THE MAXIMUM PRINCIPLE FOR MULTIPLE-VALUED ANALYTIC FUNCTIONS

BY

HAROLD WIDOM

University of California, Santa Cruz, California, U.S.A. [1]

I. Introduction

If F is a single-valued analytic function satisfying $|F(z)| \leqslant 1$ throughout a domain Ω in the Riemann sphere, then of course $|F(\zeta)| \leqslant 1$ for any particular ζ. We have $|F(\zeta)| = 1$ only if F is a constant of absolute value one. The same statements hold even if F is not necessarily single-valued but has single-valued absolute value, for log $|F|$ is still subharmonic. In particular if F is not single-valued then

$$\limsup_{z\to\partial\Omega} |F(z)| \leqslant 1$$

implies the strict inequality $|F(\zeta)| < 1$. Among the concerns of the present paper is the question of how small $|F(\zeta)|$ must be, given that F has a particular type of multiple-valued behavior.

This multiple-valued behavior may be abstracted in the following way as a character (homomorphism into the group T of complex numbers of absolute value 1) of the fundamental group of Ω. Continuation of a function element of F along a cycle γ results in multiplication by a constant of absolute value 1, which we call $\Gamma_F(\gamma)$. This constant is easily seen to be independent both of the starting point on γ and the particular element of F chosen. We may write concisely

$$\Gamma_F(\gamma) = \exp\{i \underset{\gamma}{\Delta} \arg F\}.$$

Since homotopic curves produce identical analytic continuations, Γ_F is constant on each homotopy class and may therefore be considered a function on $\pi(\Omega)$, the fundamental group of Ω. It is trivially a character.

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Given any $\Gamma \in \pi(\Omega)^*$ (the asterisk denotes character group), denote by $\mathcal{H}(\Omega, \Gamma)$ those multiple-valued analytic functions F on Ω with single-valued absolute value for which $\Gamma_F = \Gamma$. The natural question arises whether each $\mathcal{H}(\Omega, \Gamma)$ is necessarily nonempty, and it is not hard to see that the answer is yes.

First note that since the group T is abelian we may identify $\pi(\Omega)^*$ with $H_1(\Omega)^*$, the character group of the first singular homology group of Ω. This in turn may be identified with the cohomology group $H^1(\Omega, T)$. Now consider the exact sheaf sequence

$$0 \to T \to O^* \to O^*/T \to 0,$$

where O^* denotes the sheaf of germs of nonzero analytic functions on Ω (under multiplication). This induces an exact sequence of cohomology groups [2, Theorem 1]

$$H^0(\Omega, O^*/T) \to H^1(\Omega, T) \to H^1(\Omega, O^*).$$

The first of these groups is the group of sections of O^*/T and a little thought shows that each section is just an element of $\mathcal{H}(\Omega, \Gamma)$ for some Γ. The last group is 0 [2, p. 52], and this establishes the fact that each element of $H^1(\Omega, T)$, and so each $\Gamma \in \pi(\Omega)^*$, arises from a function in $\mathcal{H}(\Omega, \Gamma)$.

Another concern of this paper is the characterization of these domains Ω for which $\mathcal{H}_\infty(\Omega, \Gamma)$, the set of bounded functions of $\mathcal{H}(\Omega, \Gamma)$, is nonempty for each Γ. This is intimately connected with the question raised in the first paragraph. To see why, define for each $\Gamma \in \pi(\Omega)^*$ and $\zeta \in \Omega$

$$m(\Omega, \Gamma, \zeta) = \sup \{ |F(\zeta)| : F \in \mathcal{H}_\infty(\Omega, \Gamma), \ |F| \leqslant 1 \text{ in } \Omega \}$$

and for each $\zeta \in \Omega$

$$m(\Omega, \zeta) = \inf \{ m(\Omega, \Gamma, \zeta) : \Gamma \in \pi(\Omega)^* \}.$$

In accord with the convention of defining the supremum of an empty set of nonnegative real numbers to be zero, we set $m(\Omega, \Gamma, \zeta) = 0$ if $\mathcal{H}_\infty(\Omega, \Gamma)$ is empty.

The maximum principle for $\mathcal{H}_\infty(\Omega, \Gamma)$ is

$$|F(\zeta)| \leqslant m(\Omega, \Gamma, \zeta) \limsup_{z \to \partial\Omega} |F(z)|.$$

This inequality follows from the maximum principle for subharmonic functions and the definition of $m(\Omega, \Gamma, \zeta)$.

It is easy to see that if $\mathcal{H}_\infty(\Omega, \Gamma)$ is nonempty then each $m(\Omega, \Gamma, \zeta) > 0$; for any function in $\mathcal{H}_\infty(\Omega, \Gamma)$ may be multiplied by a rational function to produce a function of $\mathcal{H}_\infty(\Omega, \Gamma)$ not vanishing at ζ. One of the main results of the paper is that all the $\mathcal{H}_\infty(\Omega, \Gamma)$ are nonempty if and only if $m(\Omega, \zeta)$ is positive. Furthermore, we shall obtain a formula for $m(\Omega, \zeta)$

and this will give us a criterion for determing whether all the $\mathcal{H}_\infty(\Omega, \Gamma)$ are nonempty; we simply check whether $m(\Omega, \zeta)$ is positive or zero.

We should point out here that in case Ω is finitely connected, with each complementary component containing more than one point, there is no question about the existence of functions belonging to $\mathcal{H}_\infty(\Omega, \Gamma)$. For Ω is conformally equivalent to (and may therefore be assumed to be) a domain bounded by analytic Jordan curves. One can find a slightly larger domain Ω_1 (the closure of Ω lying in Ω_1) with $\pi(\Omega_1) \approx \pi(\Omega)$. We know that for each Γ there is an $F \in \mathcal{H}(\Omega_1, \Gamma)$. Then F, restricted to Ω, belongs to $\mathcal{H}_\infty(\Omega, \Gamma)$. The same thing can of course be proved without any heavy machinery.

In this case also there is a formula for $m(\Omega, \zeta)$ which was derived in [7]. The technique used in the present paper will involve approximating an arbitrary domain by appropriate finitely connected domains and showing that $m(\Omega, \zeta)$ is continuous in Ω. The formula for $m(\Omega, \zeta)$ in the finitely-connected case, which involves critical values of a certain function, must be restated in a form suitable for extension to the general case. This is accomplished by exploiting the relation between the number of critical points of the function and the connectivity characteristics of Ω.

At the end of the paper we shall give some applications to problems that motivated our work. It is interesting that although they all concern single-valued analytic functions, their investigation leads naturally to the consideration of certain multiple-valued functions.

