MODELS OF CERTAIN AUTOMORPHIC FUNCTION FIELDS

BY

KATSUYA MIYAKE

Courant Institute of Mathemathical Sciences, New York, N.Y., U.S.A.

Introduction

In his work [14] I, G. Shimura raised a question of the existence of a canonical system for a reductive $\mathbf{Q}$-group. Our purpose in this paper is to construct a canonical system for a reductive $\mathbf{Q}$-group which is obtained from a simple algebra over $\mathbf{Q}$ with a positive involution of the second kind.

To be more specific, let B be such a simple algebra over $\mathbf{Q}$, and take an involution δ of B which coincides with the positive one on the center of B. Then using a rational representation of B, we have a reductive $\mathbf{Q}$-group G and a semi-simple $\mathbf{Q}$-subgroup G^1 of G defined by

$$G_{\mathbf{Q}} = \{\alpha \in B^{\times} \mid \alpha\alpha^{\delta} = \nu(\alpha) \in \text{center of } B\};$$
$$G^1_{\mathbf{Q}} = \{\alpha \in B^{\times} \mid \alpha\alpha^{\delta} = 1 \text{ and } N(\alpha) = 1\}.$$

where N denotes the reduced norm of B over its center. The group G^1 has the following properties:

(BSD) *The homogeneous space*

$$\mathcal{H} = G^1_{\mathbf{R}}/(\text{a maximal compact subgroup})$$

defines a bounded symmetric domain;

(SL) $G^1_{\mathbf{C}}$ *is isomorphic to a direct product of copies of* $\mathbf{SL}(n, \mathbf{C})$.

It is known that an almost-simple algebraic $\mathbf{Q}$-group satisfying the conditions (BSD) and (SL) is isogenous (at least over $\mathbf{R}$) to our G^1 for suitable B and δ. For a somewhat more definite characterization of our group G^1, or of B and δ, see 1.1–2.

If $G^1_{\mathbf{R}}$ is compact, then $\mathcal{H}$ is the space consisting of only one point. Here we exclude this case.

It has been shown by Baily and Borel [3] that, for every arithmetic subgroup Γ of G^1, the quotient space $\Gamma\backslash\mathcal{H}$ is embeddable in a projective variety as a Zariski open subset. Our aim in this paper is, roughly speaking, to construct a model $(V_\Gamma, \varphi_\Gamma)$ of $\Gamma\backslash\mathcal{H}$ defined over an algebraic number field k_Γ of finite degree for every arithmetic congruence subgroup Γ of G^1, and to determine the relations between the models. Here we understand by a model of $\Gamma\backslash\mathcal{H}$ defined over k_Γ a couple $(V_\Gamma, \varphi_\Gamma)$ formed by a Zariski open subset V_Γ of a projective variety rational over k_Γ, and a holomorphic mapping φ_Γ of $\mathcal{H}$ onto V_Γ which induces a biregular isomorphism of $\Gamma\backslash\mathcal{H}$ onto V_Γ.

There are densely many special points called 'isolated fixed points' on $\mathcal{H}$. We determine certain very important properties of the point $\varphi_\Gamma(z)$ on V_Γ for every isolated fixed point z and every Γ, with which we can organize all of the model $(V_\Gamma, \varphi_\Gamma)$ of $\Gamma\backslash\mathcal{H}$ simultaneously in a canonical system. It is important to construct a canonical system, partly because it allows us to describe a group of automorphisms of a certain field of automorphic functions on $\mathcal{H}$ with respect to the arithmetic congruence subgroups of G^1 by a certain subgroup of the adelization $G_{\mathbf{A}}$ of the reductive group G, as was done by Shimura in [14] I and II.

A canonical system depends on the choice of the complex structure of $\mathcal{H}$ that makes $\mathcal{H}$ a hermitian symmetric space. Let us fix such a complex structure of $\mathcal{H}$. Then there is an algebraic number field K' of finite degree such that the field k_Γ indicated above is a finite abelian extension of K' for every Γ. We construct K' starting from the center K of B, and choose a certain (infinite) abelian extension $\mathfrak{K}$ of K' which contains every k_Γ. Actually $\mathfrak{K}$ contains the maximal abelian extension $\mathbf{Q}_{ab}$ of $\mathbf{Q}$. Now let G_∞ (resp. $G_\mathfrak{f}$) be the archimedian (resp. non-archimedian) part of $G_{\mathbf{A}}$. We identify G_∞ with $G_{\mathbf{R}}$, denote the connected component of the identity element of $G_{\mathbf{R}}$ by $G_{\mathbf{R}+}$, and put $G_{\mathbf{Q}+}=G_{\mathbf{Q}}\cap G_\mathfrak{f}G_{\mathbf{R}+}$. We take a certain closed subgroup $\mathcal{G}_+$ of $G_{\mathbf{A}}$ which contains $G_{\mathbf{Q}+}$ and $G^1_{\mathbf{A}}$, the adelization of G^1, and define an open, continuous and surjective homomorphism σ of $\mathcal{G}_+$ to Gal $(\mathfrak{K}/K')$, whose kernel is $G_{\mathbf{Q}+}G^1_{\mathbf{A}}K^{\#}G_{\mathbf{R}+}$. Here $K^{\#}$ is the closure of $K^\times K^\times_\infty$ in the idele group $K^\times_{\mathbf{A}}$ of K. The center $K^\times$ of $G_{\mathbf{Q}+}$ coincides with $G_{\mathbf{Q}+}\cap K^{\#}G_{\mathbf{R}+}$. We put $\mathfrak{A}^0=\mathcal{G}_+/K^{\#}G_{\mathbf{R}+}$. The projection of the subgroup $G_{\mathbf{Q}+}$ of $\mathcal{G}_+$ to $\mathfrak{A}^0$ is naturally identified with the quotient group $A^0_+=G_{\mathbf{Q}+}/K^\times$. We denote the open, continuous and surjective homomorphism of $\mathfrak{A}^0$ to Gal $(\mathfrak{K}/K')$ induced by σ again by σ. Then for any open compact subgroup X of $\mathfrak{A}^0$, we have a properly discontinuous group $\Gamma_X=X\cap A^0_+$ of transformations on $\mathcal{H}$ on the one hand, and, on the other, a finite abelian extension k_X of K' contained in $\mathfrak{K}$, which is determined by the open subgroup $\sigma(X)$ of Gal $(\mathfrak{K}/K')$. Let us denote the family of all the open compact subgroups of $\mathfrak{A}^0$ by $\mathfrak{Z}$. Then the family $\{\Gamma_X | X\in\mathfrak{Z}\}$ covers every arithmetic congruence subgroup of G^1 as a group of transformations on $\mathcal{H}$.

Now our main theorem states:

There exists a system $\{V_X, \varphi_X, J_{YX}(u), (X, Y \in \mathfrak{Z};\ u \in \mathfrak{A}^0)\}$ *consisting of the objects satisfying the following conditions.*

(I) *For each* $X \in \mathfrak{Z}$, *the couple* (V_X, φ_X) *is a model of* $\Gamma_X \backslash \mathcal{H}$.

(II) V_X *is rational over* k_X.

(III) *For* $u \in \mathfrak{A}^0$, $J_{YX}(u)$ *is a morphism of* V_X *onto* $V_Y^{\sigma(u)}$, *which is defined if and only if* $uXu^{-1} \subset Y$, *is rational over* k_X, *and has the following properties;*

(IIIa) $J_{XX}(u)$ *is the identity mapping of* V_X *if* $u \in X$;

(IIIb) $J_{YX}(u)^{\sigma(t)} \circ J_{XW}(t) = J_{YW}(ut)$;

(IIIc) $J_{YX}(\alpha)[\varphi_X(z)] = \varphi_Y(\alpha(z))$ *for every* $\alpha \in A_+^0$ *and every* $z \in \mathcal{H}$ *if* $\alpha X \alpha^{-1} = Y$.

(IV) *A certain reciprocity law holds at every isolated fixed point on* $\mathcal{H}$.

This is a somewhat weaker statement than what we shall actually prove. In the text, we shall take an extension $\mathfrak{A}$ of $\mathfrak{A}^0$ by a certain (finite) group of automorphisms of the center K of the algebra B in place of $\mathfrak{A}^0$.

Now let us make the property (IV) clear. For a point z of $\mathcal{H}$, put $\mathfrak{S}(z) = \{\gamma \in G_{\mathbf{Q}+} \mid \gamma(z) = z\}$. We say that z is an isolated fixed point on $\mathcal{H}$ if z is the only point on $\mathcal{H}$ that is fixed by every γ of $\mathfrak{S}(z)$. At an isolated fixed point z on $\mathcal{H}$, we have a finite algebraic extension $P(z)'$ of K' and a homomorphism η_z of $P(z)'^{\times}$ to $\mathfrak{S}(z)$, with which we construct a continuous homomorphism η_z^* of the idele group $P(z)'^{\times}_{\mathbf{A}}$ of $P(z)'$ to $\mathfrak{A}^0$. Let us denote the maximal abelian extension of $P(z)'$ by $P(z)'_{\mathbf{ab}}$ and the canonical homomorphism of $P(z)'^{\times}_{\mathbf{A}}$ onto $\mathrm{Gal}(P(z)'_{\mathbf{ab}}/P(z)')$ by $[\cdot, P(z)']$. Then (IV) can be stated as follows:

For every $X \in \mathfrak{Z}$, *the point* $\varphi_X(z)$ *on* V_X *is rational over* $P(z)'_{\mathbf{ab}}$ *and*

$$\varphi_Y(z)^{[a, P(z)']} = J_{YX}(\eta_z^*(a)^{-1})\,[\varphi_X(z)]$$

for every $a \in P(z)'^{\times}_{\mathbf{A}}$ *where* $Y = \eta_z^*(a)^{-1} X \eta_z^*(a)$.

Again this is a weaker statement than what we shall prove.

By the class field theory, every finite abelian extension of $P(z)'$ corresponds to an open subgroup of the idele group $P(z)'^{\times}_{\mathbf{A}}$ containing $P(z)'^{\times}$. Now we have:

The open subgroup of $P(z)'^{\times}_{\mathbf{A}}$ *corresponding to the finite abelian extension* $P(z)' k_Y(\varphi_Y(z))$ *of* $P(z)'$ *is*

$$P(z)'^{\times}\{a \in P(z)'^{\times}_{\mathbf{A}} \mid \eta_z^*(a) \in \pi(\mathfrak{S}(z))\, Y\}.$$

It should be noted that, for any given finite algebraic extension L of K', there exists an isolated fixed point z on $\mathcal{H}$ such that the field $P(z)'$ determined by z is linearly disjoint with L over K'.

The main theorem allows us to describe $\mathfrak{A}^0 = \mathcal{G}_+/K^{\#}G_{\mathbf{R}+}$ as a group of automorphisms of a field $\mathfrak{L}$ of automorphic functions on $\mathcal{H}$. Let $\mathfrak{L}_X$ be the field of all the meromorphic functions on $\mathcal{H}$ of the form, $f\circ\varphi_X$, with some rational function f of V_X defined on k_X, and put

$$\mathfrak{L} = \bigcup_{X\in\mathfrak{Z}}\mathfrak{L}_X.$$

For $u\in\mathfrak{A}^0$, define a mapping $\varrho(u)$ of $\mathfrak{L}$ onto itself by

$$(f\circ\varphi_X)^{\varrho(u)} = f^{\sigma(u)}\circ J_{XW}(u)\circ\varphi_W$$

for $f\circ\varphi_X \in L_X \subset \mathfrak{L}$, where $W = u^{-1}Xu$. We furnish $\mathrm{Aut}\,(\mathfrak{L}/K')$ with the topology defined by taking all the subgroups of the form,

$$\{\tau\in \mathrm{Aut}\,(\mathfrak{L}/K')\,|\,h_1^\tau = h_1, \ldots, h_n^\tau = h_n\}$$

for a finite subset $\{h_1, ..., h_n\}$ of $\mathfrak{L}$ as a basis of the neighbourhoods of the identity element. Then we have:

The mapping ϱ is an open, continuous and injective homomorphism of $\mathfrak{A}^0$ to Aut $(\mathfrak{L}/K')$, *and has the following properties:*

(i) $\varrho(u)=\sigma(u)$ *on* $\mathfrak{K}$ *for* $u\in\mathfrak{A}^0$;

(ii) $K' = \{h\in\mathfrak{L}\,|\,h^{\varrho(u)} = h$ *for every* $u\in\mathfrak{A}^0\}$;

(iii) $\varrho(X) = \mathrm{Gal}\,(\mathfrak{L}/\mathfrak{L}_X)$ *for* $X\in\mathfrak{Z}$;

(iv) $h^{\varrho(\alpha)}(z) = h(\alpha(z))$ *for* $\alpha\in A_+^0$, $h\in\mathfrak{L}$ *and* $z\in\mathcal{H}$;

(v) *Let z be an isolated fixed point on $\mathcal{H}$, and $P(z)'$ and η_z^* as above. Then $h(z)$ is rational over $P(z)'_{\mathrm{ab}}$ for every $h\in\mathfrak{L}$ defined at z. Furthermore, if we put $\tau=[a, P(z)']$ and $u=\eta_z^*(a)^{-1}$ for $a\in P(z)'^{\times}_{\mathbf{A}}$, then $h^{\varrho(u)}$ is defined at z if h is so, and $h(z)^\tau = h^{\varrho(u)}(z)$.*

(vi) *If $\Gamma_X\backslash\mathcal{H}$ is compact for some $X\in\mathfrak{Z}$, then $\varrho(\mathfrak{A}^0)$ is a subgroup of* Aut $(\mathfrak{L}/K')$ *of finite index.*

As for (vi), we shall give stronger and more precise results in the text.

After certain reduction processes, the proof of our main theorem will be done with the help of the theory of modulus-varieties of abelian varieties, which was developed by Shimura in his works [9], [10] and [11].

This paper is based on the author's doctoral dissertation submitted to Princeton University in 1969. I should like to express my deep gratitude to my teacher, Professor G. Shimura, for his guidance.

Contents

Notation and terminology

As is usual, $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$ denote respectively the ring of rational integers, and the fields of rational, real, and complex numbers. The ring of $n \times n$ matrices with entries in $\mathbf{C}$ is denoted by $\mathbf{M}(n, \mathbf{C})$, and the groups of the elements of $\mathbf{M}(n, \mathbf{C})$ with non-zero determinants and with determinant 1 respectively by $\mathbf{GL}(n, \mathbf{C})$ and by $\mathbf{SL}(n, \mathbf{C})$. The identity element of $\mathbf{GL}(n, \mathbf{C})$ is written as $\mathbf{1}_n$.

For an algebraic matrix group G defined over $\mathbf{Q}$, $G_{\mathbf{A}}$ denotes the adelization of G (over $\mathbf{Q}$), $G_{\mathfrak{f}}$ the non-archimedian (or finite) part of $G_{\mathbf{Q}}$, G_∞ the archimedian (or infinite) part of $G_{\mathbf{A}}$, so that we have $G_{\mathbf{A}} = G_{\mathfrak{f}} G_\infty$, and $G_{\infty+}$ the connected component of 1 of G_∞. We put $G_{\mathbf{A}+} = G_{\mathfrak{f}} G_{\infty+}$. By $G_{\mathbf{Q}}$, $G_{\mathbf{R}}$, and $G_{\mathbf{C}}$, we denote respectively the groups of the elements of G rational over $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$. Naturally $G_{\mathbf{R}}$ is identified with G_∞, and so, $G_{\mathbf{R}+} = G_{\infty+}$ is the connected component of 1 of $G_{\mathbf{R}}$. The group $G_{\mathbf{Q}}$ is diagonally embedded in $G_{\mathbf{A}}$ as a discrete subgroup. We put $G_{\mathbf{Q}+} = G_{\mathbf{Q}} \cap G_{\mathbf{A}+}$. In sections 1 and 2 where the adelizations does not appear, we identify $G_{\mathbf{Q}}$ with its projection to $G_\infty = G_{\mathbf{R}}$, and consider it as a subgroup of $G_{\mathbf{R}}$, so that $G_{\mathbf{Q}+} = G_{\mathbf{Q}} \cap G_{\mathbf{R}+}$.

If P is an algebraic number field of finite degree, we regard the multiplicative group $P^\times$ of P as an algebraic linear group defined over $\mathbf{Q}$ (for example, by means of the regular representation), and use the notation $P_{\mathbf{A}}^\times$, $P_{\mathfrak{f}}^\times$, $P_\infty^\times$, $P_{\infty+}^\times$ and $P_{\mathbf{A}+}^\times$. But we simply write $P^\times$ for $P_{\mathbf{Q}}^\times$. Therefore $P_+^\times$ is the group of totally positive elements of P. We say that an idele $a \in P_{\mathbf{A}}^\times$ is totally positive if $a \in P_{\mathbf{A}+}^\times$. For a Galois extension Q of P, the Galois group of Q over P is denoted by Gal (Q/P). After Weil [17], we understand by P_{ab} the maximal abelian extension of P in its algebraic closure. For $a \in P_{\mathbf{A}}^\times$, we denote by $[a, P]$ the image of a in Gal (P_{ab}/P) under the canonical homomorphism of class field theory. The closure of $P^\times P_{\infty+}^\times$ in $P_{\mathbf{A}}^\times$ is written as $P^{\neq}$. The canonical homomorphism $[\cdot, P]$ induces the isomorphism of $P_{\mathbf{A}}^\times / P^{\neq}$ onto Gal (P_{ab}/P).

For a simple algebra B over $\mathbf{Q}$, $B^\times$ denotes the multiplicative group of the invertible elements of B, and is sometimes considered as an algebraic linear group. An involution δ of B is an anti-automorphism of B of order 2, i.e. a $\mathbf{Q}$-linear automorphism of B as a vector space over $\mathbf{Q}$ such that $(ab)^\delta = b^\delta a^\delta$ and $(a^\delta)^\delta = a$ for $a, b \in B$. An involution of B is of *the first kind* or of *the second kind* according as it is trivial or not on the center of B. A *positive involution* δ of B is an involution of B such that $\mathrm{tr}\,(aa^\delta) > 0$ for $a \in B$ unless $a = 0$, where tr denotes the reduced trace of B over $\mathbf{Q}$.

1. Algebras with positive involutions of the second kind and bounded symmetric domains

1.1. Let B be a simple algebra over $\mathbf{Q}$ with an involution δ. We include the case that δ is the identity mapping. Taking a rational representation of B, we define a simply connected $\mathbf{Q}$-almost simple algebraic matrix group G^1 by

$$G^1_{\mathbf{Q}} = \{\gamma \in B^\times \mid N(\gamma) = 1, \text{ and } \gamma\gamma^\delta = 1 \text{ if } \delta \neq \text{identity}\},$$

where N denotes the reduced norm of B over its center.

It is known that every $\mathbf{Q}$-almost simple algebraic linear group is isogenous over $\mathbf{Q}$ to such G^1 with a suitable B and δ, excluding certain exceptional cases. (See Weil [16], and J. Tits, *Classification of algebraic semi-simple groups*, [1], pp. 33–62.)

Let us consider the condition:

(BSD) *The homogeneous space*

$$\mathcal{H} = G^1_{\mathbf{R}+}/(\text{a maximal compact subgroup})$$

defines a bounded symmetric domain.

Put $B_{\mathbf{R}} = B \otimes_{\mathbf{Q}} \mathbf{R}$, and extend δ $\mathbf{R}$-linearly to the involution of $B_{\mathbf{R}}$. The reduced norm N can also be extended to a multiplicative mapping of $B_{\mathbf{R}}$ to its center in the natural way. Then the group $G^1_{\mathbf{R}}$ is considered as

$$G^1_{\mathbf{R}} = \{x \in B^\times_{\mathbf{R}} \mid N(x) = 1, \text{ and } xx^\delta = 1 \text{ if } \delta \neq \text{identity}\}.$$

Hence the condition (BSD) is considered as a condition on B and δ. Let $B_{\mathbf{R}} = B_1 \oplus \dots \oplus B_g$ be the decomposition of the semi-simple algebra $B_{\mathbf{R}}$ over $\mathbf{R}$ into the direct sum of its simple components B_λ, $\lambda = 1, \dots, g$. Since δ is involutive, either δ maps a simple component onto itself, or permutes two isomorphic simple components. As is well known, it is necessary and sufficient for the condition (BSD) to be satisfied that the simple components B_λ with δ belong to the following cases:

(I) $B_\lambda \cong \mathbf{M}(2, \mathbf{R})$, and δ is the identity mapping;

(II) $B_\lambda \cong B_\mu \cong \mathbf{M}(2, \mathbf{R})$, and δ permutes B_λ and B_μ ($\lambda \neq \mu$);

(III) $B_\lambda \cong \mathbf{M}(n, \mathbf{C})$, and $x^\delta = h\,{}^t\bar{x}h^{-1}$ for $x \in B_\lambda$ where $h \in \mathbf{GL}(n, \mathbf{C})$ and ${}^t\bar{h} = h$;

(IV) $B_\lambda \cong \mathbf{M}(n, \mathbf{R})$, and $x^\delta = h\,{}^t x h^{-1}$ for $x \in B_\lambda$ where $h \in \mathbf{GL}(n, \mathbf{R})$, ${}^t h = h$ and the signature of h is $(n, 0)$, $(0, n)$, $(n-2, 2)$, or $(2, n-2)$;

(V) $B_\lambda \cong \mathbf{M}(n, \mathbf{R})$, and $x^\delta = h\,{}^t x h^{-1}$ for $x \in B_\lambda$ where $h \in \mathbf{GL}(n, \mathbf{R})$ and ${}^t h = -h$;

(VI) $B_\lambda \cong \mathbf{M}(n, \mathbf{H})$, and $x^\delta = h\,{}^t\bar{x}h^{-1}$ for $x \in B_\lambda$ where $\bar{\ }$ is the main involution of the Hamilton quaternion algebra $\mathbf{H}$, $h \in \mathbf{GL}(n, \mathbf{H})$, ${}^t\bar{h} = h$ and $\mathrm{tr}_{\mathbf{H}/\mathbf{R}}(wh\,{}^t\bar{w}) > 0$ for every non-zero n-dimensional row vector w with components in $\mathbf{H}$;

(VII) $B_\lambda \cong M(n, \mathbf{H})$, and $x^\delta = h\,{}^t\bar{x}h^{-1}$ for $x \in B_\lambda$ where $\bar{\ }$ is the main involution of the Hamilton quaternion algebra $\mathbf{H}$, $h \in \mathbf{GL}(n, \mathbf{H})$ and ${}^t\bar{h} = -h$.

1.2. A simple algebra over $\mathbf{Q}$ with a positive involution of the second kind is characterized as follows.

PROPOSITION 1. *Let B be a simple algebra over $\mathbf{Q}$ with an involution δ. Then the following three assertions are equivalent.*

(i) *The condition (BSD) is satisfied, and B and δ involve only* Case (III) *of* 1.1*;*

(ii) *B has a positive involution of the second kind which coincides with δ on the center of B;*

(iii) *B is a central simple algebra over a CM-field, and δ is an involution of the second kind which is the complex conjugation on the center of B.*

Here a CM-field is a totally imaginary quadratic extension of a totally real algebraic number field of finite degree.

The equivalence of (i) and (ii) is easily seen. For the proof of the equivalence of (ii) and (iii), see 1.2 and 1.4 of Shimura [9] and [13] respectively.

Note that, if B has a positive involution of the second kind, then B is a central simple algebra over a CM-field, and the involution coincides with the complex conjugation on the center of B.

Now let B be a simple algebra over $\mathbf{Q}$, and δ an involution of B of the second kind. If B and δ satisfy the condition (BSD), then B and δ involve only Cases (II) and (III) of 1.1. As was seen above, the algebra B must have a positive involution which coincides with δ on the center of B, if only Case (III) is involved. If Case (II) is involved, then B has no positive involution of the second kind. But we may say that this case is covered by Shimura, in the sense indicated in the following observation, since B is now a quaternion algebra over its center.

Let B be a quaternion algebra over an algebraic number field K of finite degree. We include the case where $B = \mathbf{M}(2, K)$. Suppose that B has an involution δ of the second kind, and that B and δ satisfy the condition (BSD). Then we see easily by the list of 1.1 that the center K of B is a quadratic extension of a totally real algebraic number field F which consists of all the elements of K fixed by δ. Let G and G^1 be the algebraic $\mathbf{Q}$-groups defined by

$$G_{\mathbf{Q}} = \{\alpha \in B^\times \mid \alpha\alpha^\delta = \nu(\alpha) \in F^\times\},$$
$$G^1_{\mathbf{Q}} = \{\alpha \in B^\times \mid \alpha\alpha^\delta = 1 \text{ and } N(\alpha) = 1\},$$

where N denotes the reduced norm of B over K. Let ι denote the main involution of B, and put

$$B_0 = \{\alpha \in B \mid \alpha^\delta = \alpha^\iota\}. \tag{1.2.1}$$

Then B_0 is a subalgebra of B which contains F but does not contain K. Obviously, B_0 is stable under both δ and ι. Since $\alpha\alpha^\iota = N(\alpha)$ for $\alpha \in B$, we have

$$G^1_{\mathbf{Q}} = \{\beta \in B_0^\times \mid \beta\beta^\iota = 1\}. \tag{1.2.2}$$

(Note that $\beta^\delta = \beta^{-1} = \beta^\iota$ for $\beta \in G^1_{\mathbf{Q}}$.) Therefore, especially, B_0 is a non-commutative algebra over F, and hence, must be a quaternion algebra over F such that

$$B = B_0 \otimes_F K. \tag{1.2.3}$$

Now define a $\mathbf{Q}$-subgroup G_0 of G by

$$G_{0\mathbf{Q}} = B_0^\times = \{\beta \in B_0^\times \mid \beta\beta^\delta = \beta\beta^\iota = N(\beta) \in F^\times\}. \tag{1.2.4}$$

It is obvious that G_0 is a reductive $\mathbf{Q}$-group containing G^1. Let us show that

$$G_{\mathbf{Q}} = K^\times G_{0\mathbf{Q}}. \tag{1.2.5}$$

Take an element $\zeta \in K^\times$ such that $\zeta^\delta = -\zeta$. Then we have $B = B_0 + \zeta B_0$. For $\alpha \in G_{\mathbf{Q}}$, choose elements a and b of B_0 so that $\alpha = a + \zeta b$. Since $\alpha\alpha^\delta = (a+\zeta b)(a^\iota - \zeta b^\iota) = aa^\iota - \zeta^2 bb^\iota + \zeta(ba^\iota - ab^\iota) = N(a) - \zeta^2 N(b) + \zeta(ba^\iota - ab^\iota)$ and it belongs to $F^\times$, we have $ba^\iota - ab^\iota = 0$. Then $ba^\iota = \frac{1}{2}(ba^\iota + ab^\iota) = \frac{1}{2}(ba^\iota + (ba^\iota)^\iota) = \frac{1}{2}\operatorname{tr}(ba^\iota)$ and it is an element of F. Here tr denotes the reduced trace of B over K. Put $c = ba^\iota \in F$. Then $ca = ba^\iota a = bN(a)$. If $N(a) = 0$, then we have $c = 0$ or $a = 0$ since $c \in F$. Furthermore, b must be invertible since $N(b) = -\zeta^{-2}\alpha\alpha^\delta \in F^\times$. Therefore the relation $c = ba^\iota$ implies that $c = 0$ if and only if $a = 0$. Hence, anyway, we have $\alpha = \zeta b$ with $\zeta \in K^\times$ and $b \in B_0^\times$ if $N(a) = 0$. If $N(a) \neq 0$, then $b = cN(a)^{-1}a$, and $\alpha = a + \zeta b = (1 + cN(a)^{-1}\zeta)a$ with $1 + cN(a)^{-1}\zeta \in K^\times$ and $a \in B_0^\times$. This shows that $G_{\mathbf{Q}} \subset K^\times G_{0\mathbf{Q}}$. Since the opposite inclusion is trivial, we get (1.2.5).

It can be shown that the subalgebra B_0 of B is spanned by the elements of $G^1_{\mathbf{Q}}$ over $\mathbf{Q}$. Note that B_0 depends essentially upon δ, and is not determined by the condition (1.2.3).

After the structures of B with δ, G and G^1 are thus clarified, this case is reduced to the case of a quaternion algebra B_0 over a totally real algebraic number field F of finite degree with the reductive group G_0 and the semi-simple group G^1 defined by (1.2.4) and (1.2.2) respectively, which is included, as a special case, in Shimura [14] I and II, where B with δ involving Cases (V) and (VI) of 1.1 was studied.

If K is a totally imaginary quadratic extension of F (with the above notation and assumption), then the quaternion algebra B over K has a positive involution of the second kind, and is automatically included in our case of this paper. But, in general, our results for such B (or G and G^1) in this paper are weaker than those of Shimura for B_0 (or G_0 and G^1).

If Case (I) of 1.1 is involved, then B is a quaternion algebra over a totally real algebraic number field of finite degree (if the condition (BSD) is satisfied), and hence, this case is also covered by Shimura, as is easily seen.

1.3. Hereafter we restrict ourselves to the case that B and δ involve only Case (III) of 1.1, and fix the notation as follows:

Let K be the center of B, $F=\{a\in K\,|\,a^\delta=a\}$ and D the central division algebra over K such that $B=\mathbf{M}(m, D)$ for some positive integer m. As we saw, F is a totally real algebraic number field of finite degree, and K is a totally imaginary quadratic extension of F. Put $g=[F\colon \mathbf{Q}]$ and $q^2=[D\colon K]$, $q>0$. Let D^m be the space of all m-dimensional row vectors with components in D. Then D^m is a left D- and right B-module with the canonical action of D and $B=\mathbf{M}(m, D)$. Representing B in $\mathbf{M}(2gmq^2, \mathbf{Q})$ by a fixed basis of D^m over $\mathbf{Q}$, we take an algebraic linear group G^1 so that

$$G^1_{\mathbf{Q}}=\{\gamma\in B^\times\,|\,\gamma\gamma^\delta=1 \text{ and } N(\gamma)=1\}$$

where N denotes the reduced norm of B over K. Define a reductive $\mathbf{Q}$-group G by

$$G_{\mathbf{Q}}=\{\gamma\in B^\times\,|\,\gamma\gamma^\delta=\nu(\gamma)\in F^\times\}.$$

Put $B_{\mathbf{R}}=B\otimes_{\mathbf{Q}}\mathbf{R}$ and extend δ $\mathbf{R}$-linearly to the involution of $B_{\mathbf{R}}$. The reduced norm N is also extended to a multiplicative mapping of $B_{\mathbf{R}}$ to its center. Then the groups $G^1_{\mathbf{R}}$ and $G_{\mathbf{R}}$ are regarded as

$$G^1_{\mathbf{R}}=\{x\in B^\times_{\mathbf{R}}\,|\,xx^\delta=1 \text{ and } N(x)=1\};$$

$$G_{\mathbf{R}}=\{x\in B^\times_{\mathbf{R}}\,|\,xx^\delta=\nu(x)\in\text{center of } B_{\mathbf{R}}\}.$$

Note that $G^1_{\mathbf{R}}$ is connected.

1.4. Fix a maximal compact subgroup M_0 of $G^\times_{\mathbf{R}}$ and put

$$\mathcal{H}=G_{\mathbf{R}}/M_0;$$
$$\mathfrak{M}=\mathfrak{M}_B=\{\text{all maximal compact subgroups of } G^1_{\mathbf{R}}\}.$$

As is well known, $\mathcal{M}=\{xM_0x^{-1}\,|\,x\in G^1_{\mathbf{R}}\}$. Assigning to xM_0x^{-1} the coset xM_0 of M_0 in $G^1_{\mathbf{R}}$, we have a one-to-one correspondence between $\mathcal{M}$ and the homogeneous space $\mathcal{H}$.

For $j\in B_{\mathbf{R}}$ such that $j^\delta=-j$ and $j^2=-1$, we define a subgroup $M(j)$ of $G^1_{\mathbf{R}}$ by

$$M(j)=M_B(j)=\{u\in G^1_{\mathbf{R}}\,|\,uj=ju\},$$

and put

$$\mathcal{J}=\mathcal{J}_B=\{j\in B_{\mathbf{R}}\,|\,j^\delta=-j,\ j^2=-1\quad\text{and}\quad M(j)\in\mathcal{M}\}.$$

Then for $j\in\mathcal{J}$, we have $xjx^{-1}\in\mathcal{J}$ and $M(xjx^{-1})=xM(j)x^{-1}$ for every $x\in G^1_{\mathbf{R}}$. Fix an element j_0 of $\mathcal{J}$, and put

$$\mathcal{J}(j_0)=\mathcal{J}_B(j_0)=\{xj_0x^{-1}\,|\,x\in G^1_{\mathbf{R}}\}.$$

Let ε be an element of the center of $B_{\mathbf{R}}$ such that $\varepsilon^2=1$. Then we see easily that $\varepsilon^\delta=\varepsilon$. Therefore $j\varepsilon\in\mathcal{J}$ and $M(j)=M(j\varepsilon)$ for every $j\in\mathcal{J}$. Put

$$\mathcal{E}=\mathcal{E}_B=\{\varepsilon\,|\,\varepsilon\in\text{center of } B_{\mathbf{R}} \text{ and } \varepsilon^2=1\}.$$

Obviously $\mathcal{E}$ is a multiplicative group of order 2^g where $g=[F:\mathbf{Q}]$.

PROPOSITION 2. *For every $M\in\mathcal{M}$, here exists an element $j\in\mathcal{J}$ such that $M(j)=M$. For $j, j'\in\mathcal{J}$, $M(j)=M(j')$ if and only if $j'=j\varepsilon$ for some $\varepsilon\in\mathcal{E}$. For any fixed $j_0\in\mathcal{J}$,*

$$\mathcal{J}=\textstyle\bigcup_{\varepsilon\in\mathcal{E}}\mathcal{J}(j_0\varepsilon)\quad\text{(disjoint)},$$

and the correspondence, $j\leftrightarrow M(j)$, between $\mathcal{J}(j_0\varepsilon)$ and $\mathcal{M}$ is one-to-one for each $\varepsilon\in\mathcal{E}$.

Proof. Since F is a totally real field of degree g and K is a totally imaginary quadratic extension of F, we have $B_{\mathbf{R}}=B_1\oplus\dots\oplus B_g$ where B_λ $(\lambda=1,\dots,g)$ is a simple algebra over $\mathbf{R}$ and $\mathbf{R}$-linearly isomorphic to $\mathbf{M}(mq,\mathbf{C})$. In our case, we have $B_\lambda^\delta=B_\lambda$ for $\lambda=1,\dots,g$. Let $1=\iota_1+\dots+\iota_g$, $\iota_\lambda\in B_\lambda$, be the decomposition of 1, and put

$$G^1_\lambda=\{x\in B_\lambda\,|\,xx^\delta=\iota_\lambda\quad\text{and}\quad N(x)=\iota_\lambda\}.$$

Then $G^1_{\mathbf{R}}=G^1_1\times\dots\times G^1_g$ (direct product). Moreover any maximal compact subgroup M of $G^1_{\mathbf{R}}$ is of the form $M=M_1\times\dots\times M_g$ with some maximal compact subgroups M_λ of G^1_λ, $\lambda=1,\dots,g$. Now fix one λ for a while. We know that there is an isomorphism ($\mathbf{R}$-linear) of B_λ to $\mathbf{M}(mq,\mathbf{C})$ which transform δ to the involution $X\to H^t\bar{X}H^{-1}$ of $\mathbf{M}(mq,\mathbf{C})$ with $H={}^t\bar{H}\in\mathbf{GL}(mq,\mathbf{C})$. Since H is hermitian, there is an element $Y\in\mathbf{GL}(mq,\mathbf{C})$ such that

$$YH^t\bar{Y}=J_{r,s}=\begin{bmatrix}1_r & 0\\ 0 & -1_s\end{bmatrix}$$

with some non-negative integers $r=r(\lambda)$ and $s=s(\lambda)$ such that $r+s=mq$. Twisting the isomorphism of B_λ to $\mathbf{M}(mq, \mathbf{C})$ by the inner automorphism of $\mathbf{M}(mq, \mathbf{C})$ defined by Y, we have an $\mathbf{R}$-linear isomorphism ω_λ of B_λ onto $\mathbf{M}(mq, \mathbf{C})$ such that $\omega_\lambda(x^\delta)=J_{r,s}{}^t\overline{\omega_\lambda(x)}J_{r,s}$ for $x\in B_\lambda$. Since $\omega_\lambda(N(x))=\det(\omega_\lambda(x))$, G^1_λ is isomorphic to

$$\mathbf{SU}(r, s)=\{U\in\mathbf{GL}(mq, \mathbf{C})\,|\,UJ_{r,s}{}^t\bar{U}=J_{r,s} \text{ and } \det(U)=1\}$$

through ω_λ. Put $J_0=\sqrt{-1}J_{r,s}$. Then $M'(J_0)=\{U\in\mathbf{SU}(r,s)\,|\,UJ_0=J_0U\}$ is a maximal compact subgroup of $\mathbf{SU}(r, s)$. It is well known and easily seen that, for $J\in\mathbf{GL}(mq, \mathbf{C})$ such that $J^2=-1_{mq}$ and $J_{r,s}{}^t\bar{J}J_{r,s}=-J$, the group $M'(J)=\{U\in\mathbf{SU}(r,s)\,|\,UJ=JU\}$ is a maximal compact subgroup of $\mathbf{SU}(r, s)$ if and only if either $J=XJ_0X^{-1}$ or $J=X(-J_0)X^{-1}$ for some $X\in\mathbf{SU}(r,s)$. Moreover two such J and J' define the same maximal compact subgroup $M'(J)=M'(J')$ of $\mathbf{SU}(r, s)$ if and only if either $J=J'$ or $J=-J'$. Combining these results for $\lambda=1, ..., g$, we get the proposition at once.

COROLLARY 1. *The set of pairs of non-negative integers* $\{(r(\lambda), s(\lambda))\,|\,\lambda=1, ..., g\}$ *determined in the above proof depends only on* B *and* δ. *For any fixed* $j_0\in\mathfrak{J}$, *there exist* $\mathbf{R}$-*linear isomorphisms* ω_λ *of* B_λ *onto* $\mathbf{M}(mq, \mathbf{C})$, $\lambda=1, ..., g$, *such that, for* $x\in B_\lambda$,

$$\omega_\lambda(x^\delta)=J_{r(\lambda),s(\lambda)}{}^t\overline{\omega_\lambda(x)}\,J_{r(\lambda),s(\lambda)},$$

and $\omega_\lambda(j_0\iota_\lambda)=\sqrt{-1}\,J_{r(\lambda),s(\lambda)}$ *for* $\lambda=1,\dots,g$. *Such* $\{\omega_\lambda\}$ *induces an isomorphism of* $G^1_{\mathbf{R}}$ *onto* $\prod_{\lambda=1}^g\mathbf{SU}(r(\lambda), s(\lambda))$. *Moreover the set* $\{\omega_1,\dots,\omega_g,\bar{\omega}_1,\dots,\bar{\omega}_g\}$ *is considered as a set of all the inequivalent absolutely irreducible representations of* B.

