TAUBERIAN THEOREMS FOR MULTIVALENT FUNCTIONS

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1. Introduction and background

Let
$$f(z)=\sum_{0}^{\infty} a_n z^n \tag{1.1}$$

be regular in $|z|<1$. If $f(z)$ has bounded characteristic in $|z|<1$ then it follows from classical theorems of Fatou that the Abel limit

$$f(e^{i\theta})=\lim_{r\to 1} f(re^{i\theta}) \tag{1.2}$$

exists p.p. in θ. In particular this condition is satisfied if $f(z)$ is mean p-valent in $|z|<1$ for some p.

In this paper we investigate under what conditions the power series (1.1) is summable by a Cesàro mean or is convergent at those points $e^{i\theta}$ where the Abel limit exists. It is classical that the existence of a Cesàro sum for $f(e^{i\theta})$ always implies the existence of the Abel limit (1.2) and in fact the existence of an angular limit.

The above problem was recently investigated by G. Halász [3] for univalent functions and certain subclasses of these functions. We define the αth Cesàro sums by

$$\sigma_N^{(\alpha)}(\theta)=\binom{\alpha+N}{N}^{-1}\sum_{n=0}^{N}\binom{\alpha+N-n}{N-n}a_n e^{in\theta}, \tag{1.3}$$

where $\alpha>-1$. Then Halász proved the following results.

THEOREM A. *If f is univalent in $|z|<1$ and* (1.2) *holds then, if $\alpha>2$*

$$\sigma_N^{(\alpha)}(\theta)\to f(e^{i\theta}); \tag{1.4}$$

also
$$\sigma_N^{(1)}(\theta)=O(\log N), \quad \text{as } N\to\infty. \tag{1.5}$$

It is a classical result that (1.4) implies that

$$|a_n|=o(n^{\alpha}), \quad \text{as } n\to\infty. \tag{1.6}$$

Thus (1.4) is false in general for $\alpha \leqslant 1$, when $f(z)$ is univalent. For certain subclasses of univalent function Halász was able to extend (1.4). He defined a certain class of *admissible* domains, which include star-like domains and for whose exact definition we refer the reader to [3].

Halász then proved the following results.

THEOREM B. *If $f(z)$ maps $|z|<1$ onto an admissible domain, then* (1.2) *implies*

$$\sigma_N^{(1)}(\theta)=O(1).$$

THEOREM C. *If $f(z)$ maps $|z|<1$ onto a star-like domain and $a_n \to 0$, then* (1.2) *implies that* (1.1) *converges to $f(e^{i\theta})$ for $z=e^{i\theta}$.*

THEOREM D. *The same conclusion holds if $f(z)$ maps $|z|<1$ onto an admissible domain which, for some positive δ and all large R, contains no disk with centre on $|w|=R$ and radius $(\frac{1}{2}-\delta)R$.*

2. Statement of positive results

In this paper we investigate further some questions raised by the above results. It is convenient to consider the more general class of mean p-valent functions. Results for univalent functions then arise from the special case $p=1$. In particular we can remove the hypothesis on admissible domains from the theorems of Halász and strengthen the conclusions in some of them.

We note following Halász [3] that (1.4) implies not only (1.6) but also

$$|f(z)-f(e^{i\theta})|=o\left(\frac{|e^{i\theta}-z|}{1-|z|}\right)^{\alpha+1} \tag{2.1}$$

as $|z|>1$ in any manner from $|z|<1$. From the weaker condition

$$\sigma_N^{(\alpha)}(\theta)=O(1), \tag{2.2}$$

we deduce similarly that

$$|a_n|=O(n^\alpha) \tag{2.3}$$

and

$$|f(z)|=O\left(\frac{|e^{i\theta}-z|}{1-|z|}\right)^{\alpha+1} \tag{2.4}$$

as $|z|\to 1$ in any manner, and in particular

$$f(re^{i\theta})=O(1), \quad 0<r<1. \tag{2.5}$$

If $f(z)$ is mean p-valent in $|z|<1$ then $f(z)$ satisfies (2.3) with $\alpha=2p-1$, if $p>\frac{1}{4}$[1] and (1.6) with $\alpha=-\frac{1}{2}$, if $p<\frac{1}{4}$,[2] and not in general any stronger result than this[3]. Thus (2.2) is false in general for $\alpha<\max(-\frac{1}{2}, 2p-1)$.

Our basic result shows that for mean p-valent functions the above implications are 'almost' reversible if $\alpha>-\frac{1}{2}$.

THEOREM 1. *Suppose that $\alpha>-\frac{1}{2}$ and that $f(z)$, given by* (1.1), *is mean p-valent in a neighbourhood $N_\delta(\theta)=\{z|\ |z|<1$ and $|z-e^{i\theta}|<2\delta\}$ of $z=e^{i\theta}$ in $|z|<1$ for some positive δ, p. Then if* (2.3) *holds and for some $\varepsilon>0$*

$$|f(z)|=O\left(\frac{|e^{i\theta}-z|}{1-|z|}\right)^{1+\alpha-\varepsilon}, \quad \text{as } |z|\to 1 \text{ in } N_\delta(\theta) \tag{2.4'}$$

we have (2.2), *and if* (1.6) *holds and*

$$|f(z)-f(e^{i\theta})|=o\left(\frac{|e^{i\theta}-z|}{1-|z|}\right)^{1+\alpha-\varepsilon}, \quad \text{as } |z|\to 1 \text{ in } N_\delta(\theta) \tag{2.1'}$$

we have (1.4).

If p is sufficiently small we can set $\varepsilon=0$. We have in fact

THEOREM 2. *If $p\leq\frac{1}{2}(1+\alpha)$ in Theorem 1, then* (2.5) *and* (2.3) *imply* (2.2) *and* (1.6) *and* (1.2) *imply* (1.4).

If $p>\frac{1}{2}(1+\alpha)$ a simple supplementary condition is needed. We have

THEOREM 3. *If $p>\frac{1}{2}(1+\alpha)$ in Theorem 1 and if for some constants $c<1+\alpha$ and $R>0$, we have*

$$\left|\frac{f'(z)}{f(z)}\right|\leq\frac{c}{1-|z|} \tag{2.6}$$

for all $z\in N_\delta(\theta)$ with $|f(z)|\geq R$, then again (2.5) *and* (2.3) *imply* (2.2) *and* (1.6) *and* (1.2) *imply* (1.4).

We shall show that (2.5) and (2.6) imply (2.4′) and (1.2) and (2.6) imply (2.1′) if $1+\alpha-\varepsilon>c$. Thus Theorem 3 is a simple consequence of Theorem 1. Again if $f(z)$ is mean p-valent in $N_\delta(\theta)$ then (2.5) implies (2.4′) and (1.2) implies (2.1′) if $1+\alpha-\varepsilon<2p$. Thus Theorem 2 also follows from Theorem 1 except in the case when $p=\frac{1}{2}(1+\alpha)$, which is more delicate.

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(1) [4, Theorem 3.5, p. 50]; this book is subsequently referred to as M.F.

(2) Pommerenke [5].

(3) [M.F., p. 49].

Using essentially Koebe's Theorem (as in M.F. Theorem 3.6, p. 51) we can see that the hypotheses of Theorem D imply (2.6) with $c<1$, so that Theorem D, without the assumption that the domain is admissible, follows from Theorem 3.

The above results also contain the other theorems of Halász, when we set $p=1$. In fact if $f(z)$ is univalent in $|z|<1$ and (1.2) holds then by a classical theorem we have (2.3) with $\alpha=1$. Thus Theorem 2 shows that $\sigma_N^{(1)}(\theta)=O(1)$ and that for $\alpha>1$ $\sigma_N^{(\alpha)}(\theta)\to f(e^{i\theta})$, as $N\to\infty$, which sharpens Theorem A. Also Theorem B is contained in Theorem 2 without the additional assumption that the domain is admissible.

Next under the hypotheses of theorem C it can be shown that (2.6) holds for some $c<1$, when r is sufficiently near 1. Thus Theorem C follows from Theorem 3.

For mean p-valent functions in the whole of $|z|<1$ our conclusions may be stated simply as follows.

THEOREM 4. *If $f(z)$ is mean p-valent in $|z|<1$, with $p>\frac{1}{4}$ then* (2.5) *implies* (2.2) *for $\alpha\geqslant 2p-1$ and* (1.2) *implies* (1.4) *for $\alpha>2p-1$ and for $\alpha=2p-1$ if in addition* (1.6) *holds.*

For in this case we always have (2.3) if $\alpha\geqslant 2p-1$, and hence (1.6) if $\alpha>2p-1$, as we remarked previously, so that Theorem 4 follows from Theorem 2.

2.1. A classical result of Fejér states that if $f(z)$ maps $|z|<1$ onto a Riemann-surface of finite area, so that

$$\sum_1^\infty n|a_n|^2<\infty, \tag{2.7}$$

then (1.2) implies (1.4) with $\alpha=0$, so that the series for $f(e^{i\theta})$ converges. As an easy consequence of Theorem 4 we have

THEOREM 5. *If $f(z)$ is given by* (1.1) *and* (2.7) *holds, then* (1.2) *implies* (1.4) *for $\alpha>-\frac{1}{2}$.*

We define $p=\frac{1}{2}(1+\alpha)$. It is enough to prove that for some value of w_0, $f(z)+w_0$ is mean p-valent under the hypotheses of Theorem 5. Suppose that this is false whenever $|w_0|\leqslant\varrho$ say. Then, for $|w_0|\leqslant\varrho$, there exists $R>0$, such that the area of the part of the image of $|z|<1$ by $f(z)$, which lies over the disk $|w-w_0|<R$, is at least πpR^2. The corresponding disks $|w-w_0|<R$, for varying w_0 cover $|w_0|\leqslant\varrho$ and hence by the Heine-Borel Theorem a finite subset of these disks

$$|w-w_\nu|<R_\nu, \qquad \nu=1 \text{ to } N$$

say has the same property. By a standard argument we can select a subsystem of these disks, which we may relabel

$$|w-w_\nu|<R_\nu, \qquad \nu=1 \text{ to } M,$$

which are disjoint and whose total area is at least $\frac{1}{9}$ the area of the union of the original disks and so at least $p\pi\varrho^2/9$. The total area of the image of $f(z)$ over these disks is at least $p\pi\varrho^2/9$, and this gives the required contradiction to (2.7), if ϱ is large enough. Thus Theorem 5 follows from Theorem 4.

3. Counterexamples

The above results are essentially best possible. Firstly no hypotheses of the type we have considered above will imply (C, α) summability for $\alpha \leqslant -\frac{1}{2}$. We have

THEOREM 6. *There exists $f(z)$ satisfying (2.7), having positive coefficients and continuous in $|z|<1$ such that $\sigma_N^{-(\frac{1}{2})}(0)$ is unbounded.*

Given $p>0$, we remark as in the previous section that $f(z)+w_0$ is mean p-valent for suitable w_0 so that Theorem 4 is false for any p however small if $\alpha = -\frac{1}{2}$.

It is also natural to ask whether we can take $\alpha < 2p-1$ in Theorem 4 if the coefficients are small enough. That this is false is shown by

THEOREM 7. *If $-\frac{1}{2}<\alpha<2p-1$, there exists $f(z)$ mean p-valent (even in the stricter circumferential sense*(1)*), in $|z|<1$, taking no value more than q times if $q \geqslant p$, and such that (1.6) holds and (2.2) is false for every real θ.*

In particular by choosing $\alpha=0$, $\frac{1}{2}<p<1$, we obtain a univalent function whose coefficients tend to zero and whose power series diverges everywhere on $|z|=1$. This answers in the negative a problem raised elsewhere [2].