II. Determination of $m(\Omega, \zeta)$

If Ω is finitely connected with each complementary component a continuum (compact connected set containing more than one point) then the formula for $m(\Omega, \zeta)$ given by Theorem 5.6 of [7] is

$$m(\Omega, \zeta) = \exp\{-\sum_j g(z_j, \zeta)\}, \tag{1}$$

where $g(z, \zeta)$ is Green's function for Ω with pole at ζ and the z_j are its critical points. We shall outline here the derivation of this.

The idea is to use a dual extremum problem. It can be shown that for $\zeta = \infty$ (the general case can be reduced to this) we have

$$m(\Omega, \Gamma, \infty) = \inf \frac{1}{2\pi} \int_{\partial\Omega} |f(z)|\,|dz|,$$

where the infimum is taken over all $f \in \mathcal{H}_1(\Omega, \Gamma^{-1})$ satisfying

$$f(\infty) = 0, \quad |f'(\infty)| = 1. \tag{2}$$

($\mathcal{H}_p(\Omega, \Gamma)$ for $p < \infty$ consists of those functions of $\mathcal{H}(\Omega, \Gamma)$ the pth powers of whose absolute values possess harmonic majorants. We may assume $\partial\Omega$ as smooth as desired since the problem is conformally invariant.) Therefore

$$m(\Omega, \infty) = \inf \frac{1}{2\pi} \int_{\partial\Omega} |f(z)|\,|dz|,$$

where now the infimum is taken over all f belonging to any $\mathcal{H}_1(\Omega, \Gamma)$ and satisfying (2).

Write

$$\Phi(z, \zeta) = \exp\,\{g(z, \zeta) + i\tilde{g}(z, \zeta)\},$$

where the tilde denotes harmonic conjugate. The z_j are exactly the zeros of $\Phi'(z, \infty)$. Thus for any f satisfying (2), the function

$$h(z) = \left| f(z)\, \frac{\Phi(z, \infty)}{\Phi'(z, \infty)} \prod_j \Phi(z, z_j)^{-1} \right|$$

is subharmonic in Ω and equal to

$$\left|\prod_j \Phi(\infty, z_j)^{-1}\right| = \exp\,\{-\sum_j g(z_j, \infty)\}$$

at ∞. (Here we have used the symmetry of Green's function.) Since harmonic measure at ∞ is $(2\pi)^{-1}|d\Phi(z, \infty)|$ this gives

$$\exp\,\{-\sum_j g(z_j, \infty)\} \leqslant \frac{1}{2\pi} \int_{\partial\Omega} h(z)\,|d\Phi(z, \infty)| = \frac{1}{2\pi} \int_{\partial\Omega} |f(z)|\,|dz|.$$

Equality is achieved for the function

$$f(z) = \frac{\Phi'(z, \infty)}{\Phi(z, \infty)} \prod_j \frac{\Phi(z, z_j)}{\Phi(z_j, \infty)}.$$

This indicates how (1) is derived. It is important for us to note that the number of critical points z_j is one less than the number of complementary components of Ω. For the number of critical points is the number of zeros of the single-valued function

$$\frac{d}{dz}\,\{g(z, \zeta) + i\tilde{g}(z, \zeta)\}.$$

The function has one pole, so the number of zeros is (again think of Ω as bounded)

$$1 + \frac{1}{2\pi}\,\Delta_{\partial\Omega} \arg \frac{d}{dz}\,\{g(z, \zeta) + i\tilde{g}(z, \zeta)\} = 1 + \frac{1}{2\pi}\,\Delta_{\partial\Omega} \arg \frac{|dz|}{dz}$$

$$= 1 + (\text{number of components}) - 2 \qquad (3)$$

since the outer curve of $\partial\Omega$ is described counterclockwise, the inner curves clockwise. The number of zeros is thus equal to one less then the number of complementary components, or equivalently equal to the (first) Betti number of Ω.

Consider now a general domain Ω in the sphere. Suppose Ω_1 is a subdomain of Ω each of whose complementary components contains a complementary component of Ω. (Roughly speaking each hole in Ω_1 arises from some hole in Ω.) Then every cycle in Ω_1 which is homologous to zero in Ω is also homologous to zero in Ω_1. Thus the map

$$H_1(\Omega_1)\to H_1(\Omega)$$

induced by the inclusion $\Omega_1\to\Omega$ is injective. This implies that the induced map

$$H_1(\Omega)^*\to H_1(\Omega_1)^*$$

is surjective (since a character on a subgroup extends to a character on the group). Because of the fact, already noted, that H_1 and π have the same character groups this shows that the map

$$\varphi_1\colon \pi(\Omega)^*\to\pi(\Omega_1)^* \tag{4}$$

is surjective.

A function in $\mathcal{H}_\infty(\Omega,\Gamma)$ when restricted to Ω_1 is a function in $\mathcal{H}_\infty(\Omega_1,\varphi_1(\Gamma))$. Thus since φ_1 is surjective it follows immediately that $m(\Omega,\zeta)\leqslant m(\Omega_1,\zeta)$ for each $\zeta\in\Omega_1$.

LEMMA 1.1. *Suppose Ω is the union of an increasing family of subdomains $\Omega_1, \Omega_2, \ldots$, where each Ω_n has the property that each of its complementary components contains a complementary component of Ω. Then for any ζ*

$$m(\Omega,\zeta)=\lim_{n\to\infty} m(\Omega_n,\zeta).$$

Proof. We have the maps

$$\varphi_n\colon \pi(\Omega)^*\to\pi(\Omega_n)^*.$$

Normal family considerations show that for each $\Gamma\in\pi(\Omega)^*$ one can find

$$F_n\in\mathcal{H}_\infty(\Omega_n,\varphi_n(\Gamma)) \tag{5}$$

satisfying $$|F_n|\leqslant 1 \text{ in } \Omega_n,\quad |F_n(\zeta)|=m(\Omega_n,\varphi_n(\Gamma),\zeta),$$

at least for n so large that $\zeta\in\Omega_n$. By passing to a subsequence if necessary, we may assume that $\{F_n\}$ converges uniformly on compact subsets of Ω to an F which satisfies

$$|F|\leqslant 1 \text{ in } \Omega,\quad |F(\zeta)|\geqslant\limsup_{n\to\infty} m(\Omega_n,\zeta).$$

To find Γ_F note that since each cycle γ lies in a compact subset of Ω

$$\Gamma_F(\gamma) = \exp\{i\,\Delta_\gamma \arg F\} = \lim_{n\to\infty} \exp\{i\Delta_\gamma \arg F_n\}$$

and because of (5) this is just $\Gamma(\gamma)$. Hence $F \in \mathcal{H}_\infty(\Omega, \Gamma)$, so that

$$m(\Omega, \Gamma, \zeta) \geq \limsup_{n\to\infty} m(\Omega_n, \zeta).$$

Since this holds for all Γ,

$$m(\Omega, \zeta) \geq \limsup_{n\to\infty} m(\Omega_n, \zeta).$$

But since $m(\Omega, \zeta) \leq m(\Omega_n, \zeta)$ for each n, the lemma follows.

