GROUPS OF CONTINUOUS FUNCTIONS IN HARMONIC ANALYSIS

BY

N. TH. VAROPOULOS

Institut Mittag-Leffler, Djursholm, Sweden. Faculté des Sciences, 91-Orsay, France. King's College, Cambridge, England

0. Introduction

Let G be a general abelian topological group. We shall denote by $\hat{G}$ its character group, i.e. the multiplicative group of all continuous homomorphisms

$$\chi: G \to \mathbf{T}$$

of G into $\mathbf{T}=\mathbf{R}\ (\mathrm{mod}\ 2\pi)$. For every $g \in G$ and every $\chi \in \hat{G}$ we shall denote $\langle \chi, g \rangle = \chi(g)$.

Let $K \subset G$ be a subset of G, we shall say that K is independent if:

$$n_j \in \mathbf{Z}, \quad k_j \in K, \quad j=1,2,\dots,p; \quad \sum_{j=1}^{p} n_j k_j = 0_G \ \Rightarrow \ n_j = 0, \quad j=1,2,\dots,p.$$

(0_G is of course the zero element of G).

Let $E \subset G$ be a subset of G; we shall denote by $\mathrm{Gp}\,(E)$ the subgroup algebraically generated in G by E. Let $K \subset G$ be a compact subset of G, we shall say that K is a Kronecker set if for every $f \in \mathbf{C}(K)$ such that $|f| \equiv 1$ and every $\varepsilon > 0$ we can find some $\chi \in \hat{G}$ such that:

$$\sup_{k \in K} |f(k) - \chi(k)| \leqslant \varepsilon.$$

Let $K \subset G$ be a compact subset of G, we shall say that K is an H_α set for some $\alpha \in (0, 1]$ if for every $f \in \mathbf{C}(K)$ and every $\varepsilon > 0$ we can find a sequence of characters $\{\chi_n \in \hat{G}\}_{n=1}^{\infty}$ and a sequence of complex numbers $\{\alpha_n \in \mathbf{C}\}_{n=1}^{\infty}$ such that:

$$f(k) = \sum_{j=1}^{\infty} \alpha_j \chi_j(k),\ k \in K; \qquad \sum_{j=1}^{\infty} |\alpha_j| \leqslant \alpha^{-1} \|f\|_\infty + \varepsilon.$$

We shall say that a compact set $K \subset G$ is a set of interpolation if it is an H_α set for some $\alpha \in (0, 1]$. We refer the reader to [1] and [2] for elaborations of the above definitions.

Let $K \subset G$ be a compact subset of G and let us suppose that G itself is compact. We shall then say that K is a set without true pseudomeasures if:

$$S \in \mathrm{PM}(K) \Rightarrow S \in M(K)$$

(cf. [2], ch. XI no. 3, Th. III).

We are now in a position to state the results that will be proved in this paper.

THEOREM 1. *Let G be a compact abelian group and let $K \subset G$ be a totally disconnected H_1 subset of G, let further $E \subset G$ be a compact subset of G such that $K \cap E = \emptyset$. Then for any two positive numbers $\varepsilon, \eta \in (0, 1)$ we can find some $f \in A(G)$ such that:*

(i) $\|f\|_A \leqslant 8\varepsilon^{-1}$

(ii) $|f(k) - 1| \leqslant \eta, \quad \forall k \in K$

(iii) $|f(e)| \leqslant \varepsilon, \quad \forall e \in E.$

THEOREM 2. *Let G be a compact abelian group and let $K \subset G$ be a metrisable H_1 subset of G, let further $E \subset G$ be a compact subset of G such that*

$$\mathrm{Gp}\,(K) \cap E = \emptyset.$$

Then for every $\varepsilon > 0$ we can find some $f \in A(G)$ such that:

(i) $\|f\|_A \leqslant 1$

(ii) $|f(k) - 1| \leqslant \varepsilon, \quad \forall k \in K$

(iii) $|f(e)| \leqslant \varepsilon, \quad \forall e \in E.$

COROLLARY 1. *Let G be a compact abelian group, let $K \subset G$ be a totally disconnected H_1 set and let $H \subset G$ be an H_α set for some $\alpha \in (0, 1]$. Then the set $K \cup H$ is an H_β set, where $\beta = \beta(\alpha) > 0$ depends only on α.*

COROLLARY 2. *Let G be a compact abelian group and let $K_1, K_2, \ldots, K_p$ be metrisable H_1 subsets ($p \geqslant 1$) such that their union*

$$H = K_1 \cup K_2 \cup \ldots \cup K_p$$

is independent in G. Then H is an $H_{1/p}$ set.

COROLLARY 3. *Let G be a compact abelian group, let $K \subset G$ be a Kronecker subset of G and let $L \subset G$ be a compact subset without true pseudomeasures such that $K \cup L$ is totally disconnected. Then the set $K \cup L$ is a set without true pseudomeasures.*

COROLLARY 4. *There exists $H \subset \mathbf{T}$ an $H_{(384)^{-1}}$ set such that*

$$H + H = \{h_1 + h_2;\ h_1, h_2 \in H\} = \mathbf{T}.$$

THEOREM 3. *In every compact abelian group G there exists $E \subset G$ a closed subset of spectral synthesis and $S \in \mathrm{PM}(E)$ a pseudomeasure supported by E such that no sequence of measures $\{\mu_n \in M(E)\}_{n=1}^{\infty}$ has the property:*

$$\mu_n \xrightarrow[n\to\infty]{} S \quad (\textit{in the topology } \sigma(\mathrm{PM}(G), A(G))).$$

The point of Theorem 3 is, of course, that we *can* synthesise S with a net $\{\mu_\alpha \in M(E)\}_{\alpha\in A}$ but that we must have then

$$\sup_{\alpha\in A} \|\hat{\mu}_\alpha\|_\infty = +\infty.$$

To avoid cumbersome digressions later on we shall concentrate here a few definitions and notations of a more specialized nature.

Let B be a Banach space and let $B_1 \subset B'$ be a subspace of the dual space B' of B. We shall denote by $\sigma(B, B_1)$ the topology induced on B by the family of pseudonorms

$$p_{b_1}(b) = |\langle b, b_1\rangle|, \quad b_1 \in B_1.$$

Let E be any space and $E_1 \subset E$ a subspace; we shall denote then by ξ_{E_1} the characteristic function of E_1 i.e.

$$\xi_{E_1}(e) = 1, \quad e \in E_1; \qquad \xi_{E_1}(e) = 0, \quad e \in E \setminus E_1.$$

Let G be any abelian group, let $E \subset G$ be a subset of G and let $\alpha, \beta \in \mathbf{Z}$, $\alpha, \beta \geqslant 0$ be two non negative integers s.t. $\alpha+\beta>0$. We shall denote:

$$E_{\alpha,\beta} = \{e_1+e_2+\ldots+e_\alpha-e_1'-e_2'-\ldots-e_\beta' \in G;\, e_i, e_j' \in E,\ 1\leqslant i\leqslant\alpha,\ 1\leqslant j\leqslant\beta\}.$$

We shall also set $E_{0,0} = \{0_G\}$.

Let G be any abelian group and let

$$P: G \to \mathbf{C}$$

be any complex valued function on G. We shall say that P is positive definite if for any $n \geqslant 1$ and any choice

$$g_1, g_2, \ldots, g_n \in G; \quad \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbf{C}$$

we have
$$\sum_{i,j=1}^{n} P(g_i - g_j)\,\lambda_i \bar{\lambda}_j \geqslant 0.$$

If P_1 and P_2 are two positive definite functions on G we shall say that P_1 dominates P_2 and write

$$P_1 >> P_2$$

if the difference $P_1 - P_2$ is a positive definite function on G.

For elaborations on positive definite functions we shall refer the reader to [3].

Results analogous to ours above (even sharper but in a different context) have been obtained by S. Drury [4].

Results in the opposite direction showing the sharpness of some of our theorems have been obtained in [5] and also, in a more extensive scale by T. W. Körner in [6].

It might be interesting to make a few remarks concerning the organisation of the paper.

Corollaries 1, 3 and 4 are corollaries of Theorem 1, and Corollary 2 is a corollary of Theorem 2.

The first two paragraphs, where the positive definite functions of $S(K)$ are worked out are only needed for the proof of Theorem 2. To obtain the proof of Theorem 1 one can start reading this paper from § 3 onwards. On the other hand, Theorem 3 does depend on Theorem 2.

If we are prepared to base the proof of Theorem 1 on Theorem 2 and the knowledge of positive definite functions on $S(K)$ we can obtain a proof which is perhaps more transparent and certainly lighter in notations. This was the point of view adopted in the seminars given at the Institut Mittag-Leffler. (Notes of these seminars are available.)

1. The group $S(K)$ and its characters

Let K be a compact topological space, we shall denote:

$$\mathrm{S(K)} = \{f \in \mathrm{C}(K);\ |f(k)| = 1 \forall k \in K\}.$$

$S(K)$ is then an abelian group under pointwise multiplication and is also a metric space under the metric:

$$d(f_1, f_2) = \sup_{k \in K} |f_1(k) - f_2(k)|, \quad f_1, f_2 \in S(K). \tag{1.1}$$

The above metric is translation invariant on $S(K)$, and so $S(K)$ assigned with the above metric becomes a complete metrisable abelian group.

In this and the next paragraph we shall fix once and for all K some compact space and we shall suppose that:

(i) K is metrisable,

(ii) K is totally disconnected.

In this paragraph we shall study $S(K) = S$ and $\hat{S}$ the character group of $S(K)$ for the above fixed K.

Our first task will be to introduce some notations and definitions.

Let us fix once and for all $\{K_n; \pi_n\}_{n=1}^{\infty}$ a projective family of finite spaces K_n and projections (i.e. mappings "onto")

$$\pi_n: K_{n+1} \to K_n \quad (n \geq 1) \tag{1.2}$$

such that:

$$K = \varprojlim_n K_n. \tag{1.3}$$

Such a family always exists by conditions (i) and (ii) on K. Let us denote by

$$\varphi_n: K \to K_n \quad (n \geq 1) \tag{1.4}$$

the induced canonical projections.

The mappings π_n and φ_n induce canonically by transposition

$$\check{\pi}_n: S(K_n) = S_n \to S(K_{n+1}) = S_{n+1}; \; \check{\varphi}_n: S_n \to S \quad (n \geq 1); \tag{1.5}$$

and the mappings $\check{\pi}_n$ and $\check{\varphi}_n$ are isometric embeddings that identify S_n to a closed subgroup of S_{n+1} and of S respectively. Using that identification, let us denote by:

$$H = \bigcup_{n=1}^{\infty} S_n \subset S \tag{1.6}$$

and observe that H is then a dense subgroup of S.

For every $n \geq 1$ there exists a canonical identification of S_n with $\mathbf{T}^{|K_n|}$ ($|K_n| = \text{Card } K_n$) that identifies the function $f \in S(K_n)$ with $\{f(k)\}_{k \in K_n} \in \mathbf{T}^{|K_n|}$. That identification induces also a canonical identification of $\hat{S}_n$ with $\mathbf{Z}^{|K_n|}$ and thus allows us to define uniquely for every $n \geq 1$ and every $\theta \in \hat{S}_n$

$$\text{rank } \theta = \sum_{j=1}^{|K_n|} |m_j| \geq 0,$$

where $\{m_j\}_{j=1}^{|K_n|} \in \mathbf{Z}^{|K_n|}$ is the vector in $\mathbf{Z}^{|K_n|}$ that corresponds to θ in our identification.

The mappings $\check{\pi}_n$ and $\check{\varphi}_n$ of (1.5) induce also canonically (by Pontrjagin duality) mappings

$$\hat{\pi}_n: \hat{S}_{n+1} \to \hat{S}_n; \quad \hat{\varphi}_n: \hat{S} \to \hat{S}_n \qquad (n \geq 1) \tag{1.7}$$

and it is easy to verify then that:

$$\text{rank } (\hat{\pi}_n(\theta)) \leq \text{rank } \theta; \qquad n \geq 1, \theta \in \hat{S}_{n+1}. \tag{1.8}$$

Let us denote then:

$$\Sigma = \varprojlim_n \hat{S}_n$$

the projective limit of the family $\{\hat{S}_n; \hat{\pi}_n\}_{n=1}^{\infty}$ and topologize Σ with the projective limit topology. Σ is then a topological abelian group. Let us denote for every $n \geq 1$ and every non negative integer $m \geq 0$

$$R_m^n = \{\theta \in \hat{S}_n; \text{ rank } \theta \leq m\} \subset \hat{S}_n$$

which is then a finite subset of $\hat{S}_n$. Using (1.8) and the definition of the rank it is easy to verify then that:

$$\hat{\pi}_n(R_m^{n+1}) = R_m^n; \quad n \geqslant 1, m \geqslant 0 \tag{1.9}$$

$$R_0^n = \{0_n\} \ \ (0_n = \text{the identity of } \hat{S}_n). \tag{1.10}$$

It is also easy to verify that if we set $\langle k_n, \sigma_n \rangle = \sigma_n(k_n)$ ($n \geqslant 1$, $k_n \in K_n$, $\sigma_n \in S_n$) we can identify canonically K_n with a subset K_n^* of R_1^n in such a way that the mapping $\pi_n^* = \hat{\pi}_n |_{K_n^*}$ is identified with the mapping π_n (cf. (1.2)).

This allows us to identify canonically and topologically K of (1.3) with the space

$$K^* = \varprojlim_n K_n^*$$

(the projective limit being taken for the family $\{K_n^*; \pi_n^*\}_{n=1}^{\infty}$).

K^* can in its turn be identified canonically and topologically with a closed subset of Σ. So putting these together we obtain a canonical topological identification of K with a closed subset of Σ.

Using (1.9) we shall define

$$R_m = \varprojlim_n R_m^n$$

(the limit is taken for the projective family $\{R_m^n; \hat{\pi}_n |_{R_m^n}\}_{n=1}^{\infty}$ and is assigned the canonical compact topology) and we shall identify, as we may, canonically and topologically R_m with a closed subset of Σ.

It is clear then that with the above identifications we have, using the notations of § 0:

$$R_1 = K_{0,1} \cup K_{1,0} \cup \{0_\Sigma\} \tag{1.11}$$

(0_Σ is the identity of Σ).

The following facts about the R_m^n's are trivially verifiable

$$\hat{\pi}_n\{(K_{n+1}^*)_{\alpha,\beta}\} = (K_n^*)_{\alpha,\beta};\ n \geqslant 1,\ \alpha, \beta \geqslant 0 \tag{1.12}$$

$$R_m^n = \bigcup_{\alpha+\beta \leqslant m} (K_n^*)_{\alpha,\beta}. \tag{1.13}$$

Using (1.11), (1.12) and (1.13) we deduce that

$$R_m = \bigcup_{\alpha+\beta \leqslant m} K_{\alpha,\beta} \subset \Sigma, \quad m \geqslant 1,$$

and that therefore
$$R = \bigcup_{m \geqslant 0} R_m = \operatorname{Gp}(K) \subset \Sigma,$$

so that R is a K_σ subgroup of Σ generated by the compact set K.

The mappings $\hat{\varphi}_n$ of (1.7) define "at the limit" a mapping:

$$\hat{\varphi}: \hat{S} \to \Sigma \tag{1.14}$$

and it is easy to verify (cf. [7]) that for every $\theta \in \hat{S}$ we have

$$\sup_n \operatorname{rank} \hat{\varphi}_n(\theta) < +\infty. \tag{1.15}$$

(1.15) implies then that $\hat{\varphi}(\hat{S}) \subset R$. More explicitly we have the following

THEOREM 1.1. *The mapping* $\hat{\varphi}$ *of (1.14) is* $(1-1)$ *and* $\operatorname{Im} \hat{\varphi} = R$. $\hat{\varphi}$ *identifies therefore* $\hat{S}$ *with* $R = \operatorname{Gp}(K) (\subset \Sigma)$.

Proof. The proof is an easy consequence of (1.15) and it can be found in full details in [7]. We do not give the details here to avoid repetition.

