory integrals

$$Au(x) = (2\pi)^{-n} \iint e^{i\langle x-y,\xi\rangle} \tilde{\sigma}_A(y,\xi)\, u(y)\, dy\, d\xi$$

or equivalently
$$\widehat{(Au)}(\xi) = \int e^{-i\langle y,\xi\rangle} \tilde{\sigma}_A(y,\xi)\, u(y)\, dy. \tag{2.1.7}$$

If A is of the form (2.1.1), then

$$\tilde{\sigma}_A(y,\xi) \sim (2\pi)^n \sum (-iD_\xi)^\alpha D_x^\alpha a(x,y,\xi)/\alpha!\,|_{x=y}. \tag{2.1.8}$$

Now we shall compose A with another properly supported pseudo-differential operator B. Using (2.1.5) for B and (2.1.7) for A, we obtain

$$BAu(x) = (2\pi)^{-n} \iint e^{i\langle x-y,\xi\rangle} \sigma_B(x,\xi)\, \tilde{\sigma}_A(y,\xi)\, u(y)\, dy\, d\xi, \quad x \in X,\, u \in C_0^\infty(X)$$

which proves that BA is a pseudo-differential operator. For the symbol we have by (2.1.4)

$$\sigma_{BA}(x,\eta) \sim \sum (iD_\eta)^\alpha D_y^\alpha\, \sigma_B(x,\eta) \tilde{\sigma}_A(y,\eta)/\alpha!\,|_{y=x}.$$

In view of (2.1.8) we can introduce here

$$\tilde{\sigma}_A(y,\eta) \sim \sum (-iD_\eta)^\beta D_y^\beta\, \sigma_A(y,\eta)/\beta!,$$

which gives

$$\begin{aligned}\sigma_{BA}(x,\eta) &\sim \sum (iD_\eta)^\alpha D_y^\alpha\, \sigma_B(x,\eta)\, (-iD_\eta)^\beta D_y^\beta\, \sigma_A(y,\eta)/\alpha!\,\beta!\,|_{y=x} \\ &= \sum (iD_\eta)^\alpha \sigma_B(x,\eta)\, (-iD_\eta)^\beta D_x^{\alpha+\beta} \sigma_A(x,\eta)/\alpha!\,\beta!.\end{aligned}$$

The right-hand side can be simplified by means of the binomial theorem

$$\sum_{\alpha+\beta=\gamma} \eta^\alpha \theta^\beta/\alpha!\,\beta! = (\eta+\theta)^\gamma/\gamma!$$

if we note as in the proof of Leibniz' formula that a factor iD_η to the left of a product is equivalent to the sum of a factor iD_η acting only on the first factor and one acting only on the second factor. This gives the familiar result

$$\sigma_{BA}(x,\eta) \sim \sum ((iD_\eta)^\gamma \sigma_B(x,\eta))\, D_x^\gamma\, \sigma_A(x,\eta)/\gamma!. \tag{2.1.9}$$

Obviously this remains true if only one of the operators A, B is properly supported.

We shall now consider the effect of a change of variables. Let

$$\varkappa\colon X \to X_1$$

be a diffeomorphism between open sets in $\mathbf{R}^n$ with inverse $\varkappa_1$, let $A \in L^m_{\varrho,\delta}(X)$ and set

$$A_1 u = (A(u \circ \varkappa)) \circ \varkappa_1, \quad u \in C_0^\infty(X_1).$$

This means, if A is of the form (2.1.1), that

$$A_1 u(x) = \iint e^{i\langle \varkappa_1(x)-y,\theta\rangle} a(\varkappa_1(x), y, \theta)\, u(\varkappa(y))\, dy\, d\theta$$

or after a change of variables

$$A_1 u(x) = \iint e^{i\phi(x,y,\theta)} a(\varkappa_1(x), \varkappa_1(y), \theta)\, |D\varkappa_1(y)/Dy|\, u(y)\, dy\, d\theta,$$

where $\phi(x, y, \theta) = \langle \varkappa_1(x) - \varkappa_1(y), \theta\rangle$ and $D\varkappa_1(y)/Dy = \det \varkappa_1'(y)$. This is again a Fourier integral operator but the phase function has been changed. That $A_1 \in L^m_{\varrho,\delta}(X_1)$ follows for suitable ϱ and δ from

THEOREM 2.1.2. *Let ϕ be a phase function in $X \times X \times \mathbf{R}^n$ such that $\phi(x, y, \theta)$ is a linear function of θ and $\phi'_\theta(x, y, \theta) = 0$ is equivalent to $x = y$. Every operator of the form* (1.4.1) *with $a \in S^m_{\varrho,\delta}(X \times X \times \mathbf{R}^n)$ is then in $L^m_{\varrho,\delta}(X)$ if $1-\varrho \leqslant \delta < \varrho$.*

As mentioned in the introduction we shall give a simple proof suggested by Kuranishi. The main point is the following

PROPOSITION 2.1.3. *Let ϕ be a phase function satisfying the hypotheses of Theorem 2.1.2. For some neighborhood Ω of the diagonal in $X \times X$ one can then find a C^∞ map $\psi\colon \Omega \to \mathrm{GL}(n, \mathbf{R})$ such that*

$$\phi(x, y, \psi(x, y)\theta) = \langle x - y, \theta\rangle, \quad (x, y) \in \Omega. \tag{2.1.10}$$

We have $\det \psi(x, x) \det \phi''_{x\theta}(x, y, \theta)\,|_{y=x} = 1$.

Proof. The hypothesis means that

$$\phi(x, y, \theta) = \Sigma\, \phi_j(x, y)\theta_j,$$

where $\phi_j(x, x) = 0$, but, since ϕ is a phase function,

$$\Sigma\, \theta_j \partial\phi_j(x, y)/\partial x_k \neq 0$$

for some k if $x = y$ and $\theta \neq 0$. Thus $\det(\partial\phi_j(x, y)/\partial x_k) \neq 0$ when $x = y$. By Taylor's formula we have

$$\phi_j(x, y) = \Sigma\, \phi_{kj}(x, y)(x_k - y_k),$$

where $\phi_{kj} \in C^\infty$ near the diagonal. When $x=y$ we have $\phi_{kj}(x, y) = \partial\phi_j(x, y)/\partial x_k$ thus $\det(\phi_{kj}(x, x)) = \det(\partial\phi_j(x, y)/\partial x_k) \neq 0$ then. Now we have

$$\phi(x, y, \theta) = \Sigma\, \phi_{kj}(x, y)\theta_j(x_k - y_k) = \langle x-y, \Phi(x, y)\theta\rangle.$$

If $\psi(x, y)$ is the inverse of $\Phi(x, y)$, which is well defined in a neighborhood of the diagonal, it follows that $\phi(x, y, \psi(x, y)\theta) = \langle x-y, \theta\rangle$ and the proposition is proved.

The lemma means that apart from vector bundle maps there is only one function ϕ satisfying the hypotheses of Theorem 2.1.2 near the diagonal. This will be the starting point for our definition of pseudo-differential operators on manifolds in section 2.3.

Proof of Theorem 2.1.2. In view of Proposition 1.2.4 it is no restriction to consider only operators A of the form (1.4.1) where $a(x, y, \theta) \in S^m_{\varrho,\delta}(X \times X \times \mathbf{R}^n)$ is properly supported and vanishes outside a closed subset of $X \times X$ contained in the set Ω of Proposition 2.1.3. Now a change of variables gives

$$Au(x) = \iint e^{i\langle x-y,\theta\rangle} a(x, y, \psi(x, y)\theta)\,|\det\psi(x, y)|\,u(y)\,dy\,d\theta.$$

In view of Proposition 1.1.7, part (ii), the amplitude function here belongs to $S^m_{\varrho,\delta}(X \times X \times \mathbf{R}^n)$ which proves the theorem.

We shall now return to the phase function

$$\phi(x, y, \theta) = \langle \varkappa_1(x) - \varkappa_1(y), \theta\rangle$$

which occurred in the change of variables, for we wish to determine the transformation law for the symbol. With the notations used above we obtain

$$A_1 u(x) = \iint e^{i\langle x-y,\theta\rangle} a(\varkappa_1(x), \varkappa_1(y), \psi(x, y)\,\theta)\,D(x, y)\,u(y)\,dy\,d\theta,$$

where

$$D(x, y) = |\det \varkappa_1'(x)|\,|\det\psi(x, y)|,$$

thus $D(x, x) = 1$. (That $D(x, x) = 1$ means precisely that $dy\,d\eta$ is an invariant measure on the cotangent space of X which is of course very well known.) If we take $a(x, y, \eta) = (2\pi)^{-n}\sigma_A(x, \eta)$, it follows that

$$\sigma_{A_1}(x, \eta) \sim \sum_\alpha (iD_\eta)^\alpha D_y^\alpha\, \sigma_A(\varkappa_1(x), \psi(x, y)\,\eta)\,D(x, y)/\alpha!\,|_{x=y}. \tag{2.1.11}$$

With the usual notation $\sigma_A^{(\beta)}(x, \eta) = (iD_\eta)^\beta \sigma_A(x, \eta)$,

the general term in (2.1.11) will be a linear combination of terms of the form

$$c(x, y)\, \eta^\gamma \sigma_A^{(\beta)}(\varkappa_1(x), \psi(x, y)\, \eta)$$

with
$$|\gamma| + |\alpha| \leq |\beta| \leq 2\,|\alpha|. \tag{2.1.12}$$

The second inequality is obvious. To prove the first we note that application of D_y to a function of this type does not change $|\beta| - |\gamma|$ while application of D_η increases this difference by 1. From (2.1.12) it follows that

$$|\gamma| \leq |\beta| - |\alpha| \leq |\beta| - |\beta|/2 = |\beta|/2. \tag{2.1.13}$$

Adding the terms in (2.1.11) in a different order we therefore obtain in view of (2.1.13)

$$\sigma_{A_1}(\varkappa(x), \eta) \sim \Sigma\, \sigma_A^{(\beta)}(x, {}^t\varkappa'(x)\, \eta)\, \phi_\beta(x, \eta)/\beta! \tag{2.1.14}$$

where ϕ_β is a polynomial in η of degree $\leq |\beta|/2$,

$$\phi_0(x, \eta) = 1. \tag{2.1.15}$$

Since ϕ_β does not depend on A, we can determine ϕ_β by choosing A as a differential operator. Then we have

$$\sigma_{A_1}(y, \eta)|_{y=\varkappa(x)} = e^{-i\langle y, \eta\rangle} A_1\, e^{i\langle y, \eta\rangle}\,|_{y=\varkappa(x)} = e^{-i\langle \varkappa(z), \eta\rangle}\, \sigma_A(z, D)\, e^{i\langle \varkappa(z), \eta\rangle}\,|_{z=x}.$$

Here we introduce the Taylor expansion

$$\varkappa(z) = \varkappa(x) + \varkappa'(x)\,(z-x) + \varkappa_x''(z),$$

where $\varkappa_x''(z)$ vanishes to the second order when $z = x$. We have

$$\langle \varkappa(z), \eta\rangle = \langle \varkappa(x), \eta\rangle - \langle x, {}^t\varkappa'(x)\eta\rangle + \langle z, {}^t\varkappa'(x)\eta\rangle + \langle \varkappa_x''(z), \eta\rangle,$$

so in view of Leibniz' formula we obtain (2.1.14) with

$$\phi_\beta(x, \eta) = D_z^\beta\, e^{i\langle \varkappa_x''(z), \eta\rangle}\,|_{z=x} \tag{2.1.16}$$

and with no other polynomials ϕ_β. Note in particular the first few polynomials:

$$\phi_\beta(x, \eta) = 0,\ |\beta| = 1; \quad \phi_\beta(x, \eta) = D_x^\beta\, i\,\langle \varkappa(x), \eta\rangle, \quad |\beta| = 2. \tag{2.1.17}$$

These formulas are also given in [13] where the proof is different and leads directly to (2.1.16).

The calculus we have given here is exact modulo operators in $L^{-\infty}$ and symbols in $S^{-\infty}$. However, it is complicated by the presence of infinite sums in (2.1.6), (2.1.9), (2.1.14). Now the terms with $\alpha \neq 0$ in these sums are of order $\leq m + (\delta - \varrho)$, $m + m' + (\delta - \varrho)$ and $m + 1 - 2\varrho$ respectively if $m(m')$ is the order of A (resp. B). In the case of (2.1.14) we have assumed that $\varrho + \delta \geq 1$ so $1 - 2\varrho \leq \delta - \varrho$. We can therefore obtain a simpler but cruder calculus if from the isomorphism $L^m_{\varrho,\delta}(X)/L^{-\infty}_{\varrho,\delta}(X)\to S^m_{\varrho,\delta}(X\times\mathbf{R}^n)/S^{-\infty}_{\varrho,\delta}(X\times\mathbf{R}^n)$ we deduce the isomorphism $L^m_{\varrho,\delta}(X)/L^{m+(\delta-\varrho)}_{\varrho,\delta}(X)\to S^m_{\varrho,\delta}(X\times\mathbf{R}^n)/S^{m+(\delta-\varrho)}_{\varrho,\delta}(X\times\mathbf{R}^n)$. If $A\in L^m_{\varrho,\delta}$ and $a\in S^m_{\varrho,\delta}$ we shall call a a principal symbol of A if the residue classes of A and a correspond to each other in this isomorphism. If $a(x,\xi)$ is a principal symbol of A, then $a(x,-\xi)$ is a principal symbol of ${}^tA$, if $b(x,\xi)$ is a principal symbol of A then $b(x,\xi)a(x,\xi)$ is a principal symbol of BA, and if A_1 is obtained from A by a change of variables as discussed above, then a principal symbol of A_1 is given by $a(\varkappa^{-1}(x),\, {}^t\varkappa'(x)\xi)$. In the generalizations of pseudo-differential operators which we shall discuss later on we shall only develop an analogue of this simple calculus.

2.2. The continuity of pseudo-differential operators

The estimates for operators in $L^m_{\varrho,\delta}(X)$ which we shall prove here have been given before in [13], and a variant of the proofs there has been published by Kumano-go [20]. Our purpose here is to show that they also follow quite easily from the calculus which we have established in section 2.1. We shall restrict ourselves to the L^2 continuity of operators of order 0 and refer to [13, section 5] for the continuity of operators $A\in L^m_{\varrho,\delta}$ from $H_{(s)}$ to $H_{(s-m)}$ which is an easy consequence.

We shall write

$$(u,v)=\int u\bar{v}\,dx$$

for the scalar product in $L^2(X)$ and denote the corresponding adjoint of the operator $A\in L^0_{\varrho,\delta}$ by A^*. Thus $(Au,v)=(u,A^*v)$ if $u,v\in C_0^\infty(X)$, and we have

$$\|Au\|^2=(Au,Au)=(A^*Au,u).$$

THEOREM 2.2.1. *Assume that $A\in L^0_{\varrho,\delta}(X)$ is properly supported, $\delta<\varrho$, and that for every compact set $K\subset X$*

$$\varlimsup_{\eta\to\infty}\sup_{x\in K}|\sigma_A(x,\eta)|<M. \tag{2.2.1}$$

Then there exists a self adjoint integral operator R with properly supported kernel $\in C^\infty(X\times X)$ such that

$$(Au,Au)\leqslant M^2(u,u)+(Ru,u),\quad u\in C_0^\infty(X). \tag{2.2.2}$$

Proof. We shall prove that there exists a properly supported operator $B=L^0_{\varrho,\delta}(X)$ such that

$$A^*A+B^*B-M^2=R$$

has a C^∞ kernel. Since

$$(Au, Au) + (Bu, Bu) = M^2(u, u) + (Ru, u)$$

this will prove the theorem. Now a principal symbol of $M^2 - A^*A$ is equal to $M^2 - |\sigma_A(x, \eta)|^2$, so the theorem follows from

PROPOSITION 2.2.2. *Let $C \in L^0_{\varrho,\delta}(X)$ be properly supported, $C^* = C$, and assume that for every compact set $K \subset X$*

$$\varliminf_{\eta \to \infty} \inf_{x \in K} \operatorname{Re} \sigma_C(x, \eta) > 0. \tag{2.2.3}$$

Then one can find $B \in L^0_{\varrho,\delta}(X)$ so that B is properly supported and $B^*B - C = R$ has a C^∞ kernel.

Proof. By Proposition 1.1.8 we can find a real valued symbol $b_0 \in S^0_{\varrho,\delta}(X)$ such that

$$|b_0(x, \eta)|^2 - \operatorname{Re} \sigma_C(x, \eta) = 0$$

for large $|\eta|$ when x belongs to a compact set. (Note that the symbol of $C - C^*$ is $2i \operatorname{Im} \sigma_C$ modulo $S^{\delta-\varrho}_{\varrho,\delta}$, so $\operatorname{Im} \sigma_C \in S^{\delta-\varrho}_{\varrho,\delta}$ since $C = C^*$.) Let B_0 be a properly supported operator with the symbol b_0. Then the symbol of B^*B is equal to $|b_0(x, \eta)|^2$ modulo $S^{\delta-\varrho}_{\varrho,\delta}$, so we obtain

$$C - B_0^* B_0 \in L^{\delta-\varrho}_{\varrho,\delta}(X).$$

We claim that it is possible to find successively properly supported operators $B_j \in L^{j(\delta-\varrho)}_{\varrho,\delta}(X)$ such that for $j = 1, 2, \ldots$

$$R_j = C - (B_0 + \ldots + B_{j-1})^* (B_0 + \ldots + B_{j-1}) \in L^{j(\delta-\varrho)}_{\varrho,\delta}(X).$$

We know that this is possible for $j = 1$. If $B_0, \ldots, B_{j-1}$ have already been chosen and if $B_j \in L^{j(\delta-\varrho)}_{\varrho,\delta}(X)$, then

$$C - (B_0 + \ldots + B_j)^* (B_0 + \ldots + B_j) = R_j - B_j^* B_0 - B_0^* B_j \quad \text{modulo } L^{(j+1)(\delta-\varrho)}_{\varrho,\delta}.$$

Since R_j is self adjoint we have $\operatorname{Im} \sigma_{R_j} \in S^{(j+1)(\delta-\varrho)}_{\varrho,\delta}$. If we choose B_j so that for large $|\eta|$

$$2\, \sigma_{B_j}(x, \eta)\, \sigma_{B_0}(x, \eta) = \sigma_{R_j}(x, \eta),$$

which is possible since $(\sigma_{B_0}(x, \eta))^{-1} \in S^0_{\varrho,\delta}$ for large $|\eta|$ by Proposition 1.1.8, we obtain an operator B_j with the desired properties. If we now take B so that the symbol of B is the asymptotic sum in the sense of Proposition 1.1.9 of the symbols of B_j, $j = 0, 1, 2, \ldots$, we have proved the proposition and so Theorem 2.2.1.

Theorem 2.2.1 has an important and well-known corollary.

COROLLARY 2.2.3. *Let $A \in L^0_{\varrho,\delta}(\mathbf{R}^n)$, $\delta < \varrho$, and assume that the kernel of A has compact support in $\mathbf{R}^n \times \mathbf{R}^n$, and that*

$$\overline{\lim_{\eta\to\infty}} \sup_x |\sigma_A(x,\eta)| < M.$$

Then one can find another such operator A_1 *such that* $A - A_1 \in L^{-\infty}$ *and*

$$\|A_1 u\| \leqslant M \|u\|, \quad u \in C_0^\infty(\mathbf{R}^n).$$

Proof. Let $\chi \in C_0^\infty(\mathbf{R}^n)$, $\int \chi(x)\,dx = 1$, and $0 \leqslant \hat{\chi} \leqslant 1$, and set $\chi_\varepsilon(x) = \varepsilon^{-n}\chi(x/\varepsilon)$. Then we obtain using (2.2.2) if $A_\varepsilon u = Au - A(\chi_\varepsilon * u)$

$$\|A_\varepsilon u\|^2 \leqslant M^2 \|u - \chi_\varepsilon * u\|^2 + (R(u - \chi_\varepsilon * u),\, u - \chi_\varepsilon * u) \leqslant M^2 \|u\|^2 + \|R_\varepsilon u\|\, \|u\|,$$

where R_ε is an integral operator with kernel

$$R_\varepsilon(x,y) = R(x,y) - \int R(x, y - \varepsilon z)\, \chi(z)\, dz.$$

This is arbitrarily small with ε and has support in a fixed compact set, so if $M_1 > M$ we conclude that $\|A_\varepsilon u\|^2 \leqslant M_1^2 \|u\|^2$ for small ε. Thus A_ε has the required properties then.

COROLLARY 2.2.4. *Let* $A \in L^0_{\varrho,\delta}(\mathbf{R}^n)$, $\delta < \varrho$, *and assume that the kernel of* A *has compact support in* $\mathbf{R}^n \times \mathbf{R}^n$ *and that* $\sigma_A(x,\eta) \to 0$ *when* $\eta \to \infty$, *uniformly with respect to* x. *Then the operator* A *is compact in* $L^2(\mathbf{R}^n)$.

For a converse of these corollaries we refer to [13].

2.3. Pseudo-differential operators on a manifold

Let X be a C^∞ paracompact manifold of dimension n. Using Theorem 2.1.2 we can define the space $L^m_{\varrho,\delta}(X)$ when $1 - \varrho \leqslant \delta < \varrho$ as follows: a continuous linear operator A from $C_0^\infty(X)$ to $C^\infty(X)$ belongs to $L^m_{\varrho,\delta}(X)$ if and only if for each diffeomorphism $\varkappa$ of a coordinate patch $X_\varkappa$ in X to an open set $\varkappa X_\varkappa \subset \mathbf{R}^n$ we have $A_\varkappa \in L^m_{\varrho,\delta}(\varkappa X_\varkappa)$ if $(A_\varkappa u) \circ \varkappa = A(u \circ \varkappa)$, $u \in C_0^\infty(\varkappa X_\varkappa)$. By Theorem 2.1.2 this definition agrees with the one used in section 2.1 if $X \subset \mathbf{R}^n$; moreover, it is always sufficient to require that this condition is verified for a set of coordinate systems such that the corresponding coordinate patches cover X if in addition we require that the kernel of A is C^∞ off the diagonal. The definition can also be expressed as follows: If $x_1, \ldots, x_n$ are local coordintes in an open coordinate patch X_1 of X and if $v \in C_0^\infty(X_1)$, then

$$e^{-i\langle x,\xi\rangle} A(v e^{i\langle x,\xi\rangle}) \in S^m_{\varrho,\delta}(X_1).$$

Here $\xi \in \mathbf{R}^n$ and $\langle x, \xi\rangle = x_1\xi_1 + \ldots + x_n\xi_n$.

If $u \in C_0^\infty(\varkappa_1(X_{\varkappa_1} \cap X_{\varkappa_2}))$ then

$$(A_{\varkappa_1}u)\circ\varkappa_1 = A(u\circ\varkappa_1) = A_{\varkappa_2}(u\circ\varkappa_1\circ\varkappa_2^{-1})\circ\varkappa_2$$

so if $\varkappa_{12}$ is the diffeomorphism $\varkappa_1\circ\varkappa_2^{-1}$: $\varkappa_2(X_{\varkappa_1}\cap X_{\varkappa_2})\to\varkappa_1(X_{\varkappa_1}\cap X_{\varkappa_2})$, it follows that $(A_{\varkappa_1}u)\circ\varkappa_{12} = A_{\varkappa_2}(u\circ\varkappa_{12})$. For the symbols we therefore have

$$\sigma_{A_{\varkappa_1}}(\varkappa_{12}(x), \xi) - \sigma_{A_{\varkappa_2}}(x, {}^t\varkappa_{12}'(x)\,\xi) \in S_{\varrho,\delta}^{m+(\delta-\varrho)}(\varkappa_2(X_{\varkappa_1}\cap X_{\varkappa_2})).$$

If we regard $\varkappa_j X_{\varkappa_j}\times \mathbf{R}^n$ as the cotangent space of $\varkappa_j X_{\varkappa_j}$, then $\varkappa_{12}$ maps (x_2, ξ_2) to (x_1, ξ_1) where $x_1 = \varkappa_{12}(x_2)$ and $\langle \varkappa_{12}'(x_2)t, \xi_1\rangle = \langle t, \xi_2\rangle$ for all $t\in\mathbf{R}^n$, thus $\xi_2 = {}^t\varkappa_{12}'(x_2)\,\xi_1$. If we keep Proposition 1.1.7 in mind, it follows that if using the map $\varkappa_j$ we pull $\sigma_{A_{\varkappa_j}}$ to a function σ_A^j on the cotangent space of $X_{\varkappa_j}$, then $\sigma_A^1 - \sigma_A^2 \in S_{\varrho,\delta}^{m+(\delta-\varrho)}(T^*(X_{\varkappa_1}\cap X_{\varkappa_2}))$. Using a partition of unity we can therefore piece together an element $\sigma\in S_{\varrho,\delta}^m(T^*(X))$ such that

$$\sigma - \sigma_A^1 \in S_{\varrho,\delta}^{m+(\delta-\varrho)}(T^*(X_{\varkappa_1}))$$

for any coordinate system $\varkappa_1$. We call σ a principal symbol of A. (For more details see also Atiyah and Bott [3, appendix].)

The preceding definition by localization often makes verifications of quite simple facts rather heavy notationally. A more convenient approach can be based on Theorem 2.1.2. We wish to define operators in $L_{\varrho,\delta}^m(X)$ directly as Fourier integral operators with phase function ϕ and symbol a defined on a real vector bundle E with fiber dimension n over a neighborhood Ω of the diagonal in $X\times X$. We wish ϕ to be linear in the fibers and require that the restriction of ϕ to a fiber is critical at $e\in E$ if and only if the projection πe of e on $X\times X$ belongs to the diagonal. The differential of ϕ at such a point can be regarded as a cotangent vector of $X\times X$ at $\pi e = (x, x)$ which vanishes on the tangents of the diagonal so it is of the form $(\xi, -\xi)$ where ξ is a cotangent vector of X at x. The map $E_{x,x}\ni e\to\xi\in T_x^*$ is linear and injective, hence bijective since the dimensions are equal. Thus ϕ defines over the diagonal an *isomorphism of* E *and the cotangent space* $T^*(X)$ *lifted to* $X\times X$ *by the projection* $(x, y)\to y$, and this isomorphism can be extended to a neighborhood of the diagonal.

On the other hand, if E is defined in this way then we can choose ϕ so that ϕ vanishes over the diagonal and $d\phi = \xi dx - \xi dy$ at (x, x, ξ), where $\xi\in T_x^*$. Indeed, this is possible locally and so globally by means of a partition of unity. In a neighborhood of the diagonal we cannot have any critical points along the fibers then so ϕ has the required properties. If ϕ_1 and ϕ_2 are two such functions, then $\phi_1 - \phi_2$ vanishes to the second order over the diagonal and we conclude as in the proof of Proposition 2.1.3 that

$$\phi_2(x, y, \xi) = \phi_1(x, y, \psi(x, y)\xi)$$

over a neighborhood of the diagonal where ψ is a homomorphism $E \to E$ which is the identity over the diagonal. Thus the requirements on E and ϕ determine E and ϕ essentially uniquely.

Now we can define $L^m_{\varrho,\delta}(X)$, $1-\varrho \leqslant \delta < \varrho$, as the operators which can be written as a sum of an operator with C^∞ kernel and one of the form

$$Au(x) = (2\pi)^{-n} \iint e^{i\phi(x,y,\eta)} a(x, y, \eta)\, u(y)\, dy\, d\eta, \quad u \in C_0^\infty(X),$$

where $dy\, d\eta$ is the invariant element of integration in $T^*(X)$ and $a \in S^m_{\varrho,\delta}(X \times X \times \mathbf{R}^n)$ vanishes when (x, y) is outside a suitably small neighborhood of the diagonal. A principal symbol of A is given by $a(x, x, \eta)$. The equivalence with the first definition is an immediate consequence of Theorem 2.1.2.

2.4. Oscillatory integrals with linear phase function

Theorem 2.1.2 shows that the distribution kernels of pseudo-differential operators are precisely the distributions which can be represented as oscillatory integrals with a linear phase function (with respect to θ) which is critical only over the diagonal. We shall now generalize by considering oscillatory integrals based on arbitrary linear phase functions. This is in preparation for the non-linear case which will be studied in Chapter III.

Thus let X be an open set in $\mathbf{R}^n$ and let ϕ be a phase function in $X \times \mathbf{R}^N$ which is linear with respect to the θ variables, that is,

$$\phi(x, \theta) = \langle \Phi(x), \theta \rangle$$

where Φ is a map $X \to \mathbf{R}^N$. That ϕ is a phase function means that $\langle \Phi'_{x_j}, \theta \rangle \neq 0$ for some j when $\phi'_\theta = \Phi = 0$. Thus $N \leqslant n$ (if there are such points which we assume in order to exclude a trivial case), and Φ'_x is of rank N when $\Phi(x) = 0$. This equation therefore defines a submanifold Y of codimension N.

If Φ_1 is another map $X \to \mathbf{R}^N$ such that $\Phi_1^{-1}(0) = Y$ and Φ'_{1x} is of rank N on Y, we can choose a neighborhood U of Y and a C^∞ map ψ: $U \to \mathrm{GL}(N, \mathbf{R})$ such that

$$\Phi_1(x) = \psi(x)\Phi(x),$$

thus $\phi_1(x, \theta) = \langle \Phi_1(x), \theta \rangle = \langle \Phi(x), {}^t\psi(x)\theta \rangle = \phi(x, {}^t\psi(x)\theta)$, $x \in U$. In order to construct ψ we first note that we must have $\Phi'_{1x}(x) = \psi(x)\Phi'_x(x)$ when $x \in Y$. Since the matrices Φ'_{1x} and Φ'_x have rank N and the same kernel when $x \in Y$ this condition determines ψ uniquely as a C^∞ function of $x \in Y$, which we extend to a C^∞ function in a neighborhood of Y. Then we have that

$$\Phi_1(x) - \psi(x)\Phi(x)$$

vanishes to the second order on Y. Writing $\Phi(x) = (\Phi^1(x), \ldots, \Phi^N(x))$ we obtain from Taylor's formula

$$\Phi_1^j(x) = \Sigma\, \psi_{jk}(x)\,\Phi^k(x) + \Sigma\, R_{jk}(x)\,\Phi^k(x),$$

where $R_{jk}(x)=0$ on Y. The matrix $\psi+R$ therefore has the required properties. Thus we have proved the following extension of Proposition 2.1.3:

PROPOSITION 2.4.1. *If ϕ is a linear phase function in $X\times\mathbf{R}^N$, then $\{(x,\theta); \phi'_\theta(x,\theta)=0\} = Y\times\mathbf{R}^N$ where Y is a submanifold of codimension N. If ϕ_1 is another linear phase function with the same critical points, we can find a neighborhood U of Y and a C^∞* map ψ: $U\to \mathrm{GL}\,(N,\mathbf{R})$ *such that $\phi_1(x,\theta)=\phi(x,\psi(x)\theta)$, $x\in U$.*

Conversely, let Y be a submanifold of X. Locally we can choose coordinates in X so that Y is defined by $x_1=\ldots=x_k=0$, say. Then the function $\phi(x,\theta)=\sum_1^k x_j\theta_j$ is linear with respect to θ, and the equation $\phi'_\theta=0$ is equivalent to $x\in Y$. This construction can also be made globally over a neighborhood of Y even if X and Y are manifolds, with ϕ defined in the normal bundle $N(Y)$ of Y in $T^*(X)$, lifted to a bundle E over a neighborhood U of Y by means of a C^∞ retraction $U\to Y$. We define ϕ so that if π is the projection $E\to U$, then $\phi=0$ and $d\phi(p)=\pi^*p$ if $p\in E$ and $\pi p\in Y$. Here p in the right-hand side is considered as a covector on X. This is a straightforward extension of the discussion in section 2.3 which corresponds to the diagonal in $X\times X$.

Returning to the local case where $Y\subset X\subset\mathbf{R}^n$ and ϕ is a phase function in $X\times\mathbf{R}^n$ we consider a distribution of the form

$$\langle A,u\rangle = (2\pi)^{-(n+2N)/4}\iint e^{i\phi(x,\theta)}a(x,\theta)\,u(x)\,dx\,d\theta,\quad u\in C_0^\infty(X), \tag{2.4.1}$$

where $a\in S^{m+(n-2N)/4}_{\varrho,\delta}(X\times\mathbf{R}^n)$, $1-\varrho\leqslant\delta<\varrho$. The strange normalizations made here have been chosen so that we have agreement with the representation for the kernel of a pseudo-differential operator on a manifold of dimension k derived from (2.1.5)$'$. Indeed, in that case we have $n=2k$ (the dimension of the product of the manifold by itself) and $N=k$ so we get a factor $(2\pi)^{-k}$ and $a\in S^m_{\varrho,\delta}$. A complete justification will follow from invariance properties discussed in section 3.2 and multiplicative properties proved in section 4.2.

The set of all distributions in X which modulo $C^\infty(X)$ can be represented in the form (2.4.1) with $a\in S^{m+(n-2N)/4}_{\varrho,\delta}$ will be denoted by $I^m_{\varrho,\delta}(X,Y)$. Note that in the case where a is a homogeneous function of θ the corresponding distribution is essentially a homogeneous function of the distance from X to Y depending smoothly on the nearest point in Y. By Propositions 2.4.1 and 1.2.4 the definition of $I^m_{\varrho,\delta}$ is independent of the choice of ϕ; moreover, it suffices to have ϕ defined over a neighborhood of Y if one takes a vanishing outside a smaller neighborhood.