We can now combine Lemma 1.1 with formula (1), suitably reinterpreted, to calculate $m(\Omega, \zeta)$ in the general case. This will again be given in terms of the Green function $g(z, \zeta)$ for Ω (which is identically $+\infty$ if the complement has zero logarithmic capacity).

THEOREM 1. *Denote by $B(\alpha)$ the first Betti number of the domain*

$$\Omega_\alpha = \{z \in \Omega : g(z, \zeta) > \alpha\}.$$

Then we have

$$m(\Omega, \zeta) = \exp\left\{-\int_0^\infty B(\alpha)\, d\alpha\right\}. \tag{6}$$

A remark is in order before we prove the theorem. Here is why each Ω_α is a domain, that is, a connected open set. If $g(z, \zeta) \equiv +\infty$ then $\Omega_\alpha = \Omega$ for each α. If $g(z, \zeta) \not\equiv +\infty$ and if Ω_α were disconnected it would have a component $\tilde{\Omega}$ not containing ζ, therefore throughout which $g(z, \zeta)$ is harmonic. But at each boundary point z_0 of $\tilde{\Omega}$, except for a subset of $\partial\Omega$ of logarithmic capacity zero,

$$\limsup_{z\to z_0} g(z, \zeta) \leq \alpha$$

so that we would have the incorrect inequality $g(z, \zeta) \leq \alpha$ throughout $\tilde{\Omega}$. (See [5], Theorems III. 28, 33, 36.)

To prove the theorem suppose first that Ω is finitely connected with each complementary component a continuum. Then Ω_α has Green function $g(z, \zeta) - \alpha$. Since, as we have seen, the number of critical points of Green's function for a finitely connected domain (each of whose complementary components is a continuum) is the Betti number, the number of critical values of $g(z, \zeta)$ which exceed α is $B(\alpha)$. It follows that

$$\sum g(z_j, \zeta), \tag{7}$$

the sum of the critical values of $g(z, \zeta)$, equals

$$\int_0^\infty B(\alpha)\, d\alpha. \tag{8}$$

Now take a general Ω. Find a sequence of subdomains. $\Omega_1 \subset \Omega_2 \subset \ldots \subset \Omega$ satisfying

$$\Omega = \bigcup \Omega_n$$

and such that each component of the complement of Ω_n contains a component of the complement of Ω.

Moreover, we require that all complementary domains of the Ω_n are continua. (It is a simple exercise to show that a sequence of domains with these properties exists.) If $B_n(\alpha)$ denotes the obvious function then by counting complementary components it is easy to see that for each $\alpha > 0$

$$B_n(\alpha) \leqslant B_{n+1}(\alpha), \quad \lim_{n\to\infty} B_n(\alpha) = B(\alpha).$$

The validity of (6) now follows from its validity for each Ω_n, Lemma 1.1, and the monotone convergence theorem.

COROLLARY 1.1. *If $g(z, \zeta) \equiv +\infty$ then $m(\Omega, \zeta) = 0$ unless Ω is the sphere minus at most one point in which case $m(\Omega, \zeta) = 1$.*

Proof. For each α, $\Omega_\alpha = \Omega$ so that $B(\alpha)$ is constantly equal to the first Betti number of Ω. The result follows.

In the proof of the theorem the passage from the sum of the critical values of $g(z, \zeta)$ to the integral of the Betti number $B(\alpha)$ made use of the harmonicity of g. (See (3).) However this is really not very important. Indeed Morse theory allows one to make the same passage using only the fact that g has neither a local maximum nor minimum.

The one-dimensional analogue of this, incidentally, is related to the theorem that the total variation of a real-valued continuous function on the line is the integral, over α, of the number of times the function takes the value α. The following corollary should therefore not come as a great surprise.

COROLLARY 1.2. *Let K be a compact subset of the real line, Ω the complement of K in the extended plane. Extend $g(z) = g(z, \infty)$ to K by defining it to be zero there, and let $[a, b]$ be the smallest closed interval on the line which contains K. Then*

$$m(\Omega, \infty) = \exp\{-\tfrac{1}{2}\operatorname{Var}(g; a, b)\}.$$

Proof. We may assume $g \not\equiv +\infty$. Since $g(x+iy) > g(x)$ for $y \neq 0$, each component of Ω_α (necessarily a bounded subset of the plane) contains a unique interval on the line. Hence $B(\alpha)$ is one less than the number of components of $[a, b]$ on which $g(x) \leqslant \alpha$. It follows easily that $\int_0^\infty B(\alpha)\,d\alpha$ is exactly twice the total variation of $g(x)$ on $[a, b]$.

Here is another case where m is easily computed.

COROLLARY 1.3. *Suppose the complement of* Ω *is a continuum and let* S *be a discrete subset of* Ω. *Then if* $g(z, \zeta)$ *denotes Green's function for* Ω *we have*

$$m(\Omega - S, \zeta) = \exp\{-\sum_{s \in S} g(s, \zeta)\}.$$

Proof. The domain $\Omega - S$ also has $g(z, \zeta)$ as its Green function since S, being countable, has logarithmic capacity zero. Moreover,

$$\{z \in \Omega - S : g(z, \zeta) > \alpha\}$$

is obtained from the simply connected domain

$$\{z \in \Omega : g(z, \zeta) > \alpha\}$$

by removing those points of S at which $g(s, \zeta) > \alpha$. In other words $B(\alpha)$ for $\Omega - S$ is equal to the number of points of S at which $g(s, \zeta) > \alpha$, and the result follows.

The statement of Corollary 1.3 destroys any hope one might have had that formula (1) extends to all domains. However, the extension is valid if Ω is regular. Recall that this means $g(z, \zeta)$ tends to zero as z tends to the boundary of Ω.

COROLLARY 1.4. *If* Ω *is regular then*

$$m(\Omega, \zeta) = \exp\{-\sum_j g(z_j, \zeta)\},$$

where the z_j, *possibly infinite in number, are the critical points of* $g(z, \zeta)$.

Proof. Since Ω is regular, the closure of Ω_α is a subset of Ω. Therefore Ω_α is itself finitely connected and bounded by continua. Hence, as at the beginning of the proof of the theorem, the number of critical values of $g(z, \zeta)$ which exceed α is exactly $B(\alpha)$. The result follows.