The way Theorem 1.1 is formulated in [7] is more explicit, and ought to be compared with the above (equivalent) formulation.

Remark 1.1. The condition (i) that K is metrisable can be relaxed without affecting the conclusion of Theorem 1.1.

The condition (ii), on the other hand, is essential. Indeed, as it was shown in [5] if K is not totally disconnected Theorem 1.1 is no longer valid.

Remark 1.2. Let $\sigma \in S$ we can define then

$$\mathring{\sigma}: \hat{S} \to \mathbf{T};\ \mathring{\sigma}(\theta) = \theta(\sigma) \quad \forall \theta \in \hat{S}$$

which is then an algebraic homomorphism from $\hat{S}$ to $\mathbf{T}$.

This implies at once that K identified with a subset of $\hat{S}$ is an independent subset of that group (for $\{\mathring{\sigma}|_K;\ \sigma \in S\} = S(K)$).

It is also easy to verify that for every $m \geq 0$ and every $\sigma \in S(K)$ we have

$$\sigma^{(m)} = \mathring{\sigma}|_{R_m} \in S(R_m),$$

in other words, that $\mathring{\sigma}|_{R_m}$ is continuous on R_m.

It is also clear that

$$\sigma_1, \sigma_2 \in S, d(\sigma_1, \sigma_2) \leq \varepsilon \Rightarrow \sup_{\theta \in R_m} |\sigma_1^{(m)}(\theta) - \sigma_2^{(m)}(\theta)| \leq m\varepsilon. \tag{1.16}$$

Remark 1.3. Let us suppose that G is some compact abelian group and that $K \subset G$ is some totally disconnected set of G. The canonical identification

$$\varkappa: K \to G$$

can be extended uniquely to a group homomorphism

$$\hat{\varkappa}: (S(K))^\wedge \rightarrow G$$

which is $1-1$ if K is an independent subset. $\hat{\varkappa}$ is then such that for every $m \geqslant 0$ the mapping

$$\hat{\varkappa}|_{R_m}: R_m \rightarrow G \tag{1.17}$$

is continuous, and for every $\chi \in \hat{G}$ and every $\theta \in \hat{S}$ we have:

$$\theta(\chi|_K) = \chi(\hat{\varkappa}(\theta)) \qquad (\chi|_K \in S(K)). \tag{1.18}$$

2. Positive definite functions on $S(K)$

Let K be as in the previous paragraph and let us preserve all the notations we have introduced there.

Let $M = \{\mu_n \in M^+(R_{m(n)})\}_{n=1}^{\infty}$, where $m(n) \geqslant 0$ depends on n, be a sequence of positive Radon measures defined on the compact spaces $R_{m(n)} \subset \hat{S}$; let us suppose that:

$$\|M\| = \sum_{n=1}^{\infty} \|\mu_n\| < +\infty \tag{2.1}$$

and let us set:

$$p_M(\sigma) = \sum_{n=1}^{\infty} \int_{R_{m(n)}} \theta(\sigma)\, d\mu_n(\theta) \quad \forall \sigma \in S. \tag{2.2}$$

It is clear then that $p_M(\sigma)$ is a continuous positive definite function on the group S (cf. Remark 1.2 and in particular (1.16)).

Let us now set for every Borel subset $C \subset \hat{S}$ (for the topology induced on $\hat{S}$ by Σ)

$$\tilde{M}(C) = \sum_{n=1}^{\infty} \mu_n (R_{m(n)} \cap C).$$

It is clear then that $\tilde{M}$ is an (abstract) measure defined on the Borel field of $\hat{S}$. We shall say then that the sequences

$$M = \{\mu_n \in M^+(R_{m(n)})\}_{n=1}^{\infty}, \|M\| < +\infty; \quad M' = \{\mu_n' \in M^+(R_{m'(n)})\}_{n=1}^{\infty}, \|M'\| < +\infty \tag{2.3}$$

are equivalent if $\tilde{M} = \tilde{M}'$.

It is then easy to verify that if M and M' as in (2.3) are equivalent, then $p_M = p_{M'}$ and $\|M\| = \|M'\|$.

In this paragraph we shall prove the following:

THEOREM 2.1. *Let $p: S \to \mathbf{C}$ be a continuous positive definite function on the group $S(K)$. Then there exists a unique, up to equivalence, sequence of positive Radon measures*

$$M = \{\mu_n \in M^+(R_{m(n)})\}_{n=1}^{\infty}$$

such that:

$$p = p_M; \ \|M\| = p(0_S) = \sup_{\sigma \in S} |p(\sigma)|.$$

The proof is based on a number of lemmas which we proceed to give.

Let p: $S \to \mathbf{C}$ be a continuous positive definite function on S, then for every $n \geq 1$ the function:

$$p_n = p \circ \check{\varphi}_n: S_n \to \mathbf{C} \quad (n \geq 1)$$

(cf. (1.5)) is a continuous positive definite function on the compact group S_n; we can therefore define by the classical Bochner theorem [3] a sequence of positive Radon measures $\{\lambda_n^p \in M^+(\hat{S}_n)\}_{n=1}^{\infty}$ such that for every $n \geq 1$ we have:

$$\|\lambda_n^p\| = p(0_n); \qquad p_n(\sigma) = \int_{\theta \in \hat{S}_n} \theta(\sigma)\, d\lambda_n^p(\theta),\ \sigma \in S_n. \tag{2.3}$$

The λ_n^p's are then such that if we denote by

$$\tilde{\pi}_n: M(\hat{S}_{n+1}) \to M(\hat{S}_n)$$

the mapping induced canonically on the measure spaces by the mapping $\hat{\pi}_n$ of (1.7), then we have

$$\tilde{\pi}_n(\lambda_{n+1}^p) = \lambda_n^p \quad (n \geq 1). \tag{2.4}$$

We have then:

LEMMA 2.1. *Let p_1, p_2 be two continuous positive definite functions on S, then a necessary and sufficient condition for $p_1 >> p_2$ is that*

$$\lambda_n^{p_1} \geq \lambda_n^{p_2} \quad \forall n \geq 1.$$

Proof. Clear if we use (1.6).

We must now introduce some more notations. Let:

$$I_\alpha = \{e^{i\theta} \in \mathbf{T}; |\theta| \leq \alpha\} \subset \mathbf{T}, \quad \alpha > 0$$

$$\zeta_\alpha = \frac{2\pi}{2\alpha}\, \xi_{I_\alpha} \in L^\infty(\mathbf{T})$$

$$\zeta_\alpha^{(n)} = \zeta_\alpha \otimes \zeta_\alpha \otimes \ldots \otimes \zeta_\alpha (|K_n| \text{ times}) \in L^\infty(\mathbf{T}^{|K_n|}) = L^\infty(S_n).$$

(Observe that the $L^1(S_n)$ norm of $\zeta_\alpha^{(n)}$ taken with respect to the normalised Haar measure is always 1.) We shall prove the following

Lemma 2.2. *Let α be some positive number, then for every $\varepsilon>0$ there exists some $m\geqslant 0$ (depending on α and ε only) such that:*

$$\sup_{\sigma\in\hat{S}_n\setminus R^n_m}|\zeta^{(n)}_\alpha(\sigma)|\leqslant\varepsilon,\quad \forall n\geqslant 1.$$

Proof. Observe first that there exists $\delta=\delta(\alpha)>0$ some positive number such that

$$\sup_{p\in\mathbf{Z}\setminus\{0\}}|\hat{\zeta}_\alpha(p)|\leqslant 1-\delta. \tag{2.5}$$

Let us now denote by $\{m_j(\sigma)\}_{j=1}^{|K_n|}\in\mathbf{Z}^{|K_n|}$ the vector that corresponds to $\sigma\in\hat{S}_n$ via the identification of § 1, and let us denote

$$N(\sigma)=\operatorname{Card}\{j;\, m_j(\sigma)\neq 0\}$$

the number of coordinates that are not zero and by

$$M(\sigma)=\max_{1\leqslant j\leqslant|K_n|}|m_j(\sigma)|.$$

We have then from (2.5)

$$|\zeta^{(n)}_\alpha(\sigma)|\leqslant(1-\delta)^{N(\sigma)-1}\sup_{p\geqslant M(\sigma)}|\hat{\zeta}_\alpha(p)|. \tag{2.6}$$

But it is easy to see that for every $K>0$ we can find some $m>0$ depending on K only such that:

$$\sigma\in\hat{S}_n\setminus R^n_m\Rightarrow N(\sigma)+M(\sigma)\geqslant K,\quad \forall n\geqslant 1. \tag{2.7}$$

(2.6) and (2.7) then combined with the Riemann–Lebesgue theorem give our lemma at once.

We can prove now the following key

Lemma 2.3. *Let p be a continuous positive definite function on S and let $\{\lambda^p_n\in M(\hat{S}_n)\}_{n=1}^\infty$ be the Radon measures associated to it as in (2.3). Then for every $\varepsilon>0$ there exists some non negative integer $m\geqslant 0$ (depending on p and ε only) such that*

$$\lambda^p_n(\hat{S}_n\setminus R^n_m)\leqslant\varepsilon\quad\forall n\geqslant 1.$$

Proof. We shall preserve all the notations of Lemma 2.2. Let p be as in the lemma and let us suppose without loss of generality that $p(0_S)=1$. Let us set for every $n\geqslant 1$, $m\geqslant 0$, $\alpha>0$

$$A^n_\alpha=\int_{S_n}p(\sigma)\zeta^{(n)}_\alpha(\sigma)\,dh_n(\sigma);$$

$$B_\alpha^{n,m} = \int_{R_m^n} \xi_\alpha^{(n)}(\theta)\, d\lambda_n^p(\theta);$$

$$C_\alpha^{n,m} = \int_{\hat{S}_n \setminus R_m^n} \xi_\alpha^{(n)}(\theta)\, d\lambda_n^p(\theta);$$

where h_n indicates the normalized Haar measure on S_n.

It is clear then that we have for all $\alpha > 0$, $n \geq 1$, $m \geq 0$:

$$A_\alpha^n = B_\alpha^{n,m} + C_\alpha^{n,m}. \tag{2.8}$$

Let now $\varepsilon > 0$ be as in the lemma, using then the continuity of p we can find some $\alpha(\varepsilon) > 0$ such that

$$|1 - A_{\alpha(\varepsilon)}^n| \leq \frac{\varepsilon}{2} \quad \forall n \geq 1.$$

which together with (2.8) implies that for all $n \geq 1$ and all $m \geq 0$ we have:

$$|1 - \lambda_n^p(R_m^n)| \leq |1 - B_{\alpha(\varepsilon)}^{n,m}| \leq \frac{\varepsilon}{2} + |C_{\alpha(\varepsilon)}^{n,m}| \leq \frac{\varepsilon}{2} + \sup_{\theta \in \hat{S}_n \setminus R_m^n} |\xi_{\alpha(\varepsilon)}^{(n)}(\theta)|$$

and this implies of course our lemma at once.

We shall prove finally the following

LEMMA 2.4. *Let $p \neq 0$ be a continuous positive definite function on S. Then there exists some $m \geq 0$ and a positive Radon measure $\mu \in M^+(R_m)$ such that*

$$\|\mu\| \geq \tfrac{1}{2} p(0_S) = \tfrac{1}{2} \|p\|;\ p >> \int_{\theta \in R_m} \theta(\sigma)\, d\mu(\theta).$$

Proof. We shall preserve all our previous notations. Using our previous lemma we see that we can find some $m_0 \geq 0$ such that

$$\inf_n \lambda_n^p(R_{m_0}^n) \geq \tfrac{1}{2} \|p\|.$$

It is also clear from (1.9) and (2.4) that if we denote by $\tilde{\lambda}_n = \lambda_n^p \xi_{R_{m_0}^n} \in M(R_{m_0}^n) \subset M(\hat{S}_n)$ for all $n \geq 1$, we have:

$$\tilde{\pi}_n \tilde{\lambda}_{n+1} \leq \tilde{\lambda}_n$$

$$\inf \|\tilde{\lambda}_n\| \geq \tfrac{1}{2} \|p\|.$$

This implies that there exists $\tilde{\lambda} \in M^+(R_{m_0})$ some positive Radon measure such that

$$\|\tilde{\lambda}\| \geq \tfrac{1}{2} \|p\|; \quad \tilde{\varphi}_n(\tilde{\lambda}) \leq \tilde{\lambda}_n, \tag{2.9}$$

where
$$\tilde{\varphi}_n: M(R_{m_0}) \to M(R^n_{m_0})$$
is the mapping induced on the measure spaces by
$$\hat{\varphi}_n\,|_{R_{m_0}}: R_{m_0} \to R^n_{m_0}.$$

To satisfy then the conditions of our lemma it suffices to set $m = m_0$ and $\mu = \tilde{\lambda}$ and to use Lemma 2.1 and (2.9).

We are now in a position to give the

Proof of Theorem 2.1.

(i) *Existence.* Let p be a non zero continuous positive definite function on S. Using then Lemma 2.4 we can construct inductively a sequence of measures $M = \{\mu_n \in M^+(R_{m(n)})\}_{n=1}^{\infty}$ and a sequence $\{p_n\}_{n=1}^{\infty}$ of continuous positive definite functions such that

$$p_1 = p$$
$$\|\mu_n\| \geqslant \tfrac{1}{2}\|p_n\| = \tfrac{1}{2}p_n(0_S)$$
$$p_n >> \int_{\theta \in R_{m(n)}} \theta(\sigma)\,d\mu_n(\theta); \qquad p_{n+1} = p_n - \int_{\theta \in R_{m(n)}} \theta(\sigma)\,d\mu_n(\theta), \quad n \geqslant 1.$$

It is easy to verify then that
$$\|M\| = \|p\|, \quad p = p_M$$
and this proves the first part of our theorem.

Before we prove the uniqueness part of our theorem we shall make some general considerations.

Let G be a compact abelian group and let $K \subset G$ be a compact metrisable totally disconnected subset of G, let us denote by
$$\varkappa: K \to G$$
the canonical identification and define
$$\hat{\varkappa}: (S(K))\hat{\ } \to G$$
as in § 1, Remark 1.3.

Let also $M = \{\mu_n \in M^+(R_{m(n)})\}$ be a sequence of measures that satisfies (2.1) (the R_m's and $\hat{S}$ is constructed from the group $S(K)$ with the above $K \subset G$). It is clear then that if we denote by:
$$\tilde{\varkappa}_m: M(R_m) \to M(G)$$
the mapping induced on the measure spaces by the mappings $\tilde{\varkappa}\,|_{R_m}$ of (1.17) we can define a positive Radon measure on G by setting:

$$\nu_M = \sum_{n=1}^{\infty} \tilde{\varkappa}_{m(n)}(\mu_n) \in M^+(G).$$

We have then $\|\nu_M\| = \|M\|$, and $\nu_M(G \setminus \mathrm{Gp}(K)) = 0$.

It is also clear that if M and M' are two sequences as in (2.3) and if K is an independent subset then M and M' are equivalent if and only if $\nu_M = \nu_{M'}$.

From (1.18) it follows also that for any M satisfying (2.1) we have

$$p_M(\chi|_K) = \int_G \chi(g)\, d\nu_M(g). \tag{2.10}$$

We can now complete the

Proof of Theorem 2.1.

(ii) *Uniqueness.* Let H be the group defined in (1.6) and let us assign H with the discrete topology. $\hat{H}$ its character group will then be assigned with the canonical compact topology. We can then define a mapping

$$\varkappa: K \to \hat{H}; \langle \varkappa(k), h \rangle = h(k), \quad h \in H,\ k \in K.$$

It is clear then that $\varkappa$ is continuous and $(1-1)$ and that it identifies K with a Kronecker set of $\hat{H}$. We can apply therefore the above considerations with $M = \{\mu_n \in M^+(R_{m(n)})\}_{n=1}^{\infty}$ some sequence of measures that satisfies

$$p = p_M,$$

where p is our given positive definite function on S, and $G = \hat{H}$.