Now we wish to assign a principal symbol to the distributions in $I^m_{\varrho,\delta}(X, Y)$. To do so we first recall that by Proposition 1.2.5 the distribution A is determined modulo $I^{m+(\delta-\varrho)}_{\varrho,\delta}(X, Y)$ by the restriction of a to $Y\times\mathbf{R}^N$. Thus we have a surjective map

$$S^{m+(n-2N)/4}_{\varrho,\delta}(Y\times\mathbf{R}^n)/S^{m+(n-2N)/4+(\delta-\varrho)}_{\varrho,\delta}(Y\times\mathbf{R}^n)\to I^m_{\varrho,\delta}(X,Y)/I^{m+(\delta-\varrho)}_{\varrho,\delta}(X,Y). \quad (2.4.2)$$

This is really an isomorphism. To prove this we must show that if $A=0$ then the restriction of a to $Y\times\mathbf{R}^n$ is in $S^{m+(n-2N)/4+(\delta-\varrho)}_{\varrho,\delta}(Y\times\mathbf{R}^n)$. Since χA is defined by the symbol χa if $\chi\in C^\infty(X)$, it is no restriction to assume that a vanishes for x outside such a small set that after a change of variables Y is defined by the equations $x'=(x_1, ..., x_N)=0$ and that $\phi(x,\theta)=\sum_1^N x_j\theta_j=\langle x',\theta\rangle$. Taking u as the product of a function of $x'=(x_1, ..., x_N)$ and one of $x''=(x_{N+1}, ..., x_n)$ we find that

$$\int e^{i\langle x',\theta\rangle}a(x',x'',\theta)\,u(x')\,dx'\,d\theta=0,\quad u\in C_0^\infty(\mathbf{R}^N).$$

Let $u=1$ near the origin. With $\xi\in\mathbf{R}^N$ we replace $u(x')$ by $u(x')e^{-i\langle x',\xi\rangle}$ and conclude that for all $\xi\in\mathbf{R}^N$

$$\int e^{i\langle x',\theta\rangle}a(x',x'',\xi+\theta)\,u(x')\,dx'\,d\theta=0.$$

When $\xi\to\infty$ a Taylor expansion gives in view of the Fourier inversion formula (cf. the proof of Theorem 2.1.1) that the integral is asymptotically equal to

$$\sum\int e^{i\langle x',\theta\rangle}(iD_\xi)^\alpha a(x',x'',\xi)\,\theta^\alpha u(x')\,dx'\,d\theta/\alpha!=(2\pi)^n\sum(-D_{x'})^\alpha(iD_\xi)^\alpha a(x',x'',\xi)\,|_{x'=0}/\alpha!,$$

where all terms except the first are in $S^{m+(n-2N)/4+(\delta-\varrho)}_{\varrho,\delta}(Y\times\mathbf{R}^n)$. Hence $a(0,x'',\xi)$ belongs to $S^{m+(n-2N)/4+(\delta-\varrho)}_{\varrho,\delta}(Y\times\mathbf{R}^n)$ which proves that (2.4.2) is an isomorphism.

We shall now examine to what extent (2.4.2) depends on the choice of the phase function ϕ and the local coordinates in X. To begin with we keep the local coordinates in X but replace the phase function ϕ by another ϕ_1. According to Proposition 2.4.1 we may assume that $\phi_1(x,\theta)=\phi(x,\psi(x)\theta)$, $x\in U$, where U is a neighborhood of Y and ψ a C^∞ map $U\to\mathrm{GL}(N,\mathbf{R})$. A substitution of variables now gives

$$\iint e^{i\phi(x,\theta)}a(x,\theta)\,u(x)\,dx\,d\theta=\iint e^{i\phi_1(x,\theta)}a_1(x,\theta)\,u(x)\,dx\,d\theta,$$

where
$$a_1(x,\theta)=a(x,\psi(x)\theta)\,|\det\psi(x)|. \quad (2.4.3)$$

To put this transformation law in a more natural form we first note that the map

$$Y \times \mathbf{R}^N \ni (x, \theta) \to (x, \phi'_x(x, \theta))$$

is a bijection to the normal bundle $N(Y)$ of Y in $T^*(X)$, which is linear along the fibers. We can therefore regard a as a function on $N(Y)$, and similarly for a_1. If $x \in Y$ and $(x, \phi'_x(x, \theta)) = (x, \xi)$, $(x, \phi'_{1x}(x, \theta_1)) = (x, \xi)$ we must have $\psi(x)\theta_1 = \theta$ so that $a_1(x, \theta_1) = a(x, \psi(x)\theta_1)\,|\det \psi(x)| = a(x, \theta)\,|\det \psi(x)|$. Regarded as functions on $N(Y)$ the functions a and a_1 therefore differ only by the factor $|\det \psi(x)|$. To take care of this factor we shall consider the measures defined in Y and in $N(Y)$ by the choice of ϕ.

Writing $\phi(x, \theta) = \langle \Phi(x), \theta \rangle$ we know that the map $x \to \Phi(x)$ is of rank N when $\Phi(x) = 0$. The composition $\delta(\Phi)$ where δ is the Dirac measure in $\mathbf{R}^N$ is then a well defined measure with support in Y. If $y_1, \ldots, y_{n-N}$ are local coordinates on Y and we extend them to C^∞ functions in a neighborhood of Y, then the measure is equal to

$$|D(y, \Phi)/Dx|^{-1} dy_1 \ldots dy_{n-N}.$$

Thus the measure is a density on Y, for a density in a manifold Y is a measure which in a local coordinate patch with local coordinates $y_1, \ldots, y_k$ ($k = \dim Y$) can be written in the form

$$a(y)\, dy_1 \ldots dy_k.$$

If we have an overlapping coordinate patch with local coordinates $\tilde{y}_1, \ldots, \tilde{y}_k$ the measure can also be expressed in the form $\tilde{a}(\tilde{y})\, d\tilde{y}_1 \ldots d\tilde{y}_k$, so we have the transformation law

$$\tilde{a}(\tilde{y}) = a(y)\, |Dy/D\tilde{y}|$$

in the overlap. More generally, a density of order α on Y is defined if for each choice of local coordinates we have a function $a(y)$ of the local coordinates which obeys the transformation law

$$\tilde{a}(\tilde{y}) = a(y)\, |Dy/D\tilde{y}|^\alpha.$$

Densities of order α can of course be regarded as sections of a line bundle Ω_α on Y, defined by the transition functions $|Dy/D\tilde{y}|^\alpha$, and we have $\Omega_\alpha \otimes \Omega_\beta = \Omega_{\alpha+\beta}$. A more intrinsic definition can be given as follows. The transformation law means that $a(y)\,|\det \langle t_i, dy_j \rangle|^\alpha$ where $t_1, \ldots, t_k$ are tangent vectors is independent of the choice of local coordinates; this quantity becomes $a(y)$ if we choose a dual basis in the tangent space to the basis in the cotangent space given by the differentials of the coordinates. Thus the fiber of Ω_α at y is the space of all maps a from $\Lambda^k T_y(Y) \setminus 0$ ($k = \dim Y$) to $\mathbf{C}$ such that $a(st) = |s|^\alpha a(t)$ if $s \in \mathbf{R} \setminus 0$ and $t \in \Lambda^k T_y \setminus 0$. The notions of real or positive densities are therefore well defined, and every positive density has a unique positive square root in $\Omega_{\frac{1}{2}}$. By the invariance of the definition, if $Z \to Y$ is a diffeomorphism into, then any density of order α on Y can be pulled back to a density of order α on Z. If u, v are densities of order α and $1-\alpha$ and the tensor product uv has compact support, then uv is a measure with compact support so $\int uv$ is well defined. Thus we can define the space of distributions with values in Ω_α as the dual space of $C_0^\infty(Y, \Omega_{1-\alpha})$.—Concerning the terminology we note that Atiyah and Bott [3] have called Ω_1 the volume bundle of Y.

Now let V be a cone bundle over a manifold Y, with fiber dimension N. If $t \in \mathbf{R}_+$ the group operation t: $V \to V$ defines a linear map $(\Omega_\alpha)_{tv} \to (\Omega_\alpha)_v$ so we have an equivariant action of $\mathbf{R}_+$ on Ω_α. If now $\varkappa$: $V \to \Gamma$ where Γ is an open conic set in $\mathbf{R}^n \times (\mathbf{R}^N \setminus 0)$ is a diffeomorphism commuting with the $\mathbf{R}_+$ action and preserving the fibers (see the definition of cone bundles in section 1.1) then an element $a \in S^\mu_{\varrho,\delta}(V, \Omega_\alpha)$ is transformed to one in $S^\mu_{\varrho,\delta}(\Gamma, \Omega_\alpha)$. With coordinates (y, θ) in Γ; $y \in \mathbf{R}^n$, $\theta \in \mathbf{R}^N$; this element is defined by a function $a_\varkappa(y, \theta)$, $(y, \theta) \in \Gamma$. The function $|\theta|^{-N\alpha}$ corresponds to a density of order α which is invariant under the group action. By the definitions in section 1.1 we therefore have $a \in S^\mu_{\varrho,\delta}(V, \Omega_\alpha)$ precisely when $a_\varkappa(y, \theta)|\theta|^{N\alpha} \in S^\mu_{\varrho,\delta}(\Gamma)$, that is, $a_\varkappa \in S^{\mu-N\alpha}_{\varrho,\delta}(\Gamma)$.

Using the Lebesgue measure in $\mathbf{R}^N$ we have on $Y \times \mathbf{R}^N$ a density given by $d_\phi = \delta(\Phi)d\theta_1 \dots d\theta_N$, or in terms of local coordinates $y_1, \dots, y_{n-N}$ on Y

$$|D(y, \Phi)/Dx|^{-1} dy_1 \dots dy_{n-N}\, d\theta_1 \dots d\theta_N. \tag{2.4.4}$$

This we shall map to a density on the normal bundle $N(Y)$ using the inverse of the map $\varkappa_\phi$: $Y \times \mathbf{R}^N \ni (y, \theta) \to (y, {}^t\Phi'_y \theta)$. We wish to compare d_ϕ with the density d_{ϕ_1} constructed from the phase function ϕ_1, that is, from $\Phi_1 = {}^t\psi\Phi$. In local coordinates d_{ϕ_1} is given by

$$|D(y, \Phi_1)/Dx|^{-1} dy_1 \dots dy_{n-N} d\theta_1 \dots d\theta_N = |\det \psi|^{-1} |D(y, \Phi)/Dx|^{-1} dy_1 \dots dy_{n-N} d\theta_1 \dots d\theta_N, \tag{2.4.5}$$

and d_{ϕ_1} should be mapped to a density on $N(Y)$ using the inverse of the map $\varkappa_{\phi_1}$: $Y \times \mathbf{R}^N \ni (y, \theta) \to (y, {}^t\Phi'_y \psi\theta)$. Now $\varkappa = \varkappa_{\phi_1}^{-1} \circ \varkappa_\phi$ is the map $(y, \theta) \to (y, \psi^{-1}\theta)$ so $\varkappa^* d_{\phi_1} = |\det \psi|^{-2} d_\phi$. If we recall (2.4.3), which with our present notations can be written $\varkappa^* a_1 = |\det \psi| a$, we conclude that $\varkappa^* a_1 \sqrt{d_{\phi_1}} = a\sqrt{d_\phi}$. Thus $a\sqrt{d_\phi}$ and $a_1\sqrt{d_{\phi_1}}$ define the same element in $S^{m+n/4}_{\varrho,\delta}(N(Y), \Omega_{\frac{1}{2}})$. That the order here becomes independent of N is another partial justification for the normalizations that have been made.

We can now improve (2.4.2) by stating that for $1-\varrho \leq \delta < \varrho$ there is an isomorphism

$$S^{m+n/4}_{\varrho,\delta}(N(Y), \Omega_{\frac{1}{2}}) / S^{m+n/4+(\delta-\varrho)}_{\varrho,\delta}(N(Y), \Omega_{\frac{1}{2}}) \to I^m_{\varrho,\delta}(X, Y) / I^{m+(\delta-\varrho)}_{\varrho,\delta}(X, Y)$$

which is independent of the choice of phase function ϕ. However, we have still assumed that

$X \subset \mathbf{R}^n$ so it remains to consider the effect of a change of variables before we have a truly invariant formalism.

Thus let $X \to \tilde{X}$ be a diffeomorphism between open sets in $\mathbf{R}^n$. Writing $x = x(\tilde{x})$ we transform (2.4.1) to

$$\langle A, u \rangle = (2\pi)^{-(n+2N)/4} \iint e^{i\tilde{\phi}(\tilde{x},\theta)}\, \tilde{a}(\tilde{x},\theta)\, \tilde{u}(\tilde{x})\, d\tilde{x}\, d\theta = \langle \tilde{A}, \tilde{u} \rangle. \tag{2.4.6}$$

Here $\tilde{u}(\tilde{x}) = |Dx/D\tilde{x}|^{1/2} u(x)$, that is, we regard u as a density of order $1/2$ which means that A is also transformed to $\tilde{A}$ as a density of order $1/2$. Furthermore

$$\phi(\tilde{x}, \theta) = \phi(x(\tilde{x}), \theta), \quad \tilde{a}(\tilde{x}, \theta) = a(x(\tilde{x}), \theta)\, |Dx/D\tilde{x}|^{1/2}.$$

Let $y_1, \ldots, y_{n-N}$ be local coordinates on $Y = \{x;\ \phi'_\theta(x, \theta) = 0\}$, considered as functions in X, and let $\tilde{y}_1, \ldots, \tilde{y}_{n-N}$ be the corresponding functions in $\tilde{X}$ which are thus local coordintes on $\tilde{Y} = \{\tilde{x};\ \tilde{\phi}'_\theta(\tilde{x}, \theta) = 0\}$. Clearly (x, θ) and $(\tilde{x}, \theta)$ define points in $N(Y)$ and $N(\tilde{Y})$ which correspond under the isomorphism between $T^*(X)$ and $T^*(\tilde{X})$. Now we claim that

$$a(x, \theta)\, |D(y, \Phi)/Dx|^{-1/2} = \tilde{a}(\tilde{x}, \theta)\, |D(\tilde{y}, \tilde{\Phi})/D\tilde{x}|^{-1/2} \tag{2.4.7}$$

if the two sides are evaluated at points corresponding under the isomorphism $X \to \tilde{X}$. In fact, this follows from the fact that

$$D(y, \Phi)/Dx = (D(\tilde{y}, \tilde{\Phi})/D\tilde{x})(D\tilde{x}/Dx).$$

Thus our construction is also invariant under changes of variables in X.

There is no difficulty now in proving the preceding results globally on a manifold. In order not to repeat arguments already given in section 2.3 we just state the result:

THEOREM 2.4.2. *Let X be a manifold and Y a closed submanifold. Let $I^m_{\varrho,\delta}(X, Y)$ where $1-\varrho \leqslant \delta < \varrho$ be the set of all distribution densities of order $1/2$ on X which are in $C^\infty(X \setminus Y)$ and in a neighborhood of any point in Y can be expressed in the form* (2.4.1) *where $a \in S^{m+(n-2N)/4}_{\varrho,\delta}$ and ϕ is a linear phase function which is critical along the fibers over Y and only there. Then the restriction of a to these points gives rise to an isomorphism*

$$S^{m+n/4}_{\varrho,\delta}(N(Y), \Omega_{\frac12}) / S^{m+n/4+(\delta-\varrho)}_{\varrho,\delta}(N(Y), \Omega_{\frac12}) \to I^m_{\varrho,\delta}(X, Y) / I^{m+(\delta-\varrho)}_{\varrho,\delta}(X, Y). \tag{2.4.8}$$

We shall say that a is a principal symbol of the distribution $A \in \mathcal{D}'(X, \Omega_{\frac12})$ if their residue classes correspond under this isomorphism.

2.5. The wave front set of a distribution

We shall now introduce a refinement of the notion of singular support of a distribution. To do so we let X be a manifold, $u \in \mathcal{D}'(X)$, and note that

$$\text{sing supp } u = \bigcap \{x; \phi(x) = 0\} \tag{2.5.1}$$

the intersection being taken over all $\phi \in C^\infty(X)$ with $\phi u \in C^\infty(X)$. We shall replace ϕ by a properly supported pseudodifferential operator $A \in L^0(X)$. The characteristic set of A is defined by

$$\gamma(A) = \{(x, \xi) \in T^*(X) \setminus 0; \ \varliminf_{t \to \infty} |a(x, t\xi)| = 0\}$$

if $a \in S^0(X)$ is a principal symbol for A. Clearly the choice of principal symbol is irrelevant, and since the functions $(x, \xi) \to a(x, t\xi)$ are equicontinuous the complement of the characteristic set is open so the characteristics are closed. Now set, in analogy to (2.5.1),

$$WF(u) = \bigcap_{Au \in C^\infty} \gamma(A), \tag{2.5.2}$$

where A runs over properly supported operators in $L^0(X)$. It is clear that $WF(u)$ is a closed cone in $T^*(X) \setminus 0$, and since A may be chosen as a function in $C^\infty(X)$, we have

$$\pi WF(u) \subset \text{sing supp } u$$

if π: $T^*(X) \to X$ is the projection. In fact there is equality. For if $x \notin \pi WF(u)$ it follows from the Borel–Lebesgue lemma that there are finitely many $A_1, \ldots, A_k \in L^0(X)$ such that $x \notin \pi \bigcap_1^k \gamma(A_j)$. Let $A = \Sigma A_j^* A_j$, the adjoints being taken with respect to some positive C^∞ density in X. Then $Au \in C^\infty$ and $\Sigma |a_j|^2$ is a principal symbol for A so $\pi\gamma(A)$ does not meet a neighborhood of x. We shall now appeal to the standard regularity theorem for solutions of elliptic equations.

PROPOSITION 2.5.1. *Let $B \in L^0(X)$ be properly supported and elliptic in the sense that $\gamma(B) = \emptyset$. Then one can construct a properly supported $E \in L^0(X)$ such that $EB - I$ and $BE - I$ have C^∞ kernels. Since $u = (I - EB)u + EBu$ it follows that*

$$\text{sing supp } u \subset \text{sing supp } Bu, \quad u \in \mathcal{D}'(X).$$

Proof. The principal symbol of EB is eb. So choose E_0 with principal symbol $1/b$ which is in S^0 by Proposition 1.1.8. Then

$$E_0 B = I + R_1,$$

where $R_1 \in L^{-1}$. Since $R_1^k \in L^{-k}$ it follows immediately from Proposition 1.1.9 that there is a properly supported operator $F_1 \in L^0$ such that for each $k > 0$

$$F_1 - I + R_1 - \ldots - (-R_1)^{k-1} \in L^{-k}.$$

This implies that

$$F_1 E_0 B - (I - R_1 + \ldots + (-R_1)^{k-1})(I + R_1) \in L^{-k},$$

so that $F_1 E_0 B - I \in L^{-k}$ for all k. If $E_1 = F_1 E_0$ we therefore have $E_1 B - I \in L^{-\infty}$. Similarly we construct E_2 with $BE_2 - I \in L^{-\infty}$ and conclude that

$$E_2 - E_1 = (I - E_1 B) E_2 - E_1(I - BE_2) \in L^{-\infty},$$

hence that E_1 or E_2 has the required properties.

Returning now to the discussion of singular supports, we take a function $\phi \geq 0$ which is 0 near x but 1 outside such a small neighborhood that $B = \phi + A$ is elliptic. We have $Bu = \phi u + Au \in C^\infty$ near x and it follows that $u \in C^\infty$ near x.

We sum up the preceding discussion as follows:

Definition 2.5.2. If $u \in \mathcal{D}'(X)$ the wave front set $WF(u)$ of u is the closed cone $\subset T^*(X)\setminus 0$ defined by *(2.5.2)*.

THEOREM 2.5.3. *The projection of $WF(u)$ in X is equal to* $\operatorname{sing\,supp} u$.

The preceding definition has been chosen because it is invariant. However, one can make equivalent and sometimes more useful definitions which do not involve pseudo-differential operators, and this we shall do now.

PROPOSITION 2.5.4. *If Y is an open set in X, then $WF(u|_Y) = (WF(u))|_Y$, $u \in \mathcal{D}'(X)$, where $u|_Y$ is the restriction of u to Y and $(WF(u))|_Y = (WF(u)) \cap \pi^{-1} Y$.*

Proof. If $(x, \xi) \notin WF(u)$, we can choose $A \in L^0$ with $Au \in C^\infty$, $(x, \xi) \notin \gamma(A)$. Let U be any neighborhood of x and choose $\phi, \psi \in C_0^\infty(U)$ with $\psi = 1$ near $\operatorname{supp} \phi$, $\phi = 1$ near x. If $A_1 v = \phi A \psi v$ we have $A_1 \in L^0$, $A_1 u = \phi A u - \phi A(1 - \psi) u \in C^\infty$ (by the pseudo-local property), $(x, \xi) \notin \gamma(A_1)$ and the support of the kernel of A is in $U \times U$. The fact that one may restrict attention to such operators A in (2.5.2) immediately gives the proposition.

In particular the proposition shows that $WF(u)$ and $WF(\phi u)$ agree over the set where $\phi \neq 0$ when $\phi \in C^\infty$. It remains now to study distributions of compact support in $\mathbf{R}^n$.

PROPOSITION 2.5.5. *Let $u \in \mathcal{E}'(\mathbf{R}^n)$, and let K be the closed cone obtained by the projection*

$$WF(u) \ni (x, \xi) \to \xi \in \mathbf{R}^n \setminus 0.$$

If K_1 is another closed cone $\subset \mathbf{R}^n \setminus 0$ with $K \cap K_1 = \emptyset$ then

$$|(1 + |\xi|)^N \hat{u}(\xi)| \leq C_N, \quad \xi \in K_1, \tag{2.5.3}$$

for any integer N, and K is the smallest closed cone such that (2.5.3) *holds for all disjoint closed cones K_1.*

Proof. We begin with the last statement, so we assume that (2.5.3) is valid when $K \cap K_1 = \emptyset$. Let p be a C^∞ function in R^n vanishing near the origin and near K, such that $p(t\xi) = p(\xi)$ when $t \geq 1$ and $|\xi| \geq 1$. Then the product $p(\xi)\hat{u}(\xi)$ is rapidly decreasing so $p(D)u \in C^\infty$. If $\phi, \psi \in C_0^\infty$ and $\psi = 1$ in a neighborhood of supp u, then $Au \in C^\infty$ if $Av \equiv \phi P(D)\psi v$. Since A is noncharacteristic at (x, ξ) if $\phi(x)\psi(x) \neq 0$ and $\lim_{t\to\infty} p(t\xi) \neq 0$, we conclude that $WF(u) \subset \mathbf{R}^n \times K$.

To prove the other half of the proposition we need a lemma.

LEMMA 2.5.6. *Let $u \in \mathcal{E}'(\mathbf{R}^n)$ and $A = a(x, D)$ where a is of order $-\infty$ in a conic neighborhood Γ of $WF(u)$. Then $Au \in C^\infty$.*

Proof. We may assume that $a(x, \xi) = 0$ for large x and therefore that $(x, \xi) \in \Gamma$ for large x. Choose a properly supported pseudo-differential operator A_0 with principal symbol 1 in a conic neighborhood of $WF(u)$ and over the complement of a compact set such that the full symbol is of order $-\infty$ in a cone Γ_1 with $\Gamma \cup \Gamma_1 = T^*(\mathbf{R}^n) \setminus 0$. Using the Borel–Lebesgue lemma we now choose $A_1, ..., A_k \in L^0$ such that

$$\bigcap_0^k \gamma(A_j) = \emptyset$$

and $A_j u \in C^\infty$, $j = 1, ..., k$. By Proposition 2.5.1 we can choose E so that

$$u - E \sum_0^k A_j^* A_j u \in C^\infty.$$

Thus

$$u - EA_0^* A_0 u \in C^\infty$$

and so

$$Au - AEA_0^* A_0 u \in C^\infty.$$

Because $\Gamma \cup \Gamma_1 = T^*(\mathbf{R}^n) \setminus 0$, the formulas for the symbol of a product show that $AEA_0^* A_0$ is of order $-\infty$, so $Au \in C^\infty$ as asserted.

End of proof of Proposition 2.5.5. Let $p(\xi) \in C^\infty$ be homogeneous of degree 0 outside a compact set, let $p = 0$ in a neighborhood of K and $p = 1$ at infinity in K_1. Then $p(D)$ satisfies the hypothesis of Lemma 2.5.6 so $p(D)u \in C^\infty$. Hence $p(\xi)\hat{u}(\xi)$ is rapidly decreasing so that (2.5.3) holds.

To sum up, if $u \in \mathcal{D}'(X)$, $X \subset \mathbf{R}^n$, then $(x, \xi) \notin WF(u)$ if and only if there exists a function $\phi \in C_0^\infty X)$ with $\phi(x) \neq 0$ such that the Fourier transform of ϕu is rapidly decreasing in a conic neighborhood of the half ray with direction ξ.

We shall now give an example improving Proposition 1.2.3.

PROPOSITION 2.5.7. *Let $X \subset \mathbf{R}^n$, Γ an open cone in $X \times (\mathbf{R}^N \setminus 0)$ and ϕ a phase function in Γ. If $a \in S^m_{\varrho,\delta}(X \times \mathbf{R}^N), \varrho > 0$, $\delta < 1$, vanishes near the zero section and* cone supp $a \subset \Gamma$, *then*

$$WF(A) \subset \{(x, \phi'_x);\ (x, \theta) \in \text{cone supp}\, a,\ \phi'_\theta(x, \theta) = 0\}$$

if A is the distribution $u \to I_\phi(au)$ defined by (1.2.1).

Note that in section 2.4 the principal symbol was defined in a set which by Proposition 2.5.7 contains the wave front set of the distributions in $I^m_{\varrho,\delta}(X, Y)$.

Proof. It is sufficient to show that if K_1, K_2 are disjoint closed cones in $R^n \setminus 0$ with $\phi'_x \in K_1$ as $(x, \theta) \in$ cone supp a then $\widehat{\chi A}$ is rapidly decreasing in K_2 if $\chi \in C_0^\infty(X)$. Now

$$\widehat{\chi A}(\xi) = \iint e^{i(\phi(x,\theta) - \langle x, \xi \rangle)} a(x, \theta)\, \chi(x)\, dx\, d\theta$$

and the hypotheses imply that for $(x, \theta) \in$ cone supp a and $\xi \in K_2$ we have for some $C > 0$

$$|\phi'_x(x, \theta) - \xi| \geqslant C(|\theta| + |\xi|).$$

Modification of the proof of Lemma 1.2.1 therefore gives a first order differential operator

$$L = \Sigma\, a_j\, \partial/\partial x_j + c$$

with ${}^tL \exp i(\phi(x, \theta) - \langle x, \xi \rangle) = \exp i(\phi(x, \theta) - \langle x, \xi \rangle)$ such that $a_j(x, \theta, \xi)$, $c(x, \theta, \xi)$ are C^∞ functions of x which are homogeneous of degree -1 with respect to (θ, ξ) for $(x, \theta) \in$ cone supp a and $\xi \in K_2$. It follows that for every integer $k > 0$

$$\widehat{\chi A}(\xi) = \iint e^{i(\phi(x,\theta) - \langle x, \xi \rangle)}\, L^k(a(x, \theta)\, \chi(x))\, dx\, d\theta,$$

where the integrand can be estimated by

$$(|\xi| + |\theta|)^{-k}(1 + |\theta|)^{m + \delta k}.$$

Choosing $k_0 \geqslant 0$ so that $m + (\delta - 1)k_0 \leqslant -N - 1$ we estimate this for $k > k_0$ by

$$(1 + |\theta|)^{-N-1}(1 + |\xi|)^{(k_0 - k)(1 - \delta)}$$

and conclude that
$$\widehat{\chi A}(\xi) = O(|\xi|^{-k})$$

for every k when $\xi \to \infty$ in K_2.

In particular, if A is a pseudo-differential operator in X, and if K_A is the kernel of A, we have $WF(K_A)\subset$ normal bundle of the diagonal Δ in $X\times X$ which by the projection $T^*X\times T^*X\to T^*X$ on the first factor we can identify with T^*X. Thus $WF(K_A)$ can be identified with a closed cone in $T^*X\setminus 0$ which we denote by $WF(A)$. It is clear that if $Y\subset X$ is an open set and A_Y the restriction of A to Y, then $WF(A_Y)=WF(A)\cap\pi^{-1}Y$ where π: $T^*X\to X$ is the projection. This reduces the study of $WF(A)$ to the case with $X\subset\mathbf{R}^n$ and then we shall prove

PROPOSITION 2.5.8. *If $X\subset R^n$ and A is a properly supported operator in $L^m_{\varrho,\delta}(X)$, $0\leqslant\delta<\varrho\leqslant 1$, then the complement of $WF(A)$ is the largest open cone in $T^*X\setminus 0$ where σ_A, defined by* (2.1.5), *is rapidly decreasing.*

Proof. That the complement of $WF(A)$ contains this cone is an immediate consequence of Proposition 2.5.7. To prove the opposite inclusion let $(x_0,\xi_0)\in T^*(X)\setminus(0\cup WF(A))$. We have to prove that σ_A is rapidly decreasing in a conic neighborhood. Choose $\phi,\psi\in C^\infty$ equal to 1 near x_0, but with such small support that $WF(\phi A\psi)$ does not meet $X\times\Gamma$ where Γ is a conic neighborhood of ξ_0. Write σ_1 for the symbol of $v\to\phi A\psi v$ which over a neighborhood of x_0 differs from σ_A by a rapidly decreasing function. If $\chi(x,y)=\phi(x)\psi(y)$, then

$$\widehat{\chi K_A}(\eta+\xi,\,-\xi)=\langle\phi(\cdot)\,e^{i\langle\cdot,\,-\eta-\xi\rangle},\,A\psi e^{i\langle\cdot,\,\xi\rangle}\rangle=\int e^{-i\langle x,\eta\rangle}\sigma_1(x,\xi)\,dx=\hat{\sigma}_1(\eta,\xi)$$

is by assumption rapidly decreasing if $\xi\to\infty$ in a conic neighborhood Γ_1 of ξ_0 and $|\eta|\leqslant\varepsilon|\xi|$ for a certain $\varepsilon>0$. On the other hand (cf. (2.1.3))

$$|\hat{\sigma}_1(\eta,\xi)|\leqslant C_\nu(1+|\eta|)^{-\nu}\,(1+|\xi|)^{m+\delta\nu}$$

for any ν. If follows that

$$\int_{|\eta|>\varepsilon|\xi|}|\hat{\sigma}_1(\eta,\xi)|\,d\eta=O(|\xi|^{n+m-(1-\delta)\nu})$$

is rapidly decreasing as $\xi\to\infty$ so by the Fourier inversion formula $\sigma_1(x,\xi)$ is rapidly decreasing as $\xi\to\infty$ in Γ_1. Since $\sigma_1\in S^m_{\varrho,\delta}$ it follows (see the proof of Theorem 2.9 in [13]) that all derivatives of σ_1 are also rapidly decreasing in a smaller cone which proves the statement.

Combining Proposition 2.5.8 with Lemma 2.5.6 we have proved

PROPOSITION 2.5.9. *If $u\in\mathcal{D}'(X)$ we have $Au\in C^\infty$ for all properly supported pseudo-differential operators A with*

$$WF(A)\cap WF(u)=\emptyset.$$

If Γ is a closed cone $\subset T^*(X)$ the preceding proposition leads us to define as follows a pseudo-topology in $\mathcal{D}'_\Gamma(X)=\{u\in\mathcal{D}'(X);\ WF(u)\subset\Gamma\}$: *A sequence* $u_j\in\mathcal{D}'_\Gamma(X)$ *is said to converge to* $u\in\mathcal{D}'_\Gamma(X)$ *if*

(i) $u_j\to u$ in $\mathcal{D}'(X)$ (weakly)

(ii) $Au_j\to Au$ in $C^\infty(X)$ if A is a properly supported pseudo-differential operator with

$$\Gamma\cap WF(A)=\emptyset.$$

Note that if the supports of all u_j belong to a fixed compact set $K\subset X$ we need only assume in (ii) that the convergence takes place in $C^\infty(Y)$ when Y is open and $K\subset Y\subset X$. Together with a partition of unity this allows us to consider only the case $X\subset\mathbf{R}^n$ and $u\in\mathcal{E}'(X)$ in what follows.

