DISTORTION UNDER CONFORMAL AND QUASICONFORMAL MAPPINGS

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1. Indication of results

We define a technically simple concept of a generalized quadrilateral that yields purely geometrical information concerning distortion under conformal and quasiconformal mappings.

A generalized quadrilateral Q on a simply connected plane domain D is a family of "curves" lying in D that separate or join certain boundary arcs and interior points. The conjugate family Q^* of "curves" lying in D and "crossing" every member of Q is also a generalized quadrilateral, and $(Q^*)^* = Q$. The length–width ratio of Q is

$$\varrho(Q) = \frac{l(Q)}{l(Q^*)},$$

where $l(Q)$ denotes the infimum of the euclidean lengths of the rectifiable members of Q. A K-quasiconformal mapping f (the conformal mappings are the 1-quasiconformal mappings) of D onto another plane domain D' takes Q onto a generalized quadrilateral Q' on D', and

$$\frac{1}{M}\log \varrho(Q) \leqslant \varrho(Q') \leqslant e^{M\varrho(Q)}, \tag{1.1}$$

where M is a positive constant depending only on K (M is independent of D, Q, f, and everything else, except K). In particular, the relations $\varrho(Q) \to \infty$ and (since $\varrho(Q)\varrho(Q^*) = 1$) $\varrho(Q) \to 0$ (as Q varies on a fixed D) are invariant under f.

In each of the examples suggested by Figures 1.1, 1.2, and 1.3, the families Q_s and Q_s^* of curves suggested by the solid and dotted curves, respectively, are the families of simple

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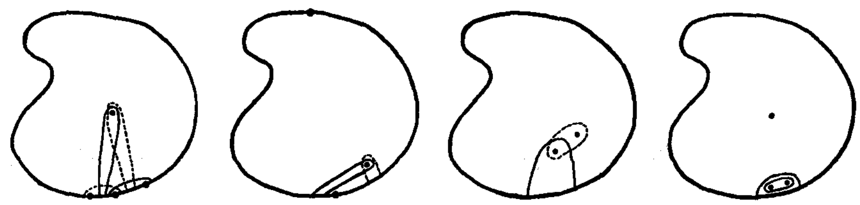

Fig. 1.1 *Fig. 1.2* *Fig. 1.3* *Fig. 1.4*

curves belonging to a generalized quadrilateral Q and its conjugate Q^*. It will follow readily from (1.1) that in each case the ratio

$$\varrho(Q_s)=\frac{l(Q_s)}{l(Q_s^*)}$$

satisfies the analogue of (1.1), and that the relations $\varrho(Q_s)\to\infty$ and $\varrho(Q_s)\to 0$ are invariant under f.

In the case of Figure 1.1 the interior point is fixed and the three boundary points vary. The ratio $\varrho(Q_s)$ measures the position of the "middle" boundary point between the two others, relative to the interior point. This leads to a characterization of the boundary functions of K-quasiconformal mappings betwen simply connected domains, which extends the Beurling–Ahlfors characterization [3] for K-quasiconformal mappings between half-planes. A less precise result is obtained as a corollary: a sense-preserving homeomorphism between the boundaries is the boundary function of a quasiconformal mapping between the interiors if and only if it preserves the relation $\varrho(Q_s)\to\infty$. The relationship between this result and Rickman's characterization [11] is discussed.

In the case of Figure 1.2 we obtain an estimate for the harmonic measure $\omega(z)$ (with respect to D) of one of the two fixed boundary arcs. We find in particular that as the distinguished interior point z tends to one of the two fixed boundary points, $\omega(z)$ remains bounded away from 0 and 1 (that is, z remains in a "conformal angle" at that boundary point) if and only if $\varrho(Q_s)$ remains bounded away from 0 and ∞. We note that a related result in one direction was given by Pommerenke [10, p. 144].

In the case of Figure 1.3 we obtain an estimate for the hyperbolic distance $h(z_1, z_2)$ with respect to D. We find in particular that as the two distinguished interior points z_1 and z_2 vary, $h(z_1, z_2)\to 0$ if and only if $\varrho(Q_s)\to\infty$. We note that Hersch [4] considered the extremal lengths of the families of Figures 1.2 and 1.3, and also gave some geometrical estimates in one direction.

In Figure 1.4 there is one fixed interior point, and there are two variable ones (near the boundary). The family of curves indicated in the figure is the family of simple curves of a generalized quadrilateral; and in this case (1.1) and its analogue with "length" replaced by "diameter" essentially become forms of a result of Lavrentieff [5] concerning a distance between the two variable points.

We also give a sufficient condition that a totally disconnected compact plane set be removable for conformal (and quasiconformal) mappings. Here we use the fact that also the ordinary annulus satisfies (1.1).

2. The generalized quadrilateral: A topological definition

Under a sense-preserving, prime-end preserving homeomorphism between simply connected plane domains D and D', generalized quadrilaterals on D correspond to generalized quadrilaterals on D'. It will be sufficient therefore (and clearly so) to define generalized quadrilaterals on the unit disc.

Let D be the open unit disc, ∂D be its boundary, and S be a finite subset of its closure $\bar{D}$. The "curves" that are members of generalized quadrilaterals on D relative to S are the members of the family $\mathcal{C}=\mathcal{C}(D, S)$, defined to be the family of all *connected*, relatively closed subsets C of D whose closures $\bar{C}$ (in $\bar{D}$) have the following properties:

(a) $\bar{C}$ is a union of finitely many Jordan arcs whose interiors (that is, the open Jordan arcs) are pairwise disjoint, and each of whose endpoints in D is an endpoint of at least two of the arcs;

(b) $\bar{C}\cap S=\emptyset$; and

(c) $\bar{C}\cap\partial D$ consists of *at most* finitely many points, called the *endpoints* of C.

Consider a particular $C\in\mathcal{C}$. We describe the separating and joining done by C in terms of the family $[C]$ of components of the relative complement $D-C$:

$$[C]=\{U\colon\ U \text{ is a component of } D-C\}.$$

Note that the relative boundary $\partial_D U$ of each $U\in[C]$ is in $\mathcal{C}$. Since we are interested only in the separating and joining of subsets of $S\cup\partial D$, it is desirable to sometimes ignore the part of a $U\in[C]$ that does not come in contact with $S\cup\partial D$. Hence we associate with each $U\in[C]$ a subset BU of $S\cup\partial D$ as follows:

$$BU=(S\cap U)\cup\Gamma,$$

where $\Gamma=\bar{U}\cap\partial D$ except in the case where $\bar{U}\cap\partial D=\partial D$ and $\partial_D U$ has exactly one endpoint. In this exceptional case Γ is ∂D minus the endpoint of $\partial_D U$. [In any case Γ is connected,

$$\bar{\Gamma}=\bar{U}\cap\partial D,$$

and BU determines the endpoints of $\partial_D U$. The separating and joining properties of C are completely determined by the family

$$B[C] = \{BU: U \in [C]\}.$$

We say that C *separates* two points of $S \cup \partial D$ provided no $BU \in B[C]$ is such that both of these points are in $\overline{BU}$. We express the requirement that C have an endpoint on a given open connected subset α of ∂D (which will satisfy $\bar{\alpha} - \alpha \subset S$) by saying that no $BU \in B[C]$ contains α.