It follows then from (2.10) that

$$p(h) = p_M(h) = \int_{\hat{h} \in \hat{H}} \langle \hat{h}, h \rangle\, d\nu_M(\hat{h}), \quad h \in H.$$

This proves that ν_M is uniquely determined from p and it implies, if we take the above considerations into account, that p determines M uniquely up to equivalence.

This completes the proof of our theorem.

3. The S^*-groups

Let $(X; \mu)$ be a positive bounded measure space ($\mu \geq 0$, $\mu(X) < +\infty$) which we shall keep fixed throughout this and the next paragraph, and let us denote by

$$S^* = S^*(X; \mu) = \{f \in L^{\infty}(X; \mu);\ |f(x)| = 1 \text{ a.e. } x \in X\}$$

which is an abelian group under pointwise multiplication. Multiplicative notations will be used for S^*.

We shall also denote by

$$U = \{f \in L^\infty(X;\mu);\ \|f\|_\infty \leqslant 1\}$$

the unit ball of L^∞.

We shall consider on $L^\infty(X;\mu)$ two topologies:

(i) The weak topology $\sigma = \sigma(L^\infty; L^1)$ of L^∞ as the dual of $L^1(X;\mu)$

(ii) The topology in measure m for which a nhd. basis of $f_0 \in L^\infty$ is given by the sets

$$W(f_0; \alpha, \beta) = \{f \in L^\infty; \mu[|f - f_0| > \alpha] < \beta\}$$

as α, β run through the positive real numbers.

It is well known and trivial that the topology m restricted on U the unit ball of L^∞ is stronger than the topology σ restricted on U. It is also evident that $m|_{S^*}$ the topology m restricted on the group S^* is compatible with the group structure. We have more precisely:

PROPOSITION 3.1. *The two topologies m and σ coincide on $S^* \subset L^\infty$ and are compatible with the group structure of S^*.*

Proof. The foregoing shows that we need only prove that the σ topology restricted on S^* is stronger than the m topology restricted on S^*.

Towards that let $f_0 \in S^*$ be fixed, we have then for every $f \in S^*$

$$\left|\int_X [\operatorname{Re}(ff_0^{-1}) - 1]\, d\mu\right| \leqslant \left|\int_X (ff_0^{-1} - 1)\, d\mu\right| = \left|\int_X (f - f_0) f_0^{-1}\, d\mu\right|$$

and therefore for arbitrary $\varepsilon < 1$ we have by Chebyshev's inequality

$$\mu[\operatorname{Re}(ff_0^{-1}) \leqslant \varepsilon] \leqslant \frac{\left|\int_X (f - f_0) f_0^{-1}\, d\mu\right|}{1 - \varepsilon}$$

which implies that for arbitrary $0 < \alpha \leqslant 2$

$$\mu[|f - f_0| > \alpha] = \mu[|ff_0^{-1} - 1| > \alpha] \leqslant \mu[\operatorname{Re}(ff_0^{-1}) < \cos\alpha] \leqslant \frac{\left|\int_X (f - f_0) f_0^{-1}\, d\mu\right|}{1 - \cos\alpha}$$

and this proves that $\sigma|_{S^*} \geqslant m|_{S^*}$ and completes the proof of our proposition.

We shall consider throughout S^* as a topological group assigned with the $\sigma|_{S^*} = m|_{S^*}$ topology.

Observe that when the measure space $(X;\mu)$ is separable (i.e. when $L^1(X;\mu)$ is a separable Banach space) then the topological group S^* is metrisable.

What we shall essentially do in the rest of this paragraph is to find all the compact subgroups of S^*. We start with the following.

PROPOSITION 3.2. *Let Γ be a discrete finitely generated abelian group and let $h: \Gamma \to S^*(X; \mu)$ be a group homomorphism. Then there exists a measurable function*

$$b: X \to \hat{\Gamma} \tag{3.1}$$

such that:

$$h(\gamma) = \langle b, \gamma \rangle \in S^*(X; \mu) \quad \forall \gamma \in \Gamma \tag{3.2}$$

(*i.e* $\forall \gamma \in \Gamma \ [h(\gamma)](x) = \langle b(x), \gamma \rangle$ *a.e.* $x \in X$).

Furthermore, the function b above is uniquely determined by (3.2) up to μ-null equivalence.

Proof. We first prove the uniqueness of b. Towards that let b_1, b_2 be as (3.1) both satisfying (3.2) then for every $\gamma \in \Gamma$ there exists $E_\gamma \subset X$ an exceptional set of μ-measure zero such that for all $x \in X \setminus E_\gamma$ we have $\langle b_1(x), \gamma \rangle = \langle b_2(x), \gamma \rangle$.

From this we conclude that

$$x \in X \setminus \left(\bigcup_{\gamma \in \Gamma} E_\gamma \right) \Rightarrow \langle b_1(x), \gamma \rangle = \langle b_2(x), \gamma \rangle, \quad \forall \gamma \in \Gamma;$$

and therefore that

$$x \in X \setminus \left(\bigcup_{\gamma \in \Gamma} E_\gamma \right) \Rightarrow b_1(x) = b_2(x);$$

and this proves the required uniqueness.

Before we prove the existence of b we shall prove the following:

LEMMA 3.1. (i) *Let Γ be a discrete countable abelian group for which the Proposition 3.2 holds, and let $\Delta \subset \Gamma$ be a subgroup of Γ. Then the Proposition 3.2 also holds for the abelian group Γ/Δ.*

(ii) *Let Γ_1, Γ_2 be two discrete countable abelian groups for which the Proposition 3.2 holds. Then the Proposition 3.2 holds for the group $\Gamma = \Gamma_1 \oplus \Gamma_2$.*

Proof of Lemma 3.1. (i) Let $p: \Gamma \to \Gamma/\Delta$ be the canonical projection, let $h: \Gamma/\Delta \to S^*(X; \mu)$ be an arbitrary group homomorphism and let $b: X \to \hat{\Gamma}$ be the measurable function which, by our hypothesis, we can associate to the group homomorphism $h \circ p: \Gamma \to S^*(X; \mu)$ as in (3.1) and (3.2). But then for every $\delta \in \Delta$ there exists $E_\delta \subset X$ an exceptional set of μ-measure zero such that for all $x \in X \setminus E_\delta$ we have $\langle b(x), \delta \rangle = 1 \in \mathbf{T}$. From this we can deduce that

$$x \in X \setminus \left(\bigcup_{\delta \in \Delta} E_\delta \right) \Rightarrow \langle b(x), \delta \rangle = 1 \in \mathbf{T} \quad \forall \delta \in \Delta$$

and so

$$x \in X \setminus \left(\bigcup_{\delta \in \Delta} E_\delta \right) \Rightarrow b(x) \in \Delta^0 \subset \hat{\Gamma},$$

where Δ^0 is the annihilator in $\hat{\Gamma}$ of Δ, but since Δ^0 can be identified canonically with $(\Gamma/\Delta)^\wedge$ we see that b defines a measurable function $b: X \to (\Gamma/\Delta)^\wedge$ for which (3.2) is valid.

(ii) Let $h: \Gamma = \Gamma_1 \oplus \Gamma_2 \to S^*(X; \mu)$ be a given group homomorphism, let $h_i = h|_{\Gamma_i}$: $\Gamma_i \to S^*(X; \mu)$ be its restriction to Γ_i $(i=1, 2)$ and let b_i: $X \to \hat{\Gamma}_i$ be the measurable function which, by hypothesis, we can associate to h_i by (3.1) and (3.2). If we set then $b(x) = (b_1(x), b_2(x)) \in \hat{\Gamma}_1 \oplus \hat{\Gamma}_2 = \hat{\Gamma}$ it is easy to see that b satisfies (3.2) with h. This completes the proof of the lemma.

End of proof of Proposition 3.2. If we use our lemma and if we take into account the fact that any finitely generated group Γ is a quotient of some group of the form $\mathbf{Z}^n$ $(n \geq 1)$ we see that it suffices to prove Proposition 3.2 for $\Gamma = \mathbf{Z}$. Towards that let $\zeta = +1 \in \mathbf{Z}$ be the positive generator of $\mathbf{Z}$ and let $\beta = h(\zeta) \in S^*(X; \mu)$, if we define then b: $X \to \mathbf{T} = \hat{\mathbf{Z}} = \hat{\Gamma}$ by setting $b(x) = \beta(x)$ $(x \in X)$ we see that:

$$h(n) = (h(\zeta))^n = \beta^n \in S^*(X; \mu) \quad \forall n \in \mathbf{Z} = \Gamma$$

which implies that:

$$[h(n)](x) = \beta^n(x) = \langle b(x), n \rangle; \quad x \in X$$

and completes the proof of the proposition.

Remark 3.1. An immediate consequence of the uniqueness of b is that if Γ_1, Γ_2 are two abelian groups and if

$$g: \Gamma_2 \to \Gamma_1 \quad h: \Gamma_1 \to S^*(X; \mu)$$

are group homomorphisms; then the functions b_1, b_2 associated to the homomorphisms h and $h \circ g$ respectively satisfy

$$b_2 = \hat{g} \circ b_1$$

where $\hat{g}$: $\hat{\Gamma}_1 \to \hat{\Gamma}_2$ is the dual homomorphism. We have now the following.

PROPOSITION 3.3. *Let G be a compact metrisable abelian group and let us suppose that there exists a sequence*

$$G_1 \subset G_2 \subset \ldots \subset G_n \subset \ldots G \tag{3.3}$$

of finite subgroups such that $H = \bigcup_{n=1}^{\infty} G_n$ is dense in G. Let further h: $G \to S^*(X; \mu)$ be a continuous group homomorphism. Then there exists

$$b: X \to \hat{G} \tag{3.4}$$

a measurable function such that

$$h(g) = \langle b, g \rangle \quad \forall g \in G \tag{3.5}$$

(i.e. $\forall g \in G$ $[h(g)](x) = \langle b(x), g \rangle$ a.e. $x \in X$).
Furthermore, b in (3.4) is uniquely determined by (3.5) up to μ-null equivalence.

Before we give the proof we observe here that the condition (3.3) is satisfied by all the "classical" groups $G = \mathbf{T}^n$ $(n \geq 1)$.

Proof. The proof of the uniqueness of b follows the same lines as that in Proposition 3.2.

Indeed let b_1, b_2 be two functions that satisfy (3.4) and (3.5) we can find then for every $k \in H = \bigcup_{n=1}^{\infty} G_n$ an exceptional set of μ-measure zero E_k such that

$$x \in X \setminus (\bigcup_{k \in H} E_k) \Rightarrow \langle b_1(x), k\rangle = \langle b_2(x), k\rangle \quad \forall k \in H$$

and H being dense in G this implies that

$$x \in X \setminus (\bigcup_{k \in H} E_k) \Rightarrow b_1(x) = b_2(x)$$

and proves the uniqueness.

Our next goal will be to construct a *well determined* function b: $X \to \hat{G}$ that will depend *only* on h: $G \to S^*(X; \mu)$ and the *sequence of subgroups* (3.3) and then prove that this function satisfies (3.4) and (3.5) [and so in particular, it is also independent of the choice of the sequence (3.3)].

Construction of b: Let

$$i_n\colon G_n \to G; \quad p_n\colon \hat{G} \to \hat{G}_n$$

be the canonical injection of G_n in G and the dual projection, let us also denote for each $n \geq 1$

$$h_n = h \circ i_n\colon G_n \to S^*(X; \mu)$$

and

$$b_n\colon X \to \hat{G}_n$$

the measurable function constructed in Proposition 3.2 that satisfies

$$h_n(g) = \langle b_n, g\rangle, \quad \forall g \in G_n, \ n \geq 1. \tag{3.6}$$

Remark 3.1 implies at once that for each $\chi \in \hat{G}$ we have the following chain of inclusions (up to μ-null equivalence).

$$b_1^{-1}[p_1(\chi)] \supset b_2^{-1}[p_2(\chi)] \supset \ldots \supset b_n^{-1}[p_n(\chi)] \supset \ldots.$$

Let us then denote for each $\chi \in \hat{G}$

$$X_\chi = \bigcap_{n=1}^{\infty} b_n^{-1}[p_n(\chi)] \subset X$$

and observe that the density of $H = \bigcup_{n=1}^{\infty} G_n$ in G implies that

$$\chi_1, \chi_2 \in \hat{G}, \chi_1 \neq \chi_2 \Rightarrow X_{\chi_1} \cap X_{\chi_2} = \emptyset \tag{3.7}$$

[up to μ-null equivalence of course].

We shall now define b: $X \to \hat{G}$ (up to μ-null equivalence) by setting

$$b(X_\chi) = \chi \quad \forall \chi \in \hat{G}; \qquad b[\complement \bigcup_{\chi \in \hat{G}} X_\chi] = 0_{\hat{G}} \in \hat{G}. \tag{3.8}$$

The definition of b in (3.8) as we said at the beginning clearly depends only on the chain (3.3) and the homomorphism $h: G \to S^*(X; \mu)$.

Let now $(X_1; \mu_1) \subset (X; \mu)$ be a submeasure space of $(X; \mu)$. [i.e. let $X_1 \subset X$ be a subset of positive measure and let μ_1 be the measure μ restricted on X_1 $(\mu_1 = \mu|_{X_1} \neq 0)$], let us further denote by $r: S^*(X; \mu) \to S^*(X_1; \mu_1)$ the restriction mapping defined by:

$$r(f) = f\,|_{X_1} \quad \forall f \in S^*(X; \mu)$$

and let us denote by $h^{(1)} = r \circ h: G \to S^*(X_1; \mu_1)$ and by $b^{(1)}: X_1 \to \hat{G}$ the measurable function constructed as in (3.8) from the chain (3.3) and the homomorphism $h^{(1)}$.

We have then:

$$b^{(1)} = b\,|_{X_1} \, (= b \text{ restricted on } X_1). \tag{3.9}$$

Indeed if we construct $b_n^{(1)}: X_1 \to \hat{G}_n (n \geq 1)$ satisfying

$$h^{(1)} \circ i_n(g) = \langle b_n^{(1)}, g \rangle \quad \forall g \in G_n$$

as in (3.6) we have by the uniqueness part of proposition 3.2 that $b_n^{(1)} = b_n\,|_{X_1}$ $(n \geq 1)$ and if we substitute this in the definition of the X_χ's and (3.7) we get (3.9) at once.

Proof of (3.5). We shall now prove that the mapping b that we constructed in (3.8) satisfies (3.5).

Towards that we claim that if suffices to prove that:

$$\int_X h(g)\, d\mu = \int_X \langle b, g \rangle\, d\mu, \quad \forall g \in G \tag{3.10}$$

$$\left(\text{i.e.} \quad \int_X [h(g)](x)\, d\mu(x) = \int_X \langle b(x), g \rangle\, d\mu(x), \quad \forall g \in G\right)$$

which is "a priori" weaker than (3.5).

Indeed if (3.10) is proved it will also follow that for every $X_1 \subset X$ of positive measure $\mu(X_1) > 0$ we have

$$\int_{X_1} h(g)\, d\mu = \int_{X_1} \langle b, g \rangle\, d\mu. \tag{3.11}$$

For, by (3.9), the relation (3.11) is no other than (3.10) applied to the homomorphism $h^{(1)} = r \circ h: G \to S^*(X_1; \mu_1)$ and the function $b^{(1)}$ constructed from it.

Since, on the other hand, X_1 in (3.11) is arbitrary, subject only to the condition $\mu(X_1) > 0$ (3.11) implies the required (3.5).

Proof of (3.10). The first thing to observe is that

$$P(g) = \int_X h(g)\, d\mu \quad g \in G$$

is a continuous positive definite function. Indeed for arbitrary $g_1, \dots, g_N \in G$ and $\lambda_1, \dots, \lambda_N \in \mathbf{C}$ we have:

$$\sum_{i,j=1}^{N} P(g_i g_j^{-1})\, \lambda_i \bar{\lambda}_j = \int_X \left(\sum_{i,j=1}^{N} h(g_i)\, \overline{h(g_j)}\, \lambda_i \bar{\lambda}_j \right) d\mu = \int_X \left| \sum_{i=1}^{N} h(g_i)\, \lambda_i \right|^2 d\mu \geqslant 0$$

and P is continuous because of the continuity of h.