This is clear by the proof of the proposition.

Remark. Let $\omega_1, ..., \omega_g$ be as in Corollary 1. For $\varepsilon\in\mathcal{E}$, put $\omega^\varepsilon_\lambda=\omega_\lambda$ if $\varepsilon\iota_\lambda=\iota_\lambda$ and $\omega^\varepsilon_\lambda=\bar{\omega}_\lambda$ if $\varepsilon\iota_\lambda=-\iota_\lambda$. Then $\omega^\varepsilon_1, ..., \omega^\varepsilon_g$ satisfy the conditions of Corollary 1 for $j_0\varepsilon\in\mathfrak{J}$.

Let $\mathrm{tr}_{B/\mathbf{Q}}$ denote the reduced trace of B over $\mathbf{Q}$ and extend it $\mathbf{R}$-linearly to an $\mathbf{R}$-linear mapping of $B_{\mathbf{R}}$ to $\mathbf{R}$. Let $\omega_1, ..., \omega_g$ be as in Corollary 1. Then we have, for $x\in B_{\mathbf{R}}$,

$$\mathrm{tr}_{B/\mathbf{Q}}(x)=\sum_{\lambda=1}^{g}[tr(\omega_\lambda(x))+\overline{tr(\omega_\lambda(x))}].$$

Therefore we see easily

COROLLARY 2. *Let* j *be an element of* $B_{\mathbf{R}}$ *such that* $j^2=-1$ *and* $j^\delta=-j$. *Then the group* $M(j)$ *is a maximal compact subgroup of* $G^1_{\mathbf{R}}$, *i.e.* j *belongs to* $\mathfrak{J}$, *if and only if* $\mathrm{tr}_{B/\mathbf{Q}}(xjx^\delta j^{-1})>0$ *for every non-zero* $x\in B_{\mathbf{R}}$.

1.5. Let us consider a subalgebra C of B which satisfies the following conditions.

(1.5.1) *C contains the center K of B;*

(1.5.2) *C is stable under δ,* i.e. $C^\delta = C$*;*

(1.5.3) *δ is a positive involution on C.*

From (1.5.3), it follows that C is a semi-simple algebra, each of whose simple components is stable under δ. Therefore the center of each simple component of C is a CM-field containing K.

Now for $j \in \mathfrak{J}$, put

$$C(j) = \{a \in B \mid aj = ja\}.$$

Then $C(j)$ satisfies the conditions (1.5.1–3). In fact, (1.5.1–2) are obviously satisfied, and (1.5.3) is assured by Corollary 2 of Proposition 2. For $\varepsilon \in \mathcal{E}$, we have $C(j) = C(j\varepsilon)$.

PROPOSITION 3. *Let C be a subalgebra of B with the conditions* (1.5.1–3) *satisfied. Then there exists an element j in $\mathfrak{J}$ such that $C(j)$ contains C.*

Proof. We know that $C_{\mathbf{R}} = C \otimes_{\mathbf{Q}} \mathbf{R}$ is semi-simple, and that δ induces a positive involution of the second kind on each simple component of $C_{\mathbf{R}}$, which is isomorphic to a full matrix algebra over $\mathbf{C}$. Regarding $C_{\mathbf{R}}$ as a subalgebra of $B_{\mathbf{R}}$, let $C_{\mathbf{R}}^{(\lambda)}$ be the projection of $C_{\mathbf{R}}$ to B_λ. Then $C_{\mathbf{R}} = C_{\mathbf{R}}^{(1)} \oplus \dots \oplus C_{\mathbf{R}}^{(g)}$. Fix one λ for a while, and let $C_{\mathbf{R}}^{(\lambda)} = X_1 \oplus \dots \oplus X_p$ be the decomposition of $C_{\mathbf{R}}^{\lambda}$ into its simple components, and $1 = e_1 + \dots + e_p$ the corresponding decomposition of the identity element of B_λ. Each X_μ is $\mathbf{R}$-linearly isomorphic to $\mathbf{M}(n_\mu, \mathbf{C})$ for some integer n_μ, and δ is a positive involution on X_μ. Therefore on account of Lemma 1 in 1.1 of Shimura [9], we see easily that there is an $\mathbf{R}$-linear isomorphism ψ of B_λ onto $\mathbf{M}(mq, \mathbf{C})$ such that $\psi(x^\delta) = \overline{{}^t\psi(x)}$ for $x \in C_{\mathbf{R}}^{(\lambda)}$. Take $J \in \mathbf{GL}(mq, \mathbf{C})$ such that ${}^t\bar{J} = J$ and $\psi(x^\delta) = J\,\overline{{}^t\psi(x)}\,J^{-1}$ for all $x \in B_\lambda$. Put $E_\mu = \psi(e_\mu)$ for $\mu = 1, \dots, p$. Then $E_\mu = \psi(e_\mu) = \psi(e_\mu^\delta) = {}^t\bar{E}_\mu$ since $e_\mu \in C_{\mathbf{R}}^{(\lambda)}$. For any $x \in C_{\mathbf{R}}^{(\lambda)}$, $\overline{{}^t\psi(x)} = \psi(x^\delta) = J\,\overline{{}^t\psi(x)}\,J^{-1}$. Therefore J belongs to the commutor algebra Y of $\psi(C_{\mathbf{R}}^{(\lambda)})$ in $\mathbf{M}(mq, \mathbf{C})$. Let Y_μ be the commutor of $\psi(X_\mu)$ in $E_\mu \mathbf{M}(mq, \mathbf{C}) E_\mu$. Then since $E_\mu \mathbf{M}(mq, \mathbf{C}) E_\mu$ is a simple algebra over $\mathbf{C}$ whose center is contained in the simple subalgebra $\psi(X_\mu)$, Y_μ is also a simple algebra over $\mathbf{C}$. Clearly $Y = Y_1 \oplus \dots \oplus Y_p$. Put $J = J_1 + \dots + J_p$ with $J_\mu \in Y_\mu$. Then $J_\mu = JE_\mu = E_\mu J$ and ${}^t\bar{J}_\mu = J_\mu$ for $\mu = 1, \dots, p$. Since $\psi(X_\mu)$ and $E_\mu \mathbf{M}(mq, \mathbf{C}) E_\mu$ is stable under ${}^t\bar{\cdot}$, this induces a positive involution on Y_μ. Therefore, for example, using a $\mathbf{C}$-linear isomorphism ψ_μ of Y_μ onto $\mathbf{M}(m_\mu, \mathbf{C})$ for some integer m_μ such that $\psi_\mu({}^t\bar{U}) = \overline{{}^t\psi_\mu(U)}$ for $U \in Y_\mu$, we can find an element H_μ in Y_μ such that $(H_\mu J_\mu {}^t\bar{H}_\mu)^2 = E_\mu$. Put $H = H_1 + \dots + H_p$ and $J_0 = \sqrt{-1}J\,{}^t\bar{H}H$. Both H and J_0 are in Y and are invertible in $\mathbf{M}(mq, \mathbf{C})$. We see easily that $J_0^2 = -1_{mq}$ and $J\,{}^t\bar{J}_0 J^{-1} = J_0$. Since $J_0 J = \sqrt{-1}J\,{}^t\bar{H}HJ = \sqrt{-1}({}^t\bar{H}H)^{-1}$, we also see that $\mathrm{tr}\,(UJ_0 J\,{}^t\bar{U}J^{-1}J_0^{-1}) > 0$ for any $U \in \mathbf{M}(mq, \mathbf{C})$ unless $U = 0$. Define $j_\lambda = \psi^{-1}(J_0)$

for every λ, $1 \leqslant \lambda \leqslant g$, and put $j = j_1 + \ldots + j_g$. On account of Corollary 2 of Proposition 2 in 1.4, we see that $j \in \mathfrak{J}$. From the definition, it is clear that $C(j)$ contains C. Q.e.d.

1.6. Let C be a subalgebra of B satisfying (1.5.1–3), $C = C_1 \oplus \ldots \oplus C_t$ where C_μ is a central simple algebra over a CM-field P_μ, $\mu = 1, \ldots, t$, and $1 = e_1 + \ldots + e_t$ with $e_\mu \in C_\mu$. Then the center of C is $P = P_1 \oplus \ldots \oplus P_t$ and contains K. Therefore each P_μ contains the field Ke_μ isomorphic to K. Let $\tilde{C}$ be the commutor of C in B. Then $\tilde{C} = \tilde{C}_1 \oplus \ldots \oplus \tilde{C}_t$ where $\tilde{C}_\mu$ is the commutor of C_μ in $e_\mu B e_\mu = e_\mu \mathbf{M}(m, D) e_\mu$, $\mu = 1, \ldots, t$. Since $e_\mu \mathbf{M}(m, D) e_\mu$ is simple, we see that $\tilde{C}_\mu$ is a central simple algebra over P_μ. It is also clear that $\tilde{C}_\mu$ is stable under δ, and that δ induces an involution of $\tilde{C}_\mu$ of the second kind which coincides with the complex conjugation on the center P_μ. Considering $\tilde{C}_\mathbf{R} = \tilde{C} \otimes_\mathbf{Q} \mathbf{R}$ as a subalgebra over $\mathbf{R}$ of $B_\mathbf{R}$, $C_\mathbf{R}$ is the commutor of $C_\mathbf{R}$ in $B_\mathbf{R}$.

Let N_μ be the reduced norm of $\tilde{C}_\mu$ over P_μ for $\mu = 1, \ldots, t$, and define algebraic matrix groups $G^1(\tilde{C})$ and G^1_μ, $\mu = 1, \ldots, t$, by

$$G^1_{\mu\mathbf{Q}} = \{a \in \tilde{C}_\mu \,|\, aa^\delta = e_\mu \text{ and } N_\mu(a) = e_\mu\};$$

$$G^1(\tilde{C}) = G^1_1 \times \ldots \times G^1_t.$$

Then $G^1(\tilde{C})$ is naturally regarded as an algebraic subgroup of G^1, and

$$G^1(\tilde{C})_\mathbf{R} = \{x \in \tilde{C}_\mathbf{R} \,|\, xx^\delta = 1 \text{ and } N_\mu(xe_\mu) = e_\mu \text{ for } \mu = 1, \ldots, t\}.$$

Put
$$\mathfrak{J}_{\tilde{C}} = C_\mathbf{R} \cap \mathfrak{J} = \{j \in \mathfrak{J} \,|\, C(j) \supset C\};$$

$$M_{\tilde{C}}(j) = \{x \in G^1(\tilde{C})_\mathbf{R} \,|\, xj = jx\}, \quad (j \in \mathfrak{J}_{\tilde{C}}).$$

Then Proposition 3 assures that $J_{\tilde{C}}$ is not empty.

Proposition 4. *Let C be a subalgebra of B satisfying the conditions* (1.5.1–3), *and the notation as above. Fix any element j_1 of $\mathfrak{J}_{\tilde{C}}$. Then $\mathfrak{J}_{\tilde{C}}$ is decomposed into a disjoint union,*

$$\mathfrak{J}_{\tilde{C}} = \bigcup_{\varepsilon \in \mathcal{E}} \mathfrak{J}_{\tilde{C}}(j_1 \varepsilon)$$

where $\mathfrak{J}_{\tilde{C}}(j_1\varepsilon) = \{xj_1\varepsilon x^{-1} \,|\, x \in G^1(\tilde{C})_\mathbf{R}\}$. Moreover the family

$$\mathfrak{M}_{\tilde{C}} = \{M_{\tilde{C}}(j) \,|\, j \in \mathfrak{J}_{\tilde{C}}\} = \{M_{\tilde{C}}(j) \,|\, j \in \mathfrak{J}_{\tilde{C}}(j_1)\}$$

coincides with the set of all maximal compact subgroups of $G^1(\tilde{C})_\mathbf{R}$.

Proof. Since $G^1(\tilde{C})_\mathbf{R}$ is a subgroup of $G^1_\mathbf{R}$, we see easily by Proposition 2 in 1.4 that $\mathfrak{J}_{\tilde{C}} \supset \bigcup_{\varepsilon \in \mathcal{E}} \mathfrak{J}_{\tilde{C}}(j_1\varepsilon)$. For any $j \in \mathfrak{J}_{\tilde{C}}$, we conclude that $M_{\tilde{C}}(j)$ is a maximal compact subgroup of $G^1(\tilde{C})_{\mathbf{R}}$, if we apply Corollary 2 of Proposition 2 to each $\tilde{C}_\mu$ and je_μ in place of B and j, and combine the results for $\mu=1, \ldots, t$. Let $\mathcal{E}_{\tilde{C}}$ be the set of all the elements ε in the center of $\tilde{C}_{\mathbf{R}}$ such that $\varepsilon^2=1$. Then $\mathcal{E}_{\tilde{C}}$ is a multiplicative group containing $\mathcal{E}=\mathcal{E}_B$ as a subgroup. By the same procedure as above using Proposition 2 in place of Corollary 2, we see that, for each j in $\mathfrak{J}_{\tilde{C}}$, there are $\varepsilon_1 \in \mathcal{E}_{\tilde{C}}$ and $x \in G^1(\tilde{C})_{\mathbf{R}}$ such that $j=xj_1\varepsilon_1x^{-1}$. On the other hand, there are $\varepsilon \in \mathcal{E}$ and $y \in G^1_{\mathbf{R}}$ such that $j=yj_1\varepsilon y^{-1}$ since both j_1 and j are in $\mathfrak{J}$. Hence we have $j_1\varepsilon_1=\varepsilon_1 j_1=zj_1\varepsilon z^{-1}$ with $z=x^{-1}y \in G^1_{\mathbf{R}}$. Put $\varepsilon_2=\varepsilon_1\varepsilon^{-1}$. Then $\varepsilon_2 j_1=zj_1z^{-1}$ since ε is in the center of $B_{\mathbf{R}}$. Take an $\mathbf{R}$-linear isomorphism ω_λ of B_λ onto $\mathbf{M}(mq, \mathbf{C})$ so that the conditions of Corollary 1 of Proposition 2 for j_1 in place of j_0 are satisfied. Then $\omega_\lambda(z^{-1})=\omega_\lambda(z^\delta)=J_\lambda {}^t\overline{\omega_\lambda(z)}J_\lambda$ and $\omega_\lambda(j_1)=\sqrt{-1}J_\lambda$ where $J_\lambda=J_{r(\lambda),s(\lambda)}$. Therefore $\omega_\lambda(\varepsilon_2)\sqrt{-1}J_\lambda=\omega_\lambda(z)\sqrt{-1}\,{}^t\overline{\omega_\lambda(z)}J_\lambda$, and so, $\omega_\lambda(\varepsilon_2)=\omega_\lambda(z){}^t\overline{\omega_\lambda(z)}$, from which follows $\omega_\lambda(\varepsilon_2)=1_{mq}$ since $\omega_\lambda(\varepsilon_2)^2=1_{mq}$. Since this is true for every $\lambda=1, \ldots, g$, we have $\varepsilon_2=1$, i.e. $\varepsilon_1=\varepsilon$. Therefore $j=xj_1\varepsilon x^{-1}$ with $x \in G^1(\tilde{C})_{\mathbf{R}}$ and $\varepsilon \in \mathcal{E}$. This proves the former half of the proposition. The latter half is clear since $M_{\tilde{C}}(j_1)$ is a maximal compact subgroup of $G^1(\tilde{C})_{\mathbf{R}}$, and every maximal compact subgroup of $G^1(\tilde{C})_{\mathbf{R}}$ is of the form of $xM(j_1)x^{-1}=M(xj_1x^{-1})$ for some $x \in G^1(\tilde{C})_{\mathbf{R}}$.

Remark. It can happen that $\mathcal{E}_{\tilde{C}} \supsetneqq \mathcal{E}=\mathcal{E}_B$. Then we cannot cover all the $j\varepsilon_1$, $\varepsilon_1 \in \mathcal{E}_{\tilde{C}}$, by $\mathfrak{J}_{\tilde{C}}$.

COROLLARY. *Let the notation and the assumptions be as in Proposition 4. Then the following four assertions are equivalent.*

(i) $\mathfrak{J}_{\tilde{C}}=\{j_1\varepsilon \mid \varepsilon \in \mathcal{E}\}$.

(ii) $G^1(\tilde{C})_{\mathbf{R}}$ *is compact.*

(iii) δ *is a positive involution on* $\tilde{C}$.

(iv) $C(j_1)$ *contains* $\tilde{C}$.

Proof. The equivalence of (i) and (ii) follows immediately from Proposition 4 and the defintion of $M_{\tilde{C}}(j_1)$. Since $G^1(\tilde{C})_{\mathbf{R}}$ is a direct product of $G^1_{\mu\mathbf{R}}$, $\mu=1, \ldots, t$, $G^1(\tilde{C})_{\mathbf{R}}$ is compact if and only if every $G^1_{\mu\mathbf{R}}$ is compact. From Corollary 1 of Proposition 2 in 1.4 applied to $\tilde{C}_\mu$ in place of B, it follows that $G^1_{\mu\mathbf{R}}$ is compact if and only if $rs=0$ for every pair of integers (r, s) determined by C_μ and δ. The definition of the pairs (r, s) shows that this is the case if and only if δ is a positive involution on $\tilde{C}_\mu$. This proves the equivalence of (ii) and (iii). Obviously (iv) implies (iii) since δ is a positive involution on $C(j_1)$. Now assume (iii). Let us consider the commutor of the center P of C in B. Write the commutor by $\tilde{P}$. Then $\tilde{P}=\tilde{P}_1 \oplus \ldots \oplus \tilde{P}_t$ where each $\tilde{P}_\mu$ is the commutor of P_μ in $e_\mu\mathbf{M}(m, D)e_\mu$. Therefore $\tilde{P}_\mu$ is nothing but $C_\mu \otimes_{P_\mu} \tilde{C}_\mu$ considered as a subalgebra of $e_\mu\mathbf{M}(m, D)e_\mu$. (Cf. Corollary 7.3G of Artin, Nesbitt and Thrall [2], p. 71.) Since δ induces positive involutions on both C_μ and

$\tilde{C}_\mu$, it is also a positive involution on $\tilde{P}_\mu = C_\mu \otimes_{P_\mu} \tilde{C}_\mu$. Therefore δ is a positive involution on $\tilde{P}$, and there is an element $j \in \mathcal{J}$ such that $C(j) \supset \tilde{P} \supset \tilde{C}$. (See Proposition 3 of 1.5.) Then j is one of $j_1\varepsilon$, $\varepsilon \in \mathcal{E}$, since we have already proved that (i) is equivalent to (iii). Hence we have $C(j_1) = C(j_1\varepsilon) = C(j) \supset \tilde{C}$. The proof is done.

1.7. At the beginning of our discussion (1.4) we have fixed a maximal compact subgroup M_0 of $G^1_{\mathbf{R}}$. On account of Proposition 2 in 1.4, there is an element j_0 in $\mathcal{J}$ such that $M_0 = M(j_0)$. We fix such j_0. Choose $\omega_1, \ldots, \omega_g$ as in Corollary 1 of Proposition 2 for this j_0. Then furnished with the complex structure obtained from that of $\prod_{\lambda=1}^{g} \mathbf{SU}(r(\lambda), s(\lambda))$ through $\{\omega_\lambda\}$, the homogeneous space $\mathcal{H} = G^1_{\mathbf{R}}/M_0$ becomes a hermitian symmetric space, and is isomorphic to a bounded symmetric domain. We denote the hermitian symmetric space thus obtained by $\mathcal{H}_{j_0}$. Then $\mathcal{H}_{j_0}$ is decomposed into a product

$$\mathcal{H}_{j_0} = \mathcal{H}^{(1)}_{j_0} \times \ldots \times \mathcal{H}^{(g)}_{j_0}$$

corresponding to the decomposition $B_{\mathbf{R}} = B_1 \oplus \ldots \oplus B_g$.

For the later use, we fix a representation of $\mathcal{H}_{j_0}$ as a bounded symmetric domain as follows.

For two non-negative integers r and s, put, as before,

$$\mathbf{SU}(r,s) = \{x \in \mathbf{M}(r+s, \mathbf{C}) \mid xJ_{r,s}{}^t\bar{x} = J_{r,s} \quad \text{and } \det(x) = 1\}$$

where $J_{r,s} = \begin{bmatrix} 1_r & 0 \\ 0 & -1_s \end{bmatrix}$, and let $\mathbf{M}(r, s; \mathbf{C})$ be the set of all $r \times s$ matrices with entries in $\mathbf{C}$. We define a bounded symmetric domain $\mathcal{H}_{r,s}$ by

$$\mathcal{H}_{r,s} = \{z \in \mathbf{M}(r, s; \mathbf{C}) \mid 1_r - z{}^t\bar{z} \text{ is positive hermitan}\},$$

and the action of an element $x = \begin{bmatrix} U & V \\ X & Y \end{bmatrix}$ of $\mathbf{SU}\,(r, s)$ on $\mathcal{H}_{r,s}$ by

$$x(z) = (\bar{U}z + \bar{V})(\bar{X}z + \bar{Y})^{-1}$$

for $z \in \mathcal{H}_{r,s}$. Note that $\bar{x} = J_{r,s}{}^t x^{-1} J_{r,s}$ for $x \in \mathbf{SU}(r, s)$. As is well known and is easily seen, this is well defined. Moreover $\mathbf{SU}(r, s)$ acts on $\mathcal{H}_{r,s}$ holomorphically and transitively. The isotropy subgroup of $\mathbf{SU}(r, s)$ at each point z of $\mathcal{H}_{r,s}$ is a maximal compact subgroup of $\mathbf{SU}(r, s)$. Especially the isotropy group at $z = 0$ coincides with the subgroup $M_0(r, s)$ of $\mathbf{SU}(r, s)$ given by

$$M_0(r,s) = \{x \in \mathbf{SU}(r,s) \mid x\sqrt{-1}\,J_{r,s} = \sqrt{-1}\,J_{r,s}\,x\}.$$

Assigning the point $x(0)$ on $\mathcal{H}_{r,s}$ to the coset $xM_0(r, s)$, we have a homeomorphic mapping of $\mathbf{SU}(r, s)/M_0(r, s)$ onto $\mathcal{H}_{r,s}$. We define the hermitian structure on $\mathbf{SU}(r, s)/M_0(r, s)$ through this mapping. Then it is easily seen that

For $x=\begin{bmatrix} U & V \\ X & Y \end{bmatrix} \in \mathbf{SU}(r, s)$ *and* $z \in \mathcal{H}_{r,s}$, *one has* $z=x(0)$ *if and only if* $x=\begin{bmatrix} 1_r & \bar{z} \\ {}^t z & 1_s \end{bmatrix}\begin{bmatrix} U & 0 \\ 0 & Y \end{bmatrix}$.

Furnishing $\mathcal{H}$ with the complex structure obtained from the structure thus defined on $\prod_{\lambda=1}^{g} \mathbf{SU}(r(\lambda), s(\lambda))/M_0(r(\lambda), s(\lambda))$ through $\{\omega_\lambda\}$, we have the hermitian symmetric space $\mathcal{H}_{j_0}$ defined clearly. Moreover, by the remark following Corollary 1 of Proposition 2 in 1.4, we see that, for $\varepsilon \in \mathcal{E}$, the space $\mathcal{H}_{j_0\varepsilon}=\mathcal{H}_{j_0\varepsilon}^{(1)}\times\ldots\times\mathcal{H}_{j_0\varepsilon}^{(g)}$ relates with $\mathcal{H}_{j_0}$ in such a way as $\mathcal{H}_{j_0\varepsilon}^{(\lambda)}=\mathcal{H}_{j_0}^{(\lambda)}$ if $\varepsilon\iota_\lambda=\iota_\lambda$ and $\mathcal{H}_{j_0\varepsilon}^{(\lambda)}=\overline{\mathcal{H}}_{j_0}^{(\lambda)}$, the space with the conjugate complex structure of the structure of $\mathcal{H}_{j_0}^{(\lambda)}$, if $\varepsilon\iota_\lambda=-\iota_\lambda$.

1.8. Let φ be an $\mathbf{R}$-linear automorphism of $B_{\mathbf{R}}$ and suppose that φ commutes with δ. Then φ induces an automorphism of $G_{\mathbf{R}}^1$, and maps $\mathcal{J}$ onto itself. Obviously $\varphi(M(j))=M(\varphi(j))$ for $j\in\mathcal{J}$, and φ permutes the sets $\mathcal{J}(j_0\varepsilon)$, $\varepsilon\in\mathcal{E}$. Through the correspondence between $\mathcal{H}$ and $\mathcal{M}$ assigning xM_0x^{-1} to xM_0 for $x\in G_{\mathbf{R}}^1$, φ induces a homeomorphism of $\mathcal{H}$ onto itself.

PROPOSITION 5. *Let φ be an $\mathbf{R}$-linear automorphism of $B_{\mathbf{R}}$ which commutes with δ. Then $\varphi(\mathcal{J}(j_0))=\mathcal{J}(j_0\varepsilon)$ for some $\varepsilon\in\mathcal{E}$, and φ induces an isomorphism of the hermitian symmetric space $\mathcal{H}_{j_0}$ onto $\mathcal{H}_{j_0\varepsilon}$. Conversely, for $\varepsilon\in\mathcal{E}$, every isomorphism of $\mathcal{H}_{j_0}$ onto $\mathcal{H}_{j_0\varepsilon}$, if exists, is obtained from such a φ.*

Proof. Let (r, s) and (r', s') be two pairs of non-negative integers such that $r+s=r'+s'$, and $\mathcal{H}_{r,s}$ and $\mathcal{H}_{r',s'}$ as in 1.7. Then Satake [7] tells us the following:

The two hermitian symmetric spaces $\mathcal{H}_{r,s}$ and $\mathcal{H}_{r',s'}$ are isomorphic to each other if and only if $(r, s)=(r', s')$ or (s', r'). Moreover, if this is the case, then the isomorphisms ψ of $\mathbf{SU}(r, s)$ to $\mathbf{SU}(r', s')$ of the form, either $\psi(x)=yxy^{-1}$ for every $x\in\mathbf{SU}(r, s)$ with some $y\in\mathbf{GL}(r+s, \mathbf{C})$ such that $yJ_{r,s}{}^t\bar{y}=J_{r',s'}$ or $\psi(x)=y\bar{x}y^{-1}$ for every $x\in\mathbf{SU}(r, s)$ with some $y\in\mathbf{GL}(r+s, \mathbf{C})$ such that $yJ_{r,s}{}^t\bar{y}=-J_{r',s'}$, induce all the isomorphisms of $\mathcal{H}_{r,s}$ to $\mathcal{H}_{r',s'}$.

Let $\omega_1, \ldots, \omega_g$ be as in Corollary 1 of Proposition 2 in 1.4 for j_0. Suppose that φ maps B_λ onto B_μ. Then the representation $\omega_\lambda^\varepsilon\circ\varphi$ of B is equivalent to either ω_μ or $\bar{\omega}_\mu$. (For the notation, see the remark in 1.4.) Here we regard ω_λ as a homomorphism of $B_{\mathbf{R}}$ to $\mathbf{M}(mq, \mathbf{C})$ defining $\omega_\lambda(B_\mu)=0$ if $\mu\neq\lambda$. Take $y_\mu\in\mathbf{GL}(mq, \mathbf{C})$ so that $\omega_\lambda^\varepsilon\circ\varphi(x)$ is equal to either $y_\mu\omega_\mu(x)y_\mu^{-1}$ or $y_\mu\overline{\omega_\mu(x)}y_\mu^{-1}$ for $x\in B_{\mathbf{R}}$. Then since φ commutes with δ, we have, for some $\nu(y_\mu)\in\mathbf{R}$,

$$y_\mu J_{r(\mu),s(\mu)}{}^t\bar{y}_\mu=\nu(y_\mu)J_{r(\lambda),s(\lambda)}.$$

Replacing y_μ by its scalar multiple, we may assume that $\nu(y_\mu)=\pm 1$. On the other hand, there exists $x\in G^1_{\mathbf{R}}$ such that $j=\varphi(j_0)=xj_0\varepsilon x^{-1}$. By the definition of $\omega^\varepsilon_\lambda$, we have

$$\omega^\varepsilon_\lambda(j)=\omega^\varepsilon_\lambda(x)\sqrt{-1}\,J_{r(\lambda),s(\lambda)}\,\omega^\varepsilon_\lambda(x)^{-1}.$$

Combining these two equalities with the fact that $\omega^\varepsilon_\lambda(j)=\omega^\varepsilon_\lambda(\varphi(j_0))$ is equal to either

$$y_\mu\,\omega_\mu(j_0)y_\mu^{-1}=y_\mu\sqrt{-1}\,J_{r(\mu),s(\mu)}\,y_\mu^{-1}$$

or

$$y_\mu\,\overline{\omega_\mu(j_0)}y_\mu^{-1}=-y_\mu\sqrt{-1}\,J_{r(\mu),s(\mu)}\,y_\mu^{-1},$$

we see easily that $\nu(y_\mu)$ is equal to 1 or -1 according as $\omega^\varepsilon_\lambda\circ\varphi$ is equivalent to ω_μ or $\bar{\omega}_\mu$. The proposition now easily follows from the results of I. Satake.

1.9. Let us consider two types of φ here. First let y be an element of the group $G_{\mathbf{R}}$ given in 1.3. The inner automorphism φ_y of $B_{\mathbf{R}}$ defined by $\varphi_y(x)=yxy^{-1}$ for $x\in B_{\mathbf{R}}$ commutes with δ. Since $\mathcal{E}$ is contained in the center of $G_{\mathbf{R}}$, the mapping of $G_{\mathbf{R}}$ to $\mathcal{E}$ assigning to $y\in G_{\mathbf{R}}$ such $\varepsilon\in\mathcal{E}$ as $y\mathcal{J}(j_0)y^{-1}=\mathcal{J}(j_0\varepsilon)$ gives a homomorphism of $G_{\mathbf{R}}$ to $\mathcal{E}$. It is easily seen that this homomorphism induces an isomorphism of the quotient group $G_{\mathbf{R}}/G_{\mathbf{R}+}$ onto the subgroup $\mathcal{E}_0$ of $\mathcal{E}$ given by

$$\mathcal{E}_0=\{\varepsilon\in\mathcal{E}\,|\,\varepsilon\iota_\lambda=\iota_\lambda \quad \text{if} \quad r(\lambda)\neq s(\lambda)\}.$$

Especially, for $y\in G_{\mathbf{R}}$, $y\mathcal{J}(j_0)y^{-1}=\mathcal{J}(j_0)$ if and only if $y\in G_{\mathbf{R}+}$. Obviously every element a of the center $G_{\mathbf{R}}$ induces the identity mapping as φ_a, and so, the identity mapping of $\mathcal{H}_{j_0}$. We define the action of $y\in G_{\mathbf{R}+}$, or y modulo the center of $G_{\mathbf{R}+}$, on $\mathcal{H}_{j_0}$ in this way through φ_y. The action of the subgroup $G^1_{\mathbf{R}}$ of $G_{\mathbf{R}+}$ thus defined coincides with the action of $G^1_{\mathbf{R}}$ on $\mathcal{H}=G^1_{\mathbf{R}}/M_0$ defined by the left transformation.

Second let A denote the group of all those $\mathbf{Q}$-linear automorphisms of B which commute with δ. Then extended $\mathbf{R}$-linearly, every element of A is regarded as an $\mathbf{R}$-linear automorphism of $B_{\mathbf{R}}$. Put

$$A'_{j_0}=\{\alpha\,|\,\alpha\in A \quad \text{and} \quad \alpha(\mathcal{J}(j_0))=\mathcal{J}(j_0)\}.$$

Then A'_{j_0} acts on $\mathcal{H}_{j_0}$ holomorphically. Put $G_{\mathbf{Q}+}=G_{\mathbf{Q}}\cap G_{\mathbf{R}+}$. Then $G_{\mathbf{Q}+}$ contains $G^1_{\mathbf{Q}}$, and the center of $G_{\mathbf{Q}+}$ is $K^\times$. For $\gamma\in G_{\mathbf{Q}+}$, the inner automorphism φ_γ of B defined by γ as $\varphi_\gamma(x)=\gamma x\gamma^{-1}$ for $x\in B$ is contained in A'_{j_0}. Put $A^0_+=G_{\mathbf{Q}+}/K^\times$. Identifying γ modulo $K^\times$ with φ_γ for $\gamma\in G_{\mathbf{Q}+}$, we consider A^0_+ as a normal subgroup of A'_{j_0}. Note that A^0_+ is of finite index in A'_{j_0}. For $\gamma\in G_{\mathbf{Q}+}$, the action of γ on $\mathcal{H}_{j_0}$ as an element of $G_{\mathbf{R}+}$ coincides with the action of γ modulo $K^\times$ as an element of A'_{j_0}.

We see the following three assertions hold:

(1.9.1) *If $G^1_{\mathbf{R}}$ is compact, then $\mathcal{H}_{j0}$ is the space consisting of just one point, and both $G_{\mathbf{R}+}$ and A'_{j_0} act trivially on $\mathcal{H}_{j0}$. If $G^1_{\mathbf{R}}$ is not compact, then an element of $G_{\mathbf{R}+}$ acts trivially on $\mathcal{H}_{j_0}$ if and only if it belongs to the center of $G_{\mathbf{R}+}$;*

(1.9.2) *If $G^1_{\mathbf{R}}$ is not compact, and $mq>2$, then the identity element is the only element of A'_{j_0} that acts trivially on $\mathcal{H}_{j_0}$;*

(1.9.3) *Suppose that B is a quaternion algebra over K, i.e., $mq=2$, and let B_0 and ι be as in* 1.2. *Then the automorphism $\delta\iota$ of B belongs to A'_{j_0} if and only if B_0 is totally indefinite,* i.e., $B_{0\mathbf{R}}=B_0\otimes_{\mathbf{Q}}\mathbf{R}$ *is isomorphic to a direct product of g copies of* $\mathbf{M}(2, \mathbf{R})$. *Furthermore, if this is the case, then the identity element and $\delta\iota$ are the only elements of A'_{j_0} that act trivially on $\mathcal{H}_{j_0}$. If $G^1_{\mathbf{R}}$ is not compact, and B_0 is not totally indefinite, then the identity element is the only element of A'_{j_0} that acts trivially on $\mathcal{H}_{j_0}$.*

The assertion (1.9.1) is easily seen and well known. Assume that $G^1_{\mathbf{R}}$ is not compact. Let φ be an $\mathbf{R}$-linear automorphism of $B_{\mathbf{R}}$ which commutes with δ, and suppose that $\varphi(\mathcal{J}(j_0))=\mathcal{J}(j_0)$, and that φ acts trivially on $\mathcal{H}_{j0}$. We suppose, moreover, that φ induces a $\mathbf{Q}$-linear automorphism of B_0 if $mq=2$, and of B if $mq>2$. Let $\mathcal{H}_{j_0}^{r}=\mathcal{H}_{j_0}^{(1)}\times\ldots\times\mathcal{H}_{j_0}^{(g)}$ be as in 1.7. If φ is not trivial on F, then it actually permutes the factors $\mathcal{H}_{j_0}^{(\lambda)}$, $\lambda=1, \ldots, g$. Therefore φ has to be trivial on F since $G^1_{\mathbf{R}}$ is not compact, and φ acts trivially on $\mathcal{H}_{j0}$. If φ is also trivial on K, then it is an inner automorphism of B defined by some element $\gamma\in G_{\mathbf{Q}+}\subset G_{\mathbf{R}+}$, i.e., $\varphi\in A^0_+$. Therefore (1.9.1) implies that φ is the identity element of A'_{j_0}. Suppose now that φ is not trivial on K. Note that, although we do not assume that φ is an automorphism of B if $mq=2$, it is meaningful to say that φ is trivial, or not on K, since B is embedded in $B_{\mathbf{R}}$. Let $\omega_1, \ldots, \omega_g$ be as in Corollary 1 of Proposition 2. Since φ is trivial on F, it induces an $\mathbf{R}$-linear automorphism of B_λ for each $\lambda=1, \ldots, g$ where $B_{\mathbf{R}}=B_1\oplus\ldots\oplus B_g$. By means of ω_λ, φ induces an $\mathbf{R}$-linear automorphism of $\mathbf{M}(mq, \mathbf{C})$ for each λ. Since φ is not trivial on K, this is not $\mathbf{C}$-linear for some λ. Fix such a λ. Then the automorphism ψ, say, is of the form $\psi(x)=y\bar{x}y^{-1}$ for some $y\in\mathbf{GL}(mq, \mathbf{C})$ such that

$$yJ_{r(\lambda),s(\lambda)}{}^t\bar{y}=-J_{r(\lambda),s(\lambda)}$$

since φ commutes with δ (see the proof of Proposition 5). In this case, we have $r(\lambda)=s(\lambda)$. We see easily that $y=u\begin{bmatrix}0 & 1_{r(\lambda)}\\ 1_{s(\lambda)} & 0\end{bmatrix}$ for some $u\in\mathbf{SU}(r(\lambda), s(\lambda))$, and that ψ induces the automorphism of $\mathcal{H}_{r(\lambda),s(\lambda)}$ of the form, $z\to u({}^tz)$. Here, since $r(\lambda)=s(\lambda)$, the mapping $z\to{}^tz$ is an automorphism of $\mathcal{H}_{r(\lambda),s(\lambda)}$, and $u({}^tz)$ means the action of u on ${}^tz\in\mathcal{H}_{r(\lambda),s(\lambda)}$. One can easily see that this mapping $z\to u({}^tz)$ is trivial if and only if $r(\lambda)=s(\lambda)=1$ and u acts trivially. Therefore, especially, if $mq>2$, then φ must be trivial on K, and (1.9.2) is proved. Now suppose that $mq=2$. Then we have $B_{0\mathbf{R}}=B_{01}+\ldots+B_{0g}$ corresponding to $B_{\mathbf{R}}=B_1\oplus\ldots\oplus B_g$, and $B_\lambda=B_{0\lambda}+\zeta B_{0\lambda}$ for each $\lambda=1,\ldots,g$ with an element ζ of $K^\times$ such that $\zeta^\delta=-\zeta$. Then (1.2.2) implies that $G^1_{\mathbf{R}}=G^1_1\times\ldots\times G^1_g$ where

$$G^1_\lambda=\{x\in B_{0\lambda}\,|\,xx^\iota=1\}.$$

Note that $B_{0\lambda}$ is isomorphic to either $\mathbf{M}(2,\mathbf{R})$ or the algebra $\mathbf{H}$ of Hamilton quaternions, and G^1_λ is compact if and only if $B_{0\lambda}$ is isomorphic to $\mathbf{H}$. Since $\varphi_\lambda=\varphi|_{B_\lambda}$ is an $\mathbf{R}$-linear automorphism of B_λ, which maps $B_{0\lambda}$ onto itself, we have $\varphi_\lambda(\zeta)=\pm\zeta$. If $\varphi_\lambda(\zeta)=\zeta$, then φ_λ is trivial on the center of B_λ, and is an inner automorphism. If $\varphi_\lambda(\zeta)=-\zeta$, then φ_λ corresponds to the automorphism,

$$y\to\begin{bmatrix}0&1\\1&0\end{bmatrix}\bar{x}\begin{bmatrix}0&1\\1&0\end{bmatrix},$$

of $\mathbf{M}(2,\mathbf{C})$ through ω_λ. Observe that this is the case if and only if $B_{0\lambda}$ is isomorphic to $\mathbf{M}(2,\mathbf{R})$. One can easily see that $\varphi_\lambda=\delta\iota$ in this case. Hence φ must be trivial on B_0 since the projection of $B_{0\mathbf{R}}$ to $B_{0\lambda}$ induces an isomorphism on B_0, and φ is either the identity mapping or $\delta\iota$ on $B_\lambda=B_{0\lambda}+\zeta B_{0\lambda}$ for at least one λ. Then φ must be the identity mapping on B_λ if $B_{0\lambda}$ is isomorphic to $\mathbf{H}$. Note that this stronger condition for φ on B_λ such that $B_{0\lambda}$ is isomorphic to $\mathbf{H}$ comes from the condition $\varphi(\mathfrak{J}(j_0))=\mathfrak{J}(j_0)$. Summing up, we get

(1.9.4) *Suppose that $mq=2$, and let the notation be as above. Let φ be an $\mathbf{R}$-linear automorphism of $B_{\mathbf{R}}$ which commutes with δ, induces a $\mathbf{Q}$-linear automorphism of B_0, and has the property that $\varphi(\mathfrak{J}(j_0))=\mathfrak{J}(j_0)$. Then φ acts trivially on $\mathcal{H}_{j_0}$ if and only if φ is the identity mapping on B_λ if $B_{0\lambda}$ is isomorphic to $\mathbf{H}$, and is either the identity mapping or $\delta\iota$ on B_λ if $B_{0\lambda}$ is isomorphic to $\mathbf{M}(2,\mathbf{R})$.*

This implies (1.9.3) immediately.