The coefficients in this example must tend to zero rather slowly. If e.g. $|a_n| = O\,(\log n)^{-2-\delta}$, where $\delta>0$, then we deduce that

$$M(r, f) = O(1-r)^{-1}\left(\log\frac{1}{1-r}\right)^{-2-\delta}$$

and hence, if $f(z)$ is mean p-valent for some finite p, we can show (2) that

$$\int |f(re^{i\theta})|\,(\log^+|f(re^{i\theta})|)^\lambda\, d\theta = O(1)$$

for $1<\lambda<1+\delta$. But now it follows from a recent extension by Sjölin [6] of a theorem of Carleson [1], that the series for $f(e^{i\theta})$ converges p.p. in θ. (This observation was made by Professor Clunie.)

(1) M.F. p. 94.

(2) By a method similar to M.F. p. 42 et seq.

The remainder of the paper is divided into two parts. In the first part we shall prove Theorems 1 and 2, followed by Theorem 3 which is an easy deduction from Theorem 1. In the second part we construct the examples needed for Theorems 6 and 7.

I. Proofs of Theorems 1 to 3

4. Localisation

In this section we show how to reduce the problem of summability for the series Σa_n to the behaviour of the function $f(z)$ in a neighbourhood of $z=1$. The method is due to W. H. Young [8] (see also [7, p. 218]). We assume, as we may do, that $\theta=0$, since otherwise we can consider $f(ze^{i\theta})$ instead of $f(z)$.

We shall denote by B constants depending on the function $f(z)$ and possibly on δ and α but not on r or N, not necessarily the same each time they occur. Particular constants will be denoted by $B_1, B_2, \ldots$ etc.

LEMMA 1. *If $f(z)$ is given by (1.1) and $\sigma_N^{(\alpha)}=\sigma_N^{(\alpha)}(0)$ by (1.3) for $\alpha>-1$, then we have for $N\geqslant 4$, $1-2/N\leqslant r\leqslant 1-1/N$, $\delta>0$ and any complex w*

$$|\sigma_N^{(\alpha)}-w|\leqslant BN^{-(\alpha+1)}\left\{\int_{-\delta}^{\delta}\frac{|f(re^{i\theta})-w|\,d\theta}{|1-re^{i\theta}|^{\alpha+2}}+\int_{-\delta}^{\delta}\frac{|f'(re^{i\theta})|\,d\theta}{|1-re^{i\theta}|^{\alpha+1}}\right\}+\varepsilon_N, \tag{4.1}$$

where $\varepsilon_N=O(1)$ or $\varepsilon_N=o(1)$ as $N\to\infty$, according as (2.3) or (1.6) holds.

We note that $$(1-z)^{-(\alpha+1)}f(z)=\sum_0^\infty\binom{N+\alpha}{N}\sigma_N^{(\alpha)}z^N,$$

$$(1-z)^{-(\alpha+1)}f'(z)+(\alpha+1)(1-z)^{-(\alpha+2)}f(z)=\sum_0^\infty N\binom{N+\alpha}{N}\sigma_N^{(\alpha)}z^{N-1}.$$

Thus

$$N\binom{N+\alpha}{N}(\sigma_N^{(\alpha)}-w)=\frac{1}{2\pi r^{N-1}}\int_{-\pi}^{\pi}\left\{\frac{f'(re^{i\theta})}{(1-re^{i\theta})^{\alpha+1}}+\frac{(\alpha+1)(f(re^{i\theta})-w)}{(1-re^{i\theta})^{\alpha+2}}\right\}e^{-i(N-1)\theta}\,d\theta. \tag{4.2}$$

We now choose the integer h, so that $h>\alpha+2$, and for $j=1, 2$ introduce the functions $\phi_j(\theta)=\phi_j(\theta,r,\delta)$ to satisfy the following conditions

(i) $\phi_1(\theta)=(1-re^{i\theta})^{-(\alpha+1)}$, $\phi_2(\theta)=(\alpha+1)(1-re^{i\theta})^{-(\alpha+2)}$, $-\pi<\theta<-\delta$ and $\delta<\theta<\pi$.

(ii) $\phi_j(\theta), \phi_j'(\theta), \ldots, \phi_j^{(h)}(\theta)$ are continuous and bounded by B for $-\pi<\theta<\pi$.

In order to satisfy (ii) we define $\phi_j(\theta)$ to be a polynomial of degree $2h+1$ in θ. This polynomial can be uniquely chosen so that $\phi_j^{(\nu)}(\theta)$ assumes preassigned values at $\theta=\pm\delta$

and $\nu=0$ to h. If the values are chosen so as to make $\phi_j^{(\nu)}(\theta)$ continuous at $\pm\delta$, subject to (i), all the values are bounded by B and hence so are the $\phi_j(\theta)$ and their first h derivatives for $|\theta|<\delta$. Thus with this definition (ii) holds.

With this definition we have

$$2\pi r^{N-1} N \binom{N+\alpha}{N} (\sigma_N^{(\alpha)} - w) = \int_{-\pi}^{\pi} \{\phi_1(\theta) f'(re^{i\theta}) + \phi_2(\theta)(f(re^{i\theta}) - w)\} e^{-i(N-1)\theta} d\theta$$
$$+ \int_{-\delta}^{\delta} \left\{ \frac{f'(re^{i\theta})}{(1-re^{i\theta})^{\alpha+1}} + \frac{(\alpha+1)(f(re^{i\theta})-w)}{(1-re^{i\theta})^{\alpha+2}} \right\} e^{-i(N-1)\theta} d\theta \quad (4.3)$$
$$- \int_{-\delta}^{\delta} \{\phi_1(\theta) f'(re^{i\theta}) + \phi_2(\theta)(f(re^{i\theta}) - w)\} e^{-i(N-1)\theta} d\theta.$$

Clearly

$$\left| \int_{-\delta}^{\delta} \{\phi_1(\theta) f'(re^{i\theta}) + \phi_2(\theta)(f(re^{i\theta}) - w)\} e^{-i(N-1)\theta} d\theta \right| \leqslant B \left| \int_{-\delta}^{\delta} \{|f'(re^{i\theta})| + |f(re^{i\theta}) - w|\} d\theta \right|$$
$$\leqslant B \int_{-\delta}^{\delta} \left\{ \frac{|f'(re^{i\theta})|}{|1-re^{i\theta}|^{\alpha+1}} + \frac{|f(re^{i\theta}) - w|}{|1-re^{i\theta}|^{\alpha+2}} \right\} d\theta.$$

A similar bound applies to the second integral on the right-hand side of (4.3). Also in view of our choice of r, we see that

$$\left\{ r^{N-1} N \binom{N+\alpha}{N} \right\}^{-1} \leqslant BN^{-(\alpha+1)}.$$

Thus to complete the proof of Lemma 1, it is sufficient to show that (4.4)

$$\int_{-\pi}^{\pi} \{\phi_1(\theta) f'(re^{i\theta}) + \phi_2(\theta)(f(re^{i\theta}) - w)\} e^{-i(N-1)\theta} d\theta = N^{\alpha+1} \varepsilon_N,$$

where ε_N satisfies the conditions of Lemma 1.

To see this we expand $f(z)$ and $f'(z)$ in terms of the power series (1.1) and integrate term by term. This gives

$$N^{\alpha+1} \varepsilon_N = \sum_{m=0}^{\infty}{}' a_m r^{m-1} \left\{ m \int_{-\pi}^{\pi} \phi_1(\theta) e^{i(m-N)\theta} d\theta + r \int_{-\pi}^{\pi} \phi_2(\theta) e^{i(m+1-N)\theta} d\theta \right\},$$

where the dash indicates that a_0 is to be replaced by $a_0 - w$.

In view of (ii) we may integrate by parts h times to obtain for $m \neq N$

$$\left| \int_{-\pi}^{\pi} \phi_j(\theta) e^{i(m-N)\theta} d\theta \right| = \left| \{i(m-N)\}^{-h} \int_{-\pi}^{\pi} \phi_j^{(h)}(\theta) e^{i(m-N)\theta} d\theta \right| \leqslant \frac{B}{|m-N|^h}.$$

This gives $$|N^{\alpha+1}\varepsilon_N| \leqslant B\left\{N|a_N| + |a_{N-1}| + \sum_{m\neq N} \frac{m|a_m|}{|m-N|^h}\right\}.$$

Suppose first that (2.3) holds. Then setting $|m-N|=\nu$, we have

$$\sum_{\substack{m\leqslant 2N\\ m\neq N}} \frac{m|a_m|}{|m-N|^h} \leqslant B(2N)^{\alpha+1}\sum_1^\infty \nu^{-h} \leqslant BN^{\alpha+1},$$

while $$\sum_{m\geqslant 2N} \frac{m|a_m|}{|m-N|^h} \leqslant B\sum_{m=2N}^\infty m^{\alpha+1-h} = O(1).$$

Thus $\varepsilon_N = O(1)$ in this case. Next if (1.6) holds, we have for $m=N+\nu$, if $1\leqslant|\nu|\leqslant\frac{1}{2}N$ and N is large, $m|a_m|<\varepsilon N^{\alpha+1}$. Thus

$$\sum_{\substack{\frac{1}{2}N\leqslant m\leqslant\frac{3}{2}N\\ m\neq N}} \frac{m|a_m|}{|m-N|^h} < 2\varepsilon N^{\alpha+1}\sum_1^\infty \nu^{-h} < B\varepsilon N^{\alpha+1},$$

while $$\sum_{|m-N|\geqslant\frac{1}{2}N} \frac{m|a_m|}{|m-N|^h} < B\sum_1^\infty m^{\alpha+1-h} = O(1),$$

so that $\varepsilon_N = o(1)$ in this case. This proves Lemma 1.

5. Preliminary estimates

We now assume that $f(z)$ satisfies the hypotheses of Theorem 1. We set $w=f(e^{i\theta})$ if (2.1′) holds and otherwise set $w=0$ in (4.1). We then suppose that $N>2\delta^{-1}$, so that for $z=re^{i\theta}$, where $|\theta|<\delta$, $r>1-2N^{-1}$, we have $|z-1|<2\delta$.

For any positive integer n, we define

$$R_n = 2^n. \tag{5.1}$$

We take for n_0 any positive integer for which $R_{n_0}>2|w|$. Also for $n\geqslant n_0$ we define E_n to be the set of all $z=re^{i\theta}$, such that

$$1-2N^{-1}<r<1-N^{-1}, \qquad |\theta|<\delta \tag{5.2}$$

and in addition $|f(z)|<R_{n_0}$, if $n=n_0$; and $R_{n-1}\leqslant|f(z)|<R_n$ if $n>n_0$. Thus the sets E_n for different n are disjoint and $E=\bigcup_{n=n_0}^\infty E_n$ is the whole set satisfying (5.2). We integrate both sides of (4.1) with respect to rdr from $r=1-2N^{-1}$ to $1-N^{-1}$, and deduce that

$$\begin{aligned} |\sigma_N^{(\alpha)}-w| &\leqslant BN^{-\alpha}\int_E\left\{\frac{|f(re^{i\theta})-w|}{|1-re^{i\theta}|^{\alpha+2}} + \frac{f'(re^{i\theta})}{|1-re^{i\theta}|^{\alpha+1}}\right\} r\,dr\,d\theta + \varepsilon_N \\ &= BN^{-\alpha}\sum_{n=n_0}^\infty \int_{E_n}\left\{\frac{|f(re^{i\theta})-w|}{|1-re^{i\theta}|^{\alpha+2}} + \frac{|f'(re^{i\theta})|}{|1-re^{i\theta}|^{\alpha+1}}\right\} r\,dr\,d\theta + \varepsilon_N, \end{aligned} \tag{5.3}$$

where ε_N satisfies the same conditions as in Lemma 1.