From Bochner's theorem [3] it follows therefore that

$$\int_X h(g)\, d\mu = P(g) = \sum_{\chi \in \hat{G}} \alpha_\chi \chi(g), \tag{3.12}$$

where

$$\alpha_\chi \geqslant 0 \ \forall\, \chi \in \hat{G};\ \sum_{\chi \in \hat{G}} \alpha_\chi = P(0_G) = \int d\mu = \|\mu\|. \tag{3.13}$$

A direct substitution in (3.12) implies then that for all $n \geqslant 1$ and all $g \in G_n$

$$\begin{aligned} \int_X h_n(g)\, d\mu &= \int_X h \circ i_n(g)\, d\mu = P(i_n(g)) = \sum_{\chi \in \hat{G}} \alpha_\chi \chi(i_n(g)) \\ &= \sum_{\chi \in \hat{G}} \alpha_\chi [p_n(\chi)]\,(g) = \sum_{\psi \in \hat{G}_n} \left(\sum_{p_n(\chi) = \psi} \alpha_\chi \right) \psi(g). \end{aligned} \tag{3.14}$$

And also from (3.6) we have:

$$\int_X h_n(g)\, d\mu = \int_X \langle b_n, g \rangle d\mu = \sum_{\psi \in \hat{G}_n} \mu[b_n^{-1}(\psi)]\, \psi(g) \tag{3.15}$$

and comparing coefficients in (3.14) and (3.15) we obtain then that for every $n \geqslant 1$ and every $\psi \in \hat{G}_n$ we have:

$$\mu[b_n^{-1}(\psi)] = \sum_{p_n(\chi) = \psi} \alpha_\chi$$

which implies that for every $\chi \in \hat{G}$ and every $n \geqslant 1$ we have

$$\mu[b_n^{-1}(p_n(\chi))] \geqslant \alpha_\chi$$

and therefore also (n being arbitrary)

$$\mu(X_\chi) \geqslant \alpha_\chi;\ \chi \in \hat{G}. \tag{3.16}$$

But (3.16) combined with (3.13) gives then

$$\mu(X_\chi) = \alpha_\chi \quad \forall \chi \in \hat{G}; \ \mu[\complement(\bigcup_{\chi \in \hat{G}} X_\chi)] = 0.$$

And this of course together with (3.8) and (3.12) proves the required (3.10).

4. The Θ-semigroups and the homomorphism of the S-groups

Let $(X; \mu)$ and U be as in § 3. We shall topologise throughout in this paragraph U with the weak topology σ which is compact on U. What will be relevant in this paragraph is that U is a semigroup under pointwise multiplication; it is indeed a topological semigroup in the weak sense that multiplication is separately continuous (i.e. $U \ni u_\alpha \xrightarrow[\alpha \in A]{} u \in U, v \in U \Rightarrow u_\alpha v \xrightarrow[\alpha \in A]{} uv$). U is also of course convex and closed under complex conjugation in L^∞. We have first the following.

PROPOSITION 4.1. *Let $V \subset U$ be a weakly closed convex subsemigroup of U, closed under complex conjugation. We can then decompose X into two disjoint measurable subsets Y and Z:*

$$X = Y \cup Z; \quad Y \cap Z = \emptyset$$

such that:

$$v \in V \Rightarrow v(x) = 1 \ a.e. \ x \in Y \tag{4.1}$$

$$\exists v_0 \in V \ such \ that \ v_0(x) = 0 \ a.e. \ x \in Z. \tag{4.2}$$

Proof. We denote by Q the class of all measurable subsets Z' of X for which there exists some $v' \in V$ (v' depending on Z') such that

$$\operatorname*{ess\,sup}_{x \in Z'} |v'(x)| < 1.$$

The fact that V is a semigroup implies then that the union of two sets of Q is still in Q.

The fact that V is convex and closed under complex conjugation implies that we can distinguish two mutually exclusive and exhaustive cases

Case A: $$v \in V \Rightarrow v(x) = 1 \text{ p.p. } x \in X$$

Case B: $$z = \sup_{Z \in Q} \mu(Z) > 0.$$

We treate these two cases separetely:

Case A: We set $Z = \emptyset$ and (4.1) (4.2) are verified trivially.

Case B: We choose for every $n \geq 1$ some $Z_n \in Q$ and some $v_n \in V$ such that

$$\mu(Z_n) \geq z - \frac{1}{n}; \ \operatorname*{ess\,sup}_{x \in Z_n} |v_n(x)| < 1 \tag{4.3}$$

and we set $Z = \bigcup_{n=1}^{\infty} Z_n$, $Y = \complement Z$.

Now (4.1) is verified trivially for if not, by the convexity and the fact that V is closed by complex conjugation it would follow that there would exist some $\tilde{Z} \in Q$ such that

$$\mu(\tilde{Z}) > 0 \quad \tilde{Z} \subset Y$$

which is impossible by the choice (4.3) (observe that we would have then $Z_n \cup \tilde{Z} \in Q \ \forall n \geqslant 1$).

Finally to verify (4.2) it is clear that it suffices to choose for v_0 some limit point in V (for the weak topology) of the set $\{v_1^n v_2^n \dots v_n^n\}_{n=1}^{\infty}$. (It is here that the weak closure of V is essential.) This terminates the proof.

We shall introduce now a construction which in some sense is the kernel of the whole paper. Towards that let us fix for the rest of this paragraph

$$\Phi, \Delta \tag{4.4}$$

two topological abelian groups, Φ being discrete, and let us also fix

$$\gamma\colon \Phi \to \Delta \quad g\colon \Phi \to S^*(X; \mu) \tag{4.5}$$

two group homomorphisms.

We shall denote then by $\mathfrak{M}$ the nhd. filter of $1 \in \Delta$ (multiplicative notations will be used for Φ and Δ). For any subset $E \subset \Delta$ we shall denote by $\tilde{E} \subset U$ the weak closure (in U) of the convex hull (in L^∞ for its vector space structure over $\mathbf{R}$) of the set $g[\gamma^{-1}(E)]\,(\subset S^* \subset U)$. We shall finally denote by $\Theta = \bigcap_{N \in \mathfrak{M}} \tilde{N}$.

We prove first the following.

LEMMA 4.1. *Let A, B, $C \subset \Delta$ be subsets such that*

$$B \cdot C = \{b \cdot c \in \Delta;\ b \in B,\ c \in C\} \subset A$$

we have then
$$\tilde{B} \cdot \tilde{C} = \{b \cdot c \in U;\ b \in \tilde{B},\ c \in \tilde{C}\} \subset \tilde{A}.$$

Proof. Indeed it is clear that

$$b \in \text{Convex hull of } g[\gamma^{-1}(B)] \subset U; \quad c \in \text{Convex hull of } g[\gamma^{-1}(C)] \subset U$$
$$\Rightarrow b \cdot c \in \text{Convex hull of } g[\gamma^{-1}(A)] \subset U$$

and since multiplication is separately continuous for the weak topology of U we see that this implies

$$b \in \tilde{B};\ c \in \text{Convex hull of } g[\gamma^{-1}(C)] \Rightarrow b \cdot c \in \tilde{A}$$

which in turn implies (because of the same reason)

$$b \in \tilde{B};\ c \in \tilde{C} \Rightarrow b \cdot c \in \tilde{A}$$

and proves our lemma.

PROPOSITION 4.2. *Θ is a weakly closed, convex, closed under complex conjugation, multiplicative subsemigroup of U containing* 1.

Proof. The facts that $1\in\Theta$ and that Θ is convex and weakly closed are immediate from the definition.

Also if we denote for any $E\subset\Delta$

$$E^{-1}=\{e^{-1};\ e\in E\}\subset\Delta,$$

and for any $T\subset S^*$
$$T^{-1}=\overline{T}=\{t^{-1}=\bar{t};\ t\in T\}\subset S^*,$$

we see that we have for any $E\subset\Delta$

$$g[\gamma^{-1}(E^{-1})]=(g[\gamma^{-1}(E)])^{-1}=\overline{g(\gamma^{-1}(E))}\subset S^*\subset U$$

and therefore also
$$(E^{-1})^{\sim}=\bar{\tilde{E}}=\{\bar{u};\ u\in\tilde{E}\}\subset U, \tag{4.6}$$

but if we take into account then that for any $N\in\mathfrak{M}$ we have $N^{-1}\in\mathfrak{M}$ we see that (4.6) implies that Θ is closed under complex conjugation.

Now for every $N\in\mathfrak{M}$ there exists some $N_1\in\mathfrak{M}$ such that $N_1^2=\{ab;\ a,\ b\in N_1\}\subset N$ and therefore also, from Lemma 4.1, such that $(\tilde{N}_1)^2\subset\tilde{N}$. N being arbitrary this implies then that Θ is closed under multiplication and is a sub-semigroup of U.

What Proposition 4.2 says is simply that Θ satisfies the conditions of Proposition 4.1.

Let now $(X_1,\mu_1)\subset(X,\mu)$ be some submeasure space of (X,μ) [i.e. $X_1\subset X$ is a measurable subset such that $\mu(X_1)>0$ and $\mu_1=\mu|_{X_1}$] and let $r\colon L^\infty(X;\mu)\to L^\infty(X_1,\mu_1)$ be the restriction mapping $r(f)=f|_{X_1}$. It is easy to verify then using the compactness of the unit ball of L^∞ for the weak topology that for every $N\in\mathfrak{M}$ the weak closure of the convex hull in $L^\infty(X_1,\mu_1)$ of $r\circ g[\gamma^{-1}(N)]$ is equal to $r(\tilde{N})=\{r(u);\ u\in\tilde{N}\}$. Starting now from the same groups Φ and Δ as in (4.4) and the homomorphisms

$$\gamma\colon\Phi\to\Delta\qquad g_1=r\circ g\colon\Phi\to S^*(X_1;\mu_1)$$

we can construct a subsemigroup Θ_1 of the unit ball U_1 of $L^\infty(X_1,\mu_1)$ as in Proposition 4.2. The above remark implies then at once that

$$\Theta_1=r(\Theta)=\{r(u);\ u\in\Theta\}\subset U_1. \tag{4.7}$$

We shall now introduce a new group Σ the closure of $\gamma(\Phi)$ in Δ, both $\gamma(\Phi)$ and Σ are subgroups of Δ and will be topologised with the restriction topology.

It is now clear that if we suppose that $\Theta=1$ then for every $x\in\Delta$ Card $[g\circ\gamma^{-1}(x)]\leqslant 1$ which implies that $\Gamma=g\circ\gamma^{-1}$ defines a well determined homomorphism:

$$\Gamma\colon\gamma(\Phi)\to S^*(X;\mu);\ \Gamma(\gamma(\varphi))=g(\varphi)\quad\forall\varphi\in\Phi.$$

Further Γ is continuous for the fact that $\bigcap_{N\in\mathfrak{m}}\tilde{N}=\Theta=\{1\}$ implies that

$$\Gamma(N\cap\gamma(\Phi))=g[\gamma^{-1}(N)]\subset\tilde{N}\cap S^*\xrightarrow[N\in\mathfrak{m}]{}1.$$

Our next aim is to extend Γ to a continuous group homomorphism from Σ to $S^*(X;\mu)$. Towards that let $\psi\in\Sigma$ be an arbitrary element we observe then that by (4.6) and Lemma 4.1 we have

$$(\psi N\cap\gamma(\Phi))^{\sim}.\overline{(\psi N\cap\gamma(\Phi))^{\sim}}\subset(NN^{-1}\cap\gamma(\Phi))^{\sim}\subset(NN^{-1})^{\sim}\subset U;N\in\mathfrak{m}.$$

This implies that if we set

$$F_\psi=\bigcap_{N\in\mathfrak{m}}(\psi N\cap\gamma(\Phi))^{\sim}\subset U$$

we have $F_\psi\cdot\overline{F}_\psi\subset\Theta=\{1\}$ which means that F_ψ reduces to a single point of $S^*(X;\mu)$. But this implies that the filter basis

$$\{\Gamma(\psi N\cap\gamma(\Phi))\}_{N\in\mathfrak{m}}=\{g\gamma^{-1}[\psi N\cap\gamma(\Phi)]\}_{N\in\mathfrak{m}}$$

converges to the single point F_ψ of $S^*(X;\mu)$. If we set then $\Gamma(\psi)=F_\psi$ we see that we have the required extension of Γ(cf. [8]), which is of course still a group homomorphism. Summing up we see that we have proved the following.

PROPOSITION 4.3. Let us suppose that $\Theta=1$; there exists then a unique continuous group homomorphism $\Gamma\colon\ \Sigma\to S^*(X;\ \mu)$ such that for every $\varphi\in\Phi$ we have $\Gamma(\gamma(\varphi))=g(\varphi)$.

5. Measures on an H_1 set

Let throughout in this paragraph G be a compact abelian group and $K\subset G$ be a totally disconnected H_1 subset of G. Let us fix also $\nu,\varkappa\in M^+(G)$ two positive Radon measures on G such that

$$\nu(K)=0;\ \operatorname{supp}\varkappa=K. \tag{5.1}$$

We shall specialize the parameters in (4.4) and (4.5) as follows

$$\begin{cases}(X;\mu)=(G;\nu);\ \Phi=\{e^{i\theta}\chi\in\mathbf{C}(G);\ \theta\in\mathbf{R},\chi\in\hat{G}\},\Delta=S^*(K;\varkappa)\\ \gamma(\varphi)=\varphi|_K\in S^*(K;\varkappa);\ g(\varphi)=\varphi\in S^*(G;\nu),\ \forall\varphi\in\Phi\end{cases} \tag{5.2}$$

and we shall denote by $\Theta(K;\varkappa,\nu)$ the semigroup of Proposition 4.2 constructed with the above choice of parameters.

It is then clear that the subgroup Σ of Proposition 4.3 constructed with the choice of parameters (5.2) is the whole group Δ [7].

We shall suppose throughout in this paragraph that K, $\varkappa$ and ν are such that

$$\Theta(K; \varkappa, \nu) = 1 \tag{5.3}$$

and under the hypothesis (5.3) draw conclusions on K, $\varkappa$ and ν.

Let now $\mathcal{D} = \{K_j \subset K\}_{j\in J}$ $|J| = |\mathcal{D}| < +\infty$ be a finite partition of the space K into disjoint open and closed subsets and let $\Delta_{\mathcal{D}} \subset \Delta$ be the closed subgroup of Δ consisting of those "functions" of $S(K) \subset \Delta = S^*(K; \varkappa)$ that are piecewise constant on the sets K_j ($j \in J$) of our partition (i.e. Card $(f(K_j)) = 1$, $j \in J$); let us also denote by

$$\Gamma_{\mathcal{D}} = \Gamma|_{\Delta_{\mathcal{D}}}: \Delta_{\mathcal{D}} \to S^*(G; \nu),$$

where Γ is of course the homomorphism of Proposition 4.3 which is now defined on the whole of $\Delta(=\Sigma)$.

$\Delta_{\mathcal{D}}$ is always a compact group that satisfies the conditions of Proposition 3.3 ($\Delta_{\mathcal{D}} \cong \mathbf{T}^{|\mathcal{D}|} = \mathbf{T}^{|J|}$). We can associate therefore to $\Gamma_{\mathcal{D}}$ a Borel function $\beta_{\mathcal{D}}: G \to \hat{\Delta}_{\mathcal{D}}$ such that

$$\Gamma_{\mathcal{D}}(\delta) = \langle \beta_{\mathcal{D}}, \delta \rangle, \quad \forall \delta \in \Delta_{\mathcal{D}}. \tag{5.4}$$

Let us denote for every $X \in \mathcal{D}$ ($X = K_j$ for some $j \in J$ in our case) θ_X: $\delta \to \langle \theta_X, \delta \rangle = \delta(X) \in \mathbf{T}$ ($\delta \in \Delta_{\mathcal{D}}$, $\theta_X \in \hat{\Delta}_{\mathcal{D}}$, δ is of course a function on K which is constant on the set X, $\delta(X)$ is simply its value on that set), let us finally denote by

$$E_{\mathcal{D}} = \bigcup_{X \in \mathcal{D}} \beta_{\mathcal{D}}^{-1}(\theta_X).$$

We have then:

LEMMA 5.1. *Let K, $\varkappa$ and ν satisfy* (5.3) *and let $\mathcal{D}_1 \leqslant \mathcal{D}_2 \leqslant \ldots$ be an increasing sequence of open and closed partitions* (*in the sense that every set of $\mathcal{D}_{n+1}$ is a subset of some set of $\mathcal{D}_n$*) *such that $\bigcup_{j=1}^{\infty} \Delta_{\mathcal{D}_j}$ is dense in $S(K)$* (*for the uniform topology*). *We have then:*

$$E_{\mathcal{D}_1} \supseteq E_{\mathcal{D}_2} \supseteq \ldots \supseteq E_{\mathcal{D}_j} \supseteq \ldots \supseteq \bigcap_{j=1}^{\infty} E_{\mathcal{D}_j} = E$$

and $\nu(E) = 0$.