III. The Classes $\mathcal{H}_\infty(\Omega, \Gamma)$

Our concern here is the characterization of those domains Ω for which $\mathcal{H}_\infty(\Omega, \Gamma)$ is nonempty for all $\Gamma \in \pi(\Omega)^*$.

THEOREM 2. *A necessary and sufficient condition that each* $\mathcal{H}_\infty(\Omega, \Gamma)$ *be nonempty is that for some (and hence all)* $\zeta \in \Omega$ *we have* $m(\Omega, \zeta) > 0$.

It is trivial that if $m(\Omega, \zeta) > 0$ for some ζ then each $\mathcal{H}_\infty(\Omega, \Gamma)$ is nonempty. We already noted in the introduction that if $\mathcal{H}_\infty(\Omega, \Gamma)$ is nonempty then $m(\Omega, \Gamma, \zeta) > 0$. Thus what is to be proved is that if $m(\Omega, \Gamma, \zeta)$ is positive for each Γ then its infimum over Γ, namely $m(\Omega, \zeta)$, is also positive. We may assume Ω possesses a Green function.

LEMMA 2.1. *Let Ω_1 be a finitely connected subdomain of Ω each of whose complementary components contain a complementary component of Ω. Let $\zeta \in \Omega_1$. Then there is a constant $m>0$ such that for any $\Gamma_1 \in \pi(\Omega_1)^*$ there is an F, defined and analytic on Ω with single-valued absolute value, satisfying*

$$|F| \leqslant 1 \text{ on } \Omega, \quad |F(\zeta)| \geqslant m, \quad \Gamma_{F|\Omega_1} = \Gamma_1.$$

Proof. We may suppose $\infty \notin \Omega$. Denote the components of the complement of Ω_1 by $C_1, \ldots, C_n, C_\infty$, where $\infty \in C_\infty$. For each $j \leqslant n$ pick a point ζ_j in C_j but not in Ω. This is possible by our assumption on Ω_1. We can also find a family of cycles $\gamma_1, \ldots, \gamma_n$ forming a homology basis for Ω_1 and for which

$$\Delta_{\gamma_j} \arg (z-\zeta_k) = 2\pi\delta_{jk}.$$

Let a_j be any one of the values of

$$(2\pi i)^{-1} \log \Gamma_1(\gamma_j)$$

and set
$$\alpha_j = \frac{n}{2\pi} \Delta_{\gamma_j} \tilde{g}(z, \infty) + a_j - \left[\frac{n}{2\pi} \Delta_{\gamma_j} \tilde{g}(z, \infty) + a_j\right]$$

(brackets denote "greatest integer in") so that $0 \leqslant \alpha_j < 1$. Finally consider

$$G(z) = \exp\{-n[g(z, \infty) + i\tilde{g}(z, \infty)]\} \prod_{j=1}^{n} (z-\zeta_j)^{\alpha_j}.$$

Then G has an upper bound on Ω, and lower bound at ζ, independent of the α_j and so of Γ_1, and clearly

$$\Gamma_{G|\Omega_1} = \Gamma_1.$$

An appropriate constant multiple of G has all the required properties.

LEMMA 2.2. *Let Ω, Ω_1 be as in Lemma 2.1, φ_1 the map (4). Then if $m(\Omega, \zeta) = 0$ we have for each $\Gamma_1 \in \pi(\Omega_1)^*$*

$$\inf\{m(\Omega, \Gamma, \zeta) : \Gamma \in \varphi_1^{-1}(\Gamma_1)\} = 0.$$

Proof. The inequality

$$m(\Omega, \bar{\Gamma}, \zeta) \geqslant m(\Omega, \Gamma, \zeta)\, m(\Omega, \bar{\Gamma}/\Gamma, \zeta)$$

holds for any $\Gamma, \bar{\Gamma} \in \pi(\Omega)^*$. By Lemma 2.1 there is, for each $\bar{\Gamma} \in \pi(\Omega)^*$, some $\Gamma \in \pi(\Omega)^*$ satisfying

$$\varphi_1(\Gamma) = \varphi_1(\bar{\Gamma})/\Gamma_1, \quad m(\Omega, \Gamma, \zeta) \geqslant m,$$

and so
$$m(\Omega, \bar{\Gamma}, \zeta) \geqslant m \cdot m(\Omega, \bar{\Gamma}/\Gamma, \zeta).$$

Consequently, since $\varphi_1(\bar{\Gamma}/\Gamma)=\Gamma_1$,

$$m(\Omega, \bar{\Gamma}, \zeta) \geqslant m \cdot \inf \{m(\Omega, \Gamma, \zeta) : \varphi_1(\Gamma) = \Gamma_1\}.$$

This holds for all $\bar{\Gamma} \in \pi(\Omega)^*$. If we take the infimum of the left side we get $m(\Omega, \zeta)$ which is zero. Therefore the right side of the inequality is zero and the lemma is established.

We shall now prove the theorem by showing that if $m(\Omega, \zeta)=0$ we can produce a $\Gamma \in \pi(\Omega)^*$ for which $m(\Omega, \Gamma, \zeta)=0$.

Let $\Omega_1, \Omega_2, \ldots$ be as in the statement of Lemma 1.1. Suppose further that if $j<k$ then each complementary component of Ω_j contains a complementary component of Ω_k. Then in addition to the surjective maps

$$\varphi_n : \pi(\Omega)^* \to \pi(\Omega_n)^*$$

we have surjective maps $\qquad \varphi_{jk} : \pi(\Omega_k)^* \to \pi(\Omega_j)^*, \quad j<k.$

By exactly the same argument as used in proving Lemma 1.1 one can show that for any fixed $\Gamma_j \in \pi(\Omega_j)^*$,

$$\inf \{m(\Omega, \Gamma, \zeta) : \Gamma \in \varphi_j^{-1}(\Gamma_j)\} = \lim_{n\to\infty} \inf \{m(\Omega_n, \Gamma_n, \zeta) : \Gamma_n \in \varphi_{jn}^{-1}(\Gamma_j)\}.$$

Thus because of our assumption that $m(\Omega, \zeta)=0$ and Lemma 2.2 we have for each j and any $\Gamma_j \in \pi(\Omega_j)^*$,

$$\lim_{n\to\infty} \inf \{m(\Omega_n, \Gamma_n, \zeta) : \Gamma_n \in \varphi_{jn}^{-1}(\Gamma_j)\} = 0.$$

One can now find inductively a sequence $\{n_j\}$ and for each i a $\Gamma_i \in \pi(\Omega_{n_i})^*$ such that

$$\varphi_{n_i, n_{i+1}}(\Gamma_{i+1}) = \Gamma_i, \quad m(\Omega_{n_i}, \Gamma_i, \zeta) \to 0. \tag{9}$$

Now the first part of (9) is just a consistency relation which guarantees the existence of a $\Gamma \in \pi(\Omega)$ such that $\varphi_{n_i}(\Gamma)=\Gamma_i$ for each i. For this Γ we have clearly

$$m(\Omega, \Gamma, \zeta) \leqslant m(\Omega_{n_i}, \Gamma_i, \zeta)$$

for each i, and so $m(\Omega, \Gamma, \zeta)=0$.