First we shall prove that $C^\infty(X)$ is sequentially dense in $\mathcal{D}'_\Gamma(X)$. To do so we take (cf. Proposition 1.1.11) a function $\chi\in\mathcal{S}(\mathbf{R}^n)$ with $\chi(0)=1$ and set for $u\in\mathcal{D}'_\Gamma(X)\cap\mathcal{E}'(X)$, $X\subset\mathbf{R}^n$,

$$u_j=\chi(D/j)u\in\mathcal{S}(R^n).$$

When $j\to\infty$ we have $u_j\to u$ in $\mathcal{S}'$, hence in $\mathcal{D}'(X)$. If A is a pseudo-differential operator with $WF(A)\cap\Gamma=\emptyset$ we must prove also that $Au_j\to Au$ in $C^\infty(X)$. In doing so we may assume that the support of u is so small that there is no $\xi\neq 0$ with $(x,\xi)\in WF(A)$, $(y,\xi)\in\Gamma$ and $x, y\in\operatorname{supp} u$. Let $K=\{\xi;\ (x,\xi)\in\Gamma$ for some $x\in\operatorname{supp} u\}$. By Proposition 2.5.5 we know that $\hat{u}$ is rapidly decreasing outside K and we have arranged so that $\operatorname{supp} u\times K$ does not meet $WF(A)$. Now

$$Au_j(x)=(2\pi)^{-n}\int e^{i\langle x,\xi\rangle}\sigma_A(x,\xi)\,\hat{u}(\xi)\,\chi(\xi/j)\,d\xi,$$

where $\sigma_A(x,\xi)\hat{u}(\xi)$ is rapidly decreasing over a neighborhood V of $\operatorname{supp} u$ because some factor is. It follows that $Au_j\to Au$ in $C^\infty(V)$. Since $\sigma_A(x,\xi)\chi(\xi/j)$ belongs to a bounded set in $S^m_{\varrho,\delta}$ the kernel of $\sigma_A(x,D)\chi(D/j)$ converges in C^∞ to that of $\sigma_A(x,D)$ outside the diagonal which proves that $Au_j=A\chi(D/j)u\to Au$ in C^∞ outside $\operatorname{supp} u$. Thus $Au_j\to Au$ in $C^\infty(X)$. In particular, $u_j\to 0$ in C^∞ outside $\operatorname{supp} u$ so we can modify the sequence so that it has support in a fixed compact set by multiplication with a C_0^∞ function which is one near $\operatorname{supp} u$.

We shall now study the multiplication of distributions. Let Γ_1, Γ_2 be two closed cones in $T^*(X)\setminus 0$ such that

$$\Gamma_1+\Gamma_2=\{(x,\xi_1+\xi_2);\ (x,\xi_j)\in\Gamma_j\}\subset T^*(X)\setminus 0. \tag{2.5.4}$$

Then $(\Gamma_1+\Gamma_2)\cup\Gamma_1\cup\Gamma_2$ is also a closed cone in $T^*(X)\setminus 0$. In fact, assume that $(x^\nu, \xi_j^\nu)\in\Gamma_j$ and that $(x^\nu, \xi_1^\nu+\xi_2^\nu)\to(x,\xi)$ where $\xi\neq 0$. If $\xi_1^\nu\to 0$ (or $\xi_2^\nu\to 0$) it follows that $(x,\xi)\in\Gamma_2$ (or Γ_1). If with respect to some Riemannian metric $|\xi_1^\nu|\to\infty$, then $(x^\nu, \xi_1^\nu/|\xi_1^\nu|)$ has a limit point $(x,\eta)\in\Gamma_1$, $|\eta|=1$, and so $(x^\nu, \xi_2^\nu/|\xi_1^\nu|)$ has the limit point $(x, -\eta)\in\Gamma_2$ in contradiction to (2.5.4). Ruling out these cases we can always pass to a subsequence such that $(x^\nu, \xi_j^\nu)\to(x,\xi_j)\in\Gamma_j$; and since $\xi=\xi_1+\xi_2$ we obtain then that $(x,\xi)\in\Gamma_1+\Gamma_2$.

THEOREM 2.5.10. *Let Γ_1, Γ_2 be two closed cones in $T^*(X)\setminus 0$ satisfying* (2.5.4). *Then the product u_1u_2 of distributions $u_j\in\mathscr{D}'_{\Gamma_j}(X)$ can be defined in one and only one way so that it is sequentially continuous with values in $\mathscr{D}'(X)$. We have*

$$WF(u_1u_2)\subset(\Gamma_1+\Gamma_2)\cup\Gamma_1\cup\Gamma_2. \tag{2.5.5}$$

Proof. The uniqueness is obvious so we need only prove the existence and (2.5.5). In doing so we may assume that $X\subset\mathbf{R}^n$ and consider u_1 and u_2 with support in such a small neighborhood V of a point in X that for the closed cones

$$K_j=\{\xi;\ (x,\xi)\in\Gamma_j \text{ for some } x\in\operatorname{supp} u_j\}$$

we have $0\notin K_1+K_2$. Let $p_j(\xi)$ be a homogeneous function of degree 0 which is 1 in a neighborhood of K_j and for which still $0\neq\xi_1+\xi_2$ if $0\neq\xi_j\in\operatorname{supp} p_j$. By Proposition 2.5.5 we know then that

$$\sup|(1-p_j(\xi))\hat u_j(\xi)|(1+|\xi|)^N<\infty$$

for any N, and if a sequence of $u_j^\nu\in\mathscr{D}'_{\Gamma_j}$ with support in V converges to 0 then this supremum converges to 0. Moreover, one then has pointwise convergence to 0 of the Fourier transforms and a uniform bound

$$|\hat u_j(\xi)|\leqslant C(1+|\xi|)^\mu.$$

(We drop ν in order not to complicate the notation.)

The convolution

$$\hat u_1*\hat u_2(\xi)=\int\hat u_1(\xi-\eta)\,\hat u_2(\eta)\,d\eta$$

is absolutely convergent. This is quite clear if we note that the right-hand side is a sum

$$\int(p_1\hat u_1)(\xi-\eta)(p_2\hat u_2)(\eta)\,d\eta+\int(p_1\hat u_1)(\xi-\eta)((1-p_2)\hat u_2)(\eta)\,d\eta$$
$$+\int((1-p_1)\hat u_1)(\xi-\eta)(p_2\hat u_2)(\eta)\,d\eta+\int((1-p_1)\hat u_1)(\xi-\eta)((1-p_2)\hat u_2)(\eta)\,d\eta.$$

The support of the first term lies in $\operatorname{supp} p_1+\operatorname{supp} p_2$ and it can be bounded by

$C'(1+|\xi|)^{2\mu+n}$. Since $(1-p_j)\hat{u}_j$ is rapidly decreasing the other integrals can be bounded by $C'(1+|\xi|)^{\mu}$. All will converge pointwise to 0 if we have sequences $u_j^\nu\to 0$ in $\mathcal{D}'_\Gamma(X)$ with support in V. This shows that if we define

$$u_1 u_2 = \mathcal{F}^{-1}\left((2\pi)^{-n}\int \hat{u}_1(\xi-\eta)\,\hat{u}_2(\eta)\,d\eta\right)$$

we obtain a continuous map into $\mathcal{D}'$ which of course is standard multiplication if $u_1, u_2 \in C^\infty$.

It remains to verify (2.5.5). The first term again vanishes outside $\operatorname{supp} p_1 + \operatorname{supp} p_2$. The last one is rapidly decreasing and the other two decrease rapidly outside $\operatorname{supp} p_1$ and $\operatorname{supp} p_2$ respectively. It follows that $WF(u_1u_2)\subset V\times((K_1+K_2)\cup K_1\cup K_2)$. If we choose a small neighborhood of x, then $(K_1+K_2)\cup K_1\cup K_2$ is as close to the fiber of $(\Gamma_1+\Gamma_2)\cup\Gamma_1\cup\Gamma_2$ as we wish which proves (2.5.5).

In particular, the preceding results allow us to discuss restrictions to submanifolds. First note that if V is the subspace $x_{k+1}=\ldots=x_n=0$ of $\mathbf{R}^n$, the distribution f_V defined in V by the restriction of a function f in $\mathbf{R}^n$ is given by

$$f_V(\phi)=\langle f\phi\sigma, 1\rangle, \quad \phi\in C_0^\infty(V)$$

where σ is the Lebesgue measure in V considered as a measure in R^n. Now the Fourier transform of $\phi\sigma$ in $\mathbf{R}^n$ is

$$\mathbf{R}^n \ni \xi \to \hat{\phi}(\xi_1, \ldots, \xi_k)$$

which is rapidly decreasing except in the directions normal to V. If $\phi^\nu\in C_0^\infty(V)$ and $\phi^\nu\to 0$ in the sense of Schwartz, then $\phi^\nu\sigma\to 0$ in $\mathcal{D}'_\Gamma(R^n)$ if Γ is the normal bundle of V.

If f is any distribution with $WF(f)\cap\Gamma=\emptyset$ the product $f(\phi\sigma)$ still makes sense and

$$\phi\to\langle f(\phi\sigma), 1\rangle \tag{2.5.6}$$

will be a continuous linear form on $C_0^\infty(V)$. We take this as our definition of the restriction f_V and have proved

THEOREM 2.5.11. *Let X be a manifold and Y a submanifold with normal bundle denoted by $N(Y)$. For every distribution f with $WF(f)\cap N(Y)=\emptyset$ the restriction of f can be uniquely defined so that it is a sequentially continuous function from $\mathcal{D}'_\Gamma(X)$ to $\mathcal{D}'(Y)$ for any closed cone $\Gamma\subset T^*(X)\setminus 0$ with $\Gamma\cap N(Y)=\emptyset$.*

Remark. This theorem is very close to well-known results on partial hypoellipticity (see Hörmander [17, Chapter IV]).

There is also a more general and precise version of Theorem 2.5.11.

THEOREM 2.5.11′. *Let X and Y be manifolds and φ: $Y \to X$ be a C^∞ map, and let*

$$N_\varphi = \{(\varphi(y), \xi) \in T^*(X);\ {}^t\varphi'_y(y)\xi = 0\}$$

be the set of normals of the map. If $f \in \mathcal{D}'(X)$ and $WF(f) \cap N_\varphi = \emptyset$ we can define the pullback $\varphi^* f$ in one and only one way so that it is equal to the composition $f \circ \varphi$ when f is a continuous function and is sequentially continuous from $\mathcal{D}'_\Gamma(X)$ to $\mathcal{D}'(Y)$ for any closed cone $\Gamma \subset T^*(X) \setminus 0$ with $\Gamma \cap N_\varphi = \emptyset$. Moreover,

$$WF(\varphi^* f) \subset \varphi^* WF(f) = \{(y, {}^t\varphi'_y(y)\xi),\ (\varphi(y), \xi) \in WF(f)\}.$$

Proof. It is sufficient to verify this locally so we assume that X and Y are open subsets in $\mathbf{R}^n$ and in $\mathbf{R}^m$ respectively. If $f \in C_0^\infty$ we have by Fourier's inversion formula

$$(\varphi^* f)(y) = (2\pi)^{-n} \int e^{i\langle \varphi(y), \xi\rangle} \hat{f}(\xi)\, d\xi.$$

If χ is a test function in Y with support near y_0 then

$$\langle \varphi^* f, \chi\rangle = (2\pi)^{-n} \int \hat{f}(\xi)\, I_\chi(\xi)\, d\xi$$

where
$$I_\chi(\xi) = \int \chi(y)\, e^{i\langle \varphi(y), \xi\rangle}\, dy$$

is rapidly decreasing in any cone where ${}^t\varphi'(y)\xi \neq 0$ when $y \in \operatorname{supp} \chi$. If V is a conic neighborhood of $\{\xi, {}^t\varphi'(y_0)\xi = 0\}$ it follows that $I_\chi(\xi)$ is rapidly decreasing outside V if the support of χ is sufficiently close to y_0. On the other hand, if V and the support of f are sufficiently small, then

$$\sup_V |\hat{f}(\xi)|\, (1 + |\xi|)^N$$

is for every N a continuous semi-norm in $\mathcal{D}'_\Gamma(X)$. It follows that $\varphi^* f$ can be extended by continuity as stated. To prove the last statement we note that

$$\mathcal{F}(\chi\varphi^* f)(\eta) = (2\pi)^{-n} \int \hat{f}(\xi)\, d\xi \int e^{i\langle \varphi(y), \xi\rangle - i\langle y, \eta\rangle} \chi(y)\, dy.$$

If (ξ, η) is outside a conic neighborhood of

$$C = \{(\xi, \eta);\ {}^t\varphi'(y_0)\xi = \eta\}$$

and the support of χ is sufficiently close to y_0, then the inner integral can be estimated by $C_N(1 + |\xi| + |\eta|)^{-N}$ for any N so the corresponding contribution to the integral is $O(1 + |\eta|)^{-N}$ for any N. On the other hand, near C we can estimate $|\eta|$ by $|\xi|$. If in addition η is outside a conic neighborhood of

$$\{{}^t\varphi'(y_0)\xi,\ \xi \in WF_{\varphi(y_0)}(f)\}$$

we must have $\hat{f}(\xi) = O(|\xi|^{-N})$ for any N if the support of f is sufficiently close to $\varphi(y_0)$. Hence $\mathcal{F}(\chi\varphi^* f)(\eta) = O(|\eta|^{-N})$ for any N outside a conic neighborhood of $\varphi^* WF(f)_{y_0}$ if supp χ and supp f are sufficiently close to y_0 and $\varphi(y_0)$ respectively. This completes the proof.

Note that we have defined the pullback $\varphi^* f$ for every f precisely when ${}^t\varphi'$ is injective, that is, φ' is surjective. In that case the definition is of course very well known.

Next we consider the linear transformation defined by a distribution $K \in \mathcal{D}'(X \times Y)$ where $X \subset \mathbf{R}^n$, $Y \subset \mathbf{R}^\nu$ are open sets. (The results have an obvious extension to manifolds if one works throughout with densities of order $1/2$ as we shall do in Chapter IV.) Then K defines a continuous map $K\colon C_0^\infty(Y) \to \mathcal{D}'(X)$,

$$\langle K\varphi, \psi\rangle = K(\psi \otimes \varphi); \quad \varphi \in C_0^\infty(Y),\ \psi \in C_0^\infty(X).$$

THEOREM 2.5.12. *For any* $u \in C_0^\infty(Y)$ *the set*

$$WF_X(K) = \{(x, \xi);\ (x, \xi, y, 0) \in WF(K) \text{ for some } y \in Y\} \tag{2.5.7}$$

contains $WF(Ku)$.

In view of Theorem 2.5.3 we obtain

COROLLARY 2.5.13. *If* $WF(K)$ *contains no point which is normal to a manifold* $x =$ *constant then* $KC_0^\infty(Y) \subset C^\infty(X)$.

Proof of Theorem 2.5.12. Since we may split K into a sum of distributions with arbitrarily small support it suffices to prove that if $K \in \mathcal{E}'$ and

$$WF(K) \subset X \times Y \times \Gamma,$$

where Γ is a closed cone in $\mathbf{R}^{n+\nu} \setminus 0$, then

$$WF(Ku) \subset X \times \Gamma_0 \quad \text{where} \quad \Gamma_0 = \{\xi,\ (\xi, 0) \in \Gamma\}.$$

To do so we note that

$$\widehat{Ku}(\xi) = (2\pi)^{-\nu} \int \hat{K}(\xi, -\eta)\,\hat{u}(\eta)\,d\eta.$$

If $\Gamma_1 \subset \mathbf{R}^n \setminus 0$ is a closed cone which does not meet Γ_0, we have for some $\varepsilon > 0$

$$|\hat{K}(\xi, -\eta)| \leqslant C_k(1 + |\xi|)^{-k}, \quad k > 0, \quad |\eta| < \varepsilon|\xi|, \quad \xi \in \Gamma_1.$$

Since $|\hat{K}(\xi, -\eta)| \leqslant C(1 + |\xi| + |\eta|)^N$, it follows that for $\xi \in \Gamma_1$,

$$|\widehat{Ku}(\xi)| \leqslant C_k'(1 + |\xi|)^{-k} + C' \int_{|\eta| > \varepsilon|\xi|} (1 + |\eta|)^N |\hat{u}(\eta)|\,d\eta$$

so $\widehat{Ku}$ is rapidly decreasing in Γ_1. Hence $WF(Ku) \subset X \times \Gamma_0$ as was to be proved.

An essentially dual question concerns the definition of Ku for general distributions u. First note that if $u \in \mathcal{D}'(Y)$ then $WF(1 \otimes u) = X \times WF(u)$. The product $K(1 \otimes u)$ is therefore well defined when $WF(K) + (X \times WF(u))$ does not meet the zero section, that is, $WF(u)$ does not meet

$$\{(y, \eta);\ (x, 0, y, -\eta) \in WF(K) \text{ for some } x\} = WF'_Y(K). \tag{2.5.8}$$

When $u \in \mathcal{E}'_\Gamma(Y)$ for some Γ not meeting $WF'_Y(K)$ the product depends continuously on u and so does the integral with respect to y. This we define to be Ku. Explicitly,

$$\langle Ku, \phi \rangle = \langle K(1 \otimes u), \phi \otimes 1 \rangle, \quad \phi \in C_0^\infty(X).$$

By what we have proved this is a continuous map from $\mathcal{E}'_\Gamma(Y)$ to $\mathcal{D}'(X)$ when Γ does not meet $WF'_Y(K)$. In particular, when the set $WF'_Y(K)$ is empty we have a continuous map $\mathcal{E}'(Y) \to \mathcal{D}'(X)$.

To estimate $WF(Ku)$ in terms of $WF(K)$ and $WF(u)$ we have to make sure that $KC_0^\infty \subset C^\infty$, so we assume that the hypotheses of Corollary 2.5.13 are fulfilled.

THEOREM 2.5.14. *Let $X \subset \mathbf{R}^n$, $Y \subset \mathbf{R}^\nu$ and $K \in \mathcal{D}'(X \times Y)$. If $WF_X(K)$ is empty, $u \in \mathcal{E}'(Y)$, and $WF(u)$ does not meet $WF'_Y(K)$ then*

$$WF(Ku) \subset WF'(K)\, WF(u), \tag{2.5.9}$$

where $WF'(K) = \{(x, \xi, y, -\eta) \in T^*(X) \times T^*(Y);\ (x, \xi, y, \eta) \in WF(K)\}$ is regarded as a relation mapping sets in $T^*(Y) \setminus 0$ to sets in $T^*(X) \setminus 0$.

Proof. Localizing by a partition of unity as indicated in the proof of Theorem 2.5.12 it suffices to prove that if K has compact support and

$$WF(K) \subset X \times Y \times \Gamma, \quad WF(u) \subset Y \times \Gamma_Y$$

where $\Gamma(\Gamma_Y)$ is a closed cone in $\mathbf{R}^{n+\nu} \setminus 0$ resp. $\mathbf{R}^\nu \setminus 0$ then

$$WF(Ku) \subset X \times Y \times (\Gamma' \Gamma_Y).$$

Here, we assume that $(\xi, \eta) \in \Gamma \Rightarrow \eta \neq 0$ and $\Rightarrow -\eta \notin \Gamma_Y$ when $\xi = 0$. To prove this we choose functions $p(\xi)$, $q(\xi, \eta)$ homogeneous of degree 0 which are equal to 1 in conical neighborhoods of Γ_Y and Γ so small that Γ, Γ_Y may be replaced by $\operatorname{supp} q \setminus \{0\}$, $\operatorname{supp} p \setminus \{0\}$ in the assumption above. Now

$$\widehat{Ku}(\xi) = (2\pi)^{-\nu} \int \hat{K}(\xi, -\eta)\, \hat{u}(\eta)\, d\eta$$

for this is true when $u\in C_0^\infty$ and the right hand side is a continuous function of $u\in\mathcal{E}'_{Y\times\Gamma_Y}$ since $\hat{K}$ is rapidly decreasing in a neighborhood of $0\times(-\Gamma_Y)$. To study $\widehat{Ku}$ we split the integral into three parts

$$I_1=\int\hat{K}(\xi,\,-\eta)\,(1-p(\eta))\,\hat{u}(\eta)\,d\eta$$

$$I_2=\int(1-q(\xi,\,-\eta))\,\hat{K}(\xi,\,-\eta)\,p(\eta)\,\hat{u}(\eta)\,d\eta$$

$$I_3=\int q(\xi,\,-\eta)\,\hat{K}(\xi,\,-\eta)\,p(\eta)\,\hat{u}(\eta)\,d\eta.$$

I_1 is rapidly decreasing since $(1-p)\hat{u}$ is rapidly decreasing and $\hat{K}$ is rapidly decreasing near $\mathbf{R}^n\times 0$ (see the proof of Theorem 2.5.12). Since $(1-q)\hat{K}$ is rapidly decreasing it is clear that I_2 is also rapidly decreasing. Finally I_3 vanishes unless for some $\eta\neq 0$ we have

$$(\xi,\,-\eta)\in\operatorname{supp}q,\quad \eta\in\operatorname{supp}p$$

which means that ξ is close to $\Gamma'\cdot\Gamma_Y$. This proves the theorem.

Remark. If A is a pseudo-differential operator in X, it follows from (2.5.9) that $WF(Au)\subset WF(u)$ if $u\in\mathcal{D}'(X)$. This improves the pseudo-local property and is of interest in connection with the following construction which seems to be the analogue for distributions of one given by Sato [27] for hyperfunctions:

If $O\subset T^*(X)\setminus 0$ is an open cone we introduce

$$\mathcal{C}_O=\mathcal{D}'(X)/\mathcal{D}'_{\complement O}(X).$$

These vector spaces form a presheaf $\mathcal{C}$ on the unit sphere bundle of $T^*(X)$; the sections of the sheaf are easily seen to be isomorphic to $\mathcal{D}'(X)/\mathcal{D}'_\emptyset(X)=\mathcal{D}'(X)/C^\infty(X)$. The preceding remark shows that all pseudo-differential operators in X (or more generally operators defined by distributions K with $WF(K)$ contained in the diagonal) define sheaf maps on $\mathcal{C}$.

Let now $X\subset\mathbf{R}^{n_X}$, $Y\subset\mathbf{R}^{n_Y}$, $Z\subset\mathbf{R}^{n_Z}$ be open sets and $K_1\in\mathcal{D}'(X\times Y)$, $K_2\in\mathcal{D}'(Y\times Z)$ be properly supported. In view of Theorem 2.5.12 the composition $K_1(K_2u)$ is then defined for $u\in C^\infty(Z)$ provided that

$$WF'_Y(K_1)\cap WF_Y(K_2)=\emptyset. \tag{2.5.10}$$

(Note that $WF_X(K_1)$, ..., $WF'_Z(K_2)$ are closed since K_1 and K_2 are properly supported.) The map $C_0^\infty(Z)\ni u\to K_1(K_2u)\in\mathcal{E}'(X)$ is then continuous so it defines a distribution

$$K=K_1\circ K_2\in\mathcal{D}'(X\times Z).$$

THEOREM 2.5.15. *When* (2.5.10) *is valid we have*

$$WF'(K_1\circ K_2)\subset WF'(K_1)\circ WF'(K_2)\cup(WF_X(K_1)\times Z)\cup(X\times WF'_Z(K_2)) \qquad (2.5.11)$$

Proof. Let us first verify that the set just defined is closed in $T^*(X\times Z)\setminus 0$. So let $(x^\nu,\xi^\nu,z^\nu,\zeta^\nu)\to(x,\xi,z,\zeta)$ where $(\xi,\zeta)\neq(0,0)$, let $(x^\nu,\xi^\nu,y^\nu,\eta^\nu)\in WF'(K_1)$ and $(y^\nu,\eta^\nu,z^\nu,\zeta^\nu)\in WF'(K_2)$. Since K_1, K_2 are proper the points y^ν belong to a fixed compact set so we may assume that $y^\nu\to y$. If $\eta^\nu\to\infty$ and η is a limit point of $\eta^\nu/|\eta^\nu|$, then $(x,0,y,\eta)\in WF'(K_1)$ and $(y,\eta,z,0)\in WF(K_2)$ which implies that $(y,\eta)\in WF'_Y(K_1)$, $(y,\eta)\in WF_Y(K_2)$ and contradicts (2.5.10). Hence we may also assume that η^ν has a finite limit η. If $\eta\neq 0$ we conclude that (x,ξ,z,ζ) is in the first set on the right, and if $\eta=0$ it is in one of the others since either ξ or ζ is $\neq 0$.

The preceding argument also shows that one can replace $WF(K_j)$ by conic neighborhoods so that the right hand side of (2.5.11) does not grow beyond a given conic neighborhood. This observation and a localization reduces the proof to showing that if K_j have compact supports and there are closed cones Γ_1 and Γ_2 in $\mathbf{R}^{n_X+n_Y}\setminus 0$ resp. $\mathbf{R}^{n_Y+n_Z}\setminus 0$ such that

$$\{\eta;\,(0,\eta)\in\Gamma_1\}\cap\{\eta;\,(\eta,0)\in\Gamma_2\}=\emptyset \qquad (2.5.12)$$

and $X\times Y\times\Gamma_1$ (resp. $Y\times Z\times\Gamma_2$) is a neighborhood of $WF'(K_1)$ (resp. $WF'(K_2)$) then

$$WF'(K_1\circ K_2)\subset X\times Z\times(\Gamma_1\circ\Gamma_2\cup\Gamma_1^0\cup\Gamma_2^0).$$

Here $$\Gamma_1^0=\{(\xi,0)\in R^{n_X+n_Z};\,(\xi,0)\in\Gamma_1\}$$

and $$\Gamma_2^0=\{(0,\zeta)\in R^{n_X+n_Z};\,(0,\zeta)\in\Gamma_2\}.$$

(Note that 0 denotes the origin in any one of the vector spaces $\mathbf{R}^{n_X}$, $\mathbf{R}^{n_Y}$, $\mathbf{R}^{n_Z}$.)

Now we have

$$(2\pi)^{n_Y}\widehat{K_1\circ K_2}(\xi,-\zeta)=\int\hat K_1(\xi,-\eta)\,\hat K_2(\eta,\zeta)\,d\eta.$$

In fact, the integral converges since $\hat K_1(\xi,-\eta)$ $(\hat K_2(\eta,-\zeta))$ is rapidly decreasing outside $\Gamma_1(\Gamma_2)$ and (2.5.12) holds; the formula is obviously valid if $K_j\in C_0^\infty$ and follows in general by continuity. The integral is a sum of the following four where p_j denotes the characteristic function of Γ_j

$$I_1 = \int p_1(\xi,\eta)\,\hat{K}_1(\xi,\,-\eta)\,p_2(\eta,\zeta)\,\hat{K}_2(\eta,\,-\zeta)\,d\eta$$

$$I_2 = \int (1-p_1(\xi,\eta))\,\hat{K}_1(\xi,\,-\eta)\,p_2(\eta,\zeta)\,\hat{K}_2(\eta,\,-\zeta)\,d\eta$$

$$I_3 = \int p_1(\xi,\eta)\,\hat{K}_1(\xi,\,-\eta)\,(1-p_2(\eta,\zeta))\,\hat{K}_2(\eta,\,-\zeta)\,d\eta$$

$$I_4 = \int (1-p_1(\xi,\eta))\,\hat{K}_1(\xi,\,-\eta)\,(1-p_2(\eta,\zeta))\,\hat{K}_2(\eta,\,-\zeta)\,d\eta.$$

Here $(1-p_1(\xi,\eta))\hat{K}_1(\xi,-\eta)$ and $(1-p_2(\eta,\zeta))\hat{K}_2(\eta,-\zeta)$ are rapidly decreasing so I_4 is rapidly decreasing. The integrand in I_3 can be estimated by

$$C_k(1+|\xi|+|\eta|)^N(1+|\eta|+|\zeta|)^{-k}$$

where N is fixed and k is arbitrary. If $\varepsilon>0$ it follows that I_3 is rapidly decreasing for $|\zeta|>\varepsilon|\xi|$ and that the contribution when $|\eta|>\varepsilon|\xi|$ is always rapidly decreasing. This implies that I_3 is rapidly decreasing outside Γ_1^0. Similarly I_2 is rapidly decreasing outside Γ_2^0. Since I_1 vanishes outside $\Gamma_1\circ\Gamma_2$ the proof is complete.

Remark. Note that if $WF_X(K_1)$, $WF'_Y(K_1)$, $WF_Y(K_2)$, $WF'_Z(K_2)$ are all empty, then (2.5.10) is automatically fulfilled and only the composition occurs in the right hand side of (2.5.11). Theorem 2.5.14 is essentially the special case when Z is a point.

III. Distributions defined by oscillatory integrals

3.0. Introduction

In this chapter and in Chapter IV we shall extend the results of section 2.4 to arbitrary non-degenerate phase functions. The main complication which occurs is that there is no simple analogue of Proposition 2.4.1. The extent to which Proposition 2.4.1 can be generalized will be determined in section 3.1. The result will not quite suffice for the proof of the transformation laws we need so the proof of these in section 3.2 will also depend on the method of stationary phase. Thus it is rather close to the earlier proofs of the invariance properties of pseudo-differential operators given in [13] for example. A consequence of this is that the principal symbols of the distributions we consider will no longer be scalars but sections of a line bundle defined by an integer cohomology class (or rather a class mod 4). This cohomology class also occurs in the work of Maslov [23] and Arnold [1]; in fact it was already introduced by Keller [18]. The geometrical interpretation of the line bundle is discussed in section 3.3 where we also establish the equivalence of our definitions with one used in [1].

3.1. Equivalence of non-degenerate phase functions

Let $X \subset \mathbf{R}^n$ and let Γ be an open conic set in $X \times \mathbf{R}^N$, ϕ a nondegenerate phase function in Γ. (For the definition see section 1.2.) We have seen in section 1.2 how a class of distributions is associated with ϕ and symbols in $S_\varrho^m(X \times \mathbf{R}^N)$ with cone support in Γ. We wish to determine to what extent this class of distributions depends on the choice of ϕ. First of all it is clear that if in Γ we have a fiber preserving diffeomorphism

$$\Gamma \ni (x, \theta) \to (x, \tilde{\theta}(x, \theta)) \in \tilde{\Gamma}$$

where $\tilde{\theta}$ is a C^∞ function, homogeneous with respect to θ of degree 1, then a change of variables in (1.2.1) gives

$$I_\phi(au) = \iint e^{i\tilde{\phi}(x, \tilde{\theta})} \tilde{a}(x, \tilde{\theta})\, u(x)\, dx\, d\tilde{\theta}$$

where $\tilde{\phi}(x, \tilde{\theta}(x, \theta)) = \phi(x, \theta)$, $\tilde{a}(x, \tilde{\theta}(x, \theta)) \,|D\tilde{\theta}/D\theta| = a(x, \theta)$. If $a \in S_\varrho^m$ it follows that $\tilde{a} \in S_\varrho^m$. It is thus clear that we can represent the same distributions in the form (1.2.1) with a symbol $\in S_\varrho^m$ (and suitable cone support) whether we use the phase function ϕ or $\tilde{\phi}$. We shall say that ϕ and $\tilde{\phi}$ are (locally) equivalent.

Our purpose in this section is to determine when two non-degenerate phase functions are equivalent. A necessary condition is obtained if we consider the map (cf. Proposition 2.5.7)

$$C \ni (x, \theta) \to (x, \phi_x') \in T^*(X) \setminus 0 \tag{3.1.1}$$

where 0 stands for the zero section and

$$C = \{(x, \theta);\ \phi_\theta'(x, \theta) = 0\}. \tag{3.1.2}$$

Since $\phi_\theta' = 0$ on C it is clear that (3.1.1) makes sense if X is a manifold and $X \times \mathbf{R}^N$ is replaced by a fiber space over X. Thus the range of (3.1.1) is the same for two equivalent phase functions. The map (3.1.1) is regular if ϕ is non-degenerate, for the tangent plane of C is defined by the equations $d\partial\phi/\partial\theta_j = 0$, $j = 1, \ldots, N$, and if in addition $dx = 0$, $d\partial\phi/\partial x_j = 0$, $j = 1, \ldots, n$, it follows that

$$\sum_k \partial^2\phi/\partial\theta_j \partial\theta_k\, d\theta_k = 0,\ j = 1, \ldots, N; \sum_k \partial^2\phi/\partial x_j \partial\theta_k\, d\theta_k = 0, \quad j = 1, \ldots, n.$$

These equations imply that $d\theta_k = 0$, $k = 1, \ldots, N$, for the differentials of $\partial\phi/\partial\theta_k$, $k = 1, \ldots, N$, are linearly independent by hypothesis. Locally the range Λ of the map (3.1.2) is thus a C^∞ manifold of dimension $\dim X$. Furthermore, Λ is conic, that is, invariant under the multiplication by positive scalars which is defined in the fibers of $T^*(X)$, and we have

$$\langle \xi, dx \rangle = \Sigma\, \xi_j dx_j = 0 \quad \text{on } \Lambda. \tag{3.1.3}$$

Here we have used local coordinates $x_1, ..., x_n$ in X and corresponding local coordinates in $T^*(X)$ obtained by taking $dx_1, .., dx_n$ as basis elements. We recall that the first order form in (3.1.3) is invariantly defined in the cotangent space of a manifold. The proof of (3.1.3) is trivial: Λ is parametrized by points on C according to (3.1.1), and we have

$$\langle \xi, dx \rangle = \langle \phi'_x, dx \rangle = d\phi - \langle \phi'_\theta, d\theta \rangle = 0$$

since $\phi'_\theta = 0$ on C and so $\phi = \langle \theta, \phi'_\theta \rangle = 0$ on C by Euler's identity. From (3.1.3) it follows by differentiation that

$$\Sigma\, d\xi_j \wedge dx_j = 0 \quad \text{on } \Lambda. \tag{3.1.4}$$

Thus Λ is a manifold of maximal dimension on which the symplectic two form of $T^*(X)$ vanishes. We shall call such a manifold *Lagrangean*, following Maslov [23]. It is classical that the stronger condition (3.1.3) is fulfilled on any conic Lagrangean manifold Λ. Indeed, if we use local coordinates, the fact that Λ is conic means that if $(x, \xi) \in \Lambda$, then $(0, \xi)$ belongs to the tangent plane of Λ at (x, ξ). Thus the tangent plane is orthogonal to $(0, \xi)$ with respect to the symplectic form, which means that (3.1.3) is valid.