Proof. The fact that the sequence $E_{\mathcal{D}_j}$ is decreasing is purely formal and evident, let us denote by $\nu_j = \nu|_{E_{\mathcal{D}_j}}$; $\nu_\infty = \nu|_E$; what we must show is that $\nu_\infty = 0$.

Let us fix once and for all, for each $j \geqslant 1$, a Borel mapping α_j: $G \to K$ such that

$$\alpha_j(\beta_{\mathcal{D}_j}^{-1}(\theta_X)) \subset X \quad \forall X \in \mathcal{D}_j \tag{5.5}$$

and which is arbitrary otherwise, let us denote by $\lambda_j = \dot{\alpha}_j(\nu_j)$ the image measure of ν_j by α_j and let $\lambda \in M(K)$ be some weak limit point of the sequence $\{\lambda_j \in M(K)\}_{j=1}^{\infty}$ (i.e. for the weak topology $\sigma(M; \mathbf{C})$).

It is then clear from (5.4) and (5.5) that for every $j \geqslant 1$ and every $\delta \in \Delta_{p_j} \subset \Delta$ we have

$$\begin{aligned}\int_G \Gamma(\delta)\,d\nu_j &= \int_G \Gamma_{p_j}(\delta)\,d\nu_j = \int_G \langle \beta_{p_j}, \delta\rangle\,d\nu_j = \int_{E_p} \langle \beta_{p_j}, \delta\rangle\,d\nu_j \\ &= \sum_{X \in p_j} \nu[\beta_{p_j}^{-1}(\theta_X)]\langle \theta_X, \delta\rangle = \sum_{X\in p_j} \nu[\beta_{p_j}^{-1}(\theta_X)]\,\delta(X) = \int_K \delta d\lambda_j = \int_K \delta(k)\,d\lambda_j(k).\end{aligned} \tag{5.6}$$

Therefore if we let $j \to \infty$ in (5.6) we obtain by the definition of ν_∞ that

$$\int_G \Gamma(\delta)\,d\nu_\infty = \int_K \delta d\lambda \quad \forall \delta \in \bigcup_{j=1}^{\infty} \Delta_{p_j} \tag{5.7}$$

but since $\bigcup_{j=1}^{\infty} \Delta_{p_j}$ is dense in $S(K)$ and since $\nu_\infty \in L^1(G;\nu)$ we deduce from (5.7) and the continuity of the homomorphism Γ that

$$\int_G \Gamma(\delta)\,d\nu_\infty = \int_K \delta d\lambda \quad \forall \delta \in S(K). \tag{5.8}$$

But if we set $\delta = \chi|_K$ ($\chi \in \hat{G}$) in (5.8) we deduce that $\hat{\nu}_\infty = \hat{\lambda}$ (cf. definition of Γ § 4), i.e. $\nu_\infty = \lambda$ which is only possible if $\nu_\infty = \lambda = 0$ since by (5.1) we have:

$$0 \leqslant \nu_\infty\,(\operatorname{supp} \lambda) \leqslant \nu(K) = 0.$$

We can now prove the following proposition:

PROPOSITION 5.1. *Let us suppose that K, $\varkappa$ and ν satisfy* (5.3) *and let $\varepsilon, \eta \in (0, 1)$ be two given real numbers. Then we can find Q a trigonometric polynomial on G such that:*

(i) $\|Q\|_A \leqslant 4\varepsilon^{-1}$

(ii) $\varkappa[k \in K;\ |Q(k) - 1| \geqslant \eta] \leqslant \eta$

(iii) $\nu[g \in G;\ |Q(g)| \geqslant \varepsilon] \leqslant \eta$.

Before we give the proof of Proposition 5.1 we shall prove the following

LEMMA 5.2. Let $\varepsilon, \eta \in (0, 1)$ *be given real numbers, let $\Xi \subset \mathbf{T}$ be a finite independent* (*over* $\mathbf{Z}$) *subset of* $\mathbf{T}$, *and let $Z \subset \mathbf{T}$ be another finite subset such that $Z \cap \Xi = \emptyset$. There exists then a trigonometric polynomial $P(\theta)$ ($\theta \in \mathbf{T}$) on $\mathbf{T}$ such that*

$$\|P\|_A \leqslant 4\,\varepsilon^{-1}; \quad \sup_{x\in\Xi} |P(x) - 1| < \eta; \quad \sup_{x \in Z} |P(x)| < \varepsilon. \tag{5.9}$$

Proof of the lemma. Using the standard construction of a Riesz product [1] on the group $\mathbf{T}$ with the discrete topology we see that we can find $\mu \in M(\bar{\mathbf{Z}})$ ($\bar{\mathbf{Z}}$ being the Bohr compactified of $\mathbf{Z}$) such that:

$$\|\mu\| \leqslant 4\,\varepsilon^{-1}; \hat{\mu}(x) = 1 \quad \forall x \in \Xi; \ \sup_{x\in Z} |\hat{\mu}(x)| \leqslant \varepsilon/2.$$

It suffices to set then $P=\hat{D}$ where $D\in M(\mathbf{Z})$ is a finite sum of Dirac δ-measures and approximates μ for the weak topology in $M(\overline{\mathbf{Z}})$.

Proof of Proposition 5.1. From the definition of $E_{\mathfrak{D}}$ and Lemma 5.1 we see that we can find some partition $\mathfrak{D}$ of K and $\theta_1, \theta_2, \dots \theta_p \in \hat{\Delta}_{\mathfrak{D}}$ finitely many characters all distinct and distinct from any of the characters $\{\theta_X; X\in\mathfrak{D}\}$ such that:

$$\nu[\complement \bigcup_{j=1}^{p} (\beta_{\mathfrak{D}}^{-1}(\theta_j))] < \eta. \tag{5.10}$$

Let us now choose $\xi\in\Delta_{\mathfrak{D}}$ some function such that the points $\xi(X)\in\mathbf{T}$ $(X\in\mathfrak{D})$ are distinct and independent (over $\mathbf{Z}$). If we take then into account that $\{\theta_X; X\in\mathfrak{D}\}\subset\hat{\Delta}_{\mathfrak{D}}$ is a set of free generators of the group $\hat{\Delta}_{\mathfrak{D}}$ we see that

$$\{\zeta_j=\langle\theta_j,\xi\rangle;\ j=1,2,\dots p\}\cap\{\xi(X);\ X\in\mathfrak{D}\}=\varnothing. \tag{5.11}$$

Let then P be the polynomial constructed in Lemma 5.2 with the ε and η of the Proposition 5.1 and with

$$\Xi=\{\xi(X);\ X\in\mathfrak{D}\};\ Z=\{\zeta_1,\dots,\zeta_p\}.$$

Let also $\{\varphi_\alpha\in\Phi\}_{\alpha\in A}$ be a net of functions of the form $e^{i\theta}\chi$ $(\chi\in\hat{G})$ such that

$$\gamma(\varphi_\alpha)=\varphi_\alpha|_K \xrightarrow[\alpha\in A]{} \xi \quad (\text{in } \Delta).$$

It is evident then by (5.9) that

$$\|P\circ\varphi_\alpha\|_{A(G)}\leqslant 4\varepsilon^{-1},\quad \alpha\in A \tag{5.12}$$

and that

$$\lim_{\alpha\in A}\varkappa[k\in K;\ |P\circ\varphi_\alpha(k)-1|\geqslant\eta]=0. \tag{5.13}$$

But we also have:

$$S^*(G;\nu)\ni\varphi_\alpha=g(\varphi_\alpha)=\Gamma(\gamma(\varphi_\alpha))\xrightarrow[\alpha\in A]{}\Gamma(\xi)\in S^*(G;\nu) \tag{5.14}$$

for the topology of $S^*(G;\nu)$ of course.

On the other hand, by the definition of $\beta_{\mathfrak{D}}$ and the ζ_j's if we denote by:

$$\beta_{\mathfrak{D}}^{-1}(\theta_j)=E_j\quad \nu_j=\nu\xi_{E_j}\quad j=1,2,\dots p,$$

we have

$$\Gamma(\xi)|_{E_j}=\Gamma_{\mathfrak{D}}(\xi)|_{E_j}=\langle\theta_j,\xi\rangle=\zeta_j\qquad j=1,2,\dots,p, \tag{5.15}$$

where of course $\Gamma(\xi)|_{E_j}$ is considered as an element of $L^\infty(E_j;\nu_j)$.

Combining (5.14) and (5.15) we conclude that

$$\varphi_\alpha|_{E_j}\xrightarrow[\alpha\in A]{}\zeta_j\quad j=1,2,\dots p \tag{5.16}$$

for the m topology (i.e. in measure) of $L^\infty(E_j; \nu_j)$. (5.9), (5.10) and (5.16) imply then that

$$\overline{\lim_{\alpha\in A}}\ \nu[g\in G;\ |P\circ\varphi_\alpha(g)|\geqslant\varepsilon]<\eta$$

and this together with (5.12) and (5.13) proves that, to satisfy the conclusions of Proposition 5.1, it suffices to set $Q=P\circ\varphi_\alpha$ for some $\alpha\in A$.

PROPOSITION 5.2. *Let* K, $\varkappa$, *and* ν *be as above satisfying* (5.3) *and let us suppose that in addition* K *is metrisable. Then*

$$\nu(G\setminus \mathrm{Gp}\,(K))=0.$$

Proof. Let us preserve all our previous notations and define for every $\sigma\in S(K)\subset S^*(K;\varkappa)$

$$p(\sigma)=\int_G \Gamma(\sigma)\,d\nu.$$

It is clear then that p is a continuous positive definite function on $S(K)$; from (2.13) it follows therefore that there exists some $\varrho\in M^+(G)$ such that:

$$p(\chi|_K)=\int_G \chi(g)\,d\varrho(g)\quad \forall\,\chi\in\hat{G};\ \varrho(G\setminus\mathrm{Gp}\,(K))=0$$

and the uniqueness of the Fourier transform proves then that $\varrho=\nu$ and completes the proof of our proposition.

Remark 5.1. It is clear that throughout this paragraph we may replace the original conditions that supp $\varkappa=K$ is a totally disconnected space by the weaker condition that the support of $\varkappa$ is totally disconnected and is contained in K.

6. Proof of Theorems 1 and 2

Proof of Theorem 1. Let G, K, and E be as in Theorem 1 and let us suppose in contradiction that there exist ε, $\eta\in(0, 1)$ two numbers for which the conclusion of Theorem 1 fails. Let us denote by

$$C(\varepsilon_1,\varepsilon_2)=\{(\varphi,\psi)\in \mathrm{C}(K)\oplus\mathrm{C}(E);\ \|\varphi-1\|_\infty\leqslant\varepsilon_1,\ \|\psi\|_\infty\leqslant\varepsilon_2\}\subseteq \mathrm{C}(K)\oplus\mathrm{C}(E)$$

$$K(\varepsilon_3)=\{(f|_K, f|_E)\in\mathrm{C}(K)\oplus\mathrm{C}(E);\ f\in A(G),\ \|f\|_A\leqslant\varepsilon_3\}\subseteq\mathrm{C}(K)\oplus\mathrm{C}(E)$$

which are both convex subsets of $\mathrm{C}(K)\oplus\mathrm{C}(E)$. Our contradictory hypothesis above is equivalent to the fact that

$$\mathrm{C}(\eta,\varepsilon)\cap K(8\varepsilon^{-1})=\varnothing. \tag{6.1}$$

A simple application of Hahn–Banach's theorem on (6.1) implies then that there exist $\varkappa \in M^+(K)$, $\nu \in M^+(G)$ two non zero positive Radon measures and φ, ψ two Borel functions of modulus $|\varphi| = |\psi| = 1$ such that for all $f \in K(8\varepsilon^{-1})$:

$$\text{either} \qquad \left|\int_G (f-1)\varphi\, d\varkappa\right| \geq \eta\|\varkappa\| \qquad\qquad \text{or} \qquad \left|\int_G f\psi\, d\nu\right| \geq \varepsilon\|\nu\|. \tag{6.2}$$

(6.2) will provide us with a contradiction.

Let us construct the semigroup $\Theta(K; \varkappa, \nu)$ as in paragraph 5 and using Proposition 4.1 let us construct $L, \Lambda \subset G$ two Borel subsets such that

$$L \cup \Lambda = G \quad L \cap \Lambda = \emptyset \tag{6.3}$$

$$f \in \Theta(K; \varkappa, \nu) \Rightarrow f(l) = 1, \quad l \in G \text{ a.e. } \nu_L = \nu\xi_L \tag{6.4}$$

$$\exists \theta \in \Theta(K; \varkappa, \nu) \text{ s.t. } \theta(\lambda) = 0, \lambda \in G \text{ a.e. } \nu_\Lambda = \nu\xi_\Lambda. \tag{6.5}$$

It is clear then from (6.4) and (4.7) that if $\nu_L \neq 0$ then $\Theta(K; \varkappa, \nu_L) = 1$. We can apply therefore Proposition 5.1 to K, $\varkappa$ and ν_L and obtain $Q \in A(G)$ some trigonometric polynomial on G such that:

$$\|Q\|_A \leq 8\varepsilon^{-1} \tag{6.6}$$

$$\varkappa[k \in K; |Q(k) - 1| \geq \delta] \leq \delta = 2^{-5}\varepsilon^2\eta \inf\{\|\nu\|, \|\varkappa\|, 1\} \tag{6.7}$$

$$\nu_L\left[|Q(g)| \geq \frac{\varepsilon}{2}\right] \leq \delta. \tag{6.8}$$

If $\nu_L \equiv 0$ we shall set $Q \equiv 1$ and (6.6), (6.7) and (6.8) are trivially verified.

On the other hand, by the definition of Θ we see that we can find a net $\{p_\gamma \in A(G)\}_{\gamma\in\Gamma}$ such that

$$\|p_\gamma\|_A \leq 1, \quad \gamma \in \Gamma, \tag{6.9}$$

$$p_\gamma \xrightarrow[\gamma\in\Gamma]{} 1 \text{ for the topology } \sigma(L^\infty(G; \varkappa), L^1(G; \varkappa)) \tag{6.10}$$

$$p_\gamma \xrightarrow[\gamma\in\Gamma]{} \theta \text{ for the topology } \sigma(L^\infty(G; \nu), L^1(G; \nu)). \tag{6.11}$$

We have then

$$\lim_{\gamma\in\Gamma}\left|\int_G Qp_\gamma\psi\, d\nu\right| = \left|\int_G Q\theta\psi\, d\nu\right| = \left|\int_L Q\theta\psi\, d\nu\right| \leq 8\varepsilon^{-1}\delta + \frac{\varepsilon}{2}\|\nu\| \leq \frac{3}{4}\varepsilon\|\nu\| \tag{6.12}$$

by (6.5), (6.6), (6.8), (6.9), and (6.11).