Now consider a family $Q \subset \mathsf{C}$, and set

$$[Q] = \bigcup_{C \in Q} [C], \quad B[Q] = \{BU: U \in [Q]\}.$$

With this notation, every $C \in Q$ separates two given points of $S \cup \partial D$ if and only if no $BU \in B[Q]$ is such that both of these points are in $\overline{BU}$; and every $C \in Q$ has an endpoint on α (given above) if and only if no $BU \in B[Q]$ contains α. For a $C \in \mathsf{C}$ the notation

$$B[C] \prec B[Q]$$

means that every member of $B[C]$ is contained in a member of $B[Q]$. We observe that if Q is the family of all members of C satisfying every requirement in *any* collection of separating and joining requirements of the above type, then for any $C \in \mathsf{C}$,

$$(\mathrm{Q}_1) \qquad C \in Q \text{ if } B[C] \prec B[Q].$$

(We check this assertion for a single separating requirement and for a single joining requirement, and note that the collection of all families Q satisfying (Q_1) is closed under the taking of arbitrary intersections.) It turns out that *the property* (Q_1) *is by itself a sufficient description of the statement that "Q is the family of all $C \in \mathsf{C}$ satisfying certain separating and joining requirements"*; and we avoid a more specific description by using (Q_1). The above discussion was given mainly to explain the property (Q_1) and is not complete: another type of separating property covered by (Q_1) will also be useful in the applications.

An *S-homeomorphism* is a sense-preserving homeomorphism of $\bar{D}$ onto itself that keeps every point of S fixed.

A *generalized quadrilateral* on D relative to S is a nonempty family $Q \subset \mathsf{C}$ with the following properties:

(Q_1) $C \in Q$ if $C \in \mathsf{C}$ and $B[C] \prec B[Q]$;

(Q_2) Q is invariant under every S-homeomorphism; and

(Q_3) no point of S is such that each of its neighorhoods contains a member of Q.

Even without the assumption (Q_2), Q would be invariant under every homeomorphism of $\bar{D}$ onto itself keeping *every* point of $S \cup \partial D$ fixed, because under such a homeomorphism each $B[C]$ $(C \in \boldsymbol{Q})$ is invariant. The assumption (Q_2) essentially says that any open connected subset α of ∂D, that is required to contain an endpoint of every member of $\boldsymbol{Q}$, has the property that $\bar{\alpha} - \alpha \subset S$. The assumption ($\mathrm{Q}_3$) says that $\boldsymbol{Q}$ is nondegenerate.

A *conjugate* family $\boldsymbol{Q}^*$ corresponds to each generalized quadrilateral $\boldsymbol{Q}$:

$$\boldsymbol{Q}^* = \{C^* : C^* \in \mathsf{C},\ C^* \cap C \neq \emptyset \text{ for every } C \in \boldsymbol{Q}\}.$$

We note that not only does a $C \in \boldsymbol{Q}$ intersect a given $C^* \in \boldsymbol{Q}^*$, but so do all of its images under S-homeomorphisms, that is, C "crosses" C^*.

PROPOSITION 1. *A generalized quadrilateral $\boldsymbol{Q}$ is symmetrical, in the sense that $\boldsymbol{Q}^*$ is a generalized quadrilateral and $(\boldsymbol{Q}^*)^* = \boldsymbol{Q}$.*

Proof. We first prove that $\boldsymbol{Q}^*$ is nonempty. It is readily seen that C has a member C^* with the property that for each $U^* \in [C^*]$ there exists a point of $\bar{D}$ every neighborhood of which contains the image of U^* under an S-homeomorphism. It follows from (Q_2) and (Q_3) that $C^* \in \boldsymbol{Q}^*$ (assume not).

Let C be any member of C such that $B[C] \prec B[\boldsymbol{Q}^*]$. We prove that $C \in \boldsymbol{Q}^*$. Assume contrary to this assertion that some $U \in [C]$ contains a $C_0 \in \boldsymbol{Q}$. To avoid difficulty in the case where $\partial_D U$ is not a simple curve, we choose a $U' \in [\mathsf{C}]$ such that $\partial_D U'$ is a simple curve, $C_0 \subset U'$, and $BU' = BU$. Since $B[C] \prec B[\boldsymbol{Q}^*]$, there exists a $U^* \in [\boldsymbol{Q}^*]$ such that $BU \subset BU^*$. Then also $BU' \subset BU^*$, and a simple argument (using the fact that $\partial_D U'$ is a simple curve) shows that U' is taken onto a subdomain of U^* by some S-homeomorphism. This S-homeomorphism takes C_0 onto a member of $\boldsymbol{Q}$ contained in U^*, contrary to the assumption $U^* \in [\boldsymbol{Q}^*]$. Thus $C \in \boldsymbol{Q}^*$.

It is obvious that $\boldsymbol{Q}^*$ is invariant under S-homeomorphisms.

We now prove that no point of S is such that each of its neighborhoods contains a member of $\boldsymbol{Q}^*$. Assume to the contrary that every neighborhood of some point of S contains a member of $\boldsymbol{Q}^*$. Then this point of S must be an endpoint of every member of $\boldsymbol{Q}$, contrary to the fact that no member of C has an endpoint in S.

Thus $\boldsymbol{Q}^*$ is a generalized quadrilateral.

Finally we prove that $(\boldsymbol{Q}^*)^* = \boldsymbol{Q}$. Clearly $\boldsymbol{Q} \subset (\boldsymbol{Q}^*)^*$. To prove the other inclusion, let $C^{**} \in (\boldsymbol{Q}^*)^*$. We prove that $B[C^{**}] \prec B[\boldsymbol{Q}]$, and thereby obtain the desired conclusion $C^{**} \in \boldsymbol{Q}$. Assume to the contrary that for some $U^{**} \in [C^{**}]$, BU^{**} is not contained in any member of $B[\boldsymbol{Q}]$. It follows that U^{**} is not contained in any member of $[\boldsymbol{Q}]$, or equivalently, that every $C \in \boldsymbol{Q}$ intersects U^{**}. From this we derive the contradictory conclusion that U^{**} contains a member of $\boldsymbol{Q}^*$. To do this we choose a $U_0 \in [\mathsf{C}]$ such that $U_0 \cup \partial_D U_0 \subset U^{**}$ and $BU_0 = BU^{**}$.

We prove that every $C \in Q$ intersects U_0. Assume the contrary. Then U_0 is contained in a member of $[Q]$ and BU_0 is contained in a member of $B[Q]$. Thus BU^{**} is contained in a member of $B[Q]$. We shall prove that $U^{**} \in [Q]$ (which is contrary to the fact that every $C \in Q$ intersects U^{**}). By (Q_2) there exists a $C \in Q$ whose intersection with C^{**} is nonempty and finite. Clearly $C \cup C^{**} \in \mathsf{C}$. We prove that $C' \in Q$, where

$$C' = (C \cup C^{**}) - U^{**},$$

and thereby conclude that $U^{**} \in [Q]$. Clearly $C' \in \mathsf{C}$. Every member of $[C']$ except U^{**} is contained in a member of $[C]$. Thus $B[C'] \prec B[Q]$, and $C' \in Q$. Thus $U^{**} \in [Q]$, and with this contradiction we conclude that every $C \in Q$ intersects U_0.

For the same reason that Q^* is nonempty, there exists a $C^* \in \mathsf{C}$ such that

$$\partial_D U_0 \subset C^* \subset U_0 \cup \partial_D U_0,$$

and such that every $C \in Q$ that intersects U_0 also intersects C^*. Thus since every $C \in Q$ intersects U_0, $C^* \in Q^*$. Since $C^* \subset U^{**}$, we have the desired contradiction, and the proof of Proposition 1 is complete.