Suppose that $mq=2$, and let the notation be as above. Let $j_0=j_{01}+\ldots+j_{0g}$ with $j_{0\lambda}\in B_\lambda$ for $\lambda=1,\ldots,g$. Fix a λ for a while, and let $j_{0\lambda}=j_1+\zeta j_2$ with j_1 and j_2 in $B_{0\lambda}$. Since $j^2_{0\lambda}=-1$ and $j^\delta_{0\lambda}=-j_{0\lambda}$, we see easily that $j_1^\delta=j_1^\iota=-j_1$, $j_2^\delta=j_2^\iota=j_2$ and $j_1j_2+j_2j_1=0$. Then $j_2=j_2^\iota=\frac{1}{2}(j_2+j_2^\iota)=\frac{1}{2}\operatorname{tr}(j_2)$ and it belongs to the center of $B_{0\lambda}$. Therefore $j_1j_2+j_2j_1=2j_1j_2=0$, and either $j_1=0$ or $j_2=0$. If $j_1=0$, then $j_{0\lambda}=\zeta j_2$ belongs to the center of B_λ, and the maximal compact subgroup $\{x\in G^1_\lambda\,|\,xj_{0\lambda}=j_{0\lambda}x\}$ of G^1_λ is G^1_λ itself, i.e., G^1_λ is compact. If $j_2=0$, then $j_{0\lambda}=j_1$. Since $j_1^\iota=-j_1$, j_1 does not belong to the center of $B_{0\lambda}$. Therefore G^1_λ is not compact since the maximal compact subgroup $\{x\in G^1_\lambda\,|\,xj_{0\lambda}=j_{0\lambda}x\}$ of G^1_λ is not the whole group G^1_λ. (Note that the element of G^1_λ spans $B_{0\lambda}$ over $\mathbf{R}$.) Now we define a group A''_{j_0} if $mq=2$ as follows:

Let B'_0 (resp. B''_0) be the direct sum of the simple components of $B_{0\mathbf{R}}$ which are isomorphic to $\mathbf{M}(2, \mathbf{R})$ (resp. $\mathbf{H}$). Then $B_{0\mathbf{R}} = B_0' \oplus B_0''$. Corresponding to this decomposition, $B_{\mathbf{R}}$ is decomposed into a direct sum, $B_{\mathbf{R}} = B' \oplus B''$, such that $B' \supset B_0'$ and $B'' \supset B_0'$. Let $\mathfrak{J}(j_0)'$ be the projection of $\mathfrak{J}(j_0)$ to B'. Then it follows from the above observation of j_0 that $\mathfrak{J}(j_0)'$ is contained in B_0'. Let β be a $\mathbf{Q}$-linear automorphism of B_0. Extending β $\mathbf{R}$-linearly to an automorphism of $B_{0\mathbf{R}}$, we have $\beta(B_0') = B_0'$ and $\beta(B_0'') = B_0''$. Put

$$A_{j_0}'' = \{\beta \mid \beta \text{ is a } \mathbf{Q}\text{-linear automorphism of } B_0 \text{ and } \beta(\mathfrak{J}(j_0)') = \mathfrak{J}(j_0)'\}.$$

Let β be an element of A_{j_0}''. Then β can always be extended to an $\mathbf{R}$-linear automorphism of $B_{\mathbf{R}}$ so that $\beta(\mathfrak{J}(j_0)) = \mathfrak{J}(j_0)$. Note that β commutes with δ. Observe that the extension is unique on B'', but not on B'. Though the extension is not unique, the holomorphic action of β on $\mathcal{H}_{j_0}$ is well defined on account of Proposition 5 and (1.9.4).

The assertion (1.9.3) shows that the quotient group $A_{j_0}'/\{1, \delta\iota\}$ is naturally considered as a subgroup of A_{j_0}'' if B_0 is totally indefinite, and so is the group A_{j_0}' itself if B_0 is not totally indefinite. In both cases, the group A_+^0 is considered as a normal subgroup of A_{j_0}''. Now put

$$A_{j_0} = \begin{cases} A_{j_0}'' & \text{if } mq = 2 \text{ and } B_0 \text{ is totally indefinite,} \\ A_{j_0}' & \text{otherwise.} \end{cases}$$

By (1.9.1–4), we see easily that the identity element is the only element of A_{j0} that acts trivially on $\mathcal{H}_{j_0}$ unless $G_{\mathbf{R}}^1$ is compact.

Hereafter *we assume that $G_{\mathbf{R}}^1$ is not compact.*

1.10. Put $D_{\mathbf{R}} = D \otimes_{\mathbf{Q}} \mathbf{R}$, and identify $D_{\mathbf{R}}^m$ with $D^m \otimes_{\mathbf{Q}} \mathbf{R}$. The module $D_{\mathbf{R}}^m$ is thus a left $D_{\mathbf{R}}$- and right $B_{\mathbf{R}}$-module. For any $j \in \mathfrak{J}$, the algebra $\mathbf{R}[j]$ generated by j in $B_{\mathbf{R}}$ over $\mathbf{R}$ is $\mathbf{R}$-linearly isomorphic to the complex number field $\mathbf{C}$ by assigning $\sqrt{-1}$ to j. The involution δ on $\mathbf{R}[j]$ corresponds to the complex conjugation. In this way, each $j \in \mathfrak{J}$ defines a complex structure on $D_{\mathbf{R}}^m$.

Let C be a subalgebra of B satisfying the conditions (1.5.1–3), and C^{-1} the reciprocal (or inverse) algebra of C. We can regard D^m (and so, $D_{\mathbf{R}}^m$) as a left $D \otimes_K C^{-1}$-module by defining $(d \otimes c) \cdot v = dvc$ for $d \in D$, $c \in C^{-1}$ and $v \in D^m$ (or $D_{\mathbf{R}}^m$).

Take $j \in \mathfrak{J}_{\tilde{C}}$. Since every element of $D \otimes_K C^{-1}$ commutes with j as $\mathbf{R}$-linear transformations of $D_{\mathbf{R}}^m$, j determines a representation Ψ_j of $D \otimes_K C^{-1}$ into $\mathbf{M}(mq^2 g, \mathbf{C})$ through the complex structure on $D_{\mathbf{R}}^m$ which j defines as above. Actually Ψ_j is a representation of $D \otimes_K C(j)^{-1}$, which contains $D \otimes_K C^{-1}$. Let $1 = \iota_1 + \dots + \iota_g$ be the decomposition of 1 of B corresponding to the decomposition $B_{\mathbf{R}} = B_1 \oplus \dots \oplus B_g$. Since each $D_{\mathbf{R}}^m \iota_\lambda$ $(1 \leqslant \lambda \leqslant g)$ is stable under the actions of j and $D \otimes_K C^{-1}$, j defines a complex structure on $D_{\mathbf{R}}^m \iota_\lambda$, and gives a representation $\Psi_j^{(\lambda)}$ of $D \otimes_K C^{-1}$ into $\mathbf{M}(mq^2, \mathbf{C})$. It is easy to see that

$$\Psi_j = \Psi_j^{(1)} + \dots + \Psi_j^{(g)}.$$

If we define a complex structure on $D_{\mathbf{R}}^{m}\iota_\lambda$ by assigning $\sqrt{-1}$ to $-j\iota_\lambda$, the structure is the complex conjugate of the structure defined by j first. For $\varepsilon \in \mathcal{E}$, let Ψ_j^{ε} be the representation of $D\otimes_K C^{-1}$ obtained by replacing $\Psi_j^{(\lambda)}$ with $\overline{\Psi}_j^{(\lambda)}$, the complex conjugate of $\Psi^{(\lambda)}$, if $\varepsilon\iota_\lambda = -\iota_\lambda$ and taking $\Psi_j^{(\lambda)}$ unchanged if $\varepsilon\iota_\lambda = \iota_\lambda$. Then we have

$$\Psi_j^{\varepsilon} = \Psi_{j\varepsilon} \quad (\varepsilon \in \mathcal{E}).$$

PROPOSITION 6. *Let $C = C_1 \oplus \ldots \oplus C_t$ be a subalgebra of B satisfying the conditions (1.5.1–3) with the simple components C_μ, $\mu = 1, \ldots, t$. For $j \in \mathfrak{J}_{\tilde{C}}$, let Ψ_j be the representation of $D\otimes_K C^{-1}$ defined above. Then, for each $\mu = 1, \ldots, t$, the restriction of the representation $\Psi_j + \overline{\Psi}_j$ to $D\otimes_K C_\mu^{-1}$ contains all the inequivalent absolutely irreducible representations of $D\otimes_K C_\mu^{-1}$ with the same multiplicity. Moreover, for $j' \in \mathfrak{J}_{\tilde{C}}$, $\Psi_{j'}$ is equivalent to Ψ_j if $j' = xjx^{-1}$ for some $x \in G^1(\tilde{C})_{\mathbf{R}}$ where $G^1(\tilde{C})$ is as in 1.6.*

Proof. The last assertion follows from immediately the definition.

Let $P = P_1 \oplus \ldots \oplus P_t$ and $1 = e_1 + \ldots + e_t$ be the decompositions of the center P of C and 1 respectively corresponding to the decomposition of C. Each P_μ $(1 \leqslant \mu \leqslant t)$ is a CM-field containing K. Fix one μ. Since e_μ commutes with every element of D as linear transformations of D^m, and D is a division algebra, the module $W_\mu = D^m e_\mu$ is isomorphic to D^{n_μ} for some integer n_μ. Moreover the $\mathbf{R}$-module $W_{\mu\mathbf{R}} = W_\mu \otimes_{\mathbf{Q}} \mathbf{R}$ is a vector space over $\mathbf{R}[j] \cong \mathbf{C}$, and gives a representation Θ_μ of $D\otimes_K C_\mu^{-1}$, since e_μ commutes with j. Let Q_μ be the field consisting of the elements of P_μ fixed by δ. Then Q_μ is totally real, and P_μ is a totally imaginary quadratic extension of Q_μ. Put $p = [Q_\mu : \mathbf{Q}]$, and take p isomorphisms $\chi_1, \ldots, \chi_p$ of P_μ into $\mathbf{C}$ such that $\chi_1, \ldots, \chi_p$ with their complex conjugates $\bar{\chi}_1, \ldots, \bar{\chi}_p$ give all the isomorphisms of P_μ into $\mathbf{C}$. Then $\chi_1, \ldots, \chi_p$ give all the isomorphisms of Q_μ into $\mathbf{C}$ (actually into $\mathbf{R}$). From the definition of Θ_μ and the fact that Q_μ is totally real, it follows that $\Theta_\mu|_{Q_\mu}$ contains all $\chi_\nu|_{Q_\mu}$, $\nu = 1, \ldots, p$, with the same multiplicity. Therefore $(\Theta_\mu + \overline{\Theta}_\mu)|_{P_\mu}$ contains all χ_ν and $\bar{\chi}_\nu$, $\nu = 1, \ldots, p$, with the same multiplicity. Here $\overline{\Theta}_\mu$ is the complex conjugate of Θ_μ. Since $D\otimes_K C_\mu^{-1}$ is a central simple algebra over P_μ, this shows that $\Theta_\mu + \overline{\Theta}_\mu$ contains all the inequivalent absolutely irreducible representations of $D\otimes_K C_\mu^{-1}$ with the same multiplicity. Identifying $W_{\mu\mathbf{R}}$ with $D_{\mathbf{R}}^{m} e_\mu$ and patching up the results for $\mu = 1, \ldots, t$, we get the proposition.

1.11. For $j_0 \in \mathfrak{J}$, let Ψ_{j_0} be the representation of $D\otimes_K C(j_0)^{-1}$ defined in 1.10. Then there is a representation Θ_{j_0} of K into $\mathbf{M}(mqg, \mathbf{C})$ such that $\Psi_{j_0}|_K$ is equivalent to $q\Theta_{j_0}$. Note that this Θ_{j_0} is quite different from Θ_μ used in the proof of Proposition 6 in 1.10. From Proposition 6, it follows that $\Theta_{j_0} + \overline{\Theta}_{j_0}$ contains all the (inequivalent) absolutely irreducible representations of K (i.e. all the isomorphisms of K into $\mathbf{C}$) with the same multiplicity. Let $(K'_{j_0}, \Theta'_{j_0})$ be the reflex of (K, Θ_{j_0}). See § 1 of Shimura [14] I for the definition. The field K'_{j_0} is generated by all the elements of $\{\mathrm{tr}(\Psi_{j_0}(a)) \mid a \in K\}$ over $\mathbf{Q}$ since $\mathrm{tr}(\Psi_{j_0}(a)) = q\ \mathrm{tr}(\Theta_{j_0}(a))$ for $a \in K$. It is known that K'_{j_0} is equal to $\mathbf{Q}$ if Θ_{j_0} is equivalent to $\overline{\Theta}_{j_0}$, the complex conjugate of Θ_{j_0}, and is a CM-field otherwise. (See 5.11–12 of Shimura [12].) Put

$$\xi_{j_0}(a') = \det(\Theta'_{j_0}(a')) \quad (a' \in K'^{\times}_{j_0}).$$

Then it is also known that ξ_{j_0} is a homomorphism of $K'^{\times}_{j_0}$ to $K^{\times}$ with the property

$$\xi_{j_0}(a')\,\xi_{j_0}(a')^{\delta} = N_{K'_{j_0}/\mathbf{Q}}(a')^{mq} \quad (a' \in K'^{\times}_{j_0}.)$$

This formula can also be derived from 1.4 of [14] I knowing that K'_{j_0} is a CM-field if $2 \nmid mq$.

Remark 1. If $j \in \mathcal{J}(j_0)$, then we see, on account of Proposition 6, that Θ_j defined by j in the same way is equivalent to Θ_{j_0}. Therefore Θ_j determines the same reflex $(K'_{j_0}, \Theta'_{j_0})$ and the same homomorphism ξ_{j_0} as Θ_{j_0} does. Hence the field $K'_{j_0\varepsilon}$ and the homomorphism $\xi_{j_0\varepsilon}$ of $K'^{\times}_{j_0\varepsilon}$ to $K^{\times}$ are determined for each $\mathcal{J}(j_0\varepsilon)$, $\varepsilon \in \mathcal{E}$, corresponding to the decomposition $\mathcal{J} = \bigcup_{\varepsilon \in \mathcal{E}} \mathcal{J}(j_0\varepsilon)$.

Remark 2. For $j_0 \in \mathcal{J}$, choose $\omega_1, \dots, \omega_g$ as in Corollary 1 of Proposition 2 in 1.4. Then $\omega_1, \dots, \omega_g$ determine g isomorphisms $\tau_1, \dots, \tau_g$ of K into $\mathbf{C}$ so that $\omega_\lambda|_K \sim mq\tau_\lambda$ for $\lambda = 1, \dots, g$. The set $\{\tau_1, \dots, \tau_g, \tau_1\delta, \dots, \tau_g\delta\}$ gives all the isomorphisms of K into $\mathbf{C}$. Let $(r(\lambda), s(\lambda))$, $\lambda = 1, \dots, g$, be the pairs of integers determined in Corollary 1 of Proposition 2. Then it can be shown that

$$\Theta_{j_0} \sim \sum_{\lambda=1}^{g} (r(\lambda)\tau_\lambda + s(\lambda)\tau_\lambda\delta).$$

Especially K'_{j_0} is equal to $\mathbf{Q}$ if and only if $r(\lambda) = s(\lambda) = mq/2$ for $\lambda = 1, \dots, g$. If this is the case, then all $\Theta_{j_0\varepsilon}$, $\varepsilon \in \mathcal{E}$, are equivalent, and all $K'_{j_0\varepsilon}$ are equal to $\mathbf{Q}$. Moreover, as we saw in 1.9, there is an element of $G_{\mathbf{R}}$ which gives an isomorphism of $\mathcal{H}_{j_0}$ onto $\mathcal{H}_{j_0\varepsilon}$ for every $\varepsilon \in \mathcal{E}$.

1.12. Let A be as in 1.9, and α any element of A, and take a $\mathbf{Q}$-linear transformation φ of D^m onto itself so that $\varphi(vx) = \varphi(v)\alpha(x)$ for $v \in D^m$ and $x \in B$. Then φ determines a $\mathbf{Q}$-linear automorphism α' of D such that $\varphi(dv) = \alpha'(d)\varphi(v)$ for $v \in D^m$ and $d \in D$. Obviously α' induces the same automorphism of K as α does. Thus we have an isomorphism $\alpha' \otimes \alpha$ of $D \otimes_K C(j_0)^{-1}$ onto $D \otimes_K C(\alpha(j_0))^{-1}$ for $j_0 \in \mathcal{J}$ since $\alpha(C(j_0)) = C(\alpha(j_0))$. We see easily that $\Psi_{\alpha(j_0)} \circ (\alpha' \otimes \alpha)$ is equivalent to Ψ_{j_0} as representations of $D \otimes_K C(j_0)^{-1}$. Therefore, especially, $\Theta_{\alpha(j_0)} \circ \alpha \sim \Theta_{j_0}$. Take $\varepsilon \in \mathcal{E}$ so that $\alpha(j_0) \in \mathcal{J}(j_0\varepsilon)$. Then $\Theta_{j_0\varepsilon} \circ \alpha \sim \Theta_{j_0}$. The fields K'_{j_0} and $K'_{j_0\varepsilon}$ are generated over $\mathbf{Q}$ by all the elements of the sets $\{\mathrm{tr}(\Theta_{j_0}(a)) \mid a \in K\}$ and $\{\mathrm{tr}(\Theta_{j_0\varepsilon}(a)) \mid a \in K\}$ respectively. Since α maps K onto itself, the equivalence of Θ_{j_0} and $\Theta_{j_0\varepsilon} \circ \alpha$ implies that K'_{j_0} is equal to $K'_{j_0\varepsilon}$.

PROPOSITION 7. *Let A be as in* 1.9, *and α any element of A. Take $\varepsilon \in \mathcal{E}$ so that $\alpha(\mathcal{J}(j_0)) = \mathcal{J}(j_0\varepsilon)$, and let $K'_{j_0\varepsilon}$, $K'_{j_0\varepsilon}$, ξ_{j_0}* and *$\xi_{j_0\varepsilon}$ be as in* 1.11 *for j_0 and $j_0\varepsilon$ respectively. Then $K'_{j_0} = K'_{j_0\varepsilon}$, and*

$$\xi_{j_0\varepsilon}(a') = \alpha(\xi_{j_0}(a')) \quad (a' \in K'^{\times}_{j_0} = K'^{\times}_{j_0\varepsilon}).$$

Proof. We have already seen that $K'_{j_0} = K'_{j_0\varepsilon}$. Put $K' = K'_{j_0}$. Let V be a (K, K')-module of type $(K, \Theta_{j_0\varepsilon})$ and of type $(K', \Theta'_{j_0\varepsilon})$. Since α is an automorphism of K, we can define another action of K on V by $a \cdot v = \alpha(a)v$ for $a \in K$ and $v \in V$ where $\alpha(a)v$ is the original action of the element $\alpha(a)$ of K on $v \in V$. Let us denote by V' the (K, K')-module V with this new action of K. Obviously V' is of type $(K, \Theta_{j_0\varepsilon} \circ \alpha)$. Let us fix a K-basis of V and represent an element a' of K' by a matrix $\Phi(a')$ with entries in K with the K-basis of V. Then Φ is equivalent to $\Theta'_{j_0\varepsilon}$. Since the fixed K-basis of V is also a K-basis of V', we have a representation Φ' of K' on V' with this K-basis. Then V' is of type (K', Φ'). Moreover, for $a' \in K'$, we have $\Phi'(a') = (\alpha^{-1}(a_{kl}))$ where $\Phi(a') = (a_{kl})$ with $a_{kl} \in K$. Now V' is of type $(K, \Theta_{j_0\varepsilon} \circ \alpha)$, and $\Theta_{j_0\varepsilon} \circ \alpha$ is equivalent to Θ_{j_0} as was seen. Therefore the uniqueness of the (K, K')-module of type (K, Θ_{j_0}), which is assured in 1.2 of [14] I, implies that V' is of type (K', Θ'_{j_0}). This means that Φ' is equivalent to Θ'_{j_0}. Hence we have $\xi_{j_0}(a') = \det(\Theta'_{j_0}(a')) = \det(\Phi'(a')) = \alpha^{-1}(\det(\Phi(a'))) = \alpha^{-1}(\det(\Theta'_{j_0\varepsilon}(a'))) = \alpha^{-1}(\xi_{j_0\varepsilon}(a'))$ for $a' \in K'$. The proof is done.

2. Isolated elements of $\mathcal{J}$ and commutative isolating subalgebras of B

2.1. For $j \in \mathcal{J}$, let $C(j)$ and $\mathcal{J}_{\tilde{C}(j)}$ be as in 1.5 and 1.6 respectively. We say that j is *isolated* if $\mathcal{J}_{\tilde{C}(j)} = \{j\varepsilon \mid \varepsilon \in \mathcal{E}\}$. The existence of an isolated element of $\mathcal{J}$ will be seen in 2.4. It follows from the corollary of Proposition 4 in 1.6 that j is isolated if and only if $C(j)$ contains $\widetilde{C(j)}$. Hence we have

(2.1.1) *An element j of $\mathcal{J}$ is isolated if and only if the commutor $\widetilde{C(j)}$ of $C(j)$ in B coincides with the center of $C(j)$.*

Let A be as in 1.9. Then the following assertion is easy to see.

(2.1.2) *If $j \in \mathcal{J}$ is isolated, then $\alpha(j)$ is also isolated for each $\alpha \in A$.*

Let C be a subalgebra of B satisfying the conditions (1.5.1–3). Then $\mathcal{J}_{\tilde{C}}$ is not empty as was seen in Proposition 3 in 1.5. We say that a subalgebra C of B is *an isolating subalgebra* of B if C satsfies the conditions (1.5.1–3) and $\mathcal{J}_{\tilde{C}} = \{j\varepsilon \mid \varepsilon \in \mathcal{E}\}$ for some j in $\mathcal{J}$. We also say that C *isolates* j, and j *is isolated by* C if C is an isolating subalgebra of B, and j belongs to $\mathcal{J}_{\tilde{C}}$. Note that every isolating subalgebra of B isolates one and only one element of $\mathcal{J}(j_0\varepsilon)$ for each $\varepsilon \in \mathcal{E}$. Obviously an element j of $\mathcal{J}$ is isolated if and only if $C(j)$ is an isolating subalgebra of B. Moreover $C(j)$ for an isolated element j of $\mathfrak{J}$ is a maximal one among isolating subalgebras of B.

Let j be an isolated element of $\mathfrak{J}$, and $P(j)$ the center of $C(j)$. Then (2.1.1) shows that the commutor $\widetilde{P(j)}$ of $P(j)$ in B is nothing but $C(j)$. Hence $P(j)$ is a commutative isolating subalgebra of B on account of the corollary of Proposition 4, and is a minimal one among isolating subalgebras of B. We see easily

(2.1.3) *For any isolated element j of $\mathfrak{J}$, the center $P(j)$ of $C(j)$ is the smallest among those isolating subalgebras of B which isolate j.*

Let P be a commutative isolating subalgebra of B, and j an element of $\mathfrak{J}$ isolated by P. Then P is a direct sum of CM-fields and contains $P(j)$. Put $C=\tilde{P}$, the commutor of P in B. Then C is contained in $C(j)$ and is an isolating subalgebra of B. Since P contains the center K of B, P is the center of C and the commutor of C in B.

2.2. PROPOSITION 8. *Let $C=C_1\oplus\dots\oplus C_t$ be a semi-simple subalgebra of B with simple components C_μ, $\mu=1,\dots,t$, and $P=P_1\oplus\dots\oplus P_t$ the center of C where P_μ is the center of C_μ for $\mu=1,\dots,t$. Suppose that the commutor of C in B coincides with P, and let C_μ^{-1} be the reciprocal (or inverse) algebra of C_μ and $q_\mu^2=[C_\mu: P_\mu]$, $q_\mu>0$ for $\mu=1,\dots,t$. Then for each μ, $D\otimes_K C_\mu^{-1}$ is P_μ-linearly isomorphic to $\mathbf{M}(qq_\mu, P_\mu)$, the full matrix algebra of size qq_μ over P_μ, and*

$$\sum_{\mu=1}^{t} qq_\mu[P_\mu: \mathbf{Q}] = [D^m: \mathbf{Q}] = 2gmq^2.$$

Proof. Let $1=e_1+\dots+e_t$ with $e_\mu\in P_\mu$, $\mu=1,\dots,t$. Then the submodule $D^m e_\mu$ of D^m is a left D-module and is isomorphic to D^{m_μ} for some integer m_μ since D is a division algebra. Therefore $e_\mu Be_\mu$ is isomorphic to $\mathbf{M}(m_\mu, D)$. From the assumption, it follows that the commutor of C_μ in $e_\mu Be_\mu$ coincides with P_μ. Hence we have

$$[C_\mu: K][P_\mu: K] = [e_\mu Be_\mu: K] = m_\mu^2[D: K].$$

Now define the action of $D\otimes_K C_\mu^{-1}$ on $D^m e_\mu$ by $(d\otimes c)\cdot v=dvc$ for $v\in D^m e_\mu$, $d\in D$ and $c\in C_\mu^{-1}$. Then $D\otimes_K C_\mu^{-1}$ acts P_μ-linearly on $D^m e_\mu$. Obviously, the action is faithful. Therefore $D\otimes_K C_\mu^{-1}$ is P_μ-linearly isomorphic to a subalgebra of $\mathbf{M}(n_\mu, P_\mu)$ where n_μ is the dimension of the vector space $D^m e_\mu$ over P_μ. We have $n_\mu=m_\mu[D: K]/[P_\mu: K]$. On the other hand, $[D\otimes_K C_\mu^{-1}: P_\mu]=[D: K][C_\mu: K]/[P_\mu: K]=m_\mu^2[D: K]^2/[P_\mu: K]^2=n_\mu^2$. This shows that $D\otimes_K C_\mu^{-1}$ is isomorphic to $\mathbf{M}(n_\mu, P_\mu)$ itself, and that $n_\mu=qq_\mu$. Since $D^m=D^m e_1\oplus\dots\oplus D^m e_t$, the last formula of the proposition is clear. The proof is done.

2.3. Let $P=P_1\oplus\dots\oplus P_t$ be a commutative isolating subalgebra of B where P_μ is a CM-field for $\mu=1,\dots,t$, and C the commutor of P in B. Then C is also an isolating subalgebra of B, and $C = C_1 \oplus \ldots \oplus C_t$ with central simple algebras C_μ over P_μ, $\mu = 1, \ldots, t$. Put $[C_\mu : P_\mu] = q_\mu^2$, $q_\mu > 0$ for $\mu = 1, \ldots, t$. This C satisfies the assumption of Proposition 8. Let j be an element of $\mathfrak{J}$ isolated by P. Then C isolates j, i.e. $j \in \mathfrak{J}_{\bar{C}}$, since P and C are both contained in $C(j)$. Let Ψ_j be the representation of $D \otimes_K C^{-1}$ defined by j in 1.10. On account of Proposition 8, we can find a representation $\Phi_{j,\mu}$ of P_μ for each μ, $1 \leqslant \mu \leqslant t$, such that

$$\Psi_j|_{P_\mu} \sim q q_\mu \Phi_{j,\mu} + (\text{zero representation}).$$

It follows from the last equality of Proposition 8, and from Proposition 6 in 1.10 that

(2.3.1) $\quad (P_\mu, \Phi_{j,\mu})$ *is a CM-type for each* $\mu = 1, \ldots, t$.

See 1.8 of [14] I for the definition of a CM-type. Let $(P'_\mu, \Phi'_{j,\mu})$ be the reflex of $(P_\mu, \Phi_{j,\mu})$. Put $R_j(P) = P'_1 \ldots P'_t$, the composite field of P'_μ, $\mu = 1, \ldots, t$. Define a mapping η_j of $R_j(P)$ to P by

$$\eta_j(a') = \sum_{\mu=1}^{t} \det(\Phi'_{j,\mu}(N_{R_j(P)/P'_\mu}(a')))\, e_\mu$$

for $a' \in R_j(P)$ where $1 = e_1 + \ldots + e_t$ with $e_\mu \in P_\mu$, $\mu = 1, \ldots, t$. It is clear that $R_j(P)$ is generated over $\mathbf{Q}$ by the elements of $\{\operatorname{tr}(\Psi_j(a)) \mid a \in P\}$. Let ε be the element of $\mathcal{E}$ such that $j \in \mathfrak{J}(j_0\varepsilon)$. Then it is easy to see that the field $K'_{j_0\varepsilon}$ defined in 1.11 is contained in $R_j(P)$, and that

$$\Theta_{j_0\varepsilon} \sim \sum_{\mu=1}^{t} q_\mu \Phi_{j,\mu}|_K.$$

PROPOSITION 9. *Let P be a commutative isolating subalgebra of B, and j the element of $\mathfrak{J}(j_0\varepsilon)$ isolated by P for $\varepsilon \in \mathcal{E}$. Let $R_j(P)$ and η_j be as above, and $K'_{j_0\varepsilon}$ and $\xi_{j_0\varepsilon}$ as in* 1.11. *Then $R_j(P)$ contains $K'_{j_0\varepsilon}$, and, for $a' \in R_j(P)^\times$,*

$$\eta_j(a')\eta_j(a')^\delta = N_{R_j(P)/\mathbf{Q}}(a');$$

$$N(\eta_j(a')) = \xi_{j_0\varepsilon}(N_{R_j(P)/K'_{j_0\varepsilon}}(a')).$$

Proof. It is enough to show the last two formulae. Since the reflex of a CM-type is again a CM-type (cf. 5.13 of [12]), the first formula is obvious. Let us show the last formula. Put $P' = R_j(P)$, and let us use the notation introduced above. For each μ, let V_μ be a (P_μ, P'_μ)-module of type $(P_\mu, \Phi_{j,\mu})$. Then $V_{\mu P'} = V_\mu \otimes_{P'_\mu} P'$ is a (P_μ, P')-module of type $(P_\mu, \Phi_{j,\mu})$. Put $W_\mu = V_{\mu P'} \times \ldots \times V_{\mu P'}$ (q_μ times). Then W_μ is a (P_μ, P')-module of type $(P_\mu, q_\mu \Phi_{j,\mu})$. Put $W = W_1 \times \ldots \times W_t$, and define the action of $P = P_1 \oplus \ldots \oplus P_t$ on W by $e_\mu W_\nu = 0$ if $\nu \neq \mu$. This action of an element $a = ae_1 + \ldots + ae_t$ of K on W makes W a (K, P')-module of type $(K, \Theta_{j_0\varepsilon})$ because of the formula followed by the proposition. Let Z be a $(K, K'_{j_0\varepsilon})$-module of type $(K, \Theta_{j_0\varepsilon})$. Then $Z_{P'} = Z \otimes_{K'_{j_0\varepsilon}} P'$ is a (K, P')-module of type

$(K, \Theta_{j_0\varepsilon})$, and hence, is isomorphic to W as (K, P')-modules. (See 1.2 of [14] I.) Take a representation Φ' of P' so that $Z_{P'}$ is of type (P', Φ'). Since $q_\mu^2=[C_\mu : P_\mu]$, the formula follows easily from computing $\det(\Phi')$ in two different ways using W and $Z_{P'}$.

Remark. From the first formula of the proposition and 1.7 of [14] I, it follows that η_j is a $\mathbf{Q}$-rational homomorphism of $R_j(P)^\times$, considered as a $\mathbf{Q}$-rational algebraic group, to G. Since $R_j(P)$ is a CM-field, we see that $\eta_j(R_j(P)^\times)\subset G_{\mathbf{Q}+}$.

PROPOSITION 10. *Let P be a commutative isolating subalgebra of B, and j an element of $\mathfrak{J}$ isolated by P. Let A be as in 1.9. Then, for any $\alpha\in A$, $\alpha(P)$ is a commutative isolating subalgebra of B, which isolates $\alpha(j)$. Moreover $R_{\alpha(j)}(\alpha(P))$ coincides with $R_j(P)$, and*

$$\eta_{\alpha(j)}(a') = \alpha(\eta_j(a')) \quad (a'\in R_j(P)^\times).$$

The proof is omitted since the proposition can be shown in a straightforward way for each simple component of P with a similar argument to that used in the proof of Proposition 7 in 1.12.

2.4. PROPOSITION 11. *Let L be any given finite algebraic extension of $K'_{j_0\varepsilon}$ for any fixed $\varepsilon\in\mathcal{E}$. Then there exists a commutative isolating subalgebra P of B such that $R_j(P)$ is linearly disjoint with L over $K'_{j_0\varepsilon}$ where j is the element of $\mathfrak{J}(j_0\varepsilon)$ isolated by P. Moreover P can be taken to be a CM-field containing K with $[P:K]=mq$.*

Proof. As we saw in Proposition 1 in 1.2, there exists a positive involution ϱ of B which coincides with δ on K. Since ϱ is of the second kind, there is an element h in B such that $h^\varrho=h$ and $x^\delta=hx^\varrho h^{-1}$ for all $x\in B$. Let $\tau_1, \ldots, \tau_g$ be g isomorphisms of K into $\mathbf{C}$ such that $\tau_1, \ldots, \tau_g, \tau_1\varrho, \ldots, \tau_g\varrho$ are all the isomorphisms of K into $\mathbf{C}$. We can choose g absolutely irreducible representations $\chi_1, \ldots, \chi_g$ of B into $\mathbf{M}(mq, \mathbf{C})$ so that, for $\lambda=1, \ldots, g$,

$$\chi_\lambda(a)=\tau_\lambda(a)1_{mq} \quad (a\in K);$$
$$\chi_\lambda(x^\varrho)={}^t\overline{\chi_\lambda(x)} \qquad (x\in B).$$

For the latter condition, see Lemma 1 of Shimura [9]. Let ζ be an element of K such that $\zeta^\varrho=-\zeta$, and take $n=(mq)^2$ elements $e_1, \ldots, e_n$ of B so that $e_\mu^\varrho=e_\mu$ for $\mu=1, \ldots, n$ and $e_1, \ldots, e_n$ span B over K. Such n elements exist since $B=B_+\oplus\zeta B_+$ where $B_+=\{x\in B \mid x^\varrho=x\}$ is a vector space of dimension n over F. Note that ${}^t\overline{\chi_\lambda(e_\mu)}=\chi_\lambda(e_\mu)$ for $\lambda=1, \ldots, g$ and $\mu=1, \ldots, n$. Let $a_1, \ldots, a_g$ be a basis of F over $\mathbf{Q}$. Take gn independent variables x_μ^ν ($\mu=1, \ldots, n$; $\nu=1, \ldots, g$) and put

$$Y_\lambda=\sum_{\nu=1}^{g}\tau_\lambda(a_\nu)\sum_{\mu=1}^{n}x_\mu^\nu\chi_\lambda(e_\mu) \quad (\lambda=1,\ldots,g).$$

Take another independent variable z, and put, for $\lambda = 1, ..., g$,

$$\psi_\lambda(z) = \psi_\lambda(z; x_\mu^\nu) = \det (z 1_{mq} - Y_\lambda \chi_\lambda(h)^t Y_\lambda).$$

If z and x_μ^ν move in $\mathbf{Q}$, then, for each λ, $\psi_\lambda(z; x_\mu^\nu)$ gives a reduced norm of an element of $\chi_\lambda(B)$ over $\tau_\lambda(K)$, and belongs to $\tau_\lambda(F)$. This shows that the polynomial $\psi_\lambda(z; x_\mu^\nu)$ has the coefficients in $\tau_\lambda(F)$. In a similar way, we see that $\psi(z) = \psi_1(z) \dots \psi_g(z)$ belongs to $\mathbf{Q}[z, x_\mu^\nu]$. Using these ψ_λ, $\lambda = 1, ..., g$, we can apply the argument of 4.10–15 and 2.1–4 of [10] I and III respectively with slight modification to our case. Although our $\chi_\lambda(h)$, $\lambda = 1, ..., g$, are not symmetric but are hermitian, we can easily derive, from 4.11 of [10] I, a similar result for a hermitian matrix, and then the rest of the argument works almost as it is. Hence we conclude that there exist an algebraic number y_1 and an element E of B of the form $E = dhd^\varrho$ with $d \in B$ such that

(i) $K(y_1)$ is a CM-field containing K with $[K(y_1): K] = mq$;

(ii) Assigning E to y_1, we have an isomorphism of $K(y_1)$ onto the subalgebra $K[E]$ of B generated by E over K;

(iii) For any such CM-type $(K(y_1), \Phi)$ as $\Phi|_K \sim \Phi_{j_0 \varepsilon}$, the field $K(y_1)'$ generated by all the elements of $\{\mathrm{tr}\,(\Phi(x)) \,|\, x \in K(y_1)\}$ over $\mathbf{Q}$ is linearly disjoint with the given field L over $K'_{j_0\varepsilon}$. Put $E_1 = d^{-1} E d = h d^\varrho d$ and $P = K[E_1]$, the subalgebra of B generated by E_1 over K. Then since $E_1^\delta = h E_1^\varrho h^{-1} = E_1$ and P is a CM-field, δ must be a positive involution on P. Moreover the commutor $\tilde{P}$ of P in B is P itself since $[P: K] = mq$. Therefore $\mathcal{J}_{\tilde{P}} = \mathcal{J}_P = \{j_1 \varepsilon \,|\, \varepsilon \in \mathcal{E}\}$ for some $j_1 \in \mathcal{J}$. Take $j \in \mathcal{J}_{\tilde{P}} \cap \mathcal{J}(j_0 \varepsilon)$, Then (iii) implies that $R_j(P)$ is linearly disjoint with L over $K'_{j_0\varepsilon}$. The proposition is proved.