The cases $n=n_0$ and $n>n_0$ will be treated slightly differently. We have first

LEMMA 2. *If $n > n_0$, we have*

$$\int_{E_n}\left\{\frac{|f(re^{i\theta})-w|}{|1-re^{i\theta}|^{\alpha+2}}+\frac{|f'(re^{i\theta})|}{|1-re^{i\theta}|^{\alpha+1}}\right\} r\,dr\,d\theta < BI_n^{\frac{1}{2}}, \tag{5.4}$$

where

$$I_n = \int_{E_n}\frac{|f(re^{i\theta})|^2 r\,dr\,d\theta}{|1-re^{i\theta}|^{2\alpha+2}}. \tag{5.5}$$

We have in E_n, $|w| < R_{n_0} < |f(re^{i\theta})|$, so that

$$|f(re^{i\theta})-w| < 2|f(re^{i\theta})|.$$

Thus by Schwarz's inequality

$$\int_{E_n}\frac{|f(re^{i\theta})-w|\,r\,dr\,d\theta}{|1-re^{i\theta}|^{\alpha+2}} \leqslant 2\int_{E_n}\frac{|f(re^{i\theta})|}{|1-re^{i\theta}|^{\alpha+2}}\,r\,dr\,d\theta \leqslant 2\left(\frac{|f(re^{i\theta})|^2 r\,dr\,d\theta}{|1-re^{i\theta}|^{2\alpha+2}}\right)^{\frac{1}{2}}\left(\int_{E_n}\frac{r\,dr\,d\theta}{|1-re^{i\theta}|^2}\right)^{\frac{1}{2}}. \tag{5.6}$$

Also
$$\int_{E_n}\frac{r\,dr\,d\theta}{|1-re^{i\theta}|^2} \leqslant \int_{1-(2/N)}^{1-(1/N)} r\,dr\int_{-\pi}^{\pi}\frac{d\theta}{|1-re^{i\theta}|^2} \leqslant \int_{1-(2/N)}^{1-(1/N)} r\,dr\,\frac{B}{1-r} \leqslant B.$$

Again

$$\int_{E_n}\frac{|f'(re^{i\theta})|}{|1-re^{i\theta}|^{\alpha+1}}\,r\,dr\,d\theta \leqslant \left(\int_{E_n}\left|\frac{f'}{f}\right|^2 r\,dr\,d\theta\right)^{\frac{1}{2}}\left(\int_{E_n}\frac{|f(re^{i\theta})|^2}{|1-re^{i\theta}|^{2\alpha+2}}\,r\,dr\,d\theta\right)^{\frac{1}{2}}. \tag{5.7}$$

Also since $f(z)$ is mean p-valent in E_n, we have

$$\int_{E_n}\left|\frac{f'}{f}\right|^2 r\,dr\,d\theta \leqslant \frac{1}{R_{n-1}^2}\int_{E_n}|f'|^2\,r\,dr\,d\theta \leqslant \frac{\pi p R_n^2}{R_{n-1}^2} = 4\pi p.$$

Now (5.4) follows from (5.6) and (5.7).

6. Estimates for $f(z)$ near $z=1$

In order to estimate I_n and the integrals corresponding to $n=n_0$ in (5.3) we need to use more strongly the fact that $f(z)$ is mean p-valent. We start by quoting the following result (M.F. Theorem 2.6, p. 32).

LEMMA 3. *Suppose that $f(z)$ is mean p-valent in a domain Δ containing k non-overlapping circles $|z-z_\nu| < r_\nu$, $1 \leqslant \nu \leqslant k$. Suppose further that $|f(z_\nu)| \leqslant \varrho_1$, $|f(z'_\nu)| \geqslant \varrho_2 > e\varrho_1$, where*

$$\delta_\nu = \frac{r_\nu - |z'_\nu - z_\nu|}{r_\nu} > 0,$$

and that $f(z) \neq 0$ for $|z-z_\nu| < \frac{1}{2} r_\nu$, $1 \leqslant \nu \leqslant k$. Then

$$\sum_{\nu=1}^{k}\left[\log\frac{A(p)}{\delta_\nu}\right]^{-1} < \frac{2p}{\log(\varrho_2/\varrho_1)-1},$$

where $A(p)$ is a constant depending on p only.

We have next

LEMMA 4. *Suppose that $f(z)$ is mean p-valent in $|\arg z| < 2\delta$, $1-\delta < |z| < 1$. Then if*

$$f(r) = O(1), \qquad 1-\delta < r < 1 \tag{6.1}$$

we have

$$f(z) = O\left(\frac{|1-z|}{1-|z|}\right)^{2p} \tag{6.2}$$

uniformly as $|z| \to 1$ for $|\arg z| < \delta$. If further

$$f(r) \to w_0, \quad as \quad r \to 1 \tag{6.3}$$

then we have

$$f(z) = w_0 + o\left(\frac{|1-z|}{1-|z|}\right)^{2p}, \tag{6.4}$$

and

$$f'(z) = o\,\frac{(|1-z|)^{2p}}{(1-|z|)^{2p+1}}, \tag{6.5}$$

uniformly as $z \to 1$ from $|z| < 1$.

Since $f(z)$ is mean p-valent for $|\arg z| < 2\delta$, $1-\delta < |z| < 1$, $f(z)$ has at most p zeros there (M. F. p. 25). Thus we may assume that $f(z) \neq 0$ for $r_0 < |z| < 1$, $|\arg z| < 2\delta$ when r_0 is sufficiently near 1. We assume also that $r_0 > 1-\delta$.

Suppose now that $\frac{1}{2}(1+r_0) < r < 1$, $|\theta| < \delta$, $z_1 = re^{i\theta}$, and $|f(z_1)| < \varrho_1$. We apply Lemma 3, with $k=1$, $r_1 = 1-r$,

$$\delta_1 = \frac{1-r-|z_1'-z_1|}{1-r}.$$

This shows that if $|z_1'-z_1| < 1-r$, and $|f(z_1')| = \varrho_2$, we have

$$\varrho_2 < e\varrho_1 \left(\frac{A(p)}{\delta_1}\right)^{2p}. \tag{6.6}$$

In particular if $z_1' = re^{i\theta'}$, where $|\theta'-\theta| < \frac{1}{2}(1-r)$, we deduce that $|f(re^{i\theta'})| < A_1(p)|f(re^{i\theta})|$. By repeating the argument a finite number of times we deduce that if $|\theta'-\theta| < K(1-r)$, $|\theta| < \delta$, $|\theta'| < \delta$, where K is a fixed positive constant then

$$|f(re^{i\theta'})| < K_1 |f(re^{i\theta})|,$$

where K_1 is a constant depending on K and p only. In view of (6.1) we deduce that

$$|f(re^{i\theta})| = O(1), \quad \tfrac{1}{2}(1+r_0) \leq r < 1, \quad |\theta| < \min\{\delta, K(1-r)\}, \tag{6.7}$$

where K is a fixed constant. Thus (6.2) holds under these hypotheses. We now define K by $K(1-r_0) = 2\delta$. Then for

$$\tfrac{1}{2}(1+r_0)<r<1, \qquad K(1-r)<|\theta|<\delta,$$

we define r_1 by
$$K(1-r_1)=|\theta|,$$

and set $z_1=r_1e^{i\theta}$, $z_1'=re^{i\theta}$, $\delta_1=(1-r)/(1-r_1)$. Then (6.6) yields

$$|f(z_1')|<A(p)|f(z_1)|\left(\frac{1-r_1}{1-r}\right)^{2p}<O\left(\frac{|\theta|}{1-r}\right)^{2p}=O\left(\frac{|1-z_1'|}{1-|z_1'|}\right)^{2p},$$

since $|f(z_1)|$ is uniformly bounded by (6.7). This completes the proof of (6.2).

Suppose next that $f(z)$ satisfies (6.3). Then, in view of (6.7) $f(z)$ is uniformly bounded as $z\to 1$ in any fixed angle $|\arg(1-z)|<\pi/2-\varepsilon$, for a fixed positive ε. Also (6.3) shows that

$$f(z)\to w_0 \tag{6.8}$$

as $z\to 1$ through real positive values. Hence in view of Montel's Theorem we deduce that (6.8) holds uniformly as $z\to 1$ in $|\arg(1-z)|<\pi/2-\varepsilon$ for a fixed positive ε, and so as $z=re^{i\theta}$ and $r\to 1$ while

$$|\theta|\leqslant K(1-r) \tag{6.9}$$

for any fixed positive K. In particular (6.4) holds as $z\to 1$ in the range (6.9).

Suppose next that

$$K(1-r)\leqslant|\theta|\leqslant\delta, \tag{6.10}$$

where K is a large fixed positive number. We define r_1 by $r_1=1-|\theta|/K$ and set $z_1=r_1e^{i\theta}$, $z_1'=re^{i\theta}$, $\delta_1=(1-r)/(1-r_1)$. Then (6.6) yields

$$|f(z_1')|<A(p)|f(z_1)|\left(\frac{1-r_1}{1-r}\right)^{2p}<A(p)(|w_0|+1)K^{-2p}\left(\frac{|\theta|}{1-r}\right)^{2p}, \quad r_1>r_1(K).$$

For we may apply (6.8) with $z_1=r_1e^{i\theta}$ instead of z. Given $\varepsilon>0$, we choose K so large that $A(p)(|w_0|+1)K^{-2p}<\varepsilon$. Then we deduce that for $|\theta|<\theta_0(\varepsilon,K)$ and $K(1-r)\leqslant|\theta|\leqslant\delta$, we have $|f(re^{i\theta})|<\varepsilon|\theta/(1-r)|^{2p}$.

This gives

$$|f(re^{i\theta})-w_0|<|w_0|+\varepsilon\left(\frac{|\theta|}{1-r}\right)^{2p}<(\varepsilon+K^{-2p}|w_0|)\left(\frac{|\theta|}{1-r}\right)^{2p}<2\varepsilon\left(\frac{|\theta|}{1-r}\right)^{2p},$$

$$|f(re^{i\theta})-w_0|<3\varepsilon\left(\frac{|1-z|}{1-r}\right)^{2p}, \quad r'<r<1,\, K(1-r)\leqslant|\theta|\leqslant\delta,$$

provided that r' is sufficiently near 1 and K is large enough. In view of what we have already proved it follows that for some $r'=r'(\varepsilon)<1$ and $\theta_0(\varepsilon)>0$, we have for $r'<r<1$, $0\leqslant|\theta|\leqslant\theta_0(\varepsilon)$, $z=re^{i\theta}$

$$|f(re^{i\theta})-w_0|<3\,\varepsilon\left(\frac{|1-z|}{1-r}\right)^{2p}.$$

This proves (6.4). Finally we have from Cauchy's inequality

$$|f'(z)|\leqslant\frac{1}{\varrho}\sup_{|\xi-z|\leqslant\varrho}|f(\xi)-w_0|.$$

Setting $\varrho=\frac{1}{2}(1-|z|)$ and using (6.4) we deduce (6.5). This completes the proof of Lemma 4.

6.1. Our next application of Lemma 3 will be needed for the proof of Theorem 2. It involves the case $k=2$.

LEMMA 5. *Suppose that $f(z)$ is mean p-valent in $|\arg z|<2\delta$, $1-\delta<|z|<1$, where $0<\delta<1$ and that* (6.1) *holds. Suppose further that for $j=1, 2$ we have*

$$1-\frac{2}{N}\leqslant r_j\leqslant 1-\frac{1}{N},\quad \varrho_j=|f(r_j e^{i\phi_j})|$$

and

$$\frac{16}{N}<4|\phi_1|\leqslant|\phi_2|\leqslant\tfrac{1}{2}\delta,\quad \varrho_2\geqslant\varrho_1.$$

Then

$$\varrho_1^{\frac{3}{2}}\varrho_2^{\frac{1}{2}}\leqslant BN^{2p}|\phi_1|^p|\phi_2|^p.$$

The estimate of Lemma 5 will be used to show that the order of magnitude implied by (6.2) cannot be attained at more than a bounded number of points on $|z|=r$ which are not too close to each other.