Example 3.1.1. If ϕ is linear with respect to θ, then as we have seen in section 2.4 the Lagrangean manifold Λ is the normal bundle of a submanifold Y of X.

Example 3.1.2. Let $H(\xi)$ be a homogeneous C^∞ function of ξ of degree 1 in a cone $\Gamma \subset \mathbf{R}^n$, and define

$$\phi(x, \xi) = \langle x, \xi \rangle - H(\xi).$$

Then the condition $\phi'_\xi = 0$ means that $x = H'(\xi)$, so ϕ is non-degenerate and

$$\Lambda = \{(H'(\xi), \xi), \xi \in \Gamma\}.$$

The preceding example essentially covers the general case, for we have

THEOREM 3.1.3. *Let $\Lambda \subset T^*(X)$ be a conic Lagrangean manifold. For every $\lambda_0 \in \Lambda$ with the local coordinates $x_1, ..., x_n$ at $\pi\lambda_0 \in X$ suitably chosen one can find a function H which is homogeneous of degree 1 in an open cone Γ in $\mathbf{R}^n$ such that if $\phi(x, \xi) = \Sigma_1^n x_j \xi_j - H(\xi)$ the Lagrangean manifold defined by ϕ is a neighborhood of λ_0 in Λ.*

Proof. The local coordinates $x_1, ..., x_n$ give rise to local coordinates $x_1, ..., x_n, \xi_1, ..., \xi_n$ in $T^*(X)$ corresponding to the expression of a covector as a sum $\Sigma\, \xi_j dx_j$. If the map $\Lambda \ni (x, \xi) \to \xi$ is regular at λ_0, then we can in a neighborhood of λ_0 define Λ by an equation

$x = X(\xi)$ where X is homogeneous of degree 0 with respect to ξ. Since Λ is Lagrangean and conic, we have $\Sigma\, \xi_j dX_j = 0$, that is,

$$d(\Sigma\, \xi_j X_j) = \Sigma\, X_j d\xi_j.$$

If $H(\xi) = \Sigma_1^n \xi_j X_j(\xi)$, it follows that H is homogeneous of degree 1 and that $\partial H/\partial \xi_j = X_j(\xi)$.

Thus it only remains to show that the map $\Lambda \ni (x, \xi) \to \xi$ is regular at λ_0 if we choose appropriate local coordinates at $x_0 = \pi\lambda_0$. Note that if $\psi(x) = \langle x, \xi_0 \rangle$ is the linear form in the coordinates such that $(x_0, \psi'_x(x_0)) = \lambda_0$, this means that the tangent plane of $\{(x, \psi'_x(x))\}$ at λ_0 shall be transversal to the tangent plane at λ_0 of Λ. But for a suitable choice of ψ the set $\{(x, \psi'_x(x))\}$ is locally an arbitrary Lagrangean manifold transversal to the fiber, for closed 1-forms are locally exact. Since one can choose such a function ψ as a local coordinate, what we have to show is that there is a Lagrange plane in $T_{\lambda_0}(T^*(X))$ transversal to two given Lagrange planes, namely the tangent space of the fiber and the tangent space of Λ. This is obvious if one has a suitable parametrization of all Lagrange planes, which we shall now introduce also as a preparation for section 3.3. (See also Arnold [1] for the following discussion.)

By x or y we denote points in $\mathbf{R}^n$ and by $X = (x, \xi)$ or $Y = (y, \eta)$ we denote points in $T^*(\mathbf{R}^n) = \mathbf{R}^n \oplus \mathbf{R}^n$. The map $(x, \xi) \to x + i\xi \in \mathbf{C}^n$ allows us to identify this space with $\mathbf{C}^n$. The hermitian scalar product in $\mathbf{C}^n$ is defined by

$$(X, Y) = \Sigma\, (x_j + i\xi_j)(y_j - i\eta_j)$$

so $\operatorname{Re}(X, Y) = \Sigma\, x_j y_j + \Sigma\, \xi_j \eta_j = (X, Y)_R$ is the Euclidean scalar product in $\mathbf{R}^{2n}$ and $\operatorname{Im}(X, Y) = \Sigma\, (\xi_j y_j - \eta_j x_j) = [X, Y]$ is the standard symplectic form which occurs in (3.1.4). Let $\Lambda(n)$ be the set of all n dimensional real subspaces λ on which $[X, Y]$ vanishes identically. This means that λ and $i\lambda$ shall be orthogonal with respect to $(\,,\,)_R$. If $U \in U(n)$, the unitary group, it is therefore clear that $\lambda \in \Lambda(n)$ implies that $U\lambda \in \Lambda(n)$. The group $U(n)$ acts transitively on $\Lambda(n)$, for if $\lambda \in \Lambda(n)$ and $e_1, \ldots, e_n$ is a real orthogonal basis for λ, we have $\operatorname{Re}(e_j, e_k) = \delta_{jk}$, $\operatorname{Im}(e_j, e_k) = 0$ so $(e_j, e_k) = \delta_{jk}$, that is, we have a complex orthogonal basis for $\mathbf{C}^n$. Thus we have a unitary map U with $U\mathbf{R}^n = \lambda$. Now $U\mathbf{R}^n = \mathbf{R}^n$ if and only if U has real coefficients, that is, U belongs to the orthogonal group $O(n)$, so we can identify $\Lambda(n)$ with $U(n)/O(n)$.

If $\lambda = U\mathbf{R}^n$ is an arbitrary Lagrange plane we claim that $\mu = D\mathbf{R}^n$ is a transversal Lagrange plane when D is a diagonal matrix satisfying an algebraic inequality. In fact, we have transversality unless $\operatorname{Im} D^{-1}U$ is singular. If the diagonal elements are $d_j' + id_j''$ this condition can be written

$$\det\,(d_j'' \operatorname{Re} u_{jk} - d_j' \operatorname{Im} u_{jk}) = 0.$$

If this equation were satisfied identically for real d_j', d_j'' we could take $d_j''=1$ and $d_j'=-i$ and conclude that det $U=0$ which is absurd. Except when the elements of the diagonal matrix D satisfy a non-trivial algebraic equation it follows that $D\mathbf{R}^n$ is transversal to any finite number of given elements of $\Lambda(n)$. This completes the proof of Theorem 3.1.3 and even shows that it is sufficient to introduce new coordinates of the form $y_k=y_k(x_k)$, $k=1, ..., n$.

Remark 1. Theorem 3.1.3 is closely related to the classical representation of a canonical transformation by a generating function. (See e.g. Carathéodory [6].)

Remark 2. Theorem 3.1.3 remains valid if we drop the assumption that Λ is conic and the conclusion that H is homogeneous. Indeed, the only change in the proof is that using (3.1.4) instead of (3.1.3) we obtain $\partial X_j/\partial\xi_k=\partial X_k/\partial\xi_j$ which implies that locally $X=H'_\xi$ for a suitable function $H(\xi)$.

Remark 3. If we regard the function ϕ in Theorem 3.1.3 as defined on $T^*(X)$ we have that $d\phi$ coincides with the form $\langle\xi, dx\rangle$ on Λ. This is an invariant statement.

Theorem 3.1.3 shows in particular that every homogeneous Lagrangean manifold can be represented by a non-degenerate phase function. Having settled this existence question we shall return to the study of the uniqueness of ϕ—in the sense of equivalence—after indicating an important relation between a Lagrangean manifold and any non-degenerate phase function defining it.

THEOREM 3.1.4. *Let ϕ be a non-degenerate phase function in a conic neighborhood of (x_0, θ_0) in $X\times\mathbf{R}^N$ with $\phi'_\theta(x_0, \theta_0)=0$, and set $\xi_0=\phi'_x(x_0, \theta_0)$ so that (x_0, ξ_0) belongs to the corresponding Lagrangean manifold Λ. Then we have*

$$N-\operatorname{rank}\phi''_{\theta\theta}(x_0, \theta_0)=n-\operatorname{rank} d\pi_\Lambda(x_0, \xi_0) \tag{3.1.5}$$

where π_Λ is the restriction to Λ of the projection $T^*(X)\to X$ and $n=\dim X$.

Proof. The right hand side is equal to the dimension of the space of tangent vectors of C (defined by (3.1.2)) at (x_0, θ_0) which are mapped to 0 by the differential of the composition of the map (3.1.1) with π_Λ, that is, the map $C\ni(x,\theta)\to x$. These are the solutions of the equations $d\phi'_\theta=0$, $dx=0$, that is, $dx=0$, $\phi''_{\theta\theta}d\theta=0$. The dimension is therefore given by the left hand side of (3.1.5).

We note that $\phi''_{\theta\theta}\theta=0$ by Euler's identity so $d\pi_\Lambda$ is never bijective, which means that Λ cannot be a section of $T^*(X)$. If $\lambda_0\in\Lambda$ and $d\pi_\Lambda$ has constant rank $=k$ in a neighborhood of

λ_0, then there is a uniquely defined manifold $Y \subset X$ of dimension k in a neighborhood of $x_0 = \pi\lambda_0$ such that $\pi\lambda \in Y$ when λ is in a neighborhood of λ_0 in Λ, and $d\pi_\Lambda$ is of course surjective to $T(Y)$. (See e.g. Sternberg [28, p. 41].) Since $\Sigma\, \xi_j dx_j = 0$ on Λ it follows that a neighborhood of λ_0 in Λ is contained in the normal bundle of Y in $T^*(X)$ and since the dimensions are equal we conclude that Λ is an open subset of the normal bundle of Y in a neighborhood of λ_0. Thus we have the situation studied in section 2.4. The hypothesis of constant rank is of course fulfilled at all points in an open set, namely the set where the rank is maximal. We give an example which shows what may happen when the rank is not constant.

Example 3.1.5. Consider in $T^*(R^2)$ the Lagrange manifold defined in Example 3.1.2 by $H(\xi) = \xi_1^3/\xi_2^2$ when (ξ_1, ξ_2) is in a conic neighborhood of $(0, \pm 1)$. We have $x_1 = 3(\xi_1/\xi_2)^2$ and $x_2 = -2(\xi_1/\xi_2)^3$ so (x_1, x_2) varies over a neighborhood of the origin on the curve $(x_1/3)^3 - (x_2/2)^2 = 0$. The manifold Λ becomes the closure of the normal bundle of the regular part of the curve.

We shall now give an answer to the question concerning equivalence of phase functions.

THEOREM 3.1.6. *Let ϕ and $\tilde{\phi}$ be non-degenerate phase functions in conic neighborhoods of $(x_0, \theta_0) \in X \times (\mathbf{R}^N \setminus 0)$ and $(x_0, \tilde{\theta}_0) \in X \times (\mathbf{R}^{\tilde{N}} \setminus 0)$ respectively. Then the functions ϕ and $\tilde{\phi}$ are equivalent in some conic neighborhoods of these points, under a diffeomorphism mapping (x_0, θ_0) to $(x_0, \tilde{\theta}_0)$, if and only if*

(i) *The elements of Lagrangean manifolds defined by ϕ and by $\tilde{\phi}$ at (x_0, θ_0) and $(x_0, \tilde{\theta}_0)$ are the same.*

(ii) *$N = \tilde{N}$.*

(iii) *$\phi''_{\theta\theta}(x_0, \theta_0)$ and $\tilde{\phi}''_{\theta\theta}(x_0, \tilde{\theta}_0)$ have the same signature.*

The necessity is obvious. Note that when (i) and (ii) are valid, it follows from Theorem 3.1.4 that the ranks of the matrices in (iii) are equal.

The proof of the sufficiency is fairly long. The first step is to show, using (i) and (ii) only, that $\tilde{\phi}$ is equivalent to a function ψ such that $\phi - \psi$ vanishes to the second order on the set C defined by (3.1.2). To do so we consider the map

$$(x, \theta) \to (x, \phi'_x, \phi'_\theta) = (x, \Phi(x, \theta)).$$

The differential of Φ for fixed x is injective, for if $\phi''_{x\theta} d\theta = \phi''_{\theta\theta} d\theta = 0$ it follows by hypothesis that $d\theta = 0$. By the implicit function theorem it follows that there is a map ψ to $\mathbf{R}^N$ from a neighborhood of $(x_0, \xi_0, 0)$ where $\xi_0 = \phi'_x(x_0, \theta'_0)$, such that

$$\theta = \Psi(x, \phi'_x, \phi'_\theta).$$

Replacing Ψ by $\Psi(x, \xi|\xi_0|/|\xi|, w)|\xi|/|\xi_0|$ we may assume that $\Psi(x, \xi, w)$ is homogeneous of degree 1 with respect to ξ.

Corresponding to $\tilde{\phi}$ we can choose a similar map $\tilde{\Psi}$. With a linear transformation $A: \mathbf{R}^N \to \mathbf{R}^N$ still to be determined we form the map

$$(x, \theta) \to (x, \tilde{\Psi}(x, \phi'_x, \phi'_\theta) + A\phi'_\theta(x, \theta)|\theta|/|\theta_0|). \tag{3.1.6}$$

It is fiber preserving and the restriction to C is the diffeomorphism onto $\tilde{C}$ such that the diagram

$$\begin{tikzcd} C \arrow[rr] \arrow[dr] & & \tilde{C} \arrow[dl] \\ & \Lambda & \end{tikzcd}$$

commutes. If the map (3.1.6) is a diffeomorphism at (x_0, θ_0), the pullback ψ of $\tilde{\phi}$ under the map will therefore have the required properties.

Writing $\phi'_x = \xi$, $\phi'_\theta = w$ we can write the differential of (3.1.6) for fixed x in the form

$$\tilde{\Psi}'_\xi \phi''_{x\theta} d\theta + \tilde{\Psi}'_w \phi''_{\theta\theta} d\theta + A\phi''_{\theta\theta} d\theta.$$

We must therefore choose A so that the matrix

$$\tilde{\Psi}'_\xi \phi''_{x\theta} + B\phi''_{\theta\theta} \tag{3.1.7}$$

is non-singular at (x_0, θ_0), where $B = A + \partial\tilde{\Psi}/\partial w$ may be any matrix at $(x_0, \xi_0, 0)$. This is possible if (and only if) $\phi''_{\theta\theta} t = 0$ implies $\tilde{\Psi}'_\xi \phi''_{x\theta} t \neq 0$ if $t \neq 0$, for then we can choose B so that the range on a space supplementary to $\operatorname{Ker} \phi''_{\theta\theta}$ is supplementary to $\tilde{\Psi}'_\xi \phi''_{x\theta} \operatorname{Ker} \phi''_{\theta\theta}$ (all computations taking place at (x_0, θ_0)). What we have to verify is thus that if $(0, t)$ is a tangent to C, then $\tilde{\Psi}'_\xi \phi''_{x\theta} t \neq 0$ if $t \neq 0$. Since $C \ni (x, \theta) \to (x, \tilde{\Psi}(x, \phi'_x, \phi'_\theta)) \in \tilde{C}$ is a diffeomorphism and the differential maps $(0, t)$ to $(0, \tilde{\Psi}'_\xi \phi''_{x\theta} t)$ the proof is complete.

Now assume that ϕ and ψ are two non-degenerate phase functions at (x_0, θ_0) such that $\phi - \psi$ vanishes to the second order on the set C defined by (3.1.2) in a neighborhood of (x_0, θ_0). This implies of course that the corresponding Lagrangean manifolds are the same, so we have now exhausted conditions (i) and (ii) in Theorem 3.1.6 completely. Using Taylor's formula we can write

$$\psi = \phi + \tfrac{1}{2} \sum b_{jk}(x, \theta) \partial\phi(x, \theta)/\partial\theta_j \partial\phi(x, \theta)/\partial\theta_k$$

where $B = (b_{jk})$ is a symmetric matrix. On C we obtain

$$\psi''_{\theta\theta} = \phi''_{\theta\theta}(I + B\phi''_{\theta\theta}), \quad \psi''_{x\theta} = \phi''_{x\theta}(I + B\phi''_{\theta\theta}) \tag{3.1.8}$$

so the non-degeneracy of ψ means precisely that $\det(I + B\phi''_{\theta\theta}) \neq 0$ at (x_0, θ_0).

We shall now prove that ψ is equivalent to ϕ in some neighborhood of (x_0, θ_0) if B is sufficiently small. To do so we first note that by Taylor's formula

$$\phi(x, \tilde{\theta}) - \phi(x, \theta) = \Sigma\, (\tilde{\theta}_j - \theta_j)\partial\phi(x, \theta)/\partial\theta_j + \Sigma\, (\tilde{\theta}_j - \theta_j)(\tilde{\theta}_k - \theta_k)\phi_{jk}(x, \theta, \tilde{\theta})$$

where ϕ_{jk} is symmetric with respect to j and k and homogeneous of degree -1 with respect to $(\theta, \tilde{\theta})$. Now put

$$\tilde{\theta}_j = \theta_j + \Sigma\, w_{jk}(x, \theta)\partial\phi(x, \theta)/\partial\theta_k$$

with some homogeneous functions w_{jk} still to be determined. Then we obtain $\phi(x, \tilde{\theta}) = \psi(x, \theta)$ if

$$w_{jk} + \Sigma\, w_{\nu j} w_{\mu k}\, \phi_{\nu\mu}(x, \theta, \tilde{\theta}) = b_{jk}/2.$$

By the implicit function theorem these equations have a unique small solution w if the matrix B is sufficiently small, and the solution must then necessarily be homogeneous of degree 1. The condition $D\tilde{\theta}/D\theta \neq 0$ at (x_0, θ_0) will also be fulfilled for sufficiently small B, so the assertion is proved.

The final part of the proof of Theorem 3.1.6 is now to prove that ϕ and ψ are equivalent at (x_0, θ_0) if $\phi - \psi$ vanishes to the second order on C and $\phi''_{\theta\theta}$ and $\psi''_{\theta\theta}$ have the same signature at (x_0, θ_0). By (3.1.8) this means that $\phi''_{\theta\theta}$ and $\phi''_{\theta\theta} + \phi''_{\theta\theta} B \phi''_{\theta\theta}$ have the same signature there. The assertion will follow from the second step in the proof if we show that there is a continuous function ψ_t of t, $0 \leqslant t \leqslant 1$, with values in the space of non-degenerate phase functions in some neighborhood of (x_0, θ_0), satisfying the hypothesis that $\psi_t - \phi$ vanishes to the second order on C, such that $\psi_0 = \phi$ and $\psi_1 = \psi$. In fact, since ψ_t is then equivalent to ψ_s if s and t are sufficiently close, the equivalence of ψ_0 and ψ_1 follows in view of the Borel-Lebesgue lemma.

To show that ψ_t can be chosen with the required properties it only remains to analyse the condition concerning signatures. Since the deformation is clearly possible if B vanishes at (x_0, θ_0), it suffices to prove the following lemma:

LEMMA 3.1.7. *Let A be a real symmetric $M \times M$ matrix and let R be the set of all real symmetric $M \times M$ matrices B such that* $\det(I + BA) \neq 0$. *Two matrices B_1, $B_2 \in R$ are then in the same component of R if and only if $A + AB_jA$ has the same signature for $j = 1, 2$; the ranks are of course equal to rank A.*

Proof. Let N be the kernel of A and N^0 the orthogonal complement. Then we have $AN = 0$ and A maps $\mathbf{R}^M$ onto N^0 with N as kernel. The operator $I + BA$ is the identity on N

so it is an isomorphism if and only if $A+ABA$ restricted to N^0 is an isomorphism. If P is the orthogonal projection on N^0, it follows that $\det(I+BA)\neq 0$ implies that $\det(I+B_tA)\neq 0$ if $B_t=(1-t)B+tPBP$, so B and PBP are in the same component of R. Moreover, B_1 and B_2 can be joined by an arc in R if and only if PB_1P and PB_2P can be joined by an arc in R consisting of operators vanishing on N and with range contained in N^0. Thus we are reduced to considering operators from N^0 to N^0, and the statement then follows from the well known fact that in the space of all non-singular $k\times k$ symmetric real matrices the components consist of matrices having the same signature. In fact, the map $B\to A+ABA$ from symmetric matrices to symmetric matrices is bijective if A is non-singular.

We have now completed the proof of Theorem 3.1.6. Note that the proof also shows that $\phi''_{\theta\theta}(x_0,\theta_0)$ can have any signature compatible with Theorem 3.1.4 when we only know the corresponding Lagrangean manifold Λ. Only in one case do we get a perfect analogue of Proposition 2.4.1:

COROLLARY 3.1.8. *Let ϕ_j be a non-degenerate phase function at (x_0,θ_j) where $\theta_j\neq 0$ and $\phi'_{j\theta}(x_0,\theta_j)=0$, $\phi''_{j\theta\theta}(x_0,\theta_j)=0$, $j=1,2$. Then it follows that ϕ_1 and ϕ_2 are equivalent at (x_0,θ_1) and (x_0,θ_2) if and only if the corresponding germs of Lagrange manifolds are the same.*

Proof. By Theorem 3.1.4 the number of θ variables in the two functions must be the same so the hypotheses of Theorem 3.1.6 are fulfilled.

We shall now discuss how to change the number of θ variables in ϕ without changing the corresponding Lagrangean manifold. In view of Corollary 3.1.8 two phase functions defining the same Lagrangean manifold will give rise to equivalent phase functions if in this way we decrease the number of θ variables in each as far as possible.

First we shall show how to increase the number of θ variables. Thus let $\phi(x,\theta)$ be a non-degenerate phase function in a conic neighborhood of (x_0,θ_0), let $\sigma\in\mathbf{R}^\nu$ and introduce

$$\phi_1(x,\theta,\sigma)=\phi(x,\theta)+A(\sigma,\sigma)/|\theta|$$

where A is a non-singular quadratic form in $\mathbf{R}^\nu$. This function is homogeneous of degree 1 in a conic neighborhood of $(x_0,\theta_0,0)$ in $X\times\mathbf{R}^{N+\nu}$. The equations

$$\partial\phi_1/\partial\theta=\partial\phi_1/\partial\sigma=0$$

mean that $\sigma=0$ and that $\partial\phi/\partial\theta=0$, so it is clear that ϕ_1 is a non-degenerate phase function defining the same Lagrangean manifold as ϕ. Thus we can always increase the number of θ variables as much as we like.

On the other hand, assuming again that ϕ is a non-degenerate phase function in a conic neighborhood of $(x_0, \theta_0) \in C$ (defined by (3.1.2)), we can decrease the fiber dimension by k units if $\phi''_{\theta\theta}(x_0, \theta_0)$ has rank at least k. For let us write $\theta' = (\theta_1, ..., \theta_{N-k})$, $\theta'' = (\theta_{N-k+1}, ..., \theta_N)$, and assume that $\det \phi''_{\theta''\theta''}(x_0, \theta_0) \neq 0$. This situation can of course be attained by a linear transformation of the θ variables. Then we have $\theta'_0 \neq 0$ for otherwise we would obtain using Euler's homogeneity relations that

$$0 = \phi''_{\theta\theta}(x_0, \theta_0)\, \theta_0 = \phi''_{\theta\theta''}(x_0, \theta_0)\, \theta''_0$$

which implies that $\theta''_0 = 0$ also, which is a contradiction. Among the equations $\phi'_\theta = 0$ we first consider the k equations $\phi'_{\theta''} = 0$ and note that they locally determine θ'' as a C^∞ function $\psi(x, \theta')$. Let now

$$\phi_1(x, \theta') = \phi(x, \theta', \psi(x, \theta')).$$

We claim that ϕ_1 is a non-degenerate phase function at (x_0, θ'_0) which defines the same Lagrangean manifold as ϕ. In proving this we may assume that $\psi = 0$ identically, for otherwise we can introduce $(\theta', \theta'' - \psi)$ as a new variable instead of θ. Then the equation $\partial\phi/\partial\theta'' = 0$ is equivalent to $\theta'' = 0$, so it follows that all mixed derivatives involving θ'' are also 0 then. The differentials of $\phi'_{\theta'}$ are therefore independent of $d\theta''$ while the differentials of $\phi'_{\theta''}$ only involve $d\theta''$ on C. Thus the non-degeneracy of ϕ implies that of ϕ_1.

3.2. Invariance under change of phase function and global definition

Using the results proved in section 3.1 we shall now show that the class of distributions which are defined according to (1.2.1) with a fixed ϕ is already determined by the Lagrangean manifold corresponding to ϕ. Let ϕ be a non-degenerate phase function in some conic neighborhood Γ of a point $(x_0, \theta_0) \in \mathbf{R}^n \times (\mathbf{R}^N \setminus 0)$, and let $a \in S^m_{\varrho, 1-\varrho}(\Gamma)$, $\varrho > \frac{1}{2}$, vanish outside a conic closed set $\subset \Gamma$ and in a neighborhood of $\mathbf{R}^n \times 0$. As in section 2.4 we modify (1.2.1) slightly and set

$$\langle A, u \rangle = (2\pi)^{-(n+2N)/4} \iint e^{i\phi(x,\theta)} a(x, \theta)\, u(x)\, dx\, d\theta, \quad u \in C_0^\infty(\mathbf{R}^n). \tag{3.2.1}$$

The reason for the factor $(2\pi)^{-(n+2N)/4}$ will become more clear when we switch to phase functions involving another number of θ variables.

Suppose now that we make a change of variables

$$x = x(\tilde{x}), \quad \theta = \theta(\tilde{x}, \tilde{\theta}).$$

In doing so we shall, as in section 2.4, transform A and u as densities of order $\frac{1}{2}$, so we set

$$\tilde{u}(\tilde{x}) = |Dx/D\tilde{x}|^{\frac{1}{2}} u(x)$$

and define $\tilde{A}$ so that $\langle \tilde{A}, \tilde{u}\rangle = \langle A, u\rangle$. Writing

$$\tilde{\phi}(\tilde{x}, \tilde{\theta}) = \phi(x(\tilde{x}), \theta(\tilde{x}, \tilde{\theta})), \quad \tilde{a}(\tilde{x}, \tilde{\theta}) = a(x(\tilde{x}), \theta(\tilde{x}, \tilde{\theta}))\, |Dx/D\tilde{x}|^{\frac{1}{2}}\, |D\theta/D\tilde{\theta}|$$

we obtain

$$\langle \tilde{A}, \tilde{u}\rangle = (2\pi)^{-(n+2N)/4} \iint e^{i\tilde{\phi}(\tilde{x}, \tilde{\theta})} \tilde{a}(\tilde{x}, \tilde{\theta})\, \tilde{u}(\tilde{x})\, d\tilde{x}\, d\tilde{\theta}, \quad \tilde{u} \in C_0^\infty. \tag{3.2.2}$$

We shall of course consider A and $\tilde{A}$ as the same distribution density of order $\frac{1}{2}$, expressed in terms of different local coordinates. Generalizing Theorem 2.4.2 we shall now show how to assign to this distribution density a principal symbol in $S_\varrho^\mu(\Lambda, \Omega_{1/2})$, $\mu = m + N/2$. (For the definition see section 1.1.)

As in section 2.4 we note that on the manifold

$$C = \{(x, \theta);\, \phi'_\theta(x, \theta) = 0\}$$

a density d_C is defined as the pullback of the Dirac measure in $\mathbf{R}^N$ under the map $\Gamma \ni (x, \theta) \to \phi'_\theta(x, \theta)$. If $\lambda_1, \ldots, \lambda_n$ are local coordinates on C extended to a neighborhood of C, the density is given by

$$d_C = |D(\lambda_1, \ldots, \lambda_n, \partial\phi/\partial\theta_1, \ldots, \partial\phi/\partial\theta_N)/D(x, \theta)|^{-1} d\lambda_1 \ldots d\lambda_n.$$

Now we claim that the density of order $\frac{1}{2}$ on Λ which is the image of $a\sqrt{d_C}$ under the map $C \ni (x, \theta) \to (x, \phi'_x) \in \Lambda$ is the same as that obtained from $\tilde{\phi}$ and $\tilde{a}$ (if we regard x and $\tilde{x}$ as local coordinates in the same manifold). Introduce $\tilde{\lambda}(\tilde{x}, \tilde{\theta}) = \lambda(x(\tilde{x}), \theta(\tilde{x}, \tilde{\theta}))$ so that λ and $\tilde{\lambda}$ correspond to the same function on Λ. What we have to prove is then that on $\tilde{C}$

$$|D(\lambda, \phi'_\theta)/D(x, \theta)|^{-1/2} a(x, \theta) = |D(\tilde{\lambda}, \tilde{\phi}'_{\tilde{\theta}})/D(\tilde{x}, \tilde{\theta})|^{-1/2} \tilde{a}(\tilde{x}, \tilde{\theta}).$$

The interpretation of this and the formulas following should of course be that x and θ are regarded as functions of $\tilde{x}$ and $\tilde{\theta}$. Since $\tilde{\phi}(\tilde{x}, \tilde{\theta}) = \phi(x, \theta)$ we have $\tilde{\phi}'_{\tilde{\theta}} = \phi'_\theta \partial\theta/\partial\tilde{\theta}$, and since $\phi'_\theta = 0$ on C we obtain

$$\begin{aligned} D(\tilde{\lambda}, \tilde{\phi}'_{\tilde{\theta}})/D(\tilde{x}, \tilde{\theta}) &= (D\theta/D\tilde{\theta})\,(D(\lambda, \phi'_\theta)/D(x, \theta))\, D(x, \theta)/D(\tilde{x}, \tilde{\theta}) \\ &= (D\theta/D\tilde{\theta})^2 (Dx/D\tilde{x})\, D(\lambda, \phi'_\theta)/D(x, \theta). \end{aligned}$$

This proves the asserted invariance if we recall the definition of $\tilde{a}$. Note that the determinant

$$D(\lambda_1, \ldots, \lambda_n, \partial\phi/\partial\theta_1, \ldots, \partial\phi/\partial\theta_N)/D(x, \theta)$$

is homogeneous of degree $n - N$ if $\lambda_1, \ldots, \lambda_n$ are homogeneous of degree 1 with respect to θ. The density of order $\frac{1}{2}$ we have defined is therefore in $S_\varrho^\mu(\Lambda, \Omega_{\frac{1}{2}})$ if $\mu = m + N/2$.

If all non-degenerate phase functions defining the same Lagrangean manifold had been equivalent, this would have finished our proof that the class of distributions defined by (3.2.1) is determined by the Lagrange manifold corresponding to ϕ. However, in order to be able to apply Corollary 3.1.8 we must also show that the class of distributions in question does not change if we restrict the number of θ variables as indicated after that result. In doing so we shall again split the θ variables in two groups

$$\theta' = (\theta_1, \ldots, \theta_{N-k}), \quad \theta'' = (\theta_{N-k+1}, \ldots, \theta_N).$$

We require that $\det \phi''_{\theta''\theta''} \neq 0$ at (x_0, θ_0). As noted in section 3.1, ϕ is then locally equivalent to a phase function such that $\phi'_{\theta''} = 0$ is equivalent to $\theta'' = 0$ so we may assume that this condition is fulfilled. Let $Q(\theta'', \theta'')$ be a quadratic form of the same signature as $\phi''_{\theta''\theta''}(x_0, \theta_0)$ and set

$$\phi_1(x, \theta) = \phi(x, \theta', 0) + Q(\theta'', \theta'')/2|\theta'|$$

which is a non-degenerate phase function equivalent to ϕ near (x_0, θ_0) in view of Theorem 3.1.6. Thus we have reduced our study to a phase function ϕ of the form

$$\phi(x, \theta) = \psi(x, \theta') + Q(\theta'', \theta'')/2|\theta'|$$

in a conic neighborhood of a point (x_0, θ_0) with $\theta''_0 = 0$. We have

$$\langle A, u\rangle = (2\pi)^{-(n+2N)/4} \iint e^{i(\psi(x,\theta') + Q(\theta'',\theta'')/2|\theta'|)} a(x, \theta', \theta'')\, u(x)\, dx\, d\theta'\, d\theta'',$$

which if we first integrate with respect to θ'' leads to

$$\langle A, u\rangle = (2\pi)^{-(n+2(N-k))/4} \iint e^{i\psi(x,\theta')} b(x, \theta')\, u(x)\, dx\, d\theta' \tag{3.2.3}$$

where
$$b(x, \theta') = (2\pi)^{-k/2} \int e^{iQ(\theta'',\theta'')/2|\theta'|} a(x, \theta', \theta'')\, d\theta''$$
$$= (2\pi)^{-k/2} |\theta'|^k \int e^{iQ(\theta'',\theta'')|\theta'|/2} a(x, \theta', |\theta'|\theta'')\, d\theta''. \tag{3.2.4}$$

Note that in the last integral we may assume that the integrand vanishes for $|\theta''| > 1$, for example.