2.5. As we saw at the beginning of 1.4 and in Proposition 2 in 1.4, there is a one-to-one correspondence between $\mathcal{J}(j_0\varepsilon)$ and $\mathcal{H}_{j_0\varepsilon}$ for each $\varepsilon \in \mathcal{E}$. We fixed such a correspondence at the beginning of 1.7, and defined the action of $G_{\mathbf{R}+}$ on $\mathcal{H}_{j_0\varepsilon}$ in 1.9.

Let z and j be the corresponding elements of $\mathcal{H}_{j_0\varepsilon}$ and of $\mathcal{J}(j_0\varepsilon)$ respectively. Then we have

$$G_{\mathbf{Q}+} \cap C(j) = \{\gamma \in G_{\mathbf{Q}+} \,|\, \gamma(z) = z\}.$$

On account of 1.6 of Shimura [13], it is clear that the elements of $G_{\mathbf{Q}+} \cap C(j)$ span $C(j)$ over $\mathbf{Q}$. We see easily, moreover, that j is an isolated element of $\mathcal{J}(j_0\varepsilon)$ if and only if

$$\{z\} = \{z' \in \mathcal{H}_{j_0\varepsilon} \,|\, \gamma(z') = z' \quad \text{for every} \quad \gamma \in G_{\mathbf{Q}+} \cap C(j)\}.$$

It follows from Proposition 2 in 1.4 and 1.9 that $G_{\mathbf{R}+}$ acts on $\mathcal{H}_{j_0\varepsilon}$ transitively. Since $G_{\mathbf{Q}+}$ is dense in $G_{\mathbf{R}+}$, the $G_{\mathbf{Q}+}$-orbit of an arbitrary point on $\mathcal{H}_{j_0\varepsilon}$ is dense in $\mathcal{H}_{j_0\varepsilon}$. Now let P be any given commutative isolating subalgebra of B, j the element of $\mathcal{J}(j_0\varepsilon)$ isolated by P, and z the isolated fixed point on $\mathcal{H}_{j_0\varepsilon}$ corresponding to j. Then since $G_{\mathbf{Q}+}$ is reduced to the subgroup $A_+^0=G_{\mathbf{Q}+}/K^\times$ of A as groups of transformations on $\mathcal{H}_{j_0\varepsilon}$, we conclude, on account of Proposition 10, that there are densely many isolated fixed points $\alpha(z)$, $\alpha\in A_+^0$, on $\mathcal{H}_{j_0\varepsilon}$ which give the same field $R_{\alpha(j)}(\alpha(P))=R_j(P)$. We use this fact with Proposition 11 later in 4.3–5 and in 5.10–12.

3. The adelization of G and the group $\mathfrak{A}_{j_0}$

Hereafter, we develop our theory on the fixed hermitian symmetric space $\mathcal{H}_{j_0}$. The theory on another $\mathcal{H}_{j_0\varepsilon}^{\mathbf{1}}$ for $\varepsilon\in\mathcal{E}$ is obtained simply by replacing j_0 by $j_0\varepsilon$.

3.1. First we show some lemmas on the idele group of an algebraic number field. Let P be an algebraic number field of finite degree.

LEMMA 1. *Let E_+ be the multiplicative group of all the totally positive units in P, $E_{+\mathfrak{f}}$ the projection of E_+ to the non-archimedian part $P_{\mathfrak{f}}^\times$ of $P_{\mathbf{A}}^\times$, and $\overline{E_{+\mathfrak{f}}}$ the closure of $E_{+\mathfrak{f}}$ in $P^\times$. Then the closure $P^{\#}$ of $P^\times P_{\infty+}^\times$ in $P_{\mathbf{A}}^\times$ is equal to $\overline{E_{+\mathfrak{f}}}P^\times P_{\infty+}^\times$. Moreover, for every positive integer n*

$$\overline{E_{+\mathfrak{f}}}=E_{+\mathfrak{f}}\overline{E_{+\mathfrak{f}}}^n;\ P^{\#}=P^\times P^{\#n};\ P^\times\cap P^{\#n}=P^{\times n}.$$

Here $\overline{E_{+\mathfrak{f}}}^n$, $P^{\#n}$, and $P^{\times n}$ are the groups of all the n-th powers of the elements of $\overline{E_{+\mathfrak{f}}}$, $P^{\#}$, and $P^\times$ respectively.

Proof. On account of 2.2 of Shimura [14] II, it is sufficient to show $P^\times\cap P^{\#n}=P^{\times n}$. Obviously $P^{\times n}$ is contained in the other. Let a be an element of $P^\times\cap P^{\#n}$. Take $b\in P^{\#}$ so that $a=b^n$. For any open subgroup U of $P_{\mathfrak{f}}^\times$, there is an element c of $P^\times$ such that bc^{-1} belongs to $UP_{\infty+}^\times$. Then $ac^{-n}=(bc^{-1})^n$ belongs to $UP_{\infty+}^\times$. Therefore, by Chevalley [4], we see that there is a totally positive element d in $P^\times$ such that $ac^{-n}=d^n$ if we take a sufficiently small open compact subgroup U of $P^\times$. Hence $a=(cd)^n\in P^{\times n}$.

Remark. Let the notation and the assumption be as in the above proof. If a is totally positive, then we can take a totally positive element as cd. In fact, if n is odd, then cd must be totally positive. If n is even, then we can take a totally positive b. Then it follows from the choice of c that c is totally positive, and so is cd.

LEMMA 2. *Let a be an element of $P^{\#}$. If $a^n=1$ for some non-zero integer n, then a belongs to $P^\times P_{\infty+}^\times$.*

Proof. Let μ be the number of the roots of 1 in P. Since $P^{\#}=P^\times P^{\#\mu\nu}$ for any positive integer ν, there are an element a_ν of $P^\times$ and an element b_ν of $P^{\#}$ sch that $a=a_\nu b_\nu^{\mu\nu}$. Then $a_\nu^n b_\nu^{n\mu\nu}=1$, and so, a_ν^n belongs to $P^\times\cap P^{\#n\mu\nu}=P^{\times n\mu\nu}$. Take $c_\nu\in P^\times$ so that $a_\nu^n=c^{n\mu\nu}$, and put $\zeta_\nu=a_\nu c_\nu^{-\mu\nu}$. Then ζ_ν is an n-th root of 1 in P. Moreover $\zeta_1^{-1}\zeta_\nu=(b_1c_1b_\nu^{-\nu}c_\nu^{-\nu})^\mu\in P^\times\cap P^{\#\mu}=$

$P^{\times\mu}$. Let $\zeta \in P^\times$ be such that $\zeta^\mu = \zeta_1^{-1}\zeta_\nu$. Obviously ζ is a root of 1 in P. Hence we have $\zeta^\mu = 1$ by the choice of μ. This means $\zeta_\nu = \zeta_1$ and $\zeta_1^{-1}a \in P^{\neq\mu\nu}$ for any positive integer ν. Since μ is even, $\zeta_1^{-1}a$ is totally positive. Let $\mathfrak{p}$ be any non-archimedian place of P, and $a_\mathfrak{p}$ the $\mathfrak{p}$-component of a. Then $(\zeta_1^{-1}a_\mathfrak{p})^n = 1$. Taking a multiple of the number of the roots of 1 in $P_\mathfrak{p}$ for ν, we see easily that $a_\mathfrak{p} = \zeta_1$. This shows that $a \in P^\times P_{\infty+}^\times$.

LEMMA 3. *For any positive integer n and any open subgroup U of $P_\mathfrak{f}^\times$, there exists an open subgroup V of $P_\mathfrak{f}^\times$ such that*

$$P^{\neq} \cap VP_{\infty+}^\times \subset \{a^n \mid a \in P^{\neq} \cap UP_{\infty+}^\times\}.$$

Proof. We may assume that U is compact. By Chevalley [4], we can find an open compact subgroup V of $P_\mathfrak{f}^\times$ such that $P^\times \cap VP_{\infty+}^\times \subset \{a^n \mid a \in P^\times \cap UP_{\infty+}^\times\}$. Then $(P^\times P_{\infty+}^\times) \cap VP_{\infty+}^\times = (P^\times \cap VP_{\infty+}^\times)P_{\infty+}^\times$ is contained in the set $\{a^n \mid a \in P^\times P_{\infty+}^\times \cap UP_{\infty+}^\times\}$. Since both $VP_{\infty+}^\times$ and $UP_{\infty+}^\times$ are open and closed, we get the lemma by taking the closures of $P^\times P_{\infty+}^\times \cap VP_{\infty+}^\times$ and $\{a^n \mid a \in P^\times P_{\infty+}^\times \cap UP_{\infty+}^\times\}$.

LEMMA 4. *Let Q be a finite algebraic extension of P. Then*

$$P_\mathbf{A}^\times \cap Q^{\neq} = P^{\neq}(P_\infty^\times \cap Q_{\infty+}^\times).$$

Proof. Obviously $P_\mathbf{A}^\times \cap Q^{\neq}$ contains the other. Let a be any element of $P_\mathbf{A}^\times \cap Q^{\neq}$, and $n = [Q:P]$. Then $a^n = N_{Q/P}(a)$ is contained in both $P^{\neq} = P^\times P^{\neq n}$ and $Q^{\neq n}$. Put $a^n = bc^n$ with $b \in P^\times$ and $c \in P^{\neq}$. Then $b = (ac^{-1})^n \in P^\times \cap Q^{\neq n} = P^\times \cap Q^{\times n}$. Take $d \in Q^\times$ so that $b = d^n$, and put $e = ac^{-1}d^{-1}$. Then $e^n = 1$ and $e \in Q^{\neq}$. Therefore e belongs to $Q^\times Q_{\infty+}^\times$. Hence we have $de = ac^{-1} \in Q^\times Q_{\infty+}^\times \cap P_\mathbf{A}^\times = P^\times(P_\infty^\times \cap Q_{\infty+}^\times)$, and so, $a = cde \in P^{\neq}(P_\infty^\times \cap Q_{\infty+}^\times)$. The proof is done.

3.2. Let K'_{j_0} and ξ_{j_0} be as in 1.11. Define a homomorphism φ of $K_{j_0}^{\prime\times}$ to $K^\times \times F^\times$ by, for $a' \in K_{j_0}^{\prime\times}$,

$$\varphi(a') = (\xi_{j_0}(a'), N_{K'_{j_0}/\mathbf{Q}}(a')). \tag{3.2.1}$$

Then from 1.7 of [14] I follows that φ extends to a continuous homomorphism of $K_{j_0\mathbf{A}}^{\prime\times}$ to $K_\mathbf{A}^\times \times F_\mathbf{A}^\times$. From the class field theory, it follows that the quotient group $(K_\mathbf{A}^\times \times F_\mathbf{A}^\times)/(K^{\neq} \times F^{\neq})$ is a compact group. Put

$$\mathfrak{W}_{j_0} = \varphi(K_{j_0\mathbf{A}}^{\prime\times})(K^{\neq} \times F^{\neq})/(K^{\neq} \times F^{\neq}).$$

Since $\varphi^{-1}(K^{\neq} \times F^{\neq})$ is a closed subgroup of $K_j^{\prime\times}$ containing $K_{j_0}^{\prime\neq}$, it defines an abelian extension of K'_{j_0}. Let $\mathfrak{K}_{j_0}$ be the subfield of $K'_{j_0\mathrm{ab}}$. Then φ induces a continuous homomorphism $\bar\varphi$ of Gal $(\mathfrak{K}_{j_0}/K'_{j_0})$ onto $\mathfrak{W}_{j_0}$ such that, for $a' \in K_{j_0\mathbf{A}}^{\prime\times}$,

$$\bar\varphi([a', K'_{j_0}]) = \varphi(a') \text{ modulo } (K^{\neq} \times F^{\neq}).$$

PROPOSITION 12. *Let the notation be as above. The homomorphism* $\bar{\varphi}$ *is a (topological) isomorphism of* Gal $(\mathfrak{K}_{j_0}/K'_{j_0})$ *onto* $\mathfrak{W}_{j_0}$. *The field* $\mathfrak{K}_{j_0}$ *is an abelian extension of* K'_{j_0} *and contains* $\mathbf{Q}_{ab}$.

Proof. It is obvious that $\bar{\varphi}$ is continuous, one-to-one and surjective. Let us show that it is open. Let U be an open subgroup of Gal $(\mathfrak{K}_{j_0}/K'_{j_0})$. Then it is compact and of finite index. Therefore $\bar{\varphi}(U)$ is compact and of finite index in $\mathfrak{W}_{j_0}$. Hence $\bar{\varphi}(U)$ must be open. Now let us show that $\mathfrak{K}_{j_0}$ contains Q_{ab}. Let a' be any element of $K'^{\times}_{j_0\mathbf{A}}$ such that $\varphi(a')$ is in $K^{\#}\times F^{\#}$. Then, especially, $N_{K'_{j_0}\mathbf{Q}}(a')$ belongs to $\mathbf{Q}^{\#}=\mathbf{Q}^{\times}_{\mathbf{A}}\cap F^{\#}$. (See Lemma 4.) Therefore $[a', K'_{j_0}]\,|\,_{\mathbf{Q}ab}=[N_{K'_{j_0}\mathbf{Q}}(a'), \mathbf{Q}]$ is the identity on $\mathbf{Q}_{ab}$. The proof is completed.

3.3. Let G be the algebraic group defined in 1.3, and $G_{\mathbf{A}}$ the adelization of G. We define a continuous homomorphism ψ of $G_{\mathbf{A}}$ to $K^{\times}_{\mathbf{A}}\times F^{\times}_{\mathbf{A}}$ by

$$\psi(x)=(N(x),\,\nu(x))\quad (x\in G_{\mathbf{A}}). \tag{3.3.1}$$

The group $G_{\mathbf{A}+}=G_{\mathfrak{f}}G_{\infty+}$ is a closed normal subgroup of $G_{\mathbf{A}}$. Let $G^{\#}$ (resp. $G^{\#}_{+}$) denote the closure of $G_{\mathbf{Q}}G_{\infty+}$ (resp. $G_{\mathbf{Q}+}G_{\infty+}$) in $G_{\mathbf{A}}$. Then $G^{\#}_{+}$ is a closed subgroup of $G_{\mathbf{A}+}$.

PROPOSITION 13. *Let the notation be as above. Then*

$$G^{\#}=K^{\#}G_{\mathbf{Q}}G^{1}_{\mathbf{A}}=\psi^{-1}(K^{\#}\times F^{\#}),$$

$$G^{\#}_{+}=K^{\#}G_{\mathbf{Q}+}\;G^{1}_{\mathbf{A}}=\psi^{-1}(K^{\#}\times F^{\#})\cap G_{\mathbf{A}+}=G^{\#}\cap G_{\mathbf{A}+}.$$

Proof. The strong approximation theorem for G^1 of Kneser [5] shows that $G^1_{\mathbf{A}}$ is contained in $G_{\mathbf{Q}+}UG_{\infty+}$ for any open subgroup U of $G_{\mathfrak{f}}$. Therefore $G^1_{\mathbf{A}}$ is contined in $G^{\#}_{+}$. Then the inclusions,

$$K^{\#}G_{\mathbf{Q}}G^{1}_{\mathbf{A}}\subset G^{\#}\subset\psi^{-1}(K^{\#}\times F^{\#}),$$

and

$$K^{\#}G_{\mathbf{Q}+}G^{1}_{\mathbf{A}}\subset G^{\#}_{+}\subset\psi^{-1}(K^{\#}\times F^{\#})\cap G_{\mathbf{A}+},$$

are clear. Since the inclusion, $K^{\#}G_{\mathbf{Q}}G^1_{\mathbf{A}}\supset\psi^{-1}(K^{\#}\times F^{\#})$, implies that $(K^{\#}G_{\mathbf{Q}}G^1_{\mathbf{A}})\cap G_{\mathbf{A}+}=K^{\#}G_{\mathbf{Q}+}G^1_{\mathbf{A}}\supset\psi^{-1}(K^{\#}\times F^{\#})\cap G_{\mathbf{A}+}$, it is enough to show $K^{\#}G_{\mathbf{Q}}G^1_{\mathbf{A}}\supset\psi^{-1}(K^{\#}\times F^{\#})$. Now let x be an element of $G_{\mathbf{A}}$ such that $\psi(x)\in K^{\#}\times F^{\#}$, and put $\psi(x)=(N(x),\nu(x))=(a,b)$. Then $aa^{\delta}=b^{mq}$. Since $K^{\#}=K^{\times}K^{\#mq}$ (see Lemma 1), we can find $c\in K^{\times}$ and $d\in K^{\#}$ so that $a=cd^{mq}$. Then $cc^{\delta}=aa^{\delta}(d^{-1}d^{-\delta})^{mq}$ belongs to $F^{\times}\cap F^{\times mq}=F^{\times mq}$. Take $e\in F^{\times}$ so that $cc^{\delta}=e^{mq}$. Put $\zeta=b^{-1}edd^{\delta}$. Then $\zeta\in F^{\#}$ and $\zeta^{mq}=1$. Hence $\zeta\in F^{\times}F^{\times}_{\infty+}$ (see Lemma 2). Since F is totally real, we have $\zeta=\pm 1$. Replacing e with ζe, we have $\psi(x)=(cd^{mq}, edd^{\delta})$ with $c\in K^{\times}$, $d\in K^{\#}$ and $e\in F^{\times}$. Since $(c,e)=\psi(xd^{-1})$, it follows from the Hasse principle for G^1 that $(c,e)=\psi(\gamma)$ with some

$\gamma \in G_{\mathbf{Q}}$. (See M. Kneser, *Hasse principle for* H^1 *of simply connected groups*, [1], pp. 159–163.) Put $y = \gamma^{-1} d^{-1} x$. Then we have $y \in G^1_{\mathbf{A}}$. This proves the proposition.

PROPOSITION 14. *$K^{\#} G_{\infty +}$ is the closure of $K^{\times} G_{\infty +}$ in $G_{\mathbf{A}}$, and contained in $G^{\#}_{+}$. Moreover, $G_{\mathbf{Q}+} \cap K^{\#} G_{\infty +} = K^{\times}$.*

This is obvious.

3.4. Let $\bar{\varphi}$ be the continuous homomorphism of $G_{\mathbf{A}}$ to the quotient group $(K^{\times}_{\mathbf{A}} \times F^{\times}_{\mathbf{A}})/(K^{\#} \times F^{\#})$ obtained from φ and the natural projection of $K^{\times}_{\mathbf{A}} \times F^{\times}_{\mathbf{A}}$ onto the quotient group. Put

$$\mathcal{G}_{j_0} = \bar{\varphi}^{-1}(\mathfrak{W}_{j_0}),$$

$$\mathcal{G}_{j_0 +} = \mathcal{G}_{j_0} \cap G_{\mathbf{A}+} = \bar{\varphi}^{-1}(\mathfrak{W}_{j_0}) \cap G_{\mathbf{A}+}.$$

Then $\mathcal{G}_{j_0}$ (resp. $\mathcal{G}_{j_0+}$) is a closed normal subgroup of $G_{\mathbf{A}}$ (resp. $G_{\mathbf{A}+}$).

PROPOSITION 15. *Let the notation be as above. Then the restriction of $\bar{\varphi}$ to $\mathcal{G}_{j_0+}$ gives an open continuous homomorphism of $\mathcal{G}_{j_0+}$ onto $\mathfrak{W}_{j_0}$, and $\bar{\varphi}$ induces a (topological) isomorphism of the quotient group $\mathcal{G}_{j_0}/G^{\#} \cong \mathcal{G}_{j_0+}/\mathcal{G}^{\#}_{+}$ onto $\mathfrak{W}_{j_0}$.*

Proof. Take any commutative isolating subalgebra P of B, and let j be the element of $\mathcal{J}(j_0)$ isolated by P, and $R_j(P)$ and η_j as in 2.3. On account of the remark following the proof of Proposition 9 in 2.3, η_j extends to a continuous homomorphism of $R_j(P)^{\times}_{\mathbf{A}}$ to $G_{\mathbf{A}+}$. Moreover we see by Proposition 9 that the image of $R_j(P)^{\times}_{\mathbf{A}}$ is contained in $\mathcal{G}_{j_0+}$, and have the formula,

$$\bar{\varphi}(\eta_j(a')) = \bar{\varphi}([N_j(a'), K'_{j_0}]) \quad (a' \in R_j(P)^{\times}_{\mathbf{A}}) \tag{3.4.1}$$

where $N_j = N_{R_j(P)/K'_0}$. From the class field theory and Proposition 12 in 3.2, it follows that the mapping $\bar{\varphi}([N_j(\cdot), K'_{j_0}])$ of $R_j(P)^{\times}_{\mathbf{A}}$ to $\mathfrak{W}_{j_0}$ is open. Since η_j is continuous, the formula (3.4.1) shows easily that $\bar{\varphi}$ is an open mapping on $\mathcal{G}_{j_0+}$. As for the surjectivity of $\bar{\varphi} | \mathcal{G}_{j_0+}$, take another commutative isolating subalgebra Q of B so that $R_{j'}(Q)$ is linearly disjoint with $R_j(P)$ over K'_{j_0} where j' is the element of $\mathcal{J}(j_0)$ isolated by Q. Then by the class field theory and the formulae (3.4.1) for j and j', we can easily see that $\bar{\varphi}(\mathcal{G}_{j_0+}) = \mathfrak{W}_{j_0}$. The rest of the proposition follows immediately from Proposition 13. Q.e.d.

Let us define a homomorphism σ of $\mathcal{G}_{j_0}$ onto Gal $(\mathfrak{K}_{j_0}/K'_{j_0})$ by

$$\sigma(x) = (\tilde{\varphi}^{-1} \circ \bar{\varphi}(x))^{-1} \quad (x \in \mathcal{G}_{j_0}). \tag{3.4.2}$$

Since Gal $(\mathfrak{K}'_{j_0}/K'_{j_0})$ is abelian, σ is well defined. Propositions 12 and 15 imply at once

PROPOSITION 15′. *Let the notation be as above. Then σ is an open continuous homomorphism of $\mathcal{G}_{j_0+}$ onto* Gal $(\mathfrak{K}_{j_0}/K'_{j_0})$, *and induces a* (*topological*) *isomorphism of the quotient group $\mathcal{G}_{j_0+}/G^{\#}_{+}$ onto* Gal $(\mathfrak{K}_{j_0}/K'_{j_0})$.

3.5. Let A be as in 1.9, and α any element of A. Then α extends to an automorphism of the adelization $B_{\mathbf{A}}=B\otimes_{\mathbf{Q}}\mathbf{Q}_{\mathbf{A}}$ of B. Obviously, α induces an automorphism of $G_{\mathbf{A}}$, and maps $G_{\mathbf{A}+}$ onto itself. We see easily that α commutes with ψ, i.e. $\alpha(\psi(x))=\psi(\alpha(x))$ for $x\in G_{\mathbf{A}}$, and maps $G^{\#}_{+}$ and $K^{\#}G_{\infty+}$ onto themselves. The algebra $B_{\mathbf{R}}=B\otimes_{\mathbf{Q}}\mathbf{R}$ over $\mathbf{R}$ is canonically identified with the archimedian (or infinite) part B_{∞} of $B_{\mathbf{A}}$. Through this identification, the action of α on $B_{\infty}=B_{\mathbf{R}}$ and, especially, on $G_{\infty}=G_{\mathbf{R}}$ coincides with that of α on them defined in 1.9.

Now let A'_{j_0} be as in 1.9, and suppose that α is in $A'^{\natural}_{j_0}$. Then Proposition 7 in 1.12 shows that α acts trivially on $\varphi(K'^{\times}_{j_0\mathbf{A}})$. Therefore α maps $\mathcal{G}^{\natural}_{j_0+}$ onto itself and induces an automorphism of $\mathcal{G}_{j_0+}/K^{\#}G_{\infty+}$. We see easily that $\sigma(\alpha(x))=\sigma(x)$ for any $x\in\mathcal{G}^{\natural}_{j_0+}$.

Suppose that $mq=2$ and that B_0 is totally indefinite. The notation being as in 1.2 and 1.9, let us define the action of $A^{\natural}_{j_0}$ on $\mathcal{G}_{j_0+}/K^{\#}G_{\infty+}$. Put $H=\psi^{-1}(\varphi(K'^{\times}_{j_0\mathbf{A}}))$. Then as is mentioned at the beginning of 3.9, we have $\mathcal{G}_{j_0+}=HG_{\mathbf{Q}+}K^{\#}G_{\infty+}$. On account of (1.2.5), we have $\mathcal{G}_{j_0+}/K^{\#}G_{\infty+}\cong HG_{0\mathbf{Q}+}/(HG_{0\mathbf{Q}+})\cap(K^{\#}G_{\infty+})$. Therefore if we show that H is contained in $G_{0\mathbf{A}}$, then we have the well defined action of A_{j_0} through this natural isomorphism. Now, in the present case, we have $K'_{j_0}=\mathbf{Q}$ as was mentioned in Remark 2 in 1.11, and easily see that φ is the diagonal embedding of $\mathbf{Q}^{\times}_{\mathbf{A}}$ into $K^{\times}_{\mathbf{A}}\times F^{\times}_{\mathbf{A}}$. On the other hand, we see easily that $G_{\mathbf{A}}=K^{\times}_{\mathbf{A}}G_{0\mathbf{A}}$. (The argument of the proof of (1.2.5) is applicable to this case with slight modification.) Let $x=ay$ be an element of H with $a\in K^{\times}_{\mathbf{A}}$ and $y\in G_{0\mathbf{A}}$. Take $b\in\mathbf{Q}^{\times}_{\mathbf{A}}$ so that $\psi(x)=\varphi(b)$. Then we have $N(x)=a^2N(y)=a^2yy^{\iota}=b$ and $xx^{\delta}=aa^{\delta}yy^{\delta}=aa^{\delta}yy^{\iota}=b$. Therefore $a=a^{\delta}$ and it belongs to $F^{\times}_{\mathbf{A}}$. This means that $x=ay\in G_{0\mathbf{A}}$.

Thus, in any case, the action of A_{j_0} on $\mathcal{G}_{j_0+}/K^{\#}G_{\infty+}$ is well defined.

Put $\mathfrak{A}^{0}_{j_0}=\mathcal{G}_{j_0+}/K^{\#}G_{\infty+}$. Then on account of Proposition 14, the subgroup $A^{0}_{+}=G_{\mathbf{Q}+}/K^{\times}$ of A_{j_0} is canonically isomorphic to the subgroup of $A^{\natural}_{j_0}$ which is the image of the subgroup $G_{\mathbf{Q}+}$ of $\mathcal{G}_{j_0+}$ under the natural projection of $\mathcal{G}_{j_0+}$ onto $\mathfrak{A}^{0}_{j_0}$. Let χ^0 denote the isomorphism. We have

$$\left\{\begin{array}{ll}\chi^0(\beta\alpha\beta^{-1})=\beta(\chi^0(\alpha)) & (\alpha\in A^{0}_{+};\ \beta\in A_{j_0});\\ \chi^0(\alpha)y=\alpha(y)\chi^0(\alpha) & (\alpha\in A^{0}_{+};\ y\in\mathfrak{A}^{0}_{j_0}).\end{array}\right. \tag{3.5.1}$$

PROPOSITION 16. *There exist a topological group $\mathfrak{A}_{j_0}$ containing $\mathfrak{A}^{0}_{j_0}$, and an injective homomorphism χ of A_{j_0} into $\mathfrak{A}_{j_0}$ satisfying the following conditions.*

(i) *$\mathfrak{A}^{0}_{j_0}$ is a closed normal subgroup of $\mathfrak{A}_{j_0}$.*

(ii) *$\chi=\chi^0$ on A^{0}_{+}.*

(iii) $\chi(A_+^0)=\chi(A_{j_0})\cap\mathfrak{A}_{j_0}^0$.

(iv) $\mathfrak{A}_{j_0}=\mathfrak{A}_{j_0}^0\chi(A_{j_0})$.

(v) $\chi(\alpha)x=\alpha(x)\chi(\alpha)$ *for* $x\in\mathfrak{A}_{j_0}$ *and* $\alpha\in A_{j_0}$.

Moreover $\mathfrak{A}_{j_0}$ *and* χ *are uniquely determined by these conditions up to isomorphisms.*

Proof. We can show the proposition by modifying 4.6 of [14] II as follows. Let $A_{j_0}=\bigcup_{\alpha\in R}A_+^0\alpha$ be the coset decomposition of A_{j0} with an arbitrarily fixed set of representations R. Note that R is a finite set. If $A_+^0\alpha\beta=A_+^0\gamma$ with α, β and γ in R, then $\alpha\beta=\zeta_{\alpha\beta,\gamma}\gamma$ with an element $\zeta_{\alpha\beta,\gamma}$ in A_+^0. Let $\mathfrak{A}_{j_0}=\mathfrak{A}_{j_0}^0\times R$, and define a group structure on $\mathfrak{A}_{j_0}$ by

$$(x,\alpha)(y,\beta)=(x\alpha(y)\chi^0(\zeta_{\alpha\beta,\gamma}),\gamma)$$

for α, β, $\gamma\in R$ and $x,y\in\mathfrak{A}_{j_0}^0$, and define $\chi\colon A_{j_0}\to\mathfrak{A}_{j_0}$ by

$$\chi(\varepsilon\alpha)=(\chi^0(\varepsilon),\alpha)$$

for $\varepsilon\in A_+^0$ and $\alpha\in R$. If we topologize $\mathfrak{A}_{j_0}$ by defining that $\mathfrak{A}_{j_0}^0\times\{\alpha\}$ is open, and furnishing $\mathfrak{A}_{j_0}^0\times\{\alpha\}$ with the topology of $\mathfrak{A}_{j_0}^0$, for each $\alpha\in R$, we can show the proposition in a stright-forward way using (3.5.1).

3.6. We fix a pair $(\mathfrak{A}_{j_0},\chi)$ which satisfies all the conditions of Proposition 16, and identify A_{j_0} with $\chi(A_{j_0})$. The closure $\overline{A_+^0}$ of A_+^0 in $\mathfrak{A}_{j_0}$ coincides with $G_+^*/K^*G_{\infty+}$. As is easily seen, moreover, $\overline{A_{j_0}}=\overline{A_+^0}A_{j_0}$ and $A_+^0=\overline{A_+^0}\cap A_{j_0}$ where $\overline{A_{j_0}}$ denotes the closure of A_{j_0} in $\mathfrak{A}_{j_0}$. Hence we have a sequence of isomorphic groups,

$$\mathfrak{A}_{j_0}/\overline{A_{j_0}}\cong\mathfrak{A}_{j_0}^0/\overline{A_+^0}\cong\mathcal{G}_{j_0+}/G_+^*\cong\mathrm{Gal}\,(\mathfrak{K}_{j_0}/K'_{j_0}).$$

Here the last isomorphism is that which is induced by σ. Combining these isomorphisms with the natural projection of $\mathfrak{A}_{j_0}$ onto $\mathfrak{A}_{j_0}/\overline{A_{j_0}}$, we get an open, continuous and surjective homomorphism of $\mathfrak{A}_{j_0}$ onto Gal $(\mathfrak{K}_{j_0}/K_{j_0})$ which coincides with the homomorphism induced by σ on $\mathfrak{A}_{j_0}^0=\mathcal{G}_{j_0+}/K^*G_{\infty+}$. We denote this homomorphism again by σ since there will be no fear of ambiguity. Summing up, we get

PROPOSITION 17. *The above defined homomorphism* σ *of* $\mathfrak{A}_{j_0}$ *to* Gal $(\mathfrak{K}_{j_0}/K'_{j_0})$ *is open, continuous and surjective, and induces a (topological) isomorphism of* $\mathfrak{A}_{j_0}/\overline{A_{j_0}}$ *onto* Gal $(\mathfrak{K}_{j_0}/K'_{j_0})$.

3.7. Let us denote by $\mathfrak{Z}_{j_0}$ the family of all the open compact subgroups of $\mathfrak{A}_{j_0}$. The following assertions are clear.

(3.7.1) *For* W, $X\in\mathfrak{Z}_{j_0}$, $W\cap X$ *again belongs to* $\mathfrak{Z}_{j_0}$.

(3.7.2) *For* $u\in\mathfrak{A}_{j_0}$, *and* $W\in\mathfrak{Z}_{j_0}$, uWu^{-1} *belongs to* $\mathfrak{Z}_{j_0}$.

(3.7.3) *Any two members of* $\mathfrak{Z}_{j_0}$ *are commensurable.*

(3.7.4) *For* $W\in\mathfrak{Z}_{j_0}$, $W\cap\mathfrak{A}_{j_0}^0$ *belongs to* $\mathfrak{Z}_{j_0}$.

It follows from Proposition 17 that $\sigma(W)$ is an open compact subgroup of Gal $(\mathfrak{K}_{j_0}/K'_{j_0})$ for each $W\in\mathfrak{Z}_{j_0}$. Let k_W denote the finite abelian extension of K'_{j_0} contained in $\mathfrak{K}_j$ corresponding to the subgroup $\sigma(W)$ of Gal $(\mathfrak{K}_{j_0}/K'_{j_0})$ for each $W\in\mathfrak{Z}_{j_0}$.

PROPOSITION 18. *For any* $W\in\mathfrak{Z}_{j_0}$ *and any* $x\in\mathfrak{A}_{j_0}$,

$$xWA_{j_0}=WxA_{j_0}=WA_{j_0}x=xA_{j_0}W=A_{j_0}xW=A_{j_0}Wx=\{y\in\mathfrak{A}_{j_0}\,|\,\sigma(y)=\sigma(x)\text{ on }k_W\}.$$

Proof. Let $W\in\mathfrak{Z}_{j_0}$. Since $\overline{A_{j_0}}$ is normal, we have $W\overline{A_{j_0}}=\overline{A_{j_0}}W$. Moreover since W is open, we have $W\overline{A_{j_0}}=WA_{j_0}=A_{j_0}W$. Obviously

$$W\overline{A_{j_0}}=\{y\in\mathfrak{A}_{j_0}\,|\,\sigma(y)\text{ is the identity on }k_W\}.$$

For $x\in\mathfrak{A}_{j_0}$, put $X=x^{-1}Wx$. Then $k_X=k_W$, and hence, $XA_{j_0}=X\overline{A_{j_0}}=W\overline{A_{j_0}}=WA_{j_0}$. Therefore we have $xWA_{j_0}=WxA_{j_0}$. The rest can be shown in a similar way.

For $W\in\mathfrak{Z}_{j_0}$, put $\Gamma_W=A_{j_0}\cap W$. It follows from Proposition 5 in 1.8 and the definition of A_{j_0} in 1.9 that every element of A_{j_0}, and so, especially, every element of Γ_W acts on $\mathcal{H}_{j_0}$ holomorphically. For each $W\in\mathfrak{Z}_{j_0}$, Γ_W is, in fact, a properly discontinuous group of transformations on $\mathcal{H}_{j_0}$. To see this, let us introduce certain subfamilies of $\mathfrak{Z}_{j_0}$.

3.8. Put $\mathcal{G}_{\mathfrak{f}}=\mathcal{G}_{j_0+}\cap G_{\mathfrak{f}}$ where $G_{\mathfrak{f}}$ is the finite part of $G_{\mathbf{A}}$. Then $\mathcal{G}_{j_0+}=\mathcal{G}_{\mathfrak{f}}G_{\infty+}$. Let π denote the natural projection of $\mathcal{G}_{j_0+}$ onto $\mathfrak{A}^0_{j_0}=\mathcal{G}_{j_0+}/K^{\#}G_{\infty+}$, and $\mathfrak{Z}^0_{j_0}$ denote the family of all the subgroups of $\mathcal{G}_{j_0+}$ of the form $S=S_{\mathfrak{f}}G_{\infty+}$ where $S_{\mathfrak{f}}$ is an open compact subgroup of $\mathcal{G}_{\mathfrak{f}}$. Then the family $\pi(\mathfrak{Z}^0_{j_0})=\{\pi(S)\,|\,S\in\mathfrak{Z}^0_{j_0}\}$ is a subfamily of $\mathfrak{Z}_{j_0}$. We see that

(3.8.1) *For* W *and* X *in* $\pi(\mathfrak{Z}^0_{j_0})$, $W\cap X$ *is again in* $\pi(\mathfrak{Z}^0_{j_0})$.
(3.8.2) *For* $u\in\mathfrak{A}_{j_0}$ *and* $W\in\pi(\mathfrak{Z}^0_{j_0})$, uWu^{-1} *belongs to* $\pi(\mathfrak{Z}^0_{j_0})$.
(3.8.3) *For any* W *in* $\mathfrak{Z}_{j_0}$, *there is a member* X *of* $\pi(\mathfrak{Z}^0_{j_0})$ *such that* X *is a normal subgroup of* W.

The assertion (3.8.1) is obvious, and (3.8.2) follows immediately from (iv) and (v) of Proposition 16 in 3.5. Let W be any member of $\mathfrak{Z}_{j_0}$, and take Y in $\pi(\mathfrak{Z}^0_{j_0})$ such that W contains Y. Let $W=\bigcup_\mu u_\mu Y$ be the coset decomposition of W. The set of representatives $\{u_\mu\}$ is a finite set. Put $X=\bigcap_\mu u_\mu Yu_\mu^{-1}$. Then this X satisfies (3.8.3) for W.