Remark 2.1. To further explain the condition (Q_1) we note one possibility that it rules out. Let α_1, α_2, and α_3 be open arcs on ∂D having disjoint closures, and let S consist of the endpoints of these arcs. Define Q to be the family of all $C \in \mathsf{C}$ having an endpoint on α_1 and having an endpoint on either α_2 or α_3. Any simple curve C having an endpoint on α_1 and separating α_2 and α_3 has the property that each member of $[C]$ is contained in a member of $[Q]$. Thus $B[C] \prec B[Q]$, and we see that (Q_1) fails to hold. (We can define a generalized quadrilateral to be the family of all $C \in \mathsf{C}$ having an endpoint on each α_j.)

3. The distortion theorem

Let D be a simply connected domain in the finite plane, which is not the whole plane, and let $\hat{D}$ denote the prime-end compactification of D. (The reader who assumes the boundary of D to be a Jordan curve in the extended plane, so that $\hat{D}$ becomes the closure of D, will loose very little.) All notation and terminology of Section 2 carry over to D, by taking a sense preserving homeomorphism of $\hat{D}$ onto the closed unit disc, and they are independent of the homeomorphism. These definitions can of course be given directly, in terms of $\hat{D}$. A generalized quadrilateral on D is symmetrical, by Proposition 1.

The length $l(Q)$ of a generalized quadrilateral Q is the infimum of the (euclidean) lengths $l(C)$ of its rectifiable members. By (Q_2) it is finite (see the proof of Lemma 2), and by (Q_3) it is positive. The length–width ratio of Q is

$$\varrho(Q) = \frac{l(Q)}{l(Q^*)}.$$

Clearly $\varrho(Q)\varrho(Q^*) = 1$, by symmetry.

We note that a generalized quadrilateral is completely determined by its minimal members, a *minimal* member being one that properly contains no other member. This holds because, as is readily seen, every member contains a minimal member, and because every member of $\mathbf{C}$ containing a member of $\mathbf{Q}$ is a member of $\mathbf{Q}$, by (Q_1). Moreover, $l(\mathbf{Q})$ is the infimum of the lengths of the rectifiable minimal members of $\mathbf{Q}$.

A K-quasiconformal homeomorphism f of D onto another plane domain D' can be extended to a homeomorphism between the prime-end compactifications $\hat{D}$ and $\hat{D}'$ (by Mori's theorem on the unit disc [1, p. 47]). Such a mapping f takes a generalized quadrilateral $\mathbf{Q}$ on D relative to S onto a generalized quadrilateral $\mathbf{Q}'$ on D' relative to $f(S)$:

$$\mathbf{Q}' = \{f(C): C \in \mathbf{Q}\}.$$

The central result concerning generalized quadrilaterals is the following distortion theorem.

THEOREM 1. *There exists $M > 0$ depending only on K such that*

$$\frac{1}{M} \log \varrho(\mathbf{Q}) \leqslant \varrho(\mathbf{Q}') \leqslant e^{M\varrho(\mathbf{Q})}.$$

The constant M is independent of D, $\mathbf{Q}$, f, and everything else, except K.

If we replace "length" by "diameter", we get another form of Theorem 1. The diameter $d(\mathbf{Q})$ of a generalized quadrilateral $\mathbf{Q}$ is the infimum of the (euclidean) diameters $d(C)$ of its members. It is finite and positive. We set

$$\varrho_d(\mathbf{Q}) = \frac{d(\mathbf{Q})}{d(\mathbf{Q}^*)}.$$

The modified version reads:

THEOREM 1'. *There exists $M > 0$ depending only on K such that*

$$\frac{1}{M} (\log \varrho_d(\mathbf{Q}))^{\frac{1}{2}} \leqslant \varrho_d(\mathbf{Q}') \leqslant e^{M\varrho_d(\mathbf{Q})^2} \quad (\varrho_d(\mathbf{Q}) \geqslant 1).$$

LEMMA 1. *Let J be a Jordan curve in the finite plane containing no point of S. (For a point $P \in S \cap \partial\hat{D}$, the assumption is that the accessible point of P, if it has one, is not on J.) There are two assertions:*

(i) *If the closure of neither component of the complement of J contains a member of $\mathbf{Q}$, then J contains a member of $\mathbf{Q}^*$.*

(ii) *If the closure of one component Δ of the complement of J contains a member of $\mathbf{Q}$, then for any $C \in \mathbf{Q}$ whose intersection with J is finite (and possibly empty), $J \cup (C \cap \Delta)$ contains a member of $\mathbf{Q}$.*

Proof. To prove (i) we assume that J contains no member of $\mathcal{Q}^*$. By (Q_2) there exists a $C \in \mathcal{Q}$ whose intersection with J is finite. Let C_1 be any component of $J \cap D$ such that C intersects both components of $D - C_1$. If there is no such component, there is nothing to prove. Clearly $C_1 \in \mathcal{C}$. Since $C_1 \notin \mathcal{Q}^*$, one component D_1 of $D - C_1$ contains a member of $\mathcal{Q}$. Clearly $C' \in \mathcal{C}$, where

$$C' = C_1 \cup (C \cap D_1).$$

Every component of $D_1 - C'$ is contained in a component of $D - C$, and since D_1 contains a member of $\mathcal{Q}$, the other component of $D - C_1$ is contained in a member of $[\mathcal{Q}]$. Thus every member of $[C']$ is contained in a member of $[\mathcal{Q}]$, and it follows that $B[C'] \prec B[\mathcal{Q}]$. This proves that $C' \in \mathcal{Q}$. For this member of $\mathcal{Q}$ there are fewer of the components C_1, and iteration of this process yields a member of $\mathcal{Q}$ contained in the closure of one component of the complement of J. This proves (i).

To prove (ii) assume that the closure of one component Δ of the complement of J contains a member of $\mathcal{Q}$, and let C be a member of $\mathcal{Q}$ whose intersection with J is finite. If $C \cap \Delta = \emptyset$, then some component C_1 of $J \cap D$ is such that the closure of each component of $D - C_1$ contains a member of $\mathcal{Q}$. (To see this consider the appearance of the components of $D - J$ after mapping $\hat{D}$ topologically onto the closed unit disc.) Thus if $C \cap \Delta = \emptyset$, $B[C_1] \prec B[\mathcal{Q}]$, and $C_1 \in \mathcal{Q}$. We assume now that $C \cap \Delta \neq \emptyset$. The obvious modification of the proof of (i) shows that $J \cup (C \cap \Delta)$ contains a member of $\mathcal{Q}$. This proves (ii).

Let h be any function defined on $\mathcal{C}$ that satisfies

$$d(C) \leqslant h(C) \leqslant l(C) \tag{3.1}$$

for each $C \in \mathcal{C}$, and set

$$h(\mathcal{Q}) = \inf \{h(C) : C \in \mathcal{Q}\}, \ \ \varrho_h(\mathcal{Q}) = \frac{h(\mathcal{Q})}{h(\mathcal{Q}^*)}.$$

LEMMA 2. *There exists an absolute constant k such that*

$$l(\mathcal{Q}) \leqslant k d(\mathcal{Q}) \max \{1, \varrho_h(\mathcal{Q})\}.$$

Remark. In the case of ordinary quadrilaterals, this lemma is essentially also given by Rickman [12, p. 390].