To evaluate (3.2.4) we must recall some well known facts concerning the method of stationary phase.

a) The Fourier transform of the function

$$\mathbf{R} \ni y \to e^{-zy^2/2}, \quad \operatorname{Re} z \geq 0,\ z \neq 0,$$

is $\eta \to e^{-\eta^2/2z}(2\pi/z)^{\frac{1}{2}}$ where $w^{\frac{1}{2}}$ is defined when $\operatorname{Re} w \geqslant 0$ so that it is equal to 1 when $w=1$. From this we conclude that the Fourier transform of

$$\mathbf{R}^k \ni y \to \exp\,(i\,\Sigma\, a_j y_j^2/2)$$

where a_j are real and $\neq 0$ is given by

$$\eta \to \exp\,(-i\,\Sigma\,\eta_j^2/2a_j)\prod\,(2\pi/-ia_j)^{\frac{1}{2}} = \exp\,(-i\,\Sigma\,\eta_j^2/2a_j)\,(2\pi)^{k/2}\exp\,(\pi i\sigma/4)\prod|a_j|^{-\frac{1}{2}}$$

where $\sigma = \Sigma\,\operatorname{sgn} a_j$. If A is a symmetric non-singular matrix it follows that the Fourier transform of

$$\mathbf{R}^k \ni y \to \exp\,(i\langle Ay, y\rangle/2)$$

is given by

$$\mathbf{R}^k \ni \eta \to \exp\,(-i\langle A^{-1}\eta, \eta\rangle/2)\,(2\pi)^{k/2}\,|\det A|^{-\frac{1}{2}}\exp\,(\pi i\,\operatorname{sgn} A/4)$$

where $\operatorname{sgn} A$ denotes the signature of A.

b) If $f \in C_0^\infty(\mathbf{R}^k)$ we obtain from Fourier's inversion formula and the preceding computation of Fourier transforms for $t > 0$

$$\int f(y)\exp\,(it\langle Ay, y\rangle/2)\,dy$$
$$= (2\pi t)^{-k/2}|\det A|^{-\frac{1}{2}}\exp\,(\pi i\,\operatorname{sgn} A/4)\int \hat{f}(\eta)\exp\,(-i\langle A^{-1}\eta, \eta\rangle/2t)\,d\eta.$$

In the right hand side we take the Taylor expansion of the exponential function. Since

$$\Big|e^{ix} - \sum_0^{\nu-1}(ix)^j/j!\Big| \leqslant |x|^\nu/\nu!, \quad x \in \mathbf{R},$$

we obtain

$$\left|\int f(y)\exp\,(it\langle Ay, y\rangle/2)\,dy - S_\nu(t)\right| \leqslant Ct^{-k/2-\nu}\sum_{|\alpha|\leqslant 2\nu+k+1}\int|D^\alpha f|\,dy \tag{3.2.5}$$

where C is independent of t and f, and

$$S_\nu(t) = (2\pi/t)^{k/2}|\det A|^{-\frac{1}{2}}\exp\,(\pi i\,\operatorname{sgn} A/4)\sum_{|\alpha|<2\nu}c_\alpha D^\alpha f(0)\,t^{-|\alpha|/2} \tag{3.2.6}$$

Here
$$c_\alpha = \frac{\partial^{|\alpha|}}{\partial\eta^\alpha}\exp\,(-i\langle A^{-1}\eta, \eta\rangle/2)/\alpha!\,\Big|_{\eta=0} \tag{3.2.7}$$

vanishes for odd $|\alpha|$, and is of course independent of t and f.

c) Let us now consider the integral (3.2.4). As already pointed out we may assume that $a(x, \theta', \theta'') = 0$ when $|\theta''| > |\theta'|$. Now apply (3.2.5). The error term will be bounded by a constant times $|\theta'|$ raised to the power

$$k/2-\nu+m+(2\nu+k+1)(1-\varrho)=m+k/2+(k+1)(1-\varrho)+\nu(1-2\varrho)\to-\infty \quad \text{as } \nu\to\infty.$$

In the sum

$$\begin{gathered} G\sum_{|\alpha|<2\nu} c_\alpha|\theta'|^{-|\alpha|/2}D^\alpha_{\theta''}a(x,\theta',|\theta'|\theta'')|_{\theta''=0}, \\ G=|\det Q|^{-\frac{1}{2}}|\theta'|^{k/2}\exp(\pi i \operatorname{sgn} Q/4) \end{gathered} \tag{3.2.8}$$

the general term is in $S^{m(\alpha)}_\varrho$ where $m(\alpha)=k/2+m+|\alpha|(\frac{1}{2}-\varrho)\to-\infty$ as $|\alpha|\to\infty$. In view of Proposition 1.1.10 it follows that $b\in S^{m+k/2}_\varrho$ and that b is the asymptotic sum of (3.2.8) when $\nu\to\infty$. Since $c_\alpha=1$ when $\alpha=0$ and $c_\alpha=0$ when $|\alpha|=1$, we have

$$b(x,\theta')-|\det Q|^{-\frac{1}{2}}|\theta'|^{k/2}\exp(\pi i\operatorname{sgn} Q/4)\,a(x,\theta',0)\in S^{m+(1-2\varrho)}_\varrho. \tag{3.2.9}$$

Let $C=\{(x,\theta);\ \phi'_\theta(x,\theta)=0\}$ and $\tilde C=\{(x,\theta');\ \psi'_{\theta'}(x,\theta')=0\}$. Since the equation $\phi'_\theta=0$ is equivalent to $\theta''=0$ and $\psi'_{\theta'}=0$, in a neighborhood of (x_0,θ_0), the map $(x,\theta')\to(x,\theta',0)$ is a diffeomorphism of $\tilde C$ onto C there, which gives a commutative diagram

$$\begin{tikzcd} \tilde C \arrow[rr] \arrow[dr] & & C \arrow[dl] \\ & \Lambda & \end{tikzcd}$$

Let $\lambda_1, \ldots, \lambda_n$ be local coordinates on C, extended to a neighborhood. Then we have on C

$$D(\lambda,\phi'_\theta)/D(x,\theta)=|\theta'|^{-k}(\det Q)\,D(\lambda,\psi'_{\theta'})/D(x,\theta').$$

This means that the densities of order $\frac{1}{2}$ on Λ defined by $a(x,\theta)\sqrt{d_C}$ and by $a(x,\theta',0)|\theta'|^{k/2}$ $|\det Q|^{-\frac{1}{2}}\sqrt{d_{\tilde C}}$ are the same. The conclusion is summed up in the following theorem, where we have adapted the normalizations to those used in section 2.4.

THEOREM 3.2.1. *Let $\phi(x,\theta)$ and $\tilde\phi(x,\tilde\theta)$ be non-degenerate phase functions in neighborhoods of $(x_0,\theta_0)\in X\times\mathbf{R}^N$ and $(x_0,\tilde\theta_0)\in X\times\mathbf{R}^{\tilde N}$ which define the same elements of Lagrange manifold Λ there. In particular, $\phi'_x(x_0,\theta_0)=\tilde\phi'_x(x_0,\tilde\theta_0)=\xi_0$. Then*

(i) *The difference*

$$\begin{gathered} \sigma=\operatorname{sgn}\phi''_{\theta\theta}(x,\theta)-\operatorname{sgn}\tilde\phi''_{\tilde\theta\tilde\theta}(x,\tilde\theta), \\ \phi'_\theta=\tilde\phi'_{\tilde\theta}=0,\quad \phi'_x=\tilde\phi'_x=\xi\in T^*_x \end{gathered} \tag{3.2.10}$$

is constant in a neighborhood of (x_0,ξ_0) in Λ.

(ii) *Every distribution which can be defined by (3.2.1) with $a\in S^{\mu+(n-2N)/4}_\varrho$, $\varrho>\frac{1}{2}$, and cone supp a in a sufficiently small conic neighborhood of (x_0,θ_0) can also be written in the same form with ϕ replaced by $\tilde\phi$ and a replaced by a function $\tilde a\in S^{\mu+(n-2\tilde N)/4}_\varrho$ with cone supp $\tilde a$ in a small conic neighborhood of $(x_0,\tilde\theta_0)$, so that moreover*

$$(\exp \pi i\sigma/4)\, a(x,\theta)\sqrt{d_C} - \tilde a(x,\tilde\theta)\sqrt{d_{\tilde C}} \in S_\varrho^{\mu+n/4+1-2\varrho}(\Lambda,\Omega_{\frac12}), \tag{3.2.11}$$

the two terms being of course in $S_\varrho^{\mu+n/4}(\Lambda,\Omega_{\frac12})$.

Proof. We have already verified the statement when ϕ and $\tilde\phi$ are equivalent or when $\tilde\phi$ is obtained from ϕ by reducing the number of θ variables. Since Corollary 3.1.8 shows that reduction of the number of θ variables in both ϕ and $\tilde\phi$ will lead to equivalent phase functions, repeated application of these special cases proves the theorem.

Since the signature is congruent to the rank mod 2, we have by Theorem 3.1.4 that

$$\sigma \equiv \operatorname{rank} \phi''_{\theta\theta} - \operatorname{rank} \tilde\phi''_{\tilde\theta\tilde\theta} = N - \tilde N.$$

Hence
$$\sigma' = (\sigma - N + \tilde N)/2 \in \mathbf{Z}. \tag{3.2.12}$$

With this notation we can rewrite (3.2.11) in the form

$$i^{\sigma'} \exp(\pi i N/4)\, a(x,\theta)\sqrt{d_C} - \exp(\pi i\tilde N/4)\,\tilde a(x,\tilde\theta)\sqrt{d_{\tilde C}} \in S_\varrho^{\mu+n/4+1-2\varrho}(\Lambda,\Omega_{\frac12}). \tag{3.2.13}$$

We are now prepared to discuss the global situation, so let X be a manifold and Λ a closed (1) conic Lagrangean submanifold of $T^*(X)\setminus 0$. Let J be the set of objects consisting of

(i) A local coordinate patch $X'\subset X$ with local coordinates $(x_1,\dots,x_n)\in\mathbf{R}^n$.

(ii) An integer $N>0$ and a non-degenerate phase function ϕ defined in a conic open subset U of $X'\times(\mathbf{R}^N\setminus 0)$ such that

$$\{(x,\theta)\in U;\ \phi'_\theta(x,\theta)=0\}\ni(x,\theta)\to(x,\phi'_x)$$

is a diffeomorphism on an open subset U^Λ of Λ.

When $j\in J$ we shall write $X'_j, \phi_j, N_j, U_j, U_j^\Lambda$ for the corresponding quantities.

Definition 3.2.2. By $I_\varrho^m(X,\Lambda)$ we shall denote the set of all $A\in\mathcal{D}'(X,\Omega_{\frac12})$ such that $A=\sum_{j\in J}A_j$ with the supports of A_j locally finite and

$$\langle A_j,u\rangle = (2\pi)^{-(n+2N_j)/4}\iint e^{i(\phi_j(x,\theta)-\pi N_j/4)} a_j(x,\theta)\,u(x)\,dx\,d\theta,\quad u\in C^\infty(X), \tag{3.2.14}$$

where dx is the Lebesgue measure with respect to the local coordinates in X'_j, $\theta\in\mathbf{R}^{N_j}$ and $a_j\in S_\varrho^{m+(n-2N_j)/4}(\mathbf{R}^n\times\mathbf{R}^{N_j})$, supp $a_j\subset\{(x,t\theta);\ t\geq 1,\ (x,\theta)\in K\}$ where K is a compact subset of the image of U_j in $\mathbf{R}^n\times\mathbf{R}^{N_j}$.

We shall now prove an analogue of Theorem 2.4.2. In doing so we must take the factor $i^{\sigma'}$ in (3.2.13) into account, so we introduce

(1) The following results remain valid if Λ is not closed provided that we only consider symbols vanishing outside closed conic subsets.

$$\sigma_{jk} = ((\operatorname{sgn} \phi''_{k\theta\theta}(x, \theta_k) - N_k) - (\operatorname{sgn} \phi''_{j\theta\theta}(x, \theta_j) - N_j))/2, \tag{3.2.15}$$

where $\phi'_{k\theta}(x, \theta_k) = \phi'_{j\theta}(x, \theta_j) = 0$ and $\phi'_{kx}(x, \theta_k) = \phi'_{jx}(x, \theta_j) = \xi \in T^*_x \cap \Lambda$. This is a locally constant integer valued function in $U^\Lambda_j \cap U^\Lambda_k$ by part (i) of Theorem 3.2.1, so we have an integral cochain defining an element $\sigma \in H^1(\Lambda, \mathbf{Z})$. Let L be the corresponding complex line bundle on Λ obtained from the cohomology class by letting $1 \in \mathbf{Z}$ act on $\mathbf{C}$ by multiplication with the imaginary unit i. (See e.g. Hirzebruch [11]. L is of course determined by the image of σ in $H^1(\Lambda, \mathbf{Z}_4)$.) L is trivial as a complex vector bundle. For since the cohomology of a fine sheaf is trivial we can find functions $\sigma_j \in C^\infty(U^\Lambda_j)$ which are homogeneous of degree 0 and satisfy the coboundary condition $\sigma_k - \sigma_j = \sigma_{jk}$ in $U^\Lambda_j \cap U^\Lambda_k$. This implies that $i^{\sigma_{jk}} = e^{-\pi i \sigma_j/2} e^{\pi i \sigma_k/2}$ which gives the asserted trivialization. However, we shall not use this since the trivialization is not natural and leads to a loss of part of the structure of L. Note that the action of $\mathbf{R}_+$ on L given by letting $\mathbf{R}_+$ act trivially on $\mathbf{C}$ in the trivialization is independent of how it is chosen, so L is an $\mathbf{R}_+$ bundle, and the spaces $S^\mu_\varrho(\Lambda, \Omega_{\frac12} \otimes L)$ are therefore well defined.

Our purpose is to establish an isomorphism

$$S^{m+n/4}_\varrho(\Lambda, \Omega_{\frac12} \otimes L)/S^{m+n/4+1-2\varrho}_\varrho(\Lambda, \Omega_{\frac12} \otimes L) \to I^m_\varrho(X, \Lambda)/I^{m+1-2\varrho}_\varrho(X, \Lambda). \tag{3.2.16}$$

The first step is to define the surjective map (3.2.16) using Theorem 3.2.1. In doing so we note that to have an element $s \in S^{m+n/4}_\varrho(\Lambda, \Omega_{\frac12} \otimes L)$ means to have an element $s_j \in S^{m+n/4}_\varrho(U^\Lambda_j, \Omega_{\frac12})$ for every j such that

$$s_j = i^{\sigma_{jk}} s_k \quad \text{in } U^\Lambda_j \cap U^\Lambda_k.$$

If cone supp $s \subset U^\Lambda_j$ we define an element $A_j = A_j(s) \in I^m_\varrho(X, \Lambda)$ by (3.2.14) with

$$a_j \in S^{m+(n-2N_j)/4}_\varrho(U_j)$$

satisfying the conditions on the support in Definition 3.2.2 and chosen so that $a_j \sqrt{d_{C_j}}$ is mapped to s_j by the map

$$C_j = \{(x, \theta) \in U_j;\ \phi'_{j\theta}(x, \theta) = 0\} \ni (x, \theta) \to (x, \phi'_{jx}) \in U^\Lambda_j.$$

This defines $a_j \in S^{m+n/4-n/2+(n-N_j)/2}_\varrho(C_j)$. An extension to a neighborhood of C_j is obtained by taking a homogeneous C^∞ retraction to C_j, and a_j is finally obtained after multiplication by a suitable homogeneous cutoff function in U_j. By Proposition 1.2.5 different choices of the extension a_j give operators differing by an element of $I^{m+n/4+(1-2\varrho)}_\varrho(X, \Lambda)$ only. Thus we have defined a map

$$S_\varrho^{m+n/4}(\Lambda, \Omega_{\frac{1}{2}} \otimes L) \ni s \to \dot{A}_j(s) \in I_\varrho^m(X, \Lambda)/I_\varrho^{m+1-2\varrho}(X, \Lambda)$$

when cone supp $s \subset U_j^\Lambda$. If in addition cone supp $s \subset U_k^\Lambda$, it follows from (3.2.13) that $\dot{A}_k(s) = \dot{A}_j(s)$. (We can split s in a finite sum of elements with supports in such small conic sets that Theorem 3.2.1 is applicable.)

Now choose a partition of unity $\Sigma\, \chi_j = 1$ in Λ such that χ_j is homogeneous of degree 0 and the sets X'_j for which $\chi_j \neq 0$ form a locally finite covering of Λ. It suffices to define χ_j on the intersection of Λ with the unit sphere bundle in $T^*(X)$ with respect to some Riemannian structure in X. This is a compact set over any compact set in X (since Λ is assumed to be closed) so the existence of such a partition of unity is clear.

If $s \in S_\varrho^{m+n/4}(\Lambda, \Omega_{\frac{1}{2}} \otimes L)$ we have $s = \Sigma\, \chi_j s$ where $\chi_j s \in S_\varrho^{m+n/4}(\Lambda, \Omega_{\frac{1}{2}} \otimes L)$ and cone supp $\chi_j s \subset U_j^\Lambda$. Thus we can define

$$A(s) = \sum \dot{A}_j(\chi_j s).$$

This definition is independent of the choice of partition of unity. For if $1 = \Sigma\, \psi_k$ is another one, we have

$$\sum_j \dot{A}_j(\chi_j s) = \sum_{j,k} \dot{A}_j(\chi_j \psi_k s) = \sum_{j,k} \dot{A}_k(\chi_j \psi_k s) = \sum_k \dot{A}_k(\psi_k s).$$

The map A from $S_\varrho^{m+n/4}(\Lambda, \Omega_{\frac{1}{2}} \otimes L)$ to $I_\varrho^m(X, \Lambda)/I_\varrho^{m+1-2\varrho}(X, \Lambda)$ is clearly surjective and it maps $S_\varrho^{m+n/4+1-2\varrho}(\Lambda, \Omega_{\frac{1}{2}} \otimes L)$ to 0. Thus it defines a surjective map (3.2.16).

To prove that (3.2.16) is injective we shall have to study how a distribution $A \in I_\varrho^m(X, \Lambda)$ acts on rapidly oscillating functions. (Compare the discussion following (2.4.2).) This will also yield a new proof of Theorem 3.2.1 which is independent of the results proved in section 3.1, and in addition we shall obtain an intrinsic definition of $I_\varrho^m(X, \Lambda)$.

We have to start by considering the local case so let $X \subset \mathbf{R}^n$ and let ϕ be a non-degenerate phase function in some conic neighborhood Γ of a point $(x_0, \theta_0) \in X \times (\mathbf{R}^N \setminus 0)$. Let $u \in C_0^\infty(X)$ and let $\psi \in C^\infty(X)$ be real valued with $\psi'_x \neq 0$ in supp u. With A defined by (3.2.1) we shall determine the asymptotic behavior of $\langle A, ue^{-it\psi}\rangle$ as $t \to \infty$. We have

$$\langle A, ue^{-it\psi}\rangle = (2\pi)^{-(n+2N)/4} \iint e^{i(\phi(x,\theta) - t\psi(x))} a(x,\theta)\, u(x)\, dx\, d\theta$$

$$= (2\pi)^{-(n+2N)/4} t^N \iint e^{it(\phi(x,\theta) - \psi(x))} a(x, t\theta)\, u(x)\, dx\, d\theta. \tag{3.2.17}$$

The important contributions must come from critical points of the exponent, that is, points where $\phi'_\theta(x,\theta) = 0$, $\phi'_x(x,\theta) = \psi'_x(x)$. This means that $(x, \phi'_x) = (x, \psi'_x)$ is an intersection between

Λ and the section $d\psi$ of $T^*(X)$. The critical point of the exponent is non-degenerate if

$$\det\begin{pmatrix}\phi''_{\theta\theta} & \phi''_{\theta x} \\ \phi''_{x\theta} & \phi''_{xx}-\psi''_{xx}\end{pmatrix} \neq 0. \tag{3.2.18}$$

This means precisely that Λ and $d\psi$ shall have a transversal intersection. In fact, the tangent plane of Λ is defined by $\{(dx, d\phi'_x);\ d\phi'_\theta=0\}$ while that of $d\psi$ is defined by $\{(dx, d\psi'_x)\}$. Transversality means that there shall exist no vector $(t, \tau)\neq 0$ in $\mathbf{R}^n \otimes \mathbf{R}^n$ such that

$$d\langle t, x\rangle + d\langle \tau, \phi'_x\rangle = 0 \quad \text{if } d\phi'_\theta = 0$$

$$d\langle t, x\rangle + d\langle \tau, \psi'_x\rangle = 0.$$

The first condition means that for some $\sigma \in \mathbf{R}^N$

$$d\langle t, x\rangle + d\langle \tau, \phi'_x\rangle + d\langle \sigma, \phi'_\theta\rangle = 0.$$

If $\tau = 0$ the second condition shows that $t=0$ also so we can eliminate t by subtraction and find that our condition means that

$$d\langle \tau, \phi'_x - \psi'_x\rangle + d\langle \sigma, \phi'_\theta\rangle = 0$$

shall imply $\tau = 0$ and therefore $\sigma = 0$ (since ϕ is non-degenerate). But this means precisely that (3.2.18) shall be valid.

We can choose ψ so that the transversality condition is fulfilled at $(x_0, \psi'_x(x_0))$ if $\phi'_x(x_0, \theta_0) = \psi'_x(x_0)$. When supp u is sufficiently close to x_0 we then know that $\phi(x,\theta)-\psi(x)$ has at most one critical point with $x \in \operatorname{supp} u$ and $(x,\theta) \in \operatorname{cone\,supp} a$, and this point is not degenerate. We denote it by (x_s, θ_s) if it exists.

In virtue of the Morse lemma it is possible to transform $\phi(x,\theta)-\psi(x)+\psi(x_s)$ to a quadratic form in a neighborhood of (x_s, θ_s), and this will allow us to apply the discussion of the method of stationary phase made before in this section. Since we shall have to consider dependence on parameters later on we pause to give a proof of the Morse lemma in the form needed here using the same argument as in the proof of Theorem 3.1.6.

LEMMA 3.2.3. *Let $h(z,w)$ ($z \in \mathbf{R}^k$, $w \in \mathbf{R}^l$) be a C^∞ function in a neighborhood of $(0, 0)$ with $h'_z(0,0)=0$ and $Q = h''_{zz}(0)$ non-singular. Then the equation $h'_z(z,w)=0$ determines in a neighborhood of the origin a C^∞, function $z(w)$ with $z(0)=0$ and we have $h(z,w)=h(z(w),w)+\langle Q(w)\zeta,\zeta\rangle/2$ where $Q(w)=h''_{zz}(z,w)|_{z=z(w)}$ and $\zeta = z - z(w) + O(|z-z(w)|^2)$ is a C^∞ function of z and w at* 0.

Proof. By the implicit function theorem the equations $h'_z(z,w)=0$ have a unique solution $z(w)$ with the stated properties. Introducing $z-z(w)$ as a new variable instead of z

we may assume that $h'_z(0, w) = 0$ for all small w. Write $\zeta = R(z, w)z$ where R is a $k \times k$ matrix to be determined with $R(0, w) = I$ so that $\langle R^*Q(w)Rz, z\rangle/2 = h(z, w) - h(0, w)$. Now we have by Taylor's formula

$$h(z, w) - h(0, w) = \tfrac{1}{2} \sum b_{jk}(z, w) z_j z_k$$

with $B = (b_{jk})$ symmetric and C^∞ as a function of z, w. Thus the problem is to choose R so that

$$R^*Q(w)R = B.$$

When $z = 0$ we have the solution $R = I$, for $B(0, w) = Q(w)$, and the differential when $R = I$ is the map

$$R \to R^*Q(w) + Q(w)R$$

which is surjective since $R^*Q + QR = C$ where C is a symmetric matrix if $R = Q^{-1}C/2$. This proves the lemma.

Choose a cutoff function $\chi \in C_0^\infty(\mathbf{R}^n \times \mathbf{R}^N)$ which is 1 in a neighborhood of (x_s, θ_s) and has support in another neighborhood where using Lemma 3.2.3 we can make a change of variables $x = x(z)$, $\theta = \theta(z)$ with z in a neighborhood of 0 in $\mathbf{R}^{n+N}$ such that

(i) $x(0) = x_s$, $\theta(0) = \theta_s$,
(ii) $\phi(x(z), \theta(z)) - \psi(x(z)) = \langle Qz, z\rangle/2 - \psi(x_s)$ where Q is the matrix in (3.2.18) at (x_s, θ_s).
(iii) $J(z) = D(x, \theta)/Dz = 1$ when $z = 0$.

We can then write

$$\langle A, ue^{-it\psi}\rangle = I_1(t) + I_2(t), \tag{3.2.19}$$

where

$$I_1(t) = t^N (2\pi)^{-(n+2N)/4} e^{-it\psi(x_s)} \int e^{it\langle Qz, z\rangle/2} a(x(z), t\theta(z))\, \chi(x(z), \theta(z))\, u(x(z))\, J(z)\, dz, \tag{3.2.20}$$

$$I_2(t) = t^N (2\pi)^{-(n+2N)/4} \iint e^{it(\phi(x,\theta) - \psi(x))} (1 - \chi(x, \theta))\, a(x, t\theta)\, u(x)\, dx\, d\theta. \tag{3.2.21}$$

We can obtain an asymptotic expansion of (3.2.20) using (3.2.5). If $a \in S_\varrho^{m+n/4-N/2}$ the error can be estimated by a constant times t to the power

$$N - (N+n)/2 - \nu + m + n/4 - N/2 + (2\nu + N + n + 1)(1 - \varrho).$$

Since $2(1-\varrho) < 1$ it will decrease like any desired power of t^{-1} if ν is chosen large enough. In view of Proposition 1.1.10 it follows that

$$e^{it\psi(x_s)} I_1(t) \sim (2\pi)^{n/4} |\det Q|^{-\frac{1}{2}} \exp(\pi i \operatorname{sgn} Q/4)\, t^{(N-n)/2}$$
$$\times \sum_\alpha c_\alpha D_z^\alpha a(x(z), t\theta(z))\, u(x(z))\, J(z)\big|_{z=0}\, t^{-|\alpha|/2},$$

where the terms are in $S^{m-n/4+|\alpha|(\frac{1}{2}-\varrho)}$ and vanish for odd $|\alpha|$. The leading term is

$$(2\pi)^{n/4}|\det Q|^{-\frac{1}{2}} \exp(\pi i \operatorname{sgn} Q/4)\, t^{(N-n)/2} a(x_s, t\theta_s)\, u(x_s).$$

The integral $I_2(t)$ decreases faster than any power of $1/t$ as $t\to\infty$. (The function χ should be omitted in case there is no critical point for the exponent in cone supp a.) This fact is very closely related to Propositions 1.2.2 and 2.5.7 and so is the method of proof. First of all, in the support of $a(x, t\theta)(1-\chi(x,\theta))$ we have for some $\varepsilon>0$ at every point either $|\phi'_\theta|>\varepsilon$ or $|\phi'_x-\psi'_x|>\varepsilon(1+|\theta|)$. Indeed, if $|\phi'_\theta|\leqslant\varepsilon$ and ε is small enough, it follows by the definition of phase functions that $|\phi'_x|>c|\theta|$ for some $c>0$ so the assertion is obviously true for large $|\theta|$. For small $|\theta|$ it is true since $\psi'_x\neq 0$ so it follows in general for reasons of continuity. Now we can split a in a sum $a=a_1+a_2$ where $a_j\in S_\varrho^{m+n/4-N/2}$, supp $a_j\subset$ supp a ($j=1,2$) and $a_1=0$ when $|\phi'_\theta|>\varepsilon$, $a_2=0$ when $|\phi'_\theta|<\varepsilon/2$. The operator corresponding to a_2 has then a C^∞ kernel according to Proposition 1.2.4 so for the corresponding operator it is clear that (3.2.17) is rapidly decreasing. We may therefore assume that $a=a_1$ so that $|\phi'_x-\psi'_x|>\varepsilon(1+|\theta|)$ in the support of $a(x,\theta)(1-\chi(x,\theta))$. Now we have $M(\phi-\psi)=-i$ if

$$M=\Sigma\, a_j\partial/\partial x_j,\quad a_j=-i\partial(\phi-\psi)/\partial x_j\,(\Sigma\,(\partial(\phi-\psi)/\partial x_k)^2)^{-1}.$$

In the support of $a(x,t\theta)(1-\chi(x,\theta))$ the estimate

$$D_x^\alpha a_j = O((1+|\theta|)^{-1}) \tag{3.2.22}$$

is valid for all α. The adjoint $L={}^tM$ is given by

$$L=\sum_1^n -a_j\partial/\partial x_j + a_0,\quad a_0=-\sum_1^n \partial a_j/\partial x_j$$

so (3.2.22) is obviously valid also for $j=0$. Now we obtain for every integer ν by repeated partial integrations

$$I_2(t)=t^N(2\pi)^{-(n+2N)/4}\iint e^{it(\phi(x,\theta)-\psi(x))}t^{-\nu}L^\nu a(x,t\theta)\,(1-\chi(x,\theta))\,u(x)\,dx\,d\theta.$$

The integrand can be bounded by a constant times

$$t^{-\nu}(1+|\theta|)^{-\nu}(1+|t\theta|)^{m+n/4-N/2+\nu(1-\varrho)}$$

which can obviously be bounded by $(1+|\theta|)^{-N-1}t^{\mu-\varrho\nu}$ for a certain constant μ which is independent of ν. Thus the integral decreases faster than any power of $1/t$ as asserted. Summing up, we have proved

THEOREM 3.2.4. *Let ϕ be a non-degenerate phase function in an open conic set $\Gamma\subset\mathbf{R}^n\times(\mathbf{R}^N\setminus 0)$ and let $a\in S_\varrho^{m+n/4-N/2}(\mathbf{R}^n\times\mathbf{R}^N)$ vanish outside $\{(x,t\theta);\, t\geqslant 1, (x,\theta)\in K\}$ for* *some compact set* $K \subset \Gamma$. *Let* $u \in C^\infty(\mathbf{R}^n)$, $\psi \in C_0^\infty(\mathbf{R}^n)$; *assume that* ψ *is real valued and that* grad $\psi \neq 0$ *in* supp u. *Then*

(i) *If there is no point* (x, θ) *in* cone supp a *with* $x \in \operatorname{supp} u$ *and* $\phi'_\theta = 0$, $\phi'_x = \psi'_x$, *then the function* $t \to \langle A, ue^{-it\psi}\rangle$ *where* A *is defined by* (3.2.1) *belongs to* $S^{-\infty}(\mathbf{R}_+)$.

(ii) *If there is precisely one point* (x_s, θ_s) *in* cone supp a *with* $x_s \in \operatorname{supp} u$ *and* $\phi'_\theta(x_s, \theta_s) = 0$, $\phi'_x(x_s, \theta_s) = \psi'_x(x_s)$, *and if* (3.2.18) *is valid there, then*

$$t \to e^{it\psi(x_s)}\langle A, ue^{-it\psi}\rangle$$

is in $S_\varrho^{m+n/4-n/2}(\mathbf{R}_+)$ *and we have*

$$e^{it\psi(x_s)}\langle A, ue^{-it\psi}\rangle - (2\pi)^{n/4}|\det Q|^{-\frac{1}{2}} \exp(\pi i \operatorname{sgn} Q/4)\, t^{(N-n)/2} a(x_s, t\theta_s)\, u(x_s) \in S_\varrho^{m-n/4+1-2\varrho}(\mathbf{R}_+) \tag{3.2.23}$$

where Q *is the matrix in* (3.2.18).

Theorem 3.2.4 can be extended to the case where ψ (and u) may depend on parameters. Thus let ψ_t be a positively homogeneous C^∞ function of a parameter $t \in \Gamma_1$, an open cone in $\mathbf{R}^M \setminus 0$, say, with values in $C^\infty(\mathbf{R}^n)$ satisfying the hypotheses of Theorem 3.2.4. (For simplicity we keep u fixed instead of as an element in $S_\varrho^k(X \times \Gamma_1)$.) Then the stationary point of $\phi - \psi_t$ is a C^∞ positively homogeneous function $(x_s(t), \theta_s(t))$ where it is defined, and we have

$$\langle A, ue^{-i\psi}t\rangle - e^{-i\psi_t(x_s(t))} b(t) \in S^{-\infty}(\Gamma_1), \tag{3.2.24}$$

where b vanishes outside the set where $x_s(t)$ is defined and

$$b(t) - (2\pi)^{n/4}\Delta(t)^{-\frac{1}{2}} \exp(\pi i \sigma/4)\, a(x_s(t), \theta_s(t))\, u(x_s(t)) \in S_\varrho^{m-n/4+1-2\varrho}(\Gamma_1). \tag{3.2.23$'$}$$

Here $\Delta(t)$ and σ are the determinant and signature of the matrix in (3.2.18) when ψ is replaced by ψ_t, evaluated at the point $(x_s(t), \theta_s(t))$. Note that $\Delta(t)$ is a homogeneous function of degree $n - N$.