We set

$$z_j=(1-\tfrac{1}{4}|\phi_j|)\,e^{i\phi_j},\quad j=1,2,\quad R=\max_{j=1,2}|f(z_j)|.$$

Then the disks

$$|z-z_j|<\tfrac{1}{4}|\phi_j|,\quad j=1,2$$

are disjoint, since

$$|z_2-z_1|>(1-\tfrac{1}{4}|\phi_1|)\,|\sin(\phi_2-\phi_1)|>\left(1-\frac{\delta}{8}\right)\frac{2}{\pi}(|\phi_2|-|\phi_1|)>\frac{|\phi_1|}{4}+\frac{|\phi_2|}{4}.$$

It follows from (6.2) that $R\leqslant B$. If $\varrho_1\leqslant eR$, then we deduce further from (6.2) that

$$\varrho_1^{\frac{3}{2}}\varrho_2^{\frac{1}{2}}\leqslant B\left(\frac{|\phi_2|}{1-r_2}\right)^p\leqslant BN^{2p}|\phi_1|^p|\phi_2|^p,$$

so that Lemma 5 holds in this case. Thus we assume that $\varrho_1>eR$. We then define r' to be the smallest number such that

$$1-\frac{|\phi_2|}{4}\leqslant r'\leqslant r_2\ \text{ and }\ |f(r'e^{i\phi_2})|=\varrho_1.$$

We then set $z_1' = r_1 e^{i\phi_1}$, $z_2' = r' e^{i\phi_2}$ and apply Lemma 3 with R, ϱ_1 instead of ϱ_1, ϱ_2 and

$$\delta_1 = \frac{4(1-r_1)}{|\phi_1|}, \qquad \delta_2 = \frac{4(1-r')}{|\phi_2|}.$$

This yields

$$\frac{2p}{\log(\varrho_1/eR)} \geqslant \left\{\log\left(\frac{A(p)}{\delta_1}\right)\right\}^{-1} + \left\{\log\left(\frac{A(p)}{\delta_2}\right)\right\}^{-1} \geqslant 4/\log\left(\frac{A(p)}{\delta_1}\cdot\frac{A(p)}{\delta_2}\right).$$

i.e.
$$\varrho_1 \leqslant A(p)\frac{R}{(\delta_1\delta_2)^{p/2}} \leqslant B\left\{\frac{|\phi_1||\phi_2|}{(1-r_1)(1-r')}\right\}^{p/2}.$$

We next apply (6.6) with z_2', $re^{i\phi_2}$ instead of z_1, z_1' and deduce

$$\varrho_2 < B\varrho_1\left(\frac{1-r'}{1-r_2}\right)^{2p}.$$

Thus
$$\varrho_1^{\frac{3}{2}}\varrho_2^{\frac{1}{2}} = \varrho_1^2\left(\frac{\varrho_2}{\varrho_1}\right)^{\frac{1}{2}} \leqslant B\left\{\frac{|\phi_1||\phi_2|}{(1-r_1)(1-r')}\right\}^p\left(\frac{1-r'}{1-r_2}\right)^p$$
$$= B\frac{|\phi_1|^p|\phi_2|^p}{(1-r_1)^p(1-r_2)^p},$$

which yields Lemma 5.

6.2. We continue to suppose that $f(z)$ is mean p-valent and $f(z) \neq 0$, $|\arg z| < 2\delta$, $r_0 \leqslant |z| < 1$, where $r_0 > 1-\delta$. In addition we now suppose that

$$|f(re^{i\theta})| < B\left|\frac{1-re^{i\theta}}{1-r}\right|^{\lambda}, \quad r_0 < r < 1, |\theta| < 2\delta. \tag{6.11}$$

This is equivalent to (2.4′) with $\lambda = 1+\alpha-\varepsilon$. In view of (6.2) we also note that (6.11) is a consequence of (6.1) when $\lambda = 2p$. Thus we suppose without loss of generality that $\lambda \leqslant 2p$.

Our aim is to deduce from these assumptions an estimate for I_n in Lemma 2. However, a direct substitution of the bound (6.11) in (5.5) gives too weak a result. A further use of Lemma 3 will show that the set of θ, for which the upper bound implied by (6.11) is attained, is relatively sparse. In this direction we prove

LEMMA 6. *Let ϕ be a positive number such that $2/N \leqslant \phi \leqslant 1-r_0$. Let k be a positive integer, such that $2p/k = \varepsilon < \lambda$ and let l_n be the length of the set of θ for which $\phi \leqslant |\theta| \leqslant 2\phi$ and $re^{i\theta} \in E_n$, where $1-2/N \leqslant r \leqslant 1-1/N$. Then we have*

$$l_n < BN^{\varepsilon/(\lambda-\varepsilon)}\phi^{\lambda/(\lambda-\varepsilon)}R_n^{-1/(\lambda-\varepsilon)}.$$

Here and subsequently B will depend on k and λ as well as the other quantities indicated above.

We define $\delta_0 = l_n/(16k)$. We may assume that

$$\delta_0 > 1 - r. \tag{6.12}$$

For otherwise it is enough to prove that

$$(1-r) < \frac{B|\phi|^{\lambda/(\lambda-\varepsilon)}}{(1-r)^{\varepsilon/(\lambda-\varepsilon)}} R_n^{-1/(\lambda-\varepsilon)},$$

i.e.

$$R_n < B\left(\frac{\phi}{1-r}\right)^{\lambda}. \tag{6.13}$$

If $l_n = 0$ our conclusion is trivial. Otherwise there exists θ, such that $|\phi| \leqslant |\theta| \leqslant 2|\phi|$ and $|f(re^{i\theta})| \geqslant R_n$. Now (6.13) follows from (6.11).

Thus we may assume that (6.12) holds. We now introduce $\theta_1, \theta_2, \dots \theta_k$, such that

$$\begin{gathered} |\theta| \leqslant |\theta_1| < |\theta_2| \leqslant |\theta_k| \leqslant 2|\phi|, \\ |\theta_{j+1}| - |\theta_j| \geqslant 4\delta_0, \quad j = 1 \text{ to } k-1 \\ 2R_{n-1} \geqslant |f(re^{i\theta_j})| \geqslant R_{n-1}. \end{gathered} \tag{6.14}$$

The numbers θ_j can be introduced in turn such that

$$\begin{gathered} \big||\theta_j| - |\theta_\nu|\big| \geqslant 4\delta_0, \quad \nu = 1 \text{ to } j-1, \\ |\phi| \leqslant |\theta_j| \leqslant 2|\phi|, \quad \text{and } re^{i\theta_j} \in E_n. \end{gathered}$$

For if θ_j did not exist for some $j \leqslant k$, the whole of E_n would be confined to the ranges

$$\big||\theta_\nu| - |\theta|\big| \leqslant 4\delta_0, \quad \nu = 1 \text{ to } j-1,$$

and so

$$l_n \leqslant 16(k-1)\delta_0,$$

which contradicts the definition of δ_0.

We now note that $\delta_0 \leqslant \phi/16 \leqslant (1/16)(1-r_0)$ and set

$$z_j = (1-\delta_0)e^{i\theta_j}, \quad z_j' = re^{i\theta_j}, \quad r_j = \delta_0, \quad j = 1 \text{ to } k$$

in Lemma 3. In view of (6.14) the disks $|z - z_j| < \delta_0$ are disjoint. Instead of ϱ_1 we take

$$M_1 = \sup_{\phi \leqslant |\theta| \leqslant 2\phi} |f((1-\delta_0)e^{i\theta})|$$

and instead of ϱ_2 we take R_n. Then either $R_n < eM_1$, or

$$k\left\{\log \frac{A(p)\delta_0}{1-r}\right\}^{-1} < \frac{2p}{\log(R_n/M_1) - 1}$$

i.e. $$R_n < eM_1 \left\{\frac{A(p)\,\delta_0}{1-r}\right\}^{2p/k} < BM_1\left(\frac{\delta_0}{1-r}\right)^{\varepsilon}.$$

This inequality is trivial if $R_n < eM_1$ and so is true generally. Also in view of (6.11) we have $M_1 < B(\phi/\delta_0)^{\lambda}$. Thus

$$R_n < B\,\frac{\phi^{\lambda}}{\delta_0^{\lambda-\varepsilon}(1-r)^{\varepsilon}} < B\,\frac{N^{\varepsilon}\phi^{\lambda}}{l_n^{\lambda-\varepsilon}}.$$

This yields Lemma 6.

7. The estimates for I_n

We deduce

LEMMA 7. *Suppose that $\lambda-\varepsilon>\frac{1}{2}$, and $\lambda/(\lambda-\varepsilon)<2\alpha+2$. Then for $n>n_0$*

$$I_n \leqslant BR_n^{(2-1/(\lambda-\varepsilon))}\,\theta_n^{\lambda/(\lambda-\varepsilon)-2\alpha-2}\,N^{\varepsilon/(\lambda-\varepsilon)-1}, \tag{7.1}$$

where θ_n is the lower bound of $|\theta|$ on E_n, and I_n is defined by (5.5).

We deduce from (6.11) that $|f(re^{i\theta})|<B_1$, $|\theta|<4(1-r)$. Also if $R_{n-1}\leqslant B_1$ we have

$$I_n \leqslant B_2\int_{1-2/N}^{1-1/N} r\,dr\int_{\theta_n}^{\pi}|1-re^{i\theta}|^{-(2\alpha+2)} \leqslant BN^{-1}\{N^{-1}+\theta_n\}^{-(2\alpha+1)},$$

which implies (7.1). We now assume that $R_{n-1}\geqslant B_1$ so that $\theta_n \geqslant 2(1-r)$. We divide E_n into the separate ranges

$$E_{n,\nu}=\{z\,|\,z=re^{i\theta},\,2^{\nu}\theta_n\leqslant|\theta|\leqslant 2^{\nu+1}\theta_n,\,z\in E_n\},\quad \nu=0 \text{ to } \infty.$$

Then we note that

$$I_{n,\nu}=\int_{E_{n,\nu}}\frac{|f(re^{i\theta})|^2 r\,dr\,d\theta}{|1-re^{i\theta}|^{2\alpha+2}} \leqslant \int_{1-2/N}^{1-1/N} dr\,\frac{Bl_{n,\nu}(r)R_n^2}{(2^{\nu}\theta_n)^{2\alpha+2}},$$

where $l_{n,\nu}(r)$ is the length of $E_{n,\nu}\cap(|z|=r)$. In view of Lemma 6 this yields

$$I_{n,\nu} < BN^{\varepsilon/(\lambda-\varepsilon)-1}R_n^{(2-1/(\lambda-\varepsilon))}(2^{\nu}\theta_n)^{(\lambda/(\lambda-\varepsilon)-2\alpha-2)}.$$

Summing from $\nu=0$ to ∞, we deduce Lemma 7.

We deduce

LEMMA 8. *If η is any positive quantity and $R_{n_0}\geqslant B_1$ then we have*

$$S=\sum_{n=n_0+1}^{\infty}(N^{-2\alpha}I_n)^{\eta} < C, \tag{7.2}$$

where C depends only on η and all the quantities that B depends on, provided that $\lambda<1+\alpha$ or $\lambda=1+\alpha=2p$.