Proof. It follows readily from (Q_2) that every member of $\mathcal{Q}$ can be "approximated" by polygonal members of $\mathcal{Q}$, a polygonal member being one that is a finite union of (finite) rectilinear segments; and for the same reason $l(\mathcal{Q})$ is the infimum of the lengths of the polygonal members of $\mathcal{Q}$. We let C be a polygonal member of $\mathcal{Q}$ such that $l(C) < l(\mathcal{Q}) + \varepsilon$, where ε is a small positive number depending on $d(\mathcal{Q})$ and $l(\mathcal{Q}^*)$. It follows easily from Lemma 1 (ii) that $d(C) < k_1 d(\mathcal{Q})$ (provided $\varepsilon < d(\mathcal{Q})$), where k_1 is an absolute constant

($k_1 = 10$ will do). Let H be a closed square of side-length $2k_1 d(Q)$ whose interior contains the closure of C. Subdivide H into m^2 nonoverlapping congruent closed squares H_j, where m is the least positive integer greater than $l(\partial H)/l(Q^*)$. Then

$$l(\partial H_j) = \frac{l(\partial H)}{m} < l(Q^*); \quad m - 1 \leqslant \frac{l(\partial H)}{l(Q^*)}.$$

We shall prove that $l(C) \leqslant 4 \sum l(\partial H_j)$, and thereby obtain the desired conclusion:

$$l(Q) \leqslant l(C) \leqslant 4 \sum l(\partial H_j) = 4m\, l(\partial H) \leqslant 4\left(1 + \frac{l(\partial H)}{l(Q^*)}\right) l(\partial H)$$

$$= 4\left(1 + 8k_1 \frac{d(Q)}{l(Q^*)}\right) 8k_1 d(Q) \leqslant 4(1 + 8k_1 \varrho_h(Q))\, 8k_1 d(Q)$$

$$\leqslant k d(Q) \max\{1, \varrho_h(Q)\}.$$

Suppose contrary to the assertion that $l(C) > 4 \sum l(\partial H_j)$. As is easily seen, H can be chosen so that $C \cap \bigcup \partial H_j$ is finite. Thus $l(C) = \sum l(C \cap H_j)$, and there exists a j (fixed forevermore) such that $l(C \cap H_j) > 4l(\partial H_j)$. Let L be a closed rectilinear segment lying in the interior of H_j, except for its endpoints which are on ∂H_j, such that $L \cap C$ is finite, and such that if X_i $(i = 1, 2)$ denote the closures of the components of $H_j - L$, then

$$l(C \cap X_i) > 2l(\partial H_j) \quad (i = 1, 2).$$

Since $l(\partial X_1) < l(\partial H_j) < l(Q^*)$, ∂X_1 contains no member of Q^*. We can require that neither ∂H_j nor L contains a point of S (in the sense explained by Lemma 1). Thus by Lemma 1 (ii), $C \cup \partial X_1$ contains a $C_0 \in Q$ that is contained in the closure of one component of the complement of ∂X_1. For this member of Q we have

$$l(C_0) \leqslant l(C) + l(\partial X_1) - 2l(\partial H_j) \leqslant l(C) - l(\partial H_j),$$

contrary to the choice of C (provided $\varepsilon < l(\partial H_j)$). The proof of Lemma 2 is complete.

LEMMA 3. *For any two functions h and h' defined on $\mathbf{C}$ and satisfying the same inequality* (3.1),

$$\varrho_{h'}(Q) \leqslant k \max\{1, \varrho_h(Q)^2\},$$

where k is an absolute constant.

Proof. Using the symmetry of Q, and applying Lemma 2 twice, we obtain independently the following two inequalities:

$$h'(Q) \leqslant k_1 h(Q) \max\{1, \varrho_h(Q)\};$$

$$h(Q^*) \leqslant k_2 h'(Q^*) \max\{1, \varrho_h(Q^*)\}.$$

Combining these, we obtain

$$\varrho_{h'}(Q) \leqslant k_1 k_2 \varrho_h(Q) \max\{1, \varrho_h(Q)\} \max\{1, \varrho_h(Q^*)\} = k \max\{1, \varrho_h(Q)^2\},$$

where $k = k_1 k_2$. This proves Lemma 3.

Proof of Theorem 1. We only need to prove the first inequality. We first prove the existence of positive absolute constants k_0 and k_1 such that

$$\frac{1}{2\pi} \log \varrho(Q) \leqslant 4K\varrho(Q') + 2\pi K + k_1 \quad \text{if} \quad \varrho(Q) > k_0. \tag{3.2}$$

Let k be the absolute constant of Lemma 3 ($k>1$) so that

$$\varrho(Q) \leqslant k \max\{1, \varrho_d(Q)^2\},$$

and set $k_0 = 100k$. If $\varrho(Q) > k_0$, then $\varrho_d(Q) > 10$, and there exists an annulus

$$A = \{z\colon R_1 < |z - z_0| < R_2\}$$

with the following properties: some member of Q^* is contained in $\{|z-z_0| < R_1\}$; every member of Q intersects both components of ∂A; and $R_2/R_1 > \varrho_d(Q)/3$.

The extremal length $\lambda(Q^*)$ of Q^* is well defined, since every member of Q^* is a union of finitely many (pairwise disjoint) curves. Any circle $\{|z-z_0| = R\}$ $(R_1 < R < R_2)$, with at most finitely many exceptions, must by Lemma 1 contain a member of Q^*. Thus we readily see by comparison that

$$\lambda(Q^*) \leqslant \frac{2\pi}{\log \dfrac{R_2}{R_1}}.$$

Since $\lambda(Q'^*) \leqslant K\lambda(Q^*)$ [1],

$$\frac{1}{K}\lambda(Q'^*) \leqslant \frac{2\pi}{\log \dfrac{\varrho_d(Q)}{3}}. \tag{3.3}$$

For any given $\varepsilon > 0$ we choose a $C' \in Q'$ such that

$$l(C') < l(Q') + \varepsilon.$$

We prove that the area of the set

$$V = \{w\colon \operatorname{dist}(w, C') < l(Q'^*)\}$$

(euclidean distance) is at most

$$2l(Q'^*)\, l(C') + \pi l(Q'^*)^2.$$

If C' is a simple curve, this is readily seen by approximating C' with polygonal Jordan arcs (not necessarily in D) and arguing by induction on the number of vertices of the approximating arcs: the term $\pi l(Q'^*)^2$ accounts for the area of the two half-discs at the ends of the "bent rectangle". Since C' is a finite union of pairwise disjoint simple curves, and since C' is connected, we can now argue by induction on the number of these simple curves. At each stage of this induction, we adjoin a "bent rectangle" and two half-discs. There is enough overlap of area to account for the area of these half-discs, and consequently the term $\pi l(Q'^*)^2$ does not need to be added after the first stage. This establishes the upper bound for the area of V.

For each $C'^* \in Q'^*$ the (total) length of $C'^* \cap V$ is at least $l(Q'^*)$, because $C'^* \cap C' \neq \emptyset$. Thus

$$\lambda(Q'^*) \geq \lambda(\{C'^* \cap V : C'^* \in Q'^*\}) \geq \frac{l(Q'^*)^2}{\text{area } V}.$$

Since this holds for each $\varepsilon > 0$,

$$\lambda(Q'^*) \geq \frac{l(Q'^*)^2}{2l(Q'^*)\, l(Q') + \pi l(Q'^*)^2} = \frac{1}{2\varrho(Q') + \pi}. \tag{3.4}$$

Combining (3.3), (3.4), and the inequality $(\varrho(Q)/k)^{\frac{1}{2}} \leq \varrho_d(Q)$, we readily obtain (3.2).