For $S\in\mathfrak{Z}^0_{j_0}$, put $\Gamma^0_S=G_{\mathbf{Q}}\cap S$. Then Γ^0_S is a subgroup of $G_{\mathbf{Q}+}$. Since S is open in $\mathcal{G}_{j_0+}$ and contains $G_{\infty+}$, $K^\times S$ is an open neighbourhood of $K^\times K^\times_{\infty+}$ in $\mathcal{G}_{j_0+}$. Therefore $K^\times S$ contains the closure $K^{\#}$ of $K^\times K^\times_{\infty+}$ in $G_{\mathbf{A}}$ since $\mathcal{G}_{j_0+}$ is closed in $G_{\mathbf{A}}$ and contains $K^{\#}$. Hence we have

(3.8.4) *If* $S\in\mathfrak{Z}^0_{j_0}$, *then* $K^{\#}S=K^\times S$, $G_{\mathbf{Q}}\cap K^{\#}S=K^\times\Gamma^0_S$ *and* $\pi(\Gamma^0_S)=\Gamma_{\pi(S)}$.

3.9. Now put $\mathcal{G}^1_{j_0} = \psi^{-1}(\varphi(K'^{\times}_{j_0\mathbf{A}}))G_{\infty+}$, and $\mathcal{G}^1_{\mathfrak{f}} = \mathcal{G}^1_j \cap G_{\mathfrak{f}}$. Then $\mathcal{G}^1_{j_0} = \mathcal{G}^1_{\mathfrak{f}} G_{\infty+}$. Obviously $\mathcal{G}^1_{j_0}$ is a normal subgroup of $\mathcal{G}_{j_0+}$. The restriction of $\bar{\psi}$ to $\mathcal{G}^1_{j_0}$ is an open, continuous and surjective homomorphism of $\mathcal{G}^1_{j_0}$ to $\mathcal{W}_{j_0}$ since the proof of Proposition 15 in 3.4 also works for $\mathcal{G}^1_{j_0}$ as it is. Observe, for instance, that $\eta_j(R_j(P)^{\times}_{\mathbf{A}})$ is actually contained in $\mathcal{G}^1_{j_0}$ where $R_j(P)$ and η_j are as in the proof Proposition 15. Hence it follows from Proposition 13 in 3.3 that $\mathcal{G}_{j_0+} = K^{\#}G_{\mathbf{Q}+}\mathcal{G}^1_{j_0}$.

PROPOSITION 19. *Let S be an arbitrary member of $\mathfrak{Z}^0_{j_0}$, and put $S' = S \cap \mathcal{G}^1_{j_0}$. Then $K^{\#}S'$ is an open subgroup of $\mathcal{G}_{j_0+}$, and $\pi(K^{\#}S')$ is compact. Moreover $G_{\mathbf{Q}} \cap K^{\#}S' = K^{\times}(G_{\mathbf{Q}} \cap S')$.*

Proof. Let U be an open compact subgroup of $K^{\times}_{\mathbf{A}}$ such that $UK^{\times}_{\infty} \subset K^{\times}_{\mathbf{A}} \cap S$, and take an open compact subgroup V of $K^{\times}_{\mathfrak{f}}$ so that

$$K^{\#} \cap VK^{\times}_{\infty} \subset \{a^{mq} \mid a \in K^{\#} \cap UK^{\times}_{\infty}\}.$$

Lemma 3 in 3.1 assures the existence of such V. Then on account of Proposition 12 in 3.2, we can find an open subgroup V' of $K'^{\times}_{j_0\mathbf{A}}$ so that $\varphi(V')$ is contained in $VK^{\times}_{\infty} \times F^{\times}_{\mathbf{A}+}$, and $\varphi(V')(K^{\#} \times F^{\#})$ is open in the subgroup $\varphi(K'^{\times}_{j_0\mathbf{A}})(K^{\#} \times F^{\#})$ of $K^{\times}_{\mathbf{A}} \times F^{\times}_{\mathbf{A}}$. Put

$$T = S \cap \psi^{-1}(\varphi(V')(K^{\#} \times F^{\#}) \cap (VK^{\times}_{\infty} \times F^{\times}_{\mathbf{A}+})).$$

Then it follows from Proposition 15 in 3.4 that T is an open subgroup of $\mathcal{G}_{j_0+}$. Let us show that $T \subset K^{\#}S'$. Let x be an element of T, and take $v \in V'$ and $(y, z) \in K^{\#} \times F^{\#}$ so that $\psi(x) = \varphi(v)(y, z)$. Put $\psi(x) = (a, b)$ and $\varphi(v) = (c, d)$. Then $y = ac^{-1} \in K^{\#} \cap VK^{\times}_{\infty}$. Take $w \in K^{\#} \cap UK^{\times}_{\infty}$ so that $y = w^{mq}$. Note that $aa^{\delta} = b^{mq}$ and $cc^{\delta} = d^{mq}$. We see that $(z^{-1}ww^{\delta})^{mq} = 1$ and $z^{-1}ww^{\delta} \in F^{\times}F^{\times}_{\infty+}$ since $z^{-1}ww^{\delta} \in F^{\#}$. (See Lemma 2 in 3.1.) Therefore $z^{-1}ww^{\delta} = 1$ since it is totally positive and F is totally real. This means the $\psi(w^{-1}x) = \varphi(v)$. Consequently, $w^{-1}x \in \mathcal{G}^1_{j_0}$. On the other hand, $w \in UK^{\times}_{\infty} \subset S$, and so, $w^{-1}x \in S$. Hence $x = w(w^{-1}x)$ belongs to $K^{\#}S'$. This proves that $K^{\#}S'$ is an open subgroup of $\mathcal{G}_{j_0+}$. Moreover $\pi(K^{\#}S')$ being open and contained in a compact group $\pi(S)$, it must be compact. Now let us show that $G_{\mathbf{Q}} \cap K^{\#}S' = K^{\times}(G_{\mathbf{Q}} \cap S')$. Obviously the latter is contained in the former. Let x be an element of $G_{\mathbf{Q}} \cap K^{\#}S'$. Then $x \in G_{\mathbf{Q}+}$. Take $y \in K^{\#}$ so that $s = y^{-1}x \in S'$. Then since $s \in \mathcal{G}^1_{j_0}$, we have $ss^{\delta} \in (\mathbf{Q}^{\times}_{\mathbf{A}}F^{\times}_{\infty+}) \cap F^{\#} = (\mathbf{Q}^{\times}_{\mathbf{A}} \cap F^{\#})F^{\times}_{\infty+}$. From Lemma 1 and 4 in 3.1 follows that $\mathbf{Q}^{\times}_{\mathbf{A}} \cap F^{\#} = \mathbf{Q}^{\#} = \mathbf{Q}^{\times}\mathbf{Q}^{\times}_{\infty+}$. Put $ss^{\delta} = ab$ with $a \in \mathbf{Q}^{\times}$ and $b \in F^{\times}_{\infty+}$. Let $c \in F^{\times}_{\infty+}$ such that $c^2 = b$. Then $a = (c^{-1}s)(c^{-1}s)^{\delta}$. It follows from Landherr [6] that there is an element $\gamma \in G_{\mathbf{Q}+}$ such that $a = \gamma\gamma^{\delta}$. (Also see Lemma 1 of M. Kneser [1], p. 160.) Then $z = \gamma^{-1}c^{-1}s \in K^{\#}G_{\mathbf{Q}+}$ and $zz^{\delta} = 1$. Let $E_+(F)$ and $E_+(K)$ be the groups of all the totally positive units of F and K respectively, and $E_+(F)_{\mathfrak{f}}$ and $E_+(K)_{\mathfrak{f}}$ their projections to the non-archimedian parts of $F^{\times}_{\mathbf{A}}$ and $K^{\times}_{\mathbf{A}}$ respectively. Since F is totally real and K is a totally imaginary quadratic extension of F, there is a positive integer n

such that $E_+(K)_{\mathfrak{f}}^n \subset E_+(F)_{\mathfrak{f}}$. Taking the closures of these sets, we have $\overline{E_+(K)_{\mathfrak{f}}^n} \subset \overline{E_+(F)_{\mathfrak{f}}}$. Then it follows from Lemma 1 that $K^{\#}$ is equal to $\overline{E_+(F)_{\mathfrak{f}}}K^\times K_\infty^\times$. Therefore $K^{\#}G_{\mathbf{Q}+} = \overline{E_+(F)_{\mathfrak{f}}}K_\infty^\times G_{\mathbf{Q}+}$. Take $d \in \overline{E_+(F)_{\mathfrak{f}}}$, $e \in K_\infty^\times$, and $\beta \in G_{\mathbf{Q}+}$ so that $z = de\beta$. Then $(de\beta)(de\beta)^\delta = d^2 ee^\delta \beta\beta^\delta = zz^\delta = 1$, and so, $d^2 \in F^{\#2} \cap F^\times F_{\infty+}^\times = (F^{\#2} \cap F^\times) F_{\infty+}^\times = F^{\times 2} F_{\infty+}^\times$. Using Lemma 2, we see easily that $z \in K_\infty^\times G_{\mathbf{Q}+}$. Therefore $s = cyz \in K_\infty^\times G_{\mathbf{Q}+}$. Put $s = f\alpha$ with $f \in K_\infty^\times$ and $\alpha \in G_{\mathbf{Q}+}$. Obviously $\alpha \in S'$. Now let us go back to $x = ys$. We have $yf = x\alpha^{-1} \in K^{\#} \cap G_{\mathbf{Q}+} = K^\times$. (See Proposition 14 in 3.3.) This means that $x = (yf)\alpha \in K^\times(G_{\mathbf{Q}} \cap S')$. The proof is completed.

COROLLARY 1. $K^{\#}G_{\mathbf{Q}} \cap \mathcal{G}_{j_0}^1 \subset G_{\mathbf{Q}+}K_\infty^\times$; $K^{\#} \cap G_{\mathbf{Q}+}\mathcal{G}_{j_0}^1 = K^\times K_\infty^\times$.

Proof. The first inclusion has been shown in the above proof since the proof works for any $s = yx$ in $\mathcal{G}_{j_0}^1$ with $y \in K^{\#}$ and $x \in G_{\mathbf{Q}}$. The second equality follows easily from the first one.

COROLLARY 2. *Let* $S \in \mathfrak{Z}_{j_0}^0$ *and* $S^1 = (S \cap K^{\#})(S \cap \mathcal{G}_{j_0}^1)$. *Then* $S^1 \in \mathfrak{Z}_{j_0}^0$. *Moreover, for any* $x \in G_{\mathbf{Q}+}\mathcal{G}_{j_0}^1$,

$$\begin{aligned} xG_{\mathbf{Q}+}(S \cap \mathcal{G}_{j_0}^1) &= G_{\mathbf{Q}+}x(S \cap \mathcal{G}_{j_0}^1) = G_{\mathbf{Q}+}(S \cap \mathcal{G}_{j_0}^1)x = x(S \cap \mathcal{G}_{j_0}^1)G_{\mathbf{Q}+} = (S \cap \mathcal{G}_{j_0}^1)xG_{\mathbf{Q}+} \\ &= (S \cap \mathcal{G}_j^1)G_{\mathbf{Q}+}x = \{y \in G_{\mathbf{Q}+}\mathcal{G}_{j_0}^1 \mid \sigma(\pi(y)) = \sigma(\pi(x)) \text{ on } k_{\pi(S^1)}\}. \end{aligned}$$

Proof. The last set contains all the others. Let $y \in G_{\mathbf{Q}+}\mathcal{G}_{j_0}^1$ such that $\sigma(\pi(y)) = \sigma(\pi(x))$ on $k_{\pi(S^1)}$. Then it follows easily from Proposition 18 in 3.7 that there are $a \in K^{\#}$, $u \in S \cap \mathcal{G}_{j_0}^1$ and $\alpha \in G_{\mathbf{Q}+}$ such that $y = x\alpha a u$. (Also see (iii) of Proposition 16 in 3.5). Then $a \in K^{\#} \cap G_{\mathbf{Q}+}\mathcal{G}_{j_0}^1 = K^\times K_\infty^\times$. Take $b \in K^\times$ and $c \in K_\infty^\times$ so that $a = bc$, and put $\beta = b\alpha$ and $v = uc$. Then $y = x\beta v \in xG_{\mathbf{Q}+}(S \cap \mathcal{G}_{j_0}^1)$. The rest can be easily seen in a similar way.

3.10. For $S \in \mathfrak{Z}_{j_0}^0$, put $S^1 = (S \cap K^{\#})(S \cap \mathcal{G}_{j_0}^1)$, and $\mathfrak{Z}_{j_0}^1 = \{S^1 \mid S \in \mathfrak{Z}_{j_0}^0\}$. Then Proposition 19 shows that $\mathfrak{Z}_{j_0}^1$ is a subfamily of $\mathfrak{Z}_{j_0}^0$. Therefore $\pi(\mathfrak{Z}_{j_0}^1) = \{\pi(S^1) \mid S^1 \in \mathfrak{Z}_{j_0}^1\}$ is a subfamily of $\mathfrak{Z}_{j_0}$. We see that

(3.10.1) *For* $x \in \mathcal{G}_{j_0+}$ *and* $S \in \mathfrak{Z}_{j_0}^0$, *we have* $xS^1x^{-1} = (xSx^{-1})^1$. *Especially,* S^1 *is a normal subgroup of* S *for every* $S \in \mathfrak{Z}_{j_0}^0$;

(3.10.2) *For* $u \in \mathfrak{A}_{j_0}$ *and* $S \in \mathfrak{Z}_{j_0}^0$, $u\pi(S^1)u^{-1}$ *belongs to* $\pi(\mathfrak{Z}_{j_0}^1)$.

The assertion (3.10.1) is clear, and (3.10.2) follows from Proposition 7 in 1.12, (v) of Proposition 16 in 3.5 and the definition of $\mathcal{G}_{j_0}^1$ in 3.9.

PROPOSITION 20. *For any sufficiently small* $S \in \mathfrak{Z}_{j_0}^0$,

$$\Gamma_{S^1}^0 = G_{\mathbf{Q}} \cap [(S \cap K^{\#})(S \cap \mathcal{G}_{j_0}^1)] \subset K^\times(S \cap G_{\mathbf{Q}}^1).$$

Proof. On account of Proposition 19, it is enough to show that $G_{\mathbf{Q}} \cap (S \cap \mathcal{G}_{j_0}^1) \subset G_{\mathbf{Q}}^1$ for sufficiently small S. Let U be an open subgroup of $K_{\mathfrak{f}}^{\times}$ such that $UK_{\infty}^{\times}$ does not contain any roots of 1 in K except 1 itself. Then $N^{-1}(UK_{\infty}^{\times})$ is an open subgroup of $G_{\mathbf{A}}$. Let S be a member of $\mathfrak{Z}_{j_0}^{0}$ contained in $N^{-1}(UK_{\infty}^{\times})$, and γ any element of $G_{\mathbf{Q}} \cap (S \cap \mathcal{G}_{j_0}^{1})$. Since the non-archimedean part of S is compact, we see that $\nu(\gamma) = \gamma\gamma^{\delta}$ is a unit of F contained in $\mathbf{Q}^{\times} = F^{\times} \cap \mathbf{Q}_{\mathbf{A}}^{\times} F_{\infty+}^{\times}$. Since $\nu(\gamma)$ is totally positive, we have $\nu(\gamma) = \gamma\gamma^{\delta} = 1$. On the other hand, $N(\gamma)$ is a unit of K, and $N(\gamma)N(\gamma)^{\delta} = \nu(\gamma)^{mq} = 1$. Therefore $N(\gamma)$ must be a root of 1 in K since K is a CM-field. From the choice of U and S, it follows that $N(\gamma) = 1$. Q.e.d.

Remark. Put $G_{\mathbf{Q}}^{u} = \{\gamma \in B \mid \gamma\gamma^{\delta} = 1\}$. Then $G_{\mathbf{Q}}^{u}$ defines an algebraic subgroup of G containing G^{1}. In the above proof, we have shown that $G_{\mathbf{Q}} \cap (S \cap \mathcal{G}_{j_0}^{1}) \subset G_{\mathbf{Q}}^{u}$ for *every* $S \in \mathfrak{Z}_{j_0}^{0}$. Moreover if we take any sufficiently small S described in the above proof, then we see that $S \cap G_{\mathbf{Q}}^{u}$ itself is contained in $G_{\mathbf{Q}}^{1}$. Since $G_{\mathbf{Q}}^{1}$ is contained in $\mathcal{G}_{j_0}^{1}$, we showed actually

(3.10.3) *For any sufficiently small* S,

$$S \cap G_{\mathbf{Q}}^{u} = S \cap G_{\mathbf{Q}}^{1} \subset \Gamma_{S^1}^{0}; \quad \pi(\Gamma_{S^1}^{0}) = \Gamma_{\pi(S^1)} = \pi(S \cap G_{\mathbf{Q}}^{u}).$$

Moreover since the subgroup $N^{-1}(UK_{\infty}^{\times})$ of $G_{\mathbf{A}}$ defined in the above proof is a normal subgroup of $G_{\mathbf{A}}$, we can assume that, if S in $\mathfrak{Z}_{j_0}^{0}$ is sufficiently small, then (3.10.3) is true for xSx^{-1} for every $x \in \mathcal{G}_{j_0+}$.

COROLLARY. *For any $W \in \mathfrak{Z}_{j_0}$, $\Gamma_W = A_{j_0} \cap W$ is a properly discontinuous group of transformations on $\mathcal{H}_{j_0}$, and $\Gamma_W \backslash \mathcal{H}_{j_0}$ can be embedded in a normal projective variety as a Zarisky open subset.*

Proof. Take $S \in \mathfrak{Z}_{j_0}^{0}$ so that Proposition 20 holds. Then $\pi(S^1)$ is a member of $\mathfrak{Z}_{j_0}$, and $\Gamma_{\pi(S^1)} = \pi(\Gamma_{S^1}^{0}) = \pi(\Gamma_{S^1}^{0}) = \pi(S \cap G_{\mathbf{Q}}^{1})$. This means that $\Gamma_{\pi(S^1)}$. coincides with the arithmetic subgroup $S \cap G_{\mathbf{Q}}^{1}$ of G^{1} as transformation groups on $\mathcal{H}_{j_0}$. Since every member W of $\mathfrak{Z}_{j_0}$ is commensurable with $\pi(S^1)$, the corollary follows from Baily and Borel [3].

4. The main theorem and reduction of the proof

4.1. In the previous section, we defined the following things related with the hermitian symmetric space $\mathcal{H}_{j_0}$ and the corresponding $\mathcal{J}(j_0)$ for a fixed element $j_0 \in \mathcal{J}$: An (infinite) abelian extension $\mathfrak{K}_{j_0}$ of K_{j_0}' in 3.2 (also see 1.11); a topological group $\mathfrak{A}_{j_0}$ in 3.5; an open, continuous and surjective homomorphism σ of $\mathfrak{A}_{j_0}$ to Gal $(\mathfrak{K}_{j_0}/K_{j_0}')$, whose kernel is the closure of the subgroup A_{j_0} of $\mathfrak{A}_{j_0}$, in 3.6; finite abelian extensions k_W of K_{j_0}' contained in $\mathfrak{K}_{j_0}$ and properly discontinuous groups Γ_W of transformations on $\mathcal{H}_{j_0}$, both parametrized by the members W of the family $\mathfrak{Z}_{j_0}$ of all the open compact subgroups of $\mathfrak{A}_{j_0}$, in 3.7.

For a commutative isolating subalgebra P, let j be the element of $\mathcal{J}(j_0)$ isolated by P.

(See 2.1.) We defined a finite algebraic extension $R_j(P)$ of K'_{j_0} and a continuous homomorphism η_j of $R_j(P)_{\mathbf{A}}^{\times}$ to $\mathcal{G}_{j_0+}$. (See 2.3 and 3.4.) We define a homomorphism η_j^* of $R_j(P)_{\mathbf{A}}^{\times}$ to $\mathfrak{A}_{j_0}$ by putting $\eta_j^* = \pi\circ\eta_j$ where π is the natural projection of $\mathcal{G}_{j_0+}$ onto the subgroup $\mathfrak{A}_{j_0}^0$ of $\mathfrak{A}_{j_0}$. (See 3.5.)

For the discontinuous group Γ_W for $W\in\mathfrak{Z}_{j_0}$, we say that a pair (V, φ) is *a model* of $\Gamma_W\backslash\mathcal{H}_{j_0}$ if V is a Zariski open subset of a normal projective variety and φ is a holomorphic mapping of $\mathcal{H}_{j_0}$ onto V such that φ induces a biregular morphism of the quotient space $\Gamma_W\backslash\mathcal{H}_{j_0}$, which also has a structure of a Zariski open subset of a normal projective variety (see the corollary of Proposition 20 in 3.10), onto V.

Now our main theorem states that

THEOREM 1. *There exists a system*

$$\{V_X, \varphi_X, J_{YX}(u), (X, Y\in\mathfrak{Z}_{j_0}; u\in\mathfrak{A}_{j_0})\}$$

consisting the objects satisfying the following conditions.

(I) *For each $X\in\mathfrak{Z}_{j_0}$, (V_X, φ_X) is a model of $\Gamma_X\backslash\mathcal{H}_{j_0}$.*

(II) *V_X is rational over k_X.*

(III) *For $u\in A_{j_0}$, $J_{YX}(u)$ is a morphism of V_X onto $V_Y^{\sigma(u)}$, which is defined if and only if $uXu^{-1}\subset Y$, is rational over k_X, and has the following properties:*

(IIIa) *$J_{XX}(u)$ is the identity mapping of V_X if $u\in X$;*

(IIIb) *$J_{YX}(u)^{\sigma(t)}\circ J_{XW}(t) = J_{YW}(ut)$;*

(IIIc) *$J_{YX}(\alpha)[\varphi_X(z)] = \varphi_Y(\alpha(z))$ for every $\alpha\in A_{j_0}$ and every $z\in\mathcal{H}_{j_0}$ if $\alpha X\alpha^{-1} = Y$.*

(IV) *Let z be an isolated fixed point on $\mathcal{H}_{j_0}$, j the corresponding element of $\mathcal{J}(j_0)$, and P a commutative isolating subalgebra of B which isolates j. Then, for every $X\in\mathfrak{Z}_{j_0}$, $\varphi_X(z)$ is rational over $R_j(P)_{\mathrm{ab}}$. Furthermore, for every $v\in R_j(P)_{\mathbf{A}}^{\times}$,*

$$\varphi_Y(z)^{\tau} = J_{YX}(\eta_j^*(v)^{-1})[\varphi_X(z)]$$

where $\tau = [v, R_j(P)]\in\mathrm{Gal}\,(R_j(P)_{\mathrm{ab}}/R_j(P))$ and $Y = \eta_j^*(v)^{-1}X\eta_j^*(v)$.

The proof will be completed in the last section with the help of the lemmas given in 4.3–7 and the theory of the modulus-varieties of PEL-structures, abelian varieties with certain additional structures.

Note that, in (IV) of Theorem 1, $\tau = [v, R_j(P)] = \sigma(\eta_j^*(v)^{-1})$ on $\mathfrak{K}_{j_0}$. (See Proposition 9 in 2.3, (3.2.1), (3.3.1) and (3.4.2).)

COROLLARY. *Let the notation and the assumptions be as in* (IV). *Put $\mathfrak{S}_{j_0}(z) = \{\alpha\in A_{j_0} \mid \alpha(z) = z\}$. Then the finite abelian extension $R_j(P)k_Y(\varphi_Y(z))$ of $R_j(P)$ corresponds to the open subgroup*

$$R_j(P)^\times\{v\in R_j(P)_{\mathbf{A}}^\times \mid \eta_j^*(v)\in\mathfrak{S}_{j_0}(z)Y\}$$

of $R_j(P)_{\mathbf{A}}^\times$ *by the class field theory.*

Proof. Suppose that $\tau=[v, R_j(P)]$ is trivial on $R_j(P)\,k_Y(\varphi_Y(z))$. Then since $[v, R_j(P)]=\sigma(\eta_j^*(v)^{-1})=$ the identity mapping on k_Y, there are $\alpha\in A_{j_0}$ and $y\in Y$ such that $\eta_j^*(v)^{-1}=y\alpha$. (See Proposition 18 in 3.7.) Then $\varphi_Y(z)^\tau=J_{YX}(\eta_j^*(v)^{-1})\circ\varphi_X(z)=J_{YX}(\alpha)\circ\varphi_X(z)=\varphi_Y(\alpha(z))$. On the other hand, we have $\varphi_Y(z)=\varphi_Y(z)^\tau$ since τ is trivial on $R_j(P)\,k_Y(\varphi_Y(z))$. Therefore $\varphi_Y(z)=\varphi_Y(\alpha(z))$, and so, there is an element $\beta\in\Gamma_Y$ such that $\beta\alpha(z)=z$. Then $\eta_j^*(v)=\alpha^{-1}y^{-1}=(\beta\alpha)^{-1}\beta y^{-1}\in\mathfrak{S}_{j_0}(z)\,Y$. The converse is easy to see.

4.2. By means of Theorem 1, we can describe the group $\mathfrak{A}_{j_0}$ as a group of automorphisms of a function field.

For $X\in\mathfrak{Z}_{j_0}$, let L_X be the field of all the rational functions on V_X defined over k_X, and put

$$\mathfrak{L}_{j_0}=\bigcup\nolimits_{X\in\mathfrak{Z}_{j_0}}\mathfrak{L}_X;\ \mathfrak{L}_X=\{f\circ\varphi_X \mid f\in L_X\}.$$

Then $\mathfrak{L}_{j_0}$ is a field of meromorphic functions on $\mathcal{H}_{j_0}$. For $u\in\mathfrak{A}_{j_0}$, define a mapping $\varrho(u)$ of $\mathfrak{L}_{j_0}$ to itself by

$$(f\circ\varphi_X)^{\varrho(u)}=f^{\sigma(u)}\circ J_{XW}(u)\circ\varphi_W$$

for $f\circ\varphi_X\in\mathfrak{L}_{j_0}$ where $W=u^{-1}Xu$. We see easily that $\varrho(u)$ is an automorphism of $\mathfrak{L}_{j_0}$ over K'_{j_0}, and that ϱ gives a homomorphism of $\mathfrak{A}_{j_0}$ to Aut $(\mathfrak{L}_{j_0}/K'_{j_0})$ as abstract groups. We regard $\mathfrak{K}_{j_0}$ as a subfield of $\mathfrak{L}_{j_0}$ in the obvious way. Then $\varrho(u)$ coincides with $\sigma(u)$ on $\mathfrak{K}_{j_0}$ for each $u\in\mathfrak{A}_{j_0}$.

Now let us topologize Aut $(\mathfrak{L}_{j_0}/K'_{j_0})$ by taking all the subgroups of the form

$$\{\tau\in\mathrm{Aut}\,(\mathfrak{L}_{j_0}/K'_{j_0}) \mid h_1^\tau=h_1,\dots,h_n^\tau=h_n\}$$

for a finite subset $\{h_1, ..., h_n\}$ of $\mathfrak{L}_{j_0}$ as a basis of the neighbourhoods of the identity. Then Aut $(\mathfrak{L}_{j_0}/K'_{j_0})$ becomes a locally compact Hausdorff-topological group. (See 1.3 of Shimura [14] II.)

THEOREM 2. *The mapping ϱ is an open, continuous and injective homomorphism of $\mathfrak{A}_{j_0}$ to* Aut $(\mathfrak{L}_{j_0}/K'_{j_0})$, *and has the following properties:*

(i) $\varrho(u)=\sigma(u)$ *on* $\mathfrak{K}_{j_0}$ *for every* $u\in\mathfrak{A}_{j_0}$;

(ii) $h^{\varrho(\alpha)}(z)=h(\alpha(z))$ *for* $\alpha\in A_{j_0}$, $h\in\mathfrak{L}_{j_0}$ *and* $z\in\mathcal{H}_{j_0}$;

(iii) *Let z, j, P, $R_j(P)$ and η_j^* be as in* (IV) *of* Theorem 1. *Then, for every $h\in\mathfrak{L}_{j_0}$ that is defined at z, $h(z)$ is rational over $R_j(P)_{\mathrm{ab}}$. Moreover, if we put $\tau=[v, R_j(P)]$ and $u=\eta_j^*(v)^{-1}$ for an arbitrary $v\in R_j(P)_{\mathbf{A}}^\times$, then $h^{\varrho(u)}$ is also defined at z, and $h(z)^\tau=h^{\varrho(u)}(z)$.*

Proof. The property (i) is clear, and (ii) and (iii) follow easily from Theorem 1. Let us show the topological properties of ϱ. First we show that

$$W=\{u\in\mathfrak{A}_{j_0}\,|\,\varrho(u) \text{ is trivial on } \mathfrak{L}_W\}$$

for any $W\in\mathfrak{Z}_{j_0}$. Then since $\mathfrak{L}_W$ is finitely generated over K'_{j_0}, we see that ϱ is continuous. Moreover since $\bigcap_{X\in\mathfrak{Z}_{j_0}}X=\{1\}$, we also see that ϱ is injective. Now, obviously, W is contained in the other. Let u be an element of $\mathfrak{A}_{j_0}$ such that $\varrho(u)$ is trivial on $\mathfrak{L}_W$. Since $\varrho(u)$ is trivial on k_W, we can find $t\in W$ and $\alpha\in A_{j_0}$ such that $u=t\alpha$. (See Proposition 18 in 3.7.) For every $h=f\circ\varphi_W\in\mathfrak{L}_W$ with $f\in L_W$, we have $f\circ\varphi_W=h=h^{\varrho(u)}=f\circ J_{WT}(t\alpha)\circ\varphi_T=f\circ\varphi_W\circ\alpha$ where $T=u^{-1}Wu$. Hence $\varphi_W=\varphi_W\circ\alpha$, and so, $\alpha\in\Gamma_W\subset W$. Thus we have $u=t\alpha\in W$, and W contains the other set. This shows that W coincides with the other. Therefore, as we have already seen, ϱ is continuous. Hence $\varrho(W)$ is a compact subgroup of Aut $(\mathfrak{L}_{j_0}/K'_{j_0})$. Then if we show

$$\mathfrak{L}_W=\{h\in\mathfrak{L}_{j_0}\,|\,h^{\varrho(u)}=h \text{ for all } u\in W\}, \tag{4.2.1}$$

we can conclude that

$$\varrho(W)=\{\tau\in\text{Aut }(\mathfrak{L}_{j_0}/K_{j_0})\,|\,\tau \text{ is trivial on } \mathfrak{L}_W\} \tag{4.2.2}$$

on account of 1.2 of [14] II. Moreover, since $\mathfrak{L}_W$ is finitely generated over K'_{j_0}, we see that $\varrho(W)$ is open in Aut $(\mathfrak{L}_{j_0}/K'_{j_0})$. Hence it is sufficient to show (4.2.1). Obviously $\mathfrak{L}_W$ is contained in the other. Let h be an element of $\mathfrak{L}_{j_0}$ such that $h^{\varrho(u)}=h$ for every $u\in W$. Then $h=f\circ\varphi_T$ with $f\in L_T$ for some $T\in\mathfrak{Z}_{j_0}$. We may assume that T is a normal subgroup of W. For any $\alpha\in\Gamma_W\subset W$, $f\circ\varphi_T=h=h^{\varrho(\alpha)}=f\circ J_{TT}(\alpha)\circ\varphi_T$. Therefore $f=f\circ J_{TT}(\alpha)$. We see easily that V_W is biregular over k_T to the quotient variety of V_T by the (finite) group of automorphisms of V_T, $\{J_{TT}(\alpha)\,|\,\alpha\in\Gamma_W\}$, since (V_W,φ_W) and (V_T,φ_T) are respectively models of $\Gamma_W\backslash\mathcal{H}_{j_0}$ and $\Gamma_T\backslash\mathcal{H}_{j_0}$. Therefore there is a rational function g on V_W defined over k_T such that $f=g\circ J_{WT}(1)$, in other words, $h=g\circ\varphi_W$. We have to show that g is defined over k_W. For any $\tau\in\text{Gal }(k_T/k_W)$, take $u\in W$ so that $\varrho(u)=\sigma(u)=\tau$ on k_T. The existence of such u follows from Proposition 18 in 3.7 at once. We have $g\circ\varphi_W=h=h^{\varrho(u)}=f^{\sigma(u)}\circ J_{TT}(u)\circ\varphi_T=g^{\sigma(u)}\circ J_{WT}(1)^{\sigma(u)}\circ J_{TT}(u)\circ\varphi_T=g^{\sigma(u)}\circ J_{WT}(u)\circ\varphi_T=g^{\sigma(u)}\circ J_{WW}(u)\circ J_{WT}(1)\circ\varphi_T=g^{\sigma(u)}\circ\varphi_W$. This means that $g=g^\tau$ for any $\tau\in\text{Gal }(k_T/k_W)$. Therefore g is defined over k_W, and $h=g\circ\varphi_W\in\mathfrak{L}_W$. Hence (4.2.1) is proved. Q.e.d.

COROLLARY 1. *The following assertions hold.*

(i) $K'_{j_0}=\{h\in\mathfrak{L}_{j_0}\,|\,h^{\varrho(u)}=h$ for every $u\in\mathfrak{A}_{j_0}\}$.

(ii) *For* $W\in\mathfrak{Z}_{j_0}$, $\varrho(W)=\{\tau\in\text{Aut }(\mathfrak{L}_{j_0}/K'_{j_0})\,|\,\tau$ *is trivial on* $\mathfrak{L}_W\}=\text{Gal }(\mathfrak{L}_{j_0}/\mathfrak{L}_W)$.

(iii) *For* $W\in\mathfrak{Z}_{j_0}$, $\mathfrak{L}_W=\{h\in\mathfrak{L}_{j_0}\,|\,h^{\varrho(u)}=h$ *for every* $u\in W\}$.

Proof. The assertions (ii) and (iii) have already been proved. Let us see (i) fold. Obviously K'_{j_0} is contained in the other. Now let h be an element of $\mathfrak{L}_{j_0}$ such that $h^{\varrho(u)}=h$ for every $u\in\mathfrak{A}_{j_0}$. Take $W\in\mathfrak{Z}_{j_0}$ and $f\in L_W$ so that $h=f\circ\varphi_W$. Then $f\circ\varphi_W=h=h^{\varrho(\alpha)}=f\circ\varphi_W\circ\alpha$ for $\alpha\in A_{j_0}$. Fix any $z\in\mathcal{H}_{j_0}$. As is seen in 2.5, we see that $\{\varphi_W(\alpha(z))\mid\alpha\in A_{j_0}\}$ is dense in V_W. Therefore f must be a constant function on V_W, i.e. $f\in k_W$. For any $\tau\in \mathrm{Gal}(k_W/K'_{j_0})$. we can find $u\in\mathfrak{A}_{j_0}$ so that $\varrho(u)=\sigma(u)=\tau$ on k_W. Then we see easily that $f=f^\tau$ since $h^{\varrho(u)}=h$. Hence $f\in K'_{j_0}$. Q.e.d.

COROLLARY 2. (i) *If $mq>2$, and $\Gamma_U\backslash\mathcal{H}_{j_0}$ is compact for some $U\in\mathfrak{Z}_{j_0}$, then ϱ is surjective.*

(ii) *If $mq=2$, and $\Gamma_U\backslash\mathcal{H}_{j_0}$ is compact for some $U\in\mathfrak{Z}_{j_0}$, then $\varrho(\mathfrak{A}_{j_0})$ is a subgroup of* $\mathrm{Aut}(\mathfrak{L}_{j_0}/K'_{j_0})$ *of finite index, and*

$$[\mathrm{Aut}(\mathfrak{L}_{j_0}/K'_{j_0})\colon \varrho(\mathfrak{A}_{j_0})]\leqslant[A''_{j_0}\colon A_{j_0}]\leqslant g=[F\colon \mathbf{Q}].$$

Here A''_{j_0} is the group defined in 1.9.

(iii) *If $mq=2$, and the algebra B_0 determined by* (1.2.1) *is isomorphic to* $\mathbf{M}(2,\mathbf{Q})$ *(therefore $F=\mathbf{Q}$, and K is an imaginary quadratic extension of $\mathbf{Q}$), then ϱ is surjective.*

Note that the compactness of $\Gamma_U\backslash\mathcal{H}_{j_0}$ does not depend on the choice of $U\in\mathfrak{Z}_{j_0}$.

Proof. For $S\in\mathfrak{Z}^0_{j_0}$, put $S^1=(S\cap K^{\neq})(S\cap G^1_{j_0})$ and $\Gamma^1_S=S\cap G^1_{\mathbf{Q}}$. Then on account of (3.10.3), we see easily that

(4.2.3) *For any sufficiently small S of $\mathfrak{Z}^0_{j_0}$,*

$$\pi(\Gamma^1_S)=\Gamma_{\pi(S^1)},$$

and $\Gamma_{\pi(S^1)}$ has no element of finite order other than the identity element.