We use Lemma 7 and (6.11) with $\theta = \theta_n$. This gives for $n > n_0$

$$R_n < B(N\theta_n)^\lambda, \tag{7.3}$$

i.e. $\theta_n^{-1} \leqslant BNR_n^{-1/\lambda}$. We substitute this in (7.1) and deduce when $\lambda < 1 + \alpha$

$$I_n \leqslant BN^{2\alpha+2-\lambda/(\lambda-\varepsilon)+\varepsilon/(\lambda-\varepsilon)-1} R_n^{2-1/(\lambda-\varepsilon)-(2\alpha+2)/\lambda+1/(\lambda-\varepsilon)} = BN^{2\alpha} R_n^a,$$

where $a = 2(1-(\alpha+1)/\lambda) < 0$ by hypothesis. Thus

$$\sum_{n=n_0+1}^{\infty} (N^{-2\alpha} I_n)^\eta \leqslant B \sum_{n=n_0+1}^{\infty} 2^{na\eta} < C$$

as required. This proves Lemma 8, when $\lambda < 1 + \alpha$.

7.1. The case $\lambda = 1 + \alpha$ is subtler and the crude inequality (7.3) is not sufficient to yield the required result in this case. We proceed to use Lemma 5 to show that R_n can attain the size indicated by (7.3) only for relatively few values of n. We set $\lambda = 1 + \alpha = 2p$, assume that $p > \frac{1}{4}$ and choose ε so small that $4p - 1 - 2\varepsilon > 0$. Then the hypotheses of Lemma 7 are satisfied and (7.1) yields

$$N^{-2\alpha} I_n < B\{R_n/(N\theta_n)^{2p}\}^{(4p-1-2\varepsilon)/(2p-\varepsilon)}$$

Thus

$$(N^{-2\alpha} I_n)^\eta < B\{R_n/(N\theta_n)^{2p}\}^{\eta_0}, \tag{7.4}$$

where

$$\eta_0 = \frac{\eta(4p-1-2\varepsilon)}{2p-\varepsilon} > 0.$$

We set $\phi_\nu = 4^\nu/N$, so that $\theta_n \geqslant \phi_0$, for $n > n_0$, and group together all those terms in the series S in (7.2) for which

$$\phi_\nu \leqslant \theta_n < \phi_{\nu+1}, \quad \nu = 0 \text{ to } \infty. \tag{7.5}$$

We denote by S_ν the sum of all these terms. If there are no such terms we set $S_\nu = 0$. If n is the biggest index of any of these terms, we have evidently, using (7.4)

$$S_\nu \leqslant B\left\{\frac{R_n}{(N\phi_\nu)^{2p}}\right\}^{\eta_0} (1 + 2^{-\eta_0} + 2^{-2\eta_0} + \dots) \leqslant C\left\{\frac{R_n}{(N\phi_\nu)^{2p}}\right\}^{\eta_0}. \tag{7.6}$$

We denote R_n by R_ν'. From the definition of θ_n it follows that θ_n increases with n, provided that $n > n_0$. Thus R_ν' is either zero or increases with ν.

It now follows from Lemma 5, that if $\mu > \nu + 1$, and R_μ', R_ν' are different from zero then

$$R_\mu'^{\frac{1}{2}} R_\nu'^{\frac{1}{2}} < BN^{2p} \phi_\mu^p \phi_\nu^p \tag{7.7}$$

and this inequality is evidently trivial if R_μ' or R_ν' is zero. We set

$$u_\nu = \frac{R_\nu'}{(N\phi_\nu)^{2p}}, \quad \text{so that} \quad S_\nu \leqslant C u_\nu^{\eta_0},$$

and deduce from (7.7) that for $\mu > \nu + 1$

$$N^{4p} u_\mu^{\frac{1}{2}} \phi_\mu^p u_\nu^{\frac{3}{2}} \phi_\nu^{3p} < B N^{2p} \phi_\mu^p \phi_\nu^p$$

so that

$$u_\mu^{\frac{1}{2}} u_\nu^{\frac{3}{2}} < \frac{B}{(N\phi_\nu)^{2p}} = B 4^{-2\nu p}. \tag{7.8}$$

Consider now first all values of ν, for which

$$u_\nu < 4^{-p\nu}. \tag{7.9}$$

Then if Σ_1 denotes the sum over all these ν, we have

$$\Sigma_1 S_\nu \leqslant C \Sigma_1 u_\nu^{\eta_0} \leqslant C \Sigma_1 4^{-\eta_0 p\nu} \leqslant C.$$

Next consider those values of ν which are odd and for which (7.9) is false. We arrange these in a sequence $\nu_1, \nu_2, \dots, \nu_k, \dots$ and deduce that $\nu_{k+1} > \nu_k + 1$. Thus (7.8) yields

$$(u_{\nu_{k+1}})^{\frac{1}{2}} < B 4^{-2\nu_k p} (u_{\nu_k})^{-\frac{3}{2}} \leqslant B 4^{-\frac{1}{2}\nu_k p},$$

since (7.9) is false for $\nu = \nu_k$. Thus if Σ_2 denotes the sum over all odd ν_k, for which (7.9) is false, we have

$$\Sigma_2 S_\nu \leqslant C \Sigma_2 u_\nu^{\eta_0} = C \sum_{k=1}^{\infty} (u_{\nu_k})^{\eta_0} \leqslant C (u_{\nu_1})^{\eta_0} + \sum_{k=1}^{\infty} C 4^{-\nu_k \eta_0 p} \leqslant C((u_{\nu_1})^{\eta_0} + 1).$$

Now if u_{ν_1} is not zero, there exists $n > n_0$ such that (7.5) holds and $R'_{\nu_1} = R_n$. Thus

$$u_{\nu_1} < \frac{4^{2p} R_n}{(N\theta_n)^{2p}}.$$

Since $\lambda = 2p$ we deduce from (7.3) $u_{\nu_1} < C$. Thus we see that $\Sigma_1 S_\nu \leqslant C$. Similarly if Σ_3 denotes the sum over all the even ν for which (7.9) is false we have $\Sigma_3 S_\nu \leqslant C$. Thus finally

$$S = \Sigma_1 S_\nu + \Sigma_2 S_\nu + \Sigma_3 S_\nu \leqslant C.$$

This proves (7.2) when $\lambda = 1 + \alpha = 2p$ and completes the proof of Lemma 8.

8. Proofs of Theorems 1 and 2

We proceed to prove Theorems 1 and 2 together and rely on the estimates (5.3), (5.4) and (7.2). We suppose first that (6.11) holds with $\lambda < 1 + \alpha$ or with $\lambda = 1 + \alpha = 2p$. From this we deduce in view of (5.3) with $w = 0$, (5.4) and (7.2) with $\eta = \frac{1}{2}$, that

$$|\sigma_N^{(\alpha)}| \leqslant B N^{-\alpha} \int_{E_{n_0}} \left\{ \frac{|f(re^{i\theta})|}{|1 - re^{i\theta}|^{\alpha+2}} + \frac{f'(re^{i\theta})}{|1 - re^{i\theta}|^{\alpha+1}} \right\} r\,dr\,d\theta + B \sum_{n_0+1}^{\infty} N^{-\alpha} I_n^{\frac{1}{2}} + O(1)$$

$$\leqslant B N^{-\alpha} \int_{E_{n_0}} \left\{ \frac{|f(re^{i\theta})|}{|1 - re^{i\theta}|^{\alpha+2}} + \frac{|f'(re^{i\theta})|}{|1 - re^{i\theta}|^{\alpha+1}} \right\} r\,dr\,d\theta + O(1), \tag{8.1}$$

where E_{n_0} denotes the subset of the region (5.2) in which

$$|f(z)| < R_{n_0} = 2^{n_0} = R'$$

say. Now

$$\int_{E_{n_0}} \frac{|f(re^{i\theta})|\, r\, dr\, d\theta}{|1-re^{i\theta}|^{\alpha+2}} < R' \int_{1-2N^{-1}}^{1-N^{-1}} r\, dr \int_{-\pi}^{\pi} |1-re^{i\theta}|^{-(\alpha+2)} d\theta = O\left\{\int_{1-2N^{-1}}^{1-N^{-1}} (1-r)^{-(\alpha+1)} dr\right\} = O\{N^{\alpha}\}.$$

Again

$$\int_{E_{n_0}} \frac{|f'(re^{i\theta})|}{|1-re^{i\theta}|^{\alpha+1}}\, r\, dr\, d\theta \leqslant \left(\int_{E_{n_0}} |f'(re^{i\theta})|^2 r\, dr\, d\theta\right)^{\frac{1}{2}} \left(\int_{E_{n_0}} |1-re^{i\theta}|^{-(2\alpha+2)} r\, dr\, d\theta\right)^{\frac{1}{2}}.$$

The first integral on the right-hand side is at most $\pi p R_{n_0}^2 = O(1)$, since $f(z)$ is mean p-valent in E_{n_0}. Again

$$\int_{E_{n_0}} |1-re^{i\theta}|^{-(2\alpha+2)} r\, dr\, d\theta \leqslant \int_{1-2N^{-1}}^{1-N^{-1}} r\, dr \int_{-\pi}^{\pi} |1-re^{i\theta}|^{-(2\alpha+2)} d\theta = O(N^{2\alpha}).$$

Thus (8.1) shows that

$$\sigma_N^{(\alpha)}(0) = O(1) \tag{8.2}$$

as $N \to \infty$, which is (2.2) with $\theta = 0$. In view of Lemma 4, (6.2) we see that (6.11) always holds with $\lambda = 2p$, when (6.1) holds. Thus if $2p = 1 + \alpha > \frac{1}{2}$, we see that (2.5) and (2.3) imply (2.2) when $\theta = 0$ and $f(z)$ is mean p-valent in $N_\delta(\theta)$ for some positive δ. The result clearly remains true for all real θ. Also, if (2.4′) holds in Theorem 1, we may take $\lambda = 1 + \alpha - \varepsilon < 1 + \alpha$ in (6.11). Thus if $\theta = 0$ and (2.3) holds we have again (8.2), which is (2.2) with $\theta = 0$ and again the result extends to arbitrary θ.

8.1. It remains to prove the parts of Theorem 1 and 2 which refer to summability. We assume consequently that (1.6) holds and in addition that $f(z)$ is mean p-valent in $N_\delta(0)$ for some $\delta > 0$ and that

$$|f(z) - w| = o\left(\frac{|1-z|}{1-|z|}\right)^{\lambda}, \tag{8.3}$$

as $|z| \to 1$ in any manner in $N_\delta(0)$. Here we have $w = f(1)$ and $\lambda = 1 + \alpha = 2p$ or $\lambda < 1 + \alpha$.

The condition (8.3) is just (2.1′) if $\lambda < 1 + \alpha$. If $\lambda = 1 + \alpha = 2p$, (8.3) is a consequence of (1.6) and (1.2). To see this we note that by Lemma 4, (1.2) with $\theta = 0$ implies (8.2) as $z \to 1$ in any manner from $|z| < 1$. In other words, given $\eta > 0$, there exist $r_0 < 1$ and $\theta_0 > 0$, such that

$$|f(re^{i\theta}) - w| < \eta \left|\frac{1-re^{i\theta}}{1-r}\right|^{2p}, \quad r_0 < r < 1,\ |\theta| < \theta_0. \tag{8.4}$$

On the other hand, we have in view of (1.6) with $\alpha = 2p - 1$

$$|f(re^{i\theta})| = O\left\{\sum_1^\infty |a_n| r^n\right\} + O(1) = o\left\{\sum_1^\infty n^{2p-1} r^n\right\} + O(1) = o(1-r)^{-2p},$$

as $r \to 1$, uniformly for $|\theta| \leqslant \pi$. Thus we can find $r_1 < 1$, such that (8.4) holds also for $r_1 < r < 1$, $\theta_0 \leqslant |\theta| \leqslant \pi$. Since η can be chosen as small as we please, we deduce that (8.3) holds as $|z| \to 1$ in any manner while $|z| < 1$, with $\lambda = 2p$.