By (3.2) there exists a $k_0' > 1$ depending only on K such that

$$\frac{1}{2\pi} \log \varrho(Q) \leq 5K\varrho(Q')$$

if $\varrho(Q) > k_0'$. Using the symmetry of Q and applying this inequality to the inverse mapping, we see that for $\varrho(Q) \geq 1$, $\varrho(Q')$ is bounded away from zero by a positive constant depending only on K. Thus there exists a positive constant M depending only on K such that $\log \varrho(Q) \leq M\varrho(Q')$ for $1 \leq \varrho(Q) \leq k_0'$; and if we require also that $M \geq 10\pi K$, we have the inequality $\log \varrho(Q) \leq M\varrho(Q')$ in any case. The proof of Theorem 1 is complete.

Theorem 1′ follows from Theorem 1, Lemma 3, and essentially the argument in the preceeding paragraph.

Remark 3.1. For ordinary quadrilaterals and a conformal f we can have

$$\frac{1}{2\pi} \log \varrho(Q) = \varrho(Q')$$

for any value of $\varrho(Q')$. We see this by mapping a radially slit circular annulus onto a rectangle.

4. Multiply connected domains

We note first that the ordinary annulus, a doubly connected domain D with non-degenerate boundary components, deserves to be called a "generalized quadrilateral".

This is because every simple curve lying in D and joining different components of $\partial\hat{D}$ is in one of the two conjugate families. These families consist of the simple curves separating and joining, respectively, the components of $\partial\hat{D}$, and either family can be Q. If we instead define $\mathcal{C}$ as before (with $S=\emptyset$) and consider the larger subfamilies of $\mathcal{C}$ separating and joining, respectively, the components of $\partial\hat{D}$, then the effect is the same, because the minimal members of these larger families are just the simple curves. It is readily seen that *all results of Section 3 hold for the ordinary annulus.*

On general finitely connected domains we consider certain families Q, *establish the main inequality* (3.2), *and thereby obtain the invariance of the relation* $\varrho(Q)\to\infty$. Let D be a finitely connected domain with only nondegenerate boundary components, and let S be a finite subset of the prime-end compactification $\hat{D}$ of D. We define the family $\mathcal{C}=\mathcal{C}(D, S)$ exactly as before. A V-side of a simple member (that is, a member that is a simple curve) $C_s\in\mathcal{C}$ is a subdomain V of D such that for some simple $C_s'\in\mathcal{C}$ homotopic to C_s in D (and having the same endpoints as C_s if C_s has endpoints), the boundary of V in $\hat{D}$ is the closure of $C_s\cup C_s'$ in $\hat{D}$. Let Q be a subfamily of $\mathcal{C}$ satisfying the analogues of (Q_2) and (Q_3) (S-homeomorphisms are now required to keep every component of $\partial\hat{D}$ fixed), and satisfying the following condition: for each simple $C_s\in\mathcal{C}$, either every member of Q intersects C_s, or for each $C\in Q$ there exists a V-side $V(C)$ of C_s such that for any V-side V of C_s contained in $V(C)$, $(C-V)\cup\partial_D V$ contains a member of Q. This condition will not in general be satisfied by Q^* even if we define Q^* as before, so we choose the following more flexible definition of Q^*: let Q^* be any subfamily of $\mathcal{C}$ such that every simple member of $\mathcal{C}$ intersecting every member of Q is in Q^*, and such that every member of Q^* intersects every member of Q. It is easy to prove Lemma 1 with a slight modification: the conclusion of part (ii) should be that $J\cup\Delta$ contains a member of Q whose length is at most $l(C\cap\Delta)+l(J)$. Lemma 2 and its proof hold without change. It is readily seen that the part of Lemma 3 used to prove the main inequality (3.2) follows directly from Lemma 2 without symmetry. Thus (3.2) holds and we have the invariance of the relation $\varrho(Q)\to\infty$. For example, we could let S consist of two points of D, let Q be the family of all $C\in\mathcal{C}$ having an endpoint on $\partial\hat{D}$ and separating the points of S, and let Q^* be the family of all Jordan curves lying in D and separating both points of S from $\partial\hat{D}$.

5. The boundary function of a K-quasiconformal mapping

Let D be a simply connected domain in the finite plane, which is not the whole plane, and let $\hat{D}$ be the prime-end compactification of D. Fix $z_0\in D$, and let P_j $(j=1, 2, 3)$ be distinct variable points of the boundary $\partial\hat{D}$. Let $\mathcal{S}$ be the family of all simple curves in

$D-\{z_0\}$ having two endpoints on $\partial\hat{D}$, neither of which is a P_j, and let α and β be the components of $(\partial\hat{D})-\{P_1, P_2, P_3\}$ such that P_1 and P_2 are the endpoints of α and P_2 and P_3 are the endpoints of β. Let Q_s be the family of all members of S having an endpoint on α and separating z_0 and β; and let Q^* be the family of all members of S having an endpoint on β and separating z_0 and α. The members of Q_s and Q_s^* are suggested by the solid and dotted curves, respectively, in Figure 1.1. Set

$$\varrho(P_1, P_2, P_3, z_0)=\frac{l(Q_s)}{l(Q_s^*)},$$

where $l(Q_s)$ denotes the infimum of the lengths of the rectifiable members of Q_s. This ratio measures the position of P_2 between P_1 and P_3, relative to z_0. We take special note of the case where D is the unit disc and z_0 is the origin. Also, we note that

$$\varrho(P_3, P_2, P_1, z_0) = 1/\varrho(P_1, P_2, P_3, z_0).$$

THEOREM. *Let φ be the boundary function of a K-quasiconformal homeomorphism f of D onto another plane domain D' with $f(z_0)=z_0'$. Then for some $M>0$ depending only on K,*

$$\varrho(P_1', P_2', P_3', z_0') \leqslant \exp\,(M\varrho(P_1, P_2, P_3, z_0))$$

for every pair of triples corresponding under φ $(P_j'=\varphi(P_j))$.

Conversely, there exists an absolute constant A such that the following holds: if φ is a sense-preserving homeomorphism of $\partial\hat{D}$ onto $\partial\hat{D}'$, and if for some $M>0$

$$\varrho(P_1, P_2, P_3, z_0) \leqslant A \text{ implies } \varrho(P_1', P_2', P_3', z_0') \leqslant M$$

for every pair of triples corresponding under φ, then φ is the boundary function of some K-quasiconformal homeomorphism f of D onto D', with $f(z_0)=z_0'$, and K depends only on M.

Proof of the first assertion. Set $S=\{P_1, P_2, P_3, z_0\}$, and note that S is the family of all simple open curves in $\mathcal{C}(D, S)$. We define a generalized quadrilateral Q to be the family of all $C\in\mathcal{C}(D, S)$ having an endpoint on α and separating z_0 and β (that is, separating z_0 and each point of β).

We first give the simple argument which shows that the minimal members of Q are those suggested by Figure 5.1. Let $C\in Q$. We find a simple $C_s\in\mathcal{C}$ that is contained in C and separates z_0 and β, as follows. Consider the $U\in[C]$ containing z_0, and consider the interior domain of every Jordan curve lying in U except for one point on C. The union of U and all these interior domains has the property that each component of its relative boundary is a simple curve, and one of these components, which we denote by C_s, separates z_0 and β. If C_s has an endpoint on α, then $C_s\in Q$. Otherwise there exists

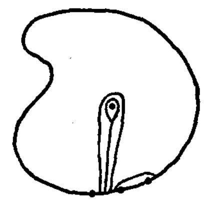

Fig. 5.1

a simple open curve $L \subset C$ that joins α to C_s and does not intersect C_s. If C_s is a closed curve, then $L \cup C_s \in Q$; and otherwise $L \cup C_s$ contains a simple member of Q. Thus the minimal members of Q are those suggested by Figure 5.1.