As is easily seen, the group $\mathrm{Aut}(\mathfrak{L}_{j_0}/\mathfrak{K}_{j_0})$ is a closed normal subgroup of $\mathrm{Aut}(\mathfrak{L}_{j_0}/K'_{j_0})$, and $\mathrm{Aut}(\mathfrak{L}_{j_0}/K'_{j_0})=\varrho(\mathfrak{A}_{j_0})\,\mathrm{Aut}(\mathfrak{L}_{j_0}/\mathfrak{K}_{j_0})$ (see Proposition 17 in 3.6). Let $\mathfrak{L}^*_W$ for $W\in\mathfrak{Z}_{j_0}$ (resp. $\mathfrak{L}^*$) be the composite of $\mathfrak{L}_W$ (resp. $\mathfrak{L}_{j_0}$) and $\mathbf{C}$. Then $\mathfrak{L}^*=\bigcup_{W\in\mathfrak{Z}_{j_0}}\mathfrak{L}^*_W$. Since $\mathfrak{L}_{j_0}$ and $\mathbf{C}$ are linearly disjoint over $\mathfrak{K}_{j_0}$, the group $\mathrm{Aut}(\mathfrak{L}_{j_0}/\mathfrak{K}_{j_0})$ can canonically be regarded as a subgroup of $\mathrm{Aut}(\mathfrak{L}^*/\mathbf{C})$. Let $\tau\in\mathrm{Aut}(\mathfrak{L}_{j_0}/\mathfrak{K}_{j_0})$. Take $S\in\mathfrak{Z}^0_{j_0}$ so that (4.2.3) holds, and put $W=\pi(S^1)$. We can find members X and Y of $\mathfrak{Z}_{j_0}$ so that $Y\subset X\subset W$, $\mathfrak{L}_W^{\tau^{-1}}\subset\mathfrak{L}_X$, $\mathfrak{L}_X^\tau\subset\mathfrak{L}_Y$, and Y is a normal subgroup of W. Then $\mathfrak{L}^*_W\subset\mathfrak{L}^{*\tau}_X\subset\mathfrak{L}^*_Y$. Let Δ be the subgroup of Γ_W such that Δ/Γ_Y corresponds to $\mathfrak{L}^{*\tau}_X$ under the isomorphism of $\mathrm{Gal}(\mathfrak{L}^*_Y/\mathfrak{L}^*_W)$ onto Γ_W/Γ_Y induced by ϱ, and put $U=\Delta Y$. Then $U\in\mathfrak{Z}_{j_0}$ and $\Gamma_U=\Delta$. Moreover τ gives an isomorphism of $\mathfrak{L}^*_X$ onto $\mathfrak{L}^*_U$. Therefore we have a birational mapping ξ of V_U to V_X such that $(f\circ\varphi_X)^\tau=f\circ\xi\circ\varphi_U$ for every $f\in L_X$. Now suppose that either $mq=2$ and $B_0\cong\mathbf{M}(2,\mathbf{Q})$, or $\Gamma_U\backslash\mathcal{H}_{j_0}$ is compact. Then there exists an $\mathbf{R}$-linear automorphism β of $B_{\mathbf{R}}$ such that β commutes with δ, $\beta(\mathcal{J}(j_0))=$

$\mathcal{J}(j_0)$ and $\xi\circ\varphi_U=\varphi_X\circ\beta$. In fact, it is enough, on account of Proposition 5 in 1.8, to show that ξ can be shifted up to a (holomorphic) automorphism of $\mathcal{H}_{j_0}$. If $mq=2$ and $B_0\cong\mathbf{M}(2, \mathbf{Q})$, we easily see this by 7.21 of [14] I. If $\Gamma_U\backslash\mathcal{H}_{j_0}$ is compact, then it follows from the choice of S that V_U and V_X are both complete non-singular minimal models, and hence, ξ is a biregular isomorphism. Therefore ξ can be shifted up to an automorphism of $\mathcal{H}_{j_0}$. As transformation groups, $\beta\Gamma_U\beta^{-1}$ coinsides with Γ_X. Note that every element of Γ_U and Γ_X is of the form $\pi(\gamma)$ with some $\gamma\in\Gamma_S^1\subset G_{\mathbf{Q}}^1$. Let γ be an element of Γ_S^1 such that $\pi(\gamma)\in\Gamma_U$. Then, as $\mathbf{R}$-linear automorphisms of $B_{\mathbf{R}}$, (and hence, as automorphisms of $\mathcal{H}_{j_0}$), $\beta\pi(\gamma)\beta^{-1}=\pi(\beta(\gamma))$ where $\beta(\gamma)$ is the image of γ under the automorphism β of $B_{\mathbf{R}}$. Take $\gamma'\in\Gamma_S^1$ so that $\pi(\beta(\gamma))=\pi(\gamma')\in\Gamma_X$. Then there is an element a of the center of $G_{\mathbf{R}}^1$ such that $\beta(\gamma)=a\gamma'$ since both $\beta(\gamma)$ and γ' are in $G_{\mathbf{R}}^1$. We see easily that $a^{mq}=1$. Therefore we have $\beta(\gamma^{mq})=\gamma'^{mq}\in\Gamma_S^1$ for every $\gamma\in\Gamma_S^1$ such that $\pi(\gamma)\in\Gamma_U$. Then by Lemma 9 in 4.8 we have $\beta(B)=B$ if $mq>2$, and $\beta(B_0)=B_0$ if $mq=2$. If $mq>2$, or if $mq=2$ and $B_0\cong\mathbf{M}(2, \mathbf{Q})$, then $\beta\in A_{j_0}$, and $\tau=\varrho(\beta)$ on $\mathfrak{L}_X$. Since X can move all the sufficiently small open compact subgroups of $\mathfrak{A}_{j_0}$, and $\varrho(\mathfrak{A}_{j_0})$ is open and closed in Aut $(\mathfrak{L}_{j_0}/K'_{j_0})$, this shows that τ belongs to $\varrho(\mathfrak{A}_{j_0})$. Thus (i) and (iii) are proved. Suppose now that $mq=2$ and $\Gamma_U\backslash\mathcal{H}_{j_0}$ is compact. Let us define a homomorphism ϱ' of A'_{j_0} to Aut $(\mathfrak{L}^*/\mathbf{C})$. Let β be an element of A''_{j_0}. For $h\in\mathfrak{L}^*$, put $h^{\varrho'(\beta)}=h\circ\beta$. Choose $S\in\mathfrak{Z}_{j_0}^0$ so that (4.3.2) holds, and $h\in\mathfrak{L}^*_{\pi(S^1)}$. Since $\beta^{-1}(S\cap G_{\mathbf{A}}^\times)$ is an open subgroup of $G_{\mathbf{A}}^\times$, there is a member T of $\mathfrak{Z}_{j_0}^0$ so that $T\subset S$ and $T\cap G_{\mathbf{A}}^1\subset\beta^{-1}(S\cap G_A^1)$. We have $\beta(\Gamma_T^1)\subset\Gamma_S^1$ since $\beta(G_{\mathbf{Q}}^1)=G_{\mathbf{Q}}^1$. Let γ be an element of Γ_T^1. Then $\beta\pi(\gamma)\beta^{-1}=\pi(\beta(\gamma))$. Therefore $h\circ\beta\circ\pi(\gamma)=h\circ\pi(\beta(\gamma))\circ\beta=h\circ\beta$, and hence, $h^{\varrho'(\beta)}=h\circ\beta\in L^*_{\pi(T^1)}\subset\mathfrak{L}^*$. This shows that ϱ' is a well defined homomorphism of A''_j to Aut $(\mathfrak{L}^*/\mathbf{C})$. Obviously, $\varrho'(\alpha)$ coincides with $\varrho(\alpha)$ considered as an element of Aut $(\mathfrak{L}^*/\mathbf{C})$ if $\alpha\in A_{j_0}$. Suppose that $\varrho'(\beta)$ is trivial on $\mathfrak{L}_X^*$ for some $X\in\mathfrak{Z}_{j_0}$. Then $\varphi_X\circ\beta=\varphi_X$, and hence, $\beta\in\Gamma_X\subset A_{j_0}$. This shows, especially, that ϱ' is injective. Now let τ be an element of Aut $(\mathfrak{L}_{j_0}/\mathfrak{K}_{j_0})$, and choose $X\in\mathfrak{Z}_{j_0}$ as above for τ and the fixed W. As we saw above, there is an element β of A''_{j_0} so that $\tau=\varrho'(\beta)$ on $\mathfrak{L}_X^*$. Once such an X is chosen, there exists an element β' of A''_{j_0} such that $\tau=\varrho'(\beta')$ on $\mathfrak{L}_{X'}^*$, for $X'\in\mathfrak{Z}_{j_0}$ if $X'\subset X$. We see easily, moreover, that the cosets βA_{j_0} and $\beta' A_{j_0}$ of A_{j_0} in A''_{j_0} coincide with each other, since $\varrho'(\beta^{-1}\beta')$ is trivial on $\mathfrak{L}_{X'}^*$. Therefore we have a well defined mapping of Aut $(\mathfrak{L}_{j_0}/\mathfrak{K}_{j_0})$ to the quotient space A''_{j_0}/A_{j_0}. Now let τ and τ' be two elements of Aut $(\mathfrak{L}_{j_0}/\mathfrak{K}_{j_0})$, and choose X, $X'\in\mathfrak{Z}_{j_0}$ and β, $\beta'\in A''_{j_0}$ so that $\tau=\varrho'(\beta)$ on $\mathfrak{L}_X^*$ and $\tau'=\varrho'(\beta')$ on $\mathfrak{L}_{X'}^*$. Suppose that $\beta A_{j_0}=\beta' A_{j_0}$. Let X'' be a member of $\mathfrak{Z}_{j_0}$ such that $X''\subset X\cap X'$, and β'' an element of A''_{j_0} such that $\tau'=\varrho'(\beta''\alpha)$ on $\mathfrak{L}_{X''}^*$. Then there is an element $\alpha\in A_{j_0}$ such that $\tau'=\varrho'(\beta''\alpha)$ on $L_{X''}^*$ since $\beta''\in\beta A_{j0}=\beta' A_{j0}$. Therefore $\tau^{-1}\tau'=\varrho'(\alpha)=\varrho(\alpha)$ on $\mathfrak{L}_{X''}^*$. Since both $\tau^{-1}\tau'$ and $\varrho(\alpha)$ belongs to Aut $(\mathfrak{L}_{j_0}/\mathfrak{K}_{j_0})$, we see that $\tau^{-1}\tau'=\varrho(\alpha)$ on $\mathfrak{L}_{X''}$. Having X'' move through all the members of $\mathfrak{Z}_{j_0}$ that are contained in $X \cap X'$, we conclude that $\tau^{-1}\tau'$ belongs to $\varrho(\mathfrak{A}_{j_0})$. Thus we get an injective mapping of the quotient space Aut $(\mathfrak{L}_{j_0}/K'_{j_0})/\varrho(\mathfrak{A}_{j_0}) \cong$ Aut $(\mathfrak{L}_{j_0}/\mathfrak{K}_{j_0})/(\mathrm{Aut}(\mathfrak{L}_{j_0}/\mathfrak{K}_{j_0}) \cap \varrho(\mathfrak{A}_{j_0}))$ into the finite set A''_{j_0}/A_{j_0}. This shows that the assertion (ii) holds, and completes the proof.

4.3. Now we proceed the reduction process of the proof of Theorem 1. Our argument is similar to that of [14] I, 3.8–11.

We say that a subfamily $\mathfrak{W}$ of $\mathfrak{Z}_{j_0}$ is *normal* if uWu^{-1} belongs to $\mathfrak{W}$ for every $W \in \mathfrak{W}$ and every $u \in \mathfrak{A}_{j_0}$. The assertions (3.8.2) and (3.10.2) show that $\pi(\mathfrak{Z}^0_{j_0})$ and $\pi(\mathfrak{Z}^1_{j_0})$ are both normal. By *a canonical system* for a normal subfamily $\mathfrak{W}$ of $\mathfrak{Z}_{j_0}$, we understand a system

$$\{V_X, \varphi_X, J_{XY}(u), (X, Y \in \mathfrak{W};\ u \in \mathfrak{A}_{j_0})\}$$

satisfying all the conditions of Theorem 1 for $\mathfrak{W}$ in place of $\mathfrak{Z}_{j_0}$.

LEMMA 5. *Let $\mathfrak{W}$ and $\mathfrak{W}'$ be normal subfamilies of $\mathfrak{Z}_{j_0}$, and suppose that there exist canonical systems*

$$\{V_S, \varphi_S, J_{TS}(u), (S, T \in \mathfrak{W};\ u \in \mathfrak{A}_{j_0})\}$$

for $\mathfrak{W}$ and

$$\{V'_L, \varphi'_L, J'_{ML}(u), (L, M \in \mathfrak{W}';\ u \in \mathfrak{A}_{j_0})\}$$

for $\mathfrak{W}'$. For $S \in \mathfrak{W}$ and $L \in \mathfrak{W}'$ such that $S \subset L$, let E_{LS} be the morphism of V_S onto V'_L defined by $\varphi'_L = E_{LS} \circ \varphi_S$. Then E_{LS} is rational over k_S and,

$$E^{\sigma(u)}_{MT} \circ J_{TS}(u) = J'_{ML}(u) \circ E_{LS} \tag{4.3.1}$$

for every $u \in \mathfrak{A}_{j_0}$ where $T = uSu^{-1} \in \mathfrak{W}$ and $M = uLu^{-1} \in \mathfrak{W}'$.

Proof. Since the sets of the points of the form $\varphi_S(z)$ and $\varphi'_L(z)$ for an isolated fixed point z on $\mathcal{H}_{j_0}$ are dense on V_S and V'_L respectively, it follows from the property (IV) of the canonical systems that E_{LS} is defined over the algebraic closure $\bar{\mathbf{Q}}$ of $\mathbf{Q}$. Let k be a finite normal algebraic extension of k_S, over which E_{LS} is rational. Take a commutative isolating subalgebra P of B so that $R_j(P)$ is linearly disjoint with k over K'_{j_0}. (Cf. Proposition 11 in 2.4.) Here j is the element of $\mathcal{J}(j_0)$ isolated by P. Let z be the corresponding isolated fixed point on $\mathcal{H}_{j_0}$. Take $u \in \mathfrak{A}_{j_0}$, and let τ be the restriction of $\sigma(u)$ to k_S. Extend τ to an automorphism ω of $\bar{\mathbf{Q}}$ over $R_j(P)$, and take $v \in R_j(P)^\times_{\mathbf{A}}$ so that $\omega = [v, R_j(P)]$ on $R_j(P)_{\mathrm{ab}}$. Since $\sigma(\eta^*_j(v)^{-1}) = [v, R_j(P)]$ on $\mathfrak{K}_{j_0}$, we see that $\sigma(\eta^*_j(v)^{-1}) = \sigma(u)$ on k_S. Hence on account of Proposition 18 in 3.7, we can find $\alpha \in A_{j_0}$ and $s \in S$ so that $\eta^*_j(v)^{-1} = \alpha us$. Put $T = uSu^{-1}$, $U = \alpha T\alpha^{-1} = \eta^*_j(v)^{-1}S\eta^*_j(v)$, $M = uLu^{-1}$, and $N = \alpha M\alpha^{-1} = \eta^*_j(v)^{-1}L\eta^*_j(v)$. Then by (IV), we have $\varphi_U(z)^\omega = J_{US}(\eta^*_j(v)^{-1}) \circ \varphi_S(z) = J_{UT}(\alpha)^\omega \circ J_{TS}(u) \circ \varphi_s(z)$. From this, we see easily that $\varphi_T(\alpha^{-1}(z))^\omega = J_{TS}(u) \circ \varphi_S(z)$. Similarly, we have $\varphi'_M(\alpha^{-1}(z))^\omega = J'_{ML}(u) \circ \varphi'_L(z)$. Therefore we conclude that

$$E^\omega_{MT} \circ J_{TS}(u) \circ \varphi_S(z) = J'_{ML}(u) \circ E_{LS}\, \varphi_S(z). \tag{4.3.2}$$

After fixing u, P, and ω, if we change z for $\beta(z)$ with $\beta\in A_+^0$, then $\beta(z)$ is an isolated fixed point on $\mathcal{H}_{j_0}$, and corresponds to $\beta(j)$ which is the element of $\mathcal{J}(j_0)$ isolated by $\beta(P)$. On account of Proposition 10 in 2.3, we have $R_j(P)=R_{\beta(j)}(\beta(P))$, and see easily that (4.3.2) holds if we replace z by $\beta(z)$ with any $\beta\in A_+^0$. Since $\{\beta(z)\,|\,\beta\in A_+^0\}$ is dense on $\mathcal{H}_{j_0}$, we obtain

$$E^{\omega}_{MT}\circ J_{TS}(u)=J'_{ML}(u)\circ E_{LS}. \tag{4.3.3}$$

Especially, if we take $u=1$, then we have $E^{\omega}_{LS}=E_{LS}$. Since ω can move all the automorphisms of k over k_S, E_{LS} should be rational over k_S. Therefore we can replace ω in (4.3.3) by $\sigma(u)$, and get the lemma.

4.4. For $W\in\mathfrak{Z}_{j_0}$, put $\mathfrak{W}(W)=\{uWu^{-1}\,|\,u\in\mathfrak{A}_{j_0}\}$. Then $\mathfrak{W}(W)$ is a normal subfamily. If there exists a canonical system for every $\mathfrak{W}(W)$, $W\in\mathfrak{Z}_{j_0}$, then Lemma 5 allows us to conclude that there is a canonical system for $\mathfrak{Z}_{j_0}$ itself, i.e. we get Theorem 1. In fact, all we have to show is (III), which is easily seen if we put $J_{LS}(1)=E_{LS}$ and $J_{MS}(u)=J'_{ML}(u)\circ E_{LS}$ with the same notation as in Lemma 5.

It should be noticed that we can easily see the uniqueness of a canonical system by Lemma 5.

4.5. LEMMA 6. *Let L and S be two members of $\mathfrak{Z}_{j_0}$ such that S is a normal subgroup of L. Then if there exists a canonical system*

$$\{V_T, \varphi_T, J_{UT}(u),\ (T,\ U\in\mathfrak{W}(S);\ u\in\mathfrak{A}_{j_0})\}$$

for $\mathfrak{W}(S)$, then there exists a canonical system

$$\{V_M, \varphi_M, J_{NM}(u),\ (M,\ N\in\mathfrak{W}(L);\ u\in\mathfrak{A}_{j_0})\}$$

for $\mathfrak{W}(L)$.

Proof. First let us construct a model (V_M, φ_M) defined over k_M for every $M\in\mathfrak{W}(L)$. Fix M, and take a normal subgroup P of M in $\mathfrak{W}(S)$. Since T is of finite index in M, the set $\{J_{TT}(\gamma)\,|\,\gamma\in\Gamma_M\}$ is a finite group of automorphisms of V_T. Each $J_{TT}(\gamma)$ and V_T are defined over k_T. Therefore there are a quotient variety V of V_T by $\{J_{TT}(\gamma)\,|\,\gamma\in\Gamma_M\}$ defined over k_T, which is a Zariski open subset of a normal projective variety, and the projection morphism E of V_T onto V defined over k_T. (Cf. Serre [8].) Let $u\in M$ and $\gamma\in\Gamma_M$. Then since $\sigma(u\gamma u^{-1})=$ the identity mapping on k_T, we can find $\beta\in A_{j_0}$ and $v\in T$ so that $u\gamma u^{-1}=\beta v$ on account of Proposition 18 in 3.7. Obviously $\beta\in\Gamma_M$. Since $J_{TT}(v)$ is the identity mapping, we have $J_{TT}(u)\circ J_{TT}(\gamma)=J_{TT}(\beta vu)=J_{TT}(\beta)^{\sigma(u)}\circ J_{TT}(u)$. If γ moves all the elements of Γ_M, then $J_{TT}(\beta)^{\sigma(u)}$ moves all the elements of $\{J_{TT}(\gamma)^{\sigma(u)}\,|\,\gamma\in\Gamma_M\}$. Therefore there is a biregular mapping $J(u)$ of V onto $V^{\sigma(u)}$ such that $J(u)\circ E = E^{\sigma(u)}\circ J_{TT}(u)$. On the other hand, it follows from Proposition 18 that σ induces an isomorphism of the quotient group $M/\Gamma_M T$ onto $\mathrm{Gal}\,(k_T/k_M)$. Let v be an element of M such that $\sigma(u)=\sigma(v)$ on k_T. Then $v=\beta wu$ with some $\beta\in\Gamma_M$ and $w\in T$. We have $J_{TT}(v)=J_{TT}(\beta)^{\sigma(u)}\circ J_{TT}(u)$. Since $E=E\circ J_{TT}(\beta)$, we have $J(v)\circ E=E^{\sigma(v)}\circ J_{TT}(v)=E^{\sigma(u)}\circ J_{TT}(u)=J(u)\circ E$, and so, $J(v)=J(u)$. This means that $J(u)$ depends only on the effect of $\sigma(u)$ on k_T. For each $\tau\in\mathrm{Gal}\,(k_T/k_M)$, put $J_\tau=J(u)$ with $u\in M$ such that $\sigma(u)=\tau$ on k_T. We see easily that $J_{\tau\xi}=J_\tau^\xi\circ J_\xi$ for $\tau,\xi\in\mathrm{Gal}\,(k_T/k_M)$. Hence on account of Weil [15], there are a variety V_M defined over k_M and a biregular morphism R of V_M onto V rational over k_T such that $J_\tau=R^\tau\circ R^{-1}$ for all $\tau\in\mathrm{Gal}(k_T/k_M)$. Put $E_{MT}=R^{-1}\circ E$ and $\varphi_M=E_{MT}\circ\varphi_T$. Then (V_M,φ_M) is a model of $\Gamma_M\backslash\mathcal{H}_{j_0}$ such that V_M is defined over k_M. Next we show

(4.5.1) *For every $U\in\mathfrak{W}(S)$ that is contained in M, the morphism E_{MU} of V_U onto V_M defined by $\varphi_M=E_{MU}\circ\varphi_U$ is rational over k_U. Moreover for any $u\in M$, $E_{MU}^{\sigma(u)}\circ J_{UW}(u)=E_{MW}$ where $W=u^{-1}Uu$.*

Let T be the member of $\mathfrak{W}(S)$ used to construct (V_M,φ_M). Then it is easily seen that (4.5.1) holds for T in place of U. Therefore, especially, $\varphi_M(z)$ is rational over the algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$ for any isolated fixed point z on $\mathcal{H}_{j_0}$ since it follows from the property (IV) of the canonical system for $\mathfrak{W}(S)$ that $\varphi_T(z)$ is rational over $\overline{\mathbf{Q}}$. Hence we conclude that E_{MU} is defined over $\overline{\mathbf{Q}}$, and so, over a finite normal algebraic extension k of K'_{j_0}. Note that $k_U=k_S$ for every $U\in\mathfrak{W}(S)$. Let P be a commutative isolating subalgebra of B such that $R_j(P)$ is linearly disjoint with kk_U over K'_{j_0} where j is the element of $\mathcal{J}(j_0)$ isolated by P. Let z be the isolated fixed point on $\mathcal{H}_{j_0}$ corresponding to j. Let τ be the restriction of $\sigma(u)$ to k_U and ω any automorphism of $\overline{\mathbf{Q}}$ over $R_j(P)$ such that $\omega=\tau$ on k_U. Take $v\in R_j(P)_{\mathbf{A}}^\times$ so that $\omega=[v,R_j(P)]$ on $R_j(P)_{\mathbf{ab}}$. Then since $\sigma(u)=\tau=[v,R_j(P)]=\sigma(\eta_j^*(v)^{-1})$ on $k_U=k_W$, there are $\alpha\in A_{j_0}$ and $w\in W$ such that $\eta_j^*(v)^{-1}=\alpha uw$. (See Proposition 18 in 3.7.) Put $X=\alpha U\alpha^{-1}=\eta_j^*(v)^{-1}W\eta_j^*(v)$, and $Y=\alpha T\alpha^{-1}$. Then $\eta_j^*(v)Y\eta_j^*(v)^{-1}=w^{-1}u^{-1}\alpha^{-1}Y\alpha uw=w^{-1}u^{-1}Tuw=T$ since $uw\in M$ and T is a normal subgroup of M. We have $\varphi_X(z)^\omega=J_{XW}(\eta_j^*(v)^{-1})\circ\varphi_W(z)=J_{XW}(\alpha uw)\circ\varphi_W(z)=J_{XU}(\alpha)^{\sigma(u)}\circ J_{UW}(u)\circ\varphi_W(z)$ and $\varphi_Y(z)^\omega=J_{YT}(\eta_j^*(v)^{-1})\circ\varphi_T(z)=J_{YT}(\alpha uw)\circ\varphi_T(z)=J_{YT}(\alpha)^{\sigma(u)}\circ J_{TT}(uw)\circ\varphi_T(z)$. Therefore $E_{MU}^\omega\circ J_{UW}(u)\circ\varphi_W(z)=E_{MU}^\omega\circ J_{UX}(\alpha^{-1})^{\sigma(u)}[\varphi_X(z)^\omega]=E_{MU}[\varphi_U(\alpha^{-1}(z))]^\omega=\varphi_M(\alpha^{-1}(z))^\omega=E_{MT}^\omega[\varphi_T(\alpha^{-1}(z))^\omega]=E_{MT}^\omega\circ J_{TY}(\alpha^{-1})^\omega[\varphi_Y(z)^\omega]=E_{MT}^\omega\circ J_{TT}(uw)\circ\varphi_T(z)$. We know that (4.5.1) holds for T in place of U. Therefore $E_{MT}^\omega\circ J_{TT}(uw)=E_{MT}^{\sigma(uw)}\circ J_{TT}(uw)=E_{MT}$ since $\omega=\tau=\sigma(u)=\sigma(uw)$ on $k_U=k_W=k_T$. Thus $E_{MT}^\omega\circ J_{TT}(uw)\circ\varphi_T(z)=E_{MT}\circ\varphi_T(z)=\varphi_M(z)=E_{MW}\circ\varphi_W(z)$, and finally we obtain

$$E_{MU}^\omega\circ J_{UW}(u)\circ\varphi_W(z)=E_{MW}\circ\varphi_W(z).$$

Using the same argument as in the proof of Lemma 5, we can derive (4.5.1) from this formula.

After constructing (V_M, φ_M) for every $M \in \mathfrak{W}(L)$, let us now construct $J_{NM}(y)$ for N, $M \in \mathfrak{W}(L)$ such that $N = yMy^{-1}$ with $y \in \mathfrak{A}_{j_0}$. Take a member T of $\mathfrak{W}(S)$ which is a normal subgroup of M and put $U = yTy^{-1}$. Then $U \in \mathfrak{W}(S)$ and U is a normal subgroup of N. Let $\gamma \in \Gamma_M$. Since $\sigma(y\gamma y^{-1}) =$identity, we have $y\gamma y^{-1} = \beta u$ with $\beta \in A_{j_0}$ and $u \in U$. Then $\beta \in \Gamma_N$, and $J_{UT}(y) \circ J_{TT}(\gamma) = J_{UT}(\beta u y) = J_{UU}(\beta)^{\sigma(y)} \circ J_{UT}(y)$. Therefore $E_{NU}^{\sigma(y)} \circ J_{UT}(y) = E_{NU}^{\sigma(y)} \circ J_{UT}(y) \circ J_{TT}(\gamma)$ for every $\gamma \in \Gamma_M$, and hence, there is a morphism $J_{NM}(y)$ of V_M onto $V_N^{\sigma(y)}$, defined over k_T, such that $J_{NM}(y) \circ E_{MT} = E_{NU}^{\sigma(y)} \circ J_{UT}(y)$. We see easily that thus defined $J_{NM}(y)$ satisfies (IIIa) and (IIIc). Let us show

(4.5.2) *For every* $W \in \mathfrak{W}(S)$ *that is contained in* M, $J_{NM}(y) \circ E_{MW} = E_{NX}^{\sigma(y)} \circ J_{XW}(y)$ *where* $X = yWy^{-1}$.

Take P, j and z as above so that $R_j(P)$ is linearly disjoint with $k_{W \cap T}$ over K'_{j_0}, and $v \in R_j(P)_{\mathbf{A}}^{\times}$ so that $\omega = [v, R_j(P)] = \sigma(y)$ on $k_{W \cap T}$. Then we can find $\alpha \in A_{j_0}$ and $w \in W \cap T$ so that $\eta_j^*(v)^{-1} = \alpha y w$. Put $Y = \alpha X \alpha^{-1} = \eta_j^*(v)^{-1} W \eta_j^*(v)$, and $Z = \alpha U \alpha^{-1} = \eta_j^*(v)^{-1} T \eta_j^*(v)$. Then we have $E_{NX}^{\sigma(y)} \circ J_{XW}(y) \circ \varphi_W(z) = E_{NU}^{\sigma(y)} \circ J_{UT}(yw) \circ \varphi_T(z)$ by a similar computation to that done above in proving (4.5.1) (but do not confuse the notation). We know that w belongs to T, and that $J_{NM}(y) \circ E_{MT} = E_{NU}^{\sigma(y)} \circ J_{UT}(y)$. Therefore we have $E_{NX}^{\sigma(y)} \circ J_{XW}(y) \circ \varphi_W(z) = J_{NM}(y) \circ E_{MT} \circ \varphi_T(z) = J_{NM}(y) \circ \varphi_M(z) = J_{NM}(y) \circ E_{MW} \circ \varphi_W(z)$. Then varying z in $\{\beta(z) \mid \beta \in A_+^0\}$, we get (4.5.2).

Now let $x \in \mathfrak{A}_{j0}$, $R = xNx^{-1}$ and $Y = xXx^{-1}$. Then we can define $J_{RN}(x)$ and $J_{RM}(xy)$. On account of (4.5.2), we get the formula,

$$J_{RM}(xy) = J_{RN}(x)^{\sigma(y)} \circ J_{NM}(y).$$

In fact, $J_{RM}(xy) \circ E_{MW} = E_{RY}^{\sigma(xy)} \circ J_{YW}(xy) = E_{RY}^{\sigma(xy)} \circ J_{YX}(x)^{\sigma(y)} \circ J_{XW}(y) = J_{RN}(x)^{\sigma(y)} \circ E_{NX}^{\sigma(y)} \circ J_{XW}(y) = J_{RN}(x)^{\sigma(y)} \circ J_{NM}(y) \circ E_{MW}$. Thus (IIIb) holds. Especially, we have $J_{NM}(uy) = J_{NM}(y)$ for $u \in N$. Therefore, if $y \in N$, we have $J_{RN}(x)^{\sigma(y)} = J_{RN}(xy) = J_{RN}(xyx^{-1}x) = J_{RN}(x)$ since $xyx^{-1} \in R$. This shows that $J_{RN}(x)$ is rational over k_N since σ induces an isomorphism of $N/\Gamma_N X$ onto Gal (k_X/k_N) and $J_{RN}(x)$ is rational over $k_X = k_S$. Finally, let z, j and P be as in (IV) of Theorem 1. Then $\varphi_M(z) = E_{MT} \circ \varphi_T(z)$ is surely rational over $R_j(P)_{\mathrm{ab}}$ for $M \in \mathfrak{W}(L)$. For $v \in R_j(P)_{\mathbf{A}}^{\times}$, put $N = \eta_j^*(v)^{-1} M \eta_j^*(v)$ and $U = \eta_j^*(v)^{-1} T \eta_j^*(v)$. Then for $\tau = [v, R_j(P)]$, $\varphi_M(z)^{\tau} = E_{MT}^{\tau}[\varphi_T(z)^{\tau}] = E_{MT}^{\tau} \circ J_{TU}(\eta_j^*(v)^{-1}) \circ \varphi_U(z)$. Since $\tau = \sigma(\eta_j^*(v)^{-1})$ on k_T, we have $\varphi_M(z)^{\tau} = E_{MT}^{\tau} \circ J_{TU}(\eta_j^*(v)^{-1}) \circ \varphi_U(z) = J_{MN}(\eta_j^*(v)^{-1}) \circ E_{NU} \circ \varphi_U(z) = J_{MN}(\eta_j^*(v)^{-1}) \circ \varphi_N(z)$. The proof is completed.

4.6. In 4.4, we reduced the proof of Theorem 1 to constructing a canonical system for $\mathfrak{W}(W)$ for each $W \in \mathfrak{Z}_{j_0}$. We may restrict ourselves to considering only the members of the subfamily $\pi(\mathfrak{Z}_{j_0}^1)$ of $\mathfrak{Z}_{j_0}$. Further, it is sufficient to show the existence of a canonical system for $\mathfrak{W}(W)$ for each member W of such a subfamily of $\pi(\mathfrak{Z}_{j_0}^1)$ as it contains a normal subgroup of every member of $\pi(\mathfrak{Z}_{j_0}^1)$. Before getting into such a subfamily of $\pi(\mathfrak{Z}_{j_0}^1)$, we proceed another type of reduction.

For a member W of $\pi(\mathfrak{Z}_{j_0}^1)$, put $\mathfrak{W}^0(W)=\{uWu^{-1}\mid u\in\mathfrak{A}_{j_0}^0\}$. We understand by *a canonical subsystem* for $\mathfrak{W}^0(W)$ a system

$$\{V_X, \varphi_X, J_{YX}(u), (X,\ Y\in\mathfrak{W}^0(W);\ u\in\mathfrak{A}_{j_0}^0)\}$$

which satisfies all the conditions of Theorem 1 for $\mathfrak{W}^0(W)$ and $\mathfrak{A}_{j_0}^0$ in place of $\mathfrak{Z}_{j_0}$ and $\mathfrak{A}_{j_0}$ respectively. Using the argument of Shimura [14] II, 5.6, let us show

LEMMA 7. *Let $W\in\pi(\mathfrak{Z}_{j_0}^1)$. If there exists a canonical subsystem for $\mathfrak{W}^0(W)$, then there exists a canonical system for $\mathfrak{W}(W)$.*

Proof. Let $\{V_X, \varphi_X, J_{YX}(u), (X,\ Y\in\mathfrak{W}^0(W);\ u\in\mathfrak{A}_{j_0}^0)\}$ be the canonical subsystem for $\mathfrak{W}^0(W)$. We see easily from Proposition 16 in 3.5 that there exists a set $\{a_1, ..., \alpha_n\}$ of elements of A_{j_0} such that

$$\mathfrak{W}(W)=\mathfrak{W}^0(W)\cup\alpha_1\mathfrak{W}^0(W)\alpha_1^{-1}\cup\ldots\cup\alpha_n\mathfrak{W}^0(W)\alpha_n^{-1}$$

(disjoint union) and that, for $X,\ Y\in\mathfrak{W}(W)$, there exists an element $u\in\mathfrak{A}_{j_0}^0$ such that $Y=uXu^{-1}$ if and only if X and Y belongs to the same $\alpha\mathfrak{W}^0(W)\alpha^{-1}$ for some $\alpha\in\{1, \alpha_1, ..., \alpha_n\}$. Fix $\alpha\in\{\alpha_1, ..., \alpha_n\}$. Let $T\in\alpha\mathfrak{W}^0(W)\alpha^{-1}$, and take $X\in\mathfrak{W}^0(W)$ so that $T=\alpha X\alpha^{-1}$. Then $k_T=k_X$ and $\Gamma_T=\alpha\Gamma_X\alpha^{-1}$. Put $V_T=V_X$ and $\varphi_T=\varphi_X\circ\alpha^{-1}$. Then (V_T, φ_T) is a model of $\Gamma_T\backslash\mathcal{H}_{j_0}$. After defining (V_T, φ_T) for all $T\in\alpha\mathfrak{W}^0(W)\alpha^{-1}$ in this way, define $J_{UT}(u)$ for $u\in\mathfrak{A}_{j_0}^0$ and $U=uTu^{-1}\in\alpha\mathfrak{W}^0(W)\alpha^{-1}$ by putting $J_{UT}(u)=J_{YX}(\alpha^{-1}u\alpha)$ where $Y=(\alpha^{-1}u\alpha)X(\alpha^{-1}u\alpha)^{-1}\in\mathfrak{W}^0(W)$. We can see in a straightforward way that $\{V_T, \varphi_T, J_{UT}(u), (T,\ U\in\alpha\mathfrak{W}^0(W)\alpha^{-1};\ u\in\mathfrak{A}_{j_0}^0)\}$ is a canonical subsystem for $\mathfrak{W}^0(\alpha W\alpha^{-1})=\alpha\mathfrak{W}^0(W)\alpha^{-1}$. To see (IV), use Proposition 10 in 2.3. Collecting the canonical subsystems for $\alpha\mathfrak{W}^0(W)\alpha^{-1}$, $\alpha\in\{1, \alpha_1, ..., \alpha_n\}$, we have a system $\{V_X, \varphi_X, J_{YX}(u), (X,\ Y\in\mathfrak{W}(W);\ u\in\mathfrak{A}_{j_0}^0)\}$ satisfying all the conditions of Theorem 1 for $\mathfrak{W}(W)$ and $\mathfrak{A}_{j_0}^0$ in place of $\mathfrak{Z}_{j_0}$ and $\mathfrak{A}_{j_0}$ respectively as is easily seen. Now all what we have to do is to define $J_{YX}(u)$ for all $u\in\mathfrak{A}_{j_0}$. Fix an element $\alpha\in A_{j_0}$ at first. For each $X\in\mathfrak{W}(W)$, put $\bar{X}=\alpha X\alpha^{-1}$, $\bar{V}_X=V_{\bar{X}}$ and $\bar{\varphi}_X=\varphi_{\bar{X}}\circ\alpha$. Since $\Gamma_X=\alpha^{-1}\Gamma_{\bar{X}}\alpha$, we see easily that $(\bar{V}_X, \bar{\varphi}_X)$ is a model of $\Gamma_X\backslash\mathcal{H}_{j_0}$. Put $\bar{J}_{YX}(u)=J_{\bar{Y}\bar{Z}}(\alpha u\alpha^{-1})$ for $u\in\mathfrak{A}_{j_0}^0$ and $Y=uXu^{-1}$. Then we see easily that the new system $\{\bar{V}_X, \bar{\varphi}_X, \bar{J}_{YX}(u), (X,\ Y\in\mathfrak{W}(W);\ u\in\mathfrak{A}_{j_0}^0)\}$ satisfies all the conditions of Theorem 1 for $\mathfrak{W}(W)$ and $\mathfrak{A}_{j_0}^0$ in place of $\mathfrak{Z}_{j_0}$ and $\mathfrak{A}_{j_0}$ respectively. To see (IV), use Proposition 10. In a similar way to that in which we showed Lemma 5, we can show the existence of a biregular morphism Q_X of V_X onto $\bar{V}_X$ for every

$X \in \mathfrak{W}(W)$, which is rational over k_X and such that $Q_X \circ \varphi_X = \bar{\varphi}_X$ and $Q_X^{\sigma(u)} \circ J_{XW}(u) = \bar{J}_{XW}(u) \circ Q_W$ for $u \in \mathfrak{A}_{j_0}^0$ with $W = u^{-1}Xu$. Put $J_{\bar{X}X}(\alpha) = Q_X$. Then we have

$$J_{\bar{X}X}(\alpha)[\varphi_X(z)] = \bar{\varphi}_X(z) = \varphi_X(\alpha(z))$$

where $\bar{X} = \alpha X \alpha^{-1}$. For $v = u\alpha \in \mathfrak{A}_{j_0}$ with $u \in \mathfrak{A}_{j_0}^0$, put $J_{YX}(v) = J_{Y\bar{X}}(u) \circ J_{\bar{X}X}(\alpha)$ where $\bar{X} = \alpha X \alpha^{-1}$ and $Y = vXv^{-1}$. Thus we obtain $J_{YX}(v)$ for all $v \in \mathfrak{A}_{j_0} = \mathfrak{A}_{j_0}^0 A_{j_0}$ (see Proposition 16), and can show in a straightforward way that these are well defined and satisfy (III a, b, c). Q.e.d.

4.7. Let D^m be the space of all the m-dimensional row vectors with components in D, as before, and fix a $\mathbf{Z}$-lattice M of D^m. For each prime integer p, let $\mathbf{Q}_p$ be the field of all the p-adic (rational) numbers and $\mathbf{Z}_p$ the ring of all the p-adic integers. Put $D_p^m = D^m \otimes_{\mathbf{Q}} \mathbf{Q}_p$ and $\mathfrak{M}_p = \mathfrak{M} \otimes_{\mathbf{Z}} \mathbf{Z}_p$. Then $\mathfrak{M}_p$ is a $\mathbf{Z}_p$-lattice of D_p^m.

For $x \in G_{\mathbf{A}}$, we define a $\mathbf{Z}$-lattice $\mathfrak{M}x$ of D^m by $\mathfrak{M}x = \bigcap_p (\mathfrak{M}_p x_p \cap D^m)$ where $\bigcap_p$ is the intersection over all prime integers, and x_p is the p-component of $x \in G_{\mathbf{A}}$. Obviously, $\mathfrak{M}x$ coincides the ordinary transformation of $\mathfrak{M}$ by x if $x \in G_{\mathbf{Q}}$.