Our proof now proceeds similarly to that in the previous section. We deduce this time from (5.3) and (5.4) that

$$|\sigma_N^{(\alpha)} - w| \leqslant BN^{-\alpha} \int_{E_{n_0}} \left\{\frac{|f(re^{i\theta}) - w|}{|1 - re^{i\theta}|^{\alpha+2}} + \frac{|f'(re^{i\theta})|}{|1 - re^{i\theta}|^{\alpha+1}}\right\} r\,dr\,d\theta + B \sum_{n_0+1}^{\infty} N^{-\alpha} I_n^{\frac{1}{2}} + \varepsilon_N, \tag{8.5}$$

where $\varepsilon_N \to 0$, as $N \to \infty$. Suppose now that $R_{n_0} \geqslant 1 + 2|w|$. Then given $\gamma > 0$, we deduce from (8.3) and $|f(re^{i\theta_n})| = R_n$, that if r is sufficiently near one and so if $N > N_0(\gamma)$, $n > n_0$, we have

$$R_n < \gamma \left(\frac{|1 - re^{i\theta_n}|}{1 - r}\right)^{\lambda}.$$

Hence if $N > N_1(\gamma)$ we have

$$R_n < \gamma (N\theta_n)^{\lambda}. \tag{8.6}$$

In view of (7.4) we now deduce that, given $\gamma_1 > 0$, we have for $n > n_0$, $N \geqslant N_1(\gamma_1)$, if $\lambda = 1 + \alpha = 2p$,

$$N^{-2\alpha} I_n < \gamma_1. \tag{8.7}$$

If $\lambda < 1 + \alpha$ we easily obtain (8.7) on replacing (7.3) by (8.6). Thus (8.7) holds generally. Hence

$$\sum_{n=n_0+1}^{\infty} N^{-\alpha} I_n^{\frac{1}{2}} \leqslant \gamma_1^{\frac{1}{4}} \sum_{n_0+1}^{\infty} (N^{-2\alpha} I_n)^{\frac{1}{4}} \leqslant C\gamma_1^{\frac{1}{4}}$$

in view of Lemma 8. On substituting this result in (8.5) we deduce that

$$|\sigma_N^{(\alpha)} - w| \leqslant B_3 N^{-\alpha} \int_{E_{n_0}} \left\{\frac{|f(re^{i\theta}) - w|}{|1 - re^{i\theta}|^{\alpha+2}} + \frac{|f'(re^{i\theta})|}{|1 - re^{i\theta}|^{\alpha+1}}\right\} r\,dr\,d\theta + \varepsilon_N. \tag{8.8}$$

We now choose for K a large positive constant and divide E_{n_0} into the two ranges

$$F = \{r, \theta \mid |\theta| < K/N\} \text{ and } G = \{r, \theta \mid K/N \leqslant |\theta| \leqslant \pi\}.$$

We suppose given a small positive quantity γ_2. Then

$$B_3 \int_G \frac{|f(re^{i\theta}) - w|}{|1 - re^{i\theta}|^{\alpha+2}} r\,dr\,d\theta \leqslant 2(R_{n_0} + |w|) B_3 \int_{1-2/N}^{1-1/N} r\,dr \int_{K/N}^{\pi} |1 - re^{i\theta}|^{-(\alpha+2)} d\theta$$

$$< 2B_3 (R_{n_0} + |w|) \int_{1-2/N}^{1-1/N} r\,dr \left(\frac{N}{K}\right)^{\alpha+1} < B_4 N^{\alpha}/K^{\alpha+1},$$

where B_3 and B_4 are independent of K. Similarly

$$B_3 \int_G \frac{|f'(re^{i\theta})|}{|1-re^{i\theta}|^{\alpha+1}} r\,dr\,d\theta \leqslant \left\{2B_3 \int_G |f'(re^{i\theta})|^2 r\,dr\,d\theta \int_{1-2/N}^{1-1/N} r\,dr\,d\theta \int_{K/N}^{\pi} |1-re^{i\theta}|^{-(2\alpha+2)} d\theta\right\}^{\frac{1}{2}}$$

$$\leqslant B_5 \{\pi p R_{n_0}^2 N^{2\alpha}/K^{2\alpha+1}\}^{\frac{1}{2}}.$$

Since by hypothesis $\alpha > -\frac{1}{2}$, both integrals can be made less than $\gamma_2 N^\alpha$ by a suitable choice of K. Having fixed K we now choose N so large that

$$KB_3|f(re^{i\theta})-w| < \gamma_2, \qquad KB_3|f'(re^{i\theta})| < N\gamma_2$$

in F. This is possible in view of Lemma 4, (6.4) and (6.5). Then

$$B_2 \int_F \left\{\frac{|f'(re^{i\theta})|}{|1-re^{i\theta}|^{\alpha+1}} + \frac{|f(re^{i\theta})-w|}{|1-re^{i\theta}|^{\alpha+2}}\right\} r\,dr\,d\theta < \frac{2\gamma_2}{K}\int_F N^{\alpha+2}\, r\,dr\,d\theta$$

$$< \frac{2\gamma_2 N^{\alpha+2}}{K}\int_{1-2/N}^{1-1/N} dr \int_{-K/N}^{K/N} d\theta = 4\gamma_2 N^\alpha.$$

Thus if N is sufficiently large we obtain finally from (8.8) $|\sigma_N^{(\alpha)} - w| < 6\gamma_2 + \varepsilon_N < 7\gamma_2$, so that

$$\sigma_N^{(\alpha)} \to w, \quad \text{as } N \to \infty.$$

This completes the proof of Theorems 1 and 2.

9. Proof of Theorem 3

We suppose that $f(z)$ is regular in $N_\delta(0)$ and satisfies (2.5) there with $\theta = 0$, so that

$$|f(r)| < M. \tag{9.1}$$

We assume that $M \geqslant R$. Let $[\theta_1, \theta_2]$ be any interval in which $|f(re^{i\theta})| \geqslant M$. Then we have

$$\log|f(re^{i\theta_2})| \leqslant \log|f(re^{i\theta_1})| + \int_{\theta_1}^{\theta_2} \left|\frac{f'(re^{i\phi})}{f(re^{i\phi})}\right| r\,d\phi \leqslant \log|f(re^{i\theta_1})| + \frac{c(\theta_2-\theta_1)}{1-r}.$$

If $re^{i\theta_2} \in N_\delta(0)$ and $|f(re^{i\theta_2})| > M$, we can take for θ_1 the largest number such that

$$|f(re^{i\theta_1})| \leqslant M \text{ and } 0 < \theta_1 < \theta_2.$$

Thus

$$\log|f(re^{i\theta_2})| \leqslant \log M + \frac{c|\theta_2|}{1-r}. \tag{9.2}$$

The inequality is trivial if $|f(re^{i\theta_2})| < M$ and is clearly valid also for negative θ_2. It is thus valid generally for $re^{i\theta_2}$ in $N_\delta(0)$. In particular (2.4′) holds for $|\theta| \leqslant 2(1-r)$, if r is sufficiently near 1.

Suppose next that $1-r \leqslant |\theta| \leqslant \frac{1}{2}\delta$ and that $|f(re^{i\theta})| > M_1 = Me^c$. Then we choose for r_1 the largest number such that $r_1 < r$, and $|f(r_1)| \leqslant M_1$. In view of (9.2) we certainly have $r_1 \geqslant 1-|\theta|$. Then $|f(te^{i\theta})| > M_1$ for $r_1 < t < r$ and so by (2.6)

$$\log|f(re^{i\theta})| \leqslant \log|f(r_1 e^{i\theta})| + \int_{r_1}^{r} \left|\frac{f'(te^{i\theta})}{f(te^{i\theta})}\right| dt < \log M_1 + \int_{r_1}^{r} \frac{c}{1-t}\, dt;$$

thus

$$|f(re^{i\theta})| \leqslant M_1 \left(\frac{1-r_1}{1-r}\right)^c = O\left(\frac{|\theta|}{1-r}\right)^c = O\left(\frac{|1-re^{i\theta}|}{1-r}\right)^c. \tag{9.3}$$

Thus (9.1) implies (9.3) in a neighbourhood $N_\delta(0)$ of $z=1$, and so we can apply Theorem 1 and deduce that (2.2) holds.

Suppose next that in addition

$$f(z) \to w_0 = f(1), \tag{9.4}$$

as $z \to 1$ through positive values. We deduce from (9.3) and Montel's theorem that (9.4) continues to hold as $z \to 1$ in the range $|\theta| < K(1-r)$ for any fixed positive K.

We choose such a value of K and suppose given a small positive quantity γ. Then we can find $r_0 = r_0(K, \gamma)$ such that we have

$$|f(re^{i\theta}) - w_0| < \gamma, \quad r_0 < r < 1, \quad |\theta| < K(1-r). \tag{9.5}$$

Suppose next that $r_0 < r < 1$, $K(1-r) < |\theta| \leqslant K(1-r_0)$. Then if $|f(re^{i\theta})| > M$ we choose the largest value $r_1 < r$ such that $|f(r_1 e^{i\theta})| \leqslant M$. We suppose that $|w_0| + \gamma < M$, so that (9.5) implies that $r_1 \geqslant 1 - |\theta|/K$. We can now apply (2.6) as before to $z = te^{i\theta}$, $r_1 < t < r$, and deduce that

$$\log|f(re^{i\theta}| \leqslant \log|f(r_1 e^{i\theta})| + c \log\left(\frac{1-r_1}{1-r}\right)$$

so that

$$|f(re^{i\theta})| \leqslant M\left(\frac{1-r_1}{1-r}\right)^c \leqslant MK^{-c}\left(\frac{|\theta|}{1-r}\right)^c \leqslant MK^{-c}\left(\frac{|1-z|}{1-|z|}\right)^c,$$

$$|f(re^{i\theta}) - w_0| \leqslant |f(re^{i\theta})| + M \leqslant 2MK^{-c}\left(\frac{|1-z|}{1-|z|}\right)^c.$$

We choose K so large that $2MK^{-c} < \gamma$, and deduce that

$$|f(re^{i\theta}) - w_0| < \gamma\left(\frac{|1-re^{i\theta}|}{1-r}\right)^c, \tag{9.6}$$

for $r_0 < r < 1$, $K(1-r) \leqslant |\theta| < K(1-r_0)$. In view of (9.5) we deduce that (9.6) is valid for $r_0 < r < 1$, $|\theta| < K(1-r_0)$.

Suppose finally that $K(1-r_0) \leqslant |\theta| < \delta_1$. We choose $c_1 = \frac{1}{2}(c+1+\alpha)$ so that $c < c_1 < 1 + \alpha$. Then if r_0 is sufficiently near 1, we have (9.3) so that

$$|f(re^{i\theta}) - w_0| < 2M_1 \left(\frac{|1-re^{i\theta}|}{1-r}\right)^c = 2M_1 \left(\frac{1-r}{|1-re^{i\theta}|}\right)^{c_1-c} \left(\frac{|1-re^{i\theta}|}{1-r}\right)^{c_1}.$$

Since $|\theta| \geq K(1-r_0)$ it follows that $|1-re^{i\theta}|$ is bounded below as $r \to 1$. Hence we can find r_1 so near 1, that for $r_1 < r < 1$, $K(1-r_0) < |\theta| < \delta_1$ we have

$$|f(re^{i\theta}) - w_0| < \gamma \left|\frac{1-re^{i\theta}}{1-r}\right|^{c_1}.$$

In view of (9.6) this inequality also holds for $|\theta| \leq K(1-r_0)$, $r_1 < r < 1$ and so holds for $r_1 < r < 1$, $|\theta| < \delta_1$. Thus (2.1′) holds with $\varepsilon = \frac{1}{2}(1+\alpha-c)$ and we can apply Theorem 1 and deduce (1.4). This completes the proof of Theorem 3.

Since Theorems 4 and 5 were deduced from Theorems 1 and 2 in the introduction, this completes the proofs of our positive theorems.