The theorem is therefore established. Since the condition $m(\Omega, \zeta)>0$ is independent of the particular choice of ζ we shall write simply $m(\Omega)>0$. Theorem 1 gives us a necessary and sufficient condition for this:

$$\int_0^\infty B(\alpha)\, d\alpha < \infty. \tag{10}$$

If $N(\alpha)=B(\alpha)+1$ then $N(\alpha)$ is just the number of complementary components of Ω_α; this is in some ways more convenient to work with than $B(\alpha)$. If $g(z,\zeta)\not\equiv+\infty$ then $B(\alpha)$ is nonincreasing and equal to zero for large α, so condition (10) is equivalent to

$$\int_0^\delta N(\alpha)\,d\alpha<\infty$$

for any positive δ.

We shall see now that the problem of whether $m(\Omega)>0$ is in some sense a local one.

THEOREM 3. *Suppose the complement of Ω is the disjoint union of the infinite closed sets $K_1, \ldots, K_n$. Let Ω_i be the complement of K_i. Then $m(\Omega)>0$ if and only if each $m(\Omega_i)>0$.*

First a lemma which enables us to take care of the problem of sets of logarithmic capacity zero.

LEMMA 3.1. *If Ω is an arbitrary domain, S a relatively closed subset of logarithmic capacity zero and having an accumulation point in Ω, then $m(\Omega-S)=0$.*

Proof. The domains Ω and $\Omega-S$ possess the same Green function $g(z,\zeta)$. Since S is totally disconnected [5, Theorem III.5] we have, by an argument like that used in the proof of Corollary 1.3,

$$m(\Omega-S,\zeta)=m(\Omega,\zeta)\exp\{-\sum_{s\in S}g(s,\zeta)\}.$$

But since S has an accumulation point in Ω, the series diverges and we have $m(\Omega-S)=0$.

To prove the theorem note first that it follows from the lemma that if any K_i has logarithmic capacity zero then both $m(\Omega)$ and $m(\Omega_i)$ are zero. We need only consider, therefore, the case where all K_i have positive logarithmic capacity, so that the Ω_i have finite Green functions $g_i(z,\zeta)$ and Ω has a finite Green function $g(z,\zeta)$.

For convenience we define g to be zero on the complement of Ω, and similarly for the g_i. Thus the complement of Ω_α is the set

$$K_\alpha=\{z:g(z,\zeta)\leqslant\alpha\}$$

and $N(\alpha)$ is the number of connected components of K_α. Analogously we have $K_{i\alpha}$ and $N_i(\alpha)$.

Since $g\leqslant g_i$ each component of $K_{i\alpha}$ is contained in some component of K_α. Every component of K_α contains a point of the complement of Ω. For a component of K_α entirely contained in Ω satisfies $g(z,\zeta)=\alpha$ at each boundary point and so throughout the component. The component therefore contains no interior point. But it is clear that in the neighborhood of each point where $g(z, \zeta) = \alpha$ there is a connected open set, containing that point in its boundary, on which $g(z, \zeta) < \alpha$. Thus our component must have interior points, a contradiction.

Since each component of K_α contains a point of some K_i it intersects, and therefore entirely contains, one of the components of $K_{i\alpha}$. Hence

$$N(\alpha) \leqslant \sum N_i(\alpha)$$

and so $m_i(\Omega) > 0$ for each i implies $m(\Omega) > 0$.

To prove the converse take any neighborhood U of K_1 whose closure is disjoint from $K_2 \cup \dots \cup K_n$. For sufficiently large M we shall have

$$g_1(z, \zeta) \leqslant Mg(z, \zeta) \tag{11}$$

throughout U. Take α so small that any component of K_α that meets U is necessarily entirely contained in U. Also for α sufficiently small, $g_1(z, \zeta) \leqslant M\alpha$ implies $z \in U$, so each component of $K_{1, M\alpha}$ is contained in U. This component contains a point of K_1, by an argument presented above, and so meets a component of K_α which is necessarily also contained in U. Since (11) holds in U this component of K_α is contained in the component of $K_{1, M\alpha}$ we started with. We have shown that for α sufficiently small each component of $K_{1, M\alpha}$ contains some component of K_α, and so

$$N_1(M\alpha) \leqslant N(\alpha).$$

This shows that $m(\Omega) > 0$ implies $m(\Omega_1) > 0$, and similarly all $m(\Omega_i) > 0$.

The analytic capacity $\gamma(K)$ of a compact set K in the plane is defined to be sup $|a_1|$ where the supremum is taken over all functions f analytic and single-valued in the complement Ω of K, with power series expansion

$$f(z) = a_0 + a_1 z^{-1} + \dots$$

near $z = \infty$, and satisfying $|f(z)| \leqslant 1$ for all $z \in \Omega$. We have already seen that except in the trivial case where K contains at most one point $m(\Omega) > 0$ implies that $C(K)$, the logarithmic capacity of K, is positive. In fact even $\gamma(K)$ must be positive. (We always have $\gamma(K) \leqslant C(K)$ [9, p. 13].) To see this consider

$$\Phi(z) = \exp \{g(z, \infty) + i\tilde{g}(z, \infty)\}.$$

For any $F \in \mathcal{H}_\infty(\Omega, \Gamma_\Phi)$ the function F/Φ is single-valued and analytic in Ω. This gives the inequality

$$\gamma(K) \geqslant C(K)\, m(\Omega, \Gamma_\Phi, \infty).$$

Certainly therefore

$$\gamma(K) \geqslant C(K)\, m(\Omega, \infty).$$

The point $\zeta=\infty$ is clearly not special. There are analogous analytic and logarithmic capacities defined with respect to any $\zeta\in\Omega$. Different ζ generally give different values for the capacities, but analytic or logarithmic capacity zero is independent of the particular point chosen.

We have the following corollary of the theorem, a strengthening of Lemma 3.1.