We identify the quotient module $D^m/\mathfrak{M}$ with the direct sum,

$$D^m/\mathfrak{M} = \textstyle\sum_p D_p^m/\mathfrak{M}_p$$

in the natural way. Here Σ_p is the direct sum over all prime integers. For $x \in G_{\mathbf{A}}$, define an isomorphism of $D^m/\mathfrak{M}$ onto $D^m/\mathfrak{M}x$ through the direct sum by the non-archimedian part of x.

For any finite number of elements $w_1, \ldots, w_t$ of D^m, put

$$S_{\mathfrak{f}}(\mathfrak{M}; w_1, \ldots, w_t) = \{x \in G_{\mathfrak{f}} \mid \mathfrak{M}x = \mathfrak{M},\ w_i x \equiv w_i \bmod \mathfrak{M},\ i = 1, \ldots, t\};$$

$$S(\mathfrak{M}; w_1, \ldots, w_t) = S_{\mathfrak{f}}(\mathfrak{M}; w_1, \ldots, w_t) G_{\infty+}.$$

Then $S(\mathfrak{M}; w_1, \ldots, w_t)$ belongs to $\mathfrak{Z}_{j_0}^0$. We have

$$x^{-1} S(\mathfrak{M}; w_1, \ldots, w_t) x = S(\mathfrak{M}x; w_1 x, \ldots, w_t x) \quad (x \in G_{j_0+}). \tag{4.7.1}$$

LEMMA 8. *Fix any* $\mathbf{Z}$*-lattice* $\mathfrak{M}$ *of* D^m*, and let the notation be as above. Suppose that, for every finite number of elements* $w_1, \ldots, w_t$*, there exists a canonical subsystem for* $\mathfrak{W}^0(\pi(S^1))$*, where* S^1 *is as in* 3.10 *for* $S = S(\mathfrak{M}; w_1, \ldots, w_t)$*. Then there exists a canonical system for* $\mathfrak{Z}_{j_0}$*.*

Proof. Let T be any member of $\mathfrak{Z}_{j_0}^0$, and let $S(\mathfrak{M})$ be the group defined above for $w_1 = \ldots = w_t = 0$. Since $T \cap S(\mathfrak{M})$ is a subgroup of T of finite index, there is a normal subgroup U of T in $\mathfrak{Z}_{j_0}^0$ which is contained in $S(\mathfrak{M})$. For example, $U = \bigcap_\mu u_\mu (T \cap S(\mathfrak{M})) u_\mu^{-1}$ where $\{u_\mu\}$ is a set of representatives of the coset decomposition $T = \bigcup_\mu u_\mu (T \cap S(\mathfrak{M}))$. Since U is a member of $\mathfrak{Z}_{j_0}^0$, there is a finite number of elements $v_1, \ldots, v_s$ of D^m such that U contains

$S(\mathfrak{M}; v_1, \ldots, v_s)$. Let $\{y_1, \ldots, y_n\}$ be a set of representatives of the coset decomposition of U by the subgroup $S(\mathfrak{M}; v_1, \ldots, v_s)$, i.e. $U = \bigcup_{\nu=1}^{n} y_\nu S(\mathfrak{M}; v_1, \ldots, v_s)$. Then we have a normal subgroup $S = \bigcap_\nu y_\nu S(\mathfrak{M}; v_1, \ldots, v_s) y_\nu^{-1}$ of U. Since each y_ν belongs to $S(\mathfrak{M})$, $y_\nu S(\mathfrak{M}; v_1, \ldots, v_s) y_\nu^{-1} = S(\mathfrak{M}; v_1 y_\nu^{-1}, \ldots, v_n y_\nu^{-1})$. Therefore, $S = S(\mathfrak{M}; w_1, \ldots, w_t)$ where $\{w_1, \ldots, w_t\} = \{v_1 y_\nu^{-1}, \ldots, v_s y_\nu^{-1} \mid \nu = 1, \ldots, n\}$. Make T^1, U^1 and S^1 from T, U and S as in 3.10. Then $\pi(U^1)$ is a normal subgroup of $\pi(T^1)$, and $\pi(S^1)$ is a normal subgroup of $\pi(U^1)$. By the assumption, there is a canonical subsystem for $\mathfrak{W}^0(\pi(S^1))$. Therefore, Lemma 7 assures the existence of a canonical system for $\mathfrak{W}(\pi(S^1))$. Using Lemma 6 successively, we see that there exists a canonical system for $\mathfrak{W}(\pi(T^1))$. Since this is true for every $\pi(T^1)$ of $\pi(\mathfrak{Z}^1_{f_0})$, we get the lemma as was seen at the beginning of 4.6.

Remark. In the above proof, we can take $S(\mathfrak{M}; v_1, \ldots, v_s)$ as small as we like, choosing a large set $v_1, \ldots, v_s$. Therefore, to see the existence of a canonical system for $\mathfrak{Z}_{f_0}$, it is sufficent to show the existence of a canonical subsystem for $\mathfrak{W}^0(\pi(S^1))$ only for every sufficiently small $S = S(\mathfrak{M}; w_1, \ldots, w_t)$. Saying "sufficiently small," we think of Proposition 20 in 3.10 and the remark following its proof.

4.8. Here we insert a lemma, which we needed to prove Corollary 2 of Theorem 2 in 4.2.

LEMMA 9. *Let U be an arbitrary open compact subgroup of $G^1_{\mathfrak{f}}$, and $\Gamma^1_U = G^1_{\mathbf{Q}} \cap U G^1_\infty$. Then for any positive integer n, the linear span of the set $\{\gamma^n \mid \gamma \in \Gamma^1_U\}$ over $\mathbf{Q}$ is equal to B if $mq > 2$, and B_0 if $mq = 2$ where B_0 is the quaternion algebra over F determined by* (1.2.1).

Proof. Let p be a prime integer in $\mathbf{Z}$ such that p decomposes completely in K, and that D is unramified at every prime factor of p in K. Then $B \otimes_{\mathbf{Q}} \mathbf{Q}_p = \mathbf{M}(m, D) \otimes_{\mathbf{Q}} \mathbf{Q}_p$ is isomorphic to the direct product of $2g$ copies of $\mathbf{M}(mq, \mathbf{Q}_p)$. Define an involution δ' of $\mathbf{M}(mq, \mathbf{Q}_p)^2$ by $(x, y)^{\delta'} = ({}^t y, {}^t x)$ for $(x, y) \in \mathbf{M}(mq, \mathbf{Q}_p)^2$, and put

$$H_{\mathbf{Q}_p} = \{(x, y) \in \mathbf{M}(mq, \mathbf{Q}_p)^2 \mid (x, y)(x, y)^{\delta'} = (1_{mq}, 1_{mq}), \text{ and } \det(x) = 1\}.$$

If we furnish $\mathbf{M}(mq, \mathbf{Q}_p)^{2g} = (M(mq, \mathbf{Q}_p)^2)^g$ with the involution defined by δ', we can find, as is well known, a $\mathbf{Q}_p$-linear isomorphism χ of $B \otimes_{\mathbf{Q}} \mathbf{Q}_p$ onto $\mathbf{M}(mq, \mathbf{Q}_p)^{2g}$ which transforms the involution δ of $B \otimes_{\mathbf{Q}} \mathbf{Q}_p$ to this involution of $\mathbf{M}(mq, \mathbf{Q}_p)^{2g}$. (Cf. M. Kneser, *Galois-Kohomologie halbeinfacher algebraischer Gruppen über $\mathfrak{p}$-adischen Körpern*, I, Math. Z. 88 (1965); II, Math. Z. 89 (1965). Also see T. A. Springer, *Galois cohomology of linear algebraic groups*, [1], pp. 149–158, and M. Kneser, *Hasse principle for H^1 of simply connected groups*, [1], p. 160.) Therefore $G^1_{\mathbf{Q}_p}$ is isomorphic to the direct product of g copies of $H_{\mathbf{Q}_p}$. For a non-negative integer e, let $H_{\mathbf{Z}_p}(e)$ be the principal congruence subgroup of $H_{\mathbf{Z}_p} = H_{\mathbf{Q}_p} \cap \mathbf{M}(mq, \mathbf{Z}_p)^2$ modulo p^e. Put $t = 2(mq)^2 g$ if $mq > 2$, and $t = 4g$ if $mq = 2$. Then for any positive integer n, and e, one can easily find t elements u_i, $i=1, \dots, t$, in $H_{\mathbf{Z}_p}(e)^g$ such that u_i^n, $i=1, \dots, t$, are linearly independent over $\mathbf{Q}_p$. Now let U be as stated in the lemma. Then we can find a positive integer e such that $U' \times \chi^{-1}(H_{\mathbf{Z}_p}(e)^g)$ is an open compact subgroup of U with some subgroup U' of $\prod_{p' \neq p} G^1_{\mathbf{Q}_{p'}}$. In fact, it is enough to choose such an e as $\{1'\} \times \chi^{-1}(H_{\mathbf{Z}_p}(e)^g)$ is contained in U where $1'$ is the identity element of $\prod_{p' \neq p} G^1_{\mathbf{Q}_{p'}}$. For given n, take u_i in $H_{\mathbf{Z}_p}(e)^g$ as above. If $mq>2$, then the module

$$\mathfrak{m} = \sum_{i=1}^{2(mq)^2 g} \mathbf{Z}_p u_i^n$$

is a $\mathbf{Z}_p$-lattice of $\mathbf{M}(mq, \mathbf{Q}_p)^{2g}$. Therefore there is some positive integer e' such that $p^{e+e'}\mathbf{M}(mq, \mathbf{Z}_p)^{2g} \subset p\mathfrak{m}$. If $mq=2$, then $H_{\mathbf{Z}_p}(e)^g$ is contained in $\chi(B_0 \otimes_{\mathbf{Q}} \mathbf{Q}_p)$ (see 1.2). Therefore the module

$$\mathfrak{m} = \sum_{i=1}^{4g} \mathbf{Z}_p u_i^n$$

is a $\mathbf{Z}_p$-lattice of $\chi(B_0 \otimes_{\mathbf{Q}} \mathbf{Q}_p)$. Hence there is a positive integer e' such that $p^{e+e'}(\chi(B_0 \otimes_{\mathbf{Q}} \mathbf{Q}_p) \cap \mathbf{M}(2, \mathbf{Z}_p)^{2g}) \subset p\mathfrak{m}$. Since $U' \times \chi^{-1}(H_{\mathbf{Z}_p}(e+e')^g)$ is an open compact subgroup of $G^1_{\mathbf{f}}$, we can find, by the strong approximation theorem, t elements γ_i, $i=1, \dots, t$, in $G^1_{\mathbf{Q}}$ such that $\gamma_i^{-1}(1', \chi^{-1}(u_i)) \in U' \times \chi^{-1}(H_{\mathbf{Z}_p}(e+e')^g)$. Then $\gamma_i \in \Gamma^1_U$. We see easily that $\chi(\gamma_i^n) - u_i^n \in p\mathfrak{m}$. This shows that $\chi(\gamma_i^n)$, $i=1, \dots, t$, are linearly independent over $\mathbf{Q}_p$. Therefore γ_i^n, $i=1, \dots, t$, are surely linearly independent over $\mathbf{Q}$. Now the lemma follows at once.

5. Modulus-varieties of PEL-structures

5.1. First we review on PEL-types and PEL-structures defined and studied by Shimura in [9], [10] and [11]. (Also see 4.1 of [12].)

Let L be a simple algebra over $\mathbf{Q}$ with a positive involution ϱ, n a positive integer such that $2n = m[L: \mathbf{Q}]$ for some integer m, and Φ a representation of L on a complex vector space $\mathbf{C}^n$ such that Φ maps the identity of L to the identity mapping of $\mathbf{C}^n$, and the direct sum of Φ and its complex conjugate Φ is equivalent to a rational representation of L. Let $L^{(m)}$ be a left L-module of dimension $2n$ over $\mathbf{Q}$, T a ϱ-antihermitian form on $L^{(m)}$, i.e. an L-valued $\mathbf{Q}$-bilinear form on $L^{(m)}$ such that

$$T(ax, by) = aT(x, y)b^{\varrho}; \quad T(x, y)^{\varrho} = -T(y, x)$$

for $a, b \in L$ and $x, y \in L^{(m)}$. We consider only a non-degenerate T. Put $L_{\mathbf{R}} = L \otimes_{\mathbf{Q}} \mathbf{R}$ and $L^{(m)}_{\mathbf{R}} = L^{(m)} \otimes_{\mathbf{Q}} \mathbf{R}$, and extend T to an $L_{\mathbf{R}}$-valued $\mathbf{R}$-bilinear form on $L^{(m)}$. Let $\mathfrak{M}$ be a Z-lattice of $L^{(m)}$ such that

$$\operatorname{tr}(T(\mathfrak{M}, \mathfrak{M})) = \mathbf{Z}$$

where tr denotes the $\mathbf{R}$-linear mapping of $L_{\mathbf{R}}$ to $\mathbf{R}$ obtained from the reduced trace of L over $\mathbf{Q}$. For any given T, a rational multiple of T satisfies this condition for any given $\mathfrak{M}$. Let $u_1, \ldots, u_t$ be elements of $L^{(m)}$. A PEL-type is a collection

$$\Omega = (L, \Phi, \varrho;\ T, \mathfrak{M};\ u_1, \ldots, u_t).$$

Two PEL-types $(L, \Phi, \varrho;\ T, \mathfrak{M};\ u_1, \ldots, u_t)$ and $(L', \Phi', \varrho';\ T', \mathfrak{M}';\ u'_1, \ldots, u'_t)$ are equivalent if $L = L'$, $\varrho = \varrho'$, $t = t'$, Φ and Φ' are equivalent, and there exists an L-linear automorphism α of $L^{(m)}$ such that $T'(\alpha(x), \alpha(y)) = T(x, y)$, $\mathfrak{M}' = \alpha(\mathfrak{M})$, and $u'_i \equiv \alpha(u_i) \bmod \mathfrak{M}'$ for $i = 1, \ldots, t$.

A PEL-structure $Q = (\mathcal{A}, \mathcal{C}, \theta;\ p_1, \ldots, p_t)$ of type $\Omega = (L, \Phi, \varrho;\ T, \mathfrak{M};\ u_1, \ldots, u_t)$ is a collection of an abelian variety $\mathcal{A}$ with a polarization $\mathcal{C}$ and points p_i, $i = 1, \ldots, t$, of finite orders, and a homomorphic embedding θ of L into $\mathrm{End}_{\mathbf{Q}}(\mathcal{A}) = \mathrm{End}(\mathcal{A}) \otimes_{\mathbf{Z}} \mathbf{Q}$ satisfying the following conditions:

There are a complex torus $\mathbf{C}^n/\overline{\mathfrak{M}}$, an $\mathbf{R}$-linear isomorphism $\mathfrak{y}$ of $L^{(m)}_{\mathbf{R}}$ onto $\mathbf{C}^n$ and a homomorphism ι of $\mathbf{C}^n$ onto $\mathcal{A}$ such that

(5.1.1) *ι induces a biregular isomorphism of $\mathbf{C}^n/\overline{\mathfrak{M}}$ to $\mathcal{A}$, and $\iota(\Phi(a)w) = \theta(a)\iota(w)$ for $a \in L \cap \theta^{-1}(\mathrm{End}(\mathcal{A}))$ and $w \in \mathbf{C}^n$;*

(5.1.2) *$\mathfrak{y}$ maps $\mathfrak{M}$ onto $\overline{\mathfrak{M}}$ and $\mathfrak{y}(ax) = \Phi(a)\mathfrak{y}(x)$ for $a \in L$ and $x \in L^{(m)}_{\mathbf{R}}$;*

(5.1.3) *$\mathcal{C}$ contains a divisor which determines a Riemannian form E on $\mathbf{C}^n/\overline{\mathfrak{M}}$ such that $E(\mathfrak{y}(x), \mathfrak{y}(y)) = \mathrm{tr}(T(x, y))$ for $x, y \in L^{(m)}_{\mathbf{R}}$;*

(5.1.4) *$\iota(\mathfrak{y}(u_i)) = p_i$ for $i = 1, \ldots, t$.*

Let $Q = (\mathcal{A}, \mathcal{C}, \theta;\ p_1, \ldots, p_t)$ be a PEL-structure, and τ an automorphism of $\mathbf{C}$. Then we get naturally a structure

$$Q^\tau = (\mathcal{A}^\tau, \mathcal{C}^\tau, \theta^\tau;\ p_1^\tau, \ldots, p_t^\tau).$$

Here θ^τ is an (injective) homomorphism of L into $\mathrm{End}_{\mathbf{Q}}(\mathcal{A}^\tau)$ defined by $\theta^\tau(a) = \theta(a)^\tau$ for $a \in L \cap \theta^{-1}(\mathrm{End}(\mathcal{A}))$.

Let $Q = (\mathcal{A}, \mathcal{C}, \theta;\ p_1, \ldots, p_t)$ and $Q' = (\mathcal{A}', \mathcal{C}', \theta';\ p'_1, \ldots, p'_t)$ are PEL-structures of types with the same (L, Φ, ϱ). We say that Q is isomorphic to Q' if there exists an isomorphism λ of the abelian variety $\mathcal{A}$ onto $\mathcal{A}'$ such that λ maps $\mathcal{C}$ into $\mathcal{C}'$, $\lambda \circ \theta(a) = \theta'(a) \circ \lambda$ for every $a \in L$ and $\lambda(p_i) = p'_i$ for $i = 1, \ldots, t$.

5.2. Let $\Omega = (L, \Phi, \varrho;\ T, \mathfrak{M};\ u_1, \ldots, u_t)$ be a PEL-type. Let $\mathfrak{o}$ be an order determined by $\mathfrak{M}$ as

$$\mathfrak{o} = \{a \in L \mid a\mathfrak{M} \subset \mathfrak{M}\},$$

and define a $\mathbf{Z}$-lattice $\mathfrak{N}$, an algebraic group $U(T)$, a Lie group $U_{\mathbf{R}}(T)$, and discrete subgroups $\Gamma(T, \mathfrak{M})$ and $\Gamma(T, \mathfrak{N}/\mathfrak{M})$ of $U_{\mathbf{R}}(T)$ by

$$\mathfrak{N} = \mathfrak{M} + \sum_{i=1}^{t} \mathfrak{o} u_i;$$

$$U(T) = \{\alpha \in \mathbf{GL}(m, L) \mid T(x\alpha, y\alpha) = T(x, y)\};$$

$$U_{\mathbf{R}}(T) = \{\alpha \in \mathbf{GL}(m, L_{\mathbf{R}}) \mid T(x\alpha, y\alpha) = T(x, y)\};$$

$$\Gamma(T, \mathfrak{M}) = \{\alpha \in U(T) \mid \mathfrak{M}\alpha = \mathfrak{M}\};$$

$$\Gamma(T, \mathfrak{N}/\mathfrak{M}) = \{\alpha \in \Gamma(T, \mathfrak{M}) \mid \mathfrak{N}(1-\alpha) \subset \mathfrak{M}\}.$$

Here we identify the algebra of all the $\mathbf{Q}$-linear endomorphisms of $L^{(m)}$ which commute with every element of L with $\mathbf{M}(m, L)$, and consider $L^{(m)}$ as a left L- and right $\mathbf{M}(m, L)$-module. This can be done since we assume that $[L^{(m)}: \mathbf{Q}] = 2n = m[L: \mathbf{Q}]$. If L is a division algebra, then $L^{(m)}$ is isomorphic to the direct sum L^m of m copies of L.

In [10] and [11] II, Shimura fixed a bounded symmetric domain $\mathcal{H}(T)$ isomorphic to the quotient space of $U_{\mathbf{R}}(T)$ by a maximal compact subgroup, and constructed a family $\sum_{\Omega} = \{Q_z \mid z \in \mathcal{H}(T)\}$ of PEL-structures of type Ω parametrized by the point z of $\mathcal{H}(T)$. We construct a family of PEL-structures of our case in 5.3–5 essentially in the same way as he did. The main theorems of the theory of the modulus-variety of PEL-structures of Shimura [12] tell:

(5.2.1) *There exists an algebraic number field $k(\Omega)$ of finite degree with the following two properties.*

(i) *Let Q be a* PEL-*structure of type Ω, and τ an automorphism of $\mathbf{C}$. Then Q^{τ} is of type Ω if and only if τ is the identity mapping on $k(\Omega)$.*

(ii) *The field $k(\Omega)$ contains* tr $(\Phi(a))$ *for every a in the center of the algebra L.*

Moreover the field $k(\Omega)$ is uniquely determined by the property (i).

(5.2.2) *There are an algebraic variety $V(\Omega)$ and an assignment $\mathfrak{v} = \mathfrak{v}_{\Omega}$ of exactly one point $\mathfrak{v}(Q)$ of $V(\Omega)$ to every* PEL-*structure Q of type Ω satisfying the following conditions.*

(i) *$V(\Omega)$ is defined over $k(\Omega)$, and is everywhere normal.*

(ii) *$\mathfrak{v}(Q) = \mathfrak{v}(Q')$ if and only if Q is isomorphic to Q'.*

(iii) *Let Q be a* PEL-*structure of type Ω, and τ an automorphism of $\mathbf{C}$ over $k(\Omega)$. Then $\mathfrak{v}(Q)^{\tau} = \mathfrak{v}(Q^{\tau})$.*

(iv) *$K(\Omega)(\mathfrak{v}(Q))$ is the field of moduli of Q,* i.e. *the subfield of $\mathbf{C}$ fixed by all such automorphisms τ of $\mathbf{C}$ as Q^{τ} is isomorphic to Q.*

(v) *There is a holomorphic mapping φ_{Ω} of $\mathcal{H}(T)$ onto $V(\Omega)$, which induces a biregular* *isomorphism of* $\Gamma(T, \mathfrak{R}/\mathfrak{W})\backslash\mathcal{H}(T)$ *onto* $V(\Omega)$, *and such that* $\mathfrak{v}(Q_z)=\varphi_\Omega(z)$ *for every member* Q_z *of* $\sum_\Omega$.

(vi) $V(\Omega)$ *is a Zariski open subset of a projective variety.*

Moreover it is seen in 3.1 of [10] III and 4.18 of [12] that

(5.2.3) *For any automorphism* τ *of* $\mathbf{C}$, *there is a* PEL-*type* Ω^τ *characterized, up to equivalence, by the condition:*

If Q *is a* PEL-*structure of type* Ω, *then* Q^τ *is of type* Ω^τ. *Moreover* $k(\Omega^\tau)=k(\Omega)^\tau$, *and* $\Omega^{(\sigma\tau)}=(\Omega^\sigma)^\tau$ *for any two automorphisms* σ *and* τ *of* $\mathbf{C}$;

(5.2.4) *For any automorphism* τ *of* $\mathbf{C}$, *there exists a biregular morphism* f_τ *of* $V(\Omega^\tau)$ *onto* $V(\Omega)^\tau$ *defined over* $k(\Omega^\tau)$ *such that, for every* PEL-*structure* Q *of type* Ω^τ, $f_\tau(\mathfrak{v}'(Q))=\mathfrak{v}(Q^{\tau^{-1}})^\tau$ *where* $\mathfrak{v}=\mathfrak{v}_\Omega$ *and* $\mathfrak{v}'=\mathfrak{v}_\Omega\tau$. *Moreover* $f_\sigma=f_\tau$ *for any automorphism* σ *of* $\mathbf{C}$ *such that* $\sigma=\tau$ *on* $k(\Omega)$.

5.3. Now we consider our case, $L=D$. Let ϱ be any positive involution of D, which coincides with the complex conjugation on K. Then the involution $x\to{}^tx^\varrho$ of $B=\mathbf{M}(m, D)$ is a positive one and coincides with δ on K. Take $h\in\mathbf{GL}(m, D)$ such that ${}^th^\varrho=h$ and $x^\delta=h^tx^\varrho h^{-1}$ for $x\in B$.

Let $\omega_1, \dots, \omega_g$ be as in Corollary 1 of Proposition 2 in 1.4 for j_0. For $\lambda=1, \dots, g$, we have

$$(5.3.1)\qquad \omega_\lambda(x^\delta)=J_\lambda{}^t\overline{\omega_\lambda(x)}J_\lambda;\quad J_\lambda=J_{r(\lambda),t(\lambda)}=\begin{bmatrix}1_{r(\lambda)} & 0\\ 0 & -1_{s(\lambda)}\end{bmatrix}.$$

Since $\omega_\lambda(x)\to\omega_\lambda({}^tx^\varrho)$ is a positive involution of $\mathbf{M}(mq, \mathbf{C})$, there is a positive definite hermitian matrix Y_λ such that

$$\omega_\lambda({}^tx^\varrho)=Y_\lambda{}^t\overline{\omega_\lambda(x)}\,Y_\lambda^{-1}$$

for all $x\in B_\mathbf{R}$. Take a positive definite hermitian matrix W_λ such that $Y_\lambda=W_\lambda^{-2}=W_\lambda^{-1}{}^t\overline{W}_\lambda^{-1}$, and put

$$(5.3.2)\qquad \psi_\lambda(x)=W_\lambda\,\omega_\lambda(x)W_\lambda^{-1}\quad(x\in B_\mathbf{R}).$$

Then we have

$$(5.3.3)\qquad \psi_\lambda({}^tx^\varrho)={}^t\overline{\psi_\lambda(x)}\quad(x\in B_\mathbf{R}).$$

Put $h_\lambda=\psi_\lambda(h)$. Then $h_\lambda={}^t\bar{h}_\lambda$. Since $h^\delta=h$, we see easily that $W_\lambda J_\lambda W_\lambda{}^t\bar{h}_\lambda=h_\lambda W_\lambda J_\lambda W_\lambda$, and so, especially, $W_\lambda^{-1}J_\lambda W_\lambda^{-1}h_\lambda$ is hermitian since so is W_λ. On the other hand, we see easily that $c_\lambda=W_\lambda^{-1}J_\lambda W_\lambda^{-1}h_\lambda$ is a scalar, if we compute $\psi_\lambda(x^\delta)=h_\lambda{}^t\overline{\psi_\lambda(x)}h_\lambda^{-1}$ through ω_λ for all $x\in B_\mathbf{R}$. Therefore c_λ is a real number. Let τ_λ be the isomorphism of K into $\mathbf{C}$ such that $\omega_\lambda(a)=\tau_\lambda(a)1_{mq}$ for $a\in K$, and let ζ be an element of $K^\times$ such that

$\zeta^{\varrho}=-\zeta$. Since ζ generates K over F, it must be totally purely imaginary. Changing ζ to its multiple of an element of $F^{\times}$ if necessary, we can take such ζ so that the real number $-\sqrt{-1}c_{\lambda}\tau_{\lambda}(\zeta)$ is poitive for all $\lambda=1, \dots, g$. Put $H=\zeta h$. Then H is an element of B such that

$$(5.3.4)\qquad {}^{t}H^{\varrho}=-H \quad \textit{and} \quad x^{\delta}=H^{t}x^{\varrho}H^{-1} \quad \textit{for every } x\in B.$$

Define a D-valued bilinear form on D^{m} by

$$(5.3.5)\qquad H(v, w)=vH^{t}w^{\varrho} \quad (v, w\in D^{m}).$$

Then, obviously, this is a non-degenerate ϱ-anti-hermitian form.

Fix a $\mathbf{Z}$-lattice $\mathfrak{M}$ of D^{m}, and take a positive rational number $\varkappa$ so that

$$(5.3.6)\qquad \mathrm{tr}\,(\varkappa H(\mathfrak{M}, \mathfrak{M}))=\mathbf{Z}.$$

Here tr denotes the reduced trace of D over $\mathbf{Q}$. Take a positive real number b_{λ} so that $b_{\lambda}^{2}=-\sqrt{-1}c_{\lambda}\tau_{\lambda}(\zeta)\varkappa$, and replace W_{λ} by the positive definite hermitian matrix $b_{\lambda}W_{\lambda}$ for $\lambda=1, \dots, g$. This does not change ψ_{λ}. We see easily that, for each $\lambda=1, \dots, g$,

$$(5.3.7)\qquad J_{\lambda}=W_{\lambda}(\sqrt{-1}\varkappa^{-1}H_{\lambda}^{-1})W_{\lambda};\ H_{\lambda}=\psi_{\lambda}(H).$$

Finally, let Ψ_{j} be as in 1.10 for each $j\in\mathcal{J}(j_{0})$, and put $\Phi_{j}=\Psi_{j}|_{D}$. Then it follows from Proposition 6 in 1.10 that Φ_{j} is equivalent to Φ_{j_0} for every $j\in\mathcal{J}(j_{0})$, and that $\Phi_{j_0}+\overline{\Phi}_{j_0}$ is equivalent to a rational representation of D.

Thus, for each finite number of elements $u_{1}, \dots, u_{t}$ of D^{m}, we have a PEL-type

$$\Omega=(D, \Phi_{j_0}, \varrho; \varkappa H, \mathfrak{M}; u_{1}, \dots, u_{t}).$$

Since D, Φ_{j_0}, and ϱ are common for all PEL-types that we consider hereafter, we write simply $\Omega=(\varkappa H, \mathfrak{M}; u_{1}, \dots, u_{t})$.

We fix H once for all. Then, for a $\mathbf{Z}$-lattice $\mathfrak{M}$, there is a positive rational number $\varkappa$ so that (5.3.6) holds. Once $\varkappa$ is so chosen, we can always find W_{λ}, $\lambda=1, \dots, g$, so that (5.3.2) and (5.3.7) hold.

5.4. Our next step is to construct a PEL-structure Q_{j} of type $\Omega=(\varkappa H, \mathfrak{M}; u_{1}, \dots, u_{t})$ for each $j\in\mathcal{J}(j_{0})$ if H, $\varkappa$, $\mathfrak{M}$ and $u_{1}, \dots, u_{t}$ are given.

Define an $\mathbf{R}$-valued $\mathbf{R}$-bilinear form E on $D_{\mathbf{R}}^{m}$ by

$$(5.4.1)\qquad E(v, w)=\mathrm{tr}(\varkappa H(v, w))=\mathrm{tr}\,(v\varkappa H^{t}w^{\varrho}) \quad (v, w\in D_{\mathbf{R}}^{m}).$$

Then E is non-degenerate and skew-symmetric.

PROPOSITION 21. *For each* $j \in \mathcal{J}(j_0)$, *the torus* $D^m_{\mathbf{R}}/\mathfrak{M}$ *with the complex structure determined by* j *in* 1.10 *defines an abelian variety* $\mathcal{A}_j$ *with the Riemannian form* $E(v, w)$, *and determines a* PEL-*structure* $Q_j = (\mathcal{A}_j, \mathcal{C}_j, \theta_j; p_1, \ldots, p_t)$ *of type* Ω.

Proof. To see that $D^m_{\mathbf{R}}/\mathfrak{M}$ with the complex structure defined by j becomes an abelian variety, it is sufficient to see that the bilinear form $E'(v, w) = E(v, wj)$ for $v, w \in D^m_{\mathbf{R}}$ is symmetric and positive definite. Take $x \in G^1_{\mathbf{R}}$ so that $j = xj_0x^{-1}$. On account of (5.3.4), we have $\varkappa H^tj^\varrho = \varkappa H^tj^\varrho H^{-1}H = \varkappa j^\delta H = -\varkappa jH = -\varkappa xj_0x^{-1}H$. Since $xx^\delta = 1$, we have $x^{-1} = x^\delta = H^tx^\varrho H^{-1}$. Hence $\varkappa H^tj^\varrho = x(-\varkappa j_0H)^tx^\varrho$. For each $\lambda = 1, \ldots, g$, it follows from the choice of ω_λ, (5.3.2) and (5.3.7) that $\psi_\lambda(-\varkappa j_0H) = -\varkappa W_\lambda(\sqrt{-1}\,J_\lambda)\,W_\lambda^{-1}H_\lambda = W_\lambda^2$. Take $y \in B_{\mathbf{R}} = B_1 \oplus \ldots \oplus B_g$ such that $\psi_\lambda(y) = W_\lambda$ for all $\lambda = 1, \ldots, g$. Since W_λ is hermitian, we see by (5.3.3) that ${}^ty^\varrho = y$. Thus we have $\varkappa H^tj^\varrho = xy^{2t}x^\varrho = (xy)^t(xy)^\varrho$. Therefore $E'(v, w) = \mathrm{tr}\,(v\varkappa H^tj^{\varrho t}w^\varrho) = \mathrm{tr}\,(v(xy)^t(xy)^{\varrho\, t}w^\varrho) = \mathrm{tr}\,((vxy)^t(wxy)^\varrho)$ for $v, w \in D^m_{\mathbf{R}}$. This shows that $E'(v, w)$ is symmetric and positive definite. Thus the torus $D^m_{\mathbf{R}}/M$ furnished with the complex structure defined by j becomes an abelian variety with the Riemannian form $E(v, w)$. Denote the abelian variety by $\mathcal{A}_j$, and the polarization of $\mathcal{A}_j$ determined by the Riemann form $E(v, w)$ by $\mathcal{C}_j$. Since the action of each element of D on $D^m_{\mathbf{R}}$ commutes with that of j, D is naturally embedded in $\mathrm{End}_{\mathbf{Q}}(\mathcal{A}_j)$. Denote this embedding by θ_j, and put $p_i \equiv u_i \bmod \mathfrak{M}$ for $i = 1, \ldots, t$. Since $\Phi_j = \Psi_j|_D$ is equivalent to $\Phi_{j_0} = \Psi_{j_0}|_D$, it is now clear that thus obtained $Q_j = (\mathcal{A}_j, \mathcal{C}_j, \theta_j; p_1, \ldots, p_t)$ is a PEL-structure of type Ω.

COROLLARY. *Let the notation be as in* Proposition 21, *and* α *any element of* $G_{\mathbf{Q}+}$. *Then* $\Omega' = (\nu(\alpha)^{-1}\varkappa H, \mathfrak{M}\alpha; u_1\alpha, \ldots, u_t\alpha)$ *is a* PEL-*type equivalent to* $\Omega = (\varkappa H, \mathfrak{M}; u_1, \ldots, u_t)$. *Moreover* α *induces an isomorphism of the* PEL-*structure* $Q_{\alpha j\alpha^{-1}}$ *of type* Ω *obtained above for* $\alpha j\alpha^{-1} \in \mathcal{J}(j_0)$ *to the* PEL-*structure* Q'_j *of type* Ω' *constructed for* j *as above.*

This is almost obvious, and the proof is omitted.

5.5. For given H, $\varkappa$, $\mathfrak{M}$ and $u_1, \ldots, u_t$, we have constructed a family $\{Q_j \mid j \in \mathcal{J}(j_0)\}$ of PEL-structures of type $\Omega = (\varkappa H, \mathfrak{M}; u_1, \ldots, u_t)$ in the previous paragraph. We say that a member Q_j of this family is *the* PEL-*structure of type* Ω *attached to* j.

The family $\{Q_j \mid j \in \mathcal{J}(j_0)\}$ is considered to be parametrized by the points of the hermitian symmetric space $\mathcal{H}_{j_0}$ through the correspondence between $\mathcal{J}(j_0)$ and $\mathcal{H}_{j_0}$ fixed in 1.7. To use the theory of the modulus-variety of PEL-structures of G. Shimura, we have to clarify the relation between Shimura's family $\sum_\Omega$ defined in [9] and [10] and ours.

Let Q_j be the PEL-structure of type Ω attached to $j \in \mathcal{J}(j_0)$. Take $x \in G^1_{\mathbf{R}}$ such that $j = xj_0x^{-1}$, and put $x_\lambda = \psi_\lambda(x)$ for $\lambda = 1, \ldots, g$. Here we use the same notation as in 5.3. Since $\psi_\lambda(j) = x_\lambda\psi_\lambda(j_0)x_\lambda^{-1} = x_\lambda W_\lambda{}'\omega_\lambda(j_0)\,W_\lambda^{-1}x_\lambda^{-1} = x_\lambda W_\lambda(\sqrt{-1}\,J_\lambda)\,W_\lambda^{-1}x_\lambda^{-1}$, we have

$$\sqrt{-1}\,J_\lambda{}^t(x_\lambda W_\lambda) = {}^t(x_\lambda W_\lambda)\,{}^t\psi_\lambda(j). \tag{5.5.1}$$

Since W_λ is hermitian, it follows from (5.3.2) that

(5.5.2) $$ {}^t(x_\lambda W_\lambda)\overline{W}_\lambda^{-1} = {}^t\omega_\lambda(x). $$

Put ${}^t\omega_\lambda(x) = \begin{bmatrix} U_\lambda & V_\lambda \\ X_\lambda & Y_\lambda \end{bmatrix}$. Since $xx^\delta = 1$, we see by (5.3.1) that

$$ {}^t\omega_\lambda(x) J_{r(\lambda), s(\lambda)} \overline{\omega_\lambda(x)} = J_{r(\lambda), s(\lambda)}, $$

that is, $$ U_\lambda {}^t\overline{U}_\lambda - V_\lambda {}^t\overline{V}_\lambda = 1_{r(\lambda)}; \quad Y_\lambda {}^t\overline{Y}_\lambda - X_\lambda {}^t\overline{X}_\lambda = 1_{s(\lambda)}; $$

$$ U_\lambda {}^t\overline{X}_\lambda - V_\lambda {}^t\overline{Y}_\lambda = 0; \quad X_\lambda {}^t\overline{U}_\lambda - Y_\lambda {}^t\overline{V}_\lambda = 0. $$

Therefore U_λ and Y_λ are invertible, and $U_\lambda^{-1} V_\lambda = {}^t\overline{(U_\lambda^{-1} X_\lambda)}$. Put $z_\lambda = U_\lambda^{-1} V_\lambda$. Then $1_{r(\lambda)} - z_\lambda {}^t\bar{z}_\lambda = U_\lambda^{-1}{}^t\bar{U}_\lambda^{-1}$ is positive hermitian. We have

$$ \omega_\lambda(x) = \begin{bmatrix} 1_{r(\lambda)} & z_\lambda \\ {}^t\bar{z}_\lambda & 1_{s(\lambda)} \end{bmatrix} \begin{bmatrix} {}^tU_\lambda & 0 \\ 0 & {}^tY_\lambda \end{bmatrix}. $$

On account of (1.7.1), this shows that the point $(z_1, \ldots, z_g)$ of the bounded symmetric domain $\mathcal{H}_{r(1),s(1)} \times \ldots \times \mathcal{H}_{r(g),s(g)}$ is the image of the point of $\mathcal{H}_{j_0}$ corresponding to $j = xj_0x^{-1} \in \mathcal{J}(j_0)$ under the mapping by which we defined the structure of the hermitian symmetric space $\mathcal{H}_{j_0}$ in 1.7. (Note the difference of the notation.)