We show that Q^* is the family of all $C \in \mathsf{C}$ having an endpoint on β and separating z_0 and α, and thus that Q is "geometrically symmetrical". Clearly every member of Q^* has an endpoint on β and separates z_0 and α (suppose not). To prove the other inclusion, we note that every member of C having an endpoint on β and separating z_0 and α intersects every minimal member of Q and is therefore in Q^* (every member of Q contains a minimal member).

The families Q_s and Q_s^* used to define $\varrho(P_1, P_2, P_3, z_0)$ are the families of simple members of Q and Q^* respectively, and the minimal members of Q and Q^* determine $\varrho(Q)$. Thus since every rectifiable minimal member $C \in Q$ can be "approximated" by a $C_s \in Q_s$ so that $l(C) \leqslant l(C_s) \leqslant 2l(C)$, and since the analogous statement holds for Q^*, the first assertion of the theorem now follows from Theorem 1.

Proof of the second assertion. Let M_1 be the absolute constant obtained from the first assertion with $K=1$. If the second assertion holds, as stated, in the special case where D and D' are the unit disc and z_0 and z_0' the origin, then in the general case we have the desired conclusion with the absolute constant $e^{M_1 A}$ (instead of A), provided

$$\varrho(P_1, P_2, P_3, z_0) \leqslant e^{M_1 A} \quad \text{implies} \quad \varrho(P_1', P_2', P_3', z_0') \leqslant \frac{1}{M_1} \log M.$$

It is therefore sufficient to prove the second assertion for the special case.

Let Z_1, Z_2, and Z_3 be any three points on the real axis $\{-\infty < Z < +\infty\}$ such that Z_2 is between Z_1 and Z_3, and set $Z_4 = \infty$. Map the upper half-plane $\{\operatorname{Im} Z > 0\}$ onto the unit disc D by a linear transformation, and let P_j be the point corresponding to Z_j. Let $[P', P'']$ be the component of $(\partial D) - \{P_1, P_2, P_3, P_4\}$ whose endpoints are P' and P''; and let $\varrho(P_1, P_2, P_3, P_4)$ be the (euclidean) diameter of the shorter of the arcs $[P_2, P_3]$ and $[P_4, P_1]$, divided by the diameter of the shorter of the arcs $[P_1, P_2]$ and $[P_3, P_4]$. Clearly $\varrho(P_1, P_2, P_3, P_4)$ is the length–width ratio of an ordinary quadrilateral. The length-width ratio of the corresponding ordinary quadrilateral on the upper half-plane is denoted by $[\varrho(Z_1, Z_2, Z_3, Z_4)$. We note that the Beurling–Ahlfors condition [3] is stated in terms of this ratio. By Theorem 1 there exists an absolute constant $M_1>0$ such that

$$\frac{1}{M_1}\log \varrho(Z_1, Z_2, Z_3, Z_4) \leqslant \varrho(P_1, P_2, P_3, P_4) \leqslant \exp\left(M_1\varrho(Z_1, Z_2, Z_3, Z_4)\right). \tag{5.1}$$

Using (5.1) we shall be able to make the transition from the Beurling–Ahlfors condition because, as is readily seen,

$$\varrho(P_1, P_2, P_3, P_4) = \max\Big\{\min\{\varrho(P_1, P_2, P_3, 0), \varrho(P_2, P_1, P_4, 0)\},\\ \min\{\varrho(P_4, P_3, P_2, 0), \varrho(P_3, P_4, P_1, 0)\}\Big\}.$$

(Each minimum takes the shorter of $[P_2, P_3]$ and $[P_4, P_1]$, and the maximum takes the shorter of $[P_1, P_2]$ and $[P_3, P_4]$.)

We now assume that Z_2 is equidistant between Z_1 and Z_3. Then $\varrho(Z_1, Z_2, Z_3, Z_4)=1$, and by (5.1), $\varrho(P_1, P_2, P_3, P_4)\leqslant e^{M_1}$. We set $A=e^{M_1}$. It follows readily from the assumption concerning M that $\varrho(P_1', P_2', P_3', P_4')\leqslant M$, where $P_j'=\varphi(P_j)$. Now map the unit disc D' onto the upper half-plane $\{\operatorname{Im} Z'>0\}$ by a linear transformation taking P_4' to ∞, and let Z_j' be the point corresponding to P_j'. By the first inequality of (5.1) applied to this mapping,

$$\varrho(Z_1', Z_2', Z_3', Z_4') \leqslant \exp\left(M_1\varrho(P_1', P_2', P_3', P_4')\right) \leqslant \exp\left(M_1 M\right).$$

Thus the correspondence between the real axes satisfies the Beurling–Ahlfors condition, and there exists a K-quasiconformal mapping between the half-planes, with K depending only on M, whose boundary function is this correspondence. This mapping yields a K-quasiconformal mapping f of D onto D' whose boundary function is φ.

We modify f so that $f(0)=0$. We can clearly do this if there exists a K_1-quasiconformal mapping f_1 of D' onto itself, with K_1 depending only on M, that keeps every point of $\partial D'$ fixed and takes $f(0)$ to 0. To prove the existence of f_1 it is sufficient to prove that $1-|f(0)|$ is bounded below by a positive constant depending only on M. To this end, assume $f(0)\neq 0$ and let α be an open arc on ∂D of length $\pi/2$ such that the radius of D' through $f(0)$ terminates at a point of the image arc $\alpha'=f(\alpha)$. Define a ratio $\delta(0, \alpha)$ to be the infimum of the lengths of the crosscuts of D separating α from 0, divided by the same for the crosscuts separating the complementary arc from 0. It follows from Theorem 1 (as we see in Section 6.1) that

$$\delta(f(0), \alpha') \leqslant e^{M'\delta(0,\alpha)},$$

where M' depends only on K, and thus only on M. In view of this inequality, the proof will be complete once we prove that the length of α' is bounded below by a positive constant depending only on M. If the length of α' is less than π, there exists an open arc β on ∂D such that $\bar{\alpha} \cap \bar{\beta}$ consists of a single point, and such that the lengths of both β and its image $\beta' = f(\beta)$ are at least π. Let P_1 and P_2 be the endpoints of α, and let P_2 and P_3 be the endpoints of β. We readily see that

$$1/\delta(0, \alpha) = \varrho(P_1, P_2, P_3, 0) \quad \text{and} \quad 1/\delta(0, \alpha') = \varrho(P_1', P_2', P_3', 0),$$

where $P_j' = f(P_j)$. Thus by assumption $1/\delta(0, \alpha') \leqslant M$ provided we require A to be larger than the absolute constant $1/\delta(0, \alpha)$. This completes the proof.

Remark 5.1. The (uniform) invariance of the relation $\varrho(P_1, P_2, P_3, z_0) \to 0$ characterizes the boundary function of a quasiconformal mapping. It is clear from the proof and Lemma 3 that the invariance of the relation $\varrho_d(P_1, P_2, P_3, z_0) \to 0$ (defined with "length" replaced by "diameter") also characterizes the boundary function.

Remark 5.2. If z_0 is replaced by a fixed point P_4 on $\partial \hat{D}$, and if we consider the ordinary quadrilaterals Q on D determined by the points P_1, P_2, P_3, and P_4 (where it is assumed that P_4 is not in the closure of $\alpha \cup \beta$), then it is clear from the proof that the invariance of the relation $\varrho(Q) \to 0$ characterizes the boundary function of a quasiconformal mapping. This fact alone follows easily from a lemma in Lehto–Virtanen's book [6, p. 25] and Rickman's lemma [12, p. 390] (or a special case of Lemma 3) to prove the necessity, and the necessity and the Beurling–Ahlfors example [3] to prove the sufficiency. This characterization with the ordinary quadrilaterals is very closely related to Rickman's characterization [11], which is however not in this convenient form.