On the other hand, we have

$$\lim_{\gamma\in\Gamma}\left|\int_G (Qp_\gamma - 1)\,\varphi\,d\varkappa\right| = \left|\int_G (Q-1)\,\varphi\,d\varkappa\right|$$
$$\leqslant \left|\int_{|Q-1|\leqslant\delta}(Q-1)\,\varphi\,d\varkappa\right| + \left|\int_{|Q-1|>\delta}(Q-1)\,\varphi\,d\varkappa\right| \tag{6.13}$$
$$\leqslant \delta\,\|\varkappa\| + \delta(8\,\varepsilon^{-1}+1) \leqslant \tfrac{3}{4}\,\eta\,\|\varkappa\|$$

by (6.6), (6.7), (6.9), and (6.10). So putting together (6.12) and (6.13) we obtain a contradiction with (6.2) and complete the proof of our theorem.

Proof of Theorem 2. Let G, K, and E be as in Theorem 2 and let us suppose in contradiction that $\varepsilon > 0$ is some positive number for which the conclusion of Theorem 2 fails. Arguing as in the proof of Theorem 1 we conclude from that that there exist two non zero positive Radon measures $\varkappa \in M^+(K)$ and $\nu \in M^+(G)$ and two Borel functions on G, φ and ψ of modulus 1 ($|\varphi| = |\psi| = 1$) such that for each f in the unit ball of $A(G)$ ($f\in A(G)$, $\|f\|_A \leqslant 1$) we have

either
$$\left|\int_G (f-1)\,\varphi\,d\varkappa\right| \geqslant \varepsilon\,\|\varkappa\| \tag{6.14}$$
or
$$\left|\int_G f\psi\,d\nu\right| \geqslant \varepsilon\,\|\nu\|.$$

We shall also suppose, as we may, that the support of $\varkappa$ is a totally disconnected set; indeed we can approach arbitrarily close any Radon measure on a metrisable compact space by another measure having as support a totally disconnected set.

Let us construct again the semigroup $\Theta(K;\varkappa,\nu)$ as in paragraph 5.

It is then clear that $0\in\Theta(K;\varkappa,\nu)$; indeed, if not by Proposition 4.1 and (4.7) we would be able to find some non zero $\nu_1 \in M^+(G)$ such that

$$0 < \nu_1 < \nu, \quad \Theta(K;\varkappa,\nu_1) = \{1\},$$

and this would contradict Proposition 5.2 (Remark 5.1) since

$$\operatorname{supp}\nu_1 \cap \operatorname{Gp}(K) = \emptyset.$$

Using the definition of $\Theta(K;\varkappa,\nu)$ we see that we can find $\{f_\alpha \in A(G)\}_{\alpha\in A}$ a net of functions such that

$$\|f_\alpha\|_A \leqslant 1, \quad \forall \alpha\in A,$$
$$f_\alpha \xrightarrow[\alpha\in A]{} 1 \quad \text{for the topology} \quad \sigma(L^\infty(G;\varkappa); L^1(G;\varkappa))$$
$$f_\alpha \xrightarrow[\alpha\in A]{} 0 \quad \text{for the topology} \quad \sigma(L^\infty(G;\nu); L^1(G;\nu)).$$

This, of course, contradicts (6.14) and proves our theorem.

7. Proof of the corollaries

Proof of Corollary 1. Let $K, H \subset G$ be as in Corollary 1 and let $\lambda \in M(K \cup H)$ be such that $\|\lambda\| = 1$. We shall distinguish two cases.

Case A.
$$|\lambda|(K \setminus H) \leqslant \frac{\alpha}{2(1+\alpha)}$$

[$|\lambda|$ denotes the total variation of λ which is a positive Radon measure on G.]

We have then

$$\|\hat{\lambda}\|_\infty \geqslant \alpha\left(1 - \frac{\alpha}{2(1+\alpha)}\right) - \frac{\alpha}{2(1+\alpha)} = \frac{\alpha}{2}.$$

Case B.
$$|\lambda|(K \setminus H) > \frac{\alpha}{2(1+\alpha)}.$$

By the hypothesis we can find then some $\chi \in \hat{G}$ such that

$$|(\lambda \xi_K)\hat{}(\chi)| > \frac{\alpha}{2(1+\alpha)}.$$

If we apply then Theorem 1 with $E \subset H \setminus K$ appropriately chosen, η sufficiently small and $\varepsilon = \frac{1}{2}\alpha(2+\alpha)^{-1}$ we can find some $f \in A(G)$ such that

$$\|f\|_A \leqslant 8\varepsilon^{-1} = 16(2+\alpha)\alpha^{-1}$$

$$\left|\int_G f\chi\,d\lambda\right| \geqslant \left|\int_K f\chi\,d\lambda\right| - \left|\int_{H\setminus K} f\chi\,d\lambda\right| \geqslant \frac{\alpha}{2(1+\alpha)} - \varepsilon\left[1 - \frac{\alpha}{2(1+\alpha)}\right] = \frac{\alpha}{4(1+\alpha)}.$$

From this we deduce at once that

$$\|\hat{\lambda}\|_\infty \geqslant \frac{\alpha^2}{64(2+\alpha)(1+\alpha)}.$$

In either Case A or Case B we have therefore

$$\|\hat{\lambda}\|_\infty \geqslant \min\left[\frac{\alpha}{2}, \frac{\alpha^2}{64(2+\alpha)(1+\alpha)}\right] = \beta(\alpha).$$

Remark 7.1. Observe that $\beta(1) = 3^{-1} \cdot 2^{-7} = (384)^{-1}$.

Proof of Corollary 2. Let $K_1, K_2, \ldots, K_p \subset G$ and $H = K_1 \cup K_2 \cup \ldots \cup K_p$ be as in Corollary 2 and let $\lambda \in M(H)$ be some Radon measure of total mass $\|\lambda\| > 1$. We must prove that $\|\hat{\lambda}\|_\infty > 1/p$.

Towards that we may suppose, by renumbering the spaces if necessary that

$$|\lambda|(K_1) > \frac{1}{p}.$$

[$|\lambda|$ denotes, as above, the total variation of λ]. This and our hypothesis implies that there exists some $\chi \in \hat{G}$ such that

$$|(\lambda \xi_{K_1})^\wedge(\chi)| > \frac{1}{p}.$$

A simple application of Theorem 2 with E an appropriate subset of $H \setminus K_1$ and ε sufficiently small implies then that there exists some $f \in A(G)$ such that

$$\|f\|_A \leqslant 1, \qquad \left| \int_G f\chi\, d\lambda \right| > \frac{1}{p}$$

and this implies our assertion that $\|\hat{\lambda}\|_\infty > 1/p$ at once and proves the corollary.

Proof of Corollary 3. We shall deduce Corollary 3 from Corollary 1 and the fact that every totally disconnected Kronecker set is a set of synthesis [9]. To do that it suffices to prove the following

LEMMA 7.1. *Let X, $Y \subset G$ be two compact subsets of the compact abelian group G and let us suppose that X and Y are both sets of synthesis of G and that the set $X \cup Y$ is a totally disconnected set of interpolation of G. Then every pseudomeasure S supported on $X \cup Y$ admits a decomposition $S = S_X + S_Y$ such that*

$$\operatorname{Supp} S_X \subset X \quad \operatorname{Supp} S_Y \subset Y. \tag{7.1}$$

Proof. Let $\Omega \supset (X \cap Y)$ be a compact nhd. of $X \cap Y$ such that the sets $X_\Omega = X \cap \complement\Omega$ and $Y_\Omega = Y \cap \complement\Omega$ are both compact. Using the total disconnectedness of X and Y we see that we can in fact find $\mathcal{N}$ some fundamental family of nhd's. Ω of $(X \cap Y)$ with the above property.

Using then the fact that $X \cup Y$ is a set of interpolation we see that we can find some $f = f_\Omega \in A(G)$ such that

$$\|f\|_A \leqslant C \quad f(x) = 0 \ \forall x \in X_\Omega; \quad f(x) = 1 \ \forall x \in (X \cup Y) \setminus X_\Omega, \tag{7.2}$$

where C is a constant independent of Ω (depending only on X and Y).

Let then $S \in \mathrm{PM}(X \cup Y)$ and let us define

$$S_Y(\Omega) = f_\Omega \cdot S; \qquad S_X(\Omega) = (1 - f_\Omega) \cdot S.$$

It is clear then by the fact that the two sets X_Ω and Y_Ω are both of synthesis that

$$\operatorname{Supp} S_Y(\Omega) \subset (X \cup Y) \setminus X_\Omega; \qquad \operatorname{Supp} S_X(\Omega) \subset (X \cup Y) \setminus Y_\Omega \tag{7.3}$$

and it is also evident from (7.2) that

$$S = S_X(\Omega) + S_Y(\Omega); \quad \|S_Y(\Omega)\|_{\mathrm{PM}} \leq C\|S\|_{\mathrm{PM}}; \quad \|S_X(\Omega)\|_{\mathrm{PM}} \leq (1+C)\|S\|_{\mathrm{PM}}. \tag{7.4}$$

But then to obtain the required decomposition (7.1) it suffices to take for S_X some weak limit point of $S_X(\Omega)$ as $\Omega \in \mathfrak{N}$, and to set $S_Y = S - S_X$; (7.3) and (7.4) imply (7.1) then at once.

Proof of Corollary 4. Taking into account Corollary 1 and Remark 7.1 we see that Corollary 4 follows from the following.

LEMMA 7.2. *There exist two Kronecker sets K_1, $K_2 \subset \mathbf{T}$ of $\mathbf{T}$ such that $K_1 + K_2 = \mathbf{T}$.*

Proof. We shall use a theorem of R. Kaufman [10] that asserts that there exists $E \subset S(D_\infty)$ (where $D_\infty = \prod_{j=1}^{\infty} \mathbf{Z}_j(2)$ is the Cantor set) a subset of 1st Baire category such that for every $f \in S(D_\infty) \setminus E$ the set $f(D_\infty) \subset \mathbf{T}$ is a Kronecker set of $\mathbf{T}$.

Let now $f_0 \in S(D_\infty)$ be such that

$$f_0(D_\infty) = \mathbf{T}$$

such a function clearly exists. Using then the above theorem of R. Kaufman we see that we can find some $f \in S(D_\infty)$ such that the following two sets

$$K_1 = f(D_\infty) \subset \mathbf{T}; \qquad (f_0 f^{-1})(D_\infty) = \{f_0(d) - f(d) \in \mathbf{T};\ d \in D_\infty\} \subset \mathbf{T}$$

are both Kronecker sets of $\mathbf{T}$.

This proves our lemma, for we have clearly then $K_1 + K_2 = \mathbf{T}$.

A close analysis of the proof of Theorem 2 shows that if the subset $K \subset G$ is supposed Kronecker the above methods yield the following more precise:

THEOREM 2′. *Let G be a compact abelian group, let $K \subset G$ be a metrisable totally disconnected Kronecker subset of G, and let $E \subset G$ be some compact subset of G such that*

$$E \cap \mathrm{Gp}\,(K) = \emptyset.$$

Then for every $\varepsilon > 0$ we can find finitely many characters $\{\chi_j \in \hat{G}\}_{j=1}^N$ and positive numbers $\{\alpha_j > 0\}_{j=1}^N$ such that

(i) $\quad |\chi_j(k) - 1| \leq \varepsilon \quad \forall k \in K, \quad j = 1, 2, \dots, N$

(ii) $\quad \sum_{j=1}^{N} \alpha_j = 1$

(iii) $\quad \left| \sum_{j=1}^{N} \alpha_j \chi_j(e) \right| \leq \varepsilon \quad \forall e \in E.$

An immediate corollary of Theorem 2′ is the following:

COROLLARY 5. *Let G be a compact abelian group and let $K \subset G$ be a compact metrisable totally disconnected Kronecker subset.*

Let also $\mu \in M(G)$ be an arbitrary Radon measure on G and let us denote by

$$\mu_K = \mu \xi_{\mathrm{Gp}(K)}$$

the restriction of μ on the Borel subset $\mathrm{Gp}(K) \subset G$.

We have then

$$\|\hat{\mu}_K\|_\infty \leqslant \|\hat{\mu}\|_\infty.$$

8. Spectral analysis in S^* and a converse of Theorem 2

In this paragraph we shall carry out a very specific construction. We shall first introduce the following definition:

Definition 8.1. Let G be some compact infinite metrisable abelian group, let $K \subset G$ be a compact totally disconnected subset of G, and let:

$$p: \mathbf{T} \times \hat{G} \to S(K)$$

be the group homomorphism defined by:

$$p[(e^{i\theta}, \chi)] = e^{i\theta} \chi|_K \in S(K).$$

We shall say that K is a Salem set of G if

$$\mathrm{Gp}\,(K) \neq G$$

and if there exists $\mu \in M^+(K)$ some probability measure on K (which we shall call a Salem measure) for which

$$\inf_{t \in \mathbf{T}, 0 \neq \chi \in \hat{G}} d_\mu(p[(t, \chi)], 1_S) > 0,$$

where d_μ is some translation invariant metric (group metric) on $S(K)$ that induces on $S(K)$ the same topology as the one induced on $S(K)$ by embedding it as a (dense) subgroup of $S^*(K; \mu)$; 1_S indicates of course the identity element of $S(K)$.

Salem's name is used because of the fact that if $K \subset G$ is some compact subset such that

$$\text{Haar measure } [\mathrm{Gp}\,(K)] = 0; \quad M_0(K) \neq \{0\} \tag{8.1}$$

(the second condition means that there exists some $0 \neq \mu \in M(K)$ such that $\hat{\mu}(x) \to 0$ when $x \to \infty$) then there exists $G_1 \subset G$ an open subgroup of G and $g_1 \in G$ such that

$$K_1 = g_1 + K \subset G_1$$

and K_1 is a Salem set of the group G_1 (in the sense of Definition 8.1). The verification is trivial and left to the reader.

Sets satisfying (8.1) have been constructed by R. Salem [11] when $G \cong \mathbf{T}$, and there are known to exist in any infinite compact metrisable abelian group [12], [13]. From this it follows easily that Salem sets in the sense of Definition 8.1 exist in every infinite metrisable compact abelian group.

Let now Γ_1 be an infinite discrete countable abelian group, let

$$K_1 \subset \hat{\Gamma}_1 = G_1$$

be some Salem set of the compact group G_1, and let $\mu_1 \in M^+(K_1)$ be some Salem measure on K_1.

By the definition of a Salem set we can identify Γ_1 with a subgroup of $S(K_1)$. Let then Γ be some countable dense (for the uniform topology) subgroup of $S(K_1)$ such that $\Gamma_1 \subset \Gamma$. Let us denote by $G = \hat{\Gamma}$ the dual group of Γ. The canonical injection

$$\Gamma \to S(K_1)$$

induces then a canonical injection

$$j\colon K_1 \to G$$

that identifies K_1 with a Kronecker subset $j(K_1) = K \subset G$ of the compact group G.

Let us denote by $\mu = j(\mu_1) \in M^+(K)$ the image by j of the measure μ_1. Let us also denote by S the group $S(K)$ with the discrete topology and by S^* the group $S(K)$ assigned with the topology induced by $S^*(K; \mu)$; Γ will then be identified to a subgroup of S. Let us also fix d some translation invariant metric on S^* which induces its topology.

Let us define

$$\pi\colon \mathbf{T} \times \Gamma \to S$$

by setting

$$\pi(e^{i\theta}, \chi) = e^{i\theta}\chi|_K \in S(K) \quad (e^{i\theta} \in \mathbf{T};\ \chi \in \hat{G} = \Gamma).$$

It follows then by the Definition 8.1 that

$$\inf_{t \in \mathbf{T};\, 1 \neq \gamma_1 \in \Gamma_1} d(\pi(t, \gamma_1), 1_S) = \delta > 0,$$

(where 1_S is the identity element of S). Let us define also

$$\Omega = \{s \in S^*;\ d(\pi(\mathbf{T}, 1_\Gamma), s) < \delta/3\}$$

which is an open nhd in S^* of the compact subgroup

$$\pi(\mathbf{T}, 1_\Gamma) = \{\pi(t, 1_\Gamma);\ t \in \mathbf{T}\} \subset S^*.$$

(We shall denote abusively the generic point of that subgroup by $\pi(e^{i\theta}) = \pi(e^{i\theta}, 1_\Gamma)$, $\theta \in \mathbf{R}$.)