COROLLARY 3.1. *If Ω is an arbitrary domain, K an infinite compact subset of Ω of analytic capacity zero, then $m(\Omega-K)=0$.*

Since for a subset K of the line $\gamma(K)>0$ is equivalent to K having positive Lebesgue measure [9, p. 14], Corollaries 1.1 and 3.1 have as a consequence the fact that if K has positive logarithmic capacity but zero Lebesgue measure then $g(z)$ is not locally of bounded variation. Of course, this can also be proved directly.

Theorem 2 tells us that if each $\mathcal{H}_\infty(\Omega,\Gamma)$ is nonempty then there is a positive lower bound $m(\Omega,\zeta)$ for the $m(\Omega,\Gamma,\zeta)$. In fact there is a positive lower bound when individual points ζ are replaced by arbitrary compact subsets of Ω. For each such set K define

$$m(\Omega,\Gamma,K)=\sup\{\inf_{\zeta\in K}|F(\zeta)|:F\in\mathcal{H}_\infty(\Omega,\Gamma),|F|\leqslant 1 \text{ in } \Omega\}$$

$$m(\Omega,K)=\inf\{m(\Omega,\Gamma,K):\Gamma\in\pi(\Omega)^*\}.$$

THEOREM 4. *If $m(\Omega)>0$ then $m(\Omega,K)>0$ for each compact subset K of Ω.*

Proof. Take any $\zeta\in K$. We know that for any $\Gamma\in\pi(\Omega)^*$ there is an $F\in\mathcal{H}_\infty(\Omega,\Gamma)$ satisfying

$$|F(z)|\leqslant 1 \quad \text{for } z\in\Omega, \quad |F(\zeta)|=m(\Omega,\zeta).$$

Let U be an open subset of Ω containing K and whose closure is contained in Ω. If z_j $(j=1,\dots,J)$ are the zeros of F in the closure of U then by the maximum principle

$$|F(z)|\leqslant\exp\left\{-\sum_{j=1}^{J}g(z_j,z)\right\}$$

and so in particular
$$\sum_{j=1}^{J}g(z_j,\zeta)\leqslant\log m(\Omega,\zeta)^{-1}.$$

Since $g(z,\zeta)$ is bounded below in the closure of U we see that the number J has a finite upper bound independent of Γ.

Pick a point $\xi\notin U$ and set

$$G(z)=F(z)\prod_{j=1}^{J}\frac{z-\xi}{z-z_j}.$$

Then G belongs to $\mathcal{H}_\infty(\Omega, \Gamma)$ and has an upper bound in Ω independent of Γ. Moreover, G does not vanish in U and has a lower bound at ζ independent of Γ. Harnack's inequality shows that G has a lower bound on all of K independent of Γ.

IV. The function $m(\Omega, \Gamma, \zeta)$

It is natural to ask (and important to know; see Section V, (D)) whether $m(\Omega, \Gamma, \zeta)$ is continuous as a function of Γ on the compact group $\pi(\Omega)^*$. That it is generally not continuous is easy to see. Let Ω be the unit disc with the origin removed, γ a circle described once counterclockwise surrounding the origin. If Γ_t with $0 \leqslant t < 1$ is determined by

$$\Gamma_t(\gamma) = e^{2\pi i t}$$

then it is a simple matter to show that $m(\Omega, \Gamma, \zeta) = |\zeta|^t$. But then

$$\Gamma_0 = \lim_{t \to 1-} \Gamma_t$$

while

$$m(\Omega, \Gamma_0, \zeta) \neq \lim_{t \to 1-} m(\Omega, \Gamma_t, \zeta).$$

Similarly one can show that for any Ω with an isolated boundary point (with the exception of the sphere minus a point) $m(\Omega, \Gamma, \zeta)$ is discontinuous. We have been unable to characterize those domains for which we have continuity. Here, however, is a class of such domains.

THEOREM 5. *Suppose Ω is the complement of a continuum K and let ζ_i be a sequence of points in Ω satisfying $\Sigma g(z, \zeta_i) < \infty$, where $g(z, \zeta)$ is Green's function for Ω. Let K_i ($i = 1, 2, \ldots$), be disjoint continua with $\zeta_i \in K_i$ such that all accumulation points of the K_i lie in K. Then*

$$m(\Omega - \bigcup K_i, \Gamma, \zeta)$$

is continuous in Γ.

Before getting into the proof of the theorem, let us see what happens when a domain may be written as the intersection of two domains Ω_1, Ω_2 whose union is the complement of a continuum. We have the Mayer–Vietoris sequence of homology groups [4, p. 189], of which a portion is the exact sequence

$$H_2(\Omega_1 \cup \Omega_2) \to H_1(\Omega_1 \cap \Omega_2) \to H_1(\Omega_1) \oplus H_1(\Omega_2) \to H_1(\Omega_1 \cup \Omega_2).$$

The groups at either end of the sequence vanish so we have

$$H_1(\Omega_1 \cap \Omega_2) \approx H_1(\Omega_1) \oplus H_1(\Omega_2)$$

from which we deduce

$$\pi(\Omega_1 \cap \Omega_2)^* \approx \pi(\Omega_1)^* \oplus \pi(\Omega_2)^*.$$

More generally we have

$$H_1(\Omega_1 \cap \ldots \cap \Omega_n) \approx H_1(\Omega_1) \oplus \ldots \oplus H_1(\Omega_n) \tag{12}$$

$$\pi(\Omega_1 \cap \ldots \cap \Omega_n)^* \approx \pi(\Omega_1)^* \oplus \ldots \oplus \pi(\Omega_n)^* \tag{13}$$

if each $\Omega_i \cup \Omega_j$ is the complement of a continuum, the same continuum for all pairs i, j. These isomorphisms will enable us to handle complicated domains built out of simpler ones.

LEMMA 5.1. *With the notation of the theorem, each* $m(\Omega - K_i, \Gamma, \zeta)$ *is continuous in* $\Gamma \in \pi(\Omega_i)^*$.