II. Proofs of Theorems 6 and 7

10. Proof of Theorem 6

We start by proving Theorem 6, which is very simple. We define for any positive integer n

$$\lambda_n = 2^{2^n}, \quad \alpha_n = \lambda_n^{-\frac{1}{2}} (\log \lambda_n)^{-\frac{3}{4}},$$

$$a_{2\lambda_n - p} = \frac{\alpha_n}{p^{\frac{1}{2}}}, \quad 1 \leq p \leq \lambda_n.$$

We also set $a_\nu = 0$, $2\lambda_n \leq \nu < \lambda_{n+1}$, and $\nu = 1, 2, 3$. Then

$$\sum_{\nu=0}^{\infty} \nu a_\nu^2 \leq \sum_{n=1}^{\infty} 2\lambda_n \alpha_n^2 \sum_{p=1}^{\lambda_n} (1/p) = A_0 \text{ say, where } A_0 < \infty.$$

Thus for any positive a_0, the image of $f(z)$ has area at most πA_0. Again

$$S = \sum_{1}^{\infty} a_\nu \leq \sum_{n=1}^{\infty} \alpha_n \sum_{p=1}^{\lambda_n} p^{-\frac{1}{2}} \leq \sum_{n=1}^{\infty} \alpha_n (2\lambda_n^{\frac{1}{2}}) < \infty.$$

Thus the series for $f(z)$ is uniformly and absolutely convergent in $|z| \leq 1$ and so $f(z)$ is continuous there.

It remains to show that $\sigma_N^{(-1/2)}(0)$ is unbounded. To see this we recall the definition (1.3) and set $\theta = 0$. Thus if $N = 2\lambda_n$

$$\sigma_N^{(-\frac{1}{2})}(0) > \binom{N-\frac{1}{2}}{N}^{-1} \sum_{p=1}^{\lambda_n} \binom{p-\frac{1}{2}}{p} a_{2\lambda_n - p} \geq AN^{\frac{1}{2}} \sum_{p=1}^{\lambda_n} \frac{\alpha_n}{p} \geq A\lambda_n^{\frac{1}{2}} \alpha_n \log \lambda_n = A(\log \lambda_n)^{\frac{1}{4}},$$

where A is an absolute constant. This completes the proof of Theorem 6.

11. Proof of Theorem 7; preliminary results

We finally prove Theorem 7. To do this we shall construct a series of Jordan polygons D_m, such that D_{m+1} is obtained from D_m by extension across a small arc of the boundary of D_m. The corresponding mapping functions $f_m(z)$, which map $|z|<1$ onto D_m converge to the univalent function $f(z)$, which maps $|z|<1$ onto D. Our counter example will then be the function

$$F(z)=e^{(\alpha+1)f(z)}.$$

The aim of the next 3 lemmas is to show that we can always choose $f_m(z)$ inductively to be large but not too large in the neighbourhood of a preassigned boundary point ξ_m of $|z|<1$ and to differ little from $f_{m-1}(z)$ at other points.

We have first

LEMMA 9. *Let γ be a crosscut in $|w|<1$ not passing through the origin and let D_0 be the subdomain of $|w|<1$, which is determined by γ and contains the origin. Suppose that $w=f(z)=\beta(z+a_2z^2+\dots)$ maps $|z|<1$ onto D_0 so that $\beta>0$, let Γ be the arc of $|z|=1$ which corresponds to γ by $f(z)$ and let δ, d be the diameters of γ, Γ respectively. Then given $\varepsilon>0$, we can choose η, such that, if either $\delta<\eta$ or $d<\eta$, we have*

$$|f(z)-z|<\varepsilon, \qquad |z|<1. \tag{11.1}$$

This follows from Lemma 6.6, p. 122 of M.F. If Lemma 1 were false, we could find a sequence $f_n(z)$ of such functions for which the corresponding values of d_n or δ_n tend to zero, while (1.1) is false. This contradicts (6.5) of Lemma 6.6, which asserts that if $d_n\to 0$ or $\delta_n\to 0$, then

$$f_n(z)\to z \tag{11.2}$$

uniformly in $|z|<1$.

LEMMA 10. *Suppose that D_0, D_1 are Jordan domains containing the origin in the w-plane and bounded by the closed Jordan curves $\gamma_0\cup\gamma$ and $\gamma_1\cup\gamma$, where γ, γ_0, γ_1 are simple Jordan arcs with the same end points but no other common points. Suppose that $D_0\subset D_1$ and that*

$$w=f_j(z)=\beta_j(z+a_2z^2+\dots),\quad \beta_j>0 \tag{11.3}$$

maps $|z|<1$ onto D_j for $j=1, 2$.

Let Γ_0 be the arc of $|z|=1$, which corresponds to γ_0 by $f_0(z)$, ξ_0 a point of Γ_0 and δ_0, d_0 the diameters of γ_0, Γ_0 respectively. Then given $\varepsilon>0$, there exists a positive ε_1 depending on ξ_0, D_0 and ε only, such that if $\delta_0<\varepsilon_1$, we have for any point in $|z|<1$, such that $f_1(z)$ lies outside D_0

$$|z-\xi_0|<\varepsilon. \tag{11.4}$$

Further
$$|f_1(z)-f_0(z)|<\varepsilon, \quad \textit{for } |z|\leqslant 1-\varepsilon. \tag{11.5}$$

The rather lengthy statement of Lemma 2 amounts to saying that if D_0 is extended in any manner across a small arc corresponding to an arc of $|z|=1$ containing a preassigned point ξ_0, then the mapping function is altered little in the interior of $|z|<1$, and only points near ξ_0 can correspond to points outside D_0. This result will enable us to construct the desired domain and the corresponding mapping function by a step-by-step process leading to a convergent sequence of mapping functions.

Set $\phi_1(z)=f_1^{-1}\{f_0(z)\}$, so that $\phi_1(z)$ maps $|z|<1$ onto a subdomain Δ_0 of $|z|<1$. Also $f_0(z)$ maps an arc of $|z|=1$ of length at least $2\pi-\pi d_0$ onto γ, and this arc is mapped back onto $|z|=1$ by $\phi_1(z)$. Thus we may apply Lemma 9 to $\phi_1(z)$ with πd_0 instead of d. In particular, given $\varepsilon_3>0$, we can find $\varepsilon_2>0$, such that
$$|\phi_1(z)-z|<\varepsilon_3, \text{ if } d_0<\varepsilon_2, \ |z|<1. \tag{11.6}$$

Suppose that $w_0=f_0(z_0)=f_1(z_1)$ is any point in D_0. Then we deduce that $z_1=\phi_1(z_0)$, so that $|z_1-z_0|<\varepsilon_3$. Also if $\varepsilon_3<\frac{1}{2}$, which we suppose, we deduce from (11.6) that
$$|\phi_1'(0)|=\left|\frac{\beta_0}{\beta_1}\right|>\tfrac{1}{2}.$$
Given $\varepsilon>0$, we may suppose that $\varepsilon_3<\frac{1}{2}\varepsilon$. Then since $f_1(z)$ is univalent we have (M. F. (1.3), p. 5) for $|z|\leqslant 1-\frac{1}{2}\varepsilon$
$$|f_1'(z)|\leqslant \beta_1\frac{1+|z|}{(1-|z|)^3}\leqslant\frac{16\,\beta_1}{\varepsilon^3}\leqslant\frac{32\,\beta_0}{\varepsilon^3}.$$
Thus if $|z_0|\leqslant 1-\varepsilon$, so that $|z_1|\leqslant 1-\frac{1}{2}\varepsilon$, we have
$$|f_1(z_0)-f_0(z_0)|=|f_1(z_0)-f_1(z_1)|\leqslant\int_{z_0}^{z_1}|f_1'(z)||dz|\leqslant\frac{32\,\beta_0\varepsilon_3}{\varepsilon^3}<\varepsilon,$$
if ε_3 is sufficiently small, which gives (11.5).

Again let ξ_1 be any point on Γ_0. Then since ξ_0 also lies on Γ_0, we have $|\xi_1-\xi_0|\leqslant d_0$. The arc Γ_0 is mapped by the continuous extension of $\phi_1(z)$ onto a crosscut Γ_0' in $|z|<1$ and in view of (11.6) we deduce that for $z=\phi_1(\xi_1)$ on this crosscut we have $|z-\xi_1|\leqslant\varepsilon_3$, so that
$$|z-\xi_0|\leqslant d_0+\varepsilon_3. \tag{11.7}$$

The set of points z, such that $f_1(z)$ lies outside D_0 forms a Jordan domain Δ_0 bounded by Γ_0' and an arc of $|z|=1$. The end points of this latter arc lie in the disk (11.7) and hence so does the arc, provided that $d_0+\varepsilon_3<\frac{1}{2}$. Thus (11.7) holds on the boundary of Δ_0 and so in the whole of Δ_0. Thus we have (11.4) provided that $d_0<\varepsilon/2$, $\varepsilon_3<\varepsilon/2$.

To complete the Lemma it is therefore, in view of (11.6), sufficient to note that $f_0^{-1}(z)$ has a continuous extension from $\bar{D}_0$ to $|z| \leqslant 1$, since D_0 is a Jordan domain. Thus the diameter d_0 of Γ_0 is small provided that the diameter δ_0 of γ_0 is sufficiently small. This completes the proof of Lemma 10.

11.1. While Lemmas 9 and 10 are very general, we now come to the heart of our construction.

LEMMA 11. *Suppose that we are given the Jordan polygon D_0, positive constants ε_0, η and K and a point ξ_0 on $|\xi_0| = 1$ and further that the closure $\bar{D}_0$ of D_0 lies in the strip $S: |v| < \frac{1}{2}\pi(1+\eta)$, where $w = u + iv$. Then we can find the Jordan polygon D_1, satisfying the conclusions of Lemma 10 with some $\varepsilon < \varepsilon_0$, such that $\bar{D}_1$ lies in S and further*

$$\mathcal{R} f_1(z) \leqslant \log \frac{K}{1-|z|}, \quad 1-\varepsilon \leqslant |z| < 1, \tag{11.8}$$

with equality for some point $z = z_1$, such that $|\xi_0 - z_1| < \varepsilon$.

We suppose that $\bar{D}_0$ lies in the rectangle

$$S_1: -a < u < a, \quad |v| < \frac{\pi}{2}(1+\eta),$$

where $w = u + iv$. Let $f_0(\xi_0) = w_0$. We then choose neighbouring points w_1, w_2 of w_0 on the boundary γ' of D_0 so close to w_0 that the polygonal arc γ_0: $w_1 w_2$ contains w_0 and has diameter less than ε_1. We then join w_1, w_2 to $u = a$ by polygonal arcs γ_1', γ_2' in S_1 which do not meet each other nor $\bar{D}_0$ except for the endpoints w_1, w_2. If $w_1' = a + ib_1$, $w_2 = a + ib_2$, are the other endpoints of γ_1', γ_2', where $b_1 > b_2$, we join w_1' to w_2' by the polygonal arc γ_3: w_1', $a + i\frac{1}{2}\pi 2(1+\frac{1}{2}\eta)$, $a_1 + i\frac{1}{2}\pi(1+\frac{1}{2}\eta)$, $a_1 - i\frac{1}{2}\pi(1+\frac{1}{2}\eta)$, $a - i\frac{1}{2}\pi(1+\frac{1}{2}\eta)$, w_2' and denote the union of γ_1', γ_3, γ_2' by γ_1. This defines the domain $D_1 = D_1(a_1)$. We assume $a_1 > a$.

The parameter a_1 is left variable. It remains to show that we can choose a_1 so that (11.8) holds. We suppose first that ε was chosen so small that $\varepsilon < \varepsilon_1$ and

$$\log \frac{K}{\varepsilon} > a + 1.$$

Then if $|z| > 1 - \varepsilon$ and $f_1(z)$ lies in D_0, we certainly have

$$\mathcal{R} f_1(z) - \log \frac{K}{1-|z|} < a - \log \frac{K}{\varepsilon} < -1. \tag{11.9}$$

Thus to prove (11.8) we may confine ourselves to those points z for which $f_1(z)$ lies outside D_0 and in fact $\mathcal{R} f_1(z) > a + 1$.