Let us fix now in an arbitrary manner

$$U \in l^{\infty}(S)$$

satisfying the following three conditions:

$$U \in \mathbf{C}_u(S^*) \quad \text{(i.e. it is a uniformly continuous function on } S^*) \tag{8.2}$$

$$U(\pi(e^{i\theta})) = e^{i\theta}, \quad \theta \in \mathbf{R}. \tag{8.3}$$

$$\operatorname{Supp} U \subset \Omega. \tag{8.4}$$

Such a choice of U is of course possible by the normality of the metrisable space S^* (a direct construction is of course also possible).

Let us fix some point $x \in G_1 \setminus \operatorname{Gp}(K_1)$, which we can, by the choice of K_1, and define $f \in l^{\infty}(\Gamma_1)$ by the properties

$$f(0) = 1 \tag{8.5}$$

$$\operatorname{Sp} f = \{x\} \tag{8.6}$$

(i.e. we set $f(\gamma_1) = \langle \gamma_1, x \rangle$, $\gamma_1 \in \Gamma_1$).

Let us now observe that, by our definition of Ω, the mapping

$$\Omega \times \Gamma_1 \rightarrow S$$

defined by $$(\omega, \gamma_1) = \omega\gamma_1 \in S \quad \omega \in \Omega,\ \gamma_1 \in \Gamma_1$$

(multiplication in S) is $(1-1)$. This and (8.4) allows us to define $F \in l^{\infty}(S)$ as following:

$$F(s) = 0 \quad \forall s \notin \Omega \cdot \Gamma_1 \subset S \tag{8.7}$$

$$F(\omega\gamma_1) = U(\omega) f(\gamma_1); \quad \omega \in \Omega,\ \gamma_1 \in \Gamma_1. \tag{8.8}$$

We see then at once from (8.2), (8.3), (8.4) and (8.5) that

$$F \in \mathbf{C}(S^*) \quad \text{(i.e. } F \text{ is uniformly continuous on } S^*) \tag{8.9}$$

$$F(\pi(e^{i\theta})) = e^{i\theta}, \quad \forall \theta \in \mathbf{R}. \tag{8.10}$$

Let us finally define $P \in \operatorname{PM}(G)$ by

$$\hat{P}(\gamma) = F(\gamma) \quad \forall \gamma \in \Gamma \subset S$$

and $$E = q^{-1}(\{x\}) \subset G,$$

where $q: G \rightarrow G_1$ is the canonical projection induced by the injection $\Gamma_1 \rightarrow \Gamma$.

We have then

PROPOSITION 8.1. *The support of P is contained in E (i.e.* $\operatorname{Supp} P \subset E$*), and* $E \cap \operatorname{Gp}(K) = \emptyset$.

Proof. The second assertion is trivial so we only have to prove the fact that $\operatorname{Supp} P \subset E$. Towards that let

$$X \in \operatorname{Supp} P \subset G = \hat{\Gamma}.$$

X considered then as an element of $l^\infty(\Gamma)$ is the limit in the weak topology $\sigma(l^\infty(\Gamma), l^1(\Gamma))$ of some net

$$\{\Psi_\alpha \in l^\infty(\Gamma)\}_{\alpha \in A},$$

where for every $\alpha \in A$

$$\Psi_\alpha(\gamma) = \sum_{\sigma \in \Gamma} b_\sigma(\alpha)\, \hat{P}(\sigma\gamma);\ \gamma \in \Gamma\ (b_\sigma(\alpha) \in \mathbf{C};\ \sigma \in \Gamma, \alpha \in A)$$

is a finite linear combination of translates of $\hat{P}$.

This implies that $q(X) = X_1 = X|_{\Gamma_1} \in l^\infty(\Gamma_1)$, the restriction of X on the subgroup Γ_1 is the limit in the weak topology $\sigma(l^\infty(\Gamma_1), l^1(\Gamma_1))$ of the net

$$\{\Psi_\alpha^{(1)} = \Psi_\alpha|_{\Gamma_1} \in l^\infty(\Gamma_1)\}_{\alpha \in A}.$$

But we have then by (8.7), (8.8) and the definition of P that

$$\Psi_\alpha^{(1)}(\gamma_1) = \sum_{\sigma \in \Gamma} b_\sigma(\alpha)\, F(\sigma\gamma_1) = \sum_{\sigma \in \Omega\Gamma_1} b_\sigma(\alpha)\, F(\sigma\gamma_1) = \sum_{\sigma_1 \in \Gamma_1,\, \omega \in \Omega} b_{\omega\sigma_1}(\alpha)\, F(\omega\sigma_1\gamma_1)$$

$$= \sum_{\sigma_1 \in \Gamma_1} \Big(\sum_{\omega \in \Omega} b_{\omega\sigma_1}(\alpha)\, U(\omega)\Big) f(\sigma_1\gamma_1) = \sum_{\sigma_1 \in \Gamma_1} b_{\sigma_1}^{(1)}(\alpha)\, f(\sigma_1\gamma_1) \quad (\gamma_1 \in \Gamma_1),$$

which allows us to conclude that $X_1 \in \operatorname{Sp} f$, in other words that $q(X) = x$, and complete the proof of the proposition.

Let us now denote

$$\Phi = \{e^{i\theta}\chi \in S(G);\ \theta \in \mathbf{R}\ \chi \in \hat{G}\} \subset S(G).$$

It is clear then that $\Phi|_K = \{\varphi|_K;\ \varphi \in \Phi\} \subset S$ is no other than $\pi(\mathbf{T} \times \Gamma) \subset S$.

We have then

PROPOSITION 8.2. *Let $\{\varphi_\nu \in \Phi\}_{\nu=1}^\infty$ be a sequence such that*

$$\varphi_\nu|_K = \pi(e^{i\theta_\nu}, \gamma_\nu) \underset{\nu\to\infty}{\longrightarrow} 1 \ \text{ in } S^* \quad (\theta_\nu \in \mathbf{R}, \gamma_\nu \in \Gamma).$$

Then

$$\langle \varphi_\nu, P\rangle \underset{\nu\to\infty}{\longrightarrow} 1,$$

where, of course, we identify φ_ν with an element of $A(G)$.

Proof. Our hypothesis implies that

$$d(\gamma_\nu, \pi(e^{-i\theta_\nu})) \underset{\nu\to\infty}{\longrightarrow} 0$$

(where we identify, as usual, $\gamma_\nu \in \Gamma$ with an element of S^*). But (8.9), (8.10) and the definition of P imply then that

$$|\hat{P}(\gamma_\nu) - e^{-i\theta_\nu}| = |F(\gamma_\nu) - e^{-i\theta_\nu}| \underset{\nu\to\infty}{\longrightarrow} 0$$

which gives, of course, at once as required

$$\langle \varphi_\nu, P\rangle = e^{i\theta_\nu}\hat{P}(\gamma_\nu) \underset{\nu\to\infty}{\longrightarrow} 1$$

and completes the proof of the proposition.

If we denote then by $\mathcal{N}$ the nhd filter of the identity $1_S \in S^*(G;\mu)$ for the topology of $S^*(G;\mu)$ we can restate what has been obtained up to now as the following:

PROPOSITION 8.3. *K is a compact totally disconnected Kronecker subset of G, $\mu \in M^+(K)$ is some probability measure on K, $P \in \mathrm{PM}\,(G)$ is a pseudomeasure on G, $E \subset G$ is the coset of some compact subgroup of G and we have*

$$\operatorname{Supp} P \subset E;\ E \cap \operatorname{Gp}(K) = \emptyset \tag{8.11}$$

$$\varphi \in \Phi,\ \varphi \underset{\mathcal{N}}{\longrightarrow} 1 \Rightarrow \langle P, \varphi\rangle \to 1. \tag{8.12}$$

Remark 8.1. We have of course an enormous freedom of choice for the group G.

If $\Gamma_1 \cong \mathbf{Z}$ we can take $\Gamma \cong \mathbf{Z}^2$ and therefore $G = \mathbf{T}^2$. If G_1 is totally disconnected we can choose Γ such that G is still totally disconnected. To see (for instance) the last point let

$$Q = \{q_j\}_{j=1}^\infty, \quad R = \{r_j\}_{j=1}^\infty$$

be two infinite sequences of (not necessarily distinct) primes such that $Q \cap R = \emptyset$. Let $\Gamma_1 \cong \sum_{j=1}^\infty \mathbf{Z}(r_j)$ $(\Gamma_1 \subset S(K_1))$ and let $\Gamma_2 \subset S(K_1)$ be some subgroup that is dense for the uniform topology of $S(K_1)$ and $\Gamma_2 \cong \sum_{j=1}^\infty \mathbf{Z}(q_j)$. We can set then

$$\Gamma = \Gamma_1 + \Gamma_2 \subset S(K_1).$$

We shall now prove the following

PROPOSITION 8.4. *Let G, K, $\mu \in M^+(G)$, $P \in \mathrm{PM}\,(G)$ be as in Proposition 8.3, and satisfy (8.11) and (8.12), let further $\{f_n \in A(G)\}_{n=1}^\infty$ be a sequence of functions such that*

$$\int f_n\, d\mu \underset{n\to\infty}{\longrightarrow} 1; \quad \langle P, f_n\rangle \underset{n\to\infty}{\longrightarrow} 0.$$

Then
$$\liminf_{n\to\infty} \|f_n\|_{A(G)} > 1.$$

Proposition 8.4 shows that Theorem 2 cannot be improved in the sense that the uniform topology involved there cannot be improved to the stronger topology of $A(E)$ (i.e. we cannot replace in general (iii) in Theorem 2 by the stronger

(iii)* $\|f|_E\|_{A(E)} \leqslant \varepsilon$.

(We can do that, however, when K is countable as is stated in Proposition 8.6 at the end of this paragraph.)

Another immediate corollary of Proposition 8.4 is the following Proposition 8.5 which will be crucial for the proof of Theorem 3.

PROPOSITION 8.5. *Let again G, K, E be as in Proposition 8.3 and let us denote by $L = K \cup E$ and by $l = \xi_K \in C(L)$ the characteristic function of K in L. We have then*

$$\|l\|_{A(L)} = \inf \{\|\tilde{l}\|_{A(G)};\ \tilde{l} \in A(G), \tilde{l}|_L = l\} > 1.$$

Proof of Proposition 8.4. Proposition 8.4 is an immediate consequence of Proposition 8.3 and the following

LEMMA 8.1. *Let G, K and $\mu \in M^+(G)$ be as in Proposition 8.3 and let $S \in PM(G)$ be such that there exists $\{f_n \in A(G)\}_{n=1}^{\infty}$ a sequence of functions such that*

$$\|f_n\|_{A(G)} \leqslant 1 + \frac{1}{n} \tag{8.13}$$

$$\int f_n \, d\mu \xrightarrow[n\to\infty]{} 1 \tag{8.14}$$

$$\langle S, f_n \rangle \xrightarrow[n\to\infty]{} 0. \tag{8.15}$$

Then for every $\varepsilon > 0$ and every $N \in \mathcal{N}$ we can find finitely many $\{\varphi_\nu \in \Phi\}_{\nu=1}^{m}$ and finitely many positive numbers $\{\alpha_\nu > 0\}_{\nu=1}^{m}$ such that

$$\Big|\sum_{\nu=1}^{m} \alpha_\nu - 1\Big| \leqslant \varepsilon \tag{8.16}$$

$$\Big|\sum_{\nu=1}^{m} \alpha_\nu \langle S, \varphi_\nu \rangle\Big| \leqslant \varepsilon \tag{8.17}$$

$$\varphi_\nu \in N;\ \nu = 1, 2, \ldots, m. \tag{8.18}$$

($\mathcal{N}$ is as before the nhd filter of 1 in $S^*(G; \mu)$.)

Proof. Let us denote by

$$f_n = \sum_{\varphi \in \Phi} \alpha_n(\varphi)\, \varphi;\ \alpha_n(\varphi) \geqslant 0,\ \varphi \in \Phi;\ \sum_{\varphi \in \Phi} \alpha_n(\varphi) \leqslant 1 + \frac{1}{n}$$

the functions that satisfy (8.13), (8.14), and (8.15) (the expansions are not unique). We have then the real part of f_n

$$\mathcal{R} f_n = \sum_{\varphi \in \Phi} \alpha_n(\varphi) \mathcal{R}\varphi$$

and, of course, also $\mathcal{R}\varphi \leqslant 1 \quad \forall \varphi \in \Phi.$

$$\int \mathcal{R} f_n d\mu = \sum_{\varphi \in \Phi} \alpha_n(\varphi) \int \mathcal{R}\varphi\, d\mu = \sum_{\varphi \in \Phi} \alpha_n(\varphi)\, \varphi^* \xrightarrow[n\to\infty]{} 1, \tag{8.19}$$

where $\varphi^* = \int \mathcal{R}\varphi\, d\mu \leqslant 1.$

Let now $\delta > 0$ be some fixed positive number and let us denote

$$A_n(\delta) = \sum_{\varphi^* < 1-\delta} \alpha_n(\varphi); \quad B_n(\delta) = \sum_{\varphi^* \geqslant 1-\delta} \alpha_n(\varphi).$$

We have then
$$A_n(\delta) + B_n(\delta) = \sum_{\varphi \in \Phi} \alpha_n(\varphi) \leqslant 1 + \frac{1}{n} \tag{8.20}$$

and we also have by (8.19) that

$$1 - \eta(n) \leqslant \sum_{\varphi \in \Phi} \alpha_n(\varphi)\, \varphi^* \leqslant (1-\delta)\, A_n(\delta) + B_n(\delta), \qquad \eta(n) \xrightarrow[n\to\infty]{} 0. \tag{8.21}$$

Adding (8.20) and (8.21) we conclude that

$$\delta A_n(\delta) \leqslant \frac{1}{n} + \eta(n)$$

which means of course that (for every fixed δ)

$$A_n(\delta) \xrightarrow[n\to\infty]{} 0. \tag{8.22}$$

If we denote then for every fixed $\delta > 0$

$$f_{n,\delta} = \sum_{\varphi^* \geqslant 1-\delta} \alpha_n(\varphi)\, \varphi$$

we conclude from (8.22) and (8.15) that

$$\langle S, f_{n,\delta} \rangle \xrightarrow[n\to\infty]{} 0 \quad (\delta \text{ fixed}). \tag{8.23}$$

On the other hand, for every $N \in \mathcal{N}$ there exists some $\delta_N > 0$ small enough such that

$$\varphi^* \geqslant 1 - \delta_N \Rightarrow \varphi \in N. \tag{8.24}$$

Also (8.13), (8.19) and (8.22) imply that

$$1 + \frac{1}{n} \geqslant \sum_{\varphi^* \geqslant 1-\delta_N} \alpha_n(\varphi) \geqslant \sum_{\varphi^* \geqslant 1-\delta_N} \alpha_n(\varphi)\, \varphi^* \xrightarrow[n\to\infty]{} 1. \tag{8.25}$$

So (8.23), (8.24) and (8.25) put together imply at once (8.17), (8.18) and (8.16) respectively. The proof of the lemma is complete.

It is worth noting, however, that when K is countable, Theorem 2 does admit an improvement, as following

PROPOSITION 8.6. *Let G be a compact abelian group, let $K \subset G$ be a countable closed independent subset and let $E \subset G$ be a compact subset such that*

$$E \cap \mathrm{Gp}(K) = \varnothing.$$

Then for every $\varepsilon > 0$ there exists $f \in A(G)$ such that

(i) $\|f\|_{A(G)} \leqslant 1 + \varepsilon$

(ii) $f(k) = 1 \quad \forall k \in K$

(iii) $f(e) = 0 \quad \forall e \in E.$

The proof is not entirely trivial but will be omitted because we feel that the proposition is not of any particular importance.

Remark 8.2. Condition (iii) in Proposition 8.6 can even be improved to

(iii)′ $f \in J(E)$ = the smallest ideal of $A(G)$ on E.