Proof. It is most convenient to think of $\Omega - K_i$ as an annulus $1 < |z| < R$ (a conformal mapping accomplishes this) with

$$K = \{z : |z| \leqslant 1\}, \quad K_i = \{z : |z| \geqslant R\}.$$

Denote by ω the harmonic function in $1 < |z| < R$ satisfying,

$$\lim_{|z| \to 1} \omega(z) = 1, \quad \lim_{|z| \to R} \omega(z) = 0.$$

Then with $g_i(z, \xi)$ the Green function for $1 < |z| < R$ and γ a circle $|z| = r (1 < r < R)$ described counterclockwise it is not hard to see that

$$\Delta_\gamma \tilde{g}_i(z, \xi) = \begin{cases} 2\pi\omega(\xi), & |\xi| > r \\ 2\pi(\omega(\xi) - 1), & |\xi| < r. \end{cases}$$

See, for example [7], p. 140. Therefore if

$$\Psi(z) = \exp\{-g_i(z, \xi) - i\tilde{g}_i(z, \xi)\}$$

then the character $\Gamma_\Psi \in \pi(\Omega - K_i)^*$ is determined by

$$\Gamma_\Psi(\gamma) = \exp\{-2\pi i \omega(\xi)\}.$$

As ξ runs along the real interval $(1, R)$, $\omega(\xi)$ decreases from 1 to 0 so $\Gamma_\Psi(\gamma)$ describes the unit circle (except for the point 1) and Γ_Ψ runs through all of $\pi(\Omega - K_i)^*$ (except for the identity character I). Now as $\Gamma_\Psi \to I$, that is as

$$\exp\{-2\pi i \omega(\xi)\} \to 1,$$

we have $|\xi| \to 1$ or R and so $g_i(z, \xi) \to 0$. This shows that for any ζ,

$$m(\Omega - K_i, \Gamma_\Psi, \zeta) \geqslant \exp\{-g_1(\zeta, \xi)\} \to 1 = m(\Omega - K_i, I, \zeta).$$

Since we always have $m \leqslant 1$ this shows that $m(\Omega - K_i, \Gamma, \zeta)$ is continuous at I.

This is enough to prove continuity everywhere. In fact, for an arbitrary domain Ω and any Γ, $\bar{\Gamma} \in \pi(\Omega)^*$ we have inequalities

$$\frac{m(\Omega, \bar{\Gamma}, \zeta)}{m(\Omega, \bar{\Gamma}/\Gamma, \zeta)} \geqslant m(\Omega, \Gamma, \zeta) \geqslant m(\Omega, \Gamma/\bar{\Gamma}, \zeta)\, m(\Omega, \bar{\Gamma}, \zeta).$$

These inequalities show that continuity of $m(\Omega, \Gamma, \zeta)$ at $\Gamma = I$ implies continuity at $\Gamma = \bar{\Gamma}$.

LEMMA 5.2. *Each* $m(\Omega - \bigcup_{i=1}^{n} K_i, \Gamma, \zeta)$ *is continuous in* Γ.

Proof. Write $\Omega_i = \Omega - K_i$. Then

$$\Omega - \bigcup K_i = \bigcap \Omega_i$$

and each $\Omega_i \cup \Omega_j$ is the complement of K, so we have the isomorphisms (12) and (13). The seconds shows that for each $\Gamma \in \pi(\bigcap \Omega_i)^*$ there are unique $\Gamma_i \in \pi(\Omega_i)^*$ such that

$$\Gamma(\gamma) = \Gamma_1(\gamma) \ldots \Gamma_n(\gamma)$$

for each cycle γ lying in $\bigcap \Omega_i$. Clearly

$$m(\bigcap \Omega_i, \Gamma, \zeta) \geqslant m(\Omega_1, \Gamma_1, \zeta) \ldots m(\Omega_n, \Gamma_n, \zeta). \tag{14}$$

It follows from (12) that any cycle γ_1 in Ω_1 is homologous in Ω_1 to a cycle γ in $\bigcap \Omega_i$ which is homologous to zero in each Ω_i with $i > 1$. Then

$$\Gamma(\gamma) = \Gamma_1(\gamma) \ldots \Gamma_n(\gamma) = \Gamma_1(\gamma_1).$$

This shows that if $\Gamma \to I$ in $\pi(\bigcap \Omega_i)^*$ then $\Gamma_1 \to I$ in $\pi(\Omega_1)^*$, and similarly for the other Γ_i. By Lemma 5.1 each $m(\Omega_i, \Gamma_i, \zeta) \to 1$ as $\Gamma_i \to I$. Hence by (14)

$$m(\bigcap \Omega_i, \Gamma, \zeta) \to 1$$

as $\Gamma \to I$ and the lemma is established.

We now complete the proof of the theorem. Write (the notation is different from the proof of Lemma 5.2)

$$\Omega_1 = \Omega - \bigcup_{i \leqslant n} K_i, \qquad \Omega_2 = \Omega - \bigcup_{i > n} K_i.$$

Then $\Omega_1 \cup \Omega_2 = \Omega$ and

$$\Omega_1 \cap \Omega_2 = \Omega - \bigcup_{i=1}^{\infty} K_i.$$

As in the proof of Lemma 5.2 any

$$\Gamma \in \pi(\Omega - \bigcup_{i=1}^{\infty} K_i)^*$$

gives rise to $\Gamma_1 \in \pi(\Omega_1)^*$ and $\Gamma_2 \in \pi(\Omega_2)^*$ and

$$m(\Omega - \bigcup_{i=1}^{\infty} K_i, \Gamma, \zeta) \geqslant m(\Omega_1, \Gamma_1, \zeta)\, m(\Omega_2, \Gamma_2, \zeta) \geqslant m(\Omega_1, \Gamma_1, \zeta)\, m(\Omega_2, \zeta).$$

Since

$$m(\Omega_2, \zeta) \geqslant m(\Omega - \{\zeta_i\}_{i>n}, \zeta)$$

(Ω_2 is a smaller domain but has the same fundamental group), Corollary 1.3 shows that

$$m(\Omega_2, \zeta) \geqslant \exp\{-\sum_{i=n+1}^{\infty} g(\zeta_i, \zeta)\}$$

which can be made as close to 1 as desired by choosing n large enough. Moreover, as in the proof of Lemma 5.2 we have $\Gamma_1 \to I$ in $\pi(\Omega_1)^*$ as $\Gamma \to I$. This shows

$$m(\Omega - \bigcup_{i=1}^{\infty} K_i, \Gamma, \zeta) \to 1$$

as $\Gamma \to I$ and the theorem is proved.

V. Applications

In this section we shall indicate some questions which lead naturally to the classes $\mathcal{H}_\infty(\Omega, \Gamma)$ and/or quantities $m(\Omega, \Gamma)$. Note that all these questions concern single-valued functions.

(A) Suppose K is a finite union of mutually exterior smooth Jordan curves. Let

$$M_n = \min_P \max_{z \in K} |P(z)|,$$

where the minimum is extended over all monic polynomials of degree n. (The extremal polynomial is the nth Tchebycheff polynomial associated with K.) The question is, how does M_n behave as $n \to \infty$?