We can now prove that the map (3.2.16) is injective. To do so we assume as in Definition 3.2.2 that $A = \sum_{j \in J} A_j$ and that for a certain point λ_0 all terms except one, say A_0, have the property that $(x, \phi'_{jx}) \neq \lambda_0$ when $(x, \theta) \in$ cone supp a_j and $\phi'_{j\theta} = 0$. We must prove that if $A = 0$ then $a_0(x, \theta) \in S_\varrho^{m+n/4-N_0/2+1-2\varrho}$ on the surface $\phi'_{0\theta}(x, \theta) = 0$ in a conic neighborhood of the point (x_0, θ_0) where $(x_0, \phi'_{0x}(x_0, \theta_0)) = \lambda_0$. In doing so we choose coordinates near x_0 according to Theorem 3.1.3, take $u \in C_0^\infty(X)$ with support close to x_0 and let Γ_1 be a small conic neighborhood of $\phi'_{0x}(x_0, \theta_0) \in \mathbf{R}^n$. With $\psi_t(x) = \langle x, t\rangle$, $t \in \Gamma_1$, we then obtain that $\langle A_j, ue^{-it\psi_t}\rangle$ is rapidly decreasing for $j \neq 0$, and since $A_0 = -\sum_{j \neq 0} A_j$ this must also be true when $j = 0$. Now the intersection between Λ and $d\psi_t$ is at $(H'(t), t)$ and it is transversal in view of Theorem 3.1.3, so we obtain

$$e^{iH(t)}\langle A_0, ue^{-i\psi_t}\rangle - (2\pi)^{n/4}\Delta(t)^{-\frac{1}{2}} \exp(\pi i\sigma/4)\, a(x_s(t), \theta_s(t))\, u(x_s(t)) \in S_\varrho^{m-n/4+1-2\varrho}$$

in Γ_1. Here $(x_s(t), \phi'_{0x}(x_s(t), \theta_s(t)) = (H'(t)), t) \in \Lambda$ is a homogeneous parametrization of Λ close to λ_0. It follows that $a(x_s(t), \theta_s(t)) \in S_\varrho^{m+n/4-N/2+1-2\varrho}(\Gamma_1)$ which proves our assertion. Summing up:

THEOREM 3.2.5. *The map* (3.2.16) *defined above is an isomorphism.*

It is clear that using (3.2.24), (3.2.23)′ with ψ_t chosen as in the preceding argument one can give a new proof of the transformation laws which we have here based on the analogue of the Morse lemma proved in section 3.1. At the same time one can obtain a characterization of $I^m(X, \Lambda)$ as the distributions for which (3.2.24) is valid when $d\psi_t$ and Λ intersect transversally. This is quite analogous to properties of pseudo-differential operators sometimes used to define them (see e.g. [12]). We leave the development of this approach to the reader but the relations between the two methods will be clarified in the next section.

The proof that (3.2.16) is injective also gives the second part of the following theorem; the first part follows from Proposition 2.5.7.

THEOREM 3.2.6. *Let $A \in I_\varrho^m(X, \Lambda)$ and let $a \in S_\varrho^{m+n/4}(\Lambda, \Omega_{\frac{1}{2}} \otimes L)$ be a principal symbol. Then $WF(A) \subset \Lambda$ and $a \in S_\varrho^{m+n/4+1-2\varrho}$ in $\Lambda \setminus WF(A)$.*

3.3. Interpretation of the line bundle L

Let again Λ be a homogeneous Lagrangean manifold $\subset T^*(X)\setminus 0$, let $\lambda_0 \in \Lambda$ and $x_0 = \pi\lambda_0$. Let φ be a non-degenerate phase function in a conic neighborhood of (x_0, θ_0) with $\varphi'_\theta = 0$ and $(x, \varphi'_x) = \lambda_0$ at (x_0, θ_0). If $\psi \in C^\infty$ at x_0, if $(x_0, \psi'_x(x_0)) = \lambda_0$ and $\{(x, \psi'_x(x))\}$ is transversal to Λ at λ_0, we shall denote by $S(\varphi, \psi)$ the signature of the non-singular matrix (3.2.18) which occurs in (3.2.23). This is of course evaluated at (x_0, θ_0) where $\varphi(x, \theta) - \psi(x)$ has a critical point, so it is clearly invariant under a change of variables in X or a fiber preserving change of θ variables. The following statement follows in part from the consistency of the main argument in section 3.2 with that outlined at the end but we give a direct proof.

PROPOSITION 3.3.1. *If φ and $\tilde{\varphi}$ are non-degenerate phase functions in neighborhoods of (x_0, θ_0) and $(x_0, \tilde{\theta}_0)$ both defining Λ at λ_0, then*

$$S(\varphi, \psi) - S(\tilde{\varphi}, \psi) = \operatorname{sgn} \varphi''_{\theta\theta}(x_0, \theta_0) - \operatorname{sgn} \tilde{\varphi}''_{\theta\theta}(x_0, \tilde{\theta}_0). \tag{3.3.1}$$

Proof. Both sides are equal to 0 if φ and $\tilde{\varphi}$ are equivalent. If φ is obtained from $\tilde{\varphi}$ by increasing the number of θ variables as described in section 3.1 it is also clear that

(3.3.1) is valid, and in view of Corollary 3.1.8 these two cases combined prove the proposition.

By our definition of L, to define an element in the fiber L_{λ_0} means to give for each phase function φ defining Λ at λ_0 a complex number z_φ such that

$$z_{\tilde{\varphi}} = z_\varphi \exp \pi i((\operatorname{sgn} \varphi''_{\theta\theta}(x_0, \theta_0) - N) - (\operatorname{sgn} \tilde{\varphi}''_{\tilde{\theta}\tilde{\theta}}(x_0, \tilde{\theta}_0) - \tilde{N}))/4$$

where N and $\tilde{N}$ denote the number of variables θ and $\tilde{\theta}$. In view of (3.3.1) this implies that

$$z_\varphi \exp \pi i(S(\varphi, \psi) - N)/4 = z_{\tilde{\varphi}} \exp \pi i(S(\tilde{\varphi}, \psi) - \tilde{N})/4,$$

so we obtain a linear isomorphism $L_{\lambda_0} \to \mathbf{C}$ independent of φ but depending on ψ, or rather on the tangent plane of $\{(x, \psi'(x))\}$ at λ_0. This is an arbitrary Lagrange plane in $T_{\lambda_0}(T^*(X))$ transversal to $T_{\lambda_0}(\Lambda)$ and to the tangent space of the fiber at λ_0. (See the proof of Theorem 3.1.3.)

Next we examine how the isomorphism depends on ψ. Passage from ψ_2 to ψ_1 will of course mean multiplication by $e^{\pi i/4}$ to the power

$$S(\varphi, \psi_1) - S(\varphi, \psi_2) = S(\tilde{\varphi}, \psi_1) - S(\tilde{\varphi}, \psi_2). \tag{3.3.2}$$

We shall give a geometric interpretation of this integer which is always even since $S(\varphi, \psi_1)$ and $S(\varphi, \psi_2)$ are both congruent to $N+n \bmod 2$. To facilitate computations we choose local coordinates according to Theorem 3.1.3 so that Λ is defined near λ_0 by means of a phase function

$$\varphi(x, \xi) = \langle x, \xi \rangle - H(\xi)$$

which is linear with respect to x. Writing $A = H''_{\xi\xi}(\xi_0)$, $B_j = \psi''_{jxx}(x_0)$ we have

$$S(\varphi, \psi_1) - S(\varphi, \psi_2) = \operatorname{sgn} \begin{pmatrix} -A & I \\ I & -B_1 \end{pmatrix} - \operatorname{sgn} \begin{pmatrix} -A & I \\ I & -B_2 \end{pmatrix}. \tag{3.3.3}$$

Note that in our local coordinates

1) the tangent plane λ_1 of the fiber is defined by $x=0$;
2) the tangent plane λ_2 of Λ is defined by $x = A\xi$;
3) the tangent plane μ_j of $\{(x, \psi'_j(x))\}$ is defined by $\xi = B_j x$.

We now recall the structure of $H^1(\Lambda(n))$ where $\Lambda(n)$ is the space of all Lagrange planes in $T^*(\mathbf{R}^n) = \mathbf{C}^n$. (We shall use integer coefficients unless other coefficients are specified.) As we have seen in section 3.1,

$$\Lambda(n) = U(n)/O(n),$$

and since $(\det O)^2 = 1$ if $O \in O(n)$, this gives a map $\det^2$: $\Lambda(n) \to S^1$. As shown by Arnold [1], $H^1(\Lambda(n))$ is the free group generated by the pullback α of the generator of $H^1(S^1)$ under this map. The class α only depends on the symplectic structure. For let E be any $2n$ dimensional vector space with a given non-singular skew symmetric bilinear form σ. We can then find linear isomorphisms γ: $E \to \mathbf{R}^n \oplus \mathbf{R}^n$ such that σ is the pullback of the standard symplectic form in $\mathbf{R}^n \oplus \mathbf{R}^n$. If we have two such maps γ_1 and γ_2, then $\gamma_1 \gamma_2^{-1}$ is a symplectic map γ: $\mathbf{R}^n \oplus \mathbf{R}^n \to \mathbf{R}^n$. Such a map is homotopic to a map $(x, \xi) \to (Ox, O\xi)$ where O is orthogonal. In fact, the Lagrange plane $\gamma(\mathbf{R}^n \oplus \{0\})$ is of the form $U\mathbf{R}^n$ where $U \in U(n)$. Since $U(n)$ is connected it follows that γ is homotopic to $U^{-1}\gamma$, which is of the form

$$\mathbf{R}^n \oplus \mathbf{R}^n \ni (x, \xi) \to (A_{11}x + A_{12}\xi, A_{22}\xi).$$

That this map is symplectic means that ${}^tA_{22}A_{11} = I$ and that ${}^tA_{22}A_{12}$ is symmetric. Replacing A_{12} by tA_{12} we conclude that the map is homotopic to one where A_{12} is 0, and since A_{11} can be connected in GL $(n, \mathbf{R})$ to an orthogonal transformation, the assertion is proved. Now a transformation $(x, \xi) \to (Ox, O\xi)$ transforms the Lagrangean plane parametrized by U to one parametrized by OU. Since this does not change the square of the determinant, we conclude that $\tilde{\gamma}^*\alpha = \alpha$ if $\tilde{\gamma}$ is the map $\Lambda(n) \to \Lambda(n)$ defined by γ. If $\tilde{\gamma}_j$ is the map $\Lambda(E) \to \Lambda(n)$ defined by γ_j, we obtain $\tilde{\gamma}_1^* \alpha = \tilde{\gamma}_2^* \alpha$. This class in $H^1(\Lambda(E))$ we shall denote by α_E. Arnold [1] proved that it is dual to the twosided cycle of all elements of $\Lambda(E)$ which are not transversal to a fixed one. This cycle was considered before in a similar context by Maslov [23] and even earlier by Keller [18]. Various alternative descriptions are discussed by Maslov [23].

Let λ_1, λ_2, μ_1, μ_2 be four elements of $\Lambda(E)$ such that λ_j and μ_k are transversal for $j = 1, 2$ and $k = 1, 2$. We can choose a path from μ_1 to μ_2 of planes transversal to λ_k, for the set of Lagrangean planes transversal to a fixed one forms a cell. In fact, the Lagrange planes in $\mathbf{R}^n \oplus \mathbf{R}^n$ which are transversal to $x = 0$ are of the form $\xi = Ax$ with A symmetric and so they form an affine space even. The homotopy class of such a curve γ_k is therefore uniquely determined and so is that of the closed curve $\gamma = \gamma_1 - \gamma_2$. We shall now compute $\langle \gamma, \alpha \rangle$ and see that we get essentially (3.3.3).

First assume that λ_1 and λ_2 are transversal and choose the coordinates so that λ_1 is defined by $x = 0$ and λ_2 by $\xi = 0$. Then

$$\mu_k = \{(x, B_k x), x \in \mathbf{R}^n\},$$

where B_k is symmetric and $\det B_k \neq 0$. To choose a curve from μ_1 to μ_2 which consists of planes transversal to λ_1 means to choose continuously a symmetric matrix B_t, $1 \leq t \leq 2$ with given values for $t = 1, 2$, which we can do. Now the plane $\xi = Bx$ is equal to $U\mathbf{R}^n$ if U is a unitary matrix with

$$U^2 = (I+iB)(I-iB)^{-1}.$$

This is clear if B has diagonal form and follows in general by diagonalization. The square of the determinant of U is $\prod(1+i\beta_t^j)/(1-i\beta_t^j)$ if U defines B_t and β_t^j are the eigenvalues of B_t. If α denotes the pullback to $\Lambda(n)$ of the invariant one form of S^1 with integral 1, then

$$\int_{\gamma_1} \alpha = \pi^{-1}(\sum \arg(1+i\,\beta_2^j) - \arg(1+i\beta_1^j)).$$

Here we have used that arg w is uniquely defined for Re $w>0$ so that it vanishes for real w.

The integral over γ_2 is similar, and we can reduce it to the case already studied by replacing B_j by $-B_j^{-1}$, for $(x, \xi)\to(\xi, -x)$ preserves α. Hence

$$\int_{\gamma} \alpha = \pi^{-1}(\sum (\arg(1+i\beta_2^j) - \arg(1-i/\beta_2^j)) - \sum(\arg((1+i\beta_1^j) - \arg(1-i/\beta_1^j)).$$

For real $t\neq 0$ we have $\arg(1+it)-\arg(1-i/t)=\pi/2$ if $t>0$, $=-\pi/2$ if $t<0$, for $it(1-i/t)=1+it$. Hence

$$\langle\gamma, \alpha\rangle = (\operatorname{sgn} B_2 - \operatorname{sgn} B_1)/2. \tag{3.3.4}$$

Assume now that λ_2 is defined by $x=A\xi$ where A is invertible and symmetric. Then the symplectic transformation $(x, \xi)\to(x, \xi-A^{-1}x)$ reduces us to the case already considered and so

$$\langle\gamma, \alpha\rangle = (\operatorname{sgn}(B_2-A^{-1}) - \operatorname{sgn}(B_1-A^{-1}))/2.$$

Now we have
$$\begin{pmatrix} -A & I \\ I & -B \end{pmatrix} = \begin{pmatrix} I & 0 \\ -A^{-1} & I \end{pmatrix} \begin{pmatrix} -A & 0 \\ 0 & A^{-1}-B \end{pmatrix} \begin{pmatrix} I & -A^{-1} \\ 0 & I \end{pmatrix}$$

so
$$\operatorname{sgn}\begin{pmatrix} -A & I \\ I & -B_j \end{pmatrix} = -(\operatorname{sgn} A + \operatorname{sgn}(B_j - A^{-1})).$$

It follows that
$$\langle\gamma, \alpha\rangle = \left(\operatorname{sgn}\begin{pmatrix} -A & I \\ I & -B_1 \end{pmatrix} - \operatorname{sgn}\begin{pmatrix} -A & I \\ I & -B_2 \end{pmatrix}\right) \Big/ 2. \tag{3.3.5}$$

This formula remains valid by continuity even if A is not invertible provided that the matrices are non-singular, which means that λ_2 is transversal to μ_1 and μ_2. We have now established the asserted connection with (3.3.3) and introduce

Definition 3.3.2. If $\lambda_1, \lambda_2, \mu_1, \mu_2$ are four Lagrange planes in a symplectic vector space E such that each of the first two is transversal to each of the last two, we define

$$\sigma(\lambda_1, \lambda_2; \mu_1, \mu_2) = \langle\gamma, \alpha_E\rangle, \tag{3.3.6}$$

where γ is a closed curve in $\Lambda(E)$ which consists of an arc from μ_1 to μ_2 of Lagrange planes transversal to λ_1 followed by an arc from μ_2 to μ_1 of Lagrange planes transversal to λ_2. Here α_E is the class of Keller, Maslov and Arnold.

It is clear that (3.3.6) gives an integer, and we have

$$\sigma(\lambda_1, \lambda_2; \mu_1, \mu_2) = -\sigma(\lambda_2, \lambda_1; \mu_1, \mu_2) = -\sigma(\mu_1, \mu_2; \lambda_1, \lambda_2). \tag{3.3.7}$$

The first equality follows immediately from the definition, for interchanging λ_1 and λ_2 means reversing the orientation of γ. To prove the second we choose coordinates so that λ_1: $x=0$; λ_2: $x=A\xi$; μ_1: $\xi=0$; μ_2: $\xi=Bx$. The symplectic transformation $(x,\xi)\to(-\xi, x)$ will interchange λ_j and μ_j, apart from a substitution of $-B$, $-A$ for A, B, so (3.3.5) reduces the statement to the obvious one

$$\operatorname{sgn}\begin{pmatrix}-A & I\\ I & -B\end{pmatrix} = -\operatorname{sgn}\begin{pmatrix}B & I\\ I & A\end{pmatrix}.$$

In addition we have of course the obvious cocycle conditions such as

$$\sigma(\lambda_1, \lambda_2; \mu_1, \mu_2) + \sigma(\lambda_1, \lambda_2; \mu_2, \mu_3) = \sigma(\lambda_1, \lambda_2; \mu_1, \mu_3). \tag{3.3.8}$$

If we sum up the discussion which led us to introduce Definition 3.3.2, we have proved

THEOREM 3.3.3. *Let λ be a point on the Lagrangean manifold $\Lambda\subset T^*(X)$. Denote by M_λ the set of all Lagrange planes in $T_\lambda(T^*X)$ which are transversal to $T_\lambda(\Lambda)$ and to the tangent space T_λ^0 of the fiber. Then the fiber L_λ of the line bundle L is naturally isomorphic to the set of all maps f: $M_\lambda\to\mathbf{C}$ such that for all $\mu_1, \mu_2\in M_\lambda$*

$$f(\mu_1) = i^{\sigma(T_\lambda^0, T_\lambda(\Lambda); \mu_1, \mu_2)} f(\mu_2). \tag{3.3.9}$$

The preceding interpretation is of course already underlying the evaluation of $A(ue^{it\psi})$ when Λ and $\{(x, \psi_x'(x))\}$ intersect transversally. An invariant interpretation of the main term in (3.2.23) is given by Theorem 3.3.3 and the fact that a density (or order $\frac{1}{2}$) on Λ together with the density u in X lifted to $\{(x, \psi_x'(x))\}$ produces a density at the intersection which by means of the symplectic measure $dx\,d\xi$ in the cotangent bundle gives a scalar. (See also the discussion following Theorem 4.2.2.)

We shall now consider the preceding construction in a general setting which seems instructive although it is not really required for the rest of the paper. Let Y be a Hausdorff topological space and E a symplectic vector bundle of fiber dimension $2n$ over Y. Then every point in Y has a neighborhood U such that there is a vector bundle isomorphism

$U \times \mathbf{C}^n \to E|_U$ preserving the symplectic structure. It is clear that the set $\Lambda(E)$ of Lagrange planes in the fibers of E form another fiber bundle over Y with fiber $\Lambda(\mathbf{C}^n)$.

We can now assign to any two continuous sections λ_1, λ_2 of $\Lambda(E)$ over Y (if such exist) an element $\alpha(\lambda_1, \lambda_2) \in H^1(Y)$. To do so we first modify the construction in Theorem 3.3.3 to avoid reduction mod 4. Thus let F be the set of all pairs $(\mu, z) \in \Lambda(E) \times \mathbf{C}$ such that μ is transversal to λ_{1x} and λ_{2x} if $\mu \in \Lambda(E_x)$, with the equivalence relation $(\mu_1, z_1) \sim (\mu_2, z_2)$ if μ_1, μ_2 are in $\Lambda(E_x)$ and $z_1 - z_2 = \sigma(\lambda_{1x}, \lambda_{2x}; \mu_1, \mu_2)$. That this is an equivalence relation follows from (3.3.7). F has a natural topology and is a fiber space with fiber $\mathbf{C}$ and structure group $\mathbf{Z}$ acting by translations so it defines an element $\alpha(\lambda_1, \lambda_2) \in H^1(Y, \mathbf{Z})$. (Similarly, by copying (3.3.9) we obtain a line bundle with structure group $\mathbf{Z}_4$ defining the reduction of $\alpha(\lambda_1, \lambda_2)$ mod 4.) Explicitly, let $\{(U_i, f_i),\ i \in I\}$ be a set of continuous integer valued sections of F over open sets $U_i \subset Y$ with $\bigcup U_i = Y$. Clearly $f_{ij} = f_j - f_i$ is a locally constant integer in $U_i \cap U_j$ so we have a 1-cocycle defining $\alpha(\lambda_1, \lambda_2)$. Note that

$$\alpha(\lambda_1, \lambda_2) = -\alpha(\lambda_2, \lambda_1); \quad \alpha(\lambda_1, \lambda_3) = \alpha(\lambda_1, \lambda_2) + \alpha(\lambda_2, \lambda_3). \tag{3.3.10}$$

The construction is obviously functorial: If $g: Z \to Y$ is a continuous map, the pullback g^*E is a symplectic vector bundle on Z with Lagrangean subbundles $g^*\lambda_1$ and $g^*\lambda_2$, and we have $\alpha(g^*\lambda_1, g^*\lambda_2) = g^*\alpha(\lambda_1, \lambda_2)$.

Example. Let E be a symplectic vector space and consider E as a symplectic vector bundle $\tilde{E}$ over $Y = \Lambda(E)$. Every point in Y is a Lagrange plane in E which gives us a section λ_2 of $\Lambda(\tilde{E})$. Let λ_1 be defined by a fixed Lagrange plane in E. Then we have

$$\alpha(\lambda_1, \lambda_2) = \alpha_E. \tag{3.3.11}$$

To prove this we may assume that $E = \mathbf{C}^n$ and that λ_1 is defined by $i\mathbf{R}^n$. Now a cohomology class in $H^1(\Lambda(E))$ is equal to α_E if and only if its restriction to some closed curve which is mapped bijectively on the unit circle by the map $\det^2$ considered above is the generator of H^1 for the curve. Such a curve is given for example by the unitary transformations $U(t)z = (e^{it\pi}z_1, z_2, \ldots, z_n)$ corresponding to the Lagrange planes $(0 \leqslant t \leqslant 1)$

$$\lambda(t):\ x_1 \sin t\pi - \xi_1 \cos t\pi = 0,\ \xi_2 = \ldots = \xi_n = 0$$

where we have used the notation $x + i\xi$ for points in $\mathbf{C}^n$. Let $V_1(V_2)$ be the part of the curve where $t \neq \frac{1}{4}\,(t \neq \frac{3}{4})$, and define a section f_j of F in V_j so that $f_1 = 0$ at $\mu(\frac{1}{4})$ and $f_2 = 0$ at $\mu(\frac{3}{4})$. Here $\mu(t)$ is defined by $x_1 \sin t\pi - \xi_1 \cos t\pi = 0$, $x_2 = \xi_2, \ldots, x_n = \xi_n$. Then $f_2 - f_1 = 0$ when $\frac{1}{4} < t < \frac{3}{4}$, and in the other interval of $V_1 \cap V_2$ we have

$$f_2 - f_1 = \sigma(\lambda_1, \lambda(0), \mu(\tfrac{1}{4}),\ \mu(\tfrac{3}{4})) = -1$$

by (3.3.4). This proves (3.3.11).

More generally, let E again be a symplectic vector bundle over Y and lift E to a symplectic vector bundle $\tilde{E}$ over $\Lambda(E)$ by means of the projection $\Lambda(E) \to Y$. Let $\tilde{\lambda}_j$ be the pullbacks of the Lagrangean subbundles λ_j of E by this projection. As above we also have a natural section S of $\Lambda(\tilde{E})$ defined by noting that every point in $\Lambda(E)$ is a Lagrange plane in the fiber over that point. In view of (3.3.10) we have

$$\alpha(\tilde{\lambda}_1, \tilde{\lambda}_2) = \alpha(\tilde{\lambda}_1, S) - \alpha(\tilde{\lambda}_2, S) = A(\lambda_1) - A(\lambda_2),$$

where we have defined $A(\lambda_j) = \alpha(\tilde{\lambda}_j, S) \in H^1(\Lambda(E))$. If s is any section of $\Lambda(E)$ over Y then the pullback s^*S is equal to s and $s^*\tilde{\lambda}_j = \lambda_j$. Choosing $s = \lambda_2$ we therefore have

$$\alpha(\lambda_1, \lambda_2) = \lambda_2^* A(\lambda_1). \tag{3.3.12}$$

Dropping the subscript we note that $A(\lambda)$ has the properties

$$\lambda^* A(\lambda) = 0, \tag{3.3.13}$$

$$i_y^* A(\lambda) = \alpha_{E_y}, \tag{3.3.14}$$

where i_y is the injection $\Lambda(E_y) \to \Lambda(E)$. In fact, (3.3.13) means just that $\alpha(\lambda, \lambda) = 0$, and (3.3.14) follows from (3.3.11) and the functorial properties. *The conditions* (3.3.13), (3.3.14) *determine* $A(\lambda)$ *uniquely*. In fact, since $H^1(\Lambda(E_y))$ is the free group generated by α_{E_y}, condition (3.3.14) means that $A(\lambda)$ is a cohomology extension of the fiber. The Leray–Hirsch theorem therefore shows that every cohomology class in $H^1(\Lambda(E))$ is of the form

$$A_1 = \pi^* c + kA(\lambda),$$

where k is an integer, $c \in H^1(Y)$ and π is the projection $\Lambda(E) \to Y$. If $\lambda^* A_1 = 0$ we obtain $0 = \lambda^* \pi^* c = (\pi\lambda)^* c = c$. If the restriction of A_1 to the fibers is equal to α_{E_y} it follows that $k = 1$ also. The class $A(\lambda)$ is a natural generalization to a symplectic vector bundle with a given Lagrangean subbundle of the class of Keller, Maslov and Arnold which corresponds to a fixed symplectic vector space. If A is the class in $H^1(\Lambda(T(T^*(X))))$ corresponding to the Lagrangean bundle given by the tangents of the fibers, (3.3.12) means that the line bundle L on a Lagrangean submanifold Λ_0 of $T^*(X)$ is defined by the pullback of A to Λ_0 by the map assigning to each point in Λ_0 its tangent space.

In this context it is very easy to give an example showing that the structure group of L cannot always be reduced further. For example, starting from Example 3.1.5 in a neighborhood of 0 we extend the curve $(x_1/3)^3 - (x_2/2)^2 = 0$ so that at infinity it becomes the axis $x_1 = 0$ with a finite segment removed. This is then a circle in $P^2_{\mathbf{R}}$, and the normal bundle Λ_0 defined according to Example 3.1.5 near the origin splits in two components, each of which is homeomorphic to $S^1 \times \mathbf{R}$. We can define Λ_0 by a linear phase function except near the origin where a defining phase function is given in Example 3.1.5. This gives easily that the cohomology class in $H_1(\Lambda_0, \mathbf{Z}_4)$ corresponding to L is a generator for $H^1(S^1 \times \mathbf{R}, \mathbf{Z}_4)$ in each of the components, thus non-trivial.

However, we know that L must be trivial in the case studied in section 2.4. Our next purpose is to relate this trivialization to the interpretation of L given by Theorem 3.3.3. As in section 2.4 let Y be a submanifold of X of codimension N and choose local coordinates $x_1, ..., x_n$ in X so that Y is defined by the equations $x' = (x_1, ..., x_N) = 0$, thus $x'' = (x_{N+1}, \dots, x_n)$ are local coordinates in Y. The phase function $\varphi(x, \theta) = \sum_1^N x_j \theta_j$ then defines the normal bundle of Y. If ψ is a function with $\lambda_0 = (x_0, \psi'(x_0)) \in N(Y) \setminus 0$ at a point $x_0 \in Y$, the matrix (3.2.18) becomes

$$\begin{pmatrix} 0 & I & 0 \\ I & -\psi''_{x'x'} & -\psi''_{x'x''} \\ 0 & -\psi''_{x''x'} & -\psi''_{x''x''} \end{pmatrix}$$

which is non-singular precisely when $\det \psi''_{x''x''} \neq 0$, that is, the restriction of ψ to Y has a non-degenerate critical point at x_0. The signature is $-\operatorname{sgn} \psi''_{x''x''}$. The isomorphism of L_{λ_0} and $\mathbf{C}$ corresponding to ψ according to Theorem 3.3.3 should therefore be multiplied by $\exp \pi i \operatorname{sgn} \psi''_{x''x''}(x_0)/4$ to agree with that used in section 2.4.

If the coordinates in $T^*(X)$ are denoted by x', x'', ξ', ξ'' the tangent plane of $N(Y)$ has the direction of the plane $x' = \xi'' = 0$. Its intersection with the tangent plane of the fiber becomes $x' = x'' = \xi'' = 0$ with orthogonal space with respect to the symplectic form given by $x' = 0$. Intersecting the plane $\xi = \psi''_{xx}(x_0)x$, which is parallel to the tangent plane of the graph of $d\psi$, with the plane $x' = 0$ and taking the quotient with respect to its orthogonal space we obtain the plane $\xi'' = \psi''_{x''x''}(x^0)x''$ defined by the non-singular matrix in which we are interested. This leads us to the following construction.

Let V be a symplectic vector space and $\lambda_1, \lambda_2, \mu$ three Lagrangean subspaces with μ transversal to λ_1 and to λ_2. If λ_1 and λ_2 are also transversal we choose coordinates so that λ_1 is defined by $x = 0$, λ_2 by $\xi = 0$ and consequently μ is defined by $\xi = Ax$ for some symmetric non-singular matrix A. A different choice of bases would give an equivalent matrix so we can set independently of the choice of basis

$$\sigma(\lambda_1, \lambda_2; \mu) = \operatorname{sgn} A. \tag{3.3.15}$$

In the general case let $\varrho = \lambda_1 \cap \lambda_2$. Then ϱ is contained in its orthogonal space $\varrho^\perp$ with respect to the symplectic form in V, and $\varrho^\perp/\varrho = V^\varrho$ is clearly a symplectic vector space with the symplectic form inherited from V. If μ is any Lagrange plane in V, then $\mu^\varrho = (\mu \cap \varrho^\perp)/\varrho$ is a Lagrange plane in V^ϱ, for

$$\dim(\mu\cap\varrho^\perp)-\dim(\mu\cap\varrho)=\dim V-\dim(\mu+\varrho)-\dim(\mu\cap\varrho)$$
$$=\dim V-\dim\mu-\dim\varrho=(\dim V-2\dim\varrho)/2=\dim V^\varrho/2.$$

When μ is transversal to λ_j it follows that $\mu\cap\varrho=\{0\}$ and therefore that μ^ϱ is transversal to λ_j^ϱ which are also mutually transversal. We can now define

$$\sigma(\lambda_1,\lambda_2;\mu)=\sigma(\lambda_1^\varrho,\lambda_2^\varrho;\mu^\varrho). \tag{3.3.16}$$

If both μ_1 and μ_2 are transversal to λ_1 and λ_2, we have

$$\sigma(\lambda_1,\lambda_2;\mu_1,\mu_2)=(\sigma(\lambda_1,\lambda_2;\mu_2)-\sigma(\lambda_1,\lambda_2;\mu_1))/2. \tag{3.3.17}$$

This follows from (3.3.4) when λ_1 and λ_2 are transversal, so all we have to prove is that with the preceding notations

$$\sigma(\lambda_1,\lambda_2;\mu_1,\mu_2)=\sigma(\lambda_1^\varrho,\lambda_2^\varrho;\mu_1^\varrho,\mu_2^\varrho). \tag{3.3.18}$$

To prove this we introduce coordinates similar to those used in the introductory arguments, so that $x=(x',x'')$, $\xi=(\xi',\xi'')$, λ_1 is defined by $x=0$ and λ_2 by $x'=\xi''=0$. We can write μ_j in the form $\xi=B^jx$ where

$$B^j=\begin{pmatrix}B_{11}^j & B_{12}^j\\ B_{21}^j & B_{22}^j\end{pmatrix}.$$

λ_2 is the limit of the plane $x'=0$, $x''=\xi''/\varepsilon^2$ when $\varepsilon\to0$, so for small ε we can write $2\sigma(\lambda_1,\lambda_2;\mu_1,\mu_2)$ as the difference of the signatures of

$$\begin{pmatrix}0 & 0 & I' & 0\\ 0 & -I''/\varepsilon^2 & 0 & I''\\ I' & 0 & -B_{11}^j & -B_{12}^j\\ 0 & I'' & -B_{21}^j & -B_{22}^j\end{pmatrix}$$

for $j=1,2$. If we multiply by ε in the second row and column the signature does not change which when $\varepsilon\to0$ shows that it is equal to that of the matrix

$$\begin{pmatrix}0 & 0 & I' & 0\\ 0 & -I'' & 0 & 0\\ I' & 0 & -B_{11}^j & -B_{12}^j\\ 0 & 0 & -B_{21}^j & -B_{22}^j\end{pmatrix}$$

which is non-singular since μ_j is transversal to λ_2. The signature is equal to $-\operatorname{sgn} I''-\operatorname{sgn} B_{22}^j$. Hence

$\sigma(\lambda_1, \lambda_2; \mu_1, \mu_2) = (\operatorname{sgn} B_{22}^2 - \operatorname{sgn} B_{22}^1)/2 = \sigma(\lambda_1^0, \lambda_2^0; \mu_1^0, \mu_2^0)$ by (3.3.4). Thus we have proved (3.3.17). Note that whereas $\sigma(\lambda_1, \lambda_2; \mu_1, \mu_2)$ is a continuous function of the four variables when λ_j is transversal to μ_k, we can only claim that $\sigma(\lambda_1, \lambda_2; \mu)$ is a *continuous* function when μ is transversal to λ_j and dim $(\lambda_1 \cap \lambda_2)$ is fixed. In that case, however, we conclude that the cohomology class $\alpha(\lambda_1, \lambda_2)$ is trivial if λ_1 and λ_2 are Lagrangean subbundles of a symplectic vector bundle E over a space Y. (Note that $(\sigma(\lambda_1, \lambda_2; \mu_j) - \dim \mu_j - \dim (\lambda_1 \cap \lambda_2))/2$ is an integer which allows us to obtain an integral 0-cochain from (3.3.17).) Summing up, we have in particular

THEOREM 3.3.4. *Assume that on the Lagrangean manifold $\Lambda \subset T^*(X)$ the intersection $T_\lambda(\Lambda) \cap T^0$ of the tangent planes of Λ and of the fiber has constant dimension. Then the bundle L is trivialized in a way compatible with the definitions used in section 2.4 if we assign to the map f: $M_\lambda \to \mathbf{C}$ in Theorem 3.3.3 the complex number*

$$f(\mu) \exp \pi i \sigma(\lambda_1, \lambda_2; \mu)/4$$

which is independent of $\mu \in M_\lambda$.