9. Bounded synthesis, tilda algebras and proof of Theorem 3

Let G be a compact group and $E \subset G$ be a compact subset of G and let us denote

$$A(E) = A(G)/I(E); \qquad I(E) = \{f \in A(G);\ f^{-1}(0) \supset E\}$$

$$\tilde{A}(E) = \{f \in \mathrm{C}(E);\ \exists\, \{f_n \in A(G)\}_{n=1}^{\infty};\ \sup_n \|f_n\|_A < +\infty,\ \sup_{f \in E} |f_n(e) - f(e)| \xrightarrow[n\to\infty]{} 0\}.$$

Both $A(E)$ and $\tilde{A}(E)$ are Banach Algebras under their canonical norm and $A(E)$ can be identified to a subalgebra of $\tilde{A}(E)$ (cf. [14]).

We introduce

Definition 9.1. We shall say that $E \subset G$ a compact subset of G is a subset of bounded synthesis of G if for every $S \in \mathrm{PM}(E)$ there exists a sequence $\{\mu_n \in M(E)\}_{n=1}^{\infty}$ of measures such that

$$\mu_n \xrightarrow[n\to\infty]{} S \quad \text{in the topology} \quad \sigma(\mathrm{PM}(G), A(G)).$$

(Observe that then by Banach–Steinhaus' theorem we have $\sup_n \|\hat{\mu}_n\|_\infty < +\infty$.)

We shall say that E is a set of 1-synthesis if for every $S \in \mathrm{PM}(E)$ there exists a sequence of measures $\{\mu_n \in M(G)\}_{n=1}^{\infty}$ such that

$$\|\hat{\mu}_n\|_\infty \leqslant \|\hat{S}\|_\infty; \quad \mu_n \xrightarrow[n\to\infty]{} S \quad \text{in the topology} \quad \sigma(\mathrm{PM}(G), A(G)).$$

We have then

PROPOSITION 9.1. *Let $E \subset G$ be a compact subset of synthesis of G (i.e. $I(E)=J(E)$). Then*

(i) *E is a subset of bounded synthesis of G if and only if $A(E)$ is a closed subalgebra of $\tilde{A}(E)$.*

(ii) *E is a subset of 1-synthesis of G if and only if the identification of $A(E)$ to a subalgebra of $\tilde{A}(E)$ is isometric.*

Proof. It is a straightforward application of Hahn–Banach's theorem and it has been outlined in [14].

Remark 9.1. Analogous definitions ($V(E)$, $\tilde{V}(E)$) exist for a compact subset $E \subset K_1 \times K_2$ of the carrier space of a tensor algebra $V = \mathrm{C}(K_1) \hat{\otimes} \mathrm{C}(K_2)$. The analogue of Proposition 9.1 also holds (cf. [14]).

PROPOSITION 9.2. *Let G_i $(i=1, 2, ..., m)$ be compact abelian groups, let $E_i \subset G_i$ $(1 \leqslant i \leqslant m)$ be a compact subset of G_i and let us denote by*

$$E = E_1 \times E_2 \times \ldots \times E_m \subset G_1 \times G_2 \times \ldots \times G_m = G$$

their cartesian product. Let us suppose that either (*) *or* (**) *below (or both) is satisfied:*

(*) *$A(E_i)$ is a closed subalgebra* of *$\tilde{A}(E_i)$ $(1 \leqslant i \leqslant m)$.*

(**) *E is a set of synthesis in G.*

Let further $f_i \in A(E_i)$ be arbitrary functions $(1 \leqslant i \leqslant m)$ and let us denote

$$f = f_1 \otimes f_2 \otimes \ldots \otimes f_m \in A(E).$$

We have then $$\|f\|_{\tilde{A}(E)} \leqslant \prod_{i=1}^{m} \|f_i\|_{\tilde{A}(E_i)}; \quad \|f\|_{A(E)} = \prod_{i=1}^{m} \|f_i\|_{A(E_i)}.$$

Proof. All but the inequality

$$\|f\|_{A(E)} \geqslant \prod_{i=1}^{m} \|f_i\|_{A(E_i)} \tag{9.1}$$

are trivial. We shall prove (9.1).

Let $S_i \in PM(E_i)$ be some synthesisable pseudomeasures on E_i such that we have for some small $\varepsilon > 0$

$$\|S_i\|_{PM} \leqslant 1; \; |\langle f_i, S_i \rangle| \geqslant \|f_i\|_{A(E_i)} - \varepsilon \quad (1 \leqslant i \leqslant m).$$

Then, provided, that either (∗) of (∗∗) holds,

$$S = S_1 \otimes S_2 \otimes \ldots \otimes S_m \in PM(E)$$

is a synthesisable pseudomeasure of $E \subset G$, which then clearly satisfies

$$\|S\| \leqslant 1, \; |\langle f, S \rangle| \geqslant \prod_{i=1}^{m} (\|f_i\|_{A(E_i)} - \varepsilon)$$

and ε being arbitrary, (9.1) follows.

Remark 9.2. The conditions (∗) and (∗∗) are of course connected with the Banach approximation property and the existence of a basis for the Banach spaces involved (cf. [15], Ch. I, § 5).

PROPOSITION 9.3. *Let $E \subset G$ be a compact subset of G that is not a set of 1-synthesis of G. Then the countable cartesian product of E with itself*

$$E^{\omega} = E \times E \times \ldots \subset G^{\omega} = G \times G \times \ldots$$

is not a set of bounded synthesis in G^{ω}.

Proof. We may suppose without loss of generality that E^{ω} is a set of synthesis in G^{ω}, for otherwise there is nothing to prove. This implies then easily that for every m $(1 \leqslant m < +\infty)$ the mth cartesian power of E

$$E^m = E \times E \times \ldots \times E \subset G^m = G \times G \times \ldots \times G$$

is a set of synthesis in G^m. Our Proposition 9.2 therefore applies. The hypothesis, on the other hand, implies that there exists some $\varepsilon > 0$ and some $f \in A(E)$ such that

$$\|f\|_{A(E)} \geqslant 1 + \varepsilon; \; \|f\|_{\tilde{A}(E)} \leqslant 1.$$

But if we denote then for every $m (1 \leqslant m < +\infty)$

$$f^{(m)} = f \otimes f \otimes \ldots \otimes f \otimes 1 \otimes 1 \otimes \ldots \in A(E^{\omega})$$

(the f appearing m times in the infinite tensor product) we see that

$$\|f^{(m)}\|_{\tilde{A}(E^{\omega})} \leqslant 1 \quad (1 \leqslant m < +\infty) \tag{9.2}$$

$$\|f^{(m)}\|_{A(E^{\omega})} \geqslant \|f \otimes f \otimes \ldots \otimes f\|_{A(E^m)} \geqslant (1 + \varepsilon)^m \tag{9.3}$$

and (9.2), (9.3) together with Proposition 9.1 prove our proposition.

PROPOSITION 9.4. *Let G be some compact abelian group, let $E \subset G$ be a compact subset of G and let us denote by*

$$E^* = \{(x, y) \in G \times G;\ x+y \in E\} \subset G \times G.$$

Then

(i) *E is a subset of synthesis for the group algebra $A(G)$ if and only if E^* is a subset of synthesis for the tensor algebra $V(G) = \mathbf{C}(G) \hat{\otimes} \mathbf{C}(G)$.*

(ii) *If $E \subset G$ is not a set of bounded synthesis of G (for the group algebra $A(G)$) then $E^* \subset G \times G$ is not a set of bounded synthesis of the tensor algebra $V(G)$.*

Proof. (i) is no other than Theorems 8.2.1 and 8.2.2 in [15], as for (ii) it is immediate and therefore left to the reader (cf. Ch. 8 in [15]).

PROPOSITION 9.5. *Let $E \subset D_\infty \times D_\infty$ (cf. Ch. 2, § 4 in [15]) be a compact subset that is of synthesis but not of bounded synthesis for the tensor algebra $V(D_\infty) = \mathbf{C}(D_\infty) \hat{\otimes} \mathbf{C}(D_\infty)$. Let G be some compact abelian group and let $K_1, K_2 \subset G$ be two compact subsets both topologically homeomorphic to D_∞ ($K_1 \cong D_\infty \cong K_2$) and such that*

$$K_1 \cap K_2 = \emptyset; \quad K_1 \cup K_2 \text{ is a Kronecker subset of } G.$$

Let us identify $D_\infty \times D_\infty$ with $K_1 + K_2 \subset G$ in such a way that $V(D_\infty)$ is identified isometrically with $A(K_1 + K_2)$ and let $E' \subset K_1 + K_2$ be the subset that corresponds to E under the above identification (cf. Ch. 4 [15]).

Then the subset E' of G is a set of synthesis of G but not of bounded synthesis.

Proof. It is a clear corollary of Theorem 4.4.2 of [15].

We are now in a position to give the

Proof of Theorem 3. Let us preserve all the notations of § 8, let G, $K \subset G$, $E \subset G$ be as in Proposition 8.5 and let us suppose in addition that G is totally disconnected (cf. Remark 8.1). It is clear then that the set $L = K \cup E \subset G$ is a set of synthesis of G but not of 1-synthesis. It is also clear that the algebra $A(L) \cong A(K) \oplus A(E)$ has a basis (qua Banach space). The argument in Ch. 1, § 5 of [15] shows then that for every $m \geq 1$ ($m \in \mathbf{Z}$) the set $L^m = L \times L \times \ldots \times L$ is a set of synthesis in G^m. This in turn, by Proposition 9.3, implies that $L^\omega = L \times L \times \ldots$ is a set of synthesis but not of bounded synthesis in the group $G^\omega = G \times G \times \ldots$.

But then Proposition 9.4 and 9.5 put together prove that in every compact abelian group that has perfect Kronecker subsets there exists some set that is of synthesis but but not of bounded synthesis (observe that G^ω is totally disconnected and therefore $V(G^\omega) \cong V(D_\infty)$). For groups with no perfect Kronecker subsets we can use K_p sets instead and the proof goes through the same way.

Addendum

While this paper was with the press S. Drury proved that the union of two countable sets of interpolation is a set of interpolation (C.R.A.S. 271 (1971), 162–163). Using one of his ideas and Theorem 1 (resp. Theorem 2) in this paper I have proved Theorem 4 below which implies of course that in general the union of two sets of interpolation is a set of interpolation (cf. § 7).

THEOREM 4. *Let G be a compact abelian group and let $K \subset G$ be a metrisable H_α $(0 < \alpha \leqslant 1)$ set of G, let also $E \subset G$ be a compact subset such that:*

$$E \cap K = \varnothing \quad (\textit{resp. } E \cap \mathrm{Gp}(K) = \varnothing)$$

Let also $0 < \varepsilon < 1$ be given. Then we can find some $f \in A(G)$ such that:

(i) $\|f\|_A \leqslant C_\alpha \varepsilon^{-1}$ (resp. $\|f\|_A \leqslant C_\alpha$),

(ii) $|f(k) - 1| \leqslant C_\alpha \varepsilon \quad \forall k \in K$,

(iii) $|f(e)| \leqslant C_\alpha \varepsilon \quad \forall e \in E$;

where C_α (here and below) denotes a constant depending only on α (and not on G, H, E or ε).

We shall give here the proof in the case $E \cap K = \varnothing$ (the case $E \cap \mathrm{Gp}(K) = \varnothing$ is treated analogously) and for the proof we shall suppose in addition that K is totally disconnected (the general case can be deduced from this as in the proof of Theorem 2).

Proof. Let K and E be as above and let $\varepsilon > 0$ be given, let also H be a torsion (i.e. every element is of finite order) subgroup of $S(K)$, dense in $S(K)$ for the uniform topology. Let us give H the discrete topology and denote by $\hat{H}$ the compact dual group of H.

For every subgroup L of H and every $k \in K$ we can define $\hat{k}_L \in \hat{L}$ a character of L by setting

$$\langle l, \hat{k}_L \rangle = l(k), \quad \forall l \in L \subset S(K).$$

We can then identify K with a Kronecker set of the group $G \times \hat{H}$ by the mapping:

$$k \to (k, \hat{k}_H) \in G \times \hat{H}.$$

Using Theorem 1 we see that we can find some $\varphi \in A(G \times \hat{H})$ such that $\|\varphi\|_A \leqslant 8\varepsilon^{-1}$, $\varphi(k, \hat{k}_H) = 1$ for all $k \in K$, and also such that $|\varphi(e, \chi)| \leqslant \varepsilon$ for all $e \in E$ and $\chi \in \hat{H}$. Let us denote by

$$\hat{\varphi}(g, h) = \int_{\hat{H}} \varphi(g, \hat{h}) \overline{\langle h, \hat{h} \rangle} \, d\hat{h} \in A(G) \hat{\otimes} l^1(H); \quad g \in G, \; h \in H$$

the partial Fourier transform of φ. It is then clear that if we chose L a "sufficiently large" finite subgroup of H and consider the function

$$\psi(g,\hat{l}) = \sum_{l\in L} \hat{\varphi}(g,l)\langle l,\hat{l}\rangle \in A(G\times \hat{L}); \quad g\in G, \hat{l}\in \hat{L},$$

then the function ψ satisfies

(α) $\|\psi\|_A \leqslant 8\varepsilon^{-1}$

(β) $|\psi(k,\hat{k}_L)-1| \leqslant \varepsilon \quad \forall k\in K$

(γ) $|\psi(e,\chi)| \leqslant 2\varepsilon \quad \forall e\in E,\ \chi\in \hat{L}.$

Let us now choose in an arbitrary fashion functions $\{f_l\in A(G)\}_{l\in L}$ such that:

$$\|f_l\|_A \leqslant C_\alpha; \quad f_l(k) = l(k) \quad \forall k\in K,\ l\in L,$$

and let us set:

$$\psi_l(g) = \frac{1}{\operatorname{Card} L} \sum_{l_1\in L} f_{ll_1^{-1}}(g) f_{l_1}(g) \in A(G); \quad g\in G,\ l\in L$$

where ll_1^{-1} is the group operation of L (this idea is borrowed from Drury's work). We have then

$$\|\psi_l\|_A \leqslant C_\alpha,\ \psi_l(k) = l(k); \quad \forall l\in L,\ k\in K$$

and (just as in Drury's case) there exists a family of measures $\{\mu_g\in M(\hat{L})\}_{g\in G}$ such that

$$\psi_l(g) = \int_{\hat{L}} \langle l,\hat{l}\rangle\, d\mu_g(\hat{l});\ \|\mu_g\| \leqslant C_\alpha; \quad g\in G,\ l\in L.$$

We shall set

$$f(g) = \sum_{l\in L} \psi_l(g)\hat{\varphi}(g,l) \in A(G), \quad g\in G$$

f satisfies all the conditions of our Theorem. Indeed it is evident from the fact that the norm of $\hat{\varphi}(g,h)$ in $A(G)\hat{\otimes} l^1(H)$ is bounded by $C_\alpha\varepsilon^{-1}$ that we have $\|f\|_A \leqslant C_\alpha\varepsilon^{-1}$. But we also have:

$$f(k) = \sum_{l\in L} \psi_l(k)\hat{\varphi}(k,l) = \sum_{l\in L} \langle l,\hat{k}_L\rangle\, \hat{\varphi}(k,l) = \psi(k,\hat{k}_L); \quad \forall k\in K$$

$$f(e) = \sum_{l\in L}\psi_l(e)\hat{\varphi}(e,l) = \sum_{l\in L}\left(\int_{\hat{L}} \langle l,\hat{l}\rangle d\mu_e(\hat{l})\right)\hat{\varphi}(e,l) = \int_{\hat{L}} \Big(\sum_{l\in L}\langle l,\hat{l}\rangle\, \hat{\varphi}(e,l)\Big) d\mu_e(\hat{l})$$

$$= \int_{\hat{L}} \psi(e,\hat{l})\, d\mu_e(\hat{l}) \quad \forall e\in E$$

and (β) and (γ) above give the required result.

Note: I gave a different proof of Theorem 4 (C.R.A.S. 271 (1970) to appear) valid only for $G = \mathbf{T}^n$ ($n\geqslant 1$). However, the constants $C_{\alpha,n}$ obtained there depend on n, and the proof there does not generalise to general groups.

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