Now the relations (5.3.3), (5.3.7), (5.5.1) and (5.5.2) make it possible to compare the PEL-structure Q_j attached above to an element j of $\mathcal{J}(j_0)$ with that which Shimura attached to a point $(z_1, \ldots, z_g)$ on the space $\mathcal{H}(\varkappa H) = \mathcal{H}_{r(1),s(1)} \times \ldots \mathcal{H}_{r(g),s(g)}$ in [9] and [10] (see 2.2–6 of [9], especially). The (possible) difference between these two occurs on the choice of the representations ψ_λ, $\lambda = 1, \ldots, g$, once $\varkappa H$, $\mathfrak{M}$ and $u_1, \ldots, u_t$ have been fixed so that (5.3.4) and (5.3.6) hold. But since, for each λ, the representation Shimura chose and our ψ_λ are equivalent, and satisfy (5.3.3), we see easily, on account of what we saw in the proof of Proposition 5 in 1.8, that

(5.5.3) *There exists such an isomorphism χ_{j_0} of $\mathcal{H}_{j_0}$ onto $\mathcal{H}(\varkappa H)$ as it maps the point on $\mathcal{H}_{j_0}$ corresponding to an element j of $\mathcal{J}(j_0)$ to the point on $\mathcal{H}(\varkappa H)$ to which the* PEL-*structure of type $\Omega = (\varkappa H, \mathfrak{M}; u_1, \ldots, u_t)$ attached by* Shimura *is isomorphic to the* PEL-*structure Q_j of type Ω attached to j in* Proposition 21.

5.6. PROPOSITION 22. *Let P be a commutative isolating subalgebra of B, and j the element of $\mathcal{J}(j_0)$ isolated by P. Let Q_j be the* PEL-*structure of type $(\varkappa H, \mathfrak{M}; u_1, \ldots, u_t)$ attached to j.*

For $a\in R_j(P)_{\mathbf{A}}^{\times}$, *let* τ *be an automorphism of* $\mathbf{C}$ *such that* $\tau=[a, R_j(P)]$ *on* $R_j(P)_{\mathrm{ab}}$. *Then* Q_j^{τ} *is isomorphic to the* PEL-*structure* Q_j' *of type*

$$(N_{R_j(P)/\mathbf{Q}}(\mathfrak{a})\varkappa H,\, \mathfrak{M}\eta_j(a)^{-1};\, u_1\eta_j(a)^{-1},\dots,u_t\eta_j(a)^{-1})$$

attached to j, *where* $\mathfrak{a}$ *is the ideal of* $R_j(P)$ *associated with* a.

Proof. Let $P=P_1\oplus\dots\oplus P_s$ where P_μ is a CM-field for $\mu=1, \dots, s$, and C the commutor of P in B. Then $C=C_1\oplus\dots\oplus C_s$ with a central simple algebra C_μ over P_μ for each $\mu=1, \dots, s$. Since the commutor of C in B is P, we see by Proposition 8 in 2.2 that $D\otimes_K C^{-1}= D\otimes_K C_1^{-1}\oplus\dots\oplus D\otimes_K C_s^{-1}$ where each $D\otimes_K C_\mu^{-1}$ is isomorphic to $M(n_\mu, P_\mu)$ with some integer n_μ such that $\sum_{\mu=1}^{s} n_\mu[P_\mu:\mathbf{Q}]=[D^m:\mathbf{Q}]$. On the other hand, C is also an isolating subalgebra of B, and isolates j, as was seen in 2.1. Therefore the representation Ψ_j' of $D\otimes_K C(j)^{-1}$ defined in 1.10 is considered as a representation of $D\otimes_K C^{-1}$. Moreover the embedding θ_j of D into $\mathrm{End}_{\mathbf{Q}}(\mathcal{A}_j)$ extends to an embedding of $D\otimes_K C^{-1}$ into $\mathrm{End}_{\mathbf{Q}}(\mathcal{A}_j)$ where $\mathcal{A}_j$ is the abelian variety of Q_j. All of these shows that Q_j is of type $(D\otimes_K C^{-1}$, Ψ_j', $\mathfrak{M}$, $\varkappa H$; $u_1, \dots, u_t)$ in the sense of 4.1 of Shimura [14] I. Hence the proposition follows at once from 4.3 of [14] I.

5.7. Let $\mathcal{G}_{j_0}^1$ be as in 3.9, where we saw $\mathcal{G}_{j_0+}=K^{\#}G_{\mathbf{Q}+}\mathcal{G}_{j_0}^1$. Define a homomorphism μ of $G_{\mathbf{Q}+}\mathcal{G}_{j_0}^1$ to $F_+^{\times}$, the multiplicative group of all the totally positive elements of F, as follows:

It follows from the definition of $\mathcal{G}_{j_0}^1$ that, for any $x\in\mathcal{G}_{j_0}^1$, the ideal $\mathfrak{x}$ of F associated with the idele $\nu(x)=xx^{\delta}$ of F is actually an ideal of $\mathbf{Q}$. Therefore there is a unique positive rational number ξ such that $\mathfrak{x}=(\xi)$, the principal ideal generated by ξ. We define $\mu(x)=\xi$. Let $y=\alpha x$ be an element of $G_{\mathbf{Q}+}\mathcal{G}_{j_0}^1$ with $\alpha\in G_{\mathbf{Q}+}$ and $x\in\mathcal{G}_{j_0}^1$. We define $\mu(y)=\nu(\alpha)\mu(x)$. Then it is easy to see that μ is a well defined homomorphism of $G_{\mathbf{Q}+}\mathcal{G}_{j_0}^1$ to $F_+^{\times}$, and that

(5.7.1) *For any* $S\in\mathfrak{Z}_{j_0}^0$, *we have* $\mu(x)=1$ *for every* $x\in S\cap\mathcal{G}_{j_0}^1$.

Let P be a commutative isolating subalgebra of B, j the element of $\mathcal{J}(j_0)$ isolated by P, and $R_j(P)$ and η_j as in 2.3. Then Proposition 9 implies at once that

(5.7.2) *For* $a\in R_j(P)_{\mathbf{A}}^{\times}$, *let* $\mathfrak{a}$ *be the ideal of* $R_j(P)$ *associated with the idele* a. *Then* $\mu(\eta_j(a))= N_{R_j(P)/\mathbf{Q}}(\mathfrak{a})$.

5.8. PROPOSITION 23. *Let* $\Omega=(\varkappa H$, $\mathfrak{M}$; $u_1, \dots, u_t)$ *be a* PEL-*type, and* $S= S(\mathfrak{M}; u_1, \dots, u_t)$ *as in* 4.7. *Then the field* $k_{\pi(S)}$ *defined in* 3.7 *coincides with the field* $k(\Omega)$ *of* (5.2.1) *for the* PEL-*type* Ω. *Moreover the* PEL-*type* $\Omega^{\sigma(\pi(x))}$ *is equivalent to* $(\mu(x^{-1})\varkappa H$, $\mathfrak{M}x$; $u_1x, \dots, u_tx)$ *for* $x\in G_{\mathbf{Q}+}\mathcal{G}_{j_0}^1$.

Proof. Take a commutative isolating subalgebra P of B so that $R_j(P)$ and $k(\Omega)k_{\pi(S)}$

are linearly disjoint over K'_{j_0} where j is the element of $\mathcal{J}(j_0)$ isolated by P. By (ii) of (5.2.1), we see that $k(\Omega)$ contains K'_{j_0}. Let x be an element of $G_{\mathbf{Q}+}\mathcal{G}^1_{j_0}$. Since $R_j(P)$ is linearly disjoint with $k_{\pi(S)}$ over K'_{j_0}, there exists an automorphism τ of $\mathbf{C}$ over $R_j(P)$ such that $\tau=\sigma(\pi(x))$ on $k_{\pi(S)}$. Take $a\in R_j(P)^\times_{\mathbf{A}}$ so that $\tau=[a, R_j(P)]$ on $R_j(P)_{\mathrm{ab}}$. Then $\sigma(\pi(x))=\tau=[a, R_j(P)]=\sigma(\pi(\eta_j(a))^{-1})$ on $k_{\pi(S)}$. Therefore it follows from Proposition 18 in 3.7 that $\pi(\eta_j(a)^{-1})=\pi(y')\pi(x)\alpha$ for some $y'\in S$ and $\alpha\in A_{j_0}$. Then (iii) of Proposition 16 in 3.5 shows that there is an element γ' in $G_{\mathbf{Q}+}$ such that $\alpha=\pi(\gamma')$. Hence we have $\pi(\eta_j(a)^{-1}\gamma'^{-1}x^{-1})=\pi(y')\in\pi(S)$. Since $K^\#G_{\infty+}S=K^\times S$, there are $b\in K^\times$ and $y\in S$ such that $\eta_j(a)^{-1}=yxb\gamma'$. Put $\gamma=b\gamma'\in G_{\mathbf{Q}+}$. Then $\eta_j(a)^{-1}=yx\gamma$. Since $\eta_j(a)^{-1}\in\mathcal{G}^1_{j_0}$, we have $y\in G_{\mathbf{Q}+}\mathcal{G}^1_{j_0}$. Let Q_j be the PEL-structure of type Ω attached to j. Then Proposition 22 tells us that Q_j^τ is the PEL-structure of type $\Omega'=(\mu(\eta_j(a))\varkappa H,\ \mathfrak{M}\eta_j(a)^{-1};\ u_1\eta_j(a)^{-1}, ..., u_t\eta_j(a)^{-1})$ attached to j. (See (5.7.2).) Then 4.1 of Shimura [11] assures that Q^τ is also of type Ω' for every PEL-structure Q of type Ω. This means that Ω^τ is quivalent to Ω'. (See (5.2.3).) Since $\Omega'=(\mu(yx\gamma)^{-1}\varkappa H,\ \mathfrak{M}yx\gamma;\ u_1yx\gamma, ..., u_tyx\gamma)$ with $y\in S\cap G_{\mathbf{Q}+}\mathcal{G}^1_{j_0}$, $\gamma\in G_{\mathbf{Q}+}$ and $\mu(\gamma)=\nu(\gamma)=\gamma\gamma^\delta=\gamma H{}^t\gamma^\varrho H^{-1}$, we see that Ω' is equivalent to the PEL-type $(\mu(x)^{-1}\varkappa H,\ \mathfrak{M}x;\ u_1x, ..., u_tx)$. (See (5.7.1).) Now take $x=1$. Then Ω^τ is equivalent to Ω for every automorphism τ of $\mathbf{C}$ over $R_j(P)$ that is the identity mapping on $k_{\pi(S)}$. Therefore, on account of (i) of (5.2.1), we have $k(\Omega)\subset k_{\pi(S)}$ since $R_j(P)$ is linearly disjoint with $k(\Omega)k_{\pi(S)}$ over K'_{j_0}. Thus for $x\in G_{\mathbf{Q}+}\mathcal{G}^1_{j_0}$, $\Omega^{\sigma(\pi(x))}$ is meaningful and equivalent to $(\mu(x)^{-1}\varkappa H,\ \mathfrak{M}x;\ u_1x, ..., u_tx)$. Suppose now that $\sigma(\pi(x))$ is trivial on $k(\Omega)$ for $x\in G_{\mathbf{Q}+}\mathcal{G}^1_{j_0}$. Then since $\Omega^{\sigma(\pi(x))}$ is equivalent to Ω, there is an element γ of $\mathbf{M}(m, D)$ such that $\gamma(\varkappa H){}^t\gamma^\varrho=\mu(x)^{-1}\varkappa H$, $\mathfrak{M}x\gamma=\mathfrak{M}$ and $u_ix\gamma\equiv u_i \bmod \mathfrak{M}$ for $i=1, ..., t$. This means that $\gamma\in G_{\mathbf{Q}+}$ and $x\gamma\in S=S(\mathfrak{M};\ u_1, ..., u_t)$. (See (5.3.4).) Hence $\sigma(\pi(x))=\sigma(\pi(x\gamma))$ is the identity mapping on $k_{\pi(S)}$. As we saw at the beginning of 3.9, $\sigma\circ\pi$ maps $G_{\mathbf{Q}+}\mathcal{G}^1_{j_0}$ onto $\mathrm{Gal}\,(\mathfrak{K}'_{j_0}/K'_{j_0})$. Since $k(\Omega)$ contains K'_{j_0} and is contained in $k_{\pi(S)}$, all automorphisms of $k_{\pi(S)}$ over $k(\Omega)$ are obtained by $\sigma(\pi(x))$ for $x\in G_{\mathbf{Q}+}\mathcal{G}^1_{j_0}$. Therefore we have $k_{\pi(S)}=k(\Omega)$. This completes the proof.

5.9. PROPOSITION 24. *Let* $\Omega=(\varkappa H,\ \mathfrak{M};\ u_1, ..., u_t)$ *be a* PEL-*type, and* $V(\Omega)$ *and* $\mathfrak{v}$ *as in* (5.2.2). *Let* P *be a commutative isolating subalgebra of* B, j *the element of* $\mathcal{J}(j_0)$ *isolated by* P, *and* Q_j *the* PEL-*structure of type* Ω *attached to* j. *Then the point* $\mathfrak{v}(Q_j)$ *of* $V(\Omega)$ *is rational over* $R_j(P)_{\mathrm{ab}}$.

Proof. It follows from Proposition 23 that the field $k(\Omega)$ is an abelian extension of K'_{j_0}, and hence, is contained in $R_j(P)_{\mathrm{ab}}$. Now let τ be an automorphism of $\mathbf{C}$ over $R_j(P)_{\mathrm{ab}}$. Then taking $a=1\in R_j(P)^\times_{\mathbf{A}}$, we see that Q_j^τ is isomorphic to Q_j. (See Proposition 22.) Therefore, on account of (ii) and (iii) of (5.2.2), we have $\mathfrak{v}(Q_j)=\mathfrak{v}(Q_j^\tau)=\mathfrak{v}(Q_j)^\tau$. This proves the proposition.

5.10. Now we restrict ourselves to the case that $S=S(\mathfrak{M};\, u_1, \ldots, u_t)$ is small enough to satisfy not only Proposition 20 but also (3.10.3). As was mentioned just after (3.10.3), we may also assume that xSx^{-1} satisfies (3.10.3) for every $x \in \mathcal{G}_{j_0+}$.

Put $S^1=(S \cap K^{\#})(S \cap \mathcal{G}^1_{j_0})$. Define a $\mathbf{Z}$-lattice $\mathfrak{N}$ of D^m and a group $\Gamma(\varkappa H, \mathfrak{N}/\mathfrak{M})$ as in 5.2. Then we see easily that $\Gamma(\varkappa H, \mathfrak{N}/M)$ coincides with $S \cap G^u_{\mathbf{Q}}$ given in (3.10.3). Therefore we have $\pi(\Gamma(\varkappa H, \mathfrak{N}/\mathfrak{M}))=\Gamma_{\pi(S^1)}$. Note that $\Gamma(\varkappa H, \mathfrak{N}/\mathfrak{M})$ coincides with $\pi(\Gamma(\varkappa H, \mathfrak{N}/\mathfrak{M}))$ as groups of transformations on $\mathcal{H}(\varkappa H)$, or on $\mathcal{H}_{j_0}$.

PROPOSITION 25. *Let $\Omega=(\varkappa H, \mathfrak{M};\, u_1, \ldots, u_t)$ and $\Omega'=(\mu(x)\varkappa H, \mathfrak{M}x^{-1};\, u_1x^{-1}, \ldots, u_tx^{-1})$ be two* PEL-*types with $x \in G_{\mathbf{Q}+}\mathcal{G}^1_{j_0}$, and $(V(\Omega), \mathfrak{v}, \varphi_\Omega)$ and $(V(\Omega'), \mathfrak{v}', \varphi_{\Omega'})$ as in* (5.2.2) *for Ω and Ω' respectively. Suppose that $\pi(S^1)=\pi(xS^1x^{-1})$ where $S^1=(S \cap K^{\#})(S \cap \mathcal{G}^1_{j_0})$ with $S=S(\mathfrak{M};\, u_1, \ldots, u_t)$. Then there is a biregular morphism $E_{\Omega\Omega'}$ of $V(\Omega')$ onto $V(\Omega)$, which is rational over $k_{\pi(S^1)}$, such that $\varphi_\Omega=E_{\Omega\Omega'}\circ\varphi_{\Omega'}$.*

Proof. As was seen above, both $\Gamma(\varkappa H, \mathfrak{N}/\mathfrak{M})$ and $\Gamma(\mu(x)\varkappa H, \mathfrak{N}x^{-1}/\mathfrak{M}x^{-1})$ coincide with $\Gamma_{\pi(S^1)}=\Gamma_{\pi(xS^1x^{-1})}$ as transformation groups on $\mathcal{H}(\varkappa H)=\mathcal{H}(\mu(x)\varkappa H)$. Therefore both $(V(\Omega), \varphi_\Omega)$ and $(V(\Omega'), \varphi_{\Omega'})$ are models of the quotient space $\Gamma(\varkappa H, \mathfrak{N}/\mathfrak{M})\backslash\mathcal{H}(\varkappa H)$, and $E_{\Omega\Omega'}$ defined by $\varphi_\Omega=E_{\Omega\Omega'}\circ\varphi_{\Omega'}$ is a biregular morphism of $V(\Omega')$ onto $V(\Omega)$. Since there are densely many isolated fixed points on $\mathcal{H}(\varkappa H)$, it follows from (v) of (5.2.2), (5.5.3) and Proposition 24 that $E_{\Omega\Omega'}$ is defined over the algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$, and so, over a finite normal algebraic extension k of $k_{\pi(S^1)}$. Note that $k_{\pi(S^1)}=k_{\pi(xS^1x^{-1})}\supset k(\Omega)=k_{\pi(S)}=k_{\pi(xSx^{-1})}=k(\Omega')$. Let P be a commutative isolating subalgebra of B such that $R_j(P)$ is linearly disjoint with k over K'_{j_0} where j is the element of $\mathcal{J}(j_0)$ isolated by P, and Q_j (resp. Q'_j) the PEL-structure of type Ω (resp. Ω') attached to j. Let τ be an automorphism of $\mathbf{C}$ over $k_{\pi(S^1)}R_j(P)$, and a an element of $R_j(P)^\times_{\mathbf{A}}$ such that $\tau=[a, R_j(P)]$ on $R_j(P)_{\mathrm{ab}}$. Since $\sigma(\eta^*_j(a)^{-1})=\tau=$ the identity mapping on $k_{\pi(S^1)}$, it follows from Corollary 2 of Proposition 19 in 3.9 that $\eta_j(a)^{-1}=y\alpha$ with $y \in S \cap \mathcal{G}^1_{j_0}$ and $\alpha \in G_{\mathbf{Q}+}$. Since $\pi(y) \in \pi(S \cap \mathcal{G}^1_{j_0})=\pi(S^1)=\pi(xS^1x^{-1})=\pi(xSx^{-1} \cap \mathcal{G}_{j_0})$, there is an element $b \in K^{\#}$ such that $b^{-1}y \in xSx^{-1} \cap \mathcal{G}^1_{j_0}$. On account of Corollary 1 of Proposition 19, we may assume that $b \in K^\times$. Then Proposition 22 in 5.6 shows that Q^τ_j (resp. Q'^τ_j) is isomorphic to the PEL-structure of type $(\nu(\alpha)^{-1}\varkappa H,$ $\mathfrak{M}\alpha;\, u_1\alpha, \ldots, u_t\alpha)$ (resp. $(\nu(b\alpha)^{-1}\mu(x)\varkappa H, \mathfrak{M}x^{-1}b\alpha;\, u_1x^{-1}b\alpha, \ldots, u_tx^{-1}b\alpha)$) attached to j. (Also see (5.7.1).) Therefore it follows from the corollary of Proposition 21 of 5.4 that Q^τ_j (resp. Q'^τ_j) is isomorphic to the PEL-structure $Q_{\alpha j\alpha^{-1}}$ (resp. $Q'_{\alpha j\alpha^{-1}}$) of type Ω (resp. Ω') attached to $\alpha j\alpha^{-1}$. (Since $b \in K^\times$, we have $\alpha j\alpha^{-1}=(b\alpha)j(b\alpha)^{-1}$.) Hence we have $\mathfrak{v}(Q_j)^\tau=\mathfrak{v}(Q^\tau_j)=\mathfrak{v}(Q_{\alpha j\alpha^{-1}})$ and $\mathfrak{v}'(Q'_j)^\tau=\mathfrak{v}'(Q'^\tau_j)=\mathfrak{v}'(Q'_{\alpha j\alpha^{-1}})$, on account of (iii) of (5.2.2). Then (v) of (5.2.2) and (5.5.3) imply that $E_{\Omega\Omega'}(\mathfrak{v}'(Q'_{\alpha j\alpha^{-1}}))=\mathfrak{v}(Q_{\alpha j\alpha^{-1}})=\mathfrak{v}(Q_j)^\tau=E^\tau_{\Omega\Omega'}(\mathfrak{v}'(Q'_j)^\tau)=E^\tau_{\Omega\Omega'}(\mathfrak{v}'(Q'_{\alpha j\alpha^{-1}}))$. As we mentioned in 2.5, this is true for every point of a dense subset

$\{\mathfrak{v}'(Q'_{\alpha\beta j\beta^{-1}\alpha^{-1}})|\beta\in G_{\mathbf{Q}+}\}$ of $V(\Omega')$ since $\beta j\beta^{-1}$ gives the same field $R_j(P)=R_{\beta j\beta^{-1}}(\beta P\beta^{-1})$ for every $\beta\in G_{\mathbf{Q}+}$. Thus we have $E_{\Omega\Omega'}=E^{\tau}_{\Omega\Omega'}$. Since τ can move all the automorphisms of k over $k_{\pi(S^1)}$, this shows that $E_{\Omega\Omega'}$ is actually rational over $k_{\pi(S^1)}$. The proof is done.

5.11. Let $\Omega=(\varkappa H, \mathfrak{M}; u_1, \dots, u_t)$ be a PEL-type, and define a PEL-type $\Omega(x)=(\mu(x)\varkappa H, \mathfrak{M}x^{-1}; u_1x^{-1}, \dots, u_tx^{-1})$ for $x\in G_{\mathbf{Q}+}\mathcal{G}^1_{j_0}$. Put $S=S(\mathfrak{M}; u_1, \dots, u_t)$ and $S^1=(S\cap K^{\#})(S\cap\mathcal{G}_{j_0})$. We assume that S is sufficiently small, as in the previous paragraph.

Now let x be an element of $G_{\mathbf{Q}+}\mathcal{G}^1_{j_0}$ such that $\pi(xS^1x^{-1})=\pi(S^1)$. Then Proposition 25 gives us a morphism $E=E_{\Omega\Omega(x)}$ of $V(\Omega(x))$ onto $V(\Omega)$. Let w be an arbitrary element of $G_{\mathbf{Q}+}\mathcal{G}^1_{j_0}$. Then we have a morphism $E'=E_{\Omega(w^{-1})\Omega(w^{-1}x)}$ of $V(\Omega(w^{-1}x))$ onto $V(\Omega(w^{-1}))$, on account of Proposition 25. On the other hand, it follows from Proposition 23 that $\Omega^{\sigma(\pi(w))}$ (resp. $\Omega(x)^{\sigma(\pi(w))}$) is equivalent to $\Omega(w^{-1})$ (resp. $\Omega(w^{-1}x)$). Therefore we have morphisms f: $V(\Omega(w^{-1}))\to V(\Omega)^{\sigma(\pi(w))}$ and f': $V(\Omega(w^{-1}x))\to V(\Omega(x))^{\sigma(\pi(w))}$ of (5.2.4).

PROPOSITION 26. *Let the notation and the assumptions be as above. Then* $E^{\sigma(\pi(w))}\circ f'=f\circ E'$.

Proof. Take a commutative isolating subalgebra P of B so that $R_j(P)$ is linearly disjoint with $k_{\pi(S^1)}$ over K'_{j_0} where j is the element of $\mathcal{J}(j_0)$ isolated by P, and take an element $a\in R_j(P)^{\times}_{\mathbf{A}}$ so that $[a, R_j(P)]=\sigma(\pi(w^{-1}))$ on $k_{\pi(S^1)}$. Note that $k(\Omega)=k(\Omega(x))=k(\Omega(w^{-1}))=k_{\pi(S)}\subset k_{\pi(S^1)}$. Let τ be an automorphism of $\mathbf{C}$ such that $\tau=[a, R_j(P)]$ on $R_j(P)_{\mathbf{ab}}$. Since $\sigma(\eta^*_j(a)^{-1})=\sigma(\pi(w)^{-1})$ on $k_{\pi(S^1)}$, there are $s\in S\cap\mathcal{G}^1_{j_0}$ and $\alpha\in G_{\mathbf{Q}+}$ such that $\eta_j(a)^{-1}=w^{-1}s\alpha$. (See Corollary 2 of Proposition 19 in 3.9.) Since $\pi(S^1)=\pi(xS^1x^{-1})$, we can find an element $b\in K^{\times}$ such that $sb^{-1}\in xSx^{-1}\cap\mathcal{G}^1_{j_0}$. (See Corollary 1 of Proposition 19.) Then on account of (5.7.1), we have $\Omega(\eta_j(a)w^{-1})=\Omega(\alpha^{-1})$ and $\Omega(\eta_j(a)w^{-1}x)=\Omega(b^{-1}\alpha^{-1}x)$. Let Q_j (resp. $\mathcal{R}_j$) be the PEL-structure of type $\Omega(w^{-1})$ (resp. $\Omega(w^{-1}x)$) attached to j. Then it follows from Proposition 22 in 5.6 that Q^{τ}_j (resp. $\mathcal{R}^{\tau}_j$) is isomorphic to the PEL-structure of type $\Omega(\alpha^{-1})$ (resp. $\Omega(b^{-1}\alpha^{-1}x)$) attached to j. Let $Q'_{\alpha j\alpha^{-1}}$ (resp. $\mathcal{R}'_{\alpha j\alpha^{-1}}$) be the PEL-structure of type Ω (resp. $\Omega(x)$) attached to $\alpha j\alpha^{-1}$. Then the corollary of Proposition 21 in 5.4 shows that Q^{τ}_j (resp. $\mathcal{R}^{\tau}_j$) is isomorphic to $Q'_{\alpha j\alpha^{-1}}$ (resp. $\mathcal{R}'_{\alpha j\alpha^{-1}}$). Hence we have $E(\mathfrak{v}'(\mathcal{R}'_{\alpha j\alpha^{-1}}))=\mathfrak{v}(Q'_{\alpha j\alpha^{-1}})$ and $E'(\mathfrak{v}'_1(\mathcal{R}_j))=\mathfrak{v}_1(Q_j)$. Since $\tau=\sigma(\pi(w))^{-1}$ on $k(\Omega(w^{-1}))=k(\Omega(w^{-1}x))$, it follows from (5.2.4) that $f(\mathfrak{v}_1(Q_j))=\mathfrak{v}(Q^{\tau}_j)^{\tau^{-1}}=\mathfrak{v}(Q'_{\alpha j\alpha^{-1}})^{\tau^{-1}}$ and $f'(\mathfrak{v}'_1(\mathcal{R}_j))=\mathfrak{v}'(\mathcal{R}^{\tau}_j)^{\tau^{-1}}=\mathfrak{v}'(\mathcal{R}'_{\alpha j\alpha^{-1}})^{\tau^{-1}}$. Thus we have $(f\circ E')(\mathfrak{v}'_1(\mathcal{R}_j))=f(\mathfrak{v}_1(Q_j))=\mathfrak{v}(Q'_{\alpha j\alpha^{-1}})^{\tau^{-1}}=E^{\tau^{-1}}(\mathfrak{v}'(\mathcal{R}'_{\alpha j\alpha})^{\tau^{-1}})=(E^{\tau^{-1}}\circ f')(\mathfrak{v}'_1(\mathcal{R}_j))$. As was mentioned in 2.5, this is true for $\beta j\beta^{-1}$ for every $\beta\in G_{\mathbf{Q}+}$, and $\{\mathfrak{v}'_j(\mathcal{R}_{\beta j\beta^{-1}})|\beta\in G_{\mathbf{Q}+}\}$ is dense in $V(\Omega(w^{-1}x))$. Therefore we have the desired result.

5.12. We are now ready to show

PROPOSITION 27. *Let $S=S(\mathfrak{M}; u_1, ..., u_t)$ be sufficiently small in the sense of the beginning of 5.10. Put $S^1=(S\cap K^{\#})(S\cap \mathcal{G}^1_{j_0})\in \mathfrak{Z}^0_{j_0}$. Then there exists a canonical subsystem for $\mathfrak{W}^0(\pi(S^1))$.*

Proof. Let W be a member of $\mathfrak{W}^0(\pi(S^1))$. Since $\pi(G_{\mathbf{Q}+}\mathcal{G}^1_{j_0})=\mathfrak{A}^0_{j_0}$, there is an element $w\in G_{\mathbf{Q}+}\mathcal{G}^1_{j_0}$ such that $W=\pi(wS^1w^{-1})$. Put $T=wSw^{-1}=S(\mathfrak{M}w^{-1};\ u_1w^{-1}, ..., u_tw^{-1})$ and $T^1=(T\cap K^{\#})(T\cap \mathcal{G}^1_{j_0})$. Then we have $W=\pi(T^1)$ by (3.10.1). For each $W\in\mathfrak{W}^0(\pi(S^1))$, fix $w\in G_{\mathbf{Q}+}\mathcal{G}^1_{j_0}$ and $T=wSw^{-1}$, and put $V_W=V(\Omega(w))$ and $\varphi_W=\varphi_{\Omega(w)}\circ\chi_{j_0}$ where $\Omega(w)$ is the PEL-type $(\mu(w)\varkappa H,\ \mathfrak{M}w^{-1};\ u_1w^{-1}, ..., u_tw^{-1})$, $(V(\Omega(w)),\ \varphi_{\Omega(w)})$ is as in (5.2.2) for $\Omega(w)$ and χ_{j_0} is as in (5.5.3). It follows from the argument given at the beginning of 5.10 that (V_W, φ_W) is a model of $\Gamma_W\backslash\mathcal{H}_{j_0}$. Since $T^1\subset T$, we have $k_W=k_{\pi(T^1)}\supset k_{\pi(T)}=k(\Omega(w))$. Therefore V_W is surely defined over k_W.

Now let us construct $J_{XW}(\pi(u))$ for $\pi(u)\in\mathfrak{A}^0_{j_0}=\pi(G_{\mathbf{Q}+}\mathcal{G}^1_{j_0})$ with $u\in G_{\mathbf{Q}+}\mathcal{G}^1_{j_0}$. Put $X=\pi(u)W\pi(u)^{-1}$, and let x be the element of $G_{\mathbf{Q}+}\mathcal{G}^1_{j_0}$ selected to define V_X and φ_X. Put $U=xSx^{-1}$ and $U^1=(U\cap K^{\#})(U\cap \mathcal{G}^1_{j_0})$. Then $X=\pi(U^1)$. Since $\Omega(x)^{\sigma(\pi(u))}$ is equivalent to $\Omega(u^{-1}x)$, we have a morphism f_u of $V(\Omega(u^{-1}))$ onto $V(\Omega(x))^{\sigma(\pi(u))}=V_X^{\sigma(\pi(u))}$ by (5.2.4). On the other hand, if we put $R=u^{-1}xSx^{-1}u$ and $R^1=(R\cap K^{\#})(R\cap \mathcal{G}^1_{j_0})$, we have $\pi(R^1)=\pi(u)^{-1}\pi(U^1)\pi(u)=\pi(u)^{-1}X\pi(u)=W=\pi(wS^1w^{-1})=\pi(wx^{-1}u(u^{-1}xS^1x^{-1}u)u^{-1}xw^{-1})=\pi(wx^{-1}\times uR^1u^{-1}xw^{-1})$. Therefore Proposition 25 applied to $\Omega(u^{-1}x)$ and $wx^{-1}u$ in place of Ω and x gives us a morphism $E_{\Omega(u^{-1}x)\Omega(w)}$ of $V(\Omega(w))=V_W$ onto $V(\Omega(u^{-1}x))$. Put

$$J_{XW}(\pi(u))=f_u\circ E_{\Omega(u^{-1}x)\Omega(w)}.$$

To see that this is well defined, let us replace u with bcu where $b\in K^\times$ and $c\in K^\times_\infty$ on account of Corollary 1 of Proposition 19 in 3.9. Since c has no effect at all, it is enough to show that

$$f_u\circ E_{\Omega(u^{-1}x)\Omega(w)}=f_{bu}\circ E_{\Omega(b^{-1}u^{-1}x)\Omega(w)}$$

for $b\in K^\times$. But this follows easily from (5.2.4), the corollary of Proposition 21 in 5.4 and Proposition 25. Since $k_{\pi(T^1)}=k_W$ contains $k(\Omega(u^{-1}x))=k_{\pi(R)}=k_{\pi(S)}=k_{\pi(T)}$, both f_u and $E_{\Omega(u^{-1}x)\Omega(w)}$ are defined over k_W, and so is $J_{XW}(\pi(u))$.

We have to show that the properties (IIIa, b, c) are satisfied. Suppose that $\pi(u)$ belongs to W. Then we may assume that $u\in T^1$, and have $x=w$. Therefore $\Omega(u^{-1}x)=\Omega(w)$. (See (5.7.1).) Moreover since $\sigma(\pi(u))$ is trivial on $k(\Omega(w))$, f_u is the identity mapping. This shows (IIIa). Let $\pi(v)$ be another element of $\mathfrak{A}^0_{j_0}$ with $v\in G_{\mathbf{Q}+}\mathcal{G}^1_{j_0}$. Put $Y=\pi(v)X\pi(v)^{-1}$, and let y be the element of $G_{\mathbf{Q}+}\mathcal{G}^1_{j_0}$ selected to define V_Y and φ_Y. We have $Y=\pi(yS^1y^{-1})$. Then

$$J_{YX}(\pi(v))^{\sigma(\pi(u))}\circ J_{XW}(\pi(u))=f_v^{\sigma(\pi(u))}\circ E^{\sigma(\pi(u))}_{\Omega(v^{-1}y)\Omega(x)}\circ f_u\circ E_{\Omega(u^{-1}x)\Omega(w)}.$$

Since $\pi(v^{-1}yS^1y^{-1}v)=\pi(xy^{-1}v(v^{-1}yS^1y^{-1}v)v^{-1}yx^{-1})$ as is easily seen, we have

$$E^{\sigma(\pi(u))}_{\Omega(v^{-1}y)\Omega(x)}\circ f_u = f'_u\circ E_{\Omega(u^{-1}v^{-1}y)\Omega(u^{-1}x)}$$

by Proposition 26 applied to $\Omega(v^{-1}y)$, $xy^{-1}v$ and u in place of Ω, x and w respectively. Now let Q be a PEL-structure of type $\Omega(u^{-1}v^{-1}y)$, and τ and ω automorphisms of $\mathbf{C}$ such that $\tau=\sigma(\pi(u))$ and $\omega=\sigma(\pi(v))$ on k_W. Then we have $f'_u(\mathfrak{v}(Q))=\mathfrak{v}'(Q^{\tau^{-1}})^\tau$. Therefore $(f_v^{\sigma(\pi(u))}\circ f'_u)(\mathfrak{v}(Q))=(f_v(\mathfrak{v}'(Q^{\tau^{-1}})))^\tau=(\mathfrak{v}''(Q^{\tau^{-1}\omega^{-1}})^\omega)^\tau=f_{vu}(\mathfrak{v}(Q))$. Hence

$$f_v^{\sigma(\pi(u))}\circ f'_u=f_{vu}.$$

The formula

$$E_{\Omega(u^{-1}v^{-1}y)\Omega(u^{-1}x)}\circ E_{\Omega(u^{-1}x)\Omega(w)}=E_{\Omega(u^{-1}v^{-1}y)\Omega(w)}$$

is easy to see. By these formulae, we see easily that

$$J_{YX}(\pi(v))^{\sigma(\pi(u))}\circ J_{XW}(\pi(u))=J_{YW}(\pi(vu))$$

for $\pi(u)$, $\pi(v)\in\mathfrak{A}^0_{j_0}=\pi(G_{\mathbf{Q}+}\,G^1_{j_0})$. This shows (IIIb). Suppose now that $u=\alpha\in G_{\mathbf{Q}+}$. Let Q_j (resp. Q'_j) be the PEL-structure of type $\Omega(w)$ (resp. $\Omega(\alpha^{-1}x)$) attached to j. Then $E_{\Omega(\alpha^{-1}x)\Omega(w)}(\mathfrak{v}(Q_j))=\mathfrak{v}'(Q'_j)$. Since Q'_j is isomorphic to the PEL-structure $Q''_{\alpha j\alpha^{-1}}$ of type $\Omega(x)$ attached to $\alpha j\alpha^{-1}$ (see the corollary of Proposition 21), $f_\alpha(\mathfrak{v}'(Q'_j))=\mathfrak{v}(Q''_{\alpha j\alpha^{-1}})$. Then combining this and (v) of (5.2.2) with (5.5.3), we see at once that

$$J_{XW}(\pi(\alpha))\circ\varphi_W=\varphi_X\circ\pi(\alpha).$$

Here the last $\pi(\alpha)\in A^0_+$ is the transformation on $\mathcal{H}_{j_0}$ corresponding to the mapping $j\to\alpha j\alpha^{-1}$ of $\mathcal{J}(j_0)$ onto itself. (See 1.9.) Thus we have (IIIc).

Finally, let P be a commutative isolating subalgebra of B, j the element of $\mathcal{J}(j_0)$ isolated by P, and Q'_j the PEL-structure of type $\Omega(w)$ attached to j. For $a\in R_j(P)^\times_{\mathbf{A}}$, let $u=\eta_j(a)^{-1}$. Let Q'_j be the PEL-structure of type $\Omega(u^{-1}x)$ attached to j. Then $E_{\Omega(u^{-1}x)\Omega(w)}(\mathfrak{v}(Q_j))=\mathfrak{v}'(Q'_j)$. Let τ be an automorphism of $\mathbf{C}$ such that $\tau=[a, R_j(P)]$ on $R_j(P)_{\mathrm{ab}}$. It follows from Proposition 22 in 5.6 that $Q_j'^{\tau^{-1}}$ is isomorphic to the PEL-structure Q''_j of type $\Omega(x)=\Omega(\eta_j(a)^{-1}u^{-1}x)$ attached to j. Therefore we have $J_{XW}(\pi(u))(\mathfrak{v}(Q_j))=f_u(\mathfrak{v}'(Q'_j))=\mathfrak{v}''(Q_j'^{\tau^{-1}})^\tau=\mathfrak{v}''(Q''_j)^\tau$. We now get the property (IV) if we describe this by the words of φ_W, φ_X and $\mathcal{H}_{j_0}$ (see (5.5.3)), and put it together with Proposition 24 in 5.9. The proof is completed.

On account of Lemma 8 in 4.7, we have also proved Theorem 1.

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