Take Ω to be the domain exterior to all the components of K, with the point at infinity included in Ω. Then Theorem 8.3 of [7] gives the asymptotic formula

$$M_n \sim C(K)^n m(\Omega, \Gamma^{-n}, \infty)^{-1},$$

where $C(K)$ is the logarithmic capacity of K and

$$\Gamma = \Gamma_{\exp\{g(z,\infty) + i\tilde{g}(z,\infty)\}}.$$

Theorem 8.4 of [7] implies that for "almost all" K (it suffices that the harmonic measures at ∞ of its components be linearly independent over the rationals) the sequence

$$M_n C(K)^{-n}$$

has the interval

$$[1, m(\Omega, \infty)^{-1}]$$

as its set of limit points.

(B) It is known [3, p. 138] that a function f in $\mathcal{H}_\infty$ of the unit disc is an extreme point of the unit ball of $\mathcal{H}_\infty$ if and only if $|f(z)| \leqslant 1$ and

$$\int \log \{1 - |f(e^{i\theta})|\}\, d\theta = -\infty. \tag{15}$$

M. Voichick [6] found a generalization of this result to certain multiply-connected domains Ω. For these domains a function F belonging to the unit ball of $\mathcal{H}_\infty(\Omega)$ is an extreme point if and only if the function $f(z)$ obtained by lifting F to the universal covering surface of Ω (the unit disc) satisfies (15). An investigation of Voichick's proof shows that his result holds for any Ω for which all the $\mathcal{H}_\infty(\Omega, \Gamma)$ are nonempty. Voichick actually stated his theorem for the domains described in Theorem 5.

(C) The n-dimensional diameter, or n-width, of a symmetric subset S of a normed linear space E is defined as

$$d_n = \inf_{E_n} \sup_{x \in S} \operatorname{dist}(x, E_n),$$

where E_n runs over all n-dimensional subspaces of E. In case E is the space of continuous functions on a compact set K and S consists of those functions analytically continuable to a domain $\Omega \supset K$ and having absolute value at most 1 there, then d_n is generally to a first approximation $\exp\{-nC(K, \Omega)\}$ for large n [8, Theorems 2, 8]; here $C(K, \Omega)$ is the capacity of K relative to the kernel $g(z, \zeta)$, Green's function for Ω. The proof of Theorem 7 of [8] shows that there is a sharp inequality

$$d_n \geqslant a \exp\{-nC(K, \Omega)\}$$

as long as $m(\Omega, K) > 0$. By Theorem 4 this holds for all K if $m(\Omega) > 0$.

(D) S. Fisher [1] proved that for any of the domains described in Theorem 5, $\mathcal{H}_\infty$ is dense in $\mathcal{H}_p$ for $p \geqslant 1$. His argument is easily reformulated to fit into the present context.

A class larger than any $\mathcal{H}_p$ is the Nevanlinna class $\mathcal{N}$. We say that $F \in \mathcal{N}(\Omega)$ if F is single-valued and analytic in Ω and $\log^+ |F|$ has a harmonic majorant. ($F \in \mathcal{H}_p$ means that $|F|^p$ has a harmonic majorant.) $\mathcal{N}(\Omega)$ is easily seen to be a linear space; it becomes a Fréchet space if we define $d(F, G)$ to be the value at a fixed $\zeta \in \Omega$ of the smallest harmonic majorant of $\log(1 + |F - G|)$.

PROPOSITION. *If $m(\Omega, \Gamma, \zeta)$ is continuous as a function of Γ then $\mathcal{H}_\infty(\Omega)$ is dense in $\mathcal{N}(\Omega)$.*

Proof. For convenience we use the same notation for a function on Ω and its lifting to the universal covering surface of Ω, the unit disc. We may assume that $\zeta \in \Omega$ corresponds to the center of the disc, so

$$d(F, G) = \frac{1}{2\pi}\int \log\{1 + |F(e^{i\theta}) - G(e^{i\theta})|\}\, d\theta,$$

where $F(e^{i\theta})$, $G(e^{i\theta})$ are the a.e. defined boundary values of F and G.

Take an $F \in \mathcal{N}(\Omega)$, let u be the smallest harmonic majorant of $\log^+|F|$ and u_n the smallest harmonic majorant of $\min(\log^+|F|, n)$. Let

$$\Gamma_n = \Gamma_{\exp\{(u-u_n)+i(u-u_n)^\sim\}}$$

and find $F_n \in \mathcal{H}_\infty(\Omega, \Gamma_n)$ satisfying

$$|F_n(z)| \leqslant 1 \text{ for } z \in \Omega, \quad |F_n(\zeta)| = m(\Omega, \Gamma_n, \zeta).$$

Let
$$G_n = F \exp\{(u_n - u) + i(u_n - u)^\sim\} F_n.$$

Clearly $G_n \in \mathcal{H}_\infty(\Omega)$ and $|G_n| \leqslant |F|$. Since $u_n(e^{i\theta}) \to u(e^{i\theta})$ in $L_1(T)$,

$$\exp\{(u_n - u) + i(u_n - u)^\sim\} \to 1$$

in measure on T and uniformly on compact subsets of Ω. This latter implies that $\Gamma_n \to I$ in $\pi(\Omega)^*$. By the continuity of m we must have $|F_n(\zeta)| \to 1$, and since $|F_n(z)| \leqslant 1$ throughout Ω a simple argument shows that $F_n(e^{i\theta}) \to 1$ in measure on T. This is enough to give

$$\int \log\{1 + |G_n(e^{i\theta}) - F(e^{i\theta})|\} \to 0$$

(take subsequences and use the dominated convergence theorem) so that $G_n \to F$ in $\mathcal{N}(\Omega)$.

The same sort of argument shows that under the same assumption $\mathcal{H}_\infty(\Omega)$ is dense in each $\mathcal{H}_p(\Omega)$, but despite this the case of $\mathcal{H}_p(\Omega)$ is quite different from the case of $\mathcal{N}(\Omega)$. If one takes Ω to be the punctured disc $0 < |z| < 1$ then all the $\mathcal{H}_p(\Omega)$ are the same as $\mathcal{H}_p$ for the disc so $\mathcal{H}_\infty(\Omega)$ is dense in $\mathcal{H}_p(\Omega)$. But $\mathcal{H}_\infty(\Omega)$ is not dense in $\mathcal{N}(\Omega)$: Since convergence in $\mathcal{N}(\Omega)$ implies uniform convergence on compact subsets, and functions in $\mathcal{H}_\infty(\Omega)$ extend analytically to $z=0$, we must have

$$\max_{|z|=r_1} |F(z)| \leqslant \max_{|z|=r_2} |F(z)|$$

for $r_1 < r_2$ whenever F is in the closure of $\mathcal{H}_\infty(\Omega)$. In particular z^{-1} belongs to $\mathcal{N}(\Omega)$ but not to the closure of $\mathcal{H}_\infty(\Omega)$.

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