The preceding discussion shows that an assertion concerning the line bundle may be perfectly obvious if L is defined in terms of phase functions while it requires some effort to prove in terms of Theorem 3.3.3. In Chapter IV we shall therefore use whichever definition of L that seems more convenient in the case at hand and omit translations to the other definition. When using the definition in terms of phase functions we shall omit the factor $e^{-\pi i N_j/4}$ from (3.2.14) and so use the transition functions (3.2.10) instead of (3.2.12). The only reason for not doing so from the beginning is that it gives the incorrect impression that the structure group of L is $\mathbf{Z}_8$ instead of $\mathbf{Z}_4$.

IV. A calculus for some classes of Fourier integral operators

4.0. Introduction

In this chapter we consider operators having a distribution kernel in one of the classes of distributions considered in Chapter III. The distribution kernel must of course be defined in a product manifold, so in section 4.1 we examine some additional structure for phase functions and Lagrangean manifolds when we are working in a product. In section 4.2 we then deduce the main results concerning adjoints and products from the theory developed in Chapter III. The resulting precise L^2 estimates are discussed in section 4.3. In view of the calculus $H_{(s)}$ estimates are immediate consequences so we shall not even state them explicitly.

4.1. Operators associated with a canonical relation

Let P and Y be two manifolds, of dimensions n_X and n_Y in general different. If A is a distribution density of order $\frac{1}{2}$ in $X \times Y$, in our notation $A \in \mathscr{D}'(X \times Y, \Omega_{\frac{1}{2}})$, then A defines a continuous bilinear form on $C_0^\infty(X, \Omega_{\frac{1}{2}}) \times C_0^\infty(Y, \Omega_{\frac{1}{2}})$ and therefore a continuous map $C_0^\infty(Y, \Omega_{\frac{1}{2}}) \to \mathscr{D}'(X, \Omega_{\frac{1}{2}})$ which we also denote by A. (Conversely, every such map is defined by a distribution $A \in \mathscr{D}'(X \times Y, \Omega_{\frac{1}{2}})$ in view of Schwartz' kernel theorem.)

If Λ is a closed conic Lagrangean submanifold of $T^*(X \times Y) \setminus 0$, we can in particular regard the space $I_\varrho^m(X \times Y, \Lambda)$ defined in section 3.2 (for $\varrho > \frac{1}{2}$) as a space of continuous linear maps from $C_0^\infty(Y)$ to $\mathscr{D}'(X)$. Since Λ can locally be defined with phase functions $\phi(x, y, \theta)$ where $\theta \in \mathbf{R}^N$, $N = n_X + n_Y$ (Theorem 3.1.3) the following is an immediate consequence of Theorem 1.4.1.

THEOREM 4.1.1. *Every element of* $I_\varrho^m(X \times Y, \Lambda)$ *is a continuous map from* $C_0^k(Y)$ *to* $\mathscr{D}'^k(X)$ *if*

$$m - k\varrho < -3(n_X + n_Y)/4. \tag{4.1.1}$$

If Λ *does not intersect* $T^*(X) \times 0_Y$ *(resp.* $0_X \times T^*(Y)$*) where* 0_Y *(resp.* 0_X*) is the zero section in* $T^*(Y)$ *(resp.* $T^*(X)$*) then every element of* $I_\varrho^m(X \times Y, \Lambda)$ *is a continuous map from* $C_0^k(Y)$ *to* $C^j(X)$ *(from* $\mathscr{E}'^j(Y)$ *to* $\mathscr{D}'^k(X)$*) if*

$$m + j - k\varrho < -3(n_X + n_Y)/4. \tag{4.1.2}$$

The kernels of all operators in $I_\varrho^m(X \times Y, \Lambda)$ *are in* C^∞ *outside the projection of* Λ *in* $X \times Y$.

All hypotheses on Λ in Theorem 4.1.1 are thus fulfilled if Λ is a conic Lagrangean submanifold of $(T^*(X) \setminus 0) \times (T^*(Y) \setminus 0)$ which is closed in $T^*(X \times Y) \setminus 0$.

If σ_X and σ_Y denote the symplectic forms in $T^*(X)$ and $T^*(Y)$ or their liftings to $T^*(X) \times T^*(Y)$, then the symplectic form in $T^*(X \times Y)$ is equal to $\sigma_X + \sigma_Y$. Thus the restriction of $\sigma_X + \sigma_Y$ to a Lagrangean submanifold equals 0. If Λ' denotes the image of Λ under the map which is the identity on $T^*(X)$ and multiplication by -1 in the fibers of $T^*(Y)$, it follows that the restriction of $\sigma_X - \sigma_Y$ to Λ' is equal to 0. Clearly, $(\Lambda')' = \Lambda$, so we have a one to one correspondence between Lagrangean manifolds satisfying the conditions in Theorem 4.1.1 and the manifolds in the following definition:

Definition 4.1.2. A closed conic submanifold C of $T^*(X \times Y) \setminus 0$ will be called a homogeneous canonical relation from $T^*(Y)$ to $T^*(X)$ if C is contained in $(T^*(X) \setminus 0) \times (T^*(Y) \setminus 0)$ and is Lagrangean with respect to $\sigma_X - \sigma_Y$, that is, C' is Lagrangean with respect to $\sigma_{X \times Y} = \sigma_X + \sigma_Y$.

The reason for the terminology is of course that if C is the graph of a function χ from $T^*(Y)$ to $T^*(X)$, we have $\chi^* \sigma_X = \sigma_Y$ which means that χ is a canonical transformation.

(X and Y have the same dimension since $\dim C = \dim X + \dim Y$.) It is usually more natural to work with the canonical relation C than with the corresponding Lagrangean manifold C'. For example, pseudo-differential operators in X are related to $\Lambda = N(\Delta)$ where Δ is the diagonal in $X \times X$, and $C = \Lambda'$ is then the graph of the identity $T^*(X) \setminus 0 \to T^*(X) \setminus 0$. This interpretation will be much more useful in studying multiplicative properties.

If C is a canonical relation from $T^*(Y)$ to $T^*(X)$ and $c_0 \in C$, then one can define C in a neighborhood of c_0 by means of a non-degenerate operator phase function ϕ in a conic neighborhood of $(x_0, y_0, \theta_0) \in X \times Y \times (\mathbf{R}^N \setminus 0)$ where (x_0, y_0) is the projection of c_0 on $X \times Y$. Thus

$$C_\phi = \{(x, y, \theta);\ \phi'_\theta(x, y, \theta) = 0\}$$

is a smooth conic manifold near (x_0, y_0, θ_0), and

$$C_\phi \ni (x, y, \theta) \to (x, \phi'_x, y, -\phi'_y)$$

is a homogeneous diffeomorphism on a conic neighborhood of c_0 in C. We shall now examine how various properties of C are reflected by properties of ϕ. In doing so we first assume that local coordinates have been chosen at x_0 and at y_0 and denote the corresponding local coordinates in $T^*(X)$ and $T^*(Y)$ by (x, ξ) and (y, η).

PROPOSITION 4.1.3. *The differential of the map $C \to T^*(X)$ is bijective if and only if $n_X = n_Y$ and*

$$D(\phi) = \det \begin{pmatrix} \phi''_{\theta\theta} & \phi''_{\theta x} \\ \phi''_{y\theta} & \phi''_{yx} \end{pmatrix} \neq 0 \quad \text{at } (x_0, y_0, \theta_0). \tag{4.1.3}$$

The map $C_\phi \to C \to T^*(X)$ then gives local coordinates (x, ξ) at (x_0, y_0, θ_0) on C_ϕ, and the density $d_{C_\phi} = \delta(\partial\phi/\partial\theta_1, \ldots, \partial\phi/\partial\theta_N)$ used in section 3.2 is equal to $|D(\phi)|^{-1} dx_1 \ldots dx_n d\xi_1 \ldots d\xi_n$.

Proof. The dimension of C is $n_X + n_Y$ so we must have $2n_X = n_X + n_Y$, that is, $n_X = n_Y$. Since the map $C_\phi \to C$ is a diffeomorphism, the bijectivity of the differential of the map $C \to T^*(X)$ means precisely that the map $C_\phi \ni (x, y, \theta) \to (x, \phi'_x)$ shall give local coordinates on C_ϕ, that is, that $(x, y, \theta) \to (x, \phi'_x, \phi'_\theta)$ shall give local coordinates at (x_0, y_0, θ_0). But

$$D(x, \phi'_x, \phi'_\theta)/D(x, y, \theta) = \det \begin{pmatrix} \phi''_{xy} & \phi''_{x\theta} \\ \phi''_{\theta y} & \phi''_{\theta\theta} \end{pmatrix} = D(\phi)$$

so this gives precisely the condition (4.1.3). The last statement follows at the same time, for

$$d_C = |D(x, \phi'_x, \phi'_\theta)/D(x, y, \theta)|^{-1} dx_1 \ldots dx_n d\xi_1 \ldots d\xi_n.$$

The condition (4.1.3) must clearly be independent of the choice of local coordinates in X and in Y and cannot change if one makes a fiber preserving substitution of the variables

(x, y, θ). However, it is not quite obvious that even the matrix in (4.1.3) is invariant so we digress to give a discussion of this point.

Let us consider a fiber space Σ over a manifold Ω. Thus Σ is a manifold and we have a projection $p: \Sigma \to \Omega$ with surjective differential. If $\sigma_0 \in \Sigma$ and $\omega_0 = p\sigma_0$, then the differential dp maps the tangent space $T_\Sigma(\sigma_0)$ onto $T_\Omega(\omega_0)$ with a kernel $T_0^\Sigma(\sigma_0)$ which is the tangent space of the fiber. Now let ϕ be a C^2 function on Σ and assume that ϕ *is stationary at* σ_0 *on the fiber* through σ_0, that is, $d\phi = 0$ on $T_\Sigma^0(\sigma_0)$. Then the linear form $d\phi$ on $T_\Sigma(\sigma_0)$ can be regarded as a linear form on $T_\Omega(\omega_0)$. We are interested in the second order derivatives of ϕ at σ_0. Let l_1 and l_2 be two vector fields on Ω, also regarded as first order differential operators, and choose vector fields L_1 and L_2 in Σ so that $(dp)L_j = l_j$, $j = 1, 2$. This can be done with $L_j(\sigma_0)$ equal to any vector t_j with $(dp)t_j = l_j(\sigma_0)$, for if (x, θ) are local coordinates in Σ such that $p(x, \theta) = x$, then the condition means precisely that

$$L_j = l_j + \sum a_{jk} \partial/\partial\theta_k, \quad j = 1, 2.$$

Note that for the commutator we have

$$[L_1, L_2] = [l_1, l_2] + \sum c_k \partial/\partial\theta_k$$

for some coefficients c_k. Thus, ϕ being stationary along the fiber,

$$[L_1, L_2]\phi(\sigma_0) = [l_1, l_2]\phi(\sigma_0) = 0 \quad \text{if } [l_1, l_2] = 0. \tag{4.1.4}$$

We shall use this observation in two ways. First, if there is a fixed coordinate system in Ω, that is, $\Omega \subset \mathbf{R}^n$, we conclude that a symmetric bilinear form on $T_\Sigma(\sigma_0)$ is invariantly defined by setting

$$B(t_1, t_2) = L_1 L_2 \phi(\sigma_0) \quad \text{if } L_j(\sigma_0) = t_j \in T_\Sigma(\sigma_0), \tag{4.1.5}$$

with L_j obtained as above from operators with constant coefficients in Ω. In fact, such operators L_j exist for any choice of t_j, and if L_1', L_2' is another choice, then $L_j - L_j' = 0$ at σ_0 so that

$$L_1 L_2 \phi(\sigma_0) = L_1' L_2 \phi(\sigma_0) = L_2 L_1' \phi(\sigma_0) = L_2' L_1' \phi(\sigma_0),$$

which at the same time proves the uniqueness of the definition of B and its symmetry. In local coordinates the matrix of B is of course the block matrix

$$\begin{pmatrix} \phi''_{\theta\theta} & \phi''_{\theta x} \\ \phi''_{x\theta} & \phi''_{xx} \end{pmatrix}.$$

Next we use the construction in the case at hand where $\Omega = X \times Y$ is the product of two manifolds with no preferred coordinate systems. Then the tangent space $T_\Sigma(\sigma_0)$

has subspaces T_Σ^X resp. T_Σ^Y consisting of vectors whose projections are tangent vectors of X resp. Y. Clearly

$$T_\Sigma = T_\Sigma^X + T_\Sigma^Y, \quad T_\Sigma^0 = T_\Sigma^X \cap T_\Sigma^Y.$$

If $t_1 \in T_\Sigma^X$ and $t_2 \in T_\Sigma^Y$ we can repeat the definition (4.1.5) where we now demand that L_1 and L_2 should correspond to vector fields l_1 and l_2 on X resp. Y. Such vector fields commute, so the preceding arguments apply. Thus B is an invariantly defined bilinear form on $T_\Sigma^X \times T_\Sigma^Y$ with symmetric restriction to $T_\Sigma^0 \times T_\Sigma^0$; if x and y are local coordinates in X and Y and (x, y, θ) are local coordinates in Σ such that $p(x, y, \theta) = (x, y)$, then the matrix of B is

$$\begin{pmatrix} \phi''_{\theta\theta} & \phi''_{\theta y} \\ \phi''_{x\theta} & \phi''_{xy} \end{pmatrix}. \tag{4.1.6}$$

Since any function of (x, y) which vanishes at $p(\sigma_0)$ is stationary along the fiber through σ_0 (if it is lifted to a function on Σ by means of the projection), it is clear that there is no sensible way of defining B on a larger space.

In what follows we denote the bilinear form just discussed by B_ϕ. When ϕ is a phase function defining the canonical relation C, there are several ranks associated with ϕ which give geometric information concerning C. We assume of course that ϕ is nondegenerate, which means that there is no element of $T^0 \setminus 0$ which is orthogonal to $T = T^X + T^Y$ with respect to B_ϕ.

PROPOSITION 4.1.4. *The rank of the differential of the projection* a) $C \to T^*(X)$, b) $C \to T^*(Y)$, c) $C \to X$, d) $C \to Y$, e) $C \to X \times Y$, *is equal to*

a) $\dim T^*(X) - \dim \{t_1 \in T^X;\ B_\phi(t_1, T^Y) = 0\} = 2 \dim X - \dim T^X + \operatorname{rank} B_\phi$,
b) $\dim T^*(Y) - \dim \{t_2 \in T^Y;\ B_\phi(T^X, t_2) = 0\} = 2 \dim Y - \dim T^Y + \operatorname{rank} B_\phi$,
c) $\dim X - \dim \{t_0 \in T^0;\ B_\phi(t_0, T^Y) = 0\}$,
d) $\dim Y - \dim \{t_0 \in T^0;\ B_\phi(T^X, t_0) = 0\}$,
e) $\dim (X \times Y) - \dim \{t_0 \in T^0;\ B_\phi(t_0, T^0) = 0\}$.

Proof. We use local coordinates throughout the computation. a) We have to determine the dimension of the vectors $(t, \tau) \in \mathbf{R}^n \times \mathbf{R}^n$, $n = n_X$, such that

$$\langle t, dx\rangle + \langle \tau, d\phi'_x\rangle = 0 \quad \text{if } d\phi'_\theta = 0.$$

This means that there shall exist a vector $a \in \mathbf{R}^N$ such that

$$\langle t, dx\rangle + \langle \tau, d\phi'_x\rangle + \langle a, d\phi'_\theta\rangle = 0.$$

Here a is uniquely determined by (t, τ) since ϕ is non-degenerate, so we may instead determine the dimension of $\{(t, \tau, a)\}$ for which this identity is valid. But t is determined by (τ, a) so we may instead determine the dimension of $\{(\tau, a)\}$ for which such a t exists, that is, all (τ, a) with

$$\langle \tau, \phi''_{xy}\, dy + \phi''_{x\theta}\, d\theta\rangle + \langle a, \phi''_{\theta\theta}\, d\theta + \phi''_{\theta y}\, dy\rangle = 0.$$

This means that $B_\phi((\tau, a), T^Y) = 0$, which proves a). The statement b) is symmetric. To prove c) we have to determine the dimension of all $t \in \mathbf{R}^n$ such that

$$\langle t, dx\rangle = 0 \text{ if } d\phi'_\theta = 0.$$

As in a) we find that this is equal to the dimension of all $a \in \mathbf{R}^N$ with $\langle a, \phi''_{\theta\theta}\, d\theta + \phi''_{\theta y}\, dy\rangle = 0$, which proves c). Since d) is symmetric and e) is included in Theorem 3.1.4, the proposition is proved.

We now return to the most regular case considered in Proposition 4.1.3.

Definition 4.1.5. A homogeneous canonical relation C from $T^*(Y)$ to $T^*(X)$ will be called a local canonical graph if the projection $C \to T^*(Y)$ and consequently the projection $C \to T^*(X)$ is a local diffeomorphism so that C is locally the graph of a canonical transformation. (This implies that $n_X = n_Y$.)

On a local canonical graph we have a density μ intrinsically defined by lifting the standard density in either $T^*(X)$ or $T^*(Y)$ by means of the map $C \to T^*(X)$ or $C \to T^*(Y)$. The results will agree since C is canonical and the canonical densities in $T^*(X)$ and $T^*(Y)$ are defined by $\sigma_X^n/n!$ and $\sigma_Y^n/n!$ where $n = n_X = n_Y$. It is clear that $\sqrt{\mu} \in S_1^{n/2}(C, \Omega_{\frac{1}{2}})$. If L is the line bundle on C obtained by transporting the line bundle associated with the Lagrange manifold $\Lambda = C'$ in section 3.2, it is clear that the map

$$S_\varrho^m(C, L) \to S_\varrho^{m+n/2}(C, \Omega_{\frac{1}{2}} \otimes L)$$

defined by multiplication with $\sqrt{\mu}$ is a bijection. For local canonical graphs it follows that we have an isomorphism

$$S_\varrho^m(C, L)/S_\varrho^{m+1-2\varrho}(C, L) \to I_\varrho^m(X \times Y, C')/I_\varrho^{m+1-2\varrho}(X \times Y, C'). \tag{4.1.7}$$

Note that the dimension of $X \times Y$ is $2n$ so the order $m + (2n)/4$ in (3.2.16) is precisely reduced to m. In particular, if $X = Y$ and C is the identity, we have recovered the approximate calculus of pseudo-differential operators outlined at the end of section 2.1 (with $\varrho + \delta = 1$ of course).

Using Proposition 4.1.3 we can make (4.1.7) quite explicit. For let ϕ be a non-degenerate phase function in a conic neighborhood U of (x_0, y_0, θ_0) in $X \times Y \times (\mathbf{R}^N \setminus 0)$ which defines a neighborhood of c_0 in C. With $a \in S_\varrho^{m+(n-N)/2}$ vanishing outside a small conic neighborhood of (x_0, y_0, θ_0) and local coordinates chosen at x_0 and y_0 we form the operator with kernel A defined by (see the concluding remarks in section 3.3)

$$\langle A, u\rangle = (2\pi)^{-(n+N)/2} \iint e^{i\phi(x,y,\theta)} a(x,y,\theta)\, u(x,y)\, dx\, dy\, d\theta, \quad u \in C_0^\infty(X \times Y).$$

Then a principal symbol for A is represented by $a\sqrt{d_{C_j}}$ transported to C by the map $C_\phi \to \Lambda \to C$. By Proposition 4.1.3 the quotient by the square root of the standard density in C corresponds to the function

$$b(x, y, \theta) = a(x, y, \theta)\,|D(\phi)|^{-\frac{1}{2}}$$

which is in S_ϱ^m if $a=0$ near $X \times Y \times 0$ because $D(\phi)$ is homogeneous of degree $n-N$. The situation is therefore as follows. Taking $b \in S_\varrho^m(X \times Y \times (\mathbf{R}^N \setminus 0))$ with cone supp $b \subset U$ and $b=0$ near $X \times Y \times 0$, we form the density of order $\frac{1}{2}$ given by

$$\langle A, u\rangle = (2\pi)^{-(n+N)/2} \iint e^{i\phi(x,y,\theta)} b(x,y,\theta)\, |D(\phi)|^{\frac{1}{2}}\, u(x,y)\, dx\, dy, \quad u \in C_0^\infty$$

or equivalently the operator from densities of order $\frac{1}{2}$ to densities of order $\frac{1}{2}$ defined locally by

$$Au(x) = (2\pi)^{-(n+N)/2} \iint e^{i\phi(x,y,\theta)} b(x,y,\theta)\, |D(\phi)|^{\frac{1}{2}}\, u(y)\, dy\, d\theta, \quad u \in C_0^\infty(Y).$$

To this operator is assigned a symbol which is the composition of b with the inverse of the map

$$C_\phi \ni (x, y, \theta) \to (x, \phi'_x, y, -\phi'_y) \in C$$

which has support in the open set $U_\phi^C \subset C$ parametrized by means of ϕ and should be regarded as an element of $S_\varrho^m(C, L)$. For the case of local canonical graphs this would give a somewhat simpler discussion than that of section 3.2, which does not involve densities on C. However, we do not wish to duplicate the arguments given there and have merely wanted to indicate a slight shortcut available in a special case including pseudo-differential operators and many other important classes of operators.

Our result will give much more complete control of operators in $I_\varrho^m(X \times Y, C)$ when C is a local canonical graph than in the general case. We shall therefore discuss now to what extent a reduction to this case is possible by considering some variables as parameters. To motivate we first consider an example related to the Cauchy problem for the wave equation.

Example 4.1.6. Let $\phi(x, y, \theta) = \langle x'-y', \theta\rangle + (x_n - y_n)|\theta|$ where $x = (x', x_n)$ and $y = (y', y_n)$ are in $\mathbf{R}^n$ and $0 \neq \theta \in \mathbf{R}^{n-1}$. Then the equation $\phi'_\theta = 0$ becomes

$$x'-y' + (x_n - y_n)\theta/|\theta| = 0.$$

If $(x, \phi'_x, y, -\phi'_y) = (x, \xi, y, \eta)$ we have $\xi = \eta = (\theta, |\theta|)$. Thus C is defined by the equations $p(x-y) = |x'-y'|^2 - (x_n - y_n)^2 = 0$, $\xi = \eta$, $p(\xi) = 0$, and that ξ is proportional to $\partial p(x-y)/\partial x$ if $x \neq y$. Thus we do not obtain a local canonical graph. However, if x_n and y_n are regarded as parameters, then $\{x', \phi'_{x'}, y', -\phi'_{y'})\} = \{(x', \theta, y', \theta)\}$ is the graph of the canonical transformation $(x', \theta) \to (x' + (x_n - y_n)\theta/|\theta|, \theta)$. Note that also in the classical energy integral method for the wave equation one usually regards the time variable as a parameter.

Let us first ignore the fact that we are working in a product manifold $X \times Y$ and consider as in Chapter III a conic Lagrangean submanifold Λ of $T^*(X)$ where X is a manifold. Let X_1 be a submanifold of X. If now ϕ is a non-degenerate phase function in an open conic neighborhood Γ of (x_0, θ_0) in $X \times (\mathbf{R}^N \setminus 0)$ defining an open subset Λ_ϕ of Λ, we let ϕ_1 denote the restriction to $\Gamma \cap (X_1 \times (\mathbf{R}^N \setminus 0))$ and ask when ϕ_1 is a non-degenerate phase function. First, that ϕ_1 is a phase function means that $\phi'_x dx$ cannot vanish on $T(X_1)$ if $\phi'_\theta = 0$, that is, that $\Lambda_\phi \cap N(X_1) = \emptyset$. Secondly, that ϕ_1 is non-degenerate means that the intersection of the tangent plane of $C_\phi = \{(x, \theta);\ \phi'_\theta(x, \theta) = 0\}$ with the tangent plane of $X_1 \times \mathbf{R}^N$ has dimension $\dim X_1$, so that the two planes are transversal. This means that the intersections of Λ and $T^*(X)|_{X_1}$ (which is of codimension equal to $\operatorname{codim}_X X_1$) are also transversal. Summing up, we have

PROPOSITION 4.1.7. *Let Λ be a conic Lagrangean submanifold of $T^*(X)$ and let X_1 be a submanifold of X. If ϕ is a non-degenerate phase function in a neighborhood of (x_0, θ_0) in $X \times (\mathbf{R}^N \setminus 0)$ where $x_0 \in X_1$, and if ϕ defines a neighborhood of $\lambda_0 \in \Lambda$, then the restriction ϕ_1 of ϕ to $X_1 \times (\mathbf{R}^N \setminus 0)$ is a non-degenerate phase function at (x_0, θ_0) if and only if*

(i) $\lambda_0 \notin N(X_1)$

(ii) Λ *intersects* $T^*(X)|_{X_1}$ *transversally at* λ_0.

The element of Lagrangean submanifold Λ_1 of $T^*(X_1)$ defined by ϕ_1 is then locally the projection of $\Lambda \cap T^*(X)|_{X_1}$ on $T^*(X_1)$.

The projection mentioned in the theorem refers of course to the exact sequence

$$0 \to N(X_1) \to T^*(X)|_{X_1} \to T^*(X_1) \to 0,$$

where the last map is the restriction of forms on $T(X)$ to $T(X_1)$. Condition (ii) implies condition (i) since the radial vector at λ_0 is symplectically orthogonal to the tangent plane of Λ and so must not be orthogonal to that of $T^*(X)|_{X_1}$. Let us also note that (ii) is equivalent to

(iii) The composed map $T_{\lambda_0}(\Lambda) \to T_{x_0}(X) \to T_{x_0}(X)/T_{x_0}(X_1)$ is surjective.

If we have two different phase functions φ and $\tilde{\varphi}$ defining a neighborhood of λ_0, we find using the proof of Proposition 3.3.1 that

$$\operatorname{sgn} \phi''_{\theta\theta} - \operatorname{sgn} \tilde{\phi}''_{\theta\theta} = \operatorname{sgn} \phi''_{1\theta\theta} - \operatorname{sgn} \tilde{\phi}''_{1\tilde{\theta}\tilde{\theta}}.$$

This shows that the pullback of the line bundle L_1 on Λ_1 under the projection mentioned in Proposition 4.1.7 can be identified in a natural way with the restriction of the line bundle L on Λ to $\Lambda \cap T^*(X)|_{X_1}$.

In particular, we can apply Proposition 4.1.7 when X and X_1 are replaced by products $X \times Y$ and $X_1 \times Y_1$ with $X_1 \subset X$ and $Y_1 \subset Y$. If C is a homogeneous canonical relation from $T^*(Y)$ to $T^*(X)$ defined in a neighborhood of $c_0 \in C$ with $\pi c_0 \in X_1 \times Y_1$ by an operator phase function ϕ, and if

(i) $c_0 \notin N(X_1) \times T^*(Y)|_{Y_1} \cup T^*(X)|_{X_1} \times N(Y_1)$,

(ii) C intersects $T^*(X \times Y)_{X_1 \times Y_1}$ transversally at c_0,

then the restriction of ϕ to $X_1 \times Y_1 \times (\mathbf{R}^N \setminus 0)$ is another operator phase function defining locally the projection of $C \cap T^*(X \times Y)|_{X_1 \times Y_1}$ into $T^*(X_1 \times Y_1)$ along $N(X_1) \times N(Y_1)$.

We shall now examine when for a given $c_0 \in C$ it is possible to choose X_1 and Y_1 with $\pi c_0 \in X_1 \times Y_1$ so that (i), (ii) are fulfilled and the local canonical relation from $T^*(Y_1)$ to $T^*(X_1)$ obtained from C at c_0 is a canonical graph. In terms of the bilinear form B_ϕ on $T^X \times T^Y$ this means that we must choose X_1, Y_1 so that (i) is fulfilled—which is usually the case—and so that B_ϕ is non-singular on $T^{X_1} \times T^{Y_1}$. (This implies that the restriction of ϕ is non-degenerate so that (ii) is fulfilled.) The situation is analyzed in the following simple

LEMMA 4.1.8. *Let V be a vector space, V_1 and V_2 two subspaces with $V_1 + V_2 = V$, and B a bilinear form on $V_1 \times V_2$. In order that there shall exist subspaces W_1, W_2 of V_1, V_2 with $W_1 \cap W_2 = V_1 \cap V_2$ such that B is non-singular on $W_1 \times W_2$ it is necessary and sufficient that no element of $V_1 \cap V_2$ is orthogonal to V_1 or to V_2. One can even choose W_1 and W_2 with dimension equal to* rank B *then.*

Proof. The necessity is trivial. To prove the sufficiency we denote the rank of B by r. Thus B defines two maps $V_1 \to V_2^*$ and $V_2 \to V_1^*$ of rank r, which are injective on $V_0 = V_1 \cap V_2$. Choose W_j with $V_0 \subset W_j \subset V_1$ of dimension r so that the maps are injective on W_j. Then B is non-singular on $W_1 \times W_2$. For if $w_2 \in W_2$ is orthogonal to W_1 with respect to B, then w_2 is orthogonal to V_1, and since the map $W_2 \to V_1^*$ is injective this proves that $w_2 = 0$.

All quantities which occur in Lemma 4.1.8 can be expressed in terms of the canonical relation if one applies Proposition 4.1.4. This gives

THEOREM 4.1.9. *Let C be a homogeneous canonical relation from $T^*(Y)$ to $T^*(X)$ and let $c_0 \in C$. Suppose that the projections*

$$C \to X, \quad C \to Y, \tag{4.1.8}$$

have surjective differentials at c_0, and let k_X (resp. k_Y) be the rank of the differential at c_0 of the projection $C \to T^*X$ minus dim X *(resp.* rank $d(C \to T^*Y)$ *minus* dim Y*). Then $k_X = k_Y = k$ and one can find submanifolds $X_1 \subset X$ and $Y_1 \subset Y$ of dimension k so that $\pi c_0 \in X_1 \times Y_1$ and a neighborhood of c_0 in C which defines a canonical graph from $T^*(Y_1)$ to $T^*(X_1)$.*

Proof. The equality $k_X = k_Y$ follows from a) and b) in Proposition 4.1.4. The hypotheses concerning (4.1.8) mean in view of c) and d) in Proposition 4.1.4 that Lemma 4.1.8 can be applied to choose the directions of X_1 and Y_1. A dense subset of directions satisfies the conditions in Lemma 4.1.8 and also condition (i) preceding it, which proves the theorem.

Somewhat loosely we can express Theorem 4.1.9 as follows. Assume that the differential of the projection $C \to T^*(X)$ always has rank $\geqslant r + \dim X$ where $r \geqslant 0$, and that the maps (4.1.8) have surjective differentials. Then one can consider C locally as a canonical graph between the cotangent spaces of manifolds of dimension r, depending on $\dim X + \dim Y - 2r$ parameters. We shall use this fact later to give L^2 estimates for the corresponding operators.

A rather complete local description of C analogous to Example 4.1.6 can be given in a neighborhood of a point $c_0 \in C$ where the maps (4.1.8) have surjective differentials and the differential of the projection $C \to T^*(X)$ has constant rank $r + \dim X$. The differential of the map $C \to T^*(Y)$ has rank $r + \dim Y$ then. Locally, the range of the projection of C in $T^*(X)$ (resp. $T^*(Y)$) is defined by $\dim X - r$ (resp. $\dim Y - r$) equations $F_j(x, \xi) = 0$, $1 \leqslant j \leqslant \dim X - r$ (resp. $G_j(y, \eta) = 0$, $1 \leqslant j \leqslant \dim Y - r$) which are homogeneous with respect to ξ (resp. η) and have linearly independent differentials. The Hamiltonian vector fields H_{F_j} (resp. H_{G_j}) in $T^*(X)$ (resp. $T^*(Y)$) corresponding to these differentials via the symplectic form are then in the tangent plane of C. Hence $H_{F_i} F_j = 0$ if all $F_k = 0$ and similarly for G, or if we introduce Poisson brackets

$$\{F_i, F_j\} = 0 \text{ when all } F_k = 0; \quad \{G_i, G_j\} = 0 \text{ when all } G_k = 0. \tag{4.1.9}$$

Now the set of points in C with fixed component in $T^*(X)$ is a manifold of dimension $\dim C - (r + \dim X) = \dim Y - r$ whose projection to $T^*(Y)$ is of the same dimension and so must be the integral of the $(\dim Y - r)$ dimensional planes spanned by the Hamiltonian vector fields H_{G_j}. The roles of X and Y can of course be reversed here. Summing up, if we choose submanifolds X_1 and Y_1 of dimension r as in Theorem 4.1.9, then C is locally obtained as follows: With $(x_1, \xi_1) \in T^*(X_1)$, $(y_1, \eta_1) \in T^*(Y_1)$ related by the canonical transformation in Theorem 4.1.9 one first solves the equations $F_j(x, \xi)=0$, $G_k(y, \eta)=0$ with $x=x_1$, $y=y_1$ so that ξ_1 (resp. η_1) is the projection of ξ (resp. η) along $N(X_1)$ (resp. $N(Y_1)$). Then we include in C the product of the "bicharacteristics" through (x, ξ) and (y, η) obtained by integrating the Hamilton–Jacobi equations with the Hamiltonians F_j and G_k using these initial data. (See e.g. Carathéodory [6].) The total dimension of C then becomes $2 \dim X_1 + (\dim X - r) + (\dim Y - r) = \dim X + \dim Y$ as it should. We leave for the reader to check that the conditions on ranks and so on required in the preceding discussion are actually verified so that the argument is valid locally. Conversely, one can also define canonical relations by starting from a canonical transformation $T^*(X_1) \to T^*(Y_1)$ and Hamilton functions satisfying (4.1.9) provided that the "bicharacteristics" are transversal to X_1 and to Y_1. An example is given in Example 4.1.6 and we shall come across a more general example of the same type in part II.