6. Estimates for the harmonic measure and hyperbolic distance

6.1. *The harmonic measure.* Let z be a variable point of the fixed simply connected domain D, and let P_1 and P_2 be distinct, fixed points of the boundary $\partial \hat{D}$. Let α and β be the components of $(\partial \hat{D}) - \{P_1, P_2\}$, and let $\omega(z)$ be the harmonic measure of α (or of β) with respect to D. Let Q_s be the family of all simple curves lying in D, having two endpoints on α, and separating z and β; and let Q_s^* be the family of all simple curves lying in D, having two endpoints on β, and separating z and α (see Figure 1.2). Set

$$\varrho(z) = \frac{l(Q_s)}{l(Q_s^*)}, \quad \tau(z) = \max\left\{\varrho(z), \frac{1}{\varrho(z)}\right\}.$$

THEOREM. *As $z \to P_1$, $\omega(z)$ remains bounded away from 0 and 1 if and only if $\tau(z)$ remains bounded. More precisely,*

$$\frac{1}{M} \log \tau(z) \leqslant \csc(\pi\omega(z)) \leqslant e^{M\tau(z)},$$

where M is a positive absolute constant.

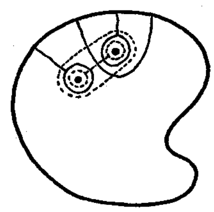

Fig. 6.1

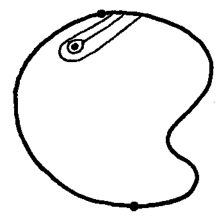

Fig. 6.2

Proof. Define a generalized quadrilateral Q on D relative to $S=\{P_1, P_2, z\}$ to be the family of all $C \in \mathsf{C}$ having an endpoint on α and separating z and β (that is, separating z and each point of β). It is readily seen that the minimal members of Q are those suggested by Figure 6.1. Every member of Q^* has an endpoint on β and separates z and α (suppose not); and every member of C with these properties intersects every minimal member of Q and is therefore in Q^*. Thus Q^* is the family of all $C \in \mathsf{C}$ having an endpoint on β and separating z and α: Q is "geometrically symmetrical". We note that Q_s and Q_s^* are the families of simple members of Q and Q^* respectively. Thus since we can "approximate" every rectifiable minimal $C \in Q$ by a $C_s \in Q_s$ so that $l(C) \leqslant l(C_s) \leqslant 2l(C)$, and since the analogous statement holds for Q^*, the theorem readily follows from Theorem 1, by mapping D conformally onto the upper half-plane so that P_1 and P_2 correspond to 0 and ∞, respectively.

6.2. *The hyperbolic distance.* Let z_1 and z_2 be distinct variable interior points of the simply connected domain D, and let $h(z_1, z_2)$ denote the hyperbolic distance with respect to D. Let Q_s be the family of all simple curves lying in D, having two endpoints on the boundary $\partial\hat{D}$, and separating z_1 and z_2; and let Q_s^* be the family of all Jordan curves lying in D and separating both z_1 and z_2 from $\partial\hat{D}$. Set

$$\varrho(z_1, z_2) = \frac{l(Q_s)}{l(Q_s^*)}.$$

THEOREM. *$h(z_1, z_2) \to 0$ if and only if $\varrho(z_1, z_2) \to \infty$. More precisely,*

$$\frac{1}{M} \log \varrho(z_1, z_2) \leqslant \frac{2}{e^{2h(z_1, z_2)} - 1} \leqslant e^{M\varrho(z_1, z_2)},$$

where M is a positive absolute constant.

Proof. Define a generalized quadrilateral Q on D relative to $S=\{z_1, z_2\}$ to be the family of all $C \in \mathsf{C}$ having an endpoint on $\partial\hat{D}$ and separating z_1 and z_2. The minimal members of Q are suggested by the solid curves in Figure 6.2. Clearly every member of Q^* separates both z_1 and z_2 from $\partial\hat{D}$. On the other hand, every member of C separating both z_1 and z_2

from $\partial\hat{D}$ intersects every minimal member of Q and is therefore in Q^*. Thus Q^* is the family of all $C \in \mathsf{C}$ separating both z_1 and z_2 from $\partial\hat{D}$. The minimal members of Q^* are suggested by the dotted curves in Figure 6.2. The families Q_s and Q_s^* are the families of simple members of Q and Q^* respectively. Thus since every rectifiable minimal $C \in Q$ can be "approximated" by a $C_s \in Q_s$ so that $l(C) \leqslant l(C_s) \leqslant 2l(C)$, and since the analogous statement holds for Q^*, the theorem readily follows from Theorem 1, by mapping D conformally onto the unit disc so that z_1 corresponds to 0.

7. The boundary metric

We consider a fixed interior point z_0 of the simply connected domain D, and two distinct variable points P_1 and P_2 in $\hat{D} - \{z_0\}$. Set $S = \{P_1, P_2, z_0\}$, and define a generalized quadrilateral Q to be the family of all $C \in \mathsf{C}$ such that no $BU \in B[C]$ contains a P_j and z_0, and such that no $BU \in B[C]$ contains a P_j and all of $\partial\hat{D}$. Members of the family Q_s of simple curves belonging to Q are suggested in the various cases by Figures 7.1, 7.2, and 7.3 (every member of Q_s is equivalent under an S-homeomorphism to one of the shown curves). By simple considerations we can determine the minimal members of Q; and by "approximating" these minimal members by members of Q_s, we see that $d(Q) = d(Q_s)$. We readily see that the function

$$d(P_1, P_2, z_0) = d(Q)$$

is a metric on $\hat{D} - \{z_0\}$. Here we use $d(Q)$ instead of $l(Q)$ only for the sake of simplicity. Essentially this metric was introduced by Lavrentieff [5] and Mazurkiewicz [8]. By Theorem 1′ applied to Q^*,

$$d(P_1', P_2', z_0') \leqslant M d(Q'^*) \left(\log \frac{d(Q^*)}{d(P_1, P_2, z_0)}\right)^{-\frac{1}{2}},$$

where P_j', z_0', and Q' correspond to P_j, z_0, and Q under a K-quasiconformal mapping of D onto D', and where M depends only on K. We readily see that if $d(P_1, P_2, z_0) < r$, where $3r$ is the euclidean distance from z_0 to ∂D, and if at most one P_j is in $\{|z - z_0| \leqslant 2r\}$, then $d(Q^*) \geqslant r$. Thus if in addition D' has the finite euclidean diameter R, then

$$d(P_1', P_2', z_0') \leqslant MR \left(\log \frac{r}{d(P_1, P_2, z_0)}\right)^{-\frac{1}{2}}.$$

This is a form of Lavrentieff's inequality [5] (extension to quasiconformal mappings is discussed by Lelong-Ferrand [7]). We get essentially the same inequality with "d" replaced by "l" (and with the $\frac{1}{2}$), but not so directly.