We set $$M(a_1) = \sup_{1-\varepsilon \leq |z| < 1} \mathcal{R}\{f_1(z)\} - \log \frac{K}{1-|z|},$$

and note that, if ε, γ_1', γ_2' are chosen as above and fixed once and for all, then $M(a_1) < 0$, if $a_1 < a+1$. Clearly the maximum $M(a_1)$ is attained for a point z_1 in $|z_1| < 1$, since

$$\mathcal{R} f_1(z) - \log \frac{1}{1-|z|} \to -\infty, \quad \text{as} \quad |z| \to 1.$$

Also a slight change in a_1 causes only a slight change in D_1 and so in $f(z_1)$ for fixed z_1. These considerations show that $M(a_1)$ is a continuous function of a_1 for $a_1 \geq a+1$. Thus it is sufficient to show that

$$M(a_1) > 0, \quad \text{for some } a_1, \tag{11.10}$$

since in this case there will certainly be a value of a_1 such that $M(a_1) = 0$.

To see this we consider first the limiting case $a_1 = \infty$, and show that

$$M(\infty) = \infty. \tag{11.11}$$

In fact when $a_1 = \infty$, the domain D_1 contains the half-strip

$$u > a, \ |v| < \frac{\pi}{2}\left(1 + \tfrac{1}{2}\eta\right),$$

and the function

$$W = U + iV = \phi(z) = \exp\left\{\frac{-f_1(z)}{1 + \frac{1}{2}\eta}\right\}$$

maps an arc of $|z| = 1$ onto a segment of the imaginary W axis, which corresponds to the arms of this half-strip at ∞. Thus by Schwarz's reflection principle W can be analytically continued across $|z| = 1$, and if $z = \xi_1$ corresponds to $W = 0$, we have

$$W \sim a(z - \xi_1), \quad \text{as } z \to \xi_1,$$

where a is a non-zero constant. Thus as $z \to \xi_1$, we have

$$f_1(z) = -\left(1 + \tfrac{1}{2}\eta\right) \log W = \left(1 + \tfrac{1}{2}\eta\right) \log\left(\frac{1}{z - \xi_1}\right) + O(1),$$

$$\mathcal{R} f_1(z) = \left(1 + \tfrac{1}{2}\eta\right) \log\left|\frac{1}{z - \xi_1}\right| + O(1) = \left(1 + \tfrac{1}{2}\eta\right) \log \frac{1}{1-|z|} + O(1),$$

if we choose z so that $\arg z = \arg \xi_1$. Thus (11.11) holds. In particular if $a_1 = \infty$, we can find z so that $1 - \varepsilon < |z| < 1$ and

$$\mathcal{R} f_1(z) - \log \frac{K}{1-|z|} > 0.$$

Now continuity considerations show that this inequality also holds for the same fixed value of z if a_1 is sufficiently large. This proves (11.10).

Thus it is possible to choose the value a_1, so that $M(a_1)=0$, and the domain D_1 is defined accordingly. We have seen that the upper bound is attained for some point z_1 in $|z_1|<1$. Since (11.9) holds, whenever $f_1(z)$ lies in D_0, it follows that $f_1(z_1)$ must lie outside D_0, so that (11.4) holds with $z=z_1$. This completes the proof of Lemma 11.

11.2. We now proceed to give our construction. Let $0, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \ldots$ be the series of rational numbers and let r_m denote the mth member of this series. Then all the rational fractions with denominator q are included in our series with $m \leqslant q^2$. Thus if $0<x<1$, we can always find a value of m such that $m \leqslant q^2$, and

$$0<|r_m-x|\leqslant \frac{1}{q}. \tag{11.12}$$

We now suppose given $\eta>0$, and define a sequence of domains as follows. We take for D_0 the square $|u|\leqslant \pi/2$, $|v|\leqslant \pi/2$. If D_{m-1} has already been constructed we construct D_m from D_{m-1} in accordance with the construction of Lemma 11 of D_1 from D_0. We take for ξ_0 the point $\xi_m=e^{2\pi i r_m}$. We take $\varepsilon=\varepsilon_m<\frac{1}{2}\varepsilon_{m-1}$, and $K_m=m^{-1/3}$, at the mth stage, and obtain a point z_m, such that $|z_m|<1$,

$$|z_m-\xi_m|<\varepsilon_m, \tag{11.13}$$

$$f_m(z_m)=\log\left(\frac{K_m}{1-|z_m|}\right)+iv. \tag{11.14}$$

Further by (11.8) we have

$$\mathfrak{R} f_m(z)\leqslant \log\left(\frac{K_m}{1-|z|}\right), \quad 1-\varepsilon_m\leqslant|z|<1. \tag{11.15}$$

In view of (11.5) we also have

$$|f_m(z)-f_{m-1}(z)|<\varepsilon_m, \quad |z|<1-\varepsilon_m. \tag{11.16}$$

In addition we assume that ε_m was chosen smaller than $1-|z_{m-1}|$, so that

$$|z_{m-1}|<1-\varepsilon_m<|z_m|. \tag{11.17}$$

The sequence of domains D_m is expanding and tends to $D=\bigcup_{m=0}^{\infty} D_m$. At the same time the sequence of functions $f_m(z)$ converges by (11.16) locally uniformly in $|z|<1$ to the univalent function $f(z)$, which maps $|z|<1$ onto D.

We note that $f(z)$ has the following properties.

$$|\mathfrak{J} f(z)|\leqslant \frac{\pi}{2}(1+\eta); \tag{11.18}$$

this is obvious from the corresponding properties for $f_m(z)$.

Next there exists a point z_m satisfying (11.13) and

$$\mathcal{R} f(z_m) > \log \left\{\frac{K_m}{1-|z_m|}\right\} - \varepsilon_m. \tag{11.19}$$

In fact by (11.14)

$$\mathcal{R} f_m(z_m) = \log \left(\frac{K_m}{1-|z_m|}\right).$$

Also by (11.17) we have for $n > m$, $|z_m| \leq |z_{n-1}| < 1-\varepsilon_n$. Thus in view of (11.16) we have, for $n > m$, $|f_n(z_m) - f_{n-1}(z_m)| < \varepsilon_n$. Thus since $\varepsilon_{n+1} < \frac{1}{2}\varepsilon_n$, we see that

$$|f(z_m) - f_m(z_m)| \leq \sum_{n=m+1}^{\infty} \varepsilon_n \leq \varepsilon_m.$$

This proves (11.19).

Finally we have for $|z| > 1 - \varepsilon_m$

$$\mathcal{R} f(z) < \log \left\{\frac{K_m}{1-|z|}\right\} + \varepsilon_m. \tag{11.20}$$

It is enough to prove (11.20) for $1-\varepsilon_m < |z| \leq 1 - \varepsilon_{m+1}$, since ε_m decreases with increasing m and so does $K_m = m^{-\frac{1}{2}}$.

In this case we have by (11.15)

$$\mathcal{R} f_m(z) \leq \log \frac{K_m}{1-|z|},$$

and by (11.16) we have, for $n > m$, $|f_n(z) - f_{n-1}(z)| < \varepsilon_n$. Thus

$$\mathcal{R} f(z) \leq \mathcal{R} f_m(z) + \sum_{n=m+1}^{\infty} |f_n(z) - f_{n-1}(z)| \leq \mathcal{R} f_m(z) + \sum_{n=m+1}^{\infty} \varepsilon_n < \log \frac{K_m}{1-|z|} + \varepsilon_m.$$

This proves (11.20).

12. Proof of Theorem 7

We can now conclude the proof of Theorem 7. We suppose $-\frac{1}{2} < \alpha < 2p-1$, and choose the positive constant η in the preceding section so small that $(\alpha+1)(1+\eta) < 2p$. We set

$$F(z) = \exp\{(\alpha+1) f(z)\}, \tag{12.1}$$

where $f(z)$ is the function constructed in the previous section. Since $f(z)$ is univalent with an image lying in the strip $|v| < (1+\eta)\pi/2$, it follows that $F(z)$ is also univalent provided that $(\alpha+1)(1+\eta) < 2$, i.e. certainly if $p \leq 1$. More generally if $p \leq q$, where q is a positive integer we see that $F(z)$ takes no value more than q times. Further the part of the Riemann surface of $F(z)$ which lies over the circle $|W| = R$, for any positive R consists of a subset of the arc $|\arg W| < (\alpha+1)(1+\eta)\pi/2 < p\pi$. Since $\arg W$ assumes no value more than once for $|W| = R$, we see that $F(z)$ is mean p-valent even in the circumferential sense.

Next it follows from (11.20) that

$$|F(z)| < e^{\varepsilon_m}\left(\frac{K_m}{1-|z|}\right)^{\alpha+1}, \quad 1-\varepsilon_m < |z| < 1.$$

Since K_m tends to zero as $m \to \infty$, we deduce that the maximum modulus $M(r, F)$ of $F(z)$ satisfies

$$M(r, F) = o(1-r)^{-\alpha-1}, \quad \text{as } r \to \infty.$$

Since $F(z)$ is mean p-valent in $|z| < 1$ and $\alpha > -\frac{1}{2}$, this implies for the coefficients a_n of $F(z)$

$$|a_n| = o(n^\alpha),$$

as required. (This is a slight extension of M.F. Theorem 3.3, p. 46 and is proved by the same method.)

Finally suppose that for some θ_0, such that $0 \leqslant \theta_0 < 2\pi$ the Césaro sums $\sigma_N^{(\alpha)}(\theta_0)$ are bounded as $N \to \infty$. In view of (2.4) this would imply

$$|F(z)| = O\left\{\frac{|e^{i\theta_0} - z|}{1-|z|}\right\}^{\alpha+1} \tag{12.2}$$

as $|z| \to 1$ in any manner. We allow z to tend to $e^{i\theta_0}$, through a subsequence $m = m_k$ of the points ξ_m, so chosen that the corresponding arguments r_m satisfy $|2\pi r_m - \theta_0| < 2\pi/\sqrt{m}$. This is possible by (11.12). Thus

$$|e^{i\theta_0} - \xi_m| = |e^{i\theta_0} - e^{2\pi i r_m}| = \frac{O(1)}{\sqrt{m}}$$

and hence in view of (11.13) we have

$$|e^{i\theta_0} - z_m| \leqslant |e^{i\theta_0} - \xi_m| + |z_m - \xi_m| < \frac{O(1)}{\sqrt{m}} + \varepsilon_m = \frac{O(1)}{\sqrt{m}}.$$

Thus (12.2) implies

$$|F(z_m)| = O\left(\frac{m^{-\frac{1}{2}}}{(1-|z_m|)}\right)^{\alpha+1}.$$

On the other hand, it follows from (12.1) and (11.19) that for all large m

$$|F(z_m)| > e^{-(\alpha+1)\varepsilon_m}\left(\frac{K_m}{1-|z_m|}\right)^{\alpha+1} > \left(\frac{\frac{1}{2}m^{-\frac{1}{3}}}{1-|z_m|}\right)^{\alpha+1}.$$

This gives a contradiction, which shows that the Césaro sums $\sigma_N^\alpha(\theta_0)$ cannot be bounded. This completes the proof of Theorem 7.

13. In conclusion I should like to express my gratitude to Dr. Halász for introducing the problem to me and allowing me to read his own paper at the proof stage.

He pointed out to me the inequality (12.2) for a function with bounded Césaro sums on which the counter example of Theorem 7 is based and showed how to use the integral representations of the sums in order to prove positive theorems. In fact the statements of nearly all the theorems arose from our discussions together and subsequent attempts by me to prove or disprove his conjectures.

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