4.2. Adjoints and products

If u and v are two densities of order $\frac{1}{2}$ in a manifold and $\operatorname{supp} u \cap \operatorname{supp} v$ is compact, we write

$$(u, v) = \langle u, \bar{v} \rangle = \int u \bar{v}.$$

The adjoint of an operator $A \in I_\varrho^m(X \times Y, C')$ where C is a homogeneous canonical relation from $T^*(Y)$ to $T^*(X)$ is defined by

$$(Au, v) = (u, A^* v), \quad v \in C_0^\infty(X, \Omega_{\frac{1}{2}}),\ u \in C_0^\infty(Y, \Omega_{\frac{1}{2}}).$$

If A is represented in the form

$$\langle A, u \rangle = (2\pi)^{-(n_X + n_Y + 2N)/4} \iiint e^{i\phi(x,y,\theta)} a(x,y,\theta)\, u(x,y)\, dx\, dy\, d\theta,\ u \in C_0^\infty,$$

in a local coordinate patch, then

$$\langle A^*, u \rangle = (2\pi)^{-(n_X + n_Y + 2N)/4} \iiint e^{-i\phi(x,y,\theta)} \overline{a(x,y,\theta)}\, u(x,y)\, dx\, dy\, d\theta, \quad u \in C_0^\infty.$$

Here $-\phi$ should be regarded as a phase function in $Y \times X \times (\mathbf{R}^N \setminus 0)$, so the corresponding canonical relation is the range of the map

$$C_\phi \ni (x, y, \theta) \to (y, -\phi'_y, x, \phi'_x)$$

which differs from C by the map $T^*(X) \times T^*(Y) \to T^*(Y) \times T^*(X)$ interchanging the two factors. The principal symbol is just changed by complex conjugation. Note that the complex conjugate of a section of the line bundle L is a section of L^{-1}. Thus we have

THEOREM 4.2.1. *If C is a homogeneous canonical relation from $T^*(Y)$ to $T^*(X)$ and $A \in I_\varrho^m(X \times Y, C')$, $\varrho > \frac{1}{2}$, then the adjoint $A^* \in I_\varrho^m(Y \times X, C_s')$ where C_s is the inverse image of C under the map s: $T^*(Y) \times T^*(X) \to T^*(X) \times T^*(Y)$ interchanging the two factors. If $a \in S_\varrho^{m+(n_X+n_Y)/4}(C, \Omega_{\frac{1}{2}} \otimes L_C)$ is a principal symbol for A, then $s^*\bar{a} \in S_\varrho^{m+(n_X+n_Y)/4}(C_s, \Omega_{\frac{1}{2}} \otimes L_{C_s})$ since there is a natural isomorphism between L_{C_s} and $s^*L_C^{-1}$, and $s^*\bar{a}$ is a principal symbol for A^*.*

The isomorphism between L_{C_s} and $s^*L_C^{-1}$ is obtained by noting that L_C is the line bundle corresponding to C and the symplectic form $\sigma_X - \sigma_Y$, so L_C^{-1} corresponds to C and the form $-(\sigma_X - \sigma_Y)$, and $s^*L_C^{-1}$ corresponds to $s^*C = C_s$ and $s^*(\sigma_Y - \sigma_X)$. But that is the definition of L_{C_s}. These are obvious consequences of Theorem 3.3.3.

We shall now discuss products, so let C_1 be a canonical relation from $T^*(Y)$ to $T^*(X)$ and C_2 another from $T^*(Z)$ to $T^*(Y)$ where X, Y, Z are three manifolds. Let $A_1 \in I_\varrho^{m_1}(X \times Y, C_1')$ and $A_2 \in I_\varrho^{m_2}(Y \times Z, C_2')$ and assume that both are properly supported so that the composition $A_1 A_2$ is defined. We wish to show that $A_1 A_2 \in I_\varrho^{m_1+m_2}(X \times Z, C')$ where C is obtained by composition of the canonical relations C_1 and C_2. (Cf. Theorem 2.5.15.) The first step in doing so is to study the composition of canonical relations.

The direct product

$$C_1 \times C_2 \subset T^*(X) \times T^*(Y) \times T^*(Y) \times T^*(Z)$$

is a symplectic manifold with respect to the symplectic form $\sigma_X - \sigma_{Y_1} + \sigma_{Y_2} - \sigma_Z$ where the two copies of Y are denoted by subscripts. The composition of C_1 and C_2 is defined as the projection in $T^*(X) \times T^*(Z)$ of the intersection of $C_1 \times C_2$ with the diagonal Δ in $T^*(X) \times T^*(Y) \times T^*(Y) \times T^*(Z)$ consisting of elements for which the two components in $T^*(Y)$ are equal. If $C_1 \times C_2$ intersects Δ transversally, then $\dim((C_1 \times C_2) \cap \Delta) = \dim(C_1 \times C_2) - \operatorname{codim} \Delta = \dim X + \dim Z$. Transversality means that there are no non-zero normals of the tangent planes of $C_1 \times C_2$ and of Δ (with respect to the symplectic form) or equivalently that there is no non-zero tangent of $C_1 \times C_2$ at an intersection which is also a tangent of Δ with zero components in $T(T^*(X))$ and $T(T^*(Z))$. Thus the projection of $(C_1 \times C_2) \cap \Delta$ in $T^*(X) \times T^*(Z)$ will then (locally) be a manifold of dimension $\dim X + \dim Z$ on which $\sigma_X - \sigma_Z$ vanishes since $-\sigma_{Y_1} + \sigma_{Y_2}$ vanishes on Δ. The projection will be a manifold and so a canonical relation $C_1 \circ C_2$ from $T^*(Z)$ to $T^*(X)$ if the map

$$(C_1 \times C_2) \cap \Delta \to T^*(X) \times T^*(Z) \setminus 0$$

is in addition injective and proper. If either C_1 or C_2 is the graph of a canonical diffeomorphism the preceding conditions are trivially fulfilled.

We shall now determine the condition for transversality in terms of local defining phase functions when local coordinates are introduced. Let ϕ_1 be a phase function near $(x_0, y_0, \theta_0) \in X \times Y \times (\mathbf{R}^{N_1} \setminus 0)$ and ϕ_2 a phase function near $(y_0, z_0, \sigma_0) \in Y \times Z \times (\mathbf{R}^{N_2} \setminus 0)$. We assume that both are non-degenerate, that they define a part of C_1 and of C_2 respectively, that $\phi'_{1\theta}(x_0, y_0, \theta_0) = 0$, $\phi'_{2\sigma}(y_0, z_0, \sigma_0) = 0$, and that when $x = x_0, \ldots, \sigma = \sigma_0$

$$(x, \phi'_{1x}(x, y, \theta), y, -\phi'_{1y}(x, y, \theta), y, \phi'_{2y}(y, z, \sigma), z, -\phi'_{2z}(y, z, \sigma)) \in \Delta$$

that is, that $\phi'_{1y}(x_0, y_0, \theta_0) + \phi'_{2y}(y_0, z_0, \sigma_0) = 0$. As noted above, transversality means that there is no vector $(0, 0, t, \tau, t, \tau, 0, 0) \neq 0$ orthogonal to the tangent plane of $C_1 \times C_2$ with respect to the symplectic form $\sigma_X - \sigma_{Y_1} + \sigma_{Y_2} - \sigma_Z$. Orthogonality to the tangent plane means that, at (x_0, y_0, θ_0) and (y_0, z_0, σ_0), $d\phi'_{1\theta}(x, y_1, \theta) = 0$, $d\phi'_{2\sigma}(y_2, z, \sigma) = 0$ implies

$$\langle dy_1, \tau \rangle - \langle dy_2, \tau \rangle + \langle d\phi'_{1y_1}, t \rangle + \langle d\phi'_{2y_2}, t \rangle = 0.$$

This is equivalent to the existence of vectors a, b such that

$$d\langle \phi'_{1\theta}(x, y_1, \theta), a \rangle + d\langle \phi'_{2\sigma}(y_2, z, \sigma), b \rangle + d\langle y_1 - y_2, \tau \rangle + d\langle \phi'_{1y_1}(x, y_1, \theta) + \phi'_{2y_2}(y_2, z, \sigma), t \rangle = 0. \tag{4.2.1}$$

Transversality thus means that this shall imply that $t = \tau = 0$ and therefore $a = b = 0$ since ϕ_1 and ϕ_2 are non-degenerate phase functions. It suffices that (4.2.1) implies that $a = b = t = 0$ for then it follows immediately that $\tau = 0$ also. Now it is clear that given a, b, t one can find τ so that (4.2.1) is valid if and only if

$$d\langle \phi'_{1\theta}(x, y, \theta), a \rangle + d\langle \phi'_{2\sigma}(y, z, \sigma), b \rangle + d\langle \phi'_{1y}(x, y, \theta) + \phi'_{2y}(y, z, \sigma), t \rangle = 0. \tag{4.2.2}$$

Transversality therefore means precisely that (4.2.2) implies $a = b = t = 0$.

Now we can consider $\phi_1(x, y, \theta) + \phi_2(y, z, \sigma)$ as a function $\phi(x, z, \omega)$ of $(x, z, \omega) \in X \times Z \times (\mathbf{R}^{N_1 + N_2 + \dim Y} \setminus 0)$ where

$$\omega = ((|\theta|^2 + |\sigma|^2)^{\frac{1}{2}} y, \theta, \sigma) \in \mathbf{R}^{N_1 + N_2 + \dim Y} \setminus 0.$$

When θ and σ vary in conic neighborhoods of θ_0 and σ_0 and y varies over a neighborhood of y_0 we obtain a diffeomorphism on a conic neighborhood of $\omega_0 = ((|\theta_0|^2 + |\sigma_0|^2)^{\frac{1}{2}} y_0, \theta_0, \sigma_0)$. It is clear that ϕ is homogeneous of degree 1 with respect to ω, and the equations $\phi'_\omega = 0$ mean precisely that

$$\phi'_{1\theta} = \phi'_{2\sigma} = \phi'_{1y} + \phi'_{2y} = 0. \tag{4.2.3}$$

At such a point the linear combinations of the differentials of the partial derivatives $\partial\phi/\partial\omega_j$ are precisely the same as the linear combinations of the differentials $\partial\phi_1/\partial\theta_j$, $\partial\phi_2/\partial\sigma_j$ and $\partial(\phi_1+\phi_2)/\partial y_j$. Thus (4.2.2) means that ϕ is a non-degenerate phase function near (x_0, z_0, ω_0). The corresponding canonical relation is

$$\{(x, \phi'_{1x}, z, -\phi'_{2z});\ \phi'_{1\theta}=\phi'_{2\sigma}=\phi'_{1y}+\phi'_{2y}=0\},$$

which means that ϕ defines $C_1 \circ C_2$ locally.

Let now

$$A_1 v(x) = (2\pi)^{-(n_X+n_Y+2N_1)/4} \iint e^{i\phi_1(x,y,\theta)} a_1(x, y, \theta)\, v(y)\, dy\, d\theta, \tag{4.2.4}$$

where $a_1 \in S_\varrho^{m_1+(n_X+n_Y-2N_1)/4}(X \times Y \times \mathbf{R}^{N_1})$ vanishes in a neighborhood of the zero section and has cone supp a_1 inside a conic set Γ_1 where ϕ_1 is a non-degenerate phase function defining part of C_1. Similarly, assume that

$$A_2 u(y) = (2\pi)^{-(n_Y+n_Z+2N_2)/4} \iint e^{i\phi_2(y,z,\sigma)} a_2(y, z, \sigma)\, u(z)\, dz\, d\sigma, \tag{4.2.5}$$

where $a_2 \in S_\varrho^{m_2+(n_Y+n_Z-2N_2)/4}(Y \times Z \times \mathbf{R}^{N_2})$ vanishes near the zero section and cone supp a_1 belongs to a conic set Γ_2 where ϕ_2 is non-degenerate and defines part of C_2. Here we are of course working with local coordinates so that the integrals are well defined. If now a_1 and a_2 vanish for large $|\theta|$ and $|\sigma|$, we obtain

$$A_1 A_2 u(x) = (2\pi)^{-(n_X+n_Z+2N)/4} \iiiint e^{i(\phi_1(x,y,\theta)+\phi_2(y,z,\sigma))} a_1(x, y, \theta)\, a_2(y, z, \sigma)\, u(z)\, dz\, dy\, d\theta\, d\sigma.$$

Here $N = N_1 + N_2 + \dim Y$ is the number of "fiber coordinates" if we regard the exponent as a phase function $\phi(x, z, \omega)$, as we did above. We would like to extend the validity of this formula to general a_j but notice that this meets the difficulty that $a_1(x, y, \theta)a_2(y, z, \sigma)$ is not quite a symbol (because differentiation with respect to θ for example improves only by a factor $(1+|\theta|)^{-\varrho}$ and not by a factor $(1+|\theta|+|\sigma|)^{-\varrho}$). In addition the cone support contains points with $\theta=0$ or $\sigma=0$ which are on the boundary of the set where the exponent is a phase function. However all essential contributions are expected from a neighborhood of points where (4.2.3) is valid and there the preceding difficulties do not occur. Motivated by this observation we argue as follows.

Since $\phi'_{1\theta}=0$ implies $\phi'_{1y} \neq 0$ we can always restrict the support of a_1 without changing the singularities of A_1 so that $|\phi'_{1y}|/|\theta|$ is bounded from above and from below in cone supp a_1. Similarly we may assume that $|\phi'_{2y}|/|\sigma|$ has fixed positive upper and lower bounds in cone supp a_2. Hence there are positive constants C_1 and C_2 such that

$$C_1|\sigma|<|\theta|<C_2|\sigma| \quad \text{when } \phi'_{1y}(x,y,\theta)+\phi'_{2y}(y,z,\sigma)=0. \tag{4.2.6}$$

We choose a homogeneous function $\chi(\theta, \sigma)$ of degree 0 which is equal to 1 when $C_1|\sigma|/2<|\theta|<2C_2|\sigma|$ and has support in the cone where $C_1|\sigma|/3<|\theta|<3C_2|\sigma|$. Thus

$$\begin{aligned} &C_1|\sigma|/3<|\theta|<3C_2|\sigma \quad \text{in supp } \chi; \\ &C_1|\sigma|/2\geqslant|\theta| \text{ or } |\theta|\geqslant 2C_2|\sigma| \quad \text{in supp } (1-\chi). \end{aligned} \tag{4.2.7}$$

Now introduce

$$Bu(x)=(2\pi)^{-(n_X+n_Z+2N)/4}\iiiint e^{i(\phi_1(x,y,\theta)+\phi_2(y,z,\sigma))}b(x,z,y,\theta,\sigma)\,u(z)\,dz\,dy\,d\theta\,d\sigma, \tag{4.2.8}$$

where $b(x,z,y,\theta,\sigma)=\chi(\theta,\sigma)a_1(x,y,\theta)a_2(y,z,\sigma)$. By the first part of (4.2.7) we have $b\in S^m_\varrho$ for $m=m_1+m_2+(n_X+n_Z+2(n_Y-N_1-N_2))/4=m_1+m_2+(n_X+n_Z-2N)/4+n_Y$, provided of course that a_1 and a_2 are symbols of the degrees indicated above. If we introduce as above a variable ω in $\mathbf{R}^N$ instead of (y,θ,σ), then $D(y,\theta,\sigma)/D\omega$ is a homogeneous function of ω of degree $-n_Y$ so it follows that (4.2.8) is an operator of order m_1+m_2 belonging to the canonical relation $C_1\circ C_2$.

Writing $r(x,z,y,\theta,\sigma)=(1-\chi(\theta,\sigma))a_1(x,y,\theta)a_2(y,z,\sigma)$ we shall now prove that

$$R(x,z,\theta,\sigma)=\int e^{i(\phi_1(x,y,\theta)+\phi_2(y,z,\sigma))}r(x,z,y,\theta,\sigma)\,dy$$

is in $S^{-\infty}$. The integrand vanishes when $|\theta|+|\sigma|$ is sufficiently small, and by (4.2.6) and the second part of (4.2.7) we have

$$|\theta|+|\sigma|\leqslant C|\phi'_{1y}(x,y,\theta)+\phi'_{2y}(y,z,\sigma)|$$

in the support of r. But repeated partial integrations with respect to y (cf. the proof of Propositions 1.2.2 and 2.5.7 then show that R can be bounded by any power of $(|\theta|+|\sigma|)^{-1}$, and the same is true for any derivative of R.

Now we claim that $A_1A_2=B+R$ where

$$Ru(x)=(2\pi)^{-(n_X+n_Z+2N)/4}\iiint R(x,z,\theta,\sigma)\,u(z)\,dz\,d\theta\,d\sigma$$

is an operator with C^∞ kernel obtained by integration with respect to θ and σ. In fact, this is obvious if a_1 and a_2 have compact support. In general we just insert cutoff functions as in (1.2.3) and obtain in the limit when it converges to one that $A_1A_2=B+R$. This proves that A_1A_2 is a Fourier integral operator, so we have proved

THEOREM 4.2.2. *Let C_1 and C_2 be homogeneous canonical relations from $T^*(Y)$ to $T^*(X)$ and from $T^*(Z)$ to $T^*(Y)$ respectively, assume that $C_1 \times C_2$ intersects the diagonal in $T^*(X) \times T^*(Y) \times T^*(Y) \times T^*(Z)$ transversally and that the projection from the intersection to $T^*(X) \times T^*(Z)$ is proper, thus gives a homogeneous canonical relation $C_1 \circ C_2$ from $T^*(Z)$ to $T^*(X)$. If $A_1 \in I_\varrho^{m_1}(X \times Y, C_1')$, $A_2 \in I_\varrho^{m_2}(Y \times Z, C_2')$ are properly supported, it follows that (for $\varrho > \frac{1}{2}$)*

$$A_1 A_2 \in I_\varrho^{m_1+m_2}(X \times Z, (C_1 \circ C_2)').$$

In order to describe the symbol of the product we must first discuss some facts concerning the line bundles of which the symbols are sections. First we discuss some properties of densities.

If E is a real vector space of dimension n, we define $\Omega_\alpha(E)$ as the space of all maps a: $\Lambda^n(E) \setminus 0 \to \mathbf{C}$ such that $a(st) = |s|^\alpha a(t)$, $t \in \Lambda^n(E) \setminus 0$, $0 \neq s \in \mathbf{R}$. Of course $\Omega_\alpha(E)$ is isomorphic to $\mathbf{C}$ but the isomorphism is not unique for $\alpha \neq 0$ unless for example we have a preferred basis in E. Clearly the tensor product $\Omega_\alpha(E) \otimes \Omega_\beta(E)$ is isomorphic to $\Omega_{\alpha+\beta}(E)$ since the product of two maps homogeneous of orders α and β is homogeneous of order $\alpha+\beta$. If E_1 is a subspace of E, of dimension n_1, then the bilinear map

$$\Lambda^{n_1}(E_1) \times \Lambda^{n-n_1}(E/E_1) \to \Lambda^n(E)$$

gives an isomorphism

$$\Omega_\alpha(E_1) \otimes \Omega_\alpha(E/E_1) \cong \Omega_\alpha(E)$$

or if we tensor with $\Omega_{-\alpha}(E/E_1)$ and note that Ω_0 is isomorphic to $\mathbf{C}$,

$$\Omega_\alpha(E_1) \cong \Omega_\alpha(E) \otimes \Omega_{-\alpha}(E/E_1).$$

If E is a vector bundle, then $\Omega_\alpha(E)$ is a line bundle with the same base, and the preceding formulas remain valid. In particular, if M is a manifold we can take $E = TM$ and obtain the bundle $\Omega_\alpha = \Omega_\alpha(M)$ over M defined also in section 2.4.

Let now M_1 and M_2 be two submanifolds of a manifold M with transversal intersection. We want to relate densities in $M_1 \cap M_2$ to densities in M_1. (We shall later take $M = T^*(X) \times T^*(Y) \times T^*(Y) \times T^*(Z)$, $M_2 = \Delta$ (the diagonal), and $M_1 = C_1 \times C_2$ where C_1 and C_2 are canonical relations. Note that the symplectic structure then gives automatically densities in M_2 and in M.) If $m \in M_1 \cap M_2$ we have by definition of transversality

$$T_m(M_1) + T_m(M_2) = T_m(M),$$

so $T_m(M_1)/T_m(M_1 \cap M_2) \cong T_m(M)/T_m(M_2)$ which gives an isomorphism

$$\Omega_\alpha(T_m(M_1 \cap M_2)) \cong \Omega_\alpha(T_m(M_1)) \otimes \Omega_{-\alpha}(T_m(M)/T_m(M_2)).$$

If we are given a positive section of $\Omega_{-\alpha}(T_m(M)/T_m(M_2))$ over M_2 we can therefore use this section to define an isomorphism

$$\Omega_\alpha(M_1 \cap M_2) \cong \Omega_\alpha(M_1)_{M_1 \cap M_2}. \tag{4.2.9}$$

Since for $m \in M_2$ we have

$$\Omega_{-\alpha}(T_m(M)/T_m(M_2)) \cong \Omega_{-\alpha}(T_m(M)) \otimes \Omega_\alpha(T_m(M_2))$$

we have such a section if there is given a positive density in M and one in M_2, for a positive density on a manifold allows one to identify Ω_1 with Ω_0 and so Ω_α with Ω_0 for all α. In particular, we can thus *define the product of a density of order $\frac{1}{2}$ on each of the canonical relations C_1 and C_2 as a density of order $\frac{1}{2}$ on $C_1 \circ C_2$ by first taking the direct product of the two densities on $C_1 \times C_2$ and then intersecting with the diagonal Δ.*

There is another way of defining a section of $\Omega_{-\alpha}(T_m(M)/T_m(M_2))$ over M which is better related to our definitions in sections 2.4 and 3.2. Suppose that we have a map γ: $M \to N$ where N is a manifold with a given positive density, $\dim N = \operatorname{codim}_M M_2$, such that γM_2 is a point n_0 and the rank of γ' is $\dim N$. Since γ' is a bijection of $T_m(M)/T_m(M_2)$ on $T_{n_0}(N)$ we can pull the element of $\Omega_1(T_{n_0}(N))$ given by the positive density in N back to the required section of $\Omega_1(T_m(M)/T_m(M_2))$. In the example where $M = T^*(X) \times T^*(Y) \times T^*(Y) \times T^*(Z)$ we have if Y is contained in a vector space an isomorphism

$$M \to \Delta \times T^*(Y)$$

preserving densities given by letting the image in Δ be defined by repetition of the first component in $T^*(Y)$ of an element in M and letting the image in $T^*(Y)$ be the second component in $T^*(Y)$ minus the first. This shows that the map $M \to T^*(Y)$ defines the same section of $\Omega_\alpha(T(M)/T(\Delta))$ as the one obtained from the symplectic measures in M and in Δ (identified of course with $T^*(X) \times T^*(Y) \times T^*(Z)$).

If in addition to the density in N and the map γ we also have a density in M, the isomorphism

$$\Omega_\alpha(T_m(M_2)) \cong \Omega_\alpha(T_m(M)) \otimes \Omega_{-\alpha}(T_m(M)/T_m(M_2))$$

gives a density on M_2. For example, if $M = \mathbf{R}^{n+N}$ and γ: $\mathbf{R}^{n+N} \to \mathbf{R}^N$ has surjective differential on $M_2 = \gamma^{-1}(0)$, the Lebesgue densities in $\mathbf{R}^{n+N}$ and $\mathbf{R}^N$ define a density on M_2. If $y_1, \ldots, y_n$ are local coordinates on M_2, considered as functions in a neighborhood of M_2, and we use $y(x), \gamma(x)$ as local coordinates in a neighborhood of M, then the density in $\mathbf{R}^{n+N}$ with respect to these local coordinates (or more precisely, the density evaluated on the dual basis of tangent vectors) is given by $|D(y, \gamma)/Dx|^{-1}$, so the density defined on M_2 agrees with the one defined in section 2.4.

This construction may be made in steps: Let γ_j: $M \to N_j$, $j=1, 2$, be C^∞ maps. Assume that positive densities are given in M, N_1, N_2 and that $\gamma = (\gamma_1, \gamma_2)$: $M \to N_1 \times N_2$ has surjective differential on $M^0 = \gamma^{-1}(n_1^0, n_2^0)$. From γ_2 and the densities in N_2 and M we then obtain a density in $M' = \gamma_2^{-1} n_2^0$ near M^0 and from this density, the density in N_1 and the map γ_1 we then obtain a density in M^0. This is the same as the one defined by the density in N and by the map γ directly. The simple verification is left for the reader.

Now $C_1 \times C_2$ is locally isomorphic to the inverse image of 0 in $\mathbf{R}^{N_1+N_2}$ for the map

$$\mathbf{R}^{n_X+n_Y+N_1+n_Y+n_Z+N_2} \ni (x, y_1, \theta, y_2, z, \sigma) \to (\phi'_{1\theta}, \phi'_{2\sigma}) \in \mathbf{R}^{N_1+N_2},$$

and this map together with the Lebesgue measures defines the density $\sqrt{d_{C_{\phi_1}}} \times \sqrt{d_{C_{\phi_2}}}$ of order $\frac{1}{2}$. The intersection of $C_1 \times C_2$ with the diagonal corresponds to the submanifold where $y_1 = y_2$ and $\phi'_{1y_1} + \phi'_{2y_2} = 0$, and the density of order $\frac{1}{2}$ which we have defined there corresponds to the map

$$(x, y_1, \theta, y_2, z, \sigma) \to (\phi'_{1\theta}, \phi'_{2\sigma}, y_1 - y_2, \phi'_{1y_1} + \phi'_{2y_2}) \in \mathbf{R}^{N_1+N_2+2n_Y}.$$

If we first consider the map to $y_1 - y_2$ which obviously gives the Lebesgue density in the plane $y_1 = y_2$, we see that an equivalent manifold and density is defined by the map

$$(x, z, y, \theta, \sigma) \to (\phi'_{1\theta}, \phi'_{2\sigma}, \phi'_{1y} + \phi'_{2y}) \in \mathbf{R}^{N_1+N_2+n_Y},$$

where now $y = y_1 = y_2$. But this is the manifold C_ϕ with the density $\sqrt{d_{C_\phi}}$ of order $\frac{1}{2}$ corresponding to $C_1 \circ C_2$. Thus the density of order $\frac{1}{2}$ on $C_1 \circ C_2$ corresponding to $A_1 A_2$ and the phase function ϕ is the product of the densities of order $\frac{1}{2}$ on C_1 and C_2 corresponding to A_1 and A_2 for the phase functions ϕ_1 and ϕ_2 as explained above.

The preceding construction simplifies considerably when C_1 (or C_2) is a locally canonical graph. For we have an isomorphism (at a point in $(C_1 \times C_2) \cap \Delta$)

$$\begin{aligned}\Omega_\alpha(T((C_1 \times C_2) \cap \Delta)) &\cong \Omega_\alpha(T(C_1 \times C_2)) \otimes \Omega_{-\alpha}(T(T^*(Y))) \\ &\cong \Omega_\alpha(T(C_1)) \otimes \Omega_{-\alpha}(T(C_1)) \otimes \Omega_\alpha(T(C_2)) = \Omega_0(T(C_1)) \otimes \Omega_\alpha(T(C_2))\end{aligned}$$

obtained from the isomorphisms of $T(M)/T(\Delta)$ and $T(C_1)$ with $T(T^*(Y))$. The sections of $\Omega_0(T(C_1))$ are just the functions on C_1 obtained by identifying densities of order $\frac{1}{2}$ with functions. Thus multiplication consists in this case simply in multiplication of the density of order $\frac{1}{2}$ on C_2 by the function on C_1 after both have been pulled back to the intersection $(C_1 \times C_2) \cap \Delta$ by means of the obvious maps to C_1 and C_2. (The second one is a local diffeomorphism.) When both C_1 and C_2 are local canonical graphs, then so is $C_1 \circ C_2$ and the multiplication becomes just multiplication of two functions.

Finally we shall relate the line boundles L_1, L_2 on C_1, C_2 to the line bundle L on $C_1 \circ C_2$. Let $\phi_1, \tilde{\phi}_1$ resp. $\phi_2, \tilde{\phi}_2$ be phase functions representing a neighborhood of a point in C_1 resp. C_2, and write

$$\phi(x, z, \omega) = \phi_1(x, y, \theta) + \phi_2(y, z, \sigma),$$

$$\tilde{\phi}(x, z, \tilde{\omega}) = \tilde{\phi}_1(x, y, \tilde{\theta}) + \tilde{\phi}_2(y, z, \tilde{\sigma}),$$

where ω and $\tilde{\omega}$ are defined as in the discussion preceding (4.2.3). These are phase functions defining $C_1 \circ C_2$, and we have

$$\operatorname{sgn} \phi''_{\omega\omega} - \operatorname{sgn} \tilde{\phi}''_{\tilde{\omega}\tilde{\omega}} = \operatorname{sgn} \phi''_{1\theta\theta} - \operatorname{sgn} \tilde{\phi}''_{1\tilde{\theta}\tilde{\theta}} + \operatorname{sgn} \phi''_{2\sigma\sigma} - \operatorname{sgn} \tilde{\phi}''_{2\tilde{\sigma}\tilde{\sigma}}. \tag{4.2.10}$$

This follows by repeating the proof of Proposition 3.3.1: the equality is trivial when ϕ_j and $\tilde{\phi}_j$ are equivalent or when they are obtained from each other by increasing the number of fiber variables as indicated in section 3.1. It follows that we have an isomorphism of the tensor product $L_1 \times L_2$ on $C_1 \times C_2$, restricted to the intersection with Δ, and L which for the trivializations of L_1, L_2, L corresponding to ϕ_1, ϕ_2 and ϕ is given by standard multiplication. It follows that we have a bilinear map from sections of $\Omega_{\frac{1}{2}} \otimes L_{C_1}$ over C_1 and sections of $\Omega_{\frac{1}{2}} \otimes L_{C_2}$ over C_2 to sections of $\Omega_{\frac{1}{2}} \otimes L_{C_1 \circ C_2}$ over $C_1 \circ C_2$. We denote it by $\times$ and have proved

THEOREM 4.2.3. *Let the hypotheses of Theorem 4.2.2 be fulfilled. If a_1 and a_2 are principal symbols of A_1 and A_2, then $a_1 \times a_2$ is a principal symbol of the product $A_1 A_2$.*

4.3. L^2 estimates

Let C be the graph of a canonical diffeomorphism $T^*(Y) \to T^*(X)$. If $A \in I^0_\varrho(X \times Y, C')$, and A is properly supported, then A^*A is a pseudo-differential operator with principal symbol $|a|^2$ if a is a principal symbol for A, regarded as a function on $T^*(Y)$. We can therefore apply the well-known results on L^2 estimates for pseudo-differential operators recalled in section 2.2 to show that A^*A is L^2 continuous (or compact, in case $a \to 0$ at ∞). Since this is equivalent to the same statements concerning A, we obtain

THEOREM 4.3.1. *Every $A \in I^0_\varrho(X \times Y, C')$ which is properly supported is continuous from $L^2_c(Y, \Omega_{\frac{1}{2}})$ to $L^2_c(X, \Omega_{\frac{1}{2}})$ where subscript c indicates compact support, and also from $L^2_{loc}(Y, \Omega_{\frac{1}{2}})$ to $L^2_{loc}(X, \Omega_{\frac{1}{2}})$, provided that C is locally a canonical graph. A maps bounded sets to compact sets if and only if a principal symbol tends to 0 at ∞ in C over compact subsets of $X \times Y$.*

We could of course also give the precise norm modulo compact operators but leave this for the reader. Instead we shall give a sufficient condition for L^2 continuity in the general case using Theorem 4.1.9. First recall that (3.2.14) defines a distribution in I^m_ϱ if

$a_j \in S_\varrho^{m+(n-2N_j)/4}$. It follows that if as in Theorem 4.1.9 we restrict to submanifolds, the order is going to increase by the codimension divided by 4. In the notation of Theorem 4.1.9 the degree will therefore increase by $(n_X+n_Y-2k)/4$. This leads immediately to

THEOREM 4.3.2. *Let C be a homogeneous canonical relation from $T^*(Y)$ to $T^*(X)$ such that the maps $C \to X$ and $C \to Y$ have surjective differentials. Let the differentials of the projections $C \to T^*(X)$ and $C \to T^*(Y)$ have rank at least $k+\dim X$ and $k+\dim Y$ respectively. Every $A \in I_\varrho^m(X \times Y, C')$ is then continuous from $L_c^2(Y, \Omega_{\frac{1}{2}})$ to $L_{\mathrm{loc}}^2(X, \Omega_{\frac{1}{2}})$ provided that $m \leqslant (2k-n_X-n_Y)/4$.*

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