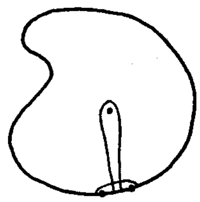

Fig. 7.1

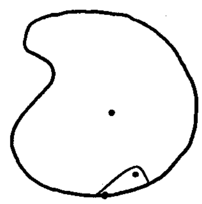

Fig. 7.2

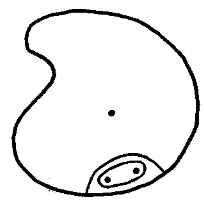

Fig. 7.3

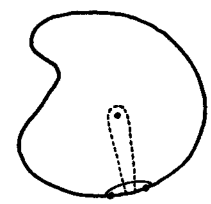

Fig. 7.4

We note the particularly simple form of this result suggested by Figure 7.4. Here Q is defined as in Section 6.1, but the interior point is fixed and the boundary arc is variable. As in Section 6.1 we see that the length and diameter ratios satisfy the inequalities of Theorem 1 and Theorem 1′, respectively. When the length ratio is less than 1, the length of the family of solid curves is the length of the family of curves suggested by Figure 7.1, and the analogous statement holds for the diameters.

8. Removable singularities

For any curvilinear annulus A in the finite plane, let $\varrho(A)$ denote the euclidean distance between the contours divided by the euclidean diameter of the inner contour. Then $\varrho(A)$ is the diameter ratio of a "generalized quadrilateral", and the relation $\varrho(A)\to 0$ is invariant under K-quasiconformal mapping (see Section 4).

We define a class $\mathcal{N}$ of totally disconnected compact sets in the finite plane as follows: $E\in\mathcal{N}$ if and only if for each $z\in E$ there exists a $\delta>0$ such that every neighborhood of z contains a curvilinear annulus A separating z and ∞ such that $A\cap E=\emptyset$ and $\varrho(A)\geqslant\delta$. To say that $E\in\mathcal{N}$ is to say that E is not too dense near any of its points.

THEOREM. *Let $E\in\mathcal{N}$ and let D be a plane domain containing E. Then every K-quasiconformal homeomorphism f of $D-E$ onto a plane domain is the restriction of a K-quasiconformal homeomorphism of D.*

Proof. It suffices to consider the case where $\infty\in D-E$ and $f(\infty)=\infty$. Let $z_0\in E$, and let $\{A_n\}$ be a sequence of curvilinear annuli contained in $D-E$ such that A_1 separates z_0 and ∞ and each A_{n+1} separates z_0 from A_n, such that $A_n\to z_0$ in the sense that every neighborhood of z_0 contains all except finitely many A_n, and such that $\liminf \varrho(A_n)>0$. Since $f(\infty)=\infty$, the image annuli $A_n'=f(A_n)$ have the property that each A_{n+1}' is contained in the bounded component Δ_n of the complement of A_n'. Since $\liminf\varrho(A_n')>0$, $\bigcap\bar{\Delta}_n$ cannot contain more than one point. Thus $\bigcap\bar{\Delta}_n=\{w_0\}$ for some w_0, and we see that $f(z)\to w_0$ as $z\to z_0$, $z\in D-E$. It follows that f is the restriction of a homeomorphism of D, which we also denote by f.

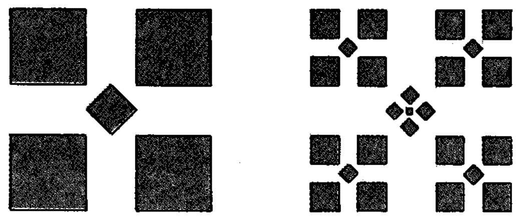

Fig. 8.1

Since the relation $\varrho(A)\to 0$ is invariant under the restricted f, $f(E)\in\mathcal{N}$. Since sets of positive area have points of metric density, every member of $\mathcal{N}$ has zero area. Thus under one-to-one conformal mapping the complement of an $E\in\mathcal{N}$ always corresponds to a domain whose complement has zero area; and by a theorem of Ahlfors and Beurling [2, p. 112], $E\in N_{\mathfrak{D}}$, that is, E is removable for functions with a bounded Dirichlet integral. (This implies that E is removable for one-to-one conformal mappings.) The statement of the theorem now follows from a result of Strebel [13] (or we could appeal less directly to [2]).

Remark 8.1. Although the sets in $\mathcal{N}$ have zero area, they can be large in the sense of linear measure. Figure 8.1 suggests the construction of a set $E\in\mathcal{N}$ with the property that E is contained in a square S and the projection of E on every straight line is the projection of S on the straight line. The limiting set E is in $\mathcal{N}$ because the second set shown is obtained from the first set by replacing each of its five squares by a set *similar* to the first set.

Remark 8.2. Let $E\in\mathcal{N}$, and let J be any Jordan curve in the plane such that $E\subset J$. Then under one-to-one conformal (or quasiconformal) mapping of a component of the complement of J onto the unit disc, E corresponds to a set of measure zero on the unit circle. This follows immediately from a result in [9, p. 57].

Remark 8.3. Let E be a set in $\mathcal{N}$ containing at least three points. We consider the unit disc Δ as the universal covering surface of the complement of E by choosing a particular projection mapping f. This defines the group G of cover transformations on Δ, and we consider any convex fundamental polygon P of G. Then $E'\in\mathcal{N}$, where $E'=\bar{P}\cap\partial\Delta$. We indicate briefly why this is true. Let $z_0\in E'$, and let w_0 be the radial limit of f at z_0. Let A be an "arbitrarily small" (open) curvilinear annulus surrounding w_0 such that $\bar{A}\cap E=\emptyset$. Only finitely many components of $f(\partial_\Delta P)\cap A$ join the two contours of A. These components divide A into (ordinary) quadrilaterals each of which lifts by the monodromy theorem onto a (schlicht) quadrilateral in Δ, and one of these quadrilaterals in Δ, which we denote by Q, separates z_0, relative to P, from a given point of P. We can use the relation $\liminf \varrho(A)>0$ to conclude that $\liminf \varrho(Q)>0$, and this readily yields the desired result. It is possible to construct an example such that $E'\in\mathcal{N}$ and $E\notin\mathcal{N}$.

References

[1]. AHLFORS, L. V., *Lectures on quasiconformal mappings*. Van Nostrand, Princeton, 1966.

[2]. AHLFORS, L. V. & BEURLING, A., Conformal invariants and function-theoretic null-sets. *Acta Math.*, 83 (1950), 101–129.

[3]. —— The boundary correspondence under quasiconformal mappings. *Acta Math.*, 96 (1956), 125–142.

[4]. HERSCH, J., Longueurs extrémales et théorie des fonctions. *Comment. Math. Helv.*, 29 (1955), 301–337.

[5]. LAVRENTIEFF, M. A., Sur la continuité des fonctions univalentes. *C. R. Acad. Sci. U.R.S.S.*, 4 (1936), 215–217.

[6]. LEHTO, O. & VIRTANEN, K. I., *Quasikonforme Abbildungen*. Springer Verlag, Berlin, 1965.

[7]. LELONG-FERRAND, J., *Représentation conforme et transformations à intégrale de Dirichlet bornée*. Gauthier-Villars, Paris, 1955.

[8]. MAZURKIEWICZ, S., Über die Definition der Primenden. *Fund. Math.*, 26 (1936), 272–279.

[9]. MCMILLAN, J. E., Boundary behavior of a conformal mapping. *Acta Math.*, 123 (1969), 43–67.

[10]. POMMERENKE, CH., Linear-invariante Familien analytischer Funktionen I. *Math. Ann.*, 155 (1964), 108–154.

[11]. RICKMAN, S., Boundary correspondence under quasiconformal mappings of Jordan domains. *J. Math. Mech.*, 18 (1968), 429–432.

[12]. —— Quasiconformally equivalent curves. *Duke Math. J.*, 36 (1969), 387–400.

[13]. STREBEL, K., On the maximal dilatation of quasiconformal mappings. *Proc. Amer. Math. Soc.*, 6 (1955), 903